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Emergent Geometry and Gauge Theory in 4 Dimensions and The Noncommutative Torus

Par

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Abstract

A detailed Monte Carlo calculation of the phase diagram of bosonic massdeformedIKKT Yang-Mills matrix models in three and six dimensions with quartic mass deformations is given. Background emergent fuzzy geometries in two and four dimensions are observed with a uctuation given by a noncommutative U(1) gauge theory very weakly coupled to normal scalar fields.

The geometry, which is determined dynamically, is given by the fuzzy spheres S_2^N and $S_2^N \times S_2^N$ respectively. The three and six matrix models are efectively

in the same universality class. For example, in two dimensions the geometry is completely stable, whereas in four dimensions the geometry is stable only in the limit $M \to \infty$, where M is the mass of the normal uctuations. The behavior of the eigenvalue distribution in the two theories is also different.

We also sketch how we can obtain a stable fuzzy four-sphere $S_2^N \times S_2^N$ in the large N limit for all values of M as well as models of topology change in which the transition between spheres of different dimensions is observed.

The stable fuzzy spheres in two and four dimensions act precisely as regulators which is the original goal of fuzzy geometry and fuzzy physics. Fuzzy physics and fuzzy theory on these spaces are briey discussed.

Key words :

Emergent geometry, Matrix Models, Fuzzy, Gauge Theory, Mulitrace, Monte Carlo, hybrid Monte Carlo.

Résumé

Un calcul de Monte Carlo détaillé pour le diagramme de phase des modéles matriciels de type Yang-Mills IKKT bosonique déformée en masse dans trois et six dimensions avec des déformations de masse quartique est aperues. Les géométries floues émergentes de fond dans deux et quatre dimensions sont observées avec une fluctuation donne par une théorie de gauge non commutative U(1) trés faiblement couplée aux champs scalaires normaux. La géométrie, déterminée dynamiquement, est donnée par les sphéres floues S_2^N et $S_2^N \times S_2^N$.
Les trois et six modéles matriciels sont effectivement dans la même classe d'universalité. Par exemple, en deux dimensions, la géométrie est complétement stable, alors que dans quatre dimensions, la géométrie est stable seulement pour la limite $M \to \infty$, ou M est la masse des fluctuations normales. Le comportement de la distribution des valeurs propres dans les deux théories est également différent. Nous décrivons également comment nous pouvons obtenir une $S_2^N \times S_2^N$ floues a quatre sphéres stable dans la grande limite N pour toutes les valeurs de M ainsi que des modéles de changement de topologie pour lesquels la transition entre les sphéres de différentes dimensions est observée. Les sphéres floues stables en deux et quatre dimensions agissent précisément comme des régulateurs qui est l'objectif initial de la géométrie floue (fuzzy geometry) et de la physique floue (fuzzy physics).

Mots clés :
La géométrie émergente, Modèles matriciels, la physique floue, Théorie de jauge, Multitrace , Monte Carlo, Hybride Monte Carlo.

ملخص

في هذا العمل قمنا بحسابات مونتي كارلو مفصلة لمنحنى الطور للبوزون مشوه-الكتلة لنماذج المصفوفات IKKT يانغ ميلز في ثلاثة و ستة أبعاد مع تشوهات كتلية من الدرجة الرابعة. الهندسات الفضائية الغامضة الناشئة من بعدين و أربعة أبعاد تمت ملاحظتها مع تقلبات معطاة بنظرية معياريةغير تبديلية U(1) مقترنة بصفة ضعيفة بحقول سلمية نظامية.
الهندسة المحددة ديناميكيا معطاة بالكرات الغامضة S_N^2 و $S_N^2 \times S_N^2$
نماذج المصفوفات ثلاثية و سداسية الأبعاد فعليا لها نفس المرتبة. مثال , في بعدين الفضاء مستقر تماما بينما في أربعة الفضاء مستقر فقط من أجل القيم الكبيرة ل M لكتلة التقلبات النظامية.
سلوك توزيعات القيم الذاتية في النظريتين مختلف أيضا. كذلك حددنا كيفية الحصول على كرات غامضة $S_N^2 \times S_N^2$ في النهايات الكبيرة ل M . من أجل كل قيم M بالاضافة الى ملاحظة نماذج تغير الطوبولوجي التي من أجلها تكون التحولات بين الكرات المختلفة الأبعاد.
الكرات الغامضة المستقرة في بعدين و أربعة أبعاد تعمل بدقة كمعدلات وهو الهدف الأصلي للهندسة الفضاء الغامضة و الفيزياء الغامضة.

الكلمات المفتاحية :

الهندسة الناشئة , نماذج المصفوفات, الفيزياء الغامضة, نظرية القياس, المونتي كارلو, المونتي كارلو الهجين.

Emergent Geometry and Gauge Theory in 4 Dimensions and The Noncommutative Torus

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Abstract

A detailed Monte Carlo calculation of the phase diagram of bosonic mass-deformed IKKT Yang-Mills matrix models in three and six dimensions with quartic mass deformations is given. Background emergent fuzzy geometries in two and four dimensions are observed with a fluctuation given by a non-commutative $U(1)$ gauge theory very weakly coupled to normal scalar fields. The geometry, which is determined dynamically, is given by the fuzzy spheres $\mathbf{S}^2_N$ and $\mathbf{S}^2_N \times \mathbf{S}^2_N$ respectively. The three and six matrix models are effectively in the same universality class. For example, in two dimensions the geometry is completely stable, whereas in four dimensions the geometry is stable only in the limit $M \longrightarrow \infty$, where M is the mass of the normal fluctuations. The behavior of the eigenvalue distribution in the two theories is also different. We also sketch how we can obtain a stable fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ in the large N limit for all values of M as well as models of topology change in which the transition between spheres of different dimensions is observed. The stable fuzzy spheres in two and four dimensions act precisely as regulators which is the original goal of fuzzy geometry and fuzzy physics. Fuzzy physics and fuzzy field theory on these spaces are briefly discussed.

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Contents

Chapter 1

Introduction and Summary

Understanding how our universe began is one of the most fundamental themes in theoretical physics, Gauge theories provide the best known description of the fundamental forces in nature. At very short distances however, physics is not known, and it seems unlikely that spacetime is a perfect continuum down to arbitrarily small scales.

Gauge theory is one of the ways to try understand this dilemma, and precisely Non-commutative gauge theory, Since its emergence in the 80's [1] [2] [3] [4] [5], noncommutative geometry has helped to reveal deep mathematical relationships between ordinary geometry and other structures, among them differential algebras and normed algebras. In particular, noncommutative geometry has shed new lights on gauge theories. Indeed, a theory of connections can be defined in great generality using the noncommutative language of associative algebra, Different approaches have been proposed to study noncommutative spaces. The theory of spectral triples, developed by Connes, emphasizes the metric structure [3] [6] [7]. On the other hand, many noncommutative spaces are studied through differential structures [8] [9] [10] [11] [12]. However, all the noncommutative gauge field theories studied so far use the same building blocks, even when there are defined through different approaches.

A gauge interaction is an implementation of the principle that the theory should be invariant under some local symmetry. In particle physics, these local symmetries take the form of functions g : $\mathcal{M} \to G$ on the space-time $\mathcal{M}$ with values in a structure group G. Electromagnetism is associated to the group G=U(1), the electroweak theory by Glashow, Weinberg and Salam uses the group $G = U(1) \times SU(2)$ and chromodynamics relies on $G = SU(3)$ [13].

Noncommutative geometry being an extension of differential geometry, it naturally generalizes this theory of fiber bundles and connections. What is astonishing is that this generalization is very elegant, very powerful and very effective, not only from a mathematical point of view, but also in its applications to physics.

To describe noncommutative spaces, the noncommutative geometry is now investigated by many authors and using this framework one can even consider the differential geometry of singular spaces like, for example, a 2-point space which has been shown to provide a geometrical interpretation of the Higgs mechanism [6] [14]. In general, such noncommutative spaces can be obtained by quantizing a given space with its Poisson structure. Furthermore, if the original space is compact one obtains a finite dimensional matrix algebra as a quantized algebra of functions over this space.

In the matrix model, matter and even spacetime are dynamically emerged out of matrices [15] [16]. Spacetime coordinates are represented by matrices and therefore noncommutative geometry appears naturally. The idea of the noncommutative geometry is to modify the microscopic structure of the spacetime. This modification is implemented by replacing fields on the spacetime by matrices. It was shown [17] [18] [19] [20] that noncommutative Yang-Mills theories in a flat background are obtained by expanding the matrix model around a flat noncommutative background. The noncommutative background is a D-brane-like background which is a solution of the equation of motion and preserves a part of supersymmetry. Various properties of noncommutative Yang-Mills have been studied from the matrix model point of view [21]. In string theory, it is discussed that the world volume theory on D-branes is described by noncommutative Yang-Mills theory [22]

A different kind of noncommutative backgrounds, a noncommutative sphere, or a fuzzy sphere is also studied in many contexts. In [23] it is discussed in the framework of matrix regularization of a membrane. In the light-cone gauge, they gave a map between functions on spherical membrane and hermitian matrices. In BFSS matrix model [24] membranes of spherical topology are considered in [25] [26] [27]. A noncommutative gauge theory on a fuzzy sphere in string theory context is discussed in [28] [29]. The approach to construct a gauge theory on the fuzzy sphere were pursued in [30] [31] [32] [33] [34] [35]

The fuzzy sphere [30] can be constructed by introducing a cut off parameter N for angular momentum of the spherical harmonics. The number of independent functions is $\sum_{l=0}^{N}(2l+1) = (N+1)$. Therefore, we can replace the functions by $(N+1)\times(N+1)$ hermitian matrices on the fuzzy sphere.

Thus, the algebra on the fuzzy sphere becomes noncommutative.

One of the principal goals of the study of field theories on fuzzy spaces is to develop an alternative non-perturbative technique [36]. To date, this new approach in the case of four dimensional field theories has been limited to studies of Euclidean field theory on $S^2 \times S^2$ [37], CP^3 [38] and S^4 [39]. S^4 is really a squashed CP^3 and includes many unwanted massive Kaluza-Klein type modes. Even $S^2 \times S^2$ is not ideal since it has curvature effects that drop off as power corrections rather than exponentially as in the case of toroidal geometries [40].

The fuzzy approach does, however, have the advantage of preserving continuous symmetries such as the $SU(2)$ symmetry of a round S^2 and does not suffer from fermion doubling [41]

In fact, this work divided into two main themes, first is Noncommutative scalar phi-four theory, and the other is Noncommutative gauge theory. In both subjects we focused on phases diagrams.

- The first theme is phase diagrams of the multitrace quartic matrix models of noncommutative Φ^4, this theory is enjoys three stable phases: (1) disordered (symmetric, one-cut, disk) phase, (2) uniform ordered (Ising, broken, asymmetric one-cut) phase and (3) non-uniform ordered (matrix, stripe, twocut, annulus) phase. This picture is expected to hold for noncommutative/fuzzy phi-four theory in any dimension, and the three phases are all stable and are expected to meet at a triple point. The triple point is identified as a termination point of the one-cut-totwo-cut transition line and is located at $(\tilde{b}, \tilde{c}) = (1.55, 0.4)$ which compares favorably with previous Monte Carlo estimate.

- The seconde theme, we study a six matrix model with global $SO(3) \times SO(3)$ symmetry containing at most quartic powers of the matrices. This theory exhibits a phase transition from a geometrical phase at low temperature to a Yang-Mills matrix phase with no background geometrical structure at high temperature. This is an exotic phase transition in the same universality class as the three matrix model but with important differences. The geometrical phase is determined dynamically, as the system cools, and is given by a fuzzy spherical background $\mathbf{S}^2_N \times \mathbf{S}^2_N$, with an Abelian gauge field which is very weakly coupled to two normal scalar fields.

Chapter 2

Noncommutative Field theory

2.1 The caconical case

Nocncommutative gauge theory in the canonical case,where the commutator of two coordinates is a constant. We will start with the most commonly used $\star$-product for the caconical case.

2.1.1 The Moyal-Weyl $\star$-product

In the canonical case, the noncommutative coordinates fulfill commutative relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \tag{2.1}$$

with the constant noncommutativity parameter $\theta \in \mathbb{R}$. The noncommutative algebra generated by the noncommutative coordinates can represented on the space of functions on $\mathbb{R}^n$ by introducing a noncommutative product, the Moyal-Weyl $\star$-product [42] [43]

$$f \star g = m.e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} f \otimes g = fg + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g + O(2) \tag{2.2}$$

with $m.(f \otimes g) = fg$ and $\partial_i = \frac{\partial}{\partial x_i}$. The product is associative,as

$$\begin{aligned}(f \star g) \star h &= m.e^{\frac{i}{2}\theta^{kl}\partial_k \otimes \partial_l}(m.e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} f \otimes g) \otimes h \\ &= m.m.e^{\frac{i}{2}\theta^{kl}(\partial_k \otimes 1 \otimes \partial_l + 1 \otimes \partial_k \otimes \partial_l)} e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j \otimes 1} f \otimes g \otimes h \\ &= m.m.e^{\frac{i}{2}\theta^{ij}(\partial_i \otimes 1 \otimes \partial_j + \partial_i \otimes \partial_j \otimes 1)} e^{\frac{i}{2}\theta^{kl} 1 \otimes \partial_k \otimes \partial_l} f \otimes g \otimes h \\ &= m.e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j}(f \otimes (m.e^{\frac{i}{2}\theta^{kl}\partial_k \otimes \partial_l} g \otimes h) \\ &= f \star (h \star h)\end{aligned} \tag{2.3}$$

and obviously reproduces (1.1). Furthermore, as θ is antisymmetric, usual complex conjugation is still an involution

$$\overline{f \star g} = m.e^{-\frac{i}{2}\theta ij \partial_i \otimes \partial_j} \bar{f} \otimes \bar{g} = \bar{g} \star \bar{f} \tag{2.4}$$

and integration has the trace property

$$\int d^n x f \star g = \int d^n x g \star f \tag{2.5}$$

if the function f and g vanish suffciently fast at infinity because $f \star g$ has to be integrable in the firs place
Differentiation on this space is an inner operation, i.e. we have

$$i\theta^{\mu\nu}\partial_\nu = [x^\mu, \cdot] \tag{2.6}$$

2.1.2 Commutative gauge theory

A non-abelian gauge theory is based on a Lie groupe with Lie algebra

$$[T^a, T^b] = i f_c^{ab} T^c \tag{2.7}$$

Matter fields transform under a Lie algebra valued infinitesimal parameter

$$\lambda = \lambda_a T^a \tag{2.8}$$

in the fundamental represntation as

$$\delta_\lambda \psi = i\lambda\psi \tag{2.9}$$

It follws that

$$(\delta_\lambda \delta_\xi - \delta_\lambda \delta_\xi) = \delta_{i[\xi,\lambda]}\psi. \tag{2.10}$$

The commutator of two consecutive infinitesimal gauge transformation closes into an infinitesimal gauge transformation.
As differentiation isn't a covariant operation, a Lie algebra valued gauge potential $a_i = a_{ia}T^a$ is introduced with the transformation property

$$\delta_\lambda a_i = \partial_i \lambda + i[\lambda, a_i]. \tag{2.11}$$

With this the covariant derivative of a field is

$$\mathcal{D}_i \psi = \partial_i \psi - i a_i \psi. \tag{2.12}$$

The field strength of the gauge potential is defined to be the commutator of two covariant derivatives

$$f_{ij} = i[\mathcal{D}_i, \mathcal{D}_j] = \partial_i a_j - \partial_j a_i - i[a_i, a_j] \tag{2.13}$$

For nonabelian gauge theory, f_{ij} is not invariant under gauge transformation, but rather transformation covariantly, i.e.

$$\partial_\lambda f = i[\lambda, f]. \tag{2.14}$$

The same is true for the $f_{ij}f^{ij}$ (Lagrangian density).In order to get a gauge invariant action, we have to use the trace over the representation of the gauge fields. As the trace is cyclic, the commutator with the gauge parameter vanishes and the action

$$\mathcal{S} = \int dx^n \mathbf{tr} f_{ij} f^{ij} \tag{2.15}$$

becomes invariant.

2.1.3 Noncommutative gauge theory

To do noncommutative gauge theory in the $\star$-product approach, we can simply mimic the commutative consrtuction, replacing the ordinary product with the $\star$-product.

Field should now transform as

$$\delta_\lambda \Psi = i\lambda \star \Psi. \tag{2.16}$$

The commutator of two such gauge transformations should again be a gauge transformation, i.e we want

$$(\delta_\Xi \delta_\Lambda - \delta_\Xi \delta_\Lambda)\Psi = \delta_{i[\Xi \overset{\star}{,} \Lambda]}\Psi, \tag{2.17}$$

which is only possible for gauge groups $U(N)$, as for $\Lambda = \Lambda_a T^a$ and $\Xi = \Xi_a T^a$ the commutator

$$[\Xi, \Lambda] = \frac{1}{2}[\Xi_a \overset{\star}{,} \Lambda_b]\{T^a, T^b\} + \frac{1}{2}\{\Xi_a \overset{\star}{,} \Lambda_b\}[T^a, T^b] \tag{2.18}$$

As coordinates do not transform under gauge transformations, multiplication from the left with coordinates no longer is a covariant operation, i.e.

$$\delta_\Lambda(x_i \star \Psi) = x_i \star \Lambda \star \Psi \neq \Lambda \star x_i \star \Psi \tag{2.19}$$

This is very much like the situation in commutative gauge theory, where acting with a derivative from the left isn't a covariant operation. Following the procedure there, we introduce covariant coordinates X^i by adding a gauge field A_i as

$$X^i = x^i + \theta^{ij} A_j \tag{2.20}$$

To make the X^i covariant, i.e. $\delta_\Lambda X^i = i[\lambda \overset{\star}{,} X^i]$, the gauge field has to transform as

$$\delta_\Lambda(\theta^{ij} A_j) = -i[x^i \overset{\star}{,} \Lambda] + i[\Lambda \overset{\star}{,} \theta^{ij} A_j] \tag{2.21}$$

and therefore

$$\delta_\Lambda A_i = \partial_i \Lambda + i[\Lambda \overset{\star}{,} A_i] \tag{2.22}$$

in exact analogy to the commutative case. The commutator with the coordinate produces the derivative on the gauge parameter, as $[x^i, f] = i\theta^{ij} f$. More generally we can introduce a covariantizer $\mathcal{D}$ that applied to a function f renders it covariant [44]

$$\delta_\Lambda(\mathcal{D}(f)) = i[\Lambda \overset{\star}{,} \mathcal{D}(f)] \tag{2.23}$$

We can now go on to formulate noncommutative gauge theory much in the same way as we formulated commutative gauge theory.

The covariant derivative $\mathcal{D}_i$ can be introduced as

$$\mathcal{D}\Psi = \partial_i \Psi - iA_i \star \Psi, \tag{2.24}$$

the filed strength F_{ij} as

$$F_{ij} = i[\mathcal{D}_i \stackrel{\star}{,} \mathcal{D}_j] = \partial_i A_j - \partial_j A_i - i[A_i \stackrel{\star}{,} A_j]. \tag{2.25}$$

The relation to the covariant coordinates subsists at this level with

$$-i([X^i \stackrel{\star}{,} X^j] - i\theta^{ij}) = \theta^{ik}\theta^{jl}F_{kl} \tag{2.26}$$

In noncommutative gauge theory,the field strength F is not gauge invariant, even for gauge group $U(1)$. It rather transforms covariantly under gauge transformations, i.e

$$\delta_\Lambda(F_{\mu\nu} \star F^{\mu\nu}) = -i[\Lambda \stackrel{\star}{,} F_{\mu\nu} \star F^{\mu\nu}]. \tag{2.27}$$

Just inserting a trace over the representation of the gauge group no longer guarantees gauge invariance. To get gauge invariant expressions, we have to use the trace property of the integral. If we set the action for non- commutative gauge theory as

$$\mathcal{S} = \int d^n x \mathbf{tr} F_{\mu\nu} \star F^{\mu\nu}, \tag{2.28}$$

2.2 The general formalism

2.2.1 Seiberg-Witten gauge theory

Noncommutative transformation properties are determined by the transformation properties of the commutative fields they depend on. Therefore the fields again transform as [45]

$$\delta_\alpha \Psi_\psi[a] = i\Lambda_\alpha[a] \star \Psi_\psi[a], \tag{2.29}$$

leading to the same consistency condition for the gauge parameter

$$i(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = i\Lambda_{-i[\alpha,\beta]} \tag{2.30}$$

The transformation law for the covariantizer is now

$$\delta_\alpha(D[a](f)) = i[\Lambda_\alpha[a] \stackrel{\star}{,} D[a](f)]. \tag{2.31}$$

The Seiberg-Witten-map can be easily extended to the derivations δ_X of the $\star$-product.The noncommutative covariant derivation $D_X[a]$ can be written with the help of a noncommutative gauge potential $A_X[a]$ now depending both on the commutative gauge potential a_i and the Poisson vector field X

$$\delta_\alpha A_X[a] = \delta_X \Lambda_\alpha[a] + i[\Lambda_\alpha[a] \stackrel{\star}{,} A_X[a]]. \tag{2.32}$$

2.2.2 Commutative actions with the frame formalism

we recall some aspects of classical differential geometry. Suppose we are working on a n-dimensional manifold M with metric $g_{\mu\nu}$.Then there are locally n derivatives ∂_μ which form a basis of the tangent space of the manifold. We can always make a local basis transformation to frame

$$e_a = e_a^\mu(x)\partial_\mu \tag{2.33}$$

with $e_a^\mu e_\nu^a = \delta_\nu^\mu$, where the metric is constant

$$\eta_{ab} = e_a^\mu e_b^\nu \tag{2.34}$$

Since forms are dual to vector fields, they may be evaluated on the frame. For the gauge field we get

$$a_a = a(e_a), \tag{2.35}$$

leading to the covariant derivate

$$D_\psi = (D\psi)(e_a) = e_a\psi - ia_a\psi. \tag{2.36}$$

The field strength becomes

$$f_{ab} = i[D_a, D_b] - iD([e_a, e_b]) = e_a a_b - e_b a_a - a([e_a, e_b]) - i[a_a, a_b] \tag{2.37}$$

Locally this means that

$$a_a = e_a^\mu a_\mu, \qquad D_a\psi = e_a^\mu D_\mu \quad and \quad f_{ab} = e_a^\mu e_b^\nu f_{\mu\nu} \tag{2.38}$$

Using these defnitions, the action for gauge theory on a curved manifold can be written in the two diferent bases as

$$\mathcal{S} = -\frac{1}{4}\int d^n x\sqrt{g}\eta^{ab}\eta^{cd} f_{ac} f_{bd} = -\frac{1}{4}\int d^n x\sqrt{g} g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} \tag{2.39}$$

where

$$\sqrt{g} = \sqrt{det(g_{\mu\nu})} = \sqrt{det(e_\mu^a e_\nu^b \eta_{ab})} = det\ e_\mu^a \tag{2.40}$$

2.2.3 Gauge theory on curved noncommutative space-time

In order to formulate gauge theory on a curved noncommutative space-time, we need a frame e_a and a Poisson structure $\{\cdot,\cdot\}_p = \pi^{\mu\nu}\partial_\mu \wedge \partial_\nu$ that are compatible with each other. Compatibility means that the frame e_a commutes with the Poisson structure $\{\cdot,\cdot\}_p$, i.e.

$$e_a\{f,g\}_p = \{e_a f, g\}_p + \{f, e_a g\}_p, \tag{2.41}$$

We can define a covariant derivative of a field by using a derivation δ_X

$$D_X \Psi_\psi = \delta_X \Psi_\psi - i A_X \star \Psi_\psi. \tag{2.42}$$

With this, a field strength could be defined as

$$-iF_{X,Y} = [D_x \stackrel{\star}{,} D_Y] - D_{[X,Y]_\star} \tag{2.43}$$

The noncommutative covariant derivative (2.14) and field strength (2.15) evaluated on the frame e a then read

$$D_a\Phi = D_{e_a}\Phi = \delta_{e_a}\Phi - iA_{e_a} \star \Phi \tag{2.44}$$

$$-iF_{ab} = -iF_{e_a,e_b} = [D_{e_a} \stackrel{\star}{,} D_{e_a}] - D_{[e_a,e_b]_\star}. \tag{2.45}$$

The field strength will transform covariantly under gauge transformations, i.e. we have

$$\delta_\Lambda = i[\Lambda \stackrel{\star}{,} F]. \tag{2.46}$$

To make the action gauge invariant, the integral has to have the trace property, a noncommuative gauge action

$$\mathcal{S} = -\frac{1}{4}\int d^n x \Omega \eta^{ab}\eta^{cd} F_{ac}F_{bd} \tag{2.47}$$

with $\Omega = \sqrt{g} + \mathcal{O}(1)$
that goes in the commutative limit

$$\mathcal{S} \to -\frac{1}{4}\int d^n x \sqrt{g}\; g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} \tag{2.48}$$

Scalars

For the noncommutative version of a scalar Lagrangian

$$\eta^{ab} D_a \bar{\phi} D_b \phi + m^2 \bar{\phi}\phi, \tag{2.49}$$

we also need an involution of the $\star$-product, i.e.

$$\overline{(f \star g)} = \bar{g} \star \bar{f}. \tag{2.50}$$

To make the NC Lagrangian invariant under NC gauge transformations, the NC gauge parameter Λ and the NC gauge field A_X have to be invariant under this involution to get

$$\delta_\Lambda \bar{\phi} = \overline{(\Lambda \star \phi)} = \bar{\phi} \star \bar{\Lambda} = \bar{\Phi} \star \Lambda \tag{2.51}$$

and

$$\overline{(A_X \star \phi)} = \bar{\phi} \star \bar{A}_x = \bar{\phi} \star A_X \tag{2.52}$$

Putting everything together, we therefore end up with an action

$$\mathcal{S} = \int d^n x \Omega (-\frac{1}{4}\eta^{ab}\eta^{cd} F_{ac} \star F_{bd} + \eta^{ab} D_a \bar{\Phi} \star D_b \Phi - m^2 \bar{\Phi} \star \Phi) \tag{2.53}$$

that is invariant under noncommutative gauge transformations

$$\delta_\Lambda S = 0 \tag{2.54}$$

and reduces in the commutative limit

$$\mathcal{S} \rightarrow \int d^n x \sqrt{g} (-\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} + g^{\nu\mu} D_\mu \bar{\phi} D_\nu \phi - m^2 \bar{\phi}\phi) \tag{2.55}$$

to scalar electrodynamics on a curved manifold

Spinors

The commutative spinor action can be written as

$$S_{spinor} = \frac{1}{2} \int d^2 x \sqrt{g} \bar{\Psi} i \gamma^a e^\mu_a (\partial_\mu - i A_\mu + m) \Psi \tag{2.56}$$

Usual gamma-matrices $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\gamma^\mu = \gamma^a e^\mu_a$, we get $\{\gamma^\mu, \gamma^{nu}\} = 2g^{\mu\nu}$. The noncommutative version (2.28) is easily constructed, and we get

$$S_{spinor} = \frac{1}{2} \int d^2 x \Omega \bar{\Psi} i \gamma^a (\delta_{e_a} - i A_{e_a} + m) \star \Psi \tag{2.57}$$

with $e_a = e^\mu_a \partial_\mu$, which is invariant under NC gauge transformations

2.3 Noncommutative scalar field theory

We are able to define a scalar field theory on this geometry. At this point we make a change of notation and introduce the short-hand notation Weyl operators $W[f] \to \hat{f}$

2.3.1 Noncommutative scalar action

We start with the ation of an Euclidean commutative $\lambda\phi^4$ theory

$$S[\phi] = \int d^dx \left(\frac{1}{2}\partial_\mu\phi(x)\partial_\mu(x) + \frac{m^2}{2}\phi^2(x) + \frac{\lambda}{4}\phi^4(x) \right), \tag{2.58}$$

where ϕ is a real valued scalar field and d is the dimension of spacetime.

To transform an ordinary scalar field theory to a noncommutative field theory we can use the Weyl quantization via Hermitian operators $\hat{\phi}$
The quantum field theory written in terms of Weyl operators $\hat{\phi}$, corresponding to a real scalar field $\phi(x)$ on $\mathbb{R}^d$

$$Z = \int d\hat{\phi} \exp\left(- S[\hat{\phi}] \right) \tag{2.59}$$

$$S[\hat{\phi}] = Tr\left(\frac{1}{2}[\hat{\partial}_\mu, \hat{\phi}]^2 + \frac{m^2}{2}\hat{\phi}^2 + \frac{\lambda}{4}\hat{\phi}^4 \right) \tag{2.60}$$

The measure $d\hat{\phi}$ is here the ordinary path integral measure for scalar fields $\mathcal{D}\phi$

This theory may be formulated in coordinate space by applying the map

$$f(x) = Tr(W[f]\Delta(x)) \tag{2.61}$$

$$W[f] = \int d^dx f(x)\Delta(x) \quad with \quad \Delta(x) = \int \frac{d^dk}{(2\pi)^d} e^{ik_\mu\hat{x}_\mu} e^{-ik_\mu x_\mu} \tag{2.62}$$

where $\Delta(x)$ is a Hermitian operator that can be understood as a mixed basis for operators of fields. and using $W[f]W[g] = W[f \star g]$,

$$S[\phi] = \int d^dx \left[\frac{1}{2}(\partial_\mu\phi(x))^2 + \frac{m^2}{2}\phi^2(x) + \frac{\lambda}{4}\phi(x) \star \phi(x) \star \phi(x) \star \phi(x) \right]. \tag{2.63}$$

the kinetic term and the mass term do not contain the star-product, because of this property $\int d^d f_1(x) \star f_2(x) = \int d^d f_1(x) f_2(x)$.

This implies that the free propagator of the quantum field theory also remains unchanged, and only the interaction part, i.e., the vertex of Feynman rules, gains extra contributions from the noncommutativity.(fig)

In particular, the commutative vertex function $i\lambda$ gains an extra phase factor of the form

$$V(k_i) = \sum_{i<j} e^{-\frac{i}{2} k_i \wedge k_j} \tag{2.64}$$

and k_i are the momenta flowing into the vertex. $V(k_i)$ is not invariant under arbitrary permutations of the momenta but only under cyclic permutations, so one has to keep track of the order in which propagators are connected to the vertices of Feynman diagrams. We can be drawn on a plane without intersecting prooagators,'planar'. Namely, we may replace every line in a planar Feynman diagram by a double line.which has only non-intersecting solid lines and loops. [**?**] (**seeFig**.(**1**)). due to the momentum conservation at the vertices and the planarity, we may label the lines of the double line notation by momenta l_i, which correspond to the original momenta via the relation $k_i = l_{i_1} - l_{i_2}$. Accordingly,when k_i $i = 1, .., 4$, are the incoming momenta for a vertex in cyclic order, the phase factor (3.7) becomes

$$e^{-\frac{i}{2} \sum_{i=1}^{4} l_{i_j} \wedge l_{i_j+1}} \tag{2.65}$$

where each of the expressions $l_{i_j} \wedge l_{i_j+1}$ corresponds to one of the incoming propagators. The over-all phase factor for a diagram is then the product of the phase factors corresponding to each of the vertices of the diagram. Therefor, by the expression (3.8), we find that for a planar diagram the factors corresponding to the internal propagators cancel out, since they contribute opposite $\pm l_{i_j} \wedge l_{i_j+1}$ to the exponent of the over-all phase factor. Consequently, we are left with the over-all phase factor

$$V(p_i) = e^{-\frac{i}{2} \sum_{i<j} p_i \wedge p_j} \tag{2.66}$$

p_i are the momenta associated to the external lines of the original single-line diagram in the cyclic order [**?**]. It immediately follows that the UV-divergencies of the commutative quantum scalar field theory, which arise from the integrals over the internal momenta of Feynman diagrams, are also present in the planar diagrams of the noncommutative theory, and therefore, due to this example, the noncommutativity of soacetime does not seem to help to naturally regularize the divergencies of quantum field theory

2.3.2 UV/IR mixing

We consider the one-particle-irreducible two point function Γ of the noncommutative $\lambda\phi^4$ scalar field theory.

$$\Gamma(p) =< \tilde{\phi}(p)\tilde{\phi}(-p) >= \sum_{n=0}^{\infty} \lambda^n \Gamma^n(p). \tag{2.67}$$

At lowest order the two-point function is given by $\Gamma^0(p) = p^2 + m^2$.

Where the first term is the planar contribution and the second term the nonplanar one. In fact, since we have only one external momentum in the case of two point function, the planar phase factor (3.9) equals unity, and thus the planar diagrams give exactly the same correction as in the commutaive case.

The one loop contribution splits topologically into two parts, one planar and one non-planar diagram [46]

$$\Gamma^1_p = \frac{\lambda}{3(2\pi)^4} \int \frac{d^4k}{k^2+m^2} \tag{2.68}$$

$$\Gamma^1_{np} = \frac{\lambda}{6(2\pi)^4} \int \frac{d^4k}{k^2+m^2} e^{ik\wedge p}, \tag{2.69}$$

In Refs [**?**] [46] it is shown that the contribution of planar diagram to noncommutattive perturbation theory is propotional to the commutative case (fig planar and non-planar)

Regularizing the momentum integrals at the energy scale Λ we find [46]

$$\Gamma^1_p = \frac{\lambda}{48\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + ...\right] \tag{2.70}$$

$$\Gamma^1_{np} = \frac{\lambda}{96\pi^2}\left[\Lambda^2_{eff} - m^2 \ln\left(\frac{\Lambda^2_{eff}}{m^2}\right) + ...\right] \tag{2.71}$$

Here we introduced the effective cutoff Λ_{eff} given by

$$\Lambda_{eff} = \frac{1}{1/\Lambda^2 + p\circ p} \quad , \quad p\circ p := -p^\mu \theta^2_{\mu\nu} p^\nu \geqslant 0 \tag{2.72}$$

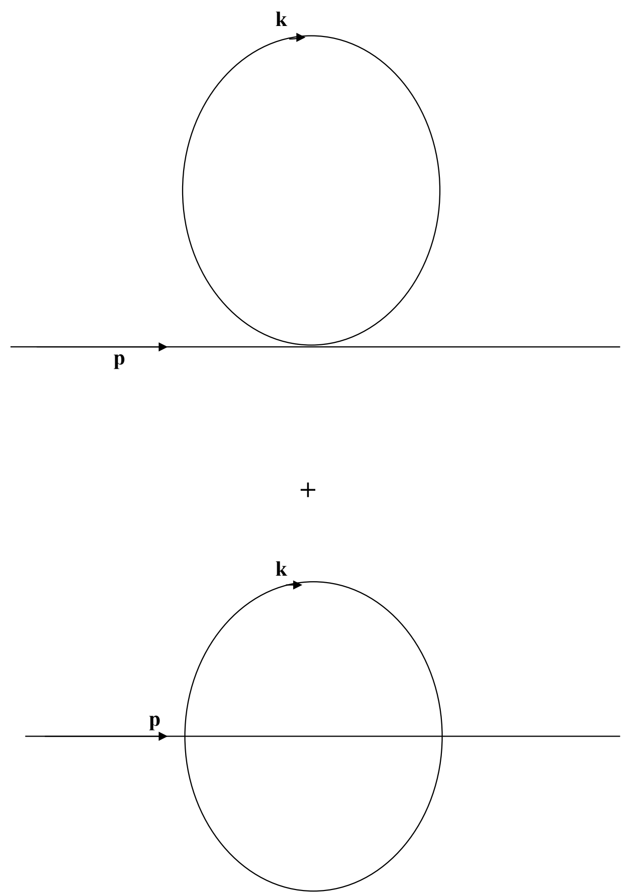


Figure 2.1: Planar and nonplanar loop diagrams, respectively.

When we take the UV-limit $\Lambda \to \infty$ of the internal momentum, we find

$$\Gamma^1_{np} \overset{\Lambda\to\infty}{\longrightarrow} \frac{\lambda}{96\pi^2}\left[\left(\frac{1}{p \circ p}\right)^2 - m^2 \ln\left(\frac{1}{m^2(p\circ p)^2}\right) + ...\right] \tag{2.73}$$

This expression diverges at the low energy limit $p \to 0$ (IR limit), The UV limit does not commute with the IR limit.

At small momenta or small non-commutativity parameter the two-point function reads

$$\Gamma(p) \simeq p^2 + m^2 + 3\lambda\Gamma^1_{np}(0) + O(\lambda^2) \tag{2.74}$$

Taking now the UV limit leads to the standard mass renormalization of the $\lambda\phi^4$ theory. Taking these limits vice versa, the effective cutoff is given by

$$\Gamma^2_{eff} = \frac{1}{\theta^2 p^2} \tag{2.75}$$

and Λ_{eff} diverges and therefore also $\Gamma^1_{np}(p)$ either in the limit $\theta \to 0$ or in the infrared limit when the incoming momentum $p \to 0$
The planar one loop contribution of $\Gamma(p)$ by defining the renormalized mass through

$$M^2_{eff} = m^2 + 2\lambda\Gamma^1_{np}(0) \tag{2.76}$$

Removing the cutoff while keeping M^2_{eff} fixed, then leads to a finite $\Gamma(p)$ for finite incoming momenta p. For zero momentum $\Gamma(p)$ diverges and the divergence at one loop is given by

$$\Gamma(p) = p^{2+} M^2_{eff} + \xi\frac{\lambda}{\theta^2 p^2} + \text{subleading terms}, \qquad \text{with } \xi = \frac{1}{96\pi^2} \tag{2.77}$$

This is not too surprising, since the nonplanar phase factor $e^{ik\wedge p}$ in (3.12), which dampens the singularity of the momentum integral, approaches unity as $p \to 0$
This exotic mixing of the high energy (UV) and low energy (IR) scales in non-commutative theories, which does not have a counter-part in commutative theories, is called UV/IR mixing

2.3.3 Phase structure of non−commutative $\lambda\phi^4$

The UV/IR mixing is one of the most intersting properties of non−commutative field theory and has no counterpat in the commutative case.

Gusber and Sondhi studied the phase diagram of 4d $\lambda\phi^4$ theory [47], based on an action of the Brazovskiia from [48]

We try summarize their resuls. At small non−commutativity parameter θ they obtained an Ising type (second order) phase to an uniformly ordered phase with $< \phi > \neq 0$.
At large θ , Leads ao an ordered phase. In this phase $< \phi >$ varies in space, which involve some non-uniform patterns like stripes.
According to Ref [47] the pahase diagram in the $\frac{m^2}{\Lambda} - \lambda$ plane is then given by Figure (2.3) , where Λ is a momentum cut-off and the λ is the coupling.

In another approach renormalization group technique were used to study the phase diagram of the $\lambda\phi^4$ model [49]. Chen and Wu obtained in $d = 4 - \epsilon$ a new IR stable fixed point, and therefore a striped phase exists. In contrast to the resuls in Ref [47], this implies that in $d = 4$ there is no stiped phase.

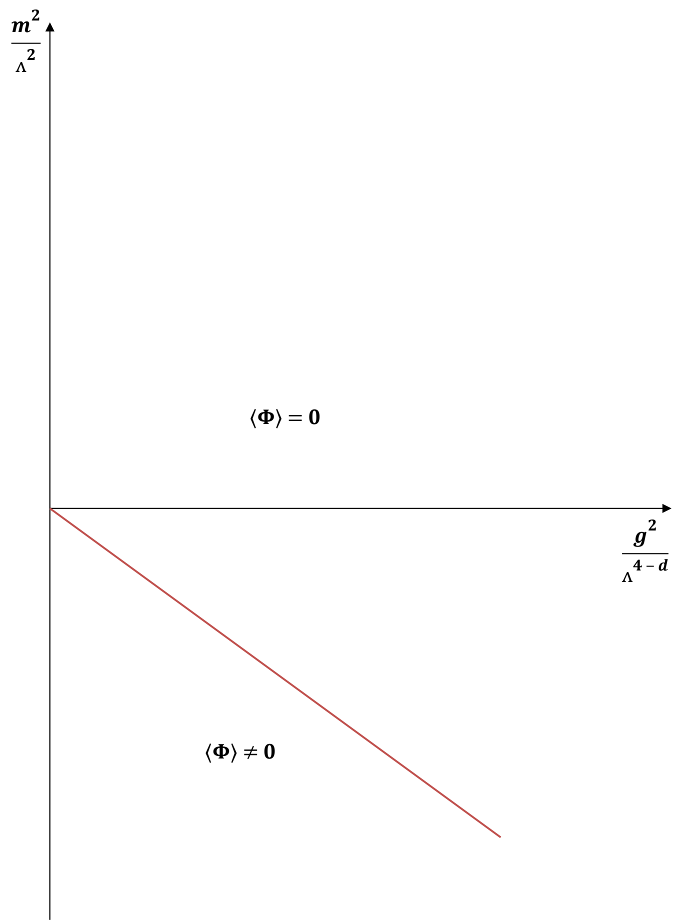


Figure 2.2: he phase diagram for the Ising scalar field theory, in d $\geq$ 1.

2.4 Noncommutative gauge theory

2.4.1 Star-gauge invariant action

To define a YangMills theory on a noncommutative plane we have to generalize the map (2.62). Let $A_\mu(x)$ be a Hermitian gauge field on $\mathbb{R}^d$, which corresponds to the unitary gauge group $U(n)$. We can introduce the Weyl operators corresponding to $A_\mu(x)$ by taking the trace of the tensor product of $\Delta(x)$ and the gauge field [50]

$$\hat{A}_\mu = \int d^d x \Delta(x) \otimes A_\mu(x) \tag{2.78}$$

the derivative of Weyl operators is equal to the Weyl operator of the usual derivative of the functions [50]

$$[\hat{\partial}_\mu, W[f]] = \int d^d x \partial_\mu f(x) \Delta(x) = W[\partial_\mu f] \tag{2.79}$$

Based on this equation a noncommutative version of the Yang-Mills action can ce defined

$$S[\hat{A}] = -\frac{1}{4g^2} \mathrm{Tr}\ \mathrm{tr}_N \Big(\big[\hat{\partial}_\mu, \hat{A}_\nu\big] - \big[\hat{\partial}_\nu, \hat{A}_\mu\big] - i\big[\hat{A}_\mu, \hat{A}_\nu\big] \Big)^2 \tag{2.80}$$

where the term in brackets is the operator analog of the field strength tensor. Here Tr is the operator is given by an integration over space-time

$$\mathrm{Tr}\ W[f] = \int d^d x\ f(x) \tag{2.81}$$

and tr_N denotes the trace in color space. This action is invariant under transformations of the form

$$\hat{A}_\mu \to \hat{U} \hat{A}_\mu \hat{U}^\dagger - i\hat{U}\big[\hat{\partial}_\mu, \hat{U}\big], \tag{2.82}$$

where $\hat{U}$ as an arbitrary unitary element of the algebra of matrix valued operators, i. e.

$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \hat{\mathbb{1}} \otimes \mathbb{1}_n \tag{2.83}$$

The symbol $\hat{\mathbb{1}}$ is here the identity on the ordinary Weyl algebra and $\mathbb{1}_n$ is a $n \times n$ unit matrix.

To set up the action in coordinate space we can construct an inverse map of (2.78). The Yang-Mills action in coordinate space the reads

$$S[A] = -\frac{1}{4g^2}\int d^dx \mathrm{tr}_N\big(F_{\mu\nu}(x)\star F_{\mu\nu}(x)\big), \tag{2.84}$$

where we introduced the noncommutative field strength tensor $F_{\mu\nu}$ given by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\big[A_\mu(x), A_\nu(x)\big]_\star. \tag{2.85}$$

The index '$\star$' indicates that the products in this commutator are star-products. According the (2.85) the simple gauge group $U(1)$ we have Yang-Mills type strucure. herefore there exist three and four point gauge interactions and noncommutative $U(1)$ theory is asymptotically free.

The action (3.54) under star-gauge transformation given by

$$A_\mu(x) \to U(x)\star A_\mu(x)\star U(x)^\dagger - iU(x)\star\partial_\mu U(x)^\dagger, \tag{2.86}$$

$U(x)$ is a star-unitary matrix field,

$$U(x)\star U(x)^\dagger = U(x)^\dagger\star U(x) = \mathbb{1}_n \tag{2.87}$$

On the classical level we considered non-commutative $U(n)$ theories which reduce to the ordinary $U(n)$ theories in the limit $\theta \to 0$.
In Ref. [51] it was shown that the gauge groups $SU(N)$, $SO(N)$ cannot be realized on a flat non-commutative manifold, while it is possible for $U(N)$

The interaction of the $SU(N)$ gauge bosons with the $U(1)$ gauge boson plays an important role in the consistency check. In particular, the $SU(N)$ theory by itself is not consistent. and when $\theta \to 0$ limit of the $U(N)$ theory does not converge to the ordinary $SU(N)\times U(1)$ commutative theory, even at the planar limit [52]

The $U(n)$ group is closed under the starproduct; the product of two starunitary matrix fields is again starunitary. In contrast to $U(n)$ the special unitary group $SU(n)$ is not closed, since in general [50]

$$\det(U)\star\det(H) \neq \det(U\star H). \tag{2.88}$$

2.4.2 Gauge-invariant observables

Let us say a few words about observables in noncommutative Yang-Mills gauge theories, since it is such a subtle subject, which clearly highlights the nonlocaliy of these theories, and is also deeply related to UV/IR mixing.

We consider an arbitrary ariented smooth contour $C \subset \mathcal{R}^d$ with smooth parametrization $\xi(t)$: [0,1]$\to \mathcal{R}^d$, and endpoints $\xi(0) = 0$, $\xi(1) = v$ in $\mathcal{R}^d$. Introduce the noncommutative parallel transport operator, construct gauge invariant observables using open Wilson lines, which are nonlocal operators defined as [53] [54] [55]:

$$\mathcal{W}(x;\xi) = P\ \exp_\star\Bigg(i\int_C d\xi^i A_i(x+\xi)\Bigg), \tag{2.89}$$

P is path ordering. The index '$\star$' at the exponential function indicateds that in the expansion of this function the star-product has to be used. The operator $\mathcal{W}(x;\xi)$ is an $n\times n$ star-unitary matrix field and transforms under the star-gauge transformation (2.86) like (it is important property of an open Wilson line)

$$\mathcal{W}(x;\xi) \to U(x)\star\mathcal{W}(x;\xi)\star U(x+v)^\dagger \tag{2.90}$$

under finite noncommutative gauge transformations [54]

$$\tilde{\mathcal{W}}(k;\xi) := \int d^4\ \mathrm{tr}\big[\mathcal{W}(x;\xi)\big]\star e^{ik.x} \tag{2.91}$$

The transformation of $\tilde{\mathcal{W}}(k;\xi)$ under a gauge transformation according to (2.90) is

$$\tilde{\mathcal{W}}(k;\xi) \to \int d^4\ \mathrm{tr}\big[U(x)\star\mathcal{W}(x;\xi)\star U^\dagger(x+v)\star e^{ik.x}\big] \tag{2.92}$$

In the traditional commutative use, gauge invariance would force us to close the contour C. But in the noncommutativr case, gauge transformation can affect translations of space-time (the translations can be arranged by (star-) multiplication with plane waves):

$$U(x+v) = e^{ik_\mu x_\mu}\star U(x)\star e^{-ik_\rho x_\rho} \tag{2.93}$$

where $k=\theta^{-1}.v$ is the total momentum of path C. With the definition of the noncommutative parallel transporter and equation (5.1) we can associate a star-gauge invariant observable with any arbitrary contour C_v by [54]

$$\mathcal{O}(C_v) = \int d^d\ \mathrm{tr}_N\big(\mathcal{W}(x;C_v)\star e^{ik_\mu x^\mu}\big). \tag{2.94}$$

We notice here that UV/IR mixing manifests itself in the property that $k \to \infty$ as $v \to \infty$

Gauge-invariant operators, generalizing the standard local gauge theory operators in the commutative limit, are local in momentum space and are given by a Fourier-type transformation [54]:

$$\hat{\mathcal{O}}(k) = \int d^d \mathcal{O} U(x; C_k) \star e^{ik_\mu x^\mu}. \tag{2.95}$$

where $\mathcal{O}(x)$ is any local gauge invariatn operator of ordinary Yang-Mills theory. In the commutative limit $\theta = 0, v = 0$; there are no gauge-invariant quantities associated with open lines in ordinary Yang-Mills theory. In that case, the totam momentum of a closed loop is unresricted, and we can remplace $e^{ik_\mu x_\mu}$ (the momentum eigenstate) by an arbitrary function $f(x)$ paricular, taking $f(x) = \delta^d(x - a)$ recovers the standard gauge-invariant Wilson loops of Yang-Mills theory. But for $\theta \neq 0$, closed loops have 0 memntum k, and only $e^{ik_\mu x_\mu} = 1$ is permitef in $\mathcal{O}(C)$ above there is no local star-gauge invariant dynamics, because evrything has to be smeared outby the Weyl operator tarce $Tr \backsim \int d^d x$. Hence the gauge dynamics below the noncommutativity scale is quite different from the commutative case. [53]

2.4.3 Application

Noncommutative QED and UV/IR Mixing

Despite the growing understanding in formulating noncommutative gauge field theories, The problem of UV/IR mixing still shows up, as expected, when one calculates the higher order diagrams of noncommutative gauge field theories.

The propagators of noncommutative gauge theories are again equal to those of their commutative counter-parts, but the vertex functions contain nontrivial phase factors, as in the case of noncommutative scalar field theories, which ultimately lead to the UV/IR mixing [56]. For noncommutative QED one obtains the vertex functions for Feynman diagrams [57].

For the correction $\Pi_\Psi^{\mu\nu}$ given by a massless fermion loop to the photon propagator one finds [58]

$$i\Pi_\Psi^{\mu\nu} = -4g^2 \int \frac{d^4}{2\pi^4} \frac{\mathrm{tr}[\gamma^\mu(\not{p} - \not{k})\gamma^\nu \not{p})]}{(p-k)^2 p^2} \sin^2(\frac{1}{2} p \wedge k) \tag{2.96}$$

Using

$$\sin^2(\frac{1}{2}p \wedge k) = \frac{1}{2}[1 - \cos(p \wedge k)] \tag{2.97}$$

we can isolate the planar and non-planar contributions. The planar part gives then the usual logarithmically UV-divergent but renormalizable contribution, whereas the nonplanar part with the dampening phase factor $\cos(p \wedge k)$ gives the leading order term [56] [58]

$$i\Pi^{\mu\nu}_{\Psi_{np}}(k) \backsim \frac{\hat{k}^{\mu}\hat{k}^{\nu}}{\hat{k}^4} \tag{2.98}$$

at the IR-limit of the external momentum, which clearly diverges quadratically as $\hat{k} \to 0$. Therefore we again encounter the UV/IR mixing, where an IR-divergence of the external momentum arises from the UV-limit of the integral over the internal loop momentum. Similar IR-divergencies arise also from other higher order corrections to propagators and vertices [56] [58].

The physical interpretation of terms like the (5.2) is very interesting. For small noncommutative momentum, the one-loop inverse propagator is given by [58]

$$\Gamma_{\mu\nu} = i\left[(k_0^2 - k_3^2 - K^2)g_{\mu\nu} - g^2\frac{\hat{k}^{\mu}\hat{k}^{\nu}}{\hat{k}^4}\right] \tag{2.99}$$

where K ($k_0^2 = k_3^2 + K^2$) represents the projection of the spatial momentum on the plane.From this one-loop inverse propagator we can read the dispersion relation for the two physical. Suppose k is along the 2direction so that $\hat{K}$ is in the 1-direction. Then the photon polarized in the direction perpendicular to $\hat{K}$ satisfies the same dispersion relation as a photon would in the commutative theory [58]

UV/IR Mixing in Noncommutative QED via Seiberg-Witten Map

In the paper [59] on the connection between noncommutative geometry and String Theory, Seiberg and Witten introduced a mapping, which relates gauge field theories in noncommutative spacetime to ordinary commutative onesknown as the $Seiberg - Witten\ map$. This mapping has virtues, since some aspects of gauge theories, such as observables and gauge fixing, are more easily understood and dealt with in the commutative theories.

for example in [60] , that the UV/IR mixing is absent in the Seiberg-Witten formalism. However, we will find that this is presumably due to the expansion in the noncommutativity parameter matrix θ in the θ-expanded $Seiberg - Witten\ map$. In the θ-exact Seiberg-Witten map for noncommutative QED the UV/IR mixing reappears, as we will demonstrate. This same argument was expressed by Schupp and You in [61], where a noncommutative model with a gauge field coupled with a spinor field in the adjoint representation was considered.

The adjoint representation of the gauge group, however, corresponds to a chargeless particle with an electric dipole moment proportional to θ, and therefore in their model the interaction vanishes at the commutative limit $\theta \to 0$. Accordingly, their model does not correspond to a noncommutative theory of electrically charged fermions, which should reduce (classically) to the commu- tative QED in the commutative limit.

Noncommutative QED via Seiberg-Witten Map

We consider exclusively the gauge group $U_*(1)$. We want to express the action of noncommutative QED,

$$S_{NCQED} = \int d^4x \left[\hat{\bar{\Psi}}(i\partial\!\!\!/ - m)\hat{\Psi} - \hat{\bar{\Psi}} \star \hat{A\!\!\!/} \star \hat{\Psi} - \frac{1}{4}\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} \right] \tag{2.100}$$

in terms of the commutative fields up to the first order in A, so that we can calculate the photon propagator correction coming from the one-loop photon self-energy diagram. Denoting the noncommutative fields by hats and dropping the lower index from θ_1. The gauge field

$$\hat{A}_\mu = A_\mu + \mathcal{O}(A^2), \tag{2.101}$$

and for the spinor field

$$\hat{\Psi} = \Psi - \frac{1}{2}\theta^{\alpha\beta}\left[A_\alpha \star_1 (\partial_\beta \Psi) + \frac{1}{2}(\partial_\beta A_\alpha) \star_1 \Psi \right] + \mathcal{O}(A^2), \tag{2.102}$$

where we use the notation

$$(f \star_1 g)(x) := \left\{ \frac{e^{\frac{i}{2}}\partial_1 \wedge \partial_2 - 1}{\frac{i}{2}\partial_1 \wedge \partial_2} f(x_1)g(x_2) \right\}_{x_1 = x_2 \equiv x} \tag{2.103}$$

Since $(f \star g)^\dagger = g^\dagger \star f^\dagger$ for any functions (or matrices) f and g, we find that

$$\hat{\bar{\Psi}} = \bar{\hat{\Psi}} = \bar{\Psi} - \frac{1}{2}\theta^{\alpha\beta}\left[(\partial_\beta \bar{\Psi}) \star_1 A_\alpha + \frac{1}{2}\bar{\Psi} \star_1 (\partial_\beta A_\alpha) \right] + \mathcal{O}(A^2), \tag{2.104}$$

Substituting (2.101), (2.102) and (2.104) into the action (2.100), we find the fermion- photon interaction term to be, up to first order in A,

$$\begin{aligned} \mathcal{L}^{(1)}_{\Psi A} = & - \bar{\Psi} \star A\!\!\!/ \star \Psi \\ & - \frac{1}{2}\theta^{\alpha\beta}\left[(\partial_\beta \bar{\Psi}) \star_1 A_\alpha + \frac{1}{2}\bar{\Psi} \star_1 (\partial_\beta A_\alpha) \right](i\partial\!\!\!/ - m)\Psi \\ & - \frac{1}{2}\theta^{\alpha\beta}\bar{\Psi}(i\partial\!\!\!/ - m)\left[A_\alpha \star_1 (\partial_\beta \Psi) + \frac{1}{2}(\partial_\beta A_\alpha) \star_1 \Psi \right] \end{aligned} \tag{2.105}$$

For the corresponding vertex function we get the expression

$$V^\mu(k_1, k_2) = -i\gamma^\mu e^{\frac{i}{2}k_1 \wedge k_2} - \frac{i}{2}(\tilde{k_1} - \tilde{k_2})^\mu (k\!\!\!/_1 + k\!\!\!/_2)\frac{e^{\frac{i}{2}k_1 \wedge k_2} - 1}{k_1 \wedge k_2} \tag{2.106}$$

where k_1 and k_2 are the incoming momenta of the outgoing and incoming fermions, respectively.

Photon Self-energy and UV/IR Mixing

Now, using the vertex function (2.106), we find the first order fermion loop correction to the photon propagator given by the one-loop photon self-energy diagram in Fig. (4.2) to be

$$\begin{aligned}\Pi^{\mu\nu}_{(1)}(k) &= -4\int\frac{d^4p}{(2\pi)^4}\times\left\{T^{\mu\nu}+\frac{i}{2}\frac{sim(\frac{1}{4}p\wedge k)}{\frac{1}{4}p\wedge k}\left[(\tilde{p}-\frac{1}{2}\tilde{k})^{\mu}k_{\rho}T^{\rho\nu}e^{-\frac{i}{4}p\wedge k}\right.\right.\\ &- \left.(\tilde{p}-\frac{1}{2}\tilde{k})^{\nu}k_{\rho}T^{\rho\mu}e^{\frac{i}{4}p\wedge k}\right]\\ &+ \left.\frac{i}{4}\frac{sim^2(\frac{1}{4}p\wedge k)}{(\frac{1}{4}p\wedge k)^2}(\tilde{p}-\frac{1}{2}\tilde{k})^{\mu}(\tilde{p}-\frac{1}{2}\tilde{k})^{\nu}k_{\rho}k_{\sigma}T^{\rho\sigma}\right\} \qquad (2.107)\end{aligned}$$

where

$$T^{\mu\nu}(k,p) := \frac{(p-k)^{\mu}p^{\nu}+p^{\mu}(p-k)^{\nu}+[m^2-(p-k).p]\eta^{\mu\nu}}{[(p-k)^2-m^2][p^2-m^2]}, \qquad (2.108)$$

which is the only term we get in the commutative case. Therefore, the first term in (2.107) is naturally understood to correspond to the planar part of the diagram, and in fact follows straightforwardly from the first terms of the vertex functions (2.106) as the phase factors cancel each other, in the same way as they do for the planar diagrams of a noncommutative scalar field theory. The other terms, on the other hand, clearly correspond to the nonplanar part with nontrivial phase factors that give rise to UV/IR mixing. Indeed, the second term in (2.106) can be shown to yield the leading order contribution (The third term leads also to similar IR-divergent terms)

$$i\Pi^{\mu\nu}_{(1)np}(k)\approx\frac{8}{\pi^2}\frac{\tilde{k}^{\mu}\tilde{k}^{\nu}}{\tilde{k}^4}\frac{4}{\pi^2}\frac{\tilde{\tilde{k}}^{\mu}k^{\nu}+k^{\mu}\tilde{\tilde{k}}^{\nu}}{\tilde{k}^4} \qquad (2.109)$$

at the IR-limit of the external momentum. The first term in (2.109) is similar to (2.98) found in the naive formulation above, whereas the second term is gauge variant and should cancel, when all the second order contributions in the coupling constant are taken into account. Therefore we conclude that also gauge field theories defined via $Seiberg-Witten$ map appear to fail to be renormalizable because of UV/IR mixing, which further shows that this is a generic property of noncommutative theories.

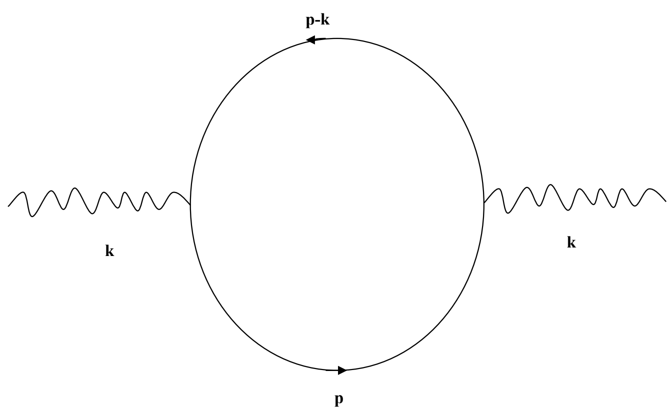


Figure 2.3: One-loop photon self-energy diagram.

2.4.4 Reduced Models and Emergent Phenomena

In this part we will work out the nonperturbative, construction definition of noncommutative Yang-Mills theory. this can be completely described in the language of matrix models (arising here as reduced models). This will also reveal some beautiful features of the vacuum structure of noncommutative gauge theories.

Will remove derivative operators ∂_i or $\hat{\partial}_i$ from the noncommutative gauge theory action. There is no analog of this manipulation in ordinary Yang-Mills theory [53].

Let us introduce covariant coordinates:

$$\hat{C}_i = (\theta^{-1})_{ij}\ \hat{x^j} + \hat{A}_i \tag{2.110}$$

Then $\hat{C}_i \to \hat{U}\hat{C}_i\hat{U}^\dagger$ under gauge transformations. We can be represented the adjoint actions

$$(\theta^{-1})_{ij}[\hat{x}^j, -] \tag{2.111}$$

i.e. $\hat{\partial}_i$ are inner derivations of the algebra $\mathcal{R}^d_\theta$. Then the entire noncommutative gauge theory can be rewritten in terms of the $\hat{C}_i$, we may rewrite the covariant derivative as:

$$\hat{\nabla}_i = \hat{\partial}'_i - i\hat{C}_i \tag{2.112}$$

where $\hat{\partial}'_i = \hat{\partial}_i + i(\theta^{-1})_{ij}\hat{x}^j$. Then using $[\hat{\partial}'_i, \hat{x}^j] = 0$, We compute:

$$[\hat{\nabla}_i, \hat{f}] = -i[\hat{C}_i, \hat{f}] \tag{2.113}$$

$$\begin{aligned} \hat{F}_{ij} &= i[\hat{\nabla}_i, \hat{\nabla}_j] \\ &= -i[\hat{C}_i, \hat{C}_j] + (\theta^{-1})_{ij} \end{aligned} \tag{2.114}$$

and consequently,

$$S_{YM} = \frac{1}{4g^2}\mathrm{Tr}\sum_{i\neq j}\left([\hat{C}_i, \hat{C}_j] + i(\theta^{-1})_{ij}\right)^2 \tag{2.115}$$

$\hat{C}_j$ are element of the $\mathcal{R}^d_\theta$, so spacetime derivatives have completely disappeared in this rewriting of noncommutative Yang-Mills theory. [53]

Flat connections $\hat{F}_{ij} = 0$ give:

$$\left[\hat{C}_i, \hat{C}_j\right] = -i(\theta^{-1})_{ij} \tag{2.116}$$

$\hat{C}_i$ are like the momentum operators $i\hat{\partial}_i$. In particular, $\hat{X}^i = \theta^{ij}\hat{C}_j$ formally represent noncommuting position operators in the ground state, wherein $\hat{X}^i = \hat{x}^i$ ($\hat{A}_i \equiv 0$). Then the noncommutative gauge degrees of freedom $\hat{C}_i$ are fluctuations around this canonical (Moyal) noncommutative spacetime. More generally noncommutative spacetime $[\hat{X}^i, \hat{X}^j] = i\Theta^{ij}\ (\hat{X})$ are obtained as non-vacuum solutions of Yang-Mills equation of motion:

$$\left[\hat{C}_i, [\hat{C}_i, \hat{C}_j]\right] = 0 \tag{2.117}$$

Thus (noncommutative) spacetime emerges as a dynamical effect in the matrix model. This is the essence of its relation to the so-called IKKT matrix model for the non-perturbative dynamics of type IIB superstrings, and also in the more recent models of emergent gravity which clarify the origin of gravity in noncommutative gauge theory. In this setting, gravity is related to quantum fluctuations Ci of spacetime at the Planck scale, while noncommutative field theory arises from field dependent fluctuations of spacetime geometry (determined via $\theta_{ij}(x)$). In particular, UV/IR mixing arises due to a non-renormalizable gravitational sector in the IR with G $\backsim$ Λ [53]

This large N matrix model is is called a twisted reduced model. The twist is $(\theta^{-1})_{ij}$. The noncommutative spacetime $\mathcal{R}^d_\theta$ is effectively hidden in the infinitelymany degrees of freedom of the large N matrices $\hat{C}_i$ and it reappears from expanding the matrix model around its classical vacuum (this is a dynamical emergent of spacetime). It is formally gotten by reduction of ordinary Yangs-Mills theory (with background flux) [53]

The (straight) open Wilson line has a particularly simple form in this matrix model formulation:

$$O(C_k) := \int d^dx\mathrm{tr}(U(x_j, C_k)) \star e^{ik.x})\mathrm{Tr}e^{ik.\theta.\hat{C}} \tag{2.118}$$

which is manifestly gauge-invariant under $\hat{C}_i \to \hat{g}\hat{C}_i\hat{g}^{-1}$
This follows easily from:

$$O(C_k) = \mathrm{Tr}\big(\hat{U}(C_k)\hat{D}(C_k)^\dagger e^{ik.x}\big) \tag{2.119}$$

2.5 The classical Lagrangian and Hamiltonian dynamics of matrix models

The fundamental idea is to set up an analog of classical dynamics in which the phase space variables are non-commutative, and the basic tool that allows one to accomplish this is cyclic invariance under a trace. Since no assumptions about commutativity of the phase space variables (such as canonical commutators/anticommutators) are made at this stage, the dynamics that we set up is not the same as standard quantum mechanics.

2.5.1 Bosonic and fermionic matrices

We shall assume finite-dimensional matrices, although ultimately an extension to the infinite-dimensional case may be needed. The matrix elements of these matrices will be constructed from ordinary complex numbers, and from complex anti-commuting Grassmann numbers. Just as a complex number can be decomposed into real and imaginaty parts

$$\mathcal{C} = \mathcal{C}_R + i\mathcal{C}_I \tag{2.120}$$

a complex Grassmann number can be decomposed into real and imaginary parts

$$\mathcal{X} = \mathcal{X}_R + i\mathcal{X}_I \tag{2.121}$$

With $\mathcal{C}_{R,I}$ and $\mathcal{X}_{R,I}$ is real. Real Grassmann numbers are built up as products of a basis of real Grassmann element $\mathcal{X}_1, \mathcal{X}_2$,.... which obey the anticommutative algebra $\{\mathcal{X}_r, \mathcal{X}_s\} = 0$

Let B_1 and B_2 be two $N \times N$ matrices with matrix elements that are even grade elements of a Grassmann algebra over the complex numbers, and let Tr be the ordinary matrix trace, which obeys the cyclic property

$$\mathrm{Tr} B_1 B_2 = \sum_{m,n} (B_1)_{mn} (B_2)_{nm} = \sum_{m,n} (B_2)_{nm} (B_1)_{mn} = \mathrm{Tr} B_2 B_1 \tag{2.122}$$

Similarly, let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two $N \times N$ matrices with matrix elements that are odd grade elements of a Grassmann algebra over the complex numbers, , which anticommute rather than commute, so that the cyclic property for these takes the form

$$\mathrm{Tr} \mathcal{X}_1 \mathcal{X}_2 = \sum_{m,n} (\mathcal{X}_1)_{mn} (\mathcal{X}_2)_{nm} = -\sum_{m,n} (\mathcal{X}_2)_{nm} (\mathcal{X}_1)_{mn} = -\mathrm{Tr} \mathcal{X}_2 \mathcal{X}_1 \tag{2.123}$$

Final bilinear cyclic identity

$$\mathrm{Tr}B\mathcal{X} = \mathrm{Tr}\mathcal{X}B. \tag{2.124}$$

We shall refer to the Grassmann even and Grassmann odd matrices B, $\mathcal{X}$ as being of bosonic and fermionic type, respectively. Clearly, operators that are of mixed bosonic and fermionic type can always be linearly decomposed into components that are purely bosonic or purely fermionic in character.

The extra minus sign that appears in the odd grade case of Eq (2.123) has implications for the adjoint properties of matrices. Letting $\mathcal{O}^g$ be a matrix of grade g, we define the adjoint by

$$(\mathcal{O}^\dagger)_{mn} = (\mathcal{O}^*)_{mn}, \tag{2.125}$$

Letting now $\mathcal{O}_1^{g_1}$ and $\mathcal{O}_2^{g_2}$ be two matrices of grade g_1 and g_2 respectively, this definition implies that

$$\begin{aligned}
(\mathcal{O}_1^{g_1}\mathcal{O}_2^{g_2})^\dagger_{mn} &= (\mathcal{O}_1^{g_1}\mathcal{O}_2^{g_2})^*_{nm} = \sum_k (\mathcal{O}_1^{g_1})^*_{nk}(\mathcal{O}_2^{g_2})^*_{km} \\
&= (-1)^{g_1 g_2}\sum_k (\mathcal{O}_2^{g_2})^*_{km}(\mathcal{O}_1^{g_1})^*_{nk} = (-1)^{g_1 g_2}\sum_k (\mathcal{O}_2^{g_2})^\dagger_{mk}(\mathcal{O}_1^{g_1})^\dagger_{kn} \\
&= (-1)^{g_1 g_2}(\mathcal{O}_2^{g_2})^\dagger\mathcal{O}_1^{g_1})^\dagger)_{mn}
\end{aligned} \tag{2.126}$$

The cyclic/anticyclic properties of Eqs (2.122,2.124) are the basic identities from which further cyclic properties can be derived. For example, from the basic bilinear identities one immediately derives the trilinear cyclic identities

$$\begin{aligned}
\mathrm{Tr}B_1[B_2,B_3] &= \mathrm{Tr}B_2[B_3,B_1] = \mathrm{Tr}B_3[B_1,B_2], \\
\mathrm{Tr}B_1\{B_2,B_3\} &= \mathrm{Tr}B_2\{B_3,B_1\} = \mathrm{Tr}B_3\{B_1,B_2\}, \\
\mathrm{Tr}B\{\mathcal{X}_1,\mathcal{X}_2\} &= \mathrm{Tr}\mathcal{X}_1[\mathcal{X}_2,B] = \mathrm{Tr}\mathcal{X}_2[\mathcal{X}_1,B], \\
\mathrm{Tr}\mathcal{X}_1\{B,\mathcal{X}_2\} &= \mathrm{Tr}\{\mathcal{X}_1,B\}\mathcal{X}_2 = \mathrm{Tr}[\mathcal{X}_1,\mathcal{X}_2]B, \\
\mathrm{Tr},\mathcal{X}[B_1,B_2] &= \mathrm{Tr}B_2[,\mathcal{X},B_1] = \mathrm{Tr}B_1[B_2,,\mathcal{X}], \\
\mathrm{Tr},\mathcal{X}\{B_1,B_2\} &= \mathrm{Tr}B_2\{,\mathcal{X},B_1\} = \mathrm{Tr}B_1\{B_2,\mathcal{X}\}, \\
\mathrm{Tr}\mathcal{X}_1\{\mathcal{X}_2,\mathcal{X}_3\} &= \mathrm{Tr}\mathcal{X}_2\{\mathcal{X}_3,\mathcal{X}_1\} = \mathrm{Tr}\mathcal{X}_3\{\mathcal{X}_1,\mathcal{X}_2\}, \\
\mathrm{Tr}\mathcal{X}_1[\mathcal{X}_2,\mathcal{X}_3] &= \mathrm{Tr}\mathcal{X}_2[\mathcal{X}_3,\mathcal{X}_1] = \mathrm{Tr}\mathcal{X}_3[\mathcal{X}_1,\mathcal{X}_2],
\end{aligned} \tag{2.127}$$

2.5.2 Derivative of a trace with respect to an operator

The basic observation of trace dynamics given the trace of a polynomial P constructed from non-commuting matrix or operator variables, one can define a derivative of the complex number TrP with respect to an operator variable $\mathcal{O}$ by varying and then cyclically permuting so that in each term the factor $\delta\mathcal{O}$ stands on the right. This gives the fundamental definition

$$\delta \mathrm{Tr} P = \mathrm{Tr}\frac{\delta \mathrm{TrP}}{\delta\mathcal{O}}\delta\mathcal{O} \tag{2.128}$$

which for arbitrary infinitesimal $\delta\mathcal{O}$ defines the operator $\delta\mathrm{TrP}/\delta\mathcal{O}$. In general we will take $\mathcal{O}$ to be either of bosonic or fermionic (but not of mixed), and we will construct δTrP to always be an even grade element of the Grassmann algebra. (When P is fermionic, we can always make it bosonic by multiplying it by a c-number auxiliary Grassmann element α). With these restrictions, for $\delta\mathcal{O}$ of the same type as $\mathcal{O}$, the operator derivative $\delta\mathrm{TrP}/\delta\mathcal{O}$ will be of the same type as $\mathcal{O}$, that is, either both will be bosonic or both will be fermionic .

Suppose that P is a bosonic monomial containing only a single factor of the operator $\mathcal{O}$, so that P has the form

$$P = A\mathcal{O}B, \tag{2.129}$$

with A and B operators that in general do not commute with each other or with $\mathcal{O}$. THen when $\mathcal{O}$ is varied, the corresponding variation of P is $\delta P = A(\delta\mathcal{O})B$, and so cyclically permuting B to the left we have

$$\delta \mathrm{Tr} P = \epsilon_B \mathrm{Tr} BA\delta\mathcal{O}, \tag{2.130}$$

$$\frac{\delta \mathrm{Tr} P}{\delta\mathcal{O}} = \epsilon_B BA, \tag{2.131}$$

$$\begin{cases} \epsilon_B = 1 & \text{when the operator B is bosonic} \\ \epsilon_B = -1 & \text{when the operator B is fremionic.} \end{cases}$$

The operator product $A\mathcal{O}$ is of the same bosonic or fermionic type as B, so we have $\epsilon_B = \epsilon_{A\mathcal{O}}$ and could equally well write

$$\frac{\delta \mathrm{Tr} P}{\delta\mathcal{O}} = \epsilon_{A\mathcal{O}} BA, \tag{2.132}$$

Suppose that P is a bosonic monomial containing two factors of the operator $\mathcal{O}$ that is being varied, and so has the general structure

$$P = A\mathcal{O}B\mathcal{O}C, \tag{2.133}$$

with A, B, and C operators that in general do not commute with each other or with $\mathcal{O}$

The variation of P is

$$\delta P = A(\delta\mathcal{O})B\mathcal{O}C + A\mathcal{O}B(\delta\mathcal{O})C \tag{2.134}$$

Thus we have in this case

$$\begin{aligned} \delta \mathrm{Tr} P &= \mathrm{Tr}\big(\epsilon_{A\mathcal{O}} B\mathcal{O}CA(\delta\mathcal{O}) + \epsilon_C CA\mathcal{O}B(\delta\mathcal{O})\big), \\ \frac{\delta \mathrm{Tr} P}{\delta\mathcal{O}} &= \epsilon_{A\mathcal{O}} B\mathcal{O}CA + \epsilon_C CA\mathcal{O}B, \end{aligned} \tag{2.135}$$

$$\begin{cases} \epsilon_C = 1(-1) & \text{According as whether C is bosonic (fermionic)} \\ \epsilon_{A\mathcal{O}} = 1(-1) & \text{According as whether the product A}\mathcal{O}\text{ is bosonic (fermionic).} \end{cases}$$

Let us expand $\delta \mathrm{Tr} P/\delta\mathcal{O}$ is the form

$$\frac{\delta \mathrm{Tr} P}{\delta\mathcal{O}} = \sum_n C_n K_n \tag{2.136}$$

with the K_n distinct Grassmann monomials that are all c-numbers (i.e., multiples of the $N\times N$ unit matrix), and with the C_n complex matrix coefficients that are unit elements in the Grassmann algebra

Let us choose $\delta\mathcal{O}$ to be an infinitesimal α times $C_p^\dagger$, with α a real number when $\mathcal{O}$ is bosonic, and with α a Grassmann element not appearing in K_p when $\mathcal{O}$ is fermionic. There must be at least one such element, or else K_p would make an identically vanishing contribution to Eq (2.128), and could not appear in the sum in Eq (2.136) We then have

$$\sum_0 \mathrm{Tr} C_p^\dagger C_n K_n \alpha = 0 \tag{2.137}$$

the coefficients of all distinct Grassmann monomials must vanish separately, we have

$$\mathrm{Tr} C_p^\dagger C_p = 0 \tag{2.138}$$

We conclude that

$$\frac{\delta \mathrm{Tr} P}{\delta\mathcal{O}} = 0 \tag{2.139}$$

When $\mathcal{O}$ is bosonic, a useful extension of the above result states that the vanishing of δTrP for all self-adjoint variations $\delta\mathcal{O}$. The C_n split into self-adjoint and anti-self adjoint parts,

$$C_n = C_n^{sa} + C_n^{asa}, \tag{2.140}$$

with $C_n^{sa} = C_n^{sa\dagger}$ and $C_n^{asa} = C_n^{asa\dagger}$. For self-adjoint $\delta\mathcal{O}$ implies that $\mathrm{Tr}C_n^{sa}\delta\mathcal{O}$ is real, and $\mathrm{Tr}C_n^{asa}\delta\mathcal{O}$ is imaginary. The vanishing $\delta\mathrm{Tr}P$ implies that both of these traces must vanish separately. Taking $\delta\mathcal{O} = C_p^{sa}$ then implies the vanishing of C_p^{sa}, while taking $\delta\mathcal{O} = iC_p^{asa}$ then implies the vanishing of C_p^{asa}.

In our applications, we shall often consider trace functionals $\delta\mathrm{Tr}P$ that are real, which will be true when the adjointness properties of the operators from which P is constructed imply that $P - P^\dagger$ is either zero or is an operator with identically vanishing trace. Real trace functionals $\delta\mathrm{Tr}P$ have the important property that when $\mathcal{O}$ is a self-adjoint bosonic operator, then $\delta\mathrm{Tr}P/\delta\mathcal{O}$ is also self-adjoint. To prove this, we make a self-adjoint variation $\mathcal{O}$, and use the reality of $\delta\mathrm{Tr}P$ to write

$$\begin{aligned} 0 \;&\equiv\; \mathrm{Im}\mathrm{Tr}\delta\mathrm{Tr}P \propto \mathrm{Tr}\left[\frac{\delta\mathrm{Tr}P}{\delta\mathcal{O}}\delta\mathcal{O} - (\delta\mathcal{O})^\dagger\left(\frac{\delta\mathrm{Tr}P}{\delta\mathcal{O}}\right)^\dagger\right] \\ &= \mathrm{Tr}\delta\mathcal{O}\left[\frac{\delta\mathrm{Tr}P}{\delta\mathcal{O}} - \left(\frac{\delta\mathrm{Tr}P}{\delta\mathcal{O}}\right)^\dagger\right]. \end{aligned} \tag{2.141}$$

2.5.3 Lagrangian and Hamiltonian dynamics of matrix models

Let $\mathrm{L}\big[\{q_r\},\{\dot{q}_r\}\big]$ be a Grassmann even polynomial function of the bosonic or fermionic operators $\{q_r\}$ and their time derivatives $\{\dot{q}_r\}$. The discrete index r labels the matrix degrees of freedom for a general matrix dynamics.

Just as a classical dynamical system can have any number of degrees of freedom, the numbers n_B and n_F of bosonic and fermionic operators $\{q_r\}$ are arbitrary, and are unrelated to the dimension N of the matrices that represent these operators. From L, we form the trace Lagrangian

$$\mathbf{L}\big[\{q_r\},\{\dot{q}_r\}\big] = \mathrm{Tr}L\big[\{q_r\},\{\dot{q}_r\}\big] \tag{2.142}$$

The corresponding action

$$\mathbf{S} = \int dt\mathbf{L} \tag{2.143}$$

Using the definition of Eq (2.128) for get variations of the action

$$0 = \mathbf{S} = \int dt \mathrm{Tr} \sum_r \left(\frac{\delta \mathbf{L}}{\delta q_r} \delta q_r + \frac{\delta \mathbf{L}}{\delta \dot{q}_r} \delta \dot{q}_r \right) \tag{2.144}$$

or after integrating by parts in the second term and discarding surface terms

$$0 = \mathbf{S} = \int dt \mathrm{Tr} \sum_r \left(\frac{\delta \mathbf{L}}{\delta q_r} - \frac{d}{dt} \frac{\delta \mathbf{L}}{\delta \dot{q}_r} \right) \delta q_r. \tag{2.145}$$

For this to hold for general same-type operator variations δq_r, the coefficient of each δq_r in Eq (2.145) must vanish for all t, giving the operator Euler−Lagrange equations

$$\frac{\delta \mathbf{L}}{\delta q_r} - \frac{d}{dt} \frac{\delta \mathbf{L}}{\delta \dot{q}_r} = 0. \tag{2.146}$$

by the definition of Eq. (2.128), we have

$$\left(\frac{\delta \mathbf{L}}{\delta q_r} \right)_{ij} = \frac{\delta \mathbf{L}}{\delta (q_r)_{ij}} = 0, \tag{2.147}$$

for each r the single EulerLagrange equation of Eq. (2.146) is equivalent to the N^2 Euler-Lagrange equations obtained by regarding L as a function of the N^2 matrix element variables $(q_r)_{ji}$.

Let us now define the momentum operator p_r conjugate to q_r by

$$P_r \equiv \frac{\delta \mathbf{L}}{\delta \dot{q}_r} \tag{2.148}$$

p_r is of the same bosonic or fermionic type as q_r . We can now introduce a trace Hamiltonian $\mathbf{H}$ by analogy with the usual definition

$$\mathbf{H} = \mathrm{Tr} \sum_r p_r \dot{q}_r - \mathbf{L} \tag{2.149}$$

The variation is

$$\begin{aligned} \delta \mathbf{H} &= \mathrm{Tr} \sum_r \big((\delta p_r) \dot{q}_r + p_r \delta \dot{q}_r \big) - \mathrm{Tr} \sum_r \left(\frac{\delta \mathbf{L}}{\delta q_r} \delta q_r + \frac{\delta \mathbf{L}}{\delta \dot{q}_r} \delta \dot{q}_r \right) \\ &= \mathrm{Tr} \sum_r ((\delta p_r) \dot{q}_r - \dot{p}_r \delta q_r) \\ &= \mathrm{Tr} \sum_r \big(\epsilon_r \dot{q}_r) \delta p_r - \dot{p}_r \delta q_r \big). \end{aligned} \tag{2.150}$$

Therefore the trace Hamiltonian $\mathbf{H}$ is a trace functional of the operators $\{q_r\}$ and $\{p_r\}$

$$\mathbf{H} = \mathbf{H}\big[\{q_r\},\{p_r\}\big], \tag{2.151}$$

with the operator derivatives

$$\frac{\delta \mathbf{H}}{\delta q_r} = -\dot{p}_r, \qquad \frac{\delta \mathbf{H}}{\delta p_r} = \epsilon_r \dot{q}_r \tag{2.152}$$

$$\begin{cases} \epsilon_r = 1 & \text{According to whether } q_r, p_r \text{ are bosonic} \\ \epsilon_r = -1 & \text{According to whether } q_r, p_r \text{ are fremionic} \end{cases}$$

2.5.4 Trace dynamics models with global supersymmetry

The Wess-Zumino model

We begin with the trace dynamics transcription of the Wess-Zumino model. The trace Lagrangian for the WessZumino

$$\begin{aligned} \mathbf{L} &= \int d^3x \mathrm{Tr}\Big(-\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \bar{\chi}\gamma^\mu \partial_\mu \chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\ &- m(AF + BG - \bar{\chi}\chi) - \lambda[(A^2 - B^2)F + G\{A,B\} - 2\bar{\chi}(A - i\gamma_5 B)\chi]\Big) \end{aligned} \tag{2.153}$$

$\gamma^{1,2,3}$ are real symmetricand γ^0, iγ^5 are real skew-symmetric. A,B,F,G self-adjoint $N \times N$ lorentz matrices and χ a Grassmann 4-component column vector spinor. The notation $\bar{\chi}$ is defined by $\bar{\chi} = \chi^T \gamma^0$. The numerical parameters λ and m are respectively the coupling constant and mass.

Taking operator variations of Eq (2.153) by using the recipe of Eq (2.128) , the Euler-Lagrange equations of Eq. (2.146) take the form

$$\begin{aligned} \partial^2 A &= mF + \lambda(\{A,F\} + \{B,G\} - 2\bar{\chi}\chi), \\ \partial^2 B &= mG + \lambda(-\{B,F\} + \{A,G\} - 2i\bar{\chi}\gamma_5\chi), \\ \gamma^\mu \partial_\mu \chi &= m\chi + \lambda(\{A,\chi\} - \{B,\gamma_5\chi\}, \\ F &= mA + \lambda(A^2, B^2), \\ G &= mB + \lambda\{A,B\}. \end{aligned} \tag{2.154}$$

Transforming to Hamiltonian form, the canonical momenta of Eq (2.148)

$$\begin{aligned} P_\chi &= -\bar{\chi}\gamma^0 = \chi^T, \\ P_A &= \partial_0 A, \\ P_B &= \partial_0 B, \end{aligned} \tag{2.155}$$

and the trace Hamiltonian is given by

$$\begin{aligned} \mathbf{H} &= \int d^3x \mathrm{Tr}\Big(\frac{1}{2}[P_A^2 + P_B^2 + (\vec{\nabla}A)^2 + (\vec{\nabla}B)^2] + P_\chi\gamma^0\vec{\gamma}.\vec{\nabla}\chi \\ &+ \frac{1}{2}(F^2 + G^2) - m\bar{\chi}\chi - \lambda P_\chi \gamma^0\{A - i\gamma_5 B, \chi\}\Big). \end{aligned} \tag{2.156}$$

The trace three-momentum $\vec{\mathbf{P}}$, wgich together with $\mathbf{H}$ forms the trace four-momentum $\mathbf{P}^\sigma$, is given by

$$\vec{\mathbf{P}} = \int d^3x \mathrm{Tr}(P_A\vec{\nabla}A + P_B\vec{\nabla}B + P_\chi\vec{\nabla}\chi). \tag{2.157}$$

Let us now perform a supersymmetry variation of the fields given by

$$\begin{aligned} \delta A &= \bar{\epsilon}\chi, \\ \delta B &= i\bar{\epsilon}\gamma_5\chi, \\ \delta\chi &= \frac{1}{2}[F + i\gamma_5 G + \gamma^\mu\partial_\mu(A + i\gamma_5 B)]\epsilon, \qquad (2.158) \\ \delta F &= \bar{\epsilon}\gamma^\mu\partial_\mu\chi, \\ \delta G &= i\bar{\epsilon}\gamma_5\gamma^\mu\partial_\mu\chi, \end{aligned} \tag{2.159}$$

with ϵ a c-number Grassmann spinor (i.e., a four-component spinor, the spin components of which are 1×1 Grassmann matrices). Substituting Eq (2.158) into the trace Lagrangian of Eq (2.153). The variation of $\mathbf{L}$ is given by

$$\delta\mathbf{L} = \int d^3 \mathrm{Tr}(\bar{J}^\mu\partial_\mu\epsilon) \tag{2.160}$$

$$\bar{J}^\mu = -\bar{\chi}\gamma^\mu[(\gamma^\nu\partial_\nu + m)(A + i\gamma_5 B) + \lambda(A^2 - B^2 + i\gamma_5\{A, B\})] \tag{2.161}$$

which identifies the trace supercharge $\mathbf{Q}_\alpha$ as

$$\begin{aligned}\mathbf{Q}_\alpha &= \int d^3 \mathrm{Tr} \bar{J}^0 \alpha \qquad (2.162)\\ &= \int d^3 \mathrm{Tr} \frac{1}{2}(P_\chi + \chi^T)[(\gamma^\nu \partial_\nu + m)(A + i\gamma_5 B) \qquad (2.163)\\ &+ \lambda(A^2 - B^2 + i\gamma_5\{A, B\})]\alpha.\end{aligned}$$

2.5.5 The supersymmetric YangMills model

As a second example of a trace dynamics model with global supersymmetry, we discuss supersymmetric YangMills theory.

We start from the trace Lagrangian

$$\mathbf{L} = \int d^3 x \mathrm{Tr}\left[\frac{1}{4g^2} F_{\mu\nu}^2 - \bar{\chi}\gamma^\mu D_\mu \chi + \frac{1}{2} D^2\right] \qquad (2.164)$$

with the field strength $F_{\mu\nu}$ and covariant derivative D_μ constructed from the gauge potential A_μ according to

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],\\ D_\mu \mathcal{O} &= \partial_\mu \mathcal{O} + [A_\mu, \mathcal{O}] \qquad (2.165)\end{aligned}$$

with

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0.$$

The Euler-Lagrange equations of motion are

$$\begin{aligned}D &= 0,\\ \gamma^\mu D_\mu \chi &= 0, \qquad (2.166)\\ D_\mu F^{\mu\nu} &= 2g^2 \bar{\chi}\gamma^\nu \chi\end{aligned}$$

as usual for a gauge system, the $\nu = 0$ component of Eq. (2.166) is not a dynamical evolution equation, but rather the constraint

$$D_l F^{l0} = 2g^2 \bar{\chi}\gamma^0 \chi. \qquad (2.167)$$

Going over to the Hamiltonian formalism, the canonical momenta are given by

$$PA_l = -\frac{1}{g^2} F_{0l}, \quad P_\chi = \chi^T, \qquad (2.168)$$

and the axial gauge trace Hamiltonian is

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_\chi, \tag{2.169}$$

with

$$\begin{aligned}\mathbf{H}_A &= \int d^3x \mathrm{Tr}\bigg(\frac{-g^2}{2}\sum_{l=1}^{2} p_{A_l}^2 - \frac{1}{2g^2}F_{03}^2 \\ &- \frac{1}{2g^2}(\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2])^2 - \frac{1}{2g^2}[\partial_3 A_1 + (\partial_3 A_2)^2]\bigg),\end{aligned} \tag{2.170}$$

$$F_{03} = \frac{1}{2}g^2 \int_{-\infty}^{\infty} dz^{'} \epsilon(z - z^{'})[-(p_\chi \chi + \chi^T p_\chi^T) + D_1 p_{A_1} + D_2 p_{A_2}]|_{z'} \tag{2.171}$$

$$\mathbf{H}_\chi = \int d^3x \mathrm{Tr}(p_\chi \gamma^0 \gamma_l D_l \chi) \tag{2.172}$$

where we have taken care to write $\mathbf{H}$ in a form symmetric in the identical quantities p_χ and χ^T, and where $\epsilon(z) = 1(-1)$ for z>0 (z<0). The trace three-momentum is

$$\mathbf{P}_m = \int d^3x \mathrm{Tr}\bigg(\sum_{l=1}^{3} F_{ml} p_{A_l} + p_\chi D_l \chi\bigg) \tag{2.173}$$

and the conserved operator C of Eq. (2.6) is given by

$$\tilde{C} = \int d^3x \bigg(\sum_{l=1}^{2}[A_l, p_{A_l}] - \chi, p_\chi\bigg) \tag{2.174}$$

with a contraction of the spinor indices in the final term of Eq. (2.174) understood. By virtue of the constraint of Eq. (2.167), the conserved operator $\tilde{C}$ can also be written as

$$\tilde{C} = \int d^3x \sum_{l=1}^{3} \partial_l p_{A_l} = -\int_{\text{sphere at }\infty} d^2 S_l p_{A_l} \tag{2.175}$$

which vanishes when the surface integral in Eq. (2.175) is zero. The corresponding conserved current $\tilde{C}^\mu$, of which $\tilde{C}$ is the charge, is given by

$$\tilde{C}^\mu = \frac{1}{g^2}[A_\nu, F^{\mu\nu}] + 2\bar{\chi}\gamma^\mu\chi. \tag{2.176}$$

The conserved trace quantity $\mathbf{N}$ and the corresponding conserved current $\mathbf{N}^\mu$ have the same form as in the WessZumino model

$$\begin{aligned} \mathbf{N} &= -i\int d^3x \mathrm{Tr}\chi^T\chi \\ \mathbf{N}^\nu &= i\mathrm{Tr}\bar{\chi}\gamma^\mu\chi \end{aligned} \tag{2.177}$$

Making now the supersymmetry variations

$$\begin{aligned} \delta A_\mu &= ig\bar{\epsilon}\gamma_\mu\chi \\ \delta\chi &= \left(\frac{i}{8g}[\gamma_\mu,\gamma_\nu]F^{\mu\nu}+\frac{i}{2}\gamma_5 D\right)\epsilon, \\ \delta D &= i\bar{\epsilon}\gamma_5\gamma^\mu D_\mu\chi \end{aligned} \tag{2.178}$$

in the trace Lagrangian, with ϵ again a c-number Grassmann spinor, we find using cyclic invariance under the trace and the γ matrix identities D that whenϵ is constant, the variation vanishes. When ϵ is not a constant, the variation of $\mathbf{L}$ is given by

$$\begin{aligned} \delta\mathbf{L} &= -i\int d^3x \mathrm{Tr}(\bar{j}\partial_\mu\epsilon), \\ \bar{j}^\mu &= \frac{i}{4g}\bar{\chi}\gamma^\mu F_{\nu\rho}[\gamma^\nu,\gamma^{rho}] \end{aligned} \tag{2.179}$$

2.6 Matrix model approach

Field theories and especially gauge theories admit many classical solutions: solitons, instantons and branes, which play important roles in nonperturbative physics. Using this approach, it was possible to study many nonperturbative features of noncommutative field theory such as solitons and instantons (for more details see [62] in references).

We will call this approach matrix model approach, as the gauge theory can be described as a matrix model having the noncommutative space as its ground state, the fluctuations creating the gauge theory. Therefore we are looking for spaces that can be represented as fnite-dimensional matrix algebras, where everything is well defned. The space on which we will base our constructions will be the fuzzy sphere [63], an N-dimensional matrix algebra corresponding to a truncation of the spherical harmonics on the sphere at angular momentum $N-1$. In four dimensions, using the product of two such fuzzy spheres $S_N^2 \times S_N^2$, generated by N^2-dimensional matrices. When the two spheres they have a same limit, this fuzzy space goes over to the product of two commutative spheres, but in a diferent limit, it also goes to noncommutative $\mathbb{R}^4$ with canonical commutation relations.

2.6.1 The Heisenberg algebra

In two dimensions, the coordinate algebra with canonical deformation

$$[x, y] = i\theta \tag{2.180}$$

The noncommutativity isn't between the coordinates and momenta, it is between the coordinares themselves. We can use the usual Fock space representation for this aglbebra.

The Fock space is given by

$$\mathcal{H} = \big\{ \mid n >, n \in \mathbb{N}_0 \big\} \tag{2.181}$$

And defining

$$x_\pm := x \pm iy \tag{2.182}$$

with

$$\big[x_+, x_-\big] = 2\theta. \tag{2.183}$$

Now define the creator and annihilation (or destruction) operators usually introduced in the process of second quantization

$$x_- \mid n >= \sqrt{2\theta}\sqrt{n+1} \mid n+1 >, \quad x_+ \mid n >= \sqrt{2\theta}\sqrt{n} \mid n-1 > \quad (2.184)$$

One can then consider a Fock space with a basis $|n>$ $(n \geqslant 0)$ provided by the eigenfunctions of the number operator N,

$$\hat{N} = \hat{x}_- \hat{x}_+ \quad (2.185)$$

$$\hat{N}|n >= n|n > \quad (2.186)$$

and the vacuum state $|0>$ defined so that

$$x_-|0 >= 0 \quad (2.187)$$

For $\theta \simeq 0$ one can write

$$\hat{N} = \hat{x}_- \hat{x}_+ \approx (x^2 + y^2)/2\theta = r^2/2\theta \quad (2.188)$$

so that configuration space at infinity can be connected with $n \to \infty$ in Fock space.

This can be generalized to higher dimensions. Any 2n-dimensional algebra with commutation relations. As we will mostly be concerned with the 4-dimensional case in the following.

The most general noncommutative $\mathcal{R}^4_\theta$ is generated by coordinates subject to the commutation relations

$$\left[x_\mu, x_\nu\right] = i\theta_{\mu\nu}, \qquad \mu,\nu \in \{1,...,4\} \quad (2.189)$$

$\theta_{\mu\nu}$ can always be cast into the form

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}$$

To simplify the following formulas, we restrict our discussion from now on to the selfdual case

$$\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma} \quad (2.190)$$

and denote

$$\theta := \theta_{12} = \theta_{34} \tag{2.191}$$

The complex coordinates

$$x_{\pm L} := x_1 \pm i x_2 \quad , \quad x_{\pm R} := x_3 \pm i x_4, \tag{2.192}$$

the commutation relations (2.189) take the form

$$[x_{+a}, x_{-b}] = 2\theta\delta_{ab}, \quad [x_{+a}, x_{+b}] = [x_{-a}, x_{-b}] = 0 \tag{2.193}$$

and the standard basis

$$\mathcal{H} = \{| \, n_1, n_2 >, \quad n_1, n_2 \in \mathbb{N}_0\} \tag{2.194}$$

with

$$\begin{aligned} x_{-L} \mid n_1, n_2 > &= \sqrt{2\theta}\sqrt{n_1 + 1} \mid n_1 + 1, n_2 > \\ x_{+L} \mid n_1, n_2 > &= \sqrt{2\theta}\sqrt{n_1} \mid n_1 - 1, n_2 > \\ x_{-R} \mid n_1, n_2 > &= \sqrt{2\theta}\sqrt{n_2 + 1} \mid n_1, n_2 + 1 > \\ x_{+R} \mid n_1, n_2 > &= \sqrt{2\theta}\sqrt{n_2} \mid n_1, n_2 - 1 > \end{aligned} \tag{2.195}$$

2.6.2 Noncommutative gauge theory

Let us start by considering the noncommutative version of a pure $U(1)$ gauge theory. We can introduce gauge theory by using a matrix action

$$S = -\frac{(2\pi)^2}{2g^2\theta^2}\mathrm{tr}([X_\mu, X_\nu] - i\theta_{\mu\nu})^2, \tag{2.196}$$

where the X_μ are infinite-dimensional matrices, and the trace is over the Fock space (2.194).

The action is invariant under unitary transformations

$$X_\mu \to U^\dagger X_\mu U \tag{2.197}$$

The fluctuation A_μ around the ground state x_μ as

$$X_\mu = x_\mu + A_\mu \tag{2.198}$$

The fluctuations A_μ are understood as infinite-dimensional matrices acting on the Fock space (2.194) as well. They have to transform as

$$X_\mu \to U^\dagger X_\mu [x_\mu, U] + U^\dagger A_\mu U \tag{2.199}$$

to make the X_μ gauge covariant. The gauge covariant field strength then reads

$$iF_{\mu\nu} = ([X_\mu, X_\nu] - i\theta_{\mu\nu}) = [x_\mu, A_\nu] - [x_\nu, A_\mu] + [A_\mu, A_\nu] \tag{2.200}$$

and the action (2.196) becomes

$$S = \frac{(2\pi)^2}{2g^2\theta^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \tag{2.201}$$

We can also use the complex covariant coordinates

$$X_{+L} = X_1 + iX_2 \quad , \quad X_{+R} = X_3 + iX_4 \tag{2.202}$$
$$X_{-L} = X_1 - iX_2 \quad , \quad X_{-R} = X_3 - iX_4 \tag{2.203}$$

and the corresponding field strength

$$F_{\alpha a, \beta b} = [X_{\alpha a}, X_{\beta b}] - 2\theta\epsilon_{\alpha\beta}\delta_{ab} \tag{2.204}$$

with a,b $\in \{L, R\}$ and $\alpha, \beta \in \{+, -\}$. The action (2.196) can xow be written in the form

$$S = \frac{(\pi)^2}{g^2\theta^2}\mathrm{tr}(\sum_a F_{+a,-a}F_{+a,-a} - \sum_{a,b} F_{+a,+b}F_{-a,-b}) \tag{2.205}$$

and the equations of motion are given by

$$\sum_{a,\alpha}[X_{\alpha a}, (F_{\alpha a, \beta b})^\dagger] = 0 \tag{2.206}$$

2.6.3 $U(1)$ instantons on $\mathcal{R}^4_\theta$

We will look for solutions of the equation of motion (2.206) which can be understood as instantons of the gauge theory

In Ref [64] they schowing it's a in noncommutative $\mathcal{R}^2_\theta$ that the noncommutative gauge theory contains the classical and quantum dynamics of all

U(N) gauge theories and that classical solutions are labeled by the rank of the gauge group and the magnetic charge. Also the BFS solutions describe variouns D-1 string attached. They can be interpreted as localized flux solutions, sometimes called fluxons.

The situation on $\mathcal{R}^4_\theta$ is more complicated, and there are diferent types of non-trivial U (1) instanton solutions on $\mathcal{R}^4_\theta$

Assuming that $\theta_{\mu\nu}$ is self-dual, there are two types of instantons: [65]

1 -There there exist straightforward generalizations of the two-dimensional localized fluxon solutions with self-dual field strength. As in the two-dimensional case, we will refer to these 4-dimensional solutions as fluxons.

2 -There are other types of U (1) instantons on $\mathcal{R}^4_\theta$, in particular anti-selfdual instantons which are much less localized than the fluxon solutions.

For the construction of the fluxons, let us consider a finite dimensional subvectorspace V_n of the Fock-space H of dimension n spanned by finite set of vectors $|n_1, n_2 >\in \mathcal{H}$

$$V_n =< \{|i_k, j_k >;\ k = 1, ..., n\} > . \tag{2.207}$$

We introduce a partial isometry S mapping $\mathcal{H}$ to $\mathcal{H}\backslash V_n$, which has

$$S^\dagger S = \mathbb{1}, \tag{2.208}$$
$$SS^\dagger = \mathbb{1} - P_{V_n} \tag{2.209}$$

where P_n: $\mathcal{H} \to V_n$ is the orthogonal projection. By unitaty gauge transformation we can assume that V_n is spanned by the vectors $|0 >, ...|n-1>$ with the projection operator onto the subspace V_n

$$P_{V_n} := \sum_{k=1}^{n} |i_k, j_k >< i_k, j_k|. \tag{2.210}$$

According [64] the solutions of the equation of motion given by

$$X^n_{+L} := Sx_{+L}S^\dagger + \sum_{k=1}^{n} \gamma^L_k |i_k, j_k >< i_k, j_k| \tag{2.211}$$
$$X^n_{+R} := Sx_{+R}S^\dagger + \sum_{k=1}^{n} \gamma^R_k |i_k, j_k >< i_k, j_k| \tag{2.212}$$

With

$$[X^n_{+L}, X^n_{+R}] = [X^n_{+L}, X^n_{-R}] = [X^n_{-L}, X^n_{+R}] = [X^n_{-L}, X^n_{-R}] = 0 \qquad (2.213)$$

and $X^n_{-a} = (X^n_{+a})^\dagger$, $\gamma_k^{L,R} \in \mathbb{C}$ determine the position of the fluxons. The field strength $F_{\mu\nu}$ fo this solution is

$$F_{\mu\nu} = P_{V_n}\theta_{\mu\nu} \qquad (2.214)$$

The action corresponding to the instanton solution is proportional to the dimension of the subspace V_n

$$S[X^n_{\pm a}] = \frac{8\pi^2}{g^2}\mathrm{tr}(P_{V_n}) = \frac{8\pi^2}{g^2}n. \qquad (2.215)$$

"pseudoparticle" solutions we mean the long range fields A_μ which minimize locally the Yang-Mills action S and for which $S(A) < \infty$. The space is euclidean and four-dimensional. All fields we are interested in satisfy the condition:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad (2.216)$$

This field $\to 0$ when $x \to \infty$.

Let us start consider S^3 is very large sphere in 4-dimensional space. From (2.216)

$$A_\mu\big|_{S^3} = \frac{1}{g}(x)\frac{\partial g(x)}{\partial x_\mu}\bigg|_{S^3} \qquad (2.217)$$

Where $g(x)$ are matrices of the gauge group. Hence every field $A_\mu(x)$ produce a certain mapping of the sphere S^3 onto the gauge group G. The phase space of the Yang-Mills fields are divided into an infinite number of components, each of which is characterized by some value of q, where q is a certain integer.

We search for the absolute minimum of the given component of the phase space. In order to do this we need the formula expressing the integer q

$$A_\mu\big|_{S^3} = \frac{1}{g}(x)\frac{\partial g(x)}{\partial x_\mu}\bigg|_{S^3} \qquad (2.218)$$

Chapter 3

Yang-Mills Matrix Theory

Modern particle theories, such as the Standard model, are quantum Yang-Mills theories. In a quantum field theory, space-time fields with relativistic field equations are quantized and, in many calculations, the quanta of the fields are interpreted as particles. In a Yang-Mills theory these fields have an internal symmetry: they are acted on by a space-time dependant non-Abelian group transformations in a way that leaves physical quantities, such as the action, invariant. These transformations are known as local gauge transformations and Yang-Mills theories are also known as non-Abelian gauge theories.

Yang-Mills theories, and especially quantum Yang-Mills theories, have many subtle and surprising properties and are still not fully understood, either in terms of their mathematically foundation or in terms of their physical predictions. However, the importance of Yang-Mills theory is clear, the Standard Model has produced calculations of amazing accuracy in particle physics and, in mathematics, ideas arising from Yang-Mills theory and from quantum field theory, are increasingly important in geometry, algebra and analysis.

Consider a complex doublet scalar field ϕ_a ; a scalar field is one that has no Lorentz index, but, as a doublet, ϕ_a transforms under a representation of $SU(2)$, the group represented by special unitary 2×2 matrices:

$$\phi_a(x) \to g_{ab}\phi_b x \tag{3.1}$$

where $g \in SU(2)$ and the repeated index is summed over. If this is a global transformation, that is, if g is independent of x, then derivative of ϕ_a have the same transformation property as a itself:

$$\frac{\partial \phi_a}{\partial x_\mu} \to \frac{\partial g_{ab}\phi_b}{\partial x_\mu} = g_{ab}\frac{\partial \phi_b}{\partial x_\mu} + \frac{\partial g_{ab}}{\partial x_\mu}\phi_b \tag{3.2}$$

In order to construct an action which includes derivatives and which is invariant under local transformations, a new derivative is defined which transforms the same way as ϕ_a :

$$D_\mu \phi_a = \frac{\partial \phi_a}{\partial x_\mu} + (A_\mu)_{ab}\phi_b \tag{3.3}$$

where A_μ is a new two-indexed space-time field, called a gauge field or gauge potential, defined to have the transformation property

$$(A_\mu)_\mu \to g_{ab}(A_\mu)_{cd}g^{-1}_{db} - \frac{\partial g_{ac}}{\partial x_\mu}g^{-1}_{cb} \tag{3.4}$$

Now, under a local transformation

$$D_\mu \phi_a \to g_{ab} D_\mu \phi_b \tag{3.5}$$

and so, $D_\mu \phi_a$ transforms in the same way as ϕ_a . This derivative is called a covariant derivative. A physical theory which includes the gauge field A should treat A_μ as a dynamical field and so the action should have a kinetic term for A_μ . In other words, the action should include derivative terms for A_μ . These terms are found in the field strength

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \partial A_\mu \partial x^\nu + [A_\mu, A_\nu] \tag{3.6}$$

which has the covariant transformation property

$$(F_{\mu\nu})_{ab} \to g_{ab}(F_{\mu\nu})_{cd}g^{-1}_{db} \tag{3.7}$$

where $[A_\mu, A_\nu]$ is the normal matrix commutator. In fact, the simplest Yang-Mills theory is pure Yang-Mills theory with action

$$S[A] = -\frac{1}{2}\int d^4x \text{ trace } F_{\mu\nu}F^{\mu\nu}. \tag{3.8}$$

and corresponding field equation

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0 \tag{3.9}$$

Solutions to this equation are known as instantons.

More generally, Yang-Mills theories contain gauge fields and matter fields like ϕ and fields with both group and Lorentz or spinor indices. Also, the group action described here can be generalized to other groups and to other representations. In the case of the Standard Model of particle physics, the gauge group is $SU(3) \times SU(2) \times$ U(1) and the group representation structure is quite intricate.

Yang-Mills theory was first discovered in the 1950s, at this time, Quantum Electrodynamics was known to describe electromagnetism. Quantum Electrodynamics is a local gauge theory, but with an abelian gauge group. It was also known that there is a approximate global non-Abelian symmetry called isospin symmetry which acts on the proton and neutron fields as a doublet and on the pion fields as a triplet. This suggested that a local version of the isospin symmetry might give a quantum field theory for the strong force with the pions fields as gauge fields [ORaifeartaigh, 1997]. This did not work because pion fields are massive whereas gauge fields are massless and the main thrust of theoretical effort in the 1950s and 1960s was directed at other models of particle physics.

However, it is now known that the proton, neutron and pion are not fundamental particles, but are composed of quarks and that there is, in fact, a quantum Yang-Mills theory of the strong force with quark fields and gauge particles called gluons. Furthermore, it is now known that it is possible to introduce a particle, called a Higgs boson, to break the non-Abelian gauge symmetry in the physics of a symmetric action and give mass terms for gauge fields. This mechanism is part of the Weinberg-Salam model, a quantum Yang-Mills theory of the electroweak force which is a component of the Standard Model and which includes both massive and massless gauge particles.

These theories were only discovered after several key experimental and theoretical breakthroughs in the late 1960s and early 1970s. After it became clear from collider experiments that proton have a substructure, the oretical study of the distance dependant properties of quantum Yang-Mills theory lead to the discovery that Yang-Mills fields are asymptotically free [Gross, 1999]. This means that the high-energy behaviour of Yang-Mills fields includes the particle like properties seen in experiments, but the low-energy behaviour may be quite different and, in fact, the quantum behaviour might not be easily deduced from the classical action. Confinement and the mass gap are examples of this. The strong force is a local gauge theory with quark fields. The quark structure of particles is observed in collider experiments, but free quarks are never detected, instead, at low-energies, they appear to bind together to form composite particles, such as neutrons, protons and pions. This is called confinement. It is possible to observe this behaviour in simulations of the quantum gauge theory of the strong force, but it has not been possible to prove mathematically that confinement is a consequence of the theory. The same is true of the mass gap, it is known that particles have non-zero mass, and this is observed in simulations, but, there is no known way of deriving the mass gap mathematically from the original theory [Clay, 2002].

The symmetries of Yang-Mills theory can be extended to include a global symmetry between the bosonic and fermionic fields called supersymmetry. While there is no direct evidence for supersymmetry in physics, the indirect case is very persuasive and it is commonly believed that direct evidence will be found in the future. Often, supersymmetric theories are more tractable, for example, Seiberg and Witten have found exact formula for many quantum properties in $N = 2$ super-Yang-Mills theory [Seiberg , Witten 1994]. It is also commonly believed by theoretical physicists that the quantum Yang-Mills theories in particle physics are in fact a limit of a more fundamental string theory.

3.1 Yang-Mills Theory in Low Dimensional

3.1.1 Bosonic Matrix Integrals

We begin our systematic study of Yang-Mills theories on low-dimensional with the simplest possible example, namely 0-dimensional Yang-Mills coupled to p adjoint scalars with a quartic interaction. There are no Yang-Mills fields in 0-dimensions and the effect of gauging is trivial, introducing only an overall factor of the gauge group volume. As a result, the partition function that we wish to study takes the form of a simple matrix integral

$$Z = \int D\ \Phi_i \mathrm{exp}\left\{ -\frac{N}{2\lambda_0}\mathrm{tr}\left(\sum_i m_i^2 \Phi_i^2 - \sum_{i<j} [\Phi_i, \Phi_j] \right) \right\} \tag{3.10}$$

We rescaled the Φ make clear that the effective couplings are $\lambda_0/m_i^2 m_j^2$, the partation function becomes

$$Z = \int D\ \phi_i \exp\left\{ -\frac{N}{2}\mathrm{tr}\left(\sum_i \frac{\tilde{\Phi}_i^2}{2} - \sum_{i<j} \frac{\lambda_0}{m_i^2 m_j^2}[\tilde{\Phi}_i, \tilde{\Phi}_j] \right) \right\} \tag{3.11}$$

3.1.2 Bosonic Yang-Mills on S^1

We now move up in dimension to consider bosonic Yang-Mills on an S^1 . The particular model that we consider is

$$S = \frac{N}{2\lambda} \int \mathrm{tr}\left(\sum_{i=1}^{p} D_0\Phi_i D_0\Phi_i + \sum_i^p M^2\Phi_i^2 - \sum_{i<j}^p [\Phi_i, \Phi_j]^2] \right) \tag{3.12}$$

From λ ,M, and the size R of the S^1 , which we will think of as an inverse temperature, we can construct two dimensionless parameters

$$\tilde{t} = (R\lambda^{1/3})^{-1} \qquad m = M\lambda^{-1/3} \tag{3.13}$$

Large masses $m >> 1$

The second regime that one can study corresponds to the limit of large mass $m >> 1$ and displays a much more interesting and nontrivial structure. Here, the theory is effectively weakly coupled and, since all mode except the zero component of the gauge field, A_0 , are massive, they can be integrated out in perturbation theory.

We consider a function of a unitary matrix U

$$Z = \int DU \exp[-N^2 S_{\mathrm{eff}(U)}] \qquad U = e^{i\oint dtA_0} \tag{3.14}$$

The computation of $S_{eff}(U)$ now proceeds exactly as in the $\mathcal{N} = 4$ theory with the identification

$$x = e^{-MR} = e^{-m/\tilde{t}}, \tag{3.15}$$

the letter partition function is rather trivial as there is only one bosonic state at energy 1 for each scalar field

$$z(x) = x \tag{3.16}$$

This leads to a simple effective action for U

$$S_{\text{eff}}(U) = p\sum_{n=1}^{\infty} \frac{x^n}{n}\text{tr}(U^n)\text{tr}(U^{-n}) \tag{3.17}$$

where the factor of p arises because S_{eff} receives contributions from p scalar fields.

Writing this in terms of moments ρ_n of the corresponding eigenvalue distribution, this becomes

$$S_{\text{eff}}(\rho_n) = \sum_{1}^{\infty} \frac{1}{n} m_n^2 |\rho_n|^2 \tag{3.18}$$

$$m_n^2 = 1 - px^n \tag{3.19}$$

where the 1 in (3.19) arises from the Vandermonde measure as usual.

$$\tilde{t}_c = m(\ln\, p)^{-1} \tag{3.20}$$

Away from the strict $m \to \infty$ limit, (3.18) will receive corrections that can potentially alter the order of the transition. In particular, as we have seen many times it is the sign of the quartic term in the effective potential for ρ_1 that is the difference between first and second order. This can be determined by computing higher-loop perturbative corrections to (3.18) and integrating out the modes ρ_i with $i > 1$ that remain massive at the transition point. The result is an effective potential for ρ_1

$$S(\rho_1) = m_1^2(x, m^{-3}|\rho_1|^2 + \frac{1}{m^6} b(x, m^{-3})|\rho_1|^4 + ... \tag{3.21}$$

The calculation of m_1 and b to the requisite orders is straightforward but tedious, yielding [66]

$$\begin{aligned} m_1^2 &= (1-px) - \frac{1}{4m^6}(p^2-p)(x^2+2x)\ln x \qquad (3.22) \\ b &= -\frac{1}{32}\frac{(p-1)\ln p}{p^3}(\ln(p)(9p^2+2p)+4p^3+7p^2-4p-4)m^{-6}+ ... \end{aligned}$$

Since $b < 0$ for all $p > 1$, we see that the phase transition is indeed of first order as one moves away from $m = \infty$. From the correction to m_1^2 in

(3.23), we can also compute the leading order shift in the phase transition temperature, obtaining [66]

$$\tilde{t}_c = m(\ln p)^{-1} + \frac{1}{4m^2}\frac{(p-1)(2p+1)}{p\ln p} + ... \tag{3.23}$$

This result will fit in nicely the picture of the $m = 0$ theory that we now describe.

3.1.3 Bosonic Yang-Mills on T^2

We now consider the first of our two-dimensional models, namely bosonic Yang-Mills on a rectangular two-torus with radii R_1, R_2

The specific model that we consider has action

$$S = \frac{N}{2\lambda}\int d^2\, x\mathrm{tr}\Bigg(F_{12}^2 + \sum_{i=1}^{p}\left[(D_\mu\Phi^I)^2 + M^2\Phi_I^2\right] - \sum_{I>J}[\Phi^I, \Phi^J]^2\Bigg) \tag{3.24}$$

and can be parametrized by the two dimensionless radii

$$r_1 = R_1\sqrt{\lambda} \qquad\qquad r_2 = R_2\sqrt{\lambda} \tag{3.25}$$

and the dimensionless mass

$$m = \frac{M}{\sqrt{\lambda}} \tag{3.26}$$

Infinite masses $m = \infty$

We begin our study of (3.24) by considering the strict limit $m = \infty$. The reason we do this is to recall some elementary facts about the exact solution of pure Yang-Mills on a two-torus that will be useful when we study large but noninfinite masses.

The exact result for the two-dimensional Yang-Mills partition function was first obtained by Migdal [67] using a lattice regularization of the theory. We consider putting unitary matrices U_L on the links of the lattice and consider the partition function

$$Z = \int \prod_L DU_L \prod_P Z_P(U_P) \tag{3.27}$$

where the product is over plaquettes P of the lattice and Z_P is a plaquette action chosen so that the continuum theory coincides with Yang-Mills. One example of a plaquette action is the usual Wilson one

$$Z_P = \exp\left\{\frac{N}{\lambda}\mathrm{tr}(U_P + U_P^{-1})\right\} \tag{3.28}$$

One can imagine integrating out some of the links to generate an effective action for the larger plaquettes that remain. Under this RG flow, the action (3.28) is not invariant, instead flowing to the plaquette action [67]

$$Z_P(U_P) = \sum_R d_R \chi_R(U_P) \exp\left\{-\frac{\lambda A}{2N} C_2(R)\right\} \tag{3.29}$$

where the representation R has dimension d_R and quadratic Casimir $C_2(R)$ and A is the area of the new plaquettes in units of the fundamental ones. It is easy to verify that (3.29) satisfies the additivity property

$$\int DU Z_P(V_1 U, A_1) Z_{P'}(U^\dagger V_2, A_2) = Z_{P+P'}(V_1 V_2, A_1 + A_2) \tag{3.30}$$

and consequently is an RG fixed point. To compute the partition function of 2-dimensional Yang-Mills, we are equally justified in using (3.28) or (3.29). The former is easier to work with as the property (3.30) can be used to integrate out all of the links except those wrapping nontrivial cycles. Restricting to the torus for simplicity, we arrive at the following expression for the partition function

$$Z = \int DUDV \sum_R d_R \exp(-\frac{r_1 r_2}{2N} C_2 R)\chi_R(UVU^{-1}V^{-1}) \tag{3.31}$$

where we have used the definitions (3.25) of r_1, r_2 . The links which remain are precisely the holonomies that we wish to study. The remaining integrals (3.31) are now sufficiently simple that we can perform them exactly to obtain

$$Z = \sum_R e^{-\frac{r_1 r_2}{2N} C_2(R)} \tag{3.32}$$

This partition function is very well-behaved and exhibits no nontrivial phase structure. As an exercise, let us obtain this conclusion in a manner more in line with the sort of analysis that we have used to study other theories above. In particular, let us consider integrating out only one holonomy, say V , from (3.31). We obtain

$$Z = \int DU \sum_R e^{-\frac{r_1 r_2}{2N} C_2(R)} \chi_R(U) \chi_R(U^\dagger) \tag{3.33}$$

Using the generalized Frobenius relations of [68], [69], [70] it is possible to write the integrand in terms of an effective action for U [67]

$$Z = \int DU \exp\Big\{ \sum_n \frac{1}{n}(-e^{-r_1 r_2 n} + 2e^{-r_1 r_2 n/2}) \mathrm{tr}(U^n)\mathrm{tr}(U^{-n}) \tag{3.34}$$

Introducing an eigenvalue density ρ as usual, we can obtain an effective action for the moments of ρ

$$S_{eff}(\rho_n) = \sum_n \frac{1}{n}(1 - e^{-r_1 r_2 n/2})^2 |\rho_n|^2 \tag{3.35}$$

We see that for all values of the coupling $\lambda A = r_1 r2$ the effective masses of the ρ_n are positive. In the limit $r_1 r_2 \to \infty$ they become arbitrarily light but the never become tachyonic. The eigenvalue distributions are thus always uniform and there is no phase transition.

3.2 Yang-Mills Theories

3.2.1 Yang-Mills Gauge Theories

The Yang-Mills matrix models are related to gauge theories by dimensional reduction. We recall the structure of Yang-Mills gauge theories

A Yang-Mills gauge theory in D dimensions has Lagrangian

$$\mathcal{L} = \frac{1}{4}F^2 - \frac{i}{2}\bar{\psi}\gamma.\mathcal{D}\psi \tag{3.36}$$

The fields in this theory are the vector potential X^a_μ , and fermions ψ^a_α . Here μ is a spacetime index running from 0 up to $D1$, and for the moment we use a Minkowski metric. The $\gamma^\mu_{\alpha\beta}$ are Dirac matrices, and $\bar{\psi}$ is defined

$$\bar{\psi} = \psi^\dagger\gamma^0 \tag{3.37}$$

The fields are in the Lie algebra of a compact semi-simple gauge group G so we can write

$$X_\mu = X^a_\mu t^a \quad\quad \psi_\alpha = \psi^a_\alpha t^a \tag{3.38}$$

where the $t^a(a = 1, .., g)$ are the generators of the Lie algebra which we choose such that

$$\mathrm{Tr} t^a t^b = 2\delta^{ab} \tag{3.39}$$

and

$$[t^a, t^b] = if^{abc}t^c. \tag{3.40}$$

The gauge field strength F is defined

$$F^a_{\mu\nu} = \partial_\mu X^a_\nu - \partial_\nu X^a_\mu + cf^{abc}X^b_\mu X^c_\nu \tag{3.41}$$

and the gauge covariant derivative is

$$(\mathcal{D}_\mu\psi)^a = \partial_\mu\psi^a + cf^{abc}X^b_\mu\psi^c. \tag{3.42}$$

The parameter c is a coupling constant. The theory is invariant under gauge transformation

$$\psi \to U\psi U^{-1}, \quad X_\mu \to UX_\mu U^{-1} - ig^{-1}(\partial U)U^{-1} \tag{3.43}$$

where $U \in G$. The fermions are optional in this model. We can define a purely bosonic gauge theory by simply omitting them.

3.3 Yang-Mills Matrix Models

To obtain a Yang-Mills matrix model, we take the Lagrangian (3.36) and assume all the fields are independent of space and time. Effectively, this means we drop all the derivative terms from (3.36)

At this stage, we also move from Minkowski to Euclidean signature. We do this by setting

$$X_0^a = iX_D^a, \quad (a = 1, .., g) \tag{3.44}$$

and taking the X_D^a real . We also set

$$\gamma^0 = i\gamma^D \tag{3.45}$$

We have been careful to leave this manipulation until last because we wish to study the Wick rotation of a Minkowski theory . This leads to a rather strange effect in the case of $D = 10$ when the fermions are Majorana. Since the Dirac matrices can no longer all be imaginary, an $SO(D)$ transformation would break the Majorana condition. However, after integrating out the fermions, full $SO(D)$ invariance is restored since we can analytically continue in X_D .

We arrive at the matrix model action

$$S_{\text{YM}} = -\text{Tr}\left(\frac{1}{4}[X_\mu, X_\nu][X_\mu, X_\nu] + \frac{1}{2}\bar{\psi}\gamma^\mu[X_\mu, \psi]\right) \tag{3.46}$$

where we have dropped the coupling constant g since it can be scaled out in a trivial manner. In those cases where the fermions were originally Majorana (before Wick rotation), we may choose the representation in which the ψ_α^a eal. In those cases where the fermions are complex, it will sometimes be convenient to rewrite the (ψ_α^a) for each a as a real vector of double the length. We can also absorb the γ^0 which appears in the definition of $\bar{\psi}$ into the γ^μ . Thus we shall sometimes write the action in the form

$$S_{\text{YM}} = -\text{Tr}\left(\frac{1}{4}[X_\mu, X_\nu][X_\mu, X_\nu] + \frac{1}{2}\psi_\alpha\Gamma^\mu_{\alpha\beta}[X_\mu, \psi_\beta]\right) \tag{3.47}$$

where the Γ^μ are some new matrices defined in terms of the γ^μ , and the ψ_α^a are now always real. In the case where the original fermions were complex, the range of the indices α and β has been doubled.

We define a partition function

$$Z_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \prod_{\alpha=1}^{\mathcal{N}} d\psi_\alpha \exp\left(\frac{1}{4}\sum_{\mu\nu} \mathrm{Tr}[X_\mu, X_\nu]^2 + \frac{1}{2}\mathrm{Tr}\psi_\alpha[\Gamma^\mu_{\alpha\beta}X_\mu, \psi_\beta]\right) \tag{3.48}$$

which we shall also sometimes refer to as the Yang-Mills integral. In principle one can integrate out the fermions to obtain

$$Z_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \mathcal{P}_{D,G}(X_\mu) \exp\left(\frac{1}{4}\sum_{\mu\nu} \mathrm{Tr}[X_\mu, X_\nu]^2\right) \tag{3.49}$$

where the Pfaffian $\mathcal{P}_{D,G}$ is a homogeneous polynomial of degree $\frac{1}{2}\mathcal{N}g$. In this representation, the gauge symmetry is

$$X_\mu \to U^\dagger X_\mu U, \quad U \in G \tag{3.50}$$

and SO(D) symmetry

$$X_\mu \to \sum_\nu Q_{\mu\nu} X_\nu, \quad Q \in SO(D) \tag{3.51}$$

In addition, we shall consider simple correlation functions

$$< c_k(X_\sigma) >= \int \prod_{\mu=1}^{D} dX_\mu C_k(X_\sigma) \mathcal{P}_{D,G}(X_\mu) \exp\left(\frac{1}{4}\sum_{\mu\nu} \mathrm{Tr}[X_\mu, X_\nu]^2\right) \tag{3.52}$$

with C_k a function of the X_μ which grows like a polynomial of degree k.

The first question one must ask about these models is whether the integrals (3.48) and (3.52) which define the partition function and correlation functions are well defined. Certainly, we must require at least that the partition function is finite for the theory to make any sense. The difficulty here is that the potential $Tr[X_\mu, X_\nu][X_\mu, X_\nu]$ has flat directions in which the matrices commute. For example, in the bosonic case, one can move to infinity along one of these directions whilst keeping the integrand constant, and thus it was widely believed that these integrals may be infinite. However, in the case of $SU(2)$ it is possible to perform the integrals for the partition function exactly. This was done originally in the supersymmetric cases [71–74] and it was found that the partition function does converge at least for $D = 4, 6, 10$. Subsequently, eigenvalue densities and some correlation functions have been calculated in [75]. It was believed that the supersymmetric versions should be more convergent than the bosonic because the contributions from the fermionic integrals would be close to zero near the flat directions. However, the $SU(2)$ bosonic partition function was calculated in [76], and was found to converge when $D \geq 5$.

The authors of [76] were able to use Monte Carlo methods to calculate the supersymmetric integrals numerically for $SU(2)$ and $SU(3)$, and the calculations have been extended to various other gauge groups, and also to the bosonic theories [77, 78]. A difficulty with numerical simulations for the supersymmetric integrals is in performing the fermionic integrations to obtain the Pfaffian, and for this reason, the exact model has only been studied for the smaller gauge groups. However, the bosonic models have now been studied for $SU(N)$ with N up to 768 [78, 79]. Analytic approximation schemes have also been constructed for the bosonic models in [80] and recently for the $D = 4$ supersymmetric model [81].

The conclusions of the numerical methods are that the supersymmetric partition function converges when $D = 4, 6, 10$ and that the bosonic partition functions converge at least when D is large enough [78].

3.4 Introduction of IKKT Models

Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT) have proposed a model $(0+0)$-dimensional matrix model should give a Paincaré invariant describing of type-IIB string theory on a flat background. Beginning with the Schild gauge-fixed form of the string action, they found that the matrix regularization of the action led to a zero dimensional matrix model equivalent to the complete dimensional reduction of ten dimensional super Yang Mills theory.

What is a matrix theory? Simply put, it is a quantum mechanics with matrix degrees of freedom. It is in general comprised of some $N \times N$ bosonic and fermionic matrices. Matrix theories have the attractive quality that they are not quantum field theories, and thus have none of the peculiarities involved with QFTs, such as renormalization. Indeed, since N is finite (though taken to be large) there are only a finite number of degrees of freedom.

Witten found that the low-energy Lagrangian which describes a system of N type-IIA D0-branes is equivalent to the dimensional reduction (the theory on a point) of ten dimensional super Yang-Mills theory to (0+1) dimensions [82].

The dimensionally reduced Hamiltonian is

$$H = \frac{R}{2}\mathrm{Tr}\left\{ P^i P^i - \frac{1}{2}[X^i, X^j][X^i, X^j] + \theta\gamma_i[X^i, \theta] \right\}. \tag{3.53}$$

X^i and P^i are $N \times N$ with bosonic entries, θ is a set of $N \times N$ matrices with fermionic entries.

Banks, Fischler, Shenker, and Susskind (BFSS) discovery after Witten that this Hamiltonian precisely describes M-theory in the light-front coordinate system, in $N = \infty$ limit of supersymmetric matrix quantum mechanics describing D0 branes [83]. The IKKT model is one proposed non-perturbative definition of string theory [84]. It is sometimes referred to as the IIB matrix model, since it is related to a gauge-fixed form of the Green-Schwarz action for the IIB string.

There are two ways of defining the model. One is as the complete dimensional reduction to 0-dimensions matrix model arising from the dimensional reduction in all ten dimebsions of $\mathcal{N} = 1$ super Yang Mills theory.

$$S = -\frac{1}{g^2}\mathrm{Tr}\left\{ \frac{1}{4}\sum_{\mu,\nu=0}^{9}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\sum_{\mu=0}^{9}\bar{\psi}\Gamma^\mu[A_\mu, \psi] \right\}. \tag{3.54}$$

The action (3.54) looks very similar to the dimensionally reduced Hamiltonian of the BFSS conjecture, Eq (3.53).This action still has SO(9.1) symmetry, as well as the SU(N) gauge symmetry. The A_μ are $N \times N$ hermitian matrix components of a vector in an SO(9.1) representation. The fermion ψ is in a Majorana-Weyl representation and $N \times N$ hermitian matrices for components [85].

The partition function for the theory is then defined via euclideanization of the action (Wick rotation of A_0 and Γ^0);

$$Z = \int dAd\psi e^{-S_E} \tag{3.55}$$

The model may also be defined in terms of a grand canonical partition function or canonical ensemble:

$$Z[\beta] = \sum_{N=1}^{\infty} \int dXd\Psi e^{-S_E^2[\beta]} \tag{3.56}$$

Here β is interpreted as a chemical potential dual to the matrix size N , and the action is

$$S_E^2 = \frac{1}{2\alpha^2\beta}\mathrm{Tr}\left\{\frac{1}{4}[A_\mu, A_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[A_\mu, \psi]\right\} + \beta N. \tag{3.57}$$

If the large N limit is smooth, we ewpect that the $\beta \to \beta_c$ limit is indentical to consider the microcanonical ensemble with fixed N and take N large [85].

From the actions (3.54) and (3.57), it is clear that the action is minimized if the bosonic matrices commute. If that is the case, they may all be simultaneously diagonalized and decomposed into a diagonal part X_μ and an off-diagonal part $\tilde{A}_\mu$;

$$A_\mu = X_\mu + \tilde{A}_\mu = \begin{pmatrix} x_\mu^1 & & & & \\ & x_\mu^2 & & & \\ & & . & & \\ & & & . & \\ & & & & x_\mu^N \end{pmatrix} + \tilde{A}_\mu$$

The fermion ψ is also decomposed into diagonal and off-diagonal composents ξ and $\tilde{\psi}$

$$\psi = \begin{pmatrix} \xi^1 & & & & \\ & \xi^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \xi^N \end{pmatrix} + \tilde{\psi}$$

where x^i_μ and ξ^i_α satisfy the constraints $\sum_{i=1}^N x^i_\mu = 0$ and $\sum_{i=1}^N \xi^i_\alpha = 0$, respectively, since we may fix the U(1) part by translation invariance. Under this transformation, the off-diagonal terms $\tilde{A}_\mu$ and ψ become massive. They may then be integrated out, and an effective action for the diagonal bosons (to be interpreted as space-time points) may be obtained by integrating out the diagonal fermions ξ^i [85].

$$\int dAd\psi e^{S[A,\psi]} = \int dXd\xi e^{-S_{\text{eff}}[X,\xi]} \tag{3.58}$$

$$= \int dXe^{-S_{\text{eff}}[X]} \tag{3.59}$$

Ishibashi et al. interpret the diagonal matrix elements as points of spacetime. By expanding the action as a perturbative series and integrating out all the other fields, an effective action for the spacetime points is found

We perform integrations over off-diagonal parts $\tilde{A}_\mu$ and $\tilde{\psi}$ by the perturbation coupling constant g^2. This scheme is valid if the spacetime points are widely separated $(|x^i - x^j|) \gg \sqrt{g}$ [85]. Residual gauge symmetries may be handled by introducing Faddeev-Popov ghosts and gauge-fixing terms

$$S_{g.f.} + S_{F.P.} = -\frac{1}{2g^2}\text{Tr}([X_\mu, A^\mu]^2) - \frac{1}{g^2}\text{Tr}([X_\mu, b][A^\mu, c]), \tag{3.60}$$

where b and c are the Faddeev-Popov ghost field.

The original action (3.54) can be expanded as follows [86]

$$S = S_2 + S_{\rm int} \tag{3.61}$$

$$S_2 = \frac{1}{g^2}\mathrm{Tr}\big(- [X_\mu, \tilde{A}_\nu][X^\mu, \tilde{A}^\nu] + [X_\mu, \tilde{A}_\nu][X^\mu, \tilde{A}^\nu] - \bar{\tilde{\psi}}\Gamma^\mu[X_\mu, \tilde{\psi}] - [\bar{\xi}, \tilde{A}_\mu]\Gamma^\mu\tilde{\psi} - \bar{\tilde{\psi}}\Gamma^\mu[\tilde{A}_\mu, \xi]\big), \tag{3.62}$$

$$S_{\rm int} = \frac{1}{g^2}\mathrm{Tr}\big(- 2[X_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu] - \frac{1}{2}[\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu] - \bar{\tilde{\psi}}\Gamma^\mu[\tilde{A}_\mu, \tilde{\psi}]\big) \tag{3.63}$$

Can be written $S_2 + S_{\rm g.f}$ in terms of the components as

$$S_2 + S_{g.f.} = \frac{1}{2g^2}\sum_{i<j}((x^i_\nu - x^j_\nu)^2 \tilde{A}^{ij*}_\mu \tilde{A}^{ij\mu} - \bar{\tilde{\psi}}^{ji}\Gamma^\mu(x^i_\mu - x^j_\mu)\tilde{\psi}^{ij}) + (\bar{\xi}^i - \bar{\xi}^j)\Gamma^\mu\tilde{\psi}^{ij}\tilde{A}^{ij*}_\mu \bar{\tilde{\psi}}^{ji}\Gamma^\mu(\xi^i - \xi^j)\tilde{A}^{ij}_\mu). \tag{3.64}$$

The first and the second terms are the kinetic terms for $\tilde{A}$ and $\tilde{\psi}$ respectively, while the last two terms are $\tilde{A}\tilde{\psi}\xi$ vertices. The basic building blocks of the Feynman rules (see [86])

3.5 The IKKT Model of IIB superstring

The supersymmetric Yang-Mills matrix theory with $D = 10$ has been proposed as a constructive definition of IIB superstring theory [24]. We give a very brief introduction here, but for a review see [87].

The idea of the IKKT conjecture is to begin with the Green-Schwarz action for the superstring in the Schild gauge:

$$S_{GS} = \int d^2\sigma \left[\sqrt{\hat{g}}\alpha \left(\frac{1}{4}\{x^\mu, x^\nu\}^2 - \frac{i}{2}\bar{\psi}\Gamma^\mu\{x^\mu, \psi\} \right) + \beta\sqrt{\hat{g}} \right] \tag{3.65}$$

Here σ are 2-dimensional world-sheet coordinates, $\hat{g} = det(\hat{g}_{ab})$ is the determinant of the world sheet metric, and α, β are parameters (which could be scaled out). The x^μ are target space coordinates, and the Poisson bracket is defined

$$\{x, y\} = \frac{1}{\sqrt{\hat{g}}}\epsilon^{ab} x\partial_a x\partial_b y \tag{3.66}$$

The theory is then regularised essentially following a method of Goldstone and Hoppe (for a review, see [88]).

1. A function y on the world sheet is replaced by an $N \times N$ traceless hermitian matrix Y , with a correspondence

$$\int d^2\sigma \sqrt{\hat{g} y} \leftrightarrow \mathrm{Tr} Y \tag{3.67}$$

2. and

$$\{x, y\} \leftrightarrow -i[X, Y] \tag{3.68}$$

Performing this regularisation, the action (3.65) becomes

$$S_{\mathrm{IKKT}} = -\alpha \left(\frac{1}{4}\mathrm{Tr}[X_\mu, X_\nu][X_\mu, X_\nu] + \frac{1}{2}\mathrm{Tr}\bar{\psi}\Gamma^\mu[X^\mu, \psi] \right) + \beta N \tag{3.69}$$

The string partition function is given by

$$\int D[x]D[\psi]\exp(-S_{GS}) \tag{3.70}$$

then becomes the matrix integral

$$\int \prod_{\mu=1}^{D} dX_\mu \prod_{\alpha=1}^{16} d\psi_\alpha \exp(-S_{IKKT}) \tag{3.71}$$

which after scaling out becomes the Yang-Mills matrix partition function $Z10, SU(N)$, with an additional factor $e^{\beta N}$.

In their original proposal, $IKKT$ interpreted the integral over the world sheet metric $\int D[\hat{g}]$ as a requirement to sum over N:

$$\mathcal{Z}_{IKKT} \sim \sum_N \mathcal{Z}_{10,SU(N)} e^{-\beta N} \tag{3.72}$$

However, in general, the matrix regularisation procedure outlined above is valid in the limit $N \to \infty$, and the partition function is often taken as the large N limit

$$\mathcal{Z}_{IKKT} \sim \mathcal{Z}_{10,SU(N)} \tag{3.73}$$

which would correspond to a more literal application of the Goldstone Hoppe regularisation. The large N limit is not yet well understood, and it is not clear exactly how to interpret the model. Nevertheless, an argument relating Wilson loops in the matrix model to string field theory in light-cone gauge provides additional evidence for the importance of the IKKT model [89].

Chapter 4

Fuzzy spaces

Studies of fuzzy spaces cross over a variety of concepts in mathematics and physics. The basic idea of fuzzy spaces is to describe compact spaces in terms of finite dimensional $(N \times N)$-matrices such that they give a concrete realization of noncommutative (NC) spaces [90, 91]. Use of fuzzy spaces in physics was suggested by Madore around 1992 [92]. Since then, fuzzy spaces have been an active area of research.

Fuzzy spaces are described by finite dimensional matrices, due to the Cayley-Hamilton theorem, there is a natural cut-off on the number of modes for matrix functions on fuzzy spaces. So one can use fuzzy spaces to construct regularized field theories in much the same way that lattice gauge theories are built. Various interesting features of field theories on fuzzy spaces have been reported; for example, existence of topological solutions such as monopoles and instantons, appearance of the so-called UV-IR mixing, and evasion of the fermion doubling problem which appears in the lattice regularization

4.1 Construction of Fuzzy Spaces

4.1.1 Construction of fuzzy CP^k

Hilbert space

A finite dimensional Hilbert space $\mathcal{H}_N$ for fuzzy $CP^k = SU(k+1)/U(k)$ (k=1,2,...) is given by holomorphic sections of a complex line bundle over CP^k, the holomorphic sections of the complex line bundle should correspond to a unitary irreducible representation $G = SU(k+1)$. Notion of holomorphicity in the representation of G can be realized by totally symmetric part of the representation, i.e., (n, 0), where n is the rank of the representation ($n = 1, 2, ...$). The other totally symmetric representation (0, n) corresponds to antiholomorphic part of the SU(k+1) representation and the (p, p)-representation gives real representation.

For SU(2) (corresponding to k=1), the representation is given by a single component, say (p), so there is no real representation. The dimension of $\mathcal{H}_N$ is then determined by that of the (n, 0)-representation for SU(k+1);

$$N^k \equiv dim(n,0) = \frac{(n+k)!}{k!\ n!} \tag{4.1}$$

Consequently, matrix algebra of fuzzy CP^k is realized by $N^{(k)} \times N^{(k)}$ -matrices. Operators or matrix functions on fuzzy CP^k are expressed by linear combinations of $N^{(k)} \times N^{(k)}$ -matrix representations of the algebra of SU(k+1) in the (n,0)-representation.

We begin with write down a holomorphic U(1) bundle $\Psi_m^{(n)}$ as

$$\Psi_m^{(n)}(g) = \sqrt{N^k}\mathcal{D}_{mN^k}^{(n,0)}(g), \tag{4.2}$$

$$\mathcal{D}_{mN^k}^{(n,0)}(g) = <(n,0), m|\hat{g}|(n,0), N^{(k)}> \tag{4.3}$$

where $|(n,0), m>$ ($m = 1, 2, , N^{(k)}$) denote the states on the Hilbert space $\mathcal{H}_N$, $|(n,0), N^{(k)}>$ is the highest or lowest weight state, g is an element of $G = SU(k+1)$ and $\hat{g}$ is a corresponding operator acting on these state. $\mathcal{D}_{mN^k}^{(n,0)}(g)$ is known as Wigner $\mathcal{D}$-functions for $SU(k+1)$ in the $(n,0)$-representation, allowing us to interoret the $\mathcal{D}$-functions as matrix elements.

Let R_A denote the right-translation operator on g;

$$R_A g = g t_A \tag{4.4}$$

where t_A are the generator of G in the fundamental representation (1,0). The element g is given by $g = exp(it_A\theta^A)$ with continuous parameters θ^A. We now consider the splitting of t_A's to those of $U(k) = SU(k) \times U(1)$ subalgebra and the rest of them, i.e., those relevant to CP^k. Let t_j ($j = 1, 2, , k^2$) and t_{k^2+2k} denote the generators of $U(k) \in SU(k+1)$, t_{k^2+2k} being a $U(1)$ element of the $U(k)$, and let $t_{\pm i}$ (i=1,2,,k) denote the rest of t_A's. One can consider $t_{\pm i}$ as a combination of rasing-type (t_{+i}) and lowering-type (t_i) operators acting on the states of $\mathcal{H}_N$. Choosing $|(n,0), N^{(k)}>$ to be the lowest weight state, we then find

$$R_j \mathcal{D}^{(n,0)}_{mN^k}(g) = 0 \quad (j = 1, 2, 3..., k^2), \tag{4.5}$$

$$R_{k^2+2k} \mathcal{D}^{(n,0)}_{mN^k}(g) = -\frac{nk}{\sqrt{2k(k+1)}} \mathcal{D}^{(n,0)}_{mN^k}(g) \tag{4.6}$$

$$R_{-i} \mathcal{D}^{(n,0)}_{mN^k}(g) = 0 \tag{4.7}$$

(5.1) and (5.2) indicate that $\Psi^{(n)}_m \sim \mathcal{D}^{(n,0)}_{mN^k}(g)$ is a $U(1)$ bundle over CP^k. One can also check that under the U(1) transformations,

$$g \to gh, \quad h = e^{it_{k^2+2k}\theta^{k^2+2k}} \tag{4.8}$$

$\Psi^{(n)}_m(g)$ transforms as

$$\Psi^{(n)}_I(g) \to \Psi^{(n)}_m(gh) = \Psi^{(n)}_m(g) exp\bigg(-i\frac{nk}{\sqrt{2k(k+1)}}\theta^{k^2+2k}\bigg). \tag{4.9}$$

In terms of geometric quantization, equation (5.3) corresponds to the polarization condition on a prequantum $U(1)$ bundle. The Hilbert space is therefore constructed as sections of the holomorphic $U(1)$ bundle Ψ^n_m . The square-integrability of $\mathcal{H}_N$ is guaranteed by the orthogonality condition of the Wigner $\mathcal{D}$ function [93];

$$\int d\mu(g) \mathcal{D}^{\star(R)}_{m,k}(g) \mathcal{D}^{\star(R')}_{m',k'}(g) = \delta^{RR'} \frac{\delta_{mm'}\delta_{kk'}}{dimR} \tag{4.10}$$

where $\mathcal{D}^{\star(R)}_{m,k}(g)\mathcal{D}^{(R)}_{m,k}(g^{-1})$, R denotes the representation of $G = SU(k+1)$, and $d\mu(g)$ is the Haar measure of G normalizsd to unity; $\int d\mu(g) = 1$. The orthogonality condition of our interest is given by

$$\int d\mu(g) \mathcal{D}^{\star(n,0)}_{m,N^k}(g) \mathcal{D}^{(n,0)}_{m',N^k}(g) = \frac{\delta_{mm'}}{N^k}. \tag{4.11}$$

Symbols and star products

We define the symbol of a matrix operator $A_{ms}(m, s = 1, 2, , N^{(k)})$ on the Hilbert space of fuzzy CP^k by [94]

$$\begin{aligned} \langle \hat{A} \rangle &\equiv \sum_{ms} \mathcal{D}^{(n,0)}_{m,N^k}(g) A_{ms} \mathcal{D}^{\star(n,0)}_{s,N^k}(g) \\ &= \langle (n,0), N^{(k)} | \hat{g}^T \hat{A} \hat{g}^\star | (n,0), N^{(k)} \rangle \end{aligned} \tag{4.12}$$

The star product of fuzzy CP^k is defined by

$$\langle \hat{A}\hat{B} \rangle \equiv \langle \hat{A} \rangle \star \langle \hat{B} \rangle \tag{4.13}$$

From (4.12) can be written

$$\langle \hat{A}\hat{B} \rangle = \sum_{msrr'p} \mathcal{D}^{(n,0)}_{m,N^k}(g) A_{mr} \mathcal{D}^{\star(n,0)}_{r,p}(g) \mathcal{D}^{(n,0)}_{r',p}(g) B_{r's} \mathcal{D}^{\star(n,0)}_{s,N^{(k)}}(g) \tag{4.14}$$

We use the relation

$$\sum_p \mathcal{D}^{\star(n,0)}_{r,p}(g) \mathcal{D}^{(n,0)}_{r',p}(g) = \delta_{rr'} \tag{4.15}$$

From (4.11), the trace of a matrix operator A can be expressed as

$$\begin{aligned} \mathrm{Tr}A &= \sum_m A_{mm} = N^k \int d\mu(g) \mathcal{D}^{(n,0)}_{m,N^{(k)}} A_{mm'} \mathcal{D}^{\star(n,0)}_{m',N^{(k)}} \\ &= N^{(k)} \int d\mu(g) \langle \hat{A} \rangle. \end{aligned} \tag{4.16}$$

The trace of the product of two matrices A, B, is also given by

$$\mathrm{Tr\ AB} = N^{(k)} \int d\mu(g) \langle \hat{A} \rangle \star \langle \hat{B} \rangle \tag{4.17}$$

Algebraic construction

Here we present the construction of fuzzy CP^k (k=1,2,...). The coordinates Q_A of fuzzy CP^k can be defined in terms of L_A which are $N^{(k)} \times N^{(k)}$-matrix representation of $SU(k+1)$ generators in the $(n,0)$-representation [94]

$$Q_A = \frac{L_A}{\sqrt{C_2^{(k)}}}, \tag{4.18}$$

where C_2^k is the quadratic Casimir. With

$$\begin{aligned} Q_A Q_A &= \mathbf{1} \qquad (4.19) \\ d_{ABC} Q_A Q_B &= c_{k,n} Q_C, \qquad (4.20) \end{aligned}$$

where d_{ABC} is the totally symmetric symbol, and $\mathbf{1}$ is the $N^{(k)} \times N^{(k)}$ identity matrix. For determine the coefficient $C_{k,n}$ in (4.20), we notice

$$\Lambda_A = a_i^\dagger (t_A)_{ij} a_j, \tag{4.21}$$

Λ_A is the $SU(k+1)$ generators in the (n,0)-representation, t_A $SU(k+1)$ generators in the fundamental representation, with normalization

$$\mathrm{tr}(t_A t_B) = \frac{1}{2\delta_{AB}}, \tag{4.22}$$

and $a_i^\dagger$,a_i are the creation and annihilation operators. Using the completeness relation

$$(t_A)_{ij}(t_A)_{kl} = \frac{1}{2}\big(\delta_{il}\delta_{jk} - \frac{1}{k+1}\delta_{ij}\delta_{kl}\Big) \tag{4.23}$$

and the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$, we can check $\Lambda_A \Lambda_A = C_2^{(k)}$. We find [94, 95]

$$\begin{aligned} d_{ABC}\Lambda_B\Lambda_C &= (k-1)\left(\frac{n}{n+1} + \frac{1}{2}\right) a_i^\dagger (t_A)_{ij} a_j \\ &= (k-1)\left(\frac{n}{n+1} + \frac{1}{2}\right)\Lambda_A \end{aligned} \tag{4.24}$$

Representing Λ_A by L_A , we can determine the coefficient $c_{k,n}$

$$C_{k,n} = \frac{(k-1)}{\sqrt{C_2^{(k)}}}\left(\frac{n}{k+1} + \frac{1}{2}\right). \tag{4.25}$$

Wher $C_2^{(k)}$ is the quadratic Casimir for $SU(k+1)$ in the (n,0)-representation

$$C_2^{(k)} = \frac{nk(n+k+1)}{2(k+1)} \tag{4.26}$$

4.1.2 Construction of fuzzy S^4

Introduction to fuzzy S^4

The idea of fuzzy S^2 has been one of the guiding forces for us io investigate fuzzy spaces [92]. For example, $\mathbf{CP}^k$ are successfully constructed in the same spirit as the fuzzy S^2. Those interested in this area, it is of great interest to obtain a four-dimensional fuzzy space. $\mathbf{CP}^2$ is not qualified for this purpose, since $\mathbf{CP}^2$ does not have a spine structure [96]. S^4 is well motivated, since naturally leads to $\mathbf{R}^4$ at a certain limit.

There have been several attempts in [97, 98] to construct fuzzy S^4 from a field theoretic. In [99] the construction is through a projection from some matrix algebra and, owing to this forcible projection, it is advocated that fuzzy S^4 obeys a nonassociative algebra.

In [98], fuzzy S^4 is alternatively considered in a way of constructing a scalar field theory on it, based on the fact that $\mathbf{CP^3}$ is a $\mathbf{CP^1}$ (or S^2) bundle over S^4. Note that the term fuzzy S^4 is also used, mainly in the context of M(atrix) theory, e.g., in [100] also showing that the existent fuzzy S^2 and S^4 models are natural candidates for the quantum geometry on the corresponding spheres in AdS/CFT correspondence. Also the construction of fuzzy S^4 is considered through fuzzy $S^2 \times S^2$. This allows one to describe fuzzy S^4 with some concrete matrix configurations [101]. However, the algebra is still non-associative and one has to deal with non-polynomial functions on fuzzy S^4. Since those functions do not naturally become polynomials on S^4 in the commutative limits. In the case of fuzzy $\mathbf{CP}^k$, the fuzzy functions are represented by full $(N \times N)$-matrices, so the product of them is given by matrix multiplication which leads to associativity of the algbra for fuzzy $\mathbf{CP}^k$. The extra constraint is expressed as an algebraic constraint such that it enables us to describe the algebra of fuzzy S^4 in terms of the algebra of $SU(4)$ in the (n,0)-representation, the algebra of fuzzy S^4 is obtained from $SU(4)$ as well with the extra constraint on top of these fuzzy $\mathbf{CP}^3$ constraints. [102]

Construction of fuzzy S^4

We begin with construction of fuzzy $\mathbf{CP}^3$. The coordinates Q_A of fuzzy $\mathbf{CP}^3$

$$Q_A = \frac{L_A}{\sqrt{C_2^{(3)}}}, \tag{4.27}$$

where L_A are $N^{(3)} \times N^{(3)}$-matrix representation of SU(4), and the coordinates satisfy the following constraints:

$$Q_A Q_A = \mathbf{1}, \tag{4.28}$$

$$d_{ABC} Q_A Q_B = c_{3,n} Q_C. \tag{4.29}$$

Where **1** is the $N^{(3)} \times N^{(3)}$ identity matrix

$$N^{(3)} = \frac{1}{6}(n+1)(n+2)(n+3), \tag{4.30}$$

and from (4.25)

$$C_{3,n} = \frac{2}{\sqrt{C_2^{(3)}}}\left(\frac{n}{4} + \frac{1}{2}\right) \tag{4.31}$$

c_2^3 is the quadratic Casimir of $SU(4)$. From (4.26)

$$C_2^{(3)} = \frac{3n(n+4)}{8} \tag{4.32}$$

for $k \ll n$ (in this case $k = 3$), we have

$$C_{k,n} \longrightarrow c_k = \sqrt{\frac{2}{k(k+1)}}(k-1) \tag{4.33}$$

$SU(4)$ is decomposition as, $SU(4) \longrightarrow SU(2) \times SU(2) \times U(1)$, where $SU(2)$ and $U(1)$ are defined

$$\begin{pmatrix} SU(2) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & SU(2) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Each $SU(2)$ is the (2×2)-matrix representation.

In order to obtain functions on fuzzy S^4, we need to require

$$[\mathcal{F}, L_\alpha] = 0 \tag{4.34}$$

where $\mathcal{F}$ denote matrix-functions of Q_As and L_α are generators of H represented by $N^{(3)} \times N^{(3)}$-matrices, H being relevant to the above decompositionof SU(4) ($H \equiv SU(2) \times U(1)$). Construction of fuzzy S^4 can be carried out by imposing the additional constraint (4.34) onto the functions on fuzzy $\mathbf{CP}^3$. What we claim is that the further condition (4.34) makes the functions $\mathcal{F}(Q_A)$ become functions on fuzzy S^4. This does not mean that fuzzy S^4 is a subset of fuzzy $\mathbf{CP}^3$. $Q_A(A = 1, ..., 15)$'s are defined in $\mathbf{R}^{15}$. While locally, say around the pole of $A = 15$, globally they are embedded in $\mathbf{R}^{15}$. The equation (4.34) is a global constraint in this sence. So the algebra of fuzzy S^4 is given by a subset of SU(4). The emerging algebraic structure of fuzzy S^4 will be clearer when we consider the commutative limit of our construction [102].

A formula for the large N-limit of matrices

In this subsection, following [103, 104]. We now consider the symbol for the product $L_B A$, where L_B are the generators of $SU(K+1)$ (in this case S^4, k=3), A being an arbitrary $N^{(3)} \times N^{(3)}$-matrix. From (4.12), the symbol of $L_B A$ is given by

$$\langle \hat{L}_B \hat{A} \rangle = \langle (n,0), N^{(3)} | \hat{g}^T \hat{L}_B A \hat{g}^\star | (n,0), N^{(3)} \rangle \tag{4.35}$$

Commutative limit

In the large n limit we can approximate Q_A to the commutative coordinates on $\mathbf{CP}^3$;

$$Q_A \approx \phi_A = -2\text{tr}(g^\dagger t_A g t_{15}) \tag{4.36}$$

With the constraints for $\mathbf{cp}^3$ according (4.28) and (4.29) is

$$\phi_A \phi_A = 1, \quad d_{ABC}\ \phi_A\ \phi_B = \sqrt{\frac{2}{3}} \phi_C \tag{4.37}$$

In (4.36), t_A is the generators of $SU(4)$, and g is a group element of $SU(4)$ given as a (4×4) matrix

Matrix-function correspondence

We studing our construction of fuzzy S^4 by confirming its matrix-function correspondence. We focus on two things:

(i) Linking between the number of matrix element for fuzzy S^4 and the number of truncated functions on 4-spheres S^4.

(ii) The relationship between the product of functions on fuzzy S^4 and that on 4-spheres S^4.

In the case of fuzzy S^2, $S^2 = SU(2)/U(1)$. We consider $\mathcal{D}^{(j)}_{mn}(g)$ is Wigner $\mathcal{D}$-functions for $SU(2)$, where the spin-j representations of an $SU(2)$ group element g and $(m, n = -j, ..., +j)$, according (4.3)

$$\mathcal{D}^{(j)}_{mn}(g) = \langle jm | \hat{g} | jn \rangle \tag{4.38}$$

Functions on S^2 can be expanding when $n = 0$ in terms of particular Wigner $\mathcal{D}$-functions $\mathcal{D}^{(j)}_{m0}(g)$, which are invariant under a $U(1)$ right-translation operator acting on g. For definition (4.4). The state $|j0\rangle$ has no $U(1)$ charge, right action of the $U(1)$ operato, $R_3\mathcal{D}^{(j)}_{m0}(g) = 0$, the $\mathcal{D}$-functions basically are spherical harmonics

$$\mathcal{D}^{(l)}_{m0}(g) = \sqrt{\frac{4\pi}{2l+1}}(-1)Y^l_{-m}, \tag{4.39}$$

and truncated expansion can be written as

$$f_{S^2} = \sum_{l=0}^{n}\sum_{m=-l}^{l} f^l_m \mathcal{D}^{(l)}_{ml}. \tag{4.40}$$

The number of f^l_m are computing by $\sum_{l=0}^{n}(2l+1)$.

This relation implements the condition (i) by defining functions on fuzzy S^2 as $(n+1)\times(n+1)$ matrices. The condition (ii). One can show an exact correspondence of products. Let $f_{mn}(m,n = 1,...,n+1)$ be an element of matrix function-operator $\hat{f}$ on fuzzy S^2. As (4.3)

$$\langle\hat{f}\rangle = \sum_{m,n} f_{mn}\mathcal{D}^{*(j)}_{mj}(g)\mathcal{D}^{(j)}_{nj}(g), \tag{4.41}$$

where $\mathcal{D}^{*(j)}_{mj}(g) = \mathcal{D}^{(j)}_{jm}(g^{-1})$. The star product of fuzzy S^2 is

$$\langle\hat{f}\hat{g}\rangle = \langle\hat{f}\rangle * \langle\hat{g}\rangle \tag{4.42}$$

from (4.57) we can write

$$\langle\hat{f}\hat{g}\rangle = \sum_{m,n,k,r,l} f_{mn}g_{kl}\mathcal{D}^{*(j)}_{mj}(g)\mathcal{D}^{(j)}_{nr}(g)\mathcal{D}^{*(j)}_{kr}(g)\mathcal{D}^{(j)}_{lj}(g) \tag{4.43}$$

with using the orthogonality of $\mathcal{D}$-function $\sum_r \mathcal{D}^{(j)}_{nr}(g)\mathcal{D}^{*(j)}_{kr}(g) = \delta_{nk}$. Let R_- the lowering operator in right action

$$R_-\mathcal{D}^{(j)}_{mn}(g)\sqrt{(j+n)(j-n+1)}\mathcal{D}^{(j)}_{mn-1}(g) \tag{4.44}$$

by iteration (4.43) [102]

$$\langle\hat{f}\hat{g}\rangle = \sum_{s=0}^{2j}(-1)^s\frac{(2j-s)!}{s!(2j)!}R^s_-\langle\hat{f}\rangle R^s_+\langle\hat{g}\rangle \tag{4.45}$$

where we use the relation $R^* = R_+$. In the large j limit, the term with s=0 in (4.45) dominates and this leads to an ordinary commutative product of $\langle \hat{f} \rangle$ and $\langle \hat{g} \rangle$. By construction, the symbols of functions on fuzzy S^2 can be regarded as commutative functions on S^2

The matrix-function correspondence for fuzzy $\mathbf{CP}^3$ can be expressed by

$$N^{(3)} \times N^{(3)} = \sum_{l=0}^{n} dim(l,l) \tag{4.46}$$

$$dim(l,l) = \frac{1}{12}(2l+3)(l+1)^2(l+2)^2 \tag{4.47}$$

where $dim(l,l)$ is the dimension of SU(4) in the (l,l)-representation. This expression indicates that the number of matrix elements coincides with the number of coefficients in an expansion series of truncated functions on $\mathbf{CP}^3 = SU(4)/U(3)$ can be ewpanded by $\mathcal{D}^{(l,l)}_{M\mathbf{0}}(g)$, Wigner-$\mathcal{D}$-functions of $SU(4)$ in the (l,l)-representation $(l = 0, 1, 2, ...)$. The lower index $M(M = 1, ..., dim(l,l)$ labels the state in the (l,l)-representation, while the index $\mathbf{0}$ represents any suitably fixed state in this representation.

Return to the conditions (i) and (ii), we trying

1. Counting the number of truncated functions on S^4

2. A one-to-one matrix-function correspondence for fuzzy 4-sphere S^4

3. Proposing a block-diagonal matrix realiazation of fuzzy S^4

- For (1), can be made in terms of the pherical harmonics $Y_{l_1 l_2 l_3 m}$ on S^4 with a truncation at $l_1 = n$ [105]

$$N^{S^4}(n) = \sum_{l_1=0}^{n}\sum_{l_2=0}^{l_1}\sum_{l_3=0}^{l_2}(2l_3+1) = \frac{1}{12}(n+1)(n+2)^2(n+3) \tag{4.48}$$

Can be regarded as an N^2 (l)-degeneracy due to an S^2 internal symmetry for the extraction of S^4 out of $\mathbf{CP}^3 \sim S^4 \times S^2$ [102]. The number of truncated functions on $\mathbf{CP}^3$ is given by (4.47), the number of those on S^4 may be calculated by

$$\begin{aligned} N^{S^4}(n) = \sum_{l=0}^{n} \frac{dim(l,l)}{N^2(l)} &= \sum_{l=0}^{n} \frac{1}{6}(l+1)(l+2)(2l+1) \\ &= \frac{1}{12}(n+1)(n+2)^2(n+3) \end{aligned} \tag{4.49}$$

- For (2). Let $(\hat{F})_{IJ}(I,J=1,2,...,N^{(3)})$ denote a matrix-function on fuzzy $\mathbf{CP}^3$, and can be defined by $\langle I|\hat{F}|J\rangle$ as matrix element of the function $\hat{F}$ on fuzzy $\mathbf{CP}^3$, also we denote $\phi_{i_1...i_n}=|i_1...i_n\rangle \equiv |I\rangle$, where $\phi_{i_1...i_n}$ is the staate of fuzzy $\mathbf{CP}^3$

  We need to find an analogous matrix expression $(\hat{F}^{S^4})_{IJ}$ for a function on fuzzy S^4, for this, we splitting each i of the $\phi_{i_1 i_2...i_n}$ into a and $\dot{a}$ as

$$\phi_{i_1 i_2...i_n}=\{\phi_{\dot{a}_1\dot{a}_2...\dot{a}_n},\phi_{a_1\dot{a}_1...\dot{a}_{n-1}},...,\phi_{a_1...a_{n-1}\dot{a}_1},\phi_{a_1a_2...a_n}\}. \tag{4.50}$$

  can obtain the states corresponding to fuzzy S^4 by imposing an additional condition on (4.50),i.e., the invariance under the transformations involving any $\dot{a}_m(m=1,,n)$. On the set of states $\phi_{\dot{a}_1...\dot{a}_n}$, which are $(n+1)$ in number, the transformations must be diagonal [102], but we can have an independent transformation for each state. The number of the states is $(n+1)$, since the sequence of $\dot{a}_m=\{3,4\}$ is in a totally symmetric order. Thus we get (n+1) different functions proportional to identity. On the set of states $\phi_{a_1\dot{a}_1...\dot{a}_{n-1}}$, we can transform the a_1 to $b_1=\{1,2\}$, corresponding to a matrix function f_{a_1,b_1} which have 2^2 independent components. But we can also choose the matrix f_{a_1,b_1} to be different for each choice of $(\dot{a}_1...\dot{a}_{n-1})$ giving $2^2\times n$ function in all, at this level. We can represent these as $f^{(\dot{a}_1...\dot{a}_{n-1})}_{a_1,b_1}$. Where we split i_m into a_m, $\dot{a}_m$ and j_m into b_m, $\dot{b}_m$. We find that the set of all functions on fuzzy S^4 is given by

$$\begin{aligned}(\hat{F}^{S^4})_{IJ}=\{&f^{(\dot{a}_1...\dot{a}_n)}\hat{\delta}_{\dot{a}_1...\dot{a}_n,\dot{b}_1...\dot{b}_n},f^{(\dot{a}_1...\dot{a}_{n-1})}_{a_1,b_1}\hat{\delta}_{\dot{a}_1...\dot{a}_n,\dot{b}_1...\dot{b}_n},\\ &f^{(\dot{a}_1...\dot{a}_{n-2})}_{a_1,a_2,b_1b_2}\hat{\delta}_{\dot{a}_1...\dot{a}_{n-2},\dot{b}_1...\dot{b}_{n-2}},....,f_{a_1...a_n,b_1...b_n}\}.\end{aligned} \tag{4.51}$$

  The structure in (4.51) shows that $\hat{F}^{S^4}$ is composed of $(l+1)\times(l+1)$-matrices. The number of matrix elements for fuzzy S^4 is counted by

$$N^{S^4}(n)=\sum_{l=0}^{n}(l+1)^2(n+1-l) \tag{4.52}$$

  with the number of these matrices for fixed l being $(n+1-l)$ and $(l=0,1,...,n)$.

The symbol of the function $\hat{F}$ On fuzzy $\mathbf{CP}^3$ can be defined as

$$\langle \hat{F} \rangle = \sum_{I,J} \langle N|g|I\rangle (\hat{F})_{IJ} \langle J|g|N\rangle \tag{4.53}$$

$|N\rangle \equiv |(n,0), N^3\rangle$ is the highest or lowest weight state of fuzzy $\mathbf{CP}^3$, $\langle J|g|N\rangle$ denote the previous $\mathcal{D}$-function.

We now consider the product of two functions on fuzzy S^4, a function on fuzzy S^4 can be described by $(l+1)\times(l+1)$-matrices. The symbol of a function on fuzzy S^4 is defined in the same way except that $(\hat{F})_{IJ}$ is replaced with $(\hat{F}^{S^4})_{IJ}$ in (4.53). The star product of fuzzy S^4 is written as

$$\langle \hat{F}^{S^4} \hat{G}^{S^4} \rangle = \sum_{IJK} (\hat{F}^{S^4})_{IJ} (\hat{G}^{S^4})_{JK} \langle N|g|I\rangle \langle K|g|N\rangle \tag{4.54}$$

We can describe $\mathbf{CP}^3$ in terms of four complex coordinates Z_i with the identification $Z_i \sim \lambda Z_i$, $(\lambda \in \mathbf{C} - \{0\})$. Following [106], write Z_i

$$Z_i = (\omega_a, \pi_{\dot{a}}) = (x_{a\dot{a}}\pi_{\dot{a}}, \pi_{\dot{a}}) \tag{4.55}$$

where ω, π is tow spinors, $a = 1,2, \dot{a} = 1,2$ and $x_{a\dot{a}}$ can be dedined with coordinate x_μ on S^4, $x_{a\dot{a}} = (\mathbf{1}x_4 - i\sigma_i x^i), (i = 1,2,3)$, σ_i beign (2×2) Pauli matrices.

The homogeneous complex coordinates of $\mathbf{CP^3}$ are defined by $Z = (z_1, z_2, z_3)^T$, T is transposition of the vector or (1×3)-matrix. We introduce the notion $U_\alpha = g_{\alpha,4}$, $g_{\alpha,4}(\alpha = 1,2,3,4)$ is matrix elements are defined the coordinates on $\mathbf{CP}^3$ with $\bar{U}U = 1$, U_α's are related to Z

$$U_\alpha = \frac{1}{\sqrt{1 + \bar{z}.z}} \tag{4.56}$$

We can parmatrize U_i by the homogeneous coordinates Z_i, *i.e*, $U_i = \frac{z_i}{\sqrt{z.\bar{z}}}$. Function on S^4 can be considered as function on $\mathbf{CP}^3$ which satisfy

$$\frac{\partial}{\partial \pi_{\dot{a}}} f_{\mathbf{CP}^3}(Z, \bar{Z}) = \frac{\partial}{\partial \bar{\pi}_{\dot{a}}} f_{\mathbf{CP}^3}(Z, \bar{Z}) = 0 \tag{4.57}$$

.

- For (3). Let us write down the equation (4.52) in the following form:

$$\begin{aligned} N^{S^4}(n) = & \quad 1 \\ + & \quad 1+2^2 \\ + & \quad 1+2^2+3^2 \\ + & \quad .... \\ + & \quad 1+2^2+3^2+4^2+...+(n+1)^2. \end{aligned} \tag{4.58}$$

Coordinates of fuzzy S^4 are then represented by these $N^{(3)} \times N^{(3)}$ block-diagonal matrices, X_A , which satisfy

$$X_A X_A \sim \mathbf{1} \tag{4.59}$$

The Algebra for Non-commutative S^4

At the beginning we will assume $SO(5)$ invariance of the algebra. the algebra of the coordinates will be given

$$[\hat{X}^a, \hat{X}^b] = \epsilon_{abcde}\hat{X}^c\hat{X}^d\hat{X}^e \tag{4.60}$$

$\hat{X}$ is $(N \times N)$ hermitian matrice and ϵ_{abcde} is the Levi-Civita symbol, with the condition

$$(\hat{X}^a)^2 - C^2\mathbf{1} = 0 \tag{4.61}$$

The algebra (4.60) can be derived as equations of motion from the following action [105].

$$S = \int dt \ \mathrm{Tr}\left\{\frac{1}{2}\beta_R(D_0\hat{X}^a)^2 + \frac{1}{4}([\hat{X}^a, \hat{X}^b] - \alpha\epsilon_{abcde}\hat{X}^c\hat{X}^d\hat{X}^e)^2\right\} \tag{4.62}$$

In fact the bosonic part of this action for Matrix theory in the pp-wave background has this form [107]. This may be regarded as the bosonic part of the M-atrix theory in some background with cencel other bosonic cordinates, $\hat{X}^6, ..., \hat{X}^9$, here $\hat{X}^a$'s $(a = 1, ..., 5)$. D_0 is the covariant derivative $\partial t + i[\hat{A}_0,]$ and β_R is plaing role the raduis of the R circle.

A representation of (4.60) in terms of 4×4 matrices is given by tensor products of Pauli matrices.

$$\begin{aligned}
\hat{X}_0^1 &= \frac{1}{3}\sigma_3 \otimes \sigma_1 \\
\hat{X}_0^2 &= \frac{1}{3}\sigma_3 \otimes \sigma_2 \\
\hat{X}_0^3 &= \frac{1}{3}\sigma_3 \otimes \sigma_3 \\
\hat{X}_0^4 &= \frac{1}{3}\sigma_1 \otimes \mathbf{1}_2 \\
\hat{X}_0^5 &= \frac{1}{3}\sigma_2 \otimes \mathbf{1}_2
\end{aligned} \tag{4.63}$$

$\mathbf{1}_2$ is 2×2 identity matrix. We can replacing one of the two Pauli matrices by a spin-j representartin of $SO(3), T^a_{(j)}$ (These are four dimensional gamma matrices. It is then $N = 2(2j + 1)$ dimensional representation). When $T^a_{(j)}$ satisfies

$$[T^a_{(j)}, T^a_{(j)}] = i\epsilon_{abc}T^c_{(j)} \tag{4.64}$$

(4.65) becomes

$$
\begin{aligned}
\hat{X}_0^1 &= \frac{2}{3}\sigma_3 \otimes T_{(j)}^1 \\
\hat{X}_0^2 &= \frac{2}{3}\sigma_3 \otimes T_{(j)}^2 \\
\hat{X}_0^3 &= \frac{2}{3}\sigma_3 \otimes T_{(j)}^3 \\
\hat{X}_0^4 &= \frac{1}{3}\sigma_1 \otimes \mathbf{1}_{2j+1} \\
\hat{X}_0^5 &= \frac{1}{3}\sigma_2 \otimes \mathbf{1}_{2j+1}
\end{aligned} \tag{4.65}
$$

All X_0^a's satisfy (4.60) except

$$
[\hat{X}^4, \hat{X}^5] = \frac{3}{4j(j+1)}\epsilon_{45cde}\hat{X}^c\hat{X}^d\hat{X}^e \tag{4.66}
$$

We will construct a non-commutative product on S^4 corresponding to the Matrix configuration in terms of the product on S^2 .

4.1.3 The Fuzzy sphere S^2_N

Every manifold comes with a naturally defined associative algebra of functions with point-wise multiplication. This algebra is generated by the coordinates of the manifold and is from the definition commutative. As it turns out, this algebra contains all the information about the original manifold and we can describe geometry of the manifold purely in terms of the algebra. Also, every commutative algebra is an algebra of functions on some manifold. therefore, what we get is

$$\text{commutative algebras} \longleftrightarrow \text{differentiable manifolds}$$

A natural question to ask is whether there is a similar expression for non-commutative algebras, or

$$\text{non-commutative algebras} \longleftrightarrow ????$$

Quite obvious answer is no, there is no space to put on the other side of the expression. Coordinates on all the manifolds commute and that is the end of the story. So, as is often the case, we define new objects, called noncommutative manifolds, that are going to fit on the right hand side. Namely we look how aspects of the regular commutative manifolds are encoded into their corresponding algebras and we call the non-commutative manifold object, that would be encoded in the same way in a non-commutative algebra.

This is going to introduce non-commutativity among the coordinates. This notion should not be completely new, as the reader probably recalls the commutation relations of the quantum mechanics

$$[x^i, p_j] = i\hbar\delta^i_j \tag{4.67}$$

In classical physics, the phase space of the theory was a regular manifold. However in quantum theory we introduce non-commutativity between (some) of the coordinate and therefore the phase space of the theory becomes non-commutative. One of the most fundamental consequences of the commutation relations is the uncertainty principle. The exact position and momentum of the particle can not be measured and therefore we can not specify a single particular point of the phase space. Similarly, if there is non-commutativity between the coordinates, there is a corresponding uncertainty principle in measurement of coordinates. The notion of a space-time point stops to make sense, since we can not exactly say, where we are. This introduces a short distance structure to the space, quantities at short distance are not well localized. Because of this the non-commutative space are sometimes called the fuzzy spaces.

In practice, we often deform a commutative space into its non-commutative analogue. In this way we get noncommutative spaces that give a desired commutative limit. An example of such deformation is already mentioned phase space of quantum mechanics, but to illustrate the idea better, and since we will need the notions later, let show how this works for a two-sphere.

The fuzzy sphere S^2_N is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2, spaned by the spherical harmonic, is truncated at a given frequency. The algebra then becomes the finite dimensional algebra of $N \times N$ matrices. More precisely, recall that the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbb{R}^3$ modulo the relation [108]

$$\sum_{i=1}^{3} x_i^2 = R^2 \tag{4.68}$$

This comes with an understood condition on commutativity of the coordinates

$$x_i x_j - x_j x_i = 0 \tag{4.69}$$

Coordinates constrained in this way generate the algebra of all the functions on the sphere. Note that this is technically not the easiest way to do so. It is easier to introduce only two coordinates θ, ϕ on the sphere and define the algebra of functions not by the generators, but by the basis, e.g. the spherical harmonics. However the two sphere defined in our way is easier deformed into the non-commutative analogue. Now we define the fuzzy two sphere by the coordinates $\hat{x}_i$, which obey the following conditions

$$\sum_{i=1}^{3} \hat{x}_i^2 = \rho^2, \tag{4.70}$$

ρ the radius of the non-commutative sphere. R did describe the regular sphere

$$\hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\Lambda_N \epsilon_{ijk} \hat{x}_k, \tag{4.71}$$

The radius of the original sphere was encoded in the sum of the squares of the coordinates. We see, that such $\hat{x}'$s are achieved by a spin-j representation of the SU(2). If we chose

$$\sum_{i=1}^{3} L_i^2 = j(j+1) \equiv \frac{N^2-1}{4}, \tag{4.72}$$

with

$$[L_i, L_j] = i\epsilon_{ijk} L_k \tag{4.73}$$

$$\hat{x} = \Lambda_N L_i, \qquad i = 1, 2, 3 \tag{4.74}$$

Λ_N is the noncommutative parameter of dimension length

$$\Lambda_N = \frac{2\rho}{\sqrt{N^2-1}}, \tag{4.75}$$

with N the dimension of the representation. Matrices $\hat{x}_i$ become coordinates on the non-commutative sphere. The limit $N \to \infty$ removes non-commutativity, since $\Lambda \to 0$, nd we recover a regular sphere with radius R

The algebra of functions S_N^2 therefore coincides with the simple matrix algebra $Mat(N, \mathbb{C})$. The normalized integral of a function $f \in S_N^2$ is given by the trace

$$\int_{S_N^2} f = \frac{4\pi\rho^2}{N} \mathrm{tr}(f). \tag{4.76}$$

The functions on the fuzzy sphere can be mapped to functions on the commutative sphere S^2 using the decomposition into harmonics under the action

$$J_i f = [L_i, f] \tag{4.77}$$

of the rotation group SU(2) . One obtains analogs of the spherical harmonics up to a maximal angular momentum $N-1$. Therefore S_N^2 is a regularization of S^2 with a UV cutff, and the commutative sphere S^2 is recovered in the limit $N \to \infty$.

4.1.4 The fuzzy 2-sphere $S^2_{N_L} \times S^2_{N_R}$

The simplest 4-dimensional generalization of the above is the product $S^2_{N_L} \times S^2_{N_R}$ of 2 such fuzzy spheres, with generally independent parameters $N_{L,R}$. It is generated by a double set of representations of $su(2)$ commuting with each other, i. e. by λ^L_i, λ^R_i satisfying

$$\begin{aligned} [\lambda^L_i, \lambda^L_j] &= i\epsilon_{ijk}\lambda^L_k, \\ [\lambda^R_i, \lambda^R_j] &= i\epsilon_{ijk}\lambda^R_k, \\ [\lambda^L_i, \lambda^R_j] &= [\lambda^R_i, \lambda^L_j] = 0 \end{aligned} \tag{4.78}$$

The Casimirs is difines

$$\begin{aligned} \sum_{i=1}^{3} \lambda^L_i \lambda^L_i &= \frac{N_L^2 - 1}{4}, \\ \sum_{i=1}^{3} \lambda^R_i \lambda^R_i &= \frac{N_R^2 - 1}{4}. \end{aligned} \tag{4.79}$$

This can be realized as a tensor product of 2 fuzzy sphere algebras

$$\lambda^L_i = \lambda_i \otimes 1_{N_R \times N_R}, \tag{4.80}$$

$$.\lambda^R_i = 1_{N_R \times N_R} \otimes \lambda_i. \tag{4.81}$$

The normalized coordinate functions are given by

$$x^{L,R}_i = \frac{2R}{\sqrt{(N^{L,R})^2 - 1}} \lambda^{L,R}_i \tag{4.82}$$

with

$$\sum (x^L_i)^2 = \sum (x^R_i)^2 = R^2 \tag{4.83}$$

In rinciple one could also introduce different radii $R^{L,R}$ for the 2 spheres, but for simplicity we will keep only one scale parameter R (and sometimes we will set R=1). This space 1 can be viewed as regularization of $S^2 \times S^2 \subset \mathbb{R}^6$, and admits the symmetry group $SU(2)_L \times SU(2)_R \subset SO(6)$. The generators $x^{L,R}_i$ should be viewed as coordinates in an embedding space $\mathbb{R}^6$. The normalized integral of a function $f \in S^2_{N_L} \times S^2_{N_R}$ is now given by

$$\int_{S^2_{N_L} \times S^2_{N_R}} f = \frac{16\pi^2 R^4}{N_R N_L} \mathrm{tr}(f), \tag{4.84}$$

where we difine the $16\pi^2 R^4$ as volume

4.1.5 The limit to the canonical case $\mathbb{R}^4_\theta$

We will mainly consider $N_L = N_R$. It is well-known [?] that if a fuzzy sphere is blown up near a given point, it can be used to obtain a (compactified) noncommutative plane with canonical commutation relations: Consider the tangential coordinates $x_{1,2}$ near the north pole $x_3 = R$. Setting

$$R^2 = N\theta/2 \tag{4.85}$$

they satisfy the commutation relations

$$[x_1, x_2] = i\frac{2R}{N}x_3 = i\frac{2R}{N}\sqrt{R^2 - x_1^2 - x_2^2} = i\theta + \mathcal{O}(1/N) \tag{4.86}$$

Therefore in the double scaling limit with $N, R \to \infty$ keeping θ fixed, we recover the commutation relation of the canonical case,

$$[x_1, x_2] = i\theta \tag{4.87}$$

up to corrections of order $1/N$. Similarty, starting with $S^2_{N_L} \times S^2_{N_R}$ and setting

$$R^2 = \frac{N_{L,R}\theta_{L,R}}{2}, \tag{4.88}$$

we obtain in the large N_L , N_R limit

$$\begin{aligned} [x_i^L, x_j^L] &= i\epsilon_{ij}\theta^L, \\ [x_i^R, x_j^R] &= i\epsilon_{ij}\theta^R, \\ [x_i^L, x_j^R] &= [x_i^R, x_j^L] = 0 \end{aligned} \tag{4.89}$$

This is the most general form of $\mathbb{R}^4_\theta$ with coordinates $(x_1, ..., x_4) \equiv (x_1^L, x_2^L, x_1^R, x_2^R)$. The integral of a function $f(x)$ then

$$\int_{S^2_{N_L} \times S^2_{N_R}} f(x) \to 4\pi^4\theta_L\theta_R \mathrm{tr}((f(x)) =: \int_{\mathbb{R}^4_\theta} f(x). \tag{4.90}$$

4.2 Fuzzification

$\mathcal{D}_{\mathbf{CP}^2}$ acts on a subspace of $\mathcal{A}_{\mathbf{CP}^2}\otimes Mat_{16}$. We can thus conceive of a fuzzy Dirac operator $\mathbf{D}_{\mathbf{CP}^2}$ which acts on a subspace of $\mathbf{A}_{\mathbf{CP}^2}\otimes Mat_{16}$, $\mathbf{A}_{\mathbf{CP}^2}$ being obtained from $\mathcal{A}_{\mathbf{CP}^2}$ by restricting "orbital" $SU(3)$ IRR's to (n,n), $n\leq N$. $\mathbf{D}_{\mathbf{CP}^2}$ is then obtained from $\mathcal{D}_{\mathbf{CP}^2}$ by projection to this subspace. $\mathcal{D}_{\mathbf{CP}^2}$ commutes only with the total $SU(3)$ Casimir J_i^2 and not with orbital $SU(3)$ Casimir $\mathcal{L}_i^2$. This causes edge effects distorting the spectrum of $\mathbf{D}_{\mathbf{CP}^2}$ for those states having (n,n) near (N,N) which $\mathcal{D}_{\mathbf{CP}^2}$ mixes with $(n^{'},n^{'})$, $n^{'}\geq N$. This particular edge phenomenon does not occur for $\mathbf{S}^2 = \mathbf{CP}^1$ where orbital angular momentum $\mathcal{L}_i^2$ commutes with the Dirac operator. A way to eliminate such problems is suggested by the work of [109–112]: We introduce the cut-off not on the orbital Casimir, but on the *total* Casimir, retaining all states upto the cut-off. That seems the best strategy as it will give a fuzzy Dirac operator $\mathbf{D}_{\mathbf{CP}^2}$ with a spectrum exactly that of the continuum operator $\mathcal{D}_{\mathbf{CP}^2}$ upto the cut-off point, and which has chirality (chirality $\Gamma_{\mathbf{CP}^2}$ of $\mathcal{D}_{\mathbf{CP}^2}$ commutes with J_i^2) and no fermion doubling. This approach is the same as the method adopted for $\mathbf{S}^2$ in [109–112] . For $\mathbf{S}^2$, the edge effect turned up as the absence of the $-E$ eigenvalue subspace for the maximum total angular momentum when the cut-off is introduced in orbital angular momentum, and attendant problems with chirality.

4.2.1 Coherent States and Star Products : The Case of $\mathbf{S}^2 \simeq \mathbf{CP}^1$

These have been treated in [111, 113, 114]. Here we summarize the main points so that we can outline the relation of wave functions and those based on matrices for fuzzy physics.

Let us first consider $\mathbf{S}^2 = \mathbf{CP}^1$ and its fuzzy versions. The algebra $\mathbf{A}$ is Mat_{2l+1}. $SU(2)$ acts on $\mathbf{A}$ on left and right with generators L_i^L and $-L_i^R$, and orbital angular momentum is $\mathcal{L}_i = L_i^L - L_i^R$. The spectrum of $\mathcal{L}^2$ is $K(K+1)$, $K = 0,1,..,2l$. We can find a basis of matrices T_M^K diagonal in $\mathcal{L}^2$ and $\mathcal{L}_3$(with eigenvalue M) and standard matrix elements for $\mathcal{L}_i$. $\mathbf{A}$ acts on a $(2l+1)-$dimensional vector space with the familiar basis $|l,m\rangle$. T_M^K are orthogonal, $K(K+1)$ and M being eigenvalues of $\mathcal{L}^2$ and $\mathcal{L}_3$:

$$(T_M^K, T_{M'}^{K'}) := Tr\, T_M^{K\dagger} T_{M'}^{K'} = \text{ constant} \times \delta_{KK'}\delta_{MM'}. \tag{4.91}$$

The above suggests that there is a way to regard $\mathbf{A}$ as "functions" on $\mathbf{S}^2$ with angular momenta cut-off at $2l$. Such functions are also represented by the linear span of spherical harmonics Y_{KM}, $K{\leq}2l$. We want to clarify the relation of Y_{KM}'s to the matrices T^K_M in $\mathbf{A}$.

Towards this end, let us introduce the coherent states

$$|g\rangle = U^{(l)}(g)|l,l\rangle \tag{4.92}$$

induced from the highest weight vector $|l;l\rangle$. $g{\rightarrow}U^{(K)}(g)$ is the angular momentum K IRR of $SU(2)$. Note the identity

$$|ge^{i\frac{\sigma_3}{2}\theta}\rangle = e^{il\theta}|g\rangle. \tag{4.93}$$

It is a theorem [113] that the diagonal matrix elements $\langle g|a|g\rangle$ completely determine the operator a. Further $\langle ge^{i\frac{\sigma_3}{2}\theta}|a|ge^{i\frac{\sigma_3}{2}\theta}\rangle = \langle g|a|g\rangle$ so that $\langle g|a|g\rangle$ depends only on

$$g\sigma_3 g^{-1} = \sigma\cdot x, \quad \sum_{i=1}^{3} x_i^2 = 1; x{\in}\mathbf{S}^2. \tag{4.94}$$

In this way, we have the map

$$\begin{aligned} &\mathbf{A}{\rightarrow}\mathbf{C}^\infty(\mathbf{S}^2),\\ &a{\rightarrow}\tilde{a};\\ where \quad &\\ &\tilde{a}(x) = \langle g|a|g\rangle. \end{aligned} \tag{4.95}$$

In this map, the image of T^K_M is Y_{KM} after a phase choice:

$$Y_{KM}(x) = \langle g|T^K_M|g\rangle. \tag{4.96}$$

For, under $g{\rightarrow}hg$, $x{\rightarrow}R(h)x$ where $h{\rightarrow}R(h)$ is the $SU(2)$ vector representation. Under this transformation, since

$$Y_{KM}(R(h)x) = \sum_{M'=-K}^{K} D^{(K)}(h)_{MM'}Y_{KM'}(x) \tag{4.97}$$

and

$$T^K_M{\longrightarrow}U^{(K)}(h)^{-1}T^K_M U^{(K)}(h) = \sum_{M'=-K}^{K} D^{(K)}(h)_{MM'}T^K_{M'}, \tag{4.98}$$

where $h \longrightarrow D^{(K)}(h)$ is the angular momentum K IRR of $SU(2)$ in a matrix representation, we have the proportionality of the two sides. (4.96) and phase conventions fix the constant of proportionality.

The map $T^K_M \rightarrow Y_{KM}$ is an isomorphism at the level of vector spaces. It can be extended to the noncommutative algebra **A** by defining a new product on Y_{KM}'s, the star product. Thus consider $\langle g|T^K_M T^L_N|g\rangle$. The functions Y_{KM} and Y_{LN} completely determine T^K_M and T^L_N, and for that reason also this matrix element. Hence it is the value of a function $Y_{KM} * Y_{LN}$, linear in each factor, at x:

$$\langle g|T^K_M T^L_N|g\rangle = [Y_{KM} * Y_{LN}](x). \tag{4.99}$$

The product $*$ here , the star product , extends by linearity to all functions with angular momenta $\leq 2l$. The resultant algebra is isomorphic to the algebra **A**.

The explicit formula for $*$ has been found by Prešnajder [111] (see also [114]). The image of $\mathcal{L}_i a$ is just $-i(\vec{x}\wedge\vec{\nabla})_i\tilde{a}$. We will use the same symbol $\mathcal{L}_i$ to denote $-i(\vec{x}\wedge\vec{\nabla})_i$ derivation. The $*$ product is covariant under the $SU(2)$ action in the sense that

$$\mathcal{L}_i(\tilde{a} * \tilde{b}) = (\mathcal{L}_i\tilde{a}) * \tilde{b} + \tilde{a} * (\mathcal{L}_i\tilde{b}). \tag{4.100}$$

It depends on l and approaches the commutative product of $\mathbf{C}^\infty(\mathbf{S}^2)$ as $l \longrightarrow \infty$. Coherent states thus give an intuitive handle on the matrix representation of functions.

But on $\mathbf{S}^2$, we also have monopole bundles. Sections of these bundles for Chern class n are spanned by the rotation matrices $D^{(j)}_{mn}$, $j \geq |n|$. They have the equivariance property

$$D^{(j)}_{mn}(ge^{i\frac{\sigma_3}{2}\theta}) = D^{(j)}_{mn}(g)e^{in\theta}. \tag{4.101}$$

This last equation is essentially a generalization of equation (4.100) , in other words one can identify $D^{(j)}_{mn}$ with $< j, m|\psi^{(j)}_n >$ where $|\psi^{(j)}_n >= D^j|j, n >$ [see equation (4.101)].

How do we represent them by matrices?

In the first instance, let $n \geq 0$ and consider the coherent states (now with an additional label)

$$\begin{aligned} |g; l+n\rangle &= U^{(l+n)}(g)|l+n, l+n\rangle \\ |g; l\rangle &= U^{(l)}(g)|l, l\rangle. \end{aligned} \tag{4.102}$$

They span vector spaces V_{l+n} and V_l. We can consider the linear operators $Hom(V_{l+n}, V_l)$ from V_{l+n} to V_l. They are $[2l+1]\times[2(l+n)+1]$ matrices in a basis of V_{l+n} and V_l, and have $U^{(l)}(g)$ acting on their left(with generators L_i^L) and $U^{(l+n)}(g)$ acting on their right (with generators $-L_i^R$). We can decompose $Hom(V_{l+n}, V_l)$ under the "orbital" angular momentum group $U^{(l)}\otimes U^{(l+n)}$ (with generators $\mathcal{L}_i = L_i^L - L_i^R$) into the direct sum $\oplus_{K=n}^{2l+n}(K)$ with the IRR K having the basis T_M^K, with $\mathcal{L}_3 T_M^K = MT_M^K$. As before, we choose T_M^K so that $\mathcal{L}_i$ follow standard phase conventions. T_M^K are orthogonal

$$Tr(T_{M'}^{K'})^\dagger T_M^K = \text{ constant} \times \delta_{K'K}\delta_{M'M}. \tag{4.103}$$

Now consider

$$\langle g; l|T_M^K|g; l+n\rangle. \tag{4.104}$$

It transforms in precisely the same manner as $D_{Mn}^{(K)}(g)$ under $g\rightarrow hg$ and $g\rightarrow ge^{i\frac{\sigma_3}{2}\theta/2}$ and hence after an overall normalisation,

$$\langle g; l|T_M^K|g; l+n\rangle = D_{Mn}^{(K)}(g). \tag{4.105}$$

Thus $Hom(V_{l+n}, V_l)$ are fuzzy versions of sections of vector bundles for Chern class $n\geq 0$. For $n < 0$, they are similarly $Hom(V_l, V_{l+|n|})$. This result is due to [115] (see also [109–112, 116]). An explicit formulae for the fuzzy version of rotation matrices can be found in [111].

It is interesting that Chern class has a clear meaning even in this matrix model: It is $|V| - |W|$ for $Hom(V, W)$, where $|V|$ and $|W|$ are dimensions of V and W.

There are two (inequivalent) fuzzy algebras acting on $Hom(V, W)$. $Mat_{|V|} = \mathbf{A}_{|V|}$ acts on the right and $Mat_{|W|} = \mathbf{A}_{|W|}$ acts on the left, where now a subscript has been introduced on $\mathbf{A}$. These left and right actions have their own $*$'s, call them $*_{|V|}$ and $*_{|W|}$: if $a\in\mathbf{A}_V$, $b\in\mathbf{A}_W$ and $\tilde{a}$ and $\tilde{b}$ are the corresponding functions, then

$$bT_M^K a \longrightarrow \tilde{b} *_{|W|} Y_{KM} *_{|V|} \tilde{a} \tag{4.106}$$

under the map of $Hom(V, W)$ to sections of bundles. There is also a fuzzy analogue for tensor products of bundles. Thus we can compose elements of $Hom(V, W)$ and $Hom(W, X)$ to get $Hom(V, X)$

$$Hom(V, X) = Hom(V, W)\otimes_{\mathbf{A}_{|W|}} Hom(W, X). \tag{4.107}$$

Its elements are ST, $S\in Hom(V,W)$, $T\in Hom(W,X)$. Its Chern class is $|V|-|X|$. If $\tilde{S}$ and $\tilde{T}$ are the representatives of S and T in terms of sections of bundles, then $ST\longrightarrow\tilde{S}*\tilde{T}$.

Tensor products $\Gamma_1\otimes\Gamma_2$ of two vector spaces Γ_1 and Γ_2 over an algebra B are defined only if $\Gamma_1(\Gamma_2)$ is a right-(left-) B-module [117]. Hence $Hom(V,W)\otimes_{\mathbf{A}_{|W|}}Hom(W',X)$ is defined only if $W=W'$. So $\tilde{S}*\tilde{T}$ is rather different in its properties from the usual tensor product of bundle sections, in particular $\tilde{T}*\tilde{S}$ makes no sense if $V\neq X$.

4.2.2 Fuzzy Dirac Spinors on $\mathbf{S}^2_F$

We can now comment on the fuzzy . Elsewhere the Watamuras [118, 119] and following them, us [120, 121], investigated the Dirac operator as acting on $\mathbf{A}\otimes C^2=\mathbf{A}^2$, $\mathbf{A}=Mat_{2l+1}$. That led to rather an elaborate formalism because of the cut-off in orbital angular momentum. So as indicated earlier, it seems more elegant to cut-off total angular momentum at some value j_0.

We can now argue such a cut-off leads to the formalism of [109–112, 122] and to supersymmetry. Thus let $T^j_{m+}\in Hom(V_{l+1/2},V_l)$ with the transformation property $< g;l|T^j_{m+}|g;l+\frac{1}{2}>\longrightarrow e^{i\frac{\theta}{2}}< g;l|T^j_{m+}|g;l+\frac{1}{2}>$ under $g\longrightarrow ge^{i\frac{\theta}{2}\sigma_3}$. One also has the transformation property

$$U^{(l)}(g)^\dagger T^j_{m+}U^{(l+\frac{1}{2})}(g)=\sum_{m'}D^{(j)}_{mm'}(g)T^j_{m'+} \tag{4.108}$$

[So $j\leq 2l+1/2$ and $j_0=2l+1/2$]. Then one can make the identification

$$D^j_{m+}(g)=\langle g;l|T^j_{m+}|g;l+\frac{1}{2}\rangle, \tag{4.109}$$

since from equation (4.101) it is easy to see that $D^j_{m+}(g)\longrightarrow e^{i\frac{\theta}{2}}D^j_{m+}(g)$ under $g\longrightarrow ge^{i\frac{\theta}{2}\sigma_3}$.

The subscript + in T^j_{m+} indicates helicity − , i.e T^j_{m+} is the fuzzy version of $< j,m|\psi^{(j)}_+ >$ of (**??**) so that it will be associated with the negative helicity part of the wave function.

For helicity +, but for same j_0, we have to consider $T^j_{m-}\in Hom(V_l,V_{l+1/2})$, with

$$U^{(l+\frac{1}{2})}(g)^\dagger T^j_{m-}U^{(l)}(g)=\sum_{m'}D^{(j)}_{mm'}(g)T^j_{m'-}. \tag{4.110}$$

Of course now ,

$$D^j_{m-}(g) =< g; l + \frac{1}{2}|T^j_{m-}|g; l >, \tag{4.111}$$

where both sides will acquire now a phase $exp(-i\frac{\theta}{2})$ under the right $U(1)$ action, namely under $g{\longrightarrow}gexp(i\frac{\theta}{2}\sigma_3)$. T^j_{m-} is then clearly the fuzzy version of $< j, m|\psi^{j)}_- >$.

This is the formalism of [109–112, 122] . As we have united $V^{(l)}$ and $V^{(l+1/2)}$, it is natural to consider $OSp(2, 1)$ or even $OSp(2, 2)$ SUSY as discovered first by Grosse et al in the second paper of [112].

Because of the mixing of l and $l + 1/2$, we have to reconsider the action of the matrix algebra $\mathbf{A}$ approximating $\mathcal{A} = C^\infty(\mathbf{S}^2)$. Mat_{2l+1} acts on $T^j_{m+}(T^j_{m-})$ on the left(right) while Mat_{2l+2} acts on $T^j_{m+}(T^j_{m-})$ on the right(left). So it is best to regard fuzzy functions to act on left(say) of T^j_{m+} and right of T^j_{m-} as Mat_{2l+1}. This suggestion is slightly different from that of [109–112, 122] where they regard the fuzzy algebra to be Mat_{2l+1} on T^j_{m+} and Mat_{2l+2} on T^j_{m-}, both acting on left. However, our proposal does not generalize to instanton (monopole) sectors.

We can restore spin parts to fuzzy wave functions. The spin wave functions for helicity $\pm$ are $T^{\frac{1}{2}}_{m_s\pm}$, where m_s denotes the two components of the spinor . The positive chirality spinors are defined by

$$< g; l|T^{\frac{1}{2}}_{m_s+}|g; l + \frac{1}{2} >= D^{\frac{1}{2}}_{m_s+} =< \frac{1}{2}, m_s|\psi^{(\frac{1}{2})}_+ >, \tag{4.112}$$

while the negative chirality spinors are defined by

$$< g; l + \frac{1}{2}|T^{\frac{1}{2}}_{m_s-}|g; l >= D^{\frac{1}{2}}_{m_s-} =< \frac{1}{2}, m_s|\psi^{(\frac{1}{2})}_- > . \tag{4.113}$$

So the two components of the total fuzzy wave functions for helicity $\pm$ are

$$< \frac{1}{2}, m_s|\psi^{\pm}_F >= \Big[\sum_{j,m}\xi^{j\pm}_m T^j_{m\mp}\Big]T^{\frac{1}{2}}_{m_s\pm}, \quad \xi^{j\pm}_m{\in}\mathbf{C}, m_s = +\frac{1}{2}, -\frac{1}{2}. \tag{4.114}$$

This is the fuzzy version of equation (**??**).

The Dirac operator $\mathbf{D}_{2g}$ is given by the truncated version of (**??**) :

$$\begin{aligned}&\rho\sum_{m_s}(\mathbf{D}_{2g})_{m'_s m_s}\{\sum_{j,m}\xi^{j+}_m T^j_{m-}T^{1/2}_{m_s+} + \sum_{j,m}\xi^{j-}_m T^j_{m+}T^{1/2}_{m_s-}\} =\\ &-\{\sum_{j,m}\xi^{j+}_m T^j_{m+}(J^{(j)}_+)_{+1/2,-1/2}\}\{T^{1/2}_{m'_s-}\} - \{\sum_{j,m}\xi^{j-}_m T^j_{m-}(J^{(j)}_-)_{-1/2,+1/2}\}\{T^{1/2}_{m'_s+}\},\\ &j{\le}2l + 1/2,\end{aligned} \tag{4.115}$$

$J_i^{(j)}$ being the angular momentum j images of $\frac{\sigma_i}{2}$.

4.2.3 The Case of $\mathbf{CP}^2$

Coherent states for $\mathbf{CP}^2$ can be defined using highest weight states. For IRR $(3,0)$, we can pick the highest weight state with $I = I_3 = 0, \quad Y = -2/3$, namely the $c-$quark: $|0,0,-2/3\rangle \equiv |0,0,-2/3;(3,0)\rangle$. Then if $g \longrightarrow U^{(3,0)}(g)$ defines the IRR, $|g;(3,0)\rangle = U^{(3,0)}(g)|0,0,-2/3;(3,0)\rangle$. For the IRR $(N,0)$, we can simply replace $|0,0,-2/3;(3,0)\rangle$ by its $N-$fold tensor product

$$|0,0,-\frac{2}{3};(3,0)\rangle \otimes |0,0,-\frac{2}{3};(3,0)\rangle \otimes ... \otimes |0,0,-\frac{2}{3};(3,0)\rangle = |0,0,-\frac{2N}{3};(N,0)\rangle, \tag{4.116}$$

and set

$$|g;(N,0)\rangle = U^{(N,0)}(g)|0,0,-\frac{2N}{3};(N,0)\rangle. \tag{4.117}$$

For $(0,N)$, we can use the $\bar{c}-$quark state $|g;(0,3)\rangle = U^{(0,3)}(g)|0,0,+2/3;(0,3)\rangle$ and its tensor product states.

The development of ideas now keep following $\mathbf{S}^2 = \mathbf{CP}^1$. Full details can be found in [114].

General theory confirms that the maps $a \longrightarrow \tilde{a}$ from matrices in the $(N,0)$ or $(0,N)$ IRR to functions on $\mathbf{CP}^2$, defined by

$$\begin{aligned} \tilde{a}(\xi) &= \langle (N,0); g|a|g;(N,0)\rangle \\ & \quad or \\ \tilde{a}(\xi) &= \langle (0,N); g|a|g;(0,N)\rangle. \end{aligned} \tag{4.118}$$

are one-to-one so that a $*-$product on $\tilde{a}$'s exists. In this map, the $SU(3)$ generators $\mathcal{L}_i$ acting on $\tilde{a}$ become the corresponding $\mathbf{CP}^2$ $SU(3)$ operators $-if_{ijk}\hat{\xi}_j\frac{\partial}{\partial\hat{\xi}_k}$. We shall use the same symbol $\mathcal{L}_i$ for these operators too. The orbital $SU(3)$ action is compatible with $*$ in the sense that $\mathcal{L}_i(\tilde{a} * \tilde{b}) = (\mathcal{L}_i\tilde{a}) * \tilde{b} + \tilde{a} * (\mathcal{L}_i\tilde{b})$. Irreducible tensor operators of $SU(3)$ are well studied [123]. With their help, fuzzy analogues of $D-$matrices can be constructed, as also sections of $U(1)$ and $U(2)$ bundles.

The fuzzy $\mathbf{CP}^2$ Dirac operator is the cut-off . We omit the details: the necessary group theory is already to be found in [116] while the rest is routine.

4.3 Gauge theory on fuzzy $S^2 \times S^2$

Now that we have the fuzzy space $S^2_{N_L} \times S^2_{N_R}$ corresponding to $(N_L N_R)^2$-dimensional matrices, we want to construct a matrix model having $S^2_{N_L} \times S^2_{N_R}$ as its ground state. As in the canonical case, the fluctuations around this ground state will produce a gauge theory. But as the matrices are now finite-dimensional, the model will be well defined and finite.

In the fuzzy case, it is natural to construct $S^2_{N_L} \times S^2_{N_R}$ as a submanifold of $\mathcal{R}^6$. We therefore consider a multi-matrix model with 6 dynamical fields (covariant coordinates) B_i^L and B_i^R , with i run from 1 to 3, which are $(N_L N_R) \times (N_L N_R)$ Hermitian matrices. As action we choose the following generalization of the action in [124],

$$S = \frac{1}{g^2} \int \frac{1}{2} F_{ia\ jb} F_{ia\ jb} + \varphi_L^2 + \varphi_R^2 \tag{4.119}$$

with $a, b = L, R$ and i,j=1,2,3; summation over repeated indices is implied. Here $\varphi_{L,R}$ are defined as

$$\begin{aligned} \varphi_L : &= \frac{1}{R^2}(B_i^L B_i^L - \frac{N_L^2 - 1}{4}) \\ \varphi_R : &= \frac{1}{R^2}(B_i^R B_i^R - \frac{N_R^2 - 1}{4}) \end{aligned} \tag{4.120}$$

and R denotes the radius of the two spheres. The field strength is defined by

$$\begin{aligned} F_{iL\ jL} &= \frac{1}{R^2}(i[B_i^L, B_j^L] + \epsilon_{ijk} B_k^L), \\ F_{iR\ jR} &= \frac{1}{R^2}(i[B_i^R, B_j^R] + \epsilon_{ijk} B_k^R), \\ F_{iL\ jR} &= \frac{1}{R^2}(i[B_i^L, B_j^R]) \end{aligned} \tag{4.121}$$

This model (4.119) is manifestly invariant under $SU(2)_L \times SU(2)_R$ rotations acting in the abvious way, and $U(N_R N_L)$ gauge transformations acting

$$B_i^{L,R} \longrightarrow U\ B_i^{L,R}\ U^{-1}, \tag{4.122}$$

if the action (4.119) is considered as a matrix model, the radius drops out using (4.84). The equations of motion for B_i^L are

$$\begin{aligned} & \{B_i^L, B_j^L B_j^L - \frac{N_L^2 - 1}{4}\} + (B_i^L + i\epsilon_{ijk} B_j^L B_k^L) \\ + \quad & i\epsilon_{ijk}[B_j^L, (B_k^L + i\epsilon_{krs} B_r^L B_s^L)] + [B_j^R, [B_j^R, B_i^L]] = 0 \end{aligned} \tag{4.123}$$

and those for B_i^R are obtained by exchanging $L \leftrightarrow R$. By construction, the minimum or ground state of the action is given by $F = \varphi = 0$, hence $B_i^{L,R} = \lambda_i^{L,R}$ as in (4.81,4.81) up to gauge transformations. [125] for a similar approach on $\mathbb{C}P^2$. We can therefore expand the covariant coordinates B_i^L and B_i^R around the ground state

$$B_i^a = \lambda_i^a + RA_i^a, \tag{4.124}$$

where $a \in \{L, R\}$ and A_i^a is very small, Then $A_i^{L,R}$ transforms under gauge transformations as

$$A_i^{L,R} \to A_i^{\prime L,R} = U\ A_i^{L,R}\ U^{-1} + U[\lambda_i^{L,R}, U^{-1}], \tag{4.125}$$

and the field strength takes a more familiar form (We do not distinguish between upper and lower indices L, R .)

$$\begin{aligned}
F_{iLjL} &= i([\frac{\lambda_i^L}{R}, A_j^L] - [\frac{\lambda_j^L}{R}, A_i^L + [A_i^L, A_j^L]), && (4.126)\\
F_{iRjR} &= i([\frac{\lambda_i^R}{R}, A_j^R] - [\frac{\lambda_j^R}{R}, A_i^L + [A_i^R, A_j^R]), && (4.127)\\
F_{iLjR} &= i([\frac{\lambda_i^L}{R}, A_j^R] - [\frac{\lambda_j^R}{R}, A_i^L + [A_i^L, A_j^R]).
\end{aligned}$$

So far, the spheres are described in terms of 3 Cartesian covariant coordinates each. In the commutative limit, we can separate the radial and tangential degrees of freedom. There are many ways to do R this; perhaps the most elegant for the present purpose is to note that the terms $\int \varphi_L + \varphi_R$ in the action imply that $\varphi_{L,R}$ is bounded for configurations with finite action. Using

$$\varphi_L = \frac{\lambda_i^L}{R} A_i^L + A_i^L \frac{\lambda_i^L}{R} + A_i^L A_i^L, \tag{4.128}$$

and similarly for φ_R it follows that

$$x_i A_i^a + A_i^a x_i = \mathcal{O}(\frac{\varphi}{N}), \tag{4.129}$$

for finite A_i^a . This means that A_i^a is tangential in the (commutative) large N limit. Alternatively, one could consider $\phi_L = N\varphi_L$, which would acquire a mass of order N and decouple from the other fields. The commutative limit

of (4.119) therefore gives the standard action for electrodynamics on $S^2 \times S^2$
,

$$S = \frac{1}{2g^2} \int_{S^2 \times S^2} F^t_{ia\ jb} F^t_{ia\ jb}, \tag{4.130}$$

with $a, b = L, R$, at the north pole $x_3^{L,R} = R$, one can replace

$$i[\frac{\lambda_i^{L,R}}{R}, .] \rightarrow -\epsilon_{ij} \frac{\partial}{\partial x_j^{L,R}} \tag{4.131}$$

In the commutative limit, so that upon identifying the commutative gauge fields A_i^{cl} via

$$A_i^{(cl)L,R} = -\epsilon_{ij} A_i^{L,R} \tag{4.132}$$

the field strength is given by the standard expression

$$F^t_{iL\ jR} = \partial_i^L A_j^{(cl)R} - \partial_j^R A_i^{(cl)L} \tag{4.133}$$

4.3.1 $U(k)$ gauge theory

The above action generalizes immediately to the nonabelian case, keeping precisely the same action (4.119,4.120) , but replacing the matrices $B_i^{L,R}$ by $k(N_R N_L) \times k(N_R N_L)$ matrices. The constraint term will then impose as ground state

$$\lambda_i^{L/R} \otimes 1_{k \times k}. \tag{4.134}$$

Expanding the covariant coordinates

$$B_i^{L,R} = \lambda_i^{L/R} \otimes 1_{k \times k} + A_{i,a}^{L/R} T^a, \tag{4.135}$$

in terms of the Hellman matrices T^a, the action (4.119) is the fuzzy version of nonabelian $U(k)$ Yang-Mills on $S^2 \times S^2$.

4.3.2 A formulation based on SO(6)

The above action can be cast into a nicer form by assembling the matrices $B_i^{L,R}$ into bigger collective matrices, following [126]. Since it is natural from the fuzzy point of view to embed $S^2 \times S^2 \subset \mathcal{R}^6$ with corresponding embedding of the symmetry group $SO(3)_L \times SO(3)_R \subset SO(6)$, we consider

$$B_\mu = (B_i^L, B_i^R), \tag{4.136}$$

to be the 6-dimensional irrep of $so(6) \cong su(4)$. Since $(4) \otimes (4) = (6) \oplus (10)$, it is natural to introduce the intertwiners

$$\gamma_\mu = (\gamma_i^L, \gamma_i^R) = (\gamma_\mu)^{\alpha,\beta} \tag{4.137}$$

We could then assemble our dynamical fields into a single $4(N_R N_L) \times 4(N_R N_L)$ matrix

$$B = B_\mu \gamma_\mu + \text{const.}\mathbb{1} \tag{4.138}$$

Of course the most general such $4(N_R N_L) \times 4(N_R N_L)$ matrix contains far too many degrees of freedom, and we have to constrain these B further. Since $SU(4)$ acts on B as

$$B \to U^T B U, \tag{4.139}$$

the γ_μ can be chosen as totally anti-symmetric matrices, which precisely singles out the $(6) \subset (4) \otimes (4)$. One can moreover impose

$$(\gamma_i^L)^\dagger = \gamma_i^L, \tag{4.140}$$
$$(\gamma_i^R)^\dagger = -\gamma_i^R, \tag{4.141}$$

and

$$\gamma_i^L \gamma_j^L = \delta_{ij} + i\epsilon_{ijk}\gamma_k^L \tag{4.142}$$
$$\gamma_i^R \gamma_j^R = -\delta_{ij} - i\epsilon_{ijk}\gamma_k^R \tag{4.143}$$
$$\gamma_i^L, \gamma_j^R] = 0 \tag{4.144}$$

the representation satisfy

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{4.145}$$

With these we define the 4 -dimensional antisymmetric matrices

$$\begin{aligned} \gamma_L^1 &= \sigma^1 \otimes \sigma^2, \quad \gamma_L^2 = \sigma^2 \otimes 1, \quad \gamma_L^3 = \sigma^3 \otimes \sigma^2, \\ \gamma_R^1 &= i\sigma^2 \otimes \sigma^1, \quad \gamma_R^2 = i1 \otimes \sigma^2, \quad \gamma_R^3 = i\sigma^2 \otimes \sigma^3 \end{aligned} \tag{4.146}$$

Where σ^1, σ^2 and σ^3 is the the Pauli matrices. We can either separate them again by introducing two $4(N_R N_L) \times 4(N_R N_L)$ matrices,

$$B^L = \frac{1}{2} + B_i^L \gamma_i^L, \quad B^R = \frac{i}{2} + B_i^R \gamma_i^R \tag{4.147}$$

we can use the γ_μ with the above properties to construct the 8×8 Gamma-matrices

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix}$$

which generate the $SO(6)$-Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = \begin{pmatrix} \gamma^\mu \gamma^{\nu\dagger} + \gamma^\nu \gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu + \gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu}$$

This suggests to consider the single Hermitian $4(N_R N_L) \times 4(N_R N_L)$ matrix

$$C = \Gamma^\mu B_\mu + C_0 = \begin{pmatrix} 0 & B^L \\ B^L & 0 \end{pmatrix} + \begin{pmatrix} 0 & B^R \\ -B^R & 0 \end{pmatrix} =: C^L + C^R \tag{4.148}$$

where $C_0 = C_0^L + C_0^R$ denote the constant 8×8 -matrices

$$C_0^L = -\frac{i}{2}\Gamma_1^L \Gamma_2^L \Gamma_3^L = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C_0^L = -\frac{i}{2}\Gamma_1^R \Gamma_2^R \Gamma_3^R = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the above basis. Using the Clifford algebra and the above definitions one then finds

$$C^2 = B_\mu B_\mu + \frac{1}{2} - \frac{i}{4}[\Gamma_\mu, \Gamma_\nu] F_{\mu\nu} \tag{4.149}$$

The field strength $F_{\mu\nu}$ coincides with the definition in (4.121) if written in the $L - R$ notation,

$$F_{ia\ jb} = i[B_{ia}, B_{jb}] + \delta_{ab}\epsilon_{ijk} B_{ka}. \tag{4.150}$$

Therefore the action

$$S = \mathrm{Tr}((c^2 - \frac{N^2}{2})^2) = 8\mathrm{tr}(B_\mu B_\mu - \frac{N^2-1}{2})^2 + 4\mathrm{tr}F_{\mu\nu}F_{\mu\nu} \qquad (4.151)$$

with

$$(B_\mu B_\mu - \frac{N^2-1}{2})^2 =: (B_{iL}B_{iL} - \frac{N_L^2-1}{4})^2 + (B_{iR}B_{iR} - \frac{N_L^2-1}{4})^2, \qquad (4.152)$$

and

$$N^2 = \frac{N_L^2 + N_R^2}{2} \qquad (4.153)$$

4.3.3 Stability analysis of the SO(6)

Consider the action (4.151) We will split off the radial degrees of freedom for large N by setting

$$B_{iL} = \lambda_{iL} + A_{iL} = \lambda_{iL} + \mathcal{A}_{iL} + x_{iL}\Phi_L \qquad (4.154)$$

and similarly for B_{iR}. The stability of our geometry will depend on the behavior of Φ^L and Φ^R . We calculate that

$$\begin{aligned} B_\mu B_\mu - \frac{N^2-1}{2} &= N(\Phi_L + \Phi_R) + \Phi_L\Phi_L + \Phi_R\Phi_R \\ &+ \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + \mathcal{O}(1/N), \end{aligned} \qquad (4.155)$$

where we used that $\lambda_{ia}\mathcal{A}_{ia} = 0$ and therfore both $,\mathcal{A}_{ia}x_{ia} = \mathcal{O}(1/N)$ and $\mathcal{A}_{ia}[\lambda_{ia}, .] = \mathcal{O}(1/N)$ for $a = L, R$. Setting

$$\begin{aligned} \Phi_L + \Phi_R &= \Phi_1, \\ \Phi_L - \Phi_R &= \Phi_2 \end{aligned} \qquad (4.156)$$

we get

$$\begin{aligned} B_\mu B_\mu - \frac{N^2-1}{2} &= N\Phi_1 + \Phi_1\Phi_1 + \Phi_2\Phi_2 \\ &+ \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + \mathcal{O}(1/N), \end{aligned} \qquad (4.157)$$

4.3.4 Breaking $SO(6) \to SO(3) \times SO(3)$

To obtain the original action (4.119) for $S^2 \times S^2$, we have to break the $SO(6)-$symmetry down to $SO(3) \times SO(3)$. We can do this by using the left and right gauge fields C^L and C^R introduced in (4.148) separately. Their squares are

$$C_L^2 = B_{iL}B_{iL} + \frac{1}{4} + \begin{pmatrix} \gamma_L^i & 0 \\ 0 & \gamma_L^i \end{pmatrix} (B_{iL} + i\epsilon_{ijk}B_{jL}B_{kL}),$$

$$C_R^2 = B_{iR}B_{iR} + \frac{1}{4} - \begin{pmatrix} \gamma_R^i & 0 \\ 0 & \gamma_R^i \end{pmatrix} (B_{iR} + i\epsilon_{ijk}B_{jR}B_k).$$

As both γ_L^i, γ_R^i are traceless, we have

$$\begin{aligned} S_{break} &:= 2\mathrm{Tr}((C_L^2 - \frac{N_L^2}{4})(C_R^2 - \frac{N_R^2}{4})) \qquad (4.158) \\ &= 16\mathrm{Tr}((B_{iL}B_{iL} - \frac{N_L^2 - 1}{4})(B_{iR}B_{iR} - \frac{N_R^2 - 1}{4})). \end{aligned}$$

With these terms we can recover our action as

$$\begin{aligned} S &= S - S_{break} := 2\mathrm{Tr}((C^2 - \frac{N^2}{2}) - 2(C_L^2 - \frac{N_L^2}{4})(C_R^2 - \frac{N_R^2}{4})) \qquad (4.159) \\ &= 8\mathrm{tr}((B_{iL}B_{iL} - \frac{N_L^2 - 1}{4})^2 + (B_{iR}B_{iR} - \frac{N_R^2 - 1}{4})^2 + \frac{1}{2}F_{\mu\nu}F_{\mu\nu}). \qquad (4.160) \end{aligned}$$

4.4 Fuzzy Tori

We consider a toroidal lattice with lattice spacing a and N sites in every dimensions. The lattice size is then $L = Na$. We consider the unitary operators

$$Z_i = \exp(i\frac{2\pi}{L}x_i) \ , \ Z_i^N = 1. \tag{4.161}$$

The second condition simply restricts the points to $x_i \in a\mathbf{Z}$. We have immediately the commutation relations

$$[x_i, x_j] = i\theta_{ij} \Leftrightarrow Z_i Z_j = \exp(-2\pi i\Theta_{ij}) Z_j Z_i \ , \ \Theta = \frac{2\pi}{L^2}\theta. \tag{4.162}$$

We consider the case $\theta_{ij} = \theta Q_{ij}$ in two and four dimensions where

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.163}$$

The momentum in each direction will be assumed to have the usual periodicity, viz

$$k_i = \frac{2\pi m_i}{aN}. \tag{4.164}$$

The period of m_i is exactly N. The range of m_i is $[0, N-1]$ or equivalently $[-(N-1)/2, +(N-1)/2]$ and hence we obtain in the large lattice limit $L \longrightarrow \infty$ the cutoff

$$\Lambda = \frac{\pi}{a}. \tag{4.165}$$

The quantization of the noncommutativity parameters θ and Θ are given by

$$\theta = \frac{Na^2}{\pi} \ , \ \Theta = \frac{2}{N}. \tag{4.166}$$

In other words, we have rational noncommutativity Θ, for $N > 2$, and hence a finite dimensional representation of the algebra of the noncommutative torus exists. In general we require N to be odd for Θ to come out rational and thus be guaranteed the existence of the fuzzy torus. The cutoff in this case becomes

$$\Lambda = \sqrt{\frac{N\pi}{\theta}}. \tag{4.167}$$

This is consistent with the result of the fuzzy $\mathbf{CP}^n$.

The full Heisenberg algebra of the noncommutative torus includes also the fuzzy derivative operators

$$D_j = \exp(a\partial_j)\ ,\ D_j Z_i D_j^+ = \exp(\frac{2\pi i\delta_{ij}}{N})Z_i. \tag{4.168}$$

In two dimensions a finite dimensional $N\times N$ representation is given in terms of the clock and shift operators (with $\omega = \exp(2\pi i\Theta)$)

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ & & . & . & & & \\ & & & . & . & & \\ & & & & . & . & \\ & & & & & 0 & 1 \\ 1 & . & . & . & & & 0 \end{pmatrix}\ ,\ \Gamma_2 = \begin{pmatrix} 1 & & & & & & \\ & \omega & & & & & \\ & & \omega^2 & & & & \\ & & & \omega^3 & & & \\ & & & & . & & \\ & & & & & . & \\ & & & & & & . \end{pmatrix}, \tag{4.169}$$

by

$$Z_1 = \Gamma_2\ ,\ Z_2 = \Gamma_1\ ,\ D_1 = (\Gamma_1)^{\frac{N+1}{2}}\ ,\ D_2 = (\Gamma_2^+)^{\frac{N+1}{2}}. \tag{4.170}$$

The solution in four dimensions is obtained by taking tensor products of these. Thus a real scalar field Φ on the fuzzy torus is a hermitean $\mathcal{N}\times\mathcal{N}$ matrix where $\mathcal{N} = N^{d/2}$, i.e. the space of functions on the fuzzy torus is $\mathrm{Mat}(\mathcal{N},\mathbf{C})$. Furthermore, the integral is defined by the usual formula

$$\int_{\text{fuzzy torus}} = (2\pi\theta)^{d/2} Tr. \tag{4.171}$$

A basis of $\mathrm{Mat}(\mathcal{N},\mathbf{C})$ is given by the plane waves on the fuzzy torus defined by

$$\phi_{\vec{m}} = \frac{1}{N^{d/4}}\prod_{i=1}^{d} Z_i^{m_i}\prod_{i<j}\exp(\frac{2\pi i}{N}Q_{ij}m_i m_j) \equiv \frac{1}{N^{d/4}}\exp(ik_i x_i). \tag{4.172}$$

They satisfy

$$\phi^+_{\vec{m}} = \phi_{-\vec{m}} \ , \ Tr\phi^+_{\vec{m}}\phi_{\vec{m}'} = \delta_{\vec{m}\vec{m}'}. \tag{4.173}$$

A noncommutative Φ^4 theory on the fuzzy torus is given by

$$S = (2\pi\theta)^{d/2} Tr\left[\frac{1}{2a^2}\sum_i (D_i\Phi D_i^+ - \Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4\right]. \tag{4.174}$$

We expand the scalar field Φ in the plane waves $\phi_{\vec{m}}$ as

$$\Phi = \sum_{\vec{m}} \Phi_{\vec{m}}\phi_{\vec{m}}. \tag{4.175}$$

We compute immediately

$$\begin{aligned} D_i\Phi D_i^+ &= \sum_{\vec{m}} \Phi_{\vec{m}} D_i\phi_{\vec{m}} D_i^+ \\ &= \sum_{\vec{m}} \Phi_{\vec{m}}\phi_{\vec{m}} \exp(\frac{2\pi i m_i}{N}). \end{aligned} \tag{4.176}$$

Hence

$$\begin{aligned} Tr(D_i\Phi D_i^+)^2 &= \sum_{\vec{m}} \Phi_{\vec{m}}\Phi^+_{\vec{m}} \\ &= Tr\Phi^2. \end{aligned} \tag{4.177}$$

Thus the action can be rewritten as

$$S = (2Na^2)^{d/2} Tr\left[\frac{1}{a^2}\sum_i (\Phi^2 - D_i\Phi D_i^+\Phi) + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4\right]. \tag{4.178}$$

We compute the kinetic term and the propagator given respectively by

$$\frac{1}{2}\sum_{\vec{m}} \Phi_{\vec{m}}\Phi^+_{\vec{m}}\left(\frac{2}{a^2}\sum_i (1-\cos ak_i) + m^2\right). \tag{4.179}$$

$$<\Phi_{\vec{m}}\Phi^+_{\vec{m}'}>= \frac{\delta_{\vec{k}}\delta_{\vec{k}'}}{\frac{2}{a^2}\sum_i (1-\cos ak_i) + m^2}. \tag{4.180}$$

Thus the behavior of the propagator for large momenta is different and as a consequence the calculation of α_0^2 on fuzzy tori will be different from the result obtained using a sharp cutoff. We get [127]

for $d = 2$

$$< \int_{\text{fuzzy torus}} d^2x\Phi^2(x) >= V \int_0^\pi \frac{d^2r}{(2\pi)^2} \frac{1}{\sum_i(1-\cos r_i) + m^2a^2/2}, \quad (4.181)$$

for $d = 4$

$$< \int_{\text{fuzzy torus}} d^4x\Phi^2(x) >= \frac{V\Lambda^2}{\pi} \int_0^\pi \frac{d^4r}{(2\pi)^4} \frac{1}{\sum_i(1-\cos r_i) + m^2a^2/2}. \quad (4.182)$$

Chapter 5

Emerent Geometry in Yang-Mills Matrix Models and Fuzzy Sphere S^2 and $S^2 \times S^2$

5.1 The Model

A commutative/noncommutative space in Connes' approach to geometry is given in terms of a spectral triple $(\mathcal{A}, \Delta, \mathcal{H})$ rather than in terms of a set of points [128]. $\mathcal{A}$ is the algebra of functions or bounded operators on the space, Δ is the Laplace operator or, in the case of spinors, the Dirac operator, and $\mathcal{H}$ is the Hilbert space on which the algebra of bounded operators and the differential operator Δ are represented.

In the IKKT model the geometry is in a precise sense emergent. The algebra $\mathcal{A}$ is given, in the large N limit, by Hermitian matrices with smooth eigenvalue distribution and bounded square trace [129]. The Laplacian/Dirac operator is given in terms of the background solution while the Hilbert space $\mathcal{H}$ is given by the adjoint representation of the gauge group $U(N)$.

In this article we will study IKKT Yang-Mills matrix models with quartic mass deformations in three and six dimensions with $SO(3)$ and $SO(3) \times SO(3)$ symmetries which will lead naturally to the fuzzy two-sphere $\mathbf{S}^2_N$ and to the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ respectively.

Noncommutative gauge theory on the fuzzy two-sphere [130, 131] was introduced in [132]. It was derived as the low energy dynamics of open strings moving in a background magnetic field with S^3 metric in [133]. This theory consists of the Yang-Mills term which can be obtained from the reduction to zero dimensions of ordinary $U(N)$ Yang-Mills theory in 3 dimensions and a Chern-Simons term due to Myers effect [134]. The model was studied perturbatively in [135] and [136] and nonperturbatively in [137]. This model contains beside the usual two-dimensional gauge field a scalar fluctuation normal to the sphere. In [138] a generalized model was proposed and studied in which this normal scalar field was suppressed by giving it a potential with very large mass. This was studied further in [139, 140] where the instability of the sphere was interpreted along the lines of an emergent geometry phenomena.

In [141] an elegant random matrix model with a single matrix was shown to be equivalent to a gauge theory on the fuzzy sphere with a very particular form of the potential which in the large N limit leads to a decoupled normal scalar fluctuation. In [142–145] an alternative model of gauge theory on the fuzzy sphere was proposed in which field configurations live in the Grassmannian manifold $U(2N)/(U(N+1)\times U(N-1))$. In [142, 143] this model was shown to possess the same partition function as the commutative model via the application of the powerful localization techniques.

Noncommutative gauge theory on the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ which is given by a six matrix model with global $SO(3)\times SO(3)$ symmetry containing at most quartic powers of the matrices was proposed in [146]. The value $M = 1/2$ of the mass deformation parameter corresponds to the model studied [147] which can also be shown to correspond to a random matrix model with two matrices. This theory involves two normal scalar fields plus a four-dimensional gauge field. Again the mass deformation parameter M is essentially the mass of these normal fluctuations and thus for large M these scalar fields become weakly coupled. In [148] an interpretation of these normal scalar fields as dark energy as dark energy is put forward.

Before we give a summary of the main results reported in this article we will first summarize the main models studied here and their normalization. The most general mass-deformed IKKT Yang-Mills matrix model in three dimensions up to quartic power in the gauge field X_a is given by [139]

$$S[X] = NTr\Big[-\frac{1}{4}[X_a,X_b]^2+\frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c+M(X_a^2)^2+\beta X_a^2\Big].$$

The Steinacker model [141] corresponds to $M = 1/2$ and

$$\beta = \frac{2\alpha^2}{9}(1 - 4c_2 M).$$

The model with $M = 0$ will correspond to a perfect square action (exact Yang-Mills term $\sim F_{ab}^2$). We will study a one-parameter family (labeled by M) with β fixed as above. We may also use the parameters m^2 and μ defined by $m^2 = 2c_2 M$ and $\beta = -\alpha^2\mu$ especially in one-loop calculations. The coefficient c_2 is nothing else but the $SU(2)$ second Casimir, viz $c_2 = L_a^2 = (N^2 - 1)/4$.

The above model will lead to an emergent two-sphere $\mathbf{S}_N^2$, i.e. $X_a \sim L_a$. To obtain an emergent $\mathbf{S}_{N_0}^2 \times \mathbf{S}_{N_0}^2$, with $N = N_0^2$, we consider two copies of the above model with gauge fields X_a and Y_a respectively, and then couple X_a and Y_a in the usual way, i.e. via commutators squared. The only modification required in the normalization explained above is to replace the quadratic Casimir as $c_2 \longrightarrow c_2^0 = (N_0^2 - 1)/4$. Thus, we consider the mass-deformed IKKT Yang-Mills model in six dimensions given by

$$S[X, Y] = S[X] + S[Y] - \frac{N}{2} Tr[X_a, Y_b]^2.$$

In the above models α will play the role of the gauge coupling constant. More precisely, the inverse gauge coupling constant is found to be given by the combination

$$\widetilde{\alpha}^2 = \alpha^2 N \sim \frac{1}{g}.$$

In order to take the commutative large N limit we will need to study carefully the plots of various observables as functions of $\widetilde{\alpha}$, and determine from their behavior under N, the scaled quantities which are stable in the strict $N \longrightarrow \infty$ limit. These scaled observables will be termed collapsed and it may even happen that the inverse gauge coupling constant itself $\widetilde{\alpha}^2$ will be required to be collapsed, i.e. scaled with N, appropriately.

Indeed, it is found in the case of $\mathbf{S}_N^2$ that scaled observables are actually functions of $\bar{\alpha}$ and not of $\widetilde{\alpha}$, where $\bar{\alpha}$ the is the so-called collapsed coupling constant defined by

$$\bar{\alpha} = \widetilde{\alpha}\sqrt{N} = \alpha N.$$

This can also be confirmed by one-loop calculation.

The main results of this article are as follows:

- The dynamically emergent geometry, which is given by a fuzzy two-sphere $\mathbf{S}^2_N$, in the 3−dimensional mass-deformed IKKT matrix models, is found to be stable for all values of the deformation parameter M. The critical gauge coupling constant $\widetilde{\alpha}$ is found to scale as in equation (5.24), i.e. as $\widetilde{\alpha} \sim 1/\sqrt{N}$. The sphere-to-matrix transition line is pushed to 0 and only one phase survives.

- The 6−dimensional mass-deformed IKKT matrix model exhibits a phase transition from a geometrical phase at low temperature, given by a fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ background, to a Yang-Mills matrix phase with no background geometrical structure at high temperature. The inverse temperature β is here identified with the gauge coupling constant $\widetilde{\alpha}$.

- The transition is exotic in the sense that we observe, for small values of M, a discontinuous jump in the entropy, characteristic of a 1st order transition, yet with divergent critical fluctuations and a divergent specific heat with critical exponent $\alpha = 1/2$. The critical gauge coupling constant is pushed downwards as the scalar field mass is increased.

- For small M, the system in the Yang-Mills phase is well approximated by 6 decoupled matrices with a joint eigenvalue distribution which is uniform inside a ball in $\mathbf{R}^6$. This gives what we call the $d = 6$ law given by equation (5.47). For large M, the transition from the four-sphere phase $\mathbf{S}^2_N \times \mathbf{S}^2_N$ to the Yang-Mills matrix phase turns into a crossover and the eigenvalue distribution in the Yang-Mills matrix phase changes from the $d = 6$ law to a uniform distribution.

- In the Yang-Mills matrix phase the specific heat is equal to 3/2 which coincides with the specific heat of 6 independent matrix models with quartic potential in the high temperature limit. Once the geometrical phase is well established the specific heat takes the value 5/2 with the gauge field contributing 3/2 and the two scalar fields contributing 1.

This article is organized as follows. In section 2 we review the construction of noncommutative fuzzy gauge theory on the fuzzy sphere from random matrix theory, and write down our generalized three matrix model. In section 3 we show by means of Monte Carlo that the emergent fuzzy sphere in this three matrix model is stable for all values of the mass deformation parameter

M. In section 4 we write down the analogous six matrix model, and present its one-loop quantization, and then discuss in great detail the calculation of the phase diagram by means of Monte Carlo. It is shown here that the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ is only stable for large values of the mass deformation parameter M. In section 5 we give a detailed discussion of the phases of the model using the eigenvalue distribution. We also revisit in this section the effective potential in order to prove the critical behavior of the theory. In section 6 various related topics are discussed briefly such as: emergent gauge theory in two dimensions, monopoles and instantons, topology change and stability of the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$, critical behavior, Dirac operators, random matrix theory formulation, and generalization to fuzzy four sphere $\mathbf{S}^4$ and fuzzy $\mathbf{CP}^n$. Section 7 contains our conclusion.

5.2 The 3−dimensional mass deformed Yang-Mills matrix model

A $U(n)$ gauge action on the fuzzy sphere can be derived from a simple 1−matrix model as follows [141]. We introduce Pauli matrices τ_a and the three $N \times N$ matrices L_a, which are the $SU(2)$ generators in the irreducible representation of spin $s = (N-1)/2$, and define the matrix

$$\bar{C} = (\frac{1}{2} + \tau_a L_a) \otimes \mathbf{1}_n. \tag{5.1}$$

It is a trivial exrecise to check that

$$\bar{C} = (j(j+1) - (\frac{N}{2})^2) \otimes \mathbf{1}_n, \tag{5.2}$$

where j is the eigenvalue of the operator $\vec{J} = \vec{L} + \vec{\sigma}/2$ which takes the two values $N/2$ and $(N-2)/2$. The eigenvalues of $\bar{C}$ are therefore $N/2$ with multiplicity $n(N+1)$ and $-N/2$ with multiplicity $n(N-1)$. Hence

$$\bar{C}^2 = (\frac{N}{2})^2 \mathbf{1}_{2nN}. \tag{5.3}$$

As it turns out this matrix $\bar{C}$ can be obtained as a classical configuration of the following $2nN-$dimensional 1−matrix action

$$S[C] = \frac{1}{4g^2} \frac{1}{N} Trtr_2 \bigg(C^2 - \bigg(\frac{N}{2}\bigg)^2 \bigg)^2. \tag{5.4}$$

Indeed, the equations of motion derived from this action reads

$$C(C^2 - \frac{N^2}{4}) = 0. \tag{5.5}$$

It is easy to see that $\bar{C}$ solves this equation of motion and that the value of the action in this configuration is identically zero ,i.e. $S[C = \bar{C}] = 0$.

Expanding around the vacuum $\bar{C}$ by writing

$$C = \frac{1}{2} + C_0 + \sigma_a C_a, \tag{5.6}$$

where C_0 and C_a are $nN \times nN$ matrices and imposing the condition

$$C_0 = 0, \tag{5.7}$$

we get

$$C^2|_{C_0=0} = \frac{1}{4} + C_a^2 + \frac{1}{2}\epsilon_{abc}\sigma_c F_{ab}. \tag{5.8}$$

The curvature F_{ab} is given in terms of C_a by

$$F_{ab} = i[C_a, C_b] + \epsilon_{abc}C_c. \tag{5.9}$$

Hence, we obtain the action

$$S[C] = \frac{1}{g^2}\frac{1}{N}\left[\frac{1}{4}TrF_{ab}^2 + \frac{1}{2}Tr(C_a^2 - \frac{N^2-1}{4})^2\right]. \tag{5.10}$$

The normal scalar field Φ on the fuzzy sphere is given in terms of C_a by

$$\Phi = \frac{C_a^2 - L_a^2}{2\sqrt{c_2}} \ , \ c_2 = L_a^2 = \frac{N^2-1}{4}. \tag{5.11}$$

The $U(n)$ gauge action becomes

$$S[C] = \frac{1}{g^2}\frac{1}{N}\left[\frac{1}{4}TrF_{ab}^2 + 2m^2 Tr\Phi^2\right]. \tag{5.12}$$

The mass m^2 is given by the value of the Casimir, viz

$$m^2 = c_2. \tag{5.13}$$

We can bring this action into the form (with $X_a = 2\alpha C_a/3$)

$$S[X] = NTr\bigg[-\frac{1}{4}[X_a, X_b]^2 + \frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c + M(X_a^2)^2 + \beta X_a^2\bigg]$$
$$M = \frac{m^2}{2c_2}\ ,\ \beta = -\alpha^2\mu = \frac{2\alpha^2}{9}(1-2m^2). \qquad (5.14)$$

We will also use extensively the parameter $\widetilde{\alpha}$ defined by

$$\widetilde{\alpha}^4 = \alpha^4 N^2 = \left(\frac{3}{2}\right)^4 \frac{1}{g^2}. \qquad (5.15)$$

The parameter g is the gauge coupling constant. The classical absolute minimum of the model is given by the fuzzy sphere configurations $C_a = L_a$. Expanding around this solution by writing $C_a = L_a + A_a$ yields a $U(n)$ theory with a 3−component gauge field $\vec{A}$ where the extra normal component is given by Φ. In the commutative limit $N \longrightarrow \infty$ the field $\Phi = n_a A_a$ is infinitely heavy and hence it decouples. The curvature in terms of A_a is given by $F_{ab} = i[L_a, A_b] - i[L_b, A_a] + \epsilon_{abc}A_c + i[A_a, A_b] \longrightarrow i\mathcal{L}_a A_b - i\mathcal{L}_b A_a + \epsilon_{abc}A_c$. The pure gauge action $S[C]$ becomes

$$S[C] \ = \ \frac{1}{4g^2N}TrF_{ab}^2 + \frac{2c_2}{g^2N}Tr\Phi^2 \longrightarrow \frac{1}{4g^2}\int\frac{d\Omega}{4\pi}F_{ab}^2 + \frac{N^2}{2g^2}\int\frac{d\Omega}{4\pi}\Phi^2. \qquad (5.16)$$

This is the action of a pure $U(n)$ gauge theory on the ordinary sphere. The action (5.14) defines therefore a pure $U(n)$ gauge theory on the fuzzy sphere at least in the region of the phase space where the vacuum configuration $C_a = L_a$ is stable. In this study we will deal mostly with $n = 1$ and hence $Tr\mathbf{1} = N$.

The action (5.14) should be compared with the action (3.8) of [139]. Here $m^2 = c_2$ whereas m^2 is a free parameter in [139]. Furthermore, β is fixed here as $\beta = 2\alpha^2/9$ for $m^2 = 0$ whereas β is a free parameter for $m^2 = 0$ in [139]. This action is also slightly different from the action (1) studied in [138]. Here, the $m^2 = 0$ theory is the perfect square action $F_{ab}^2/2$ whereas the $m^2 = 0$ limit of the action (1) studied in [138] is the Alekseev-Recknagel-Schomerus (ARS) stringy action given by $F_{ab}^2/2+$ Chern-Simons or equivalently [133]

$$S[X] = NTr\bigg[-\frac{1}{4}[X_a, X_b]^2 + \frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c\bigg]. \qquad (5.17)$$

The perfect square action $F_{ab}^2/2$ and the ARS action are connected by the critical line in the plane $\widetilde{\alpha} - \beta$ of the model with $M = 0$ computed in [149].

5.3 An emergent stable fuzzy sphere S^2

5.3.1 Results

The case of a single sphere contained in the three dimensional mass-deformed IKKT matrix model which is studied in this section is much simpler and differs from the case of $\mathbf{S}^2 \times \mathbf{S}^2$ studied in the next section in two important aspects.

1. There exists for every finite N only two distinct phases regardless of the value of the mass parameter M. The fuzzy sphere phase $\mathbf{S}^2_N$ and the Yang-Mills matrix phase. This is confirmed by the eigenvalue distribution.

2. But since the critical value separating the two phases is found in one-loop and Monte Carlo to scale as $1/\sqrt{N}$ the Yang-Mills phase is vanishingly small in the limit $N \longrightarrow \infty$.

Thus, effectively the three dimensional mass-deformed IKKT model (5.14) contains really a single phase.

Let us also mention that the critical points are estimated in three different independent ways in both the three and the six mass-deformed IKKT matrix models. These are:

1. From the intersection point of the expectation value of the actions for different values of N.

2. From the peak or minimum of the specfic heat.

3. From the discontinuity or maximum of the radius defined below.

5.3.2 Discussion

We discuss now these issues in more detail. We want then to study, by means of the Monte Carlo method , the phase diagram in the plane $\widetilde{\alpha}-M$ of the model (5.14) which we will rewrite again as

$$S[X]=NTr\bigg[-\frac{1}{4}[X_a,X_b]^2+\frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c+M(X_a^2)^2+\beta X_a^2\bigg]. \quad (5.18)$$

The parameter M is taken in the range between $M=0$ (perfect square action) and $M=1/2$ (Steinacker's action) and even beyond whereas β is given in terms of M by

$$\beta=-\alpha^2\mu=\frac{2\alpha^2}{9}(1-4c_2M). \quad (5.19)$$

The parameters β is therefore not independent and the two independent parameters of the model are α and M.

The background minimal solution of this model is $X_a=\alpha\phi L_a$. In the limit $m^2\longrightarrow\infty$, we have $\mu\longrightarrow m^2$, and we find a critical line separating the fuzzy sphere phase solution with $\phi\neq 0$, from the Yang-Mills matrix phase solution with $\phi=0$, given by the critical line derived originally from a one-loop analysis in [135]. Explicitly this is given in terms of the inverse gauge coupling constant $\widetilde{\alpha}^4\sim 1/g^2$ by

$$\widetilde{\alpha}^4=\frac{81}{2m^2}. \quad (5.20)$$

We use the Metropolis algorithm for the Monte Carlo update.

For each value of the mass parameter M, we have measured the critical point $\widetilde{\alpha}_*$ at the minimum of the specific heat

$$C_v=<S^2>-<S>^2\ . \quad (5.21)$$

with error bars estimated by the value of the step. See figure (5.2). The results are shown on table (5.1). We observe from table (5.2) that the collapsed coupling constant is not $\widetilde{\alpha}$ but it is given by

$$\bar{\alpha}=\widetilde{\alpha}\sqrt{N}=\alpha N. \quad (5.22)$$

Another measurement of the critical value $\bar{\alpha}_*$ is given by intersection point of the actions with different values of N. See figure (5.4). The results are shown on table (5.3).

$M/\widetilde{\alpha}$	$N=4$	$N=6$	$N=9$	$N=16$	$N=25$
0.5	1.9	1.5	1.3	1.0	0.8
1	1.7	1.45	1.2	0.9	0.7
2	1.6	1.4	1.0	0.8	0.6
5	1.3	1.1	0.9	0.65	0.55
10	1.2	1	0.8	0.6	0.5
50	0.8	0.7	0.5	0.4	0.3
100	0.7	0.5	0.4	0.3	0.3

Table 5.1: The critical values $\widetilde{\alpha}_*$ for different values of M and N.

The two different measurements, which give an upper and lower estimates, of the critical point $\bar{\alpha}_*$ are included on the phase diagram (5.1). The data are fitted to straight lines and compared to theory in table (5.4). The theoretical prediction (5.20) in terms of the collapsed coupling constant $\bar{\alpha}$ is given by

$$\bar{\alpha} = \frac{3}{M^{1/4}}. \tag{5.23}$$

This shows explicitly that the fuzzy sphere is absolutely stable in this model for all values of M, including the Steinacker value $M = 1/2$, since the inverse of the critical gauge coupling constant behaves as

$$\widetilde{\alpha} = \frac{3}{N^{1/2}M^{1/4}} \longrightarrow 0. \tag{5.24}$$

The sphere-to-matrix transition line is pushed to 0 and only one phase survives. This is the result advocated originally by Steinacker in [141].

For completeness we also measure the radius of the sphere R as a function of the mass M and N. This is defined by the formula

$$\frac{1}{R} = \frac{1}{\phi^2\widetilde{\alpha}^2 c_2} TrX_a^2 \ , \ c_2 = \frac{N^2-1}{4}, \phi = \frac{2}{3}. \tag{5.25}$$

The results are shown on figure (5.5).

$M/\bar{\alpha}$	$N=4$	$N=6$	$N=9$	$N=16$	$N=25$
0.5	3.8	3.7	3.9	4.0	4.0
1	3.4	3.55	3.6	3.6	3.5
2	3.2	3.4	3	3.2	3
5	2.6	2.7	2.7	2.6	2.75
10	2.4	2.4	2.4	2.4	2.5
50	1.6	1.7	1.5	1.6	1.5
100	1.4	1.22	1.2	1.2	1.5

Table 5.2: The critical values $\bar{\alpha}_*$ for different values of M and N.

M	$\bar{\alpha}$
0.5	3.125
1	2.55
2	2.1
5	1.65
10	1.35
50	0.9
100	0.75

Table 5.3: The critical values $\bar{\alpha}_*$ from the intersection point of the actions.

method	a	b
Minimum C_v	-0.22 ± 0.014	1.29 ± 0.036
Theory	-0.25	0.48
Intersection S	-0.268 ± 0.003	0.936 ± 0.007

Table 5.4: The slop and the intercept of the straight lines used to fit the data with the theoretical prediction.

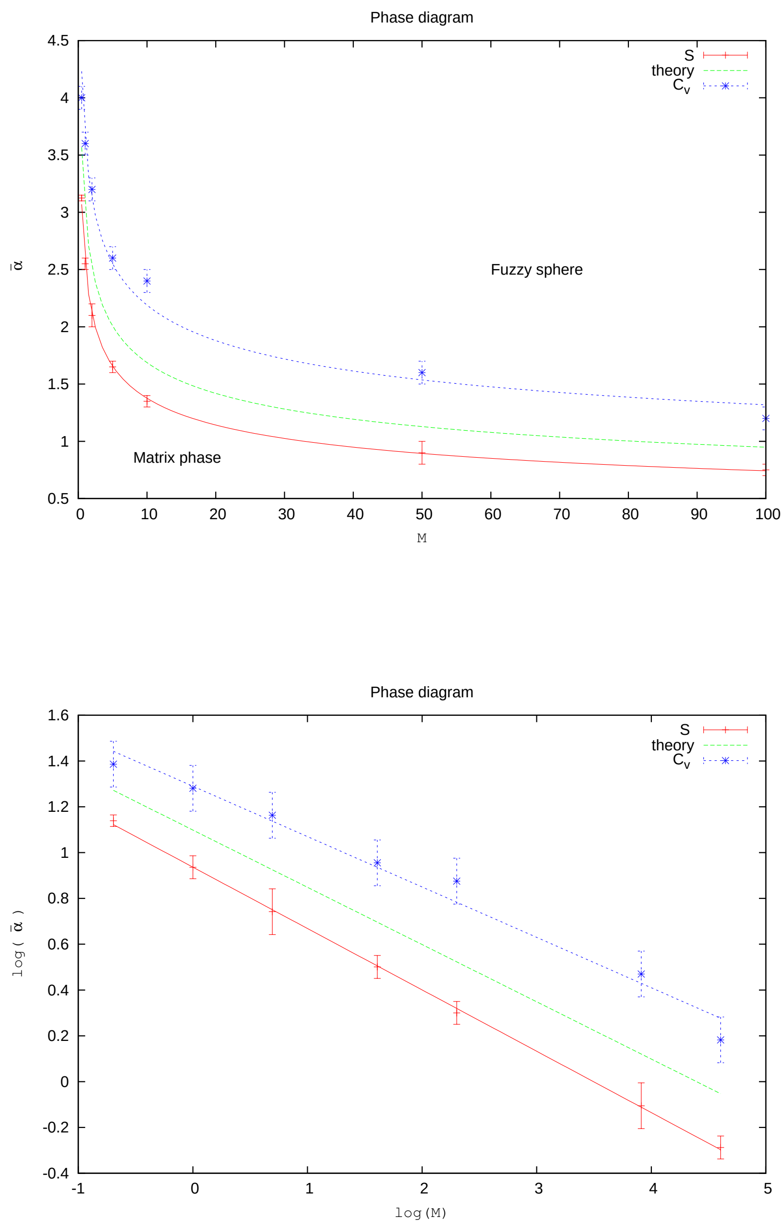
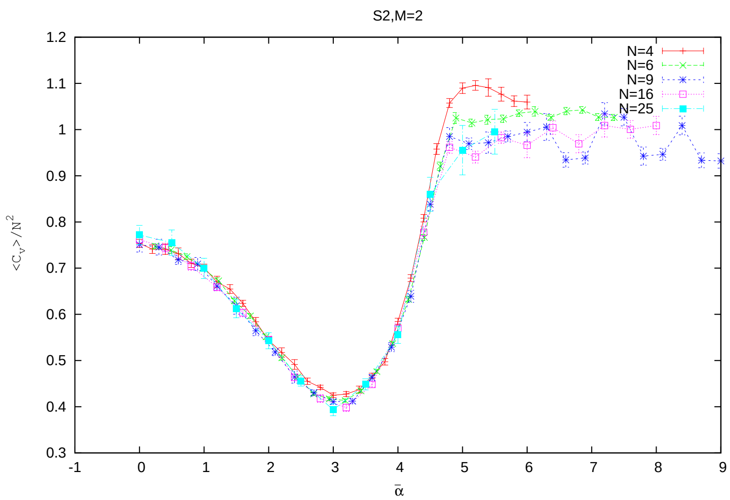


Figure 5.1: The phase diagram of the three dimensional Yang-Mills matrix model.



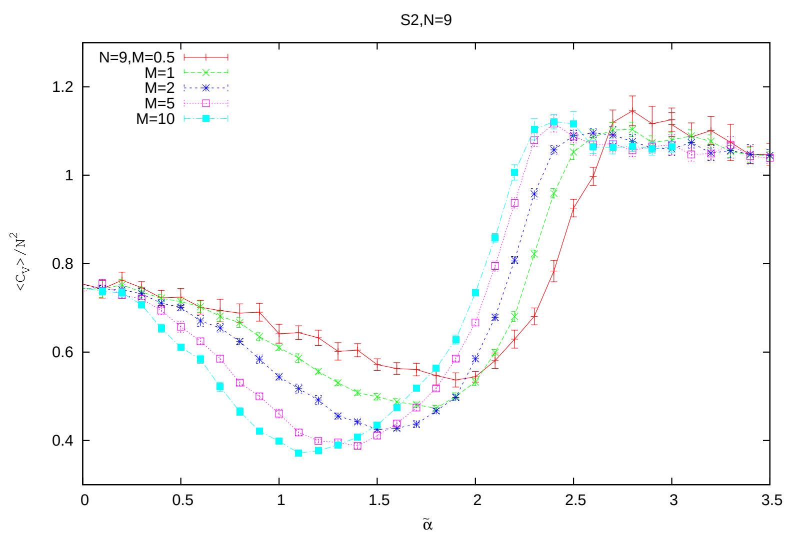


Figure 5.2: The specific heat in the three dimensional Yang-Mills matrix model.

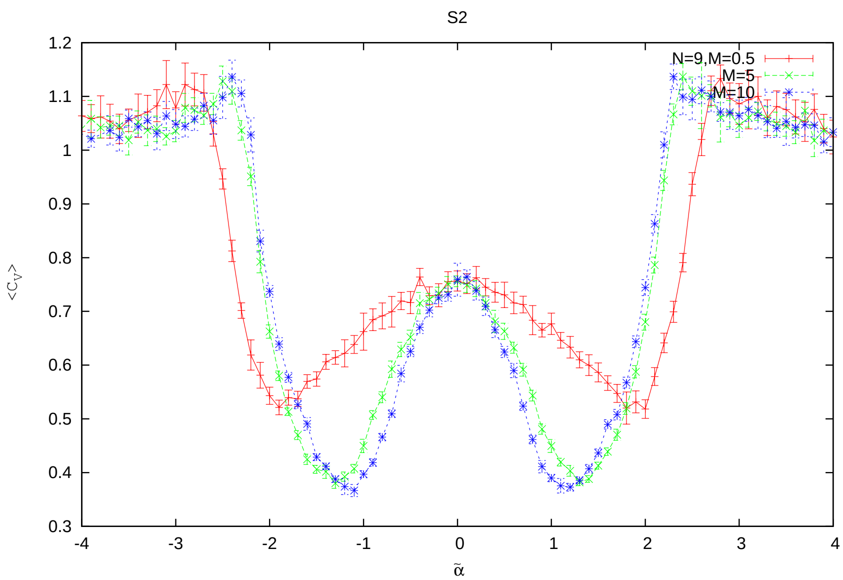
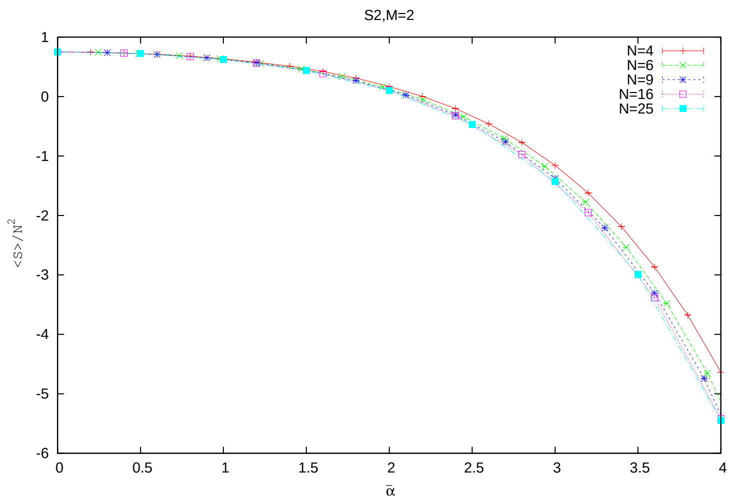


Figure 5.3: C_v



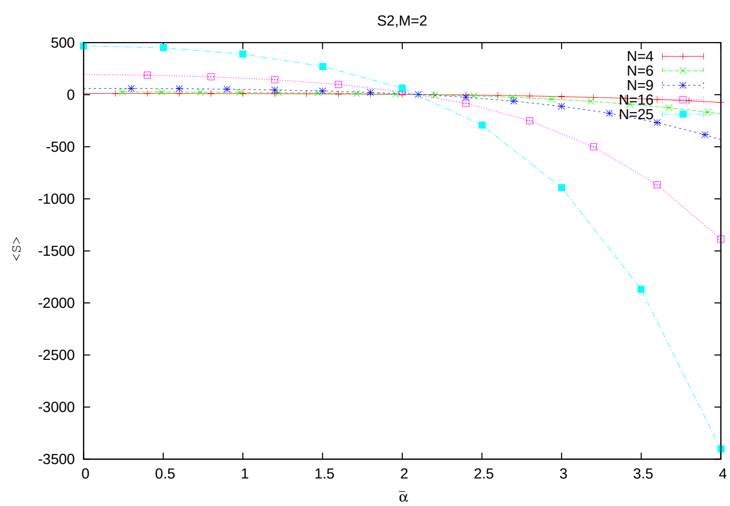


Figure 5.4: The action in the three dimensional Yang-Mills matrix model.

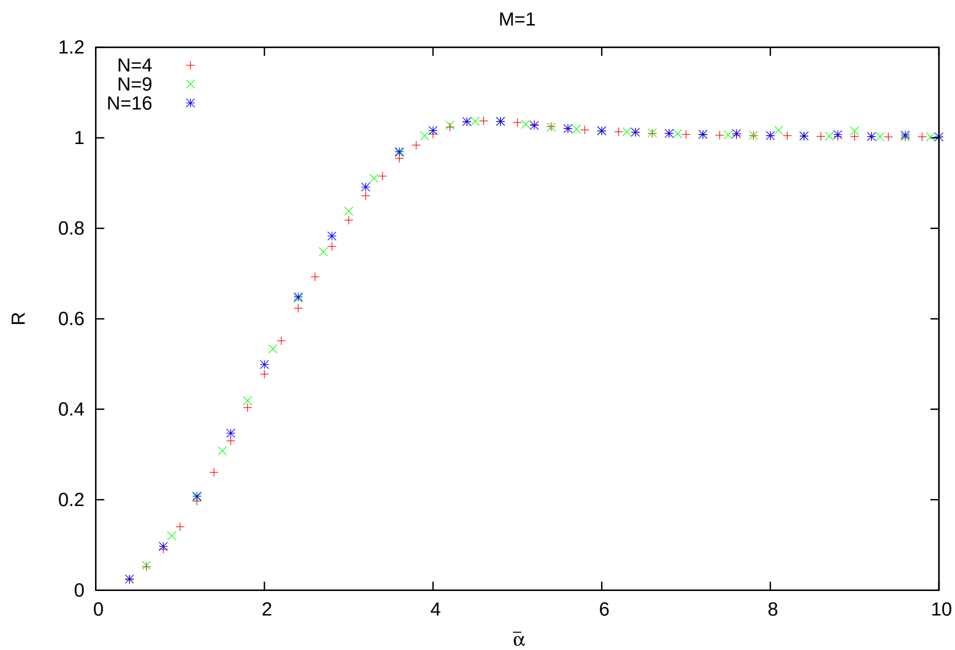


Figure 5.5: The radius in the three dimensional Yang-Mills matrix model.

5.4 The emergent fuzzy $\mathbf{S}^2 \times \mathbf{S}^2$ is stable only in the limit $M \longrightarrow \infty$

The six dimensional mass-deformed IKKT model, which will contain $\mathbf{S}^2 \times \mathbf{S}^2$, is obtained by joining together two copies of the three dimensional mass-deformed IKKT model (5.14) for gauge fields X_a and Y_a respectively, then coupling the matrices X_a and Y_a in the usual way, i.e. by adding a term proportional to the commutator squared.

The case of $\mathbf{S}^2_{N_0} \times \mathbf{S}^2_{N_0}$, with $N = N_0^2$, studied in this section is characterized by three phases as opposed to the effectively one phase characterizing $\mathbf{S}^2_N$. We have, for every value of the mass parameter M, the fuzzy $\mathbf{S}^2_{N_0} \times \mathbf{S}^2_{N_0}$ phase as expected for large values of $\widetilde{\alpha}$, and a Yang-Mills matrix phase for small values of $\widetilde{\alpha}$. In this case, in contrast with $\mathbf{S}^2_N$, the Yang-Mills phase is stable for every value of M as we take $N \longrightarrow \infty$, and the transition line goes as $\widetilde{\alpha}_* \sim 1/M^{0.25}$, and as a consequence the fuzzy $\mathbf{S}^2_{N_0} \times \mathbf{S}^2_{N_0}$ is stable only in the limit $M \longrightarrow \infty$.

However, the most distinct difference between the single fuzzy $\mathbf{S}^2_N$ case and the case of fuzzy $\mathbf{S}^2_{N_0} \times \mathbf{S}^2_{N_0}$ is the appearance of a third phase between the above standard phases for large M.

Indeed, there seems to exist another transition in M around the Steinacker's value $M = 0.5$ where the profile of the eigenvalue distribution, for $\widetilde{\alpha} = 0$, changes from the $d = 6$ law (small values of M) to a uniform distribution (large value of M). Thus, the phase diagram develops a distinct third phase between the fuzzy $\mathbf{S}^2_{N_0} \times \mathbf{S}^2_{N_0}$ phase and the Yang-Mills matrix phase for large values of M which looks like a cross-over. The transition line between the $\mathbf{S}^2_{N_0} \times \mathbf{S}^2_{N_0}$ phase and this crossover phase is determined at a saturated line around $\widetilde{\alpha}_* \sim 4.2$. Another possible interpretation of this phase is that of a strongly coupled gauge theory on the emergent background geometry and this is also supported by the measurement of the radius in this region.

5.4.1 The 6−dimensional mass deformed Yang-Mills matrix model

Generalization of the above construction is straightforward to four dimensions, i.e. to fuzzy $\mathbf{S}^2 \times \mathbf{S}^2$, and yields immediately the 6−matrix action

$$\begin{aligned} S[X,Y] &= NTr\bigg[-\frac{1}{4}[X_a,X_b]^2+\frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c+M(X_a^2)^2+\beta X_a^2\bigg] \\ &+ NTr\bigg[-\frac{1}{4}[Y_a,Y_b]^2+\frac{2i\alpha}{3}\epsilon_{abc}Y_aY_bY_c+M(Y_a^2)^2+\beta Y_a^2\bigg] \\ &+ NTr\bigg[-\frac{1}{2}[X_a,Y_b]^2\bigg]. \end{aligned} \quad (5.26)$$

We choose N to be a perfect square such that

$$N = N_0^2. \quad (5.27)$$

The parameters of the model are given by

$$M = \frac{m^2}{2c_2^0}\ ,\ \beta = -\alpha^2\mu\ ,\ \mu = \frac{2}{9}(4c_2^0M-1)\ ,\ c_2^0 = \frac{N_0^2-1}{4}. \quad (5.28)$$

Before we report the Monte Carlo data we discuss the one-loop effective potential of this theory and its phase structure.

The background solution (absolute minimum) of this model is by construction given by

$$X_a = \alpha\phi L_a\otimes\mathbf{1}\ ,\ Y_a = \alpha\phi\mathbf{1}\otimes L_a. \quad (5.29)$$

The fuzzy four-sphere is given explicitly by

$$x_1^2+x_2^2+x_3^2 = 1\ ,\ [x_a,x_b] = \frac{i}{\sqrt{c_2^0}}\epsilon_{abc}x_c, \quad (5.30)$$

$$y_1^2+y_2^2+y_3^2 = 1\ ,\ [y_a,y_b] = \frac{i}{\sqrt{c_2^0}}\epsilon_{abc}y_c, \quad (5.31)$$

$$[x_a,y_b] = 0, \quad (5.32)$$

where $x_a = L_a \otimes \mathbf{1}/\sqrt{c_2^0}$ and $y_a = \mathbf{1} \otimes L_a/\sqrt{c_2^0}$.

The L_a are the $SU(2)$ generators in the irreducible representation of spin $s = (N_0 - 1)/2$, and the background value of ϕ is $2/3$. By expanding around this solution, as $X_a = 2\alpha C_a^{(1)}/3$, $Y_a = 2\alpha C_a^{(2)}/3$, $C_a^{(i)} = L_a^{(i)} + A_a^{(i)}$, we obtain in the limit $N \longrightarrow \infty$ a $U(1)$ gauge theory on $\mathbf{S}^2 \times \mathbf{S}^2$ given by the action

$$S[C^{(1)}, C^{(2)}] = \frac{1}{g^2}\frac{1}{N}Tr\left[\frac{1}{4}F_{ab}^{(1)2} + \frac{1}{4}F_{ab}^{(2)2} + \frac{1}{2}F_{ab}^{(12)2} + 2m^2\Phi^{(1)2} + 2m^2\Phi^{(2)2}\right] \quad (5.33)$$

By comparing with equation (20) of [147] it seems to us that $m^2 = c_2$ as in the case of the single fuzzy sphere. The definition of the components of curvature tensor and the two scalar fields are obviously given by

$$F_{ab}^{(i)} = i[C_a^{(i)}, C_b^{(i)}] + \epsilon_{abc}C_c^{(i)} \;,\; F_{ab}^{(12)} = i[C_a^{(1)}, C_b^{(2)}]. \quad (5.34)$$

$$\Phi^{(i)} = \frac{C_a^{(i)2} - c_2^0}{2\sqrt{c_2^0}}. \quad (5.35)$$

5.4.2 Quantization at one-loop

For simplicity let us go back to the case of a single sphere and discuss the derivation of the effective potential there first [135].

Quantization around this background, using the background field method gives, after gauge fixing in the Lorentz gauge, the effective action

$$\Gamma[X_a] \;=\; S[X_a] + \frac{1}{2}Tr\log\Omega - Tr\log\mathcal{X}^2, \quad (5.36)$$

where the Laplacian operator Ω is given explicitly by the formula

$$\Omega_{ab} \;=\; \mathcal{X}_c^2\delta_{ab} - 2\mathcal{F}_{ab} + 4M(X_c^2 - c_2\alpha^2)\delta_{ab} + 8MX_aX_b + 2(\beta + 2c_2\alpha^2 M)\delta_{ab}. \quad (5.37)$$

The first term in (5.36) is due to the gauge field while the second term is due to the ghost field. In above the notation $\mathcal{X}_a$ and $\mathcal{F}_{ab}$ means that the covariant derivative X_a and the curvature $F_{ab} = i[X_a, X_b] + \alpha\epsilon_{abc}X_c$ act by commutators, i.e $\mathcal{X}_a(A) = [X_a, A]$, $\mathcal{F}_{ab}(A) = [F_{ab}, A]$ where $A \in Mat_N$. Similarly, $\mathcal{X}^2(A) = [X_a, [X_a, A]]$.

The UV-IR mixing behavior on the fuzzy four-sphere in the model with $M = \beta = 0$ was studied in great detail in [146].

It can be shown that the contributions from all the terms in the Laplacian Ω except the first are subleading [135, 150]. Thus, the effective potential for $X_a = \alpha\phi L_a$ is given by

$$\frac{V}{2c_2} = \tilde{\alpha}^4\left[\frac{\phi^4}{4} - \frac{\phi^3}{3} + m^2\frac{\phi^4}{4} - \mu\frac{\phi^2}{2}\right] + \log\phi^2. \tag{5.38}$$

The calculation on the fuzzy four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ proceeds in exactly the same way [146, 150]. First we fix the Lorentz gauge, then compute the effective action, and then substitute the background matrices $X_a = \alpha\phi L_a \otimes \mathbf{1}$, $Y_a = \alpha\phi\mathbf{1} \otimes L_a$. The calculation of the classical part of the effective potential is trivial. The quantum part goes along the same lines as above. In particular, the contributions from all the terms in the Laplacian Ω are negligible except the first one. Thus we get the logarithmic potential

$$\frac{1}{2}Tr_d Tr\log\phi^2 - Tr\phi^2 = \frac{d}{2}N^2\log\phi^2 - N^2\log\phi^2. \tag{5.39}$$

On $\mathbf{S}^2$ we have $d = 3$ whereas on $\mathbf{S}^2 \times \mathbf{S}^2$ we have $d = 6$. We get then on $\mathbf{S}^2 \times \mathbf{S}^2$ the effective potential

$$\begin{aligned}\frac{V}{2N^2} &= 2c_2^0\alpha^4\left[\frac{\phi^4}{4} - \frac{\phi^3}{3} + m^2\frac{\phi^4}{4} - \mu\frac{\phi^2}{2}\right] + \log\phi^2 \\ &= \tilde{\alpha}_0^4\left[\frac{\phi^4}{4} - \frac{\phi^3}{3} + m^2\frac{\phi^4}{4} - \mu\frac{\phi^2}{2}\right] + \log\phi^2,\end{aligned} \tag{5.40}$$

where we have redefined the coupling constant by

$$\frac{N_0^2}{2}\alpha^4 = \widetilde{\alpha}_0^4. \tag{5.41}$$

The difference between the result on $\mathbf{S}^2$ and this result lies in the replacement $\widetilde{\alpha} \longrightarrow \widetilde{\alpha}_0$ and the replacement $c_2 \longrightarrow c_2^0$ in the definition of μ. The analysis of the phase structure is therefore identical (see section 5.2).

The equation of motion reads then

$$\frac{V'}{2N^2} = \tilde{\alpha}_0^4\left[\phi^3 - \phi^2 + m^2\phi^3 - \mu\phi\right] + \frac{2}{\phi}. \tag{5.42}$$

In the limit $m^2 \longrightarrow \infty$, we have $\mu \longrightarrow 4m^2/9$, and we find a critical line separating the fuzzy sphere phase solution with $\phi \neq 0$, from the Yang-Mills matrix phase solution with $\phi = 0$, given by the formula

$$(1+m^2)\phi_* = \frac{3}{8}\big(1+\sqrt{1+\frac{32t}{9}}\big)$$
$$\frac{1}{\widetilde{\alpha}_{0*}^4} = \frac{\phi_*^2(\phi_*+2\mu)}{8}$$
$$t = \mu(1+m^2). \tag{5.43}$$

More detail on the derivation of this formula and the analysis of the phase diagram from the effective potential can be found in section 5.2.

The small mass limit $M \longrightarrow 0$ gives

$$\widetilde{\alpha} \sim N \tag{5.44}$$

which diverges with N. This is precisely the divergent asymptotic behavior of the critical line of the model $M = 0$ observed in [149].

The large mass limit $M \longrightarrow \infty$ (or equivalently $m^2 \longrightarrow \infty$) of these equations is given in terms of $\widetilde{\alpha}$ by

$$\widetilde{\alpha} = 3\big(\frac{2}{M}\big)^{1/4}. \tag{5.45}$$

Thus as opposed to the case of a single sphere the collapsed coupling constant is $\tilde{\alpha}$ and not $\bar{\alpha}$ which signals the persistence of the instability of the emergent four-sphere geometry as we will now show with the Monte Carlo results.

5.4.3 Monte Carlo calculation

We perform Monte Carlo simulation of the above six dimensional matrix model using the Metropolis algorithm. We vary M for different values of N. We work with $N = 4 - 25$ and $M = 0.05 - 50$.

5.4.4 The action:

- The intersection point of the average actions (entropies) for various values of N leads in this case to a robust measurement of the transition point between the fuzzy four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ phase and the matrix Yang-Mills phase. See (5.6). This measurement in comparison with the theoretical prediction value (6.34) gives an under estimation of the critical point. The results are included on table (5.5).

- For small values of M we also observe a discrete jump in the entropy at the transition point between the fuzzy four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ phase and the matrix Yang-Mills phase. See the second graph of(5.6). The physics of this discontinuity is discussed in more detail in section 5 using the approximation of the effective potential.

- The behavior of the action which couples the two spheres, viz

$$S_{12} = \mathrm{YM}_{12} = -\frac{N}{2} Tr[X_a, Y_b]^2, \tag{5.46}$$

  for small and large values of M, is shown on figure (5.7). We choose $M = 0.1$, and $M = 120$, for $N = 16$. We also plot S_1 and S_2, which are obviously defined, for comparison.

  We observe that for small values of M the action YM_{12} is comparable to S_i, whereas for large values of M it becomes quite negligible. This means in particular, that for large values of M, the approximation of the effective potential is expected to work well.

5.4.5 The specific heat:

- For small values of M, the geometric phase transition from the fuzzy four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ phase to the matrix Yang-Mills phase is marked by, and is measured at, the peak of the specific heat. See figure (5.8). The peak becomes harder to resolve for larger values of M and N. This is in contrast with the case of the fuzzy two-sphere $\mathbf{S}^2$ where a peak in the specific heat is observed only for very small values of M. The results are included on table (5.6).

- There seems to exist a new transition in M, around $M \sim 0.5$ which is the Steinacker's value, beyond which the peak in C_v ceases from marking the transition from the fuzzy four-sphere phase to the Yang-Mills matrix phase, and the specific heat develops a minimum where the transition actually occurs. See figure (5.9) and the results are included on table (5.7). Thus, for larger values of M above $M \sim 0.5$ the geometric phase transition from the fuzzy four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ phase to the matrix Yang-Mills phase is marked by, and is measured at, the minimum of C_v.

- We also observe that the value at the peak of the specific heat saturates at around $\widetilde{\alpha} \sim 4.1$ for larger values of M. This peak corresponds in this case to a transition from a fuzzy four-sphere to a crossover phase as we will describe further shortly.

5.4.6 The radius:

We also measure the radii of the two spheres given in terms of TrX_a^2 and TrY_a^2 respectively by the formula (5.25) with the substitution $N \longrightarrow N_0$. From the Monte Carlo data we observe no difference, beyond and above statistical fluctuations, between the two radii. This is obvious from the figure (5.10).

We observe for small values of M a discontinuity in the radius at the transition point as seen neatly on figure (5.10). We note that in this case it becomes harder to thermalize the system in the fuzzy four-sphere phase due to the zero modes of the matrices X_a and Y_a.

For medium and larger values of M the discontinuity is smoothed out as shown on figure (5.12). The transition point in this case is taken, for medium values of M, at the maximum reached by the radius R in the fuzzy four-sphere phase before decreasing to zero in the Yang-mills matrix phase. For larger values of M, the transition point is taken at the point where the radius R drops below one. In summary,

$$R \longrightarrow \begin{cases} 1 \ , \ \widetilde{\alpha} >> \widetilde{\alpha}_* & \text{fuzzy four-sphere phase} \\ 0 \ , \ \widetilde{\alpha} << \widetilde{\alpha}_* & \text{Yang-Mills matrix phase.} \end{cases}$$

This consists an independent measurement of the critical point between the fuzzy four-sphere phase and the Yang-Mills matrix phase. The data for different values of N is well collapsed and thus one can obtain from the radius a single estimate for the critical point shown on table (5.8).

5.4.7 The phase diagram

The critical line, and as a consequence the phase diagram, from the measurements of the action (intersection point or jump), the specific heat (peak and minimum) and the radius (jump, maximum and dropping below 1), together with the theoretical calculation given by equation (6.34), are shown on figure (5.13).

In summary, we observe roughly three phases. The fuzzy four-sphere phase as expected for large values of $\widetilde{\alpha}$, and a Yang-Mills matrix phase for small values of $\widetilde{\alpha}$, and this is the case for every value of the mass parameter M.

Therefore, in this case the fuzzy four-sphere phase is only stable in the limit $M \longrightarrow \infty$ since the collapsed gauge coupling constant is $\widetilde{\alpha}$ and not $\bar{\alpha}$, in contrast to what happens in the case of a single sphere.

As we will discuss, in the next section, the Yang-Mills phase in this case is characterized by different eigenvalue distributions for large and small values of M. There seems to exist another transition in M around the Steinacker's value $M = 0.5$ where the profile of the eigenvalue distribution, for $\widetilde{\alpha} = 0$, changes from the $d = 6$ law (small values of M) to a uniform distribution (large value of M). See below for a detailed discussion.

Also, we observe that the phase diagram develops a distinct third phase between the fuzzy four-sphere phase and the Yang-Mills matrix phase for large values of M. This looks like a crossover phase. The critical line between the fuzzy four-sphere phase and this new phase, as measured by the peak of C_v, saturates around $\widetilde{\alpha} = 4.2$. Another possible interpretation of this phase is that of a strongly coupled gauge theory on the emergent background geometry and this is supported by the measurement of the radius in this region.

M	$\widetilde{\alpha}$
$M = 0.05$	8.50
$M = 0.1$	7.20
$M = 0.5$	3.75
$M = 1.0$	2.80
$M = 10$	1.55
$M = 20$	1.17
$M = 25$	1.11
$M = 30$	1.06
$M = 35$	1.02
$M = 40$	0.98
$M = 60$	0.89
$M = 80$	0.83
$M = 100$	0.78

Table 5.5: The critical values $\widetilde{\alpha}_*$ from the intersection of the action.

$M/\widetilde{\alpha}$	$N = 4$	$N = 9$	$N = 16$	$N = 25$	Extrapolation
$M = 0.05$	7.50	7.60	8.20	8.70	8.5614 ± 0.345
$M = 0.1$	6.10	6.70	6.80	7.00	7.0262 ± 0.1505
$M = 0.5$	4.90	5.00	5.20	5.30	5.3040 ± 0.0813
$M = 1.0$	4.60	4.70	4.70	4.80	4.7914 ± 0.0360
$M = 10$	4.20	4.30	4.20	4.30	4.2851 ± 0.0534
$M = 20$	4.20	4.20	4.30	4.20	4.2482 ± 0.0492
$M = 25$	4.30	4.30	4.30	4.20	4.2419 ± 0.0449
$M = 30$	4.30	4.20	4.30	4.20	4.2149 ± 0.0534
$M = 35$	4.30	4.20	4.20	4.20	4.1666 ± 0.0168
$M = 40$	4.10	4.30	4.30	4.20	4.3087 ± 0.0717
$M = 60$	4.20	4.30	4.20	4.20	4.2271 ± 0.0531
$M = 80$	4.10	4.10	4.20	4.20	4.2063 ± 0.0374
$M = 100$	4.20	4.30	4.10	4.20	4.1788 ± 0.0849

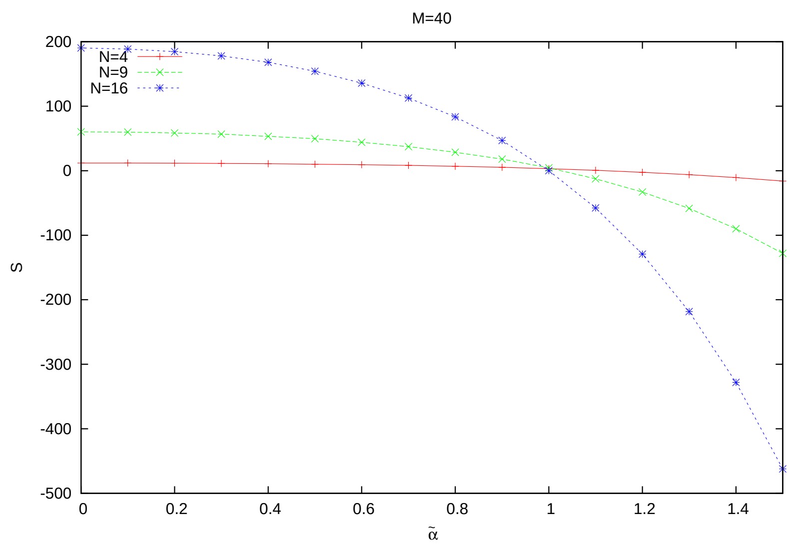
Table 5.6: The critical values $\widetilde{\alpha}_*$ from the peak in C_v.

$M/\widetilde{\alpha}$	$N=4$	$N=9$	$N=16$	$N=25$	Extrapolation
$M=0.05$	7.50	7.60	8.20	8.70	8.5614 ± 0.345
$M=0.1$	6.10	6.70	6.80	7.00	7.0262 ± 0.1505
$M=0.5$	3.70	3.70	3.85	3.90	3.8885 ± 0.0664
$M=1.0$	4.60	4.70	4.70	4.80	4.7914 ± 0.0360
$M=10$	2.50	2.30	2.30	2.35	2.2622 ± 0.05124
$M=20$	2.20	2.00	1.90	2.00	1.8850 ± 0.0571
$M=25$	2.10	1.90	1.90	2.00	1.8912 ± 0.0717
$M=30$	2.10	1.80	1.70	1.80	1.5404 ± 0.0233
$M=35$	2.00	1.80	1.65	1.65	1.4917 ± 0.0516
$M=40$	1.90	1.70	1.60	1.55	1.4979 ± 0.0155
$M=60$	1.70	1.50	1.55	1.50	1.4573 ± 0.0433
$M=80$	1.60	1.50	1.50	1.45	1.4376 ± 0.0168
$M=100$	1.60	1.40	1.45	1.40	1.3573 ± 0.0823

Table 5.7: The critical values $\widetilde{\alpha}_*$ from the minimum in C_v.

M	$\widetilde{\alpha}$
$M=0.05$	8.10
$M=0.1$	6.70
$M=0.5$	4.60
$M=1.0$	4.20
$M=10$	3.50
$M=20$	2.50
$M=25$	2.50
$M=30$	2.45
$M=35$	2.40
$M=40$	2.40

Table 5.8: The critical values $\widetilde{\alpha}_*$ from the radius.



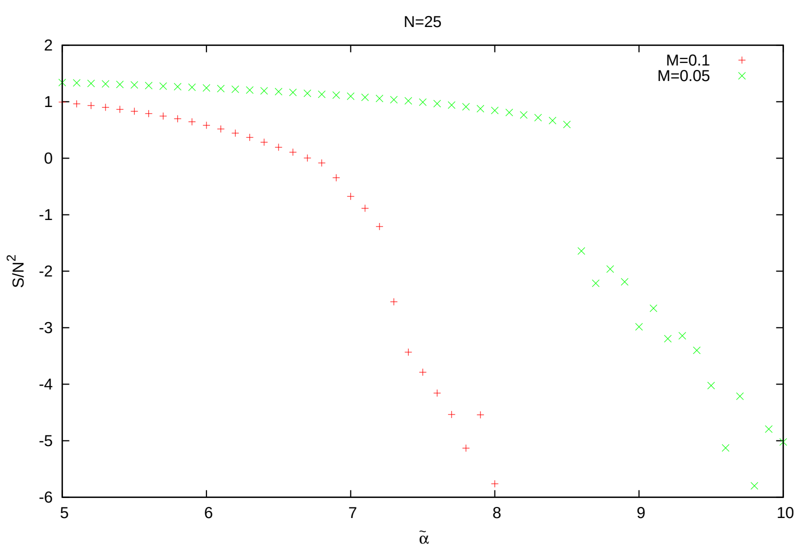
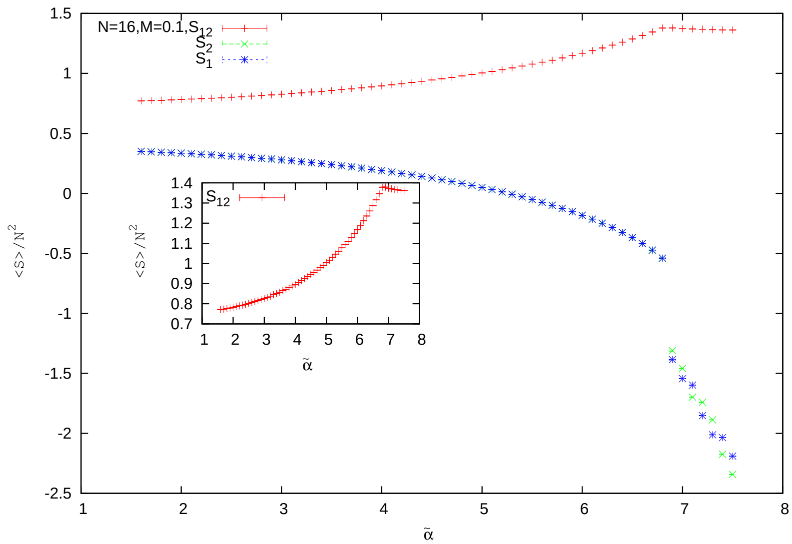


Figure 5.6: The intersection point of the action in the six dimensional Yang-Mills matrix model.



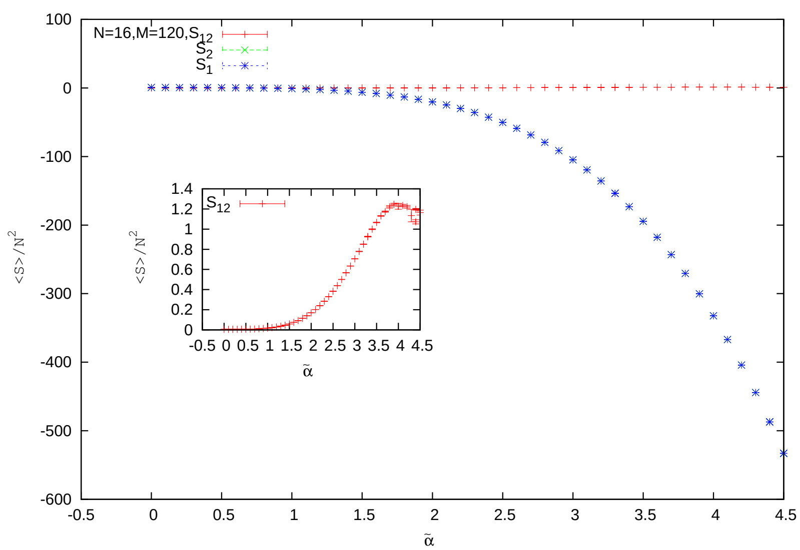
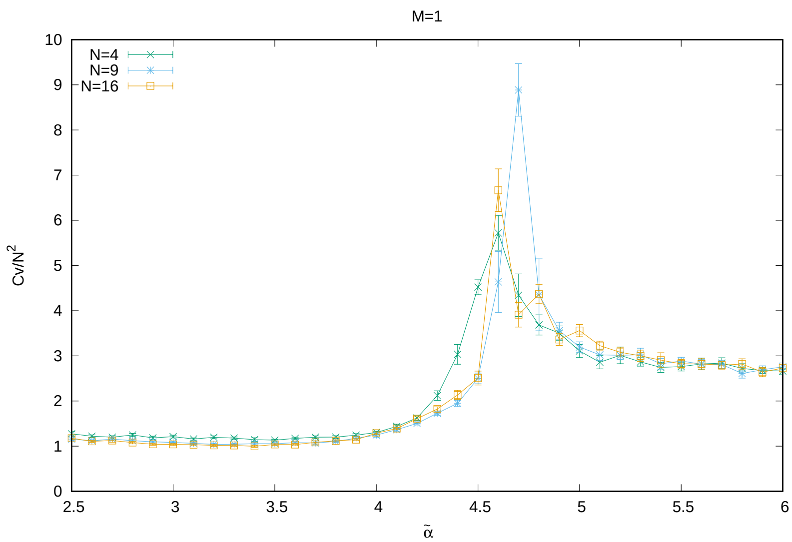


Figure 5.7: The action YM_{12}.



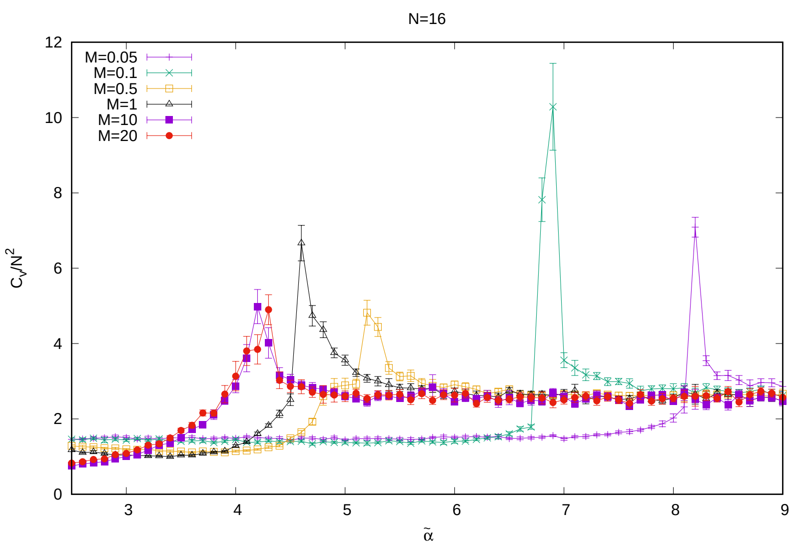


Figure 5.8: The specific heat (peak) in the six dimensional Yang-Mills matrix model.

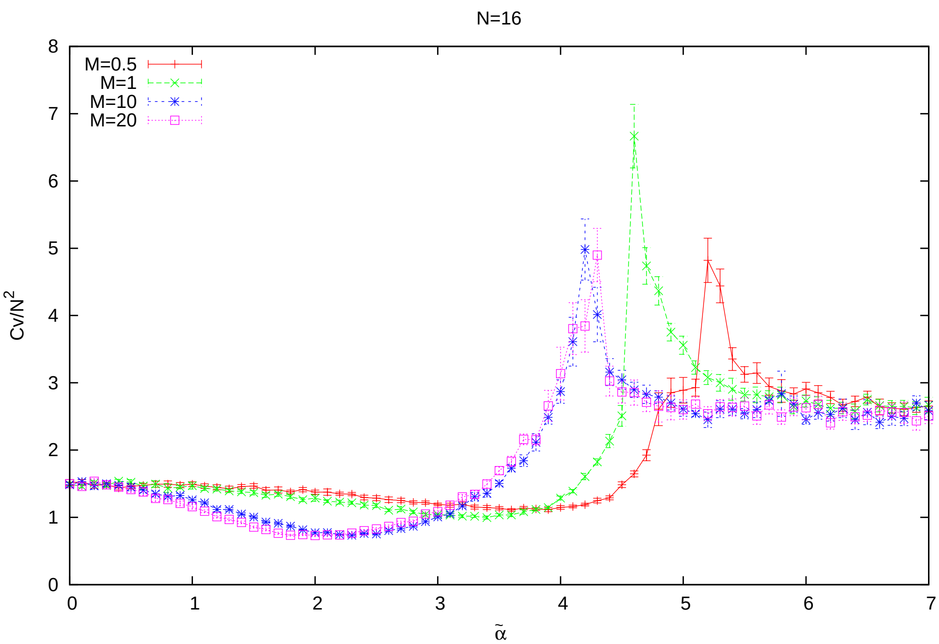
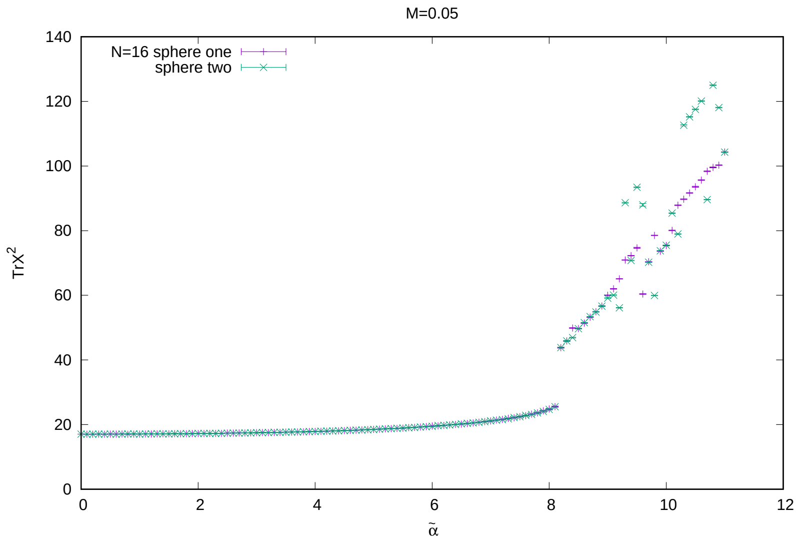


Figure 5.9: The specific heat (minimum) in the six dimensional Yang-Mills matrix model.



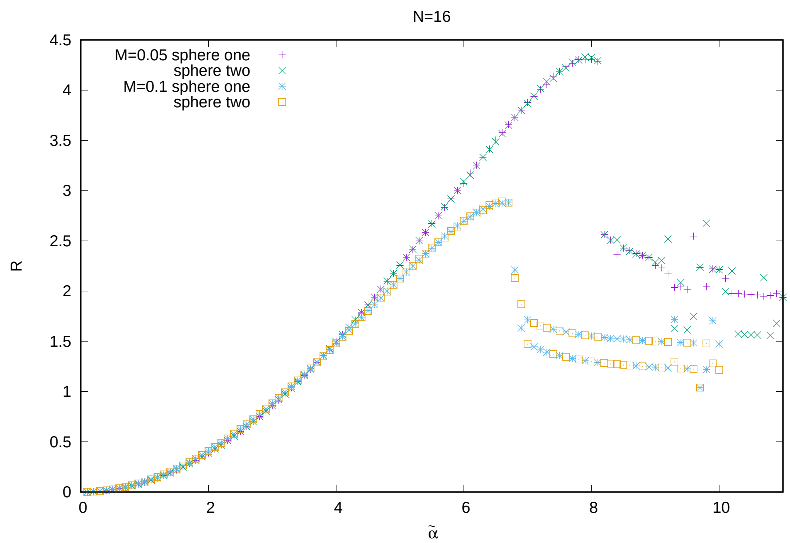
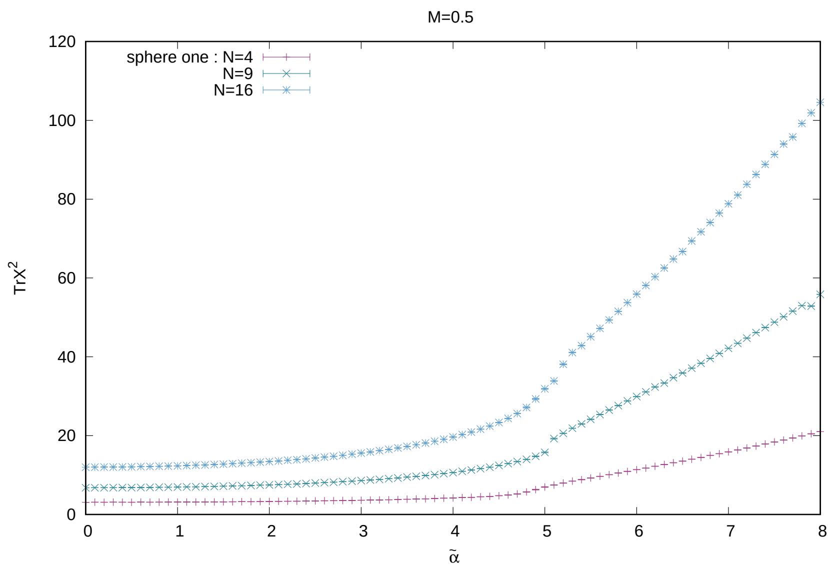


Figure 5.10: The radius in the six dimensional Yang-Mills matrix model for small values of M.



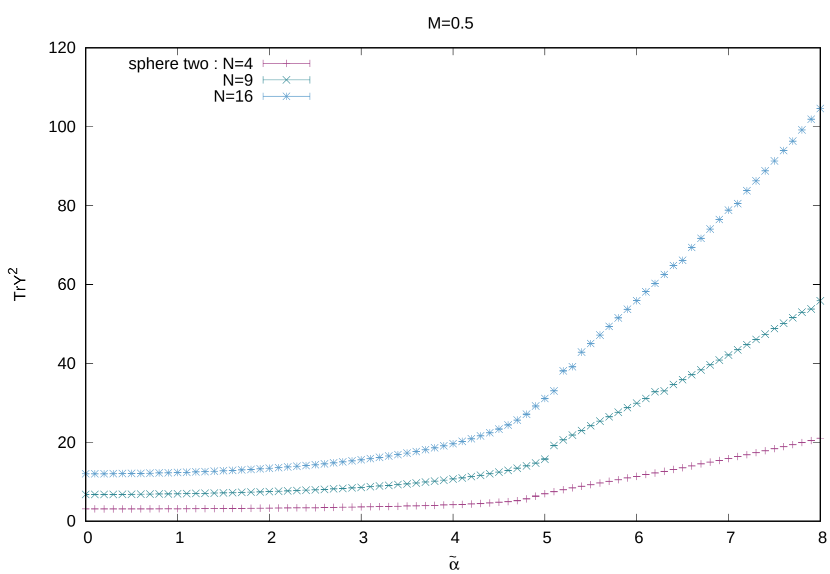


Figure 5.11: The radius in the two sphers.

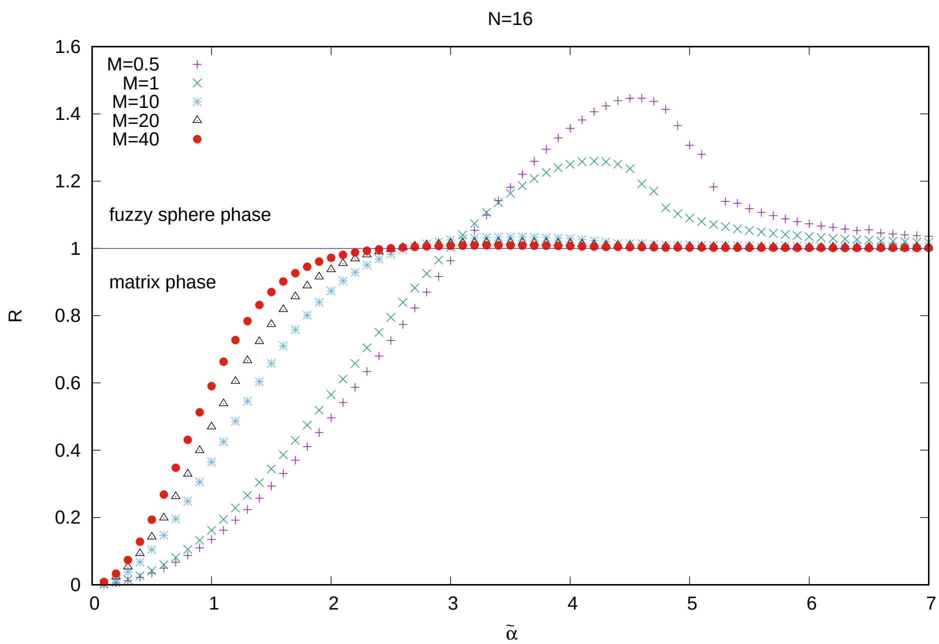


Figure 5.12: The radius in the six dimensional Yang-Mills matrix model.

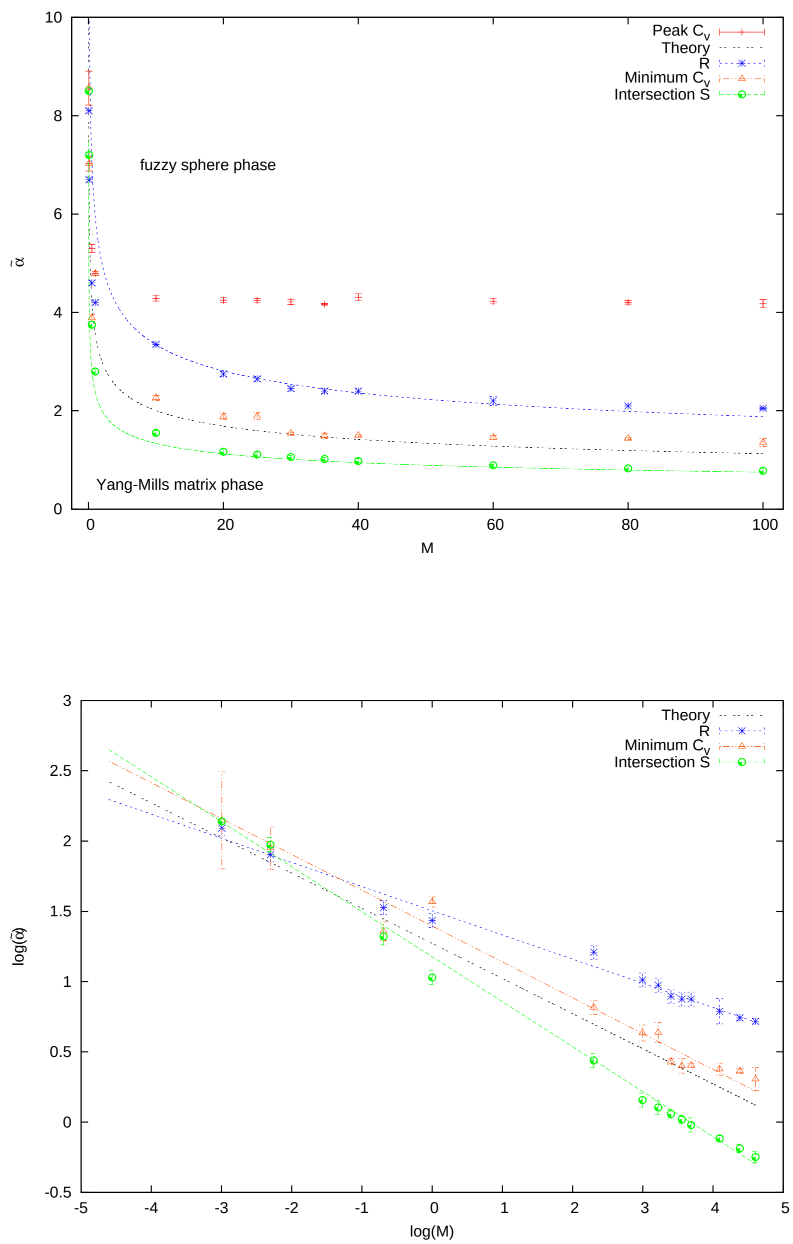


Figure 5.13: The phase diagram of the six dimensional Yang-Mills matrix model.

5.5 Eigenvalues distributions and critical behavior

5.5.1 Eigenvalues distributions from Monte Carlo

The phase transition between the fuzzy four-sphere phase and the Yang-Mills phase can be characterized fully by the behavior of the eigenvalue distribution across the transition line. This is by far the most detailed order parameter at our disposal.

5.5.2 Small M:

The behavior of the eigenvalue distributions for small values of M, such as $M = 0.05$ and $M = 0.01$, is depicted on figure (5.14). The two limiting behaviors are as follows:

- For large values of the gauge coupling constant we observe a point spectrum given by the eigenvalues of the $SU(2)$ generators in the largest irreducible representation which is of size N.

- Motivated by the work [151] it was conjectured in [152] that the joint eigenvalues distribution of d matrices X_1, X_2,...X_d with dynamics given by a reduced Yang-Mills action should be uniform inside a solid ball of some radius R. See also [**?**,**?**,**?**]. Let $\rho(x_1, ..., x_d)$ be the joint eigenvalues distribution of the d matrices X_1, X_2, ... and X_d. We assume that $\rho(x_1, ..., x_4)$ is uniform inside a four dimensional ball of radius r. The eigenvalues distribution of a single matrix, say X_d, which is induced by integrating out the other $d-1$ matrices is given by

$$\rho(\lambda) = \frac{\Omega_{d-1}}{V_d(d-1)}(r^2 - \lambda^2)^{(d-1)/2}. \tag{5.47}$$

- For small values of the gauge coupling constant $\widetilde{\alpha} \longrightarrow 0$ we observe a very good agreement with this law with $d = 6$.We also include in the second graph of figure (5.14), the $d = 4$ law and the $d = 3$ parabolic law, for comparison. The fit for $N = 16$, $M = 0.01$, $\widetilde{\alpha} = 0$ gives a value of the radius r of the distribution given by

$$r \simeq 2. \tag{5.48}$$

We believe that this is a universal behavior at small $\widetilde{\alpha}$ and small M.

5.5.3 Large M:

Some data is included on figures (5.15) and (5.16).

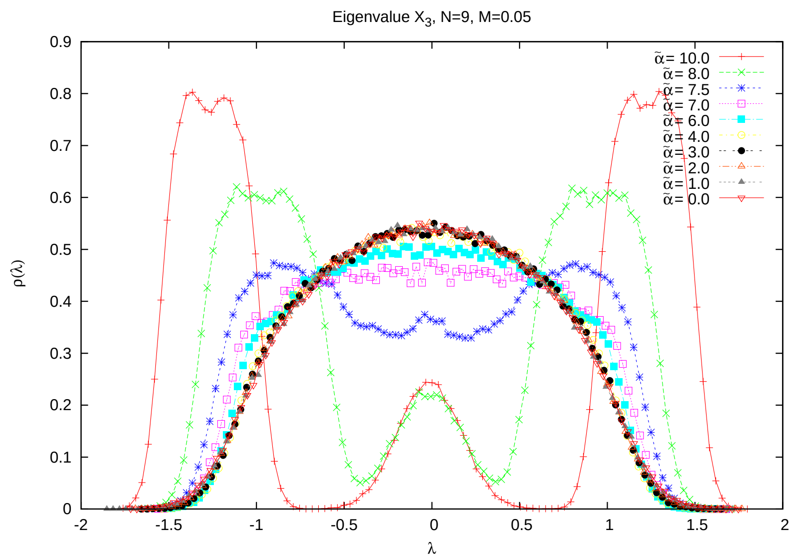
- For large values of the gauge coupling constant we still observe a point spectrum given by the eigenvalues of the $SU(2)$ generators.
- For small values of the gauge coupling constant $\widetilde{\alpha} \longrightarrow 0$ we observe approximately a uniform distribution. This is neatly shown for $M = 30$, $N = 9$ and $M = 1$, $N = 25$ on figure (5.15). The Yang-Mills matrix phase is then characterized, for large values of M, by this one-cut uniform distribution as opposed to the $d = 6$ law which characterizes the Yang-Mills phase for small values of M. The transition from the fuzzy four-sphere to the uniform distribution goes through a new phase or a crossover as we will now discuss.
- **A new phase or a crossover:** There seems to be another phase appearing for large values of M between the fuzzy four-sphere phase and the Yang-Mills phase. This is indicated by the transition from the distinct point spectrum in the fuzzy four-sphere phase to a phase where a strong gauge field is superimposed on the four-sphere background in such a way that the middle peaks flatten then disappears slowly in favor of the uniform distribution. The last peaks to go are the maximum and the minimum of the $SU(2)$ configuration. See figure (5.16) for $N = 16, 9$ and $M = 10$ where this transition occurs at $\widetilde{\alpha} = 4.2$ at the peak of C_v. Recall that the peak of the specfic heat saturates for large values of $\widetilde{\alpha}$ at $\widetilde{\alpha} = 4.2$. However, this new phase may only be a crossover transition. Indeed, on the second graph of (5.17) we plot the behavior of X_a^2 as a function of $\widetilde{\alpha}$ for $N = 9$ and $M = 30$. We observe that the profile of the eigenvalue distribution changes drastically only at the transition point to the uniform distribution.
- **The coupling between the two spheres:** The coupling between the two spheres can be probed by the eigenvalue distribution of the commutator $i[X_1, Y_1]$. A sample is shown on figure (5.19) for $N = 9$ and $M = 10$. We observe deep inside the fuzzy four-sphere phase that the commutator $i[X_1, Y_1]$ is very different from the rotationally identical commutators $i[X_1, X_2]$ and $i[Y_1, Y_2]$, whereas the three commutators behave indistinguishably from each other deep inside the Yang-Mills matrix phase. In the middle phase, during the crossover, the eigenvalue

distribution of $i[X_1, Y_1]$ approaches quickly the profile of the other two. The widths, for example, become less and less different, and shrink to a minimum value in the limit $\widetilde{\alpha} \longrightarrow 0$. This is shown explicitly for the commutator $i[X_1, X_2]$ for $N = 9$ and $M = 30$.

- **Steinacker's value $M = 0.5$ and transition point in M:** See (5.18). The behavior of the eigenvalue distribution deep inside the Yang-Mills matrix phase for $M = 0.5$, and other medium values of M, is neither given by the $d = 6$ law, nor it is given by a uniform distribution. This indicates that this value is near the triple point where the transition line between the fuzzy four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ and the Yang-Mills matrix phase becomes a crossover phase.

- **Rotational symmetry:** The eigenvalue distributions of X_3, Y_3, and the commutators $i[X_1, X_2]$, $i[Y_1, Y_2]$, and the squares X_a^2 and Y_a^2, in the fuzzy four-sphere phase are shown on the first graph of (5.17). This shows explicitly the rotational symmetry between the two spheres.

The case of a single sphere: The physics in this case is very similar to the case of the fuzzy four-sphere and a sample of the eigenvalue distributions is included on figures (5.20) and (5.21).



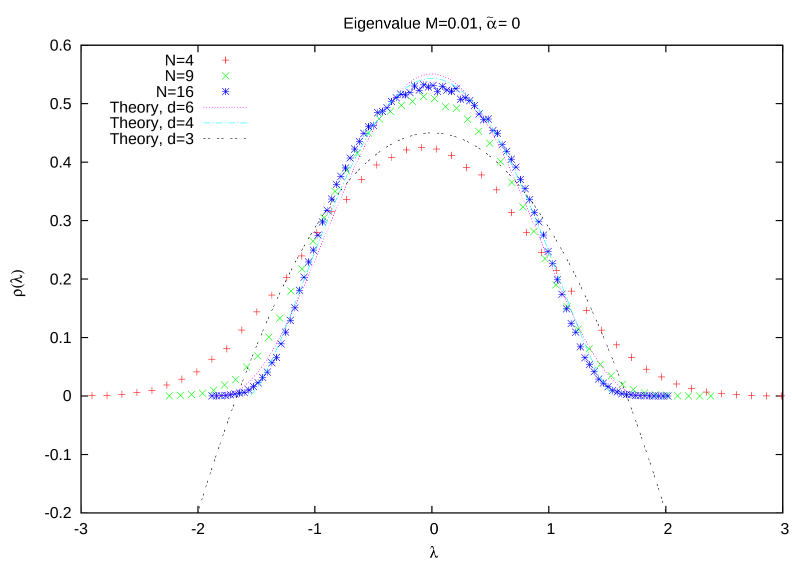
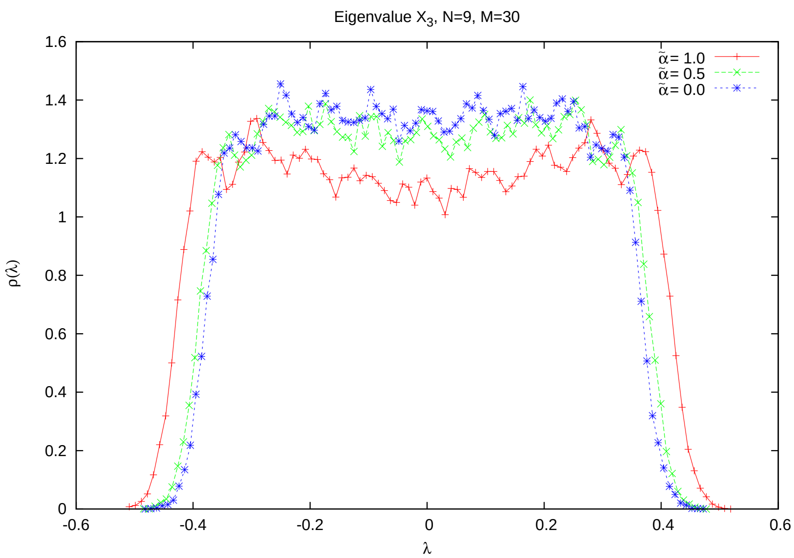


Figure 5.14: The eigenvalue distributions for small values of M.



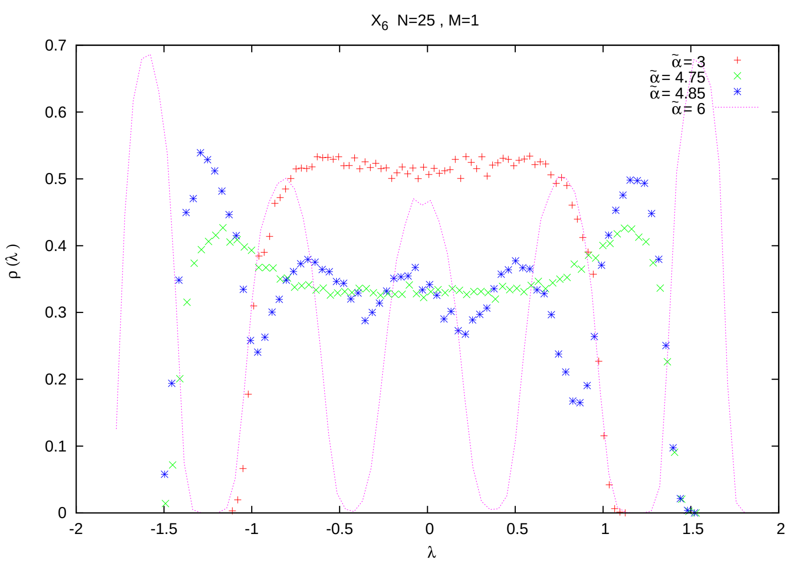


Figure 5.15: The eigenvalue distributions for large values of M.

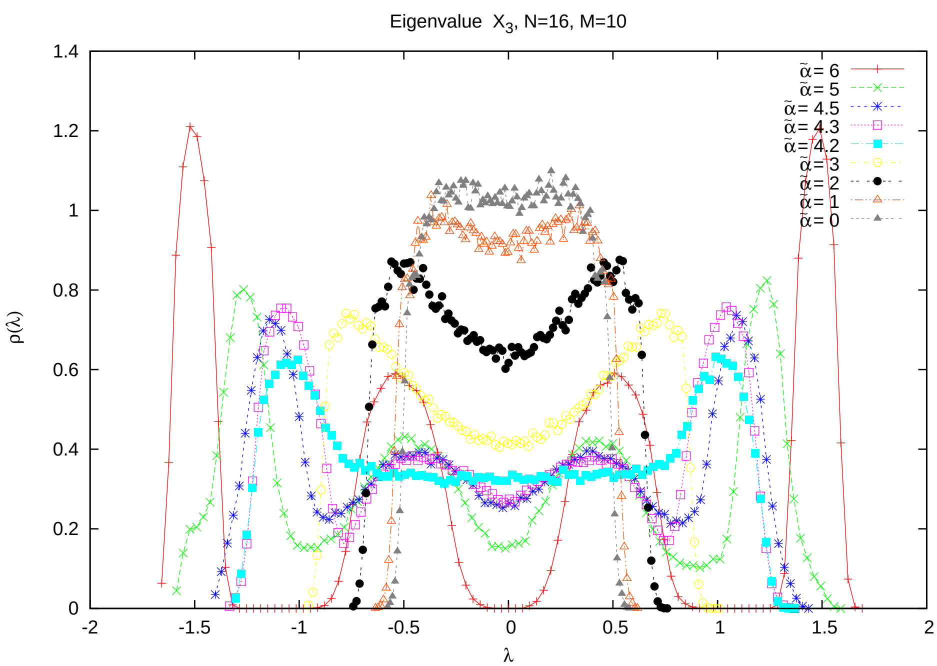
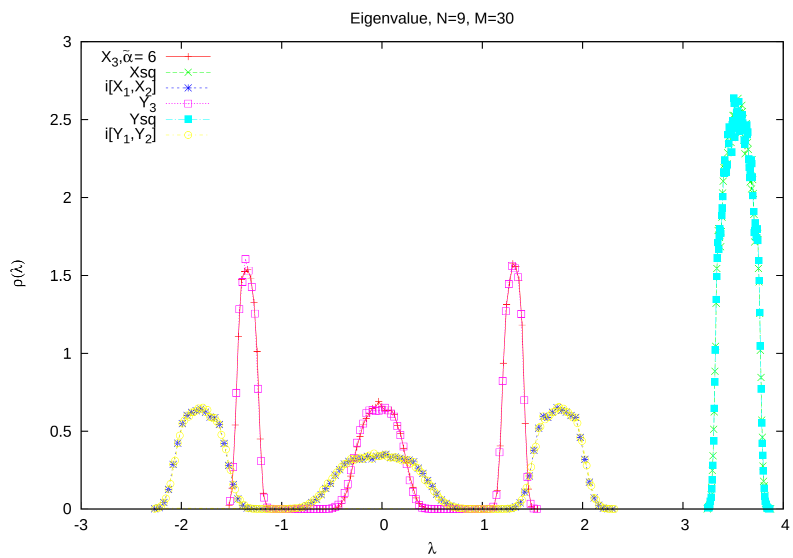


Figure 5.16: The eigenvalue distributions for large values of M.



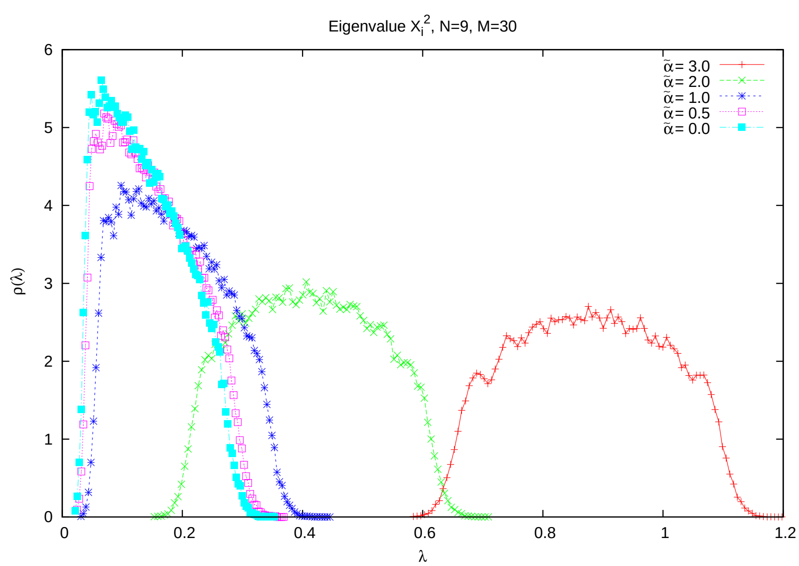
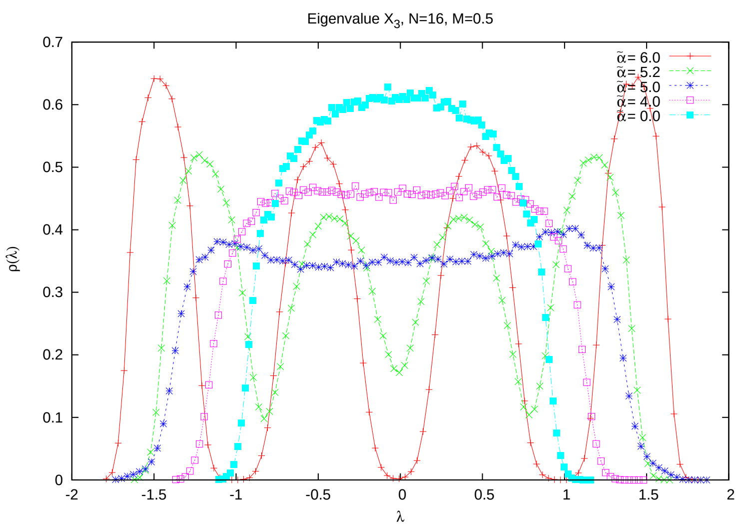


Figure 5.17: Rotational invariance in the eigenvalue distributions for large values of M.



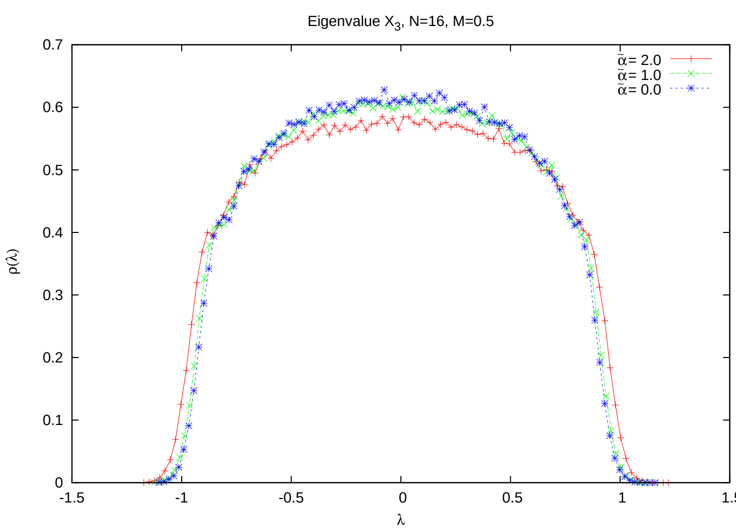


Figure 5.18: The eigenvalue distributions for the Steinacker value $M = 1/2$.

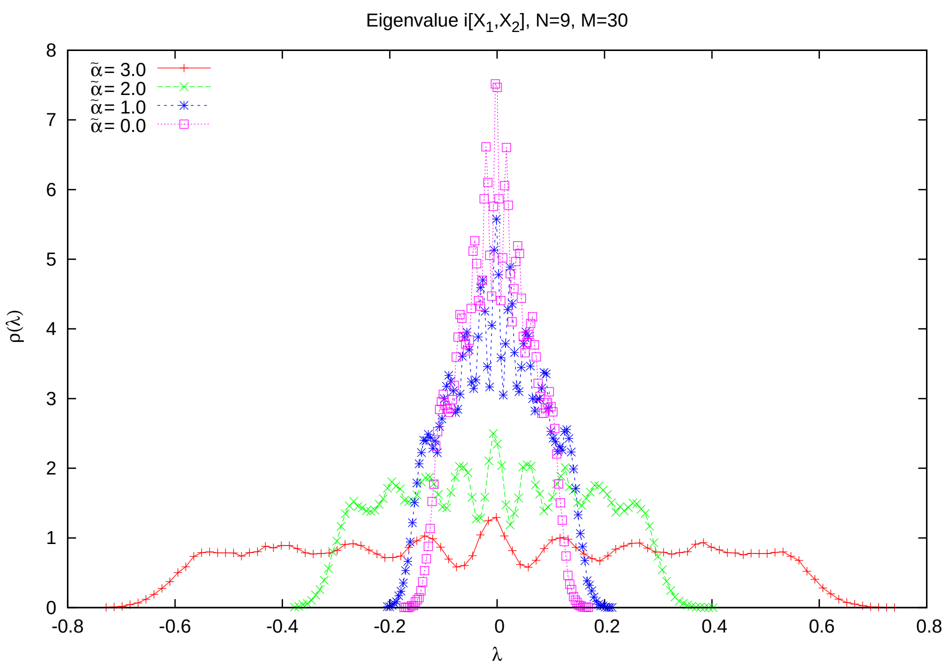
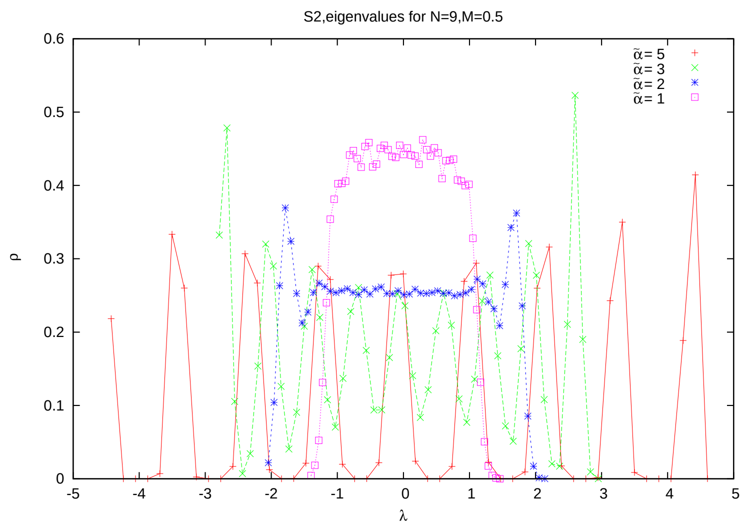


Figure 5.19: The eigenvalue distributions of the commutators $i[X_1, Y_1]$, etc.



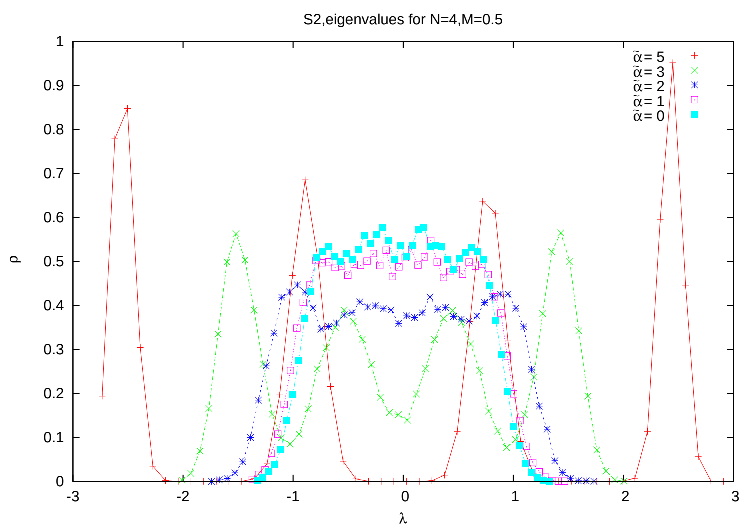


Figure 5.20: The eigenvalue distributions on a single sphere.

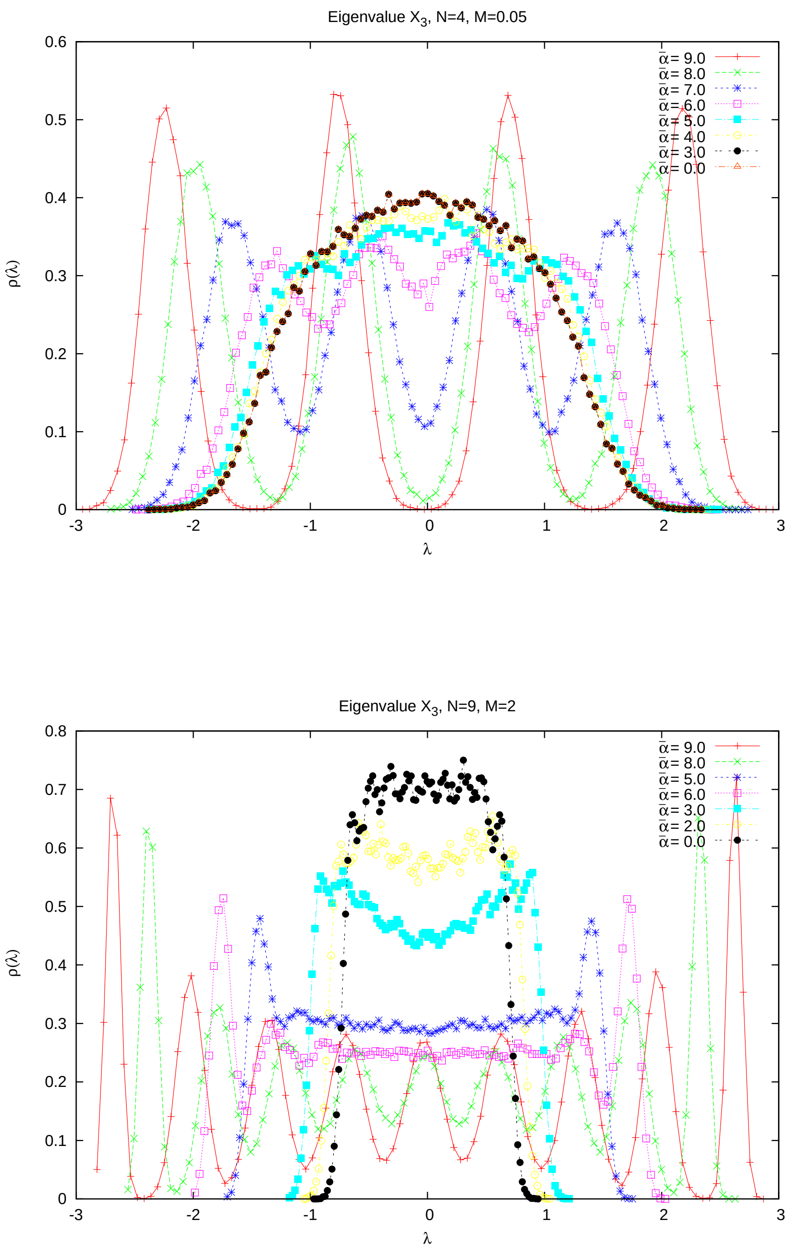


Figure 5.21: The eigenvalue distributions on a single sphere (more).

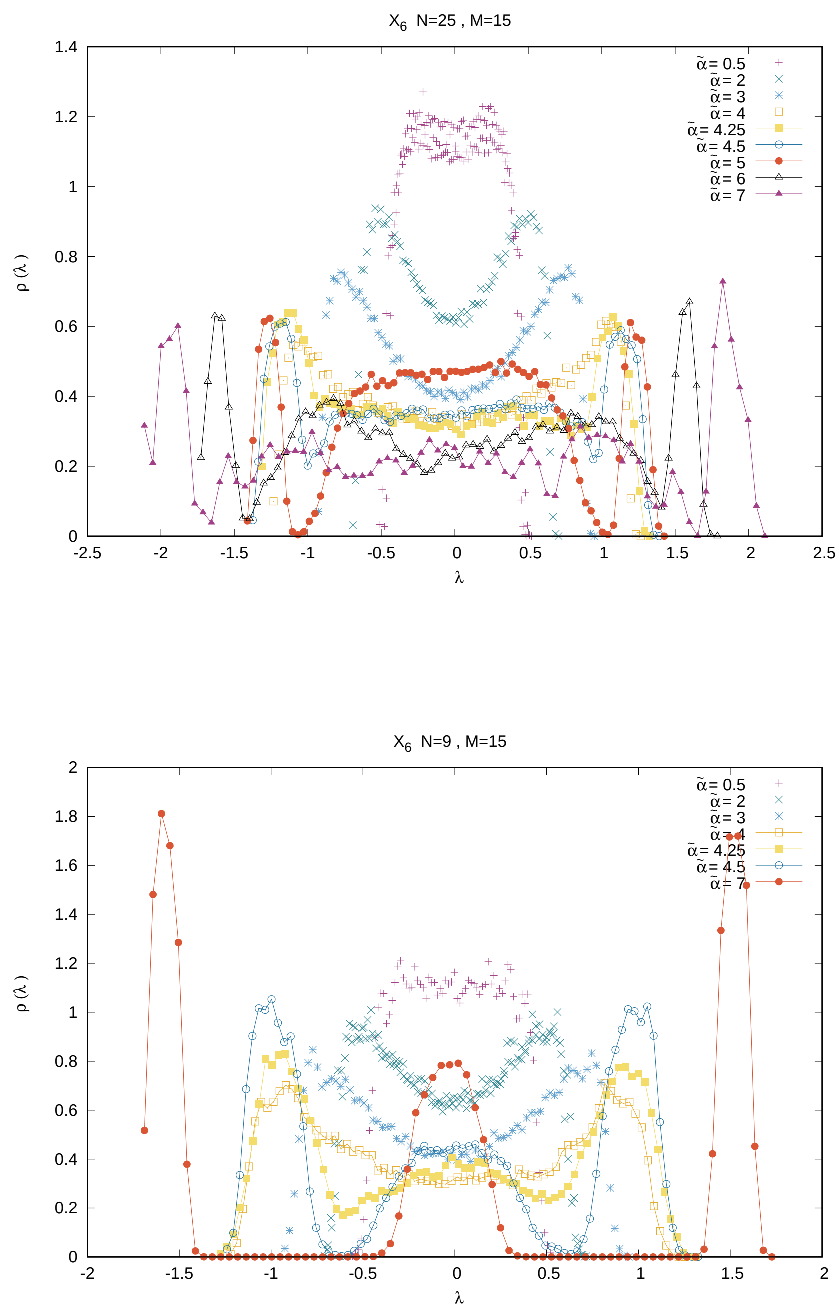


Figure 5.22: The eigenvalue distribution for X_6 across the transition line.

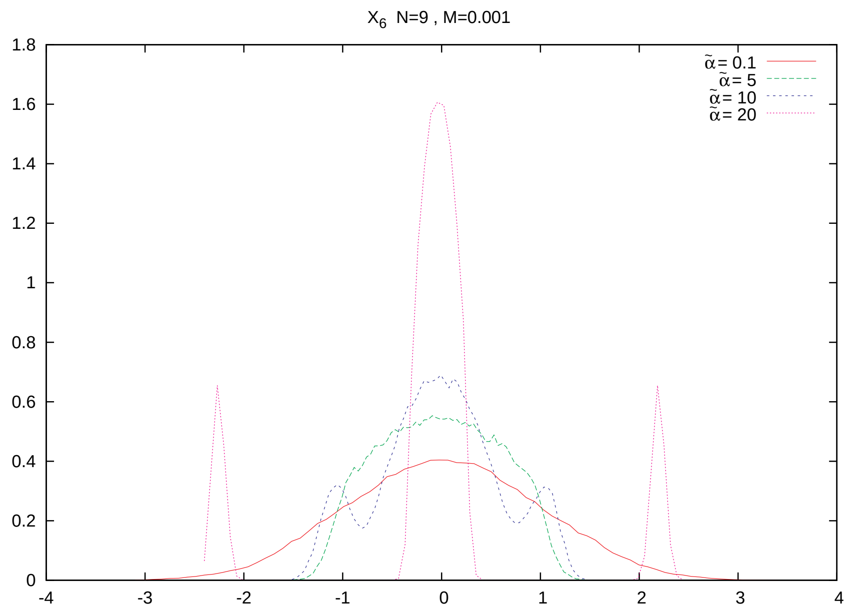
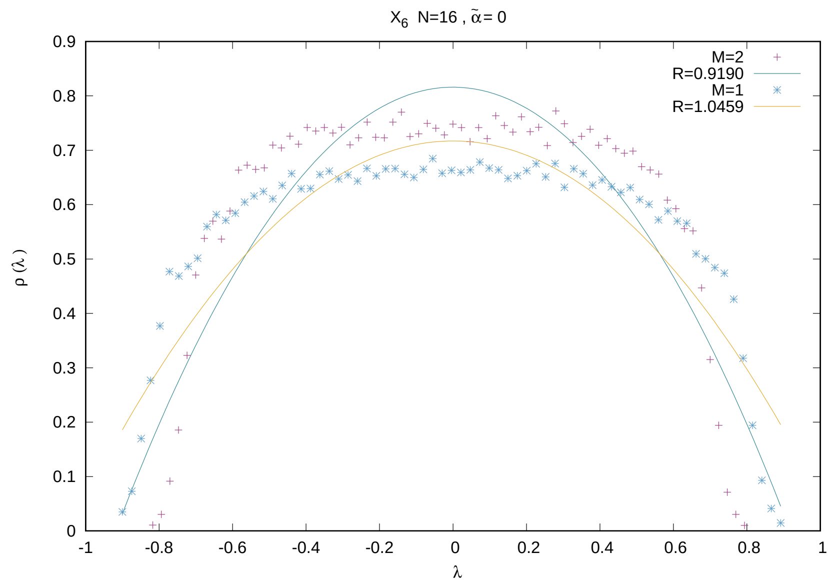


Figure 5.23: The eigenvalue distribution for X_6 across the transition line.



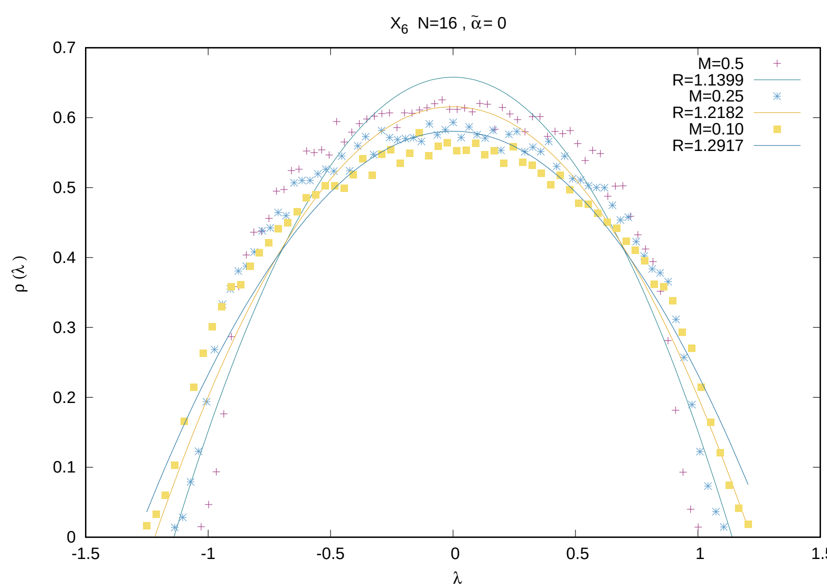


Figure 5.24: The eigenvalue distributions for X_6 for small and large values of M and $\tilde{\alpha} = 0$.

5.6 The effective potential revisited

We recall the formula for the effective potential

$$\frac{V}{2N^2} = \widetilde{\alpha}_0^4\left[\frac{\phi^4}{4}-\frac{\phi^3}{3}+m^2\frac{\phi^4}{4}-\mu\frac{\phi^2}{2}\right]+\log\phi^2. \tag{5.49}$$

The classical solutions are given by the condition

$$\frac{V'}{2N^2} = 0. \tag{5.50}$$

Clearly the effective potential is not bounded at $\phi = 0$. However, this should not pose any problem since this potential is valid for large values of $\widetilde{\alpha}$ where we know that the fuzzy sphere exists. For smaller values of $\widetilde{\alpha}$ the fuzzy sphere configuration ceases to exist and we enter the matrix phase as we have seen in Monte Carlo data. The classical potential admits as solutions $\phi = 0$ (local minimum) together with the fuzzy four-sphere solution ϕ_+ (global minimum), and a maximum of the barrier between them denoted by ϕ_- (local maximum). The solutions $\phi_\pm$ exist for $t > -1/4$. As we decrease μ towards negative values the global minimum ϕ_+ becomes degenerate with $\phi = 0$ at $t = \mu(1+m^2) = -2/9$ and the height at the maximum of the barrier becomes $\widetilde{\alpha}^4/324(1+m^2)^3$. There exists therefore a first order transition in μ, at $\mu = -2/9(1+m^2)$ for every fixed m^2, which is of the same character as the one at $m^2 = 0$, from a fuzzy-four sphere phase for $\mu > -2/9(1+m^2)$ to $X_a = Y_a = 0$ for $\mu < -2/9(1+m^2)$. For our value $\mu = 2(2m^2-1)/9$ the global minimum is always $\phi = 2/3$ and it is separated from $\phi = 0$ by a barrier. The effective potential admits four real solutions. The largest positive solution is ϕ_+ while the other positive solution gives the local maximum and it will determine the height of the barrier in the effective potential. At the critical point these two solutions merge and the barrier disappears. This is different from the classical solution where the barrier never disappears. The solution ϕ_+ ceases to exist at the critical value determined by the condition

$$\frac{V''}{2N^2} = 0. \tag{5.51}$$

We recall also that the critical value is found by solving (5.50) and (5.51) and is given implicitly by

$$(1+m^2)\phi_* = \frac{3}{8}\big(1+\sqrt{1+\frac{32t}{9}}\big)$$
$$\frac{1}{\widetilde{\alpha}_{0*}^4} = \frac{\phi_*^2(\phi_*+2\mu)}{8}$$
$$t = \mu(1+m^2). \tag{5.52}$$

The critical value is sent to infinity for $\phi_* = -2\mu$ which is equivalent to $t = -1/4$. Thus quantum mechanically the fuzzy four-sphere may exist for $t > -1/4$ which is to be compared with the classical prediction $t > -2/9$. In this region the classical potential is always positive and thus one should consider all $SU(2)$ representations which are degenerate with $X_a = Y_a = 0$, i.e. for which $\sum_i n_i c_2(n_i)/N \longrightarrow 0$ in the limit $N \longrightarrow 0$ with $\sum_i n_i = N$. However, for large $\widetilde{\alpha}$ the ground state is dominated by the representation with the smallest Casimir. The fuzzy four-sphere is therefore not stable in this regime and the critical line between the fuzzy four-sphere phase and the Yang-Mills matrix phase asymptotes the line $t = -2/9$ as shown in [149]. What interests us the most in this section is the expansion of the solution ϕ around the critical value ϕ^*. This is given in [139]. The only difference between the result on $\mathbf{S}^2$ given in [139] and our result her lies in the replacement

$$\widetilde{\alpha} \longrightarrow \widetilde{\alpha}_0 \tag{5.53}$$

and the replacement

$$c_2 \longrightarrow c_2^0 \tag{5.54}$$

in the definition of μ. We get the solution

$$\phi = \phi_* + \frac{4}{\widetilde{\alpha}_{0*}^{\frac{5}{2}}}\frac{1}{\sqrt{3\phi_*+4\mu}}\sqrt{\widetilde{\alpha}_0 - \widetilde{\alpha}_{0*}} + ... \tag{5.55}$$

This takes the form

$$\phi = \phi_* + \sigma \ , \ \sigma = \frac{4(2N)^{1/2}}{\widetilde{\alpha}_*^{\frac{5}{2}}}\frac{1}{\sqrt{3\phi_*+4\mu}}\sqrt{\widetilde{\alpha} - \widetilde{\alpha}_*} + ... \tag{5.56}$$

In the large N limit we have

$$\mu \longrightarrow \frac{2NM}{9}. \tag{5.57}$$

$$\phi_* \longrightarrow \sqrt{\frac{\mu}{NM}} = \frac{\sqrt{2}}{3}. \tag{5.58}$$

$$\alpha_*^4 \longrightarrow \frac{8N^2M}{\mu^2} = \frac{162}{M}. \tag{5.59}$$

Thus

$$\phi = \phi_* + \sigma \ , \ \sigma = \frac{6}{\tilde{\alpha}_*^{\frac{5}{2}}\sqrt{M}}\sqrt{\tilde{\alpha} - \tilde{\alpha}_*} + ... \tag{5.60}$$

5.7 Critical behavior from one-loop effective potential

Let us start by noting the Schwinger-Dyson identity

$$\begin{aligned} & 4 < \mathrm{YM}_1 > +4 < \mathrm{YM}_2 > +4 < \mathrm{YM}_{12} > +3 < \mathrm{CS}_1 > +3 < \mathrm{CS}_2 > \\ + \ & 4 < \mathrm{Quar}_1 > +4 < \mathrm{Quar}_2 > +2 < \mathrm{Quad}_1 > +2 < \mathrm{Quad}_2 >= 6N^2. \end{aligned} \tag{5.61}$$

We have the obvious definitions for the various observables

$$\mathrm{YM}_1 = NTr\Big[-\frac{1}{4}[X_a, X_b]^2\Big] \ , \ \mathrm{YM}_2 = NTr\Big[-\frac{1}{4}[Y_a, Y_b]^2\Big] \ , \ \mathrm{YM}_{12} = NTr\Big[-\frac{1}{4}[X_a, Y_b]^2\Big]. \tag{5.62}$$

$$\mathrm{CS}_1 = NTr\Big[\frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c\Big] \ , \ \mathrm{CS}_2 = NTr\Big[\frac{2i\alpha}{3}\epsilon_{abc}Y_aY_bY_c\Big]. \tag{5.63}$$

$$\mathrm{Quar}_1 = NMTr(X_a^2)^2 \ , \ \mathrm{Quar}_2 = NMTr(Y_a^2)^2. \tag{5.64}$$

$$\mathrm{Quad}_1 = N\beta TrX_a^2 \ , \ \mathrm{Quad}_2 = N\beta TrY_a^2. \tag{5.65}$$

We obtain immediately a formula for the average action given by

$$\frac{S}{N^2}=\frac{3}{2}+\frac{1}{4N^2}<\mathrm{CS}_1>+\frac{1}{4N^2}<\mathrm{CS}_2>+\frac{1}{2N^2}<\mathrm{Quad}_1>+\frac{1}{2N^2}<\mathrm{Quad}_2> \quad (5.66)$$

We compute

$$\frac{1}{4N^2}<\mathrm{CS}_1>=\frac{1}{4N^2}<\mathrm{CS}_2>=-\frac{\widetilde{\alpha}^4\phi^3}{24N}. \quad (5.67)$$

$$\frac{1}{2N^2}<\mathrm{Quad}_1>=\frac{1}{2N^2}<\mathrm{Quad}_2>=-\frac{\mu\widetilde{\alpha}^4\phi^2}{8N}. \quad (5.68)$$

We also note here the formula for the related radius (with $\phi_0=2/3$)

$$\frac{1}{R}=<\frac{1}{\phi_0^2\widetilde{\alpha}^2c_2^0}TrX_a^2>=<\frac{1}{\phi_0^2\widetilde{\alpha}^2c_2^0}TrY_a^2>=\left(\frac{3}{2}\right)^2\phi^2. \quad (5.69)$$

Thus

$$\begin{aligned}\frac{S}{N^2} &= \frac{3}{2}-\frac{\widetilde{\alpha}^4}{4N}\left(\frac{1}{3}\phi^3+\mu\phi^2\right)\\ &= \frac{3}{2}-\frac{\widetilde{\alpha}^4}{4N}\left(\frac{1}{3}\phi_*^3+\mu\phi_*^2\right)-\frac{\widetilde{\alpha}_*^4}{4N}(\phi_*^2+2\mu\phi_*)\sigma\\ &= \frac{S_*}{N^2}-\frac{4}{\phi_*}\sigma. \end{aligned} \quad (5.70)$$

$$\begin{aligned}\frac{S_*}{N^2} &= \left(\frac{3}{2}-\frac{4}{3}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha}_*}\right)^4-\frac{\widetilde{\alpha}^4\mu}{12N}\phi_*^2\right)_*\\ &= \frac{3}{2}-\frac{4}{3}\frac{\phi_*+3\mu}{\phi_*+2\mu}\\ &\longrightarrow -\frac{1}{2}. \end{aligned} \quad (5.71)$$

This is the value of the average action or entropy in the fuzzy four-sphere phase at the transition point exactly for M large. For M small we get instead the (constant) value

$$\frac{S_*}{N^2}\longrightarrow\frac{1}{6}. \quad (5.72)$$

In the Yang-Mills matrix phase we go back to the first line of equation (5.70) and set $\phi = 0$ to get the value $3/2$ at $\widetilde{\alpha} = 0$. The transition to the Yang-Mills phase occurs quite suddenly for small M. Thus the entropy for small M has a discrete jump given by

$$\frac{\Delta S}{N^2} \longrightarrow \frac{4}{3}. \tag{5.73}$$

The average action can also be derived using the formula

$$\frac{< S >}{N^2} = \frac{3}{2} + \widetilde{\alpha}^4 \frac{d}{d\widetilde{\alpha}^4}\Big(\frac{F}{N^2}\Big), \tag{5.74}$$

where (with $X_a = \alpha D_a$, $\widetilde{\alpha}^4 \hat{S} = S$)

$$Z = \exp(-F) = \int dD_a \exp\Big(\frac{3N^2}{2}\ln\widetilde{\alpha}^4 - \widetilde{\alpha}^4\hat{S}\Big). \tag{5.75}$$

The free energy is given by

$$\frac{F}{2N^2} = \frac{V}{2N^2} + \frac{1}{2}\ln\widetilde{\alpha}^4 + \text{constant}. \tag{5.76}$$

Then we can compute the specific heat by the formula

$$\begin{aligned} C_v &= \frac{< S^2 > - < S >^2}{N^2} \\ &= \frac{< S >}{N^2} - \widetilde{\alpha}^4 \frac{d}{d\widetilde{\alpha}^4}\Big(\frac{< S >}{N^2}\Big). \end{aligned} \tag{5.77}$$

We compute immediately

$$C_v = \frac{3}{2} + \frac{\widetilde{\alpha}^5}{16N}\phi(\phi + 2\mu)\frac{d\phi}{d\widetilde{\alpha}}. \tag{5.78}$$

From this equation we can derive immediately the divergent part of the specific heat to be given by

$$\begin{aligned} C_v &= C_v^B + \frac{1}{4(2N)^{1/2}}\frac{\phi_*(\phi_* + 2\mu)}{\sqrt{3\phi_* + 4\mu}}\frac{\widetilde{\alpha}_*^{5/2}}{\sqrt{\widetilde{\alpha} - \widetilde{\alpha}_*}} \\ &= C_v^B + \frac{3^{1/2}}{2^{7/8}M^{1/8}}\frac{1}{\sqrt{\widetilde{\alpha} - \widetilde{\alpha}_*}}. \end{aligned} \tag{5.79}$$

The critical exponent of the specific heat is 1/2 which is precisely the value obtained in two dimensions. Similarly, the critical value $\widetilde{\alpha}_*$ and the coefficient of the singularity become vanishingly small when $M \longrightarrow \infty$. The behavior of the background specific heat C_v^B deep inside the fuzzy four-sphere phase is computed as follows. The solution ϕ of the equation of motion for large values of $\widetilde{\alpha}$ is found to be given by

$$\phi = \frac{2}{3} - \frac{27}{2M\widetilde{\alpha}^4} + ... \tag{5.80}$$

By substitution in (5.78) then using $\phi \sim 2/3$ since we are only interested in the background value of the specfic heat, and also using (5.57), we get

$$\begin{aligned} C_v^B &= \frac{3}{2} + \frac{\widetilde{\alpha}^5}{16N}\phi(\phi + 2\mu)\frac{54}{M\widetilde{\alpha}^5} \\ &= \frac{3}{2} + 1 \\ &= \frac{5}{2}. \end{aligned} \tag{5.81}$$

5.8 Related topics

5.8.1 Emergent gauge theory in two dimensions:

We have established that the fuzzy sphere is completely stable in the three matrix model (5.18) for any value of M. This includes Steinacker's value $M = 1/2$. We can now speak in a consistent way about constructing $U(n)$ gauge theory on fuzzy $\mathbf{S}^2$. This entails the construction of monopoles sectors and the direct evaluation of the partition function on the fuzzy sphere, or alternatively its evaluation by means of localization technique, as a sum over instantons contributions. This has been done for Steinacker's value in [141] and [143] respectively. The result in this case was found to be identical to the result on the ordinary sphere. This shows that the fuzzy sphere is acting here as a regularized version of the sphere, and also shows that this matrix method is potentially a powerful new method for gauge theory. The construction of fuzzy monopoles and instantons and the corresponding Ginsparg-Wilson fermions on this stable fuzzy sphere can also be carried out along the lines of [153–155].

5.8.2 A stable four-sphere $\mathbf{S}^2 \times \mathbf{S}^2$ and topology change:

The main conclusion we have reached in this article is the difference between fuzzy $\mathbf{S}^2_N$ and fuzzy $\mathbf{S}^2_N \times \mathbf{S}^2_N$. The fuzzy four-sphere is only stable in the large M limit. A simple modification of the six matrix model may lead to a stable four-dimensional geometry for any M. This consists in adding the terms

$$N\beta_1(TrX_a^2 + TrY_a^2), \tag{5.82}$$

with a particular coefficient β_1. This will modify the critical line (6.34) in such a way that $\widetilde{\alpha}$ is replaced by $\bar{\alpha}$. Topology change can also be easily obtained in the above six matrix model by setting zero the mass parameters M and β on one of the spheres. This way we will have the possibility of a transition between the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ phase and the fuzzy sphere $\mathbf{S}^2_N$ phase. This is what we call a topology change. This scenario was previously observed on fuzzy $\mathbf{CP}^n$ [156].

Critical behavior: The critical behavior of the above stable fuzzy two-sphere $\mathbf{S}^2_N$ and also the critical behavior of the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ can also be determined more carefully along the line of [157].

Comparison with [147], the $2-$matrix model and instanton calculus: The six matrix model (5.26) with $M = 1/2$ is precisely the action considered in [147]. Since this model is only stable for large M, the corresponding instanton calculus should only be expected to be valid deep inside the fuzzy four-sphere phase. A $2-$matrix model associated with the $6-$dimensional Yang-Mills matrix model (5.26) can also be constructed starting from an $SO(6)$ formulation. This should be contrasted with the $SO(3)$ formulation on the fuzzy sphere which leads to (5.4). In the case of the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ the matrices C and D corresponding to X_a and Y_a respectively are found to be highly constrained [147] as opposed to the single constraint $C_0 = 0$ found on the fuzzy sphere. The six matrix model considered in this article is also closely related to the action studied in [158].

Dirac operator: There are two seemingly different formulations of the Dirac operator on the fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$. The one presented in [147] is again based on $SO(6)$ whereas the one presented in [159] is based on $SO(3) \times SO(3)$.

5.8.3 Fuzzy four-sphere S^4 and fuzzy $\mathbf{CP}^2$:

The above analysis is expected to hold without much change on fuzzy $\mathbf{S}^4$ [160–162] and fuzzy $\mathbf{CP}^2$ [163–166]. The analogous actions on fuzzy $\mathbf{CP}^n$ are given by [156]

$$S = \frac{1}{g^2N}Tr\bigg[-\frac{1}{4}[D_a,D_b]^2+i\epsilon_{abc}D_aD_bD_c\bigg]+\frac{3n}{4g^2N}Tr\Phi+\frac{M_0^2}{N}Tr\Phi^2+\frac{M^2}{N}\frac{(2n+3)^2}{36n}Tr\Phi. \tag{5.83}$$

5.9 Conclusion

In this article we have studied IKKT Yang-Mills matrix models with mass deformations in three and six dimensions. The $3-$dimensional IKKT matrix models considered here are very similar to the ones studied in [135, 138]. However, the dynamically emergent geometry, which is given by a fuzzy two-sphere $\mathbf{S}^2_N$, is found to be stable for all values of the deformation parameter M. This was anticipated previously for $M = 1/2$ in [141]. Indeed, the critical gauge coupling constant $\widetilde{\alpha}$ is found to scale as in equation (5.24), i.e. as $\widetilde{\alpha} \sim 1/\sqrt{N}$. The sphere-to-matrix transition line is pushed to 0 and only one phase survives. In this case the fuzzy sphere acts then as a regulator of the commutative sphere, and as a consequence, fuzzy field theory and fuzzy physics, based on this emergent fuzzy sphere, makes full sense for all values of the gauge coupling constant. We have also studied in this article $6-$dimensional IKKT matrix models, with global $SO(3) \times SO(3)$ symmetry, containing at most quartic powers of the matrices proposed in [146]. The value $M = 1/2$ of the deformation corresponds to the model of [147]. This theory exhibits a phase transition from a geometrical phase at low temperature, given by a fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ background, to a Yang-Mills matrix phase with no background geometrical structure at high temperature. The geometry as well as an Abelian gauge field and two scalar fields are determined dynamically as the temperature is decreases and the fuzzy four-sphere condenses. The transition is exotic in the sense that we observe, for small values of M, a discontinuous jump in the entropy, characteristic of a 1st order transition, yet with divergent critical fluctuations and a divergent specific heat with critical exponent $\alpha = 1/2$. The critical temperature is pushed upwards as the scalar field mass is increased. For small M, the system in the Yang-Mills phase is well approximated by 6 decoupled matrices with a joint eigenvalue distribution which is uniform inside a ball in $\mathbf{R}^6$. This gives what we call the $d = 6$ law given by equation (5.47). For large M, the transition from the four-sphere phase to the Yang-Mills matrix phase turns into a crossover and the eigenvalue distribution in the Yang-Mills matrix phase changes from the $d = 6$ law to a uniform distribution.

In the Yang-Mills matrix phase the specific heat is equal to 3/2 which coincides with the specific heat of 6 independent matrix models with quartic potential in the high temperature limit and is therefore consistent with this interpretation. Once the geometrical phase is well established the specific heat takes the value 5/2 with the gauge field contributing 3/2 and the two scalar fields contributing 1. This should be contrasted with the case of $\mathbf{S}^2_N$ in which the specific heat in the Yang-Mills matrix phase is equal to 3/4, coinciding with the specific heat of 3 independent matrix models with quartic potential in the high temperature limit, while in the geometrical phase the specific heat takes the value 1 divided equally, i.e. with the gauge field contributing 1/2 and the two scalar fields contributing 1/2. The counting on $\mathbf{S}^2_N$ is clear cut. In the sphere phase which coincides with the perturbative region of the theory the gauge field contributes 1/2 [167] and the scalar field contribute 1/2 (trivial to check in the quartic matrix model for very large positive values of the mass parameter [168, 169]). The counting on $\mathbf{S}^2_N \times \mathbf{S}^2_N$ is more involved. There are here two scalar fields and one gauge field with 4 components. Again, it is very natural to suppose that each scalar field will contribute 1/2 (since they are free and dimension does not enter in the quartic matrix model). Therefore, the gauge field will contribute 3/2. This is the picture we also get, at least formally, by assuming that the gauge field is free and Abelian (which is true deep in the sphere phase in the large N limit) and then fixing the gauge in the axial gauge. The $6-$dimensional IKKT Yang-Mills matrix models studied here present thus an appealing picture of a $4-$dimensional geometrical phase emerging as the system cools and suggests a scenario for the emergence of geometry in the early universe. See [148] and references therein.

Chapter 6

Emergent Geometry in the multitrace quartic matrix models

6.1 Introduction and Motivation

The original motivation for this work is the theory of noncommutative Φ^4 which we now briefly describe. A scalar phi-four theory on a non-degenerate noncommutative Euclidean spacetime is a a three-parameter matrix model of the generic form

$$S = \mathrm{Tr}_H\big(aM\Delta M + bM^2 + cM^4\big). \tag{6.1}$$

The Laplacian Δ captures precisely the underlying geometry, i.e. the metric, of the noncommutative Euclidean spacetime in the sense of [170, 171]. This theory can be regularized non-perturbatively using $N \times N$ matrices in an almost obvious way, i.e. the Hilbert space H can be taken to be finite dimensional of size N. This theory exhibits the following three known phases:

- The usual 2nd order Ising phase transition between disordered $< M >= 0$ and uniform ordered $< M >\sim \mathbf{1}_N$ phases. This appears for small values of c. This is the only transition observed in commutative phi-four, and thus it can be accessed in a small noncommutativity parameter expansion.

- A matrix transition between disordered $< M >= 0$ and non-uniform ordered $< M >\sim \gamma$ phases with $\gamma^2 = \mathbf{1}_N$. This transition coincides, for very large values of c, with the 3rd order transition of the real quartic matrix model, i.e. the model with $a = 0$, which occurs at $b = -2\sqrt{Nc}$. In terms of $\tilde{b} = bN^{-3/2}$ and $\tilde{c} = cN^{-2}$ this reads

$$\tilde{b} = -2\sqrt{\tilde{c}}. \tag{6.2}$$

This is therefore a transition from a one-cut (disc) phase to a two-cut (annulus) phase [168,169]. See also [172,173].

- A transition between uniform ordered $< M >\sim \mathbf{1}_N$ and non-uniform ordered $< M >\sim \gamma$ phases. The non-uniform phase, in which translational/rotational invariance is spontaneously broken, is absent in the commutative theory. The non-uniform phase is essentially the stripe phase observed originally on Moyal-Weyl spaces in [174,175].

Thus, the uniform ordered phase $< \Phi >\sim \mathbf{1}_N$ is stable in the theory (6.1). This fact is in contrast with the case of the real quartic matrix model $V = \mathrm{Tr}_H(bM^2 + cM^4)$ in which this solution becomes unstable for all values of the couplings. The source of this stability is obviously the addition of the kinetic term to the action.

The non-uniform ordered phase [176] is a full blown nonperturbative manifestation of the perturbative UV-IR mixing effect [177] which is due to the underlying highly non-local matrix degrees of freedom of the noncommutative scalar field.

The above picture of the phase diagram holds for noncommutative phi-four in any dimension, and the three phases are all stable, and are expected to meet at a triple point. The phase structure in four dimensions was discussed using the Hartree-Fock approximation in [174] and studied by means of the Monte Carlo method, employing the fuzzy torus [178] as regulator, in [175].

In two dimensions the noncommutative phi-four theory is renormalizable [179]. The regularized theory on the fuzzy sphere [130,131] is given by the action (6.1) with a finite dimensional Hilbert space H of size N and a Laplacian $\Delta = [L_a, [L_a, ..]]$ where L_a are the generators of $SU(2)$ in the IRR of spin $(N-1)/2$.

The above phase structure was confirmed in two dimensions by means of Monte Carlo simulations on the fuzzy sphere in [180,181]. Indeed, fuzzy scalar phi-four theory enjoys three stable phases: i) disordered (symmetric, one-cut, disk) phase, ii) uniform ordered (Ising, broken, asymmetric one-cut) phase and iii) non-uniform ordered (matrix, stripe, two-cut, annulus) phase.

The phase diagram is shown on the two graphs of figure (6.1) which were generated using the Metropolis algorithm.

The problem of the phase structure of fuzzy phi-four was also studied by means of the Monte Carlo method in [182–186]. The analytic derivation of the phase diagram of noncommutative phi-four on the fuzzy sphere was attempted in [127, 187–193].

The related problem of Monte Carlo simulation of noncommutative phi-four on the fuzzy torus, and the fuzzy disc was considered in [175], [194], and [195] respectively. For a recent study see [196].

In [186] the phase diagram of fuzzy phi-four theory was computed by Monte Carlo sampling of the eigenvalues λ_i of the scalar field M. This was possible by coupling the scalar field M to a $U(1)$ gauge field X_a on the fuzzy sphere which then allowed us, by employing the $U(N)$ gauge symmetry, to reduce scalar phi-four theory to only its eigenvalues. The pure gauge term is such that the gauge field X_a is fluctuating around $X_a = L_a$.

Another powerful method which allows us to reduce noncommutative scalar phi-four theory to only its eigenvalues, without the additional dynamical gauge field, is the multitrace approach. The multitrace approach was initiated in [187, 188]. See also [193] for a review and an extension of this method to the noncommutative Moyal-Weyl plane. For an earlier approach see [127] and for a similar more non-perturbative approach see [189–192]. The multitrace expansion is the analogue of the Hopping parameter expansion on the lattice in the sense that we perform a small kinetic term expansion, i.e. expanding in the parameter a of (6.1), while treating the potential exactly. This should be contrasted with the small interaction expansion of the usual perturbation theory. The effective action obtained in this approach is a matrix model which can be expressed solely in terms of the eigenvalues λ_i and which, on general grounds, can only be a function of the combinations $T_{2n} \propto \sum_{i\neq j}(\lambda_i - \lambda_j)^{2n}$. To the lowest non-trivial order we get an effective action of the form [188, 189, 193]

$$\begin{aligned} S_{\rm eff} &= \sum_i (b\lambda_i^2 + c\lambda_i^4) - \frac{1}{2}\sum_{i\neq j}\ln(\lambda_i-\lambda_j)^2 \\ &+ \Big[\frac{aN}{4}v_{2,1}\sum_{i\neq j}(\lambda_i-\lambda_j)^2 + \frac{a^2N^2}{12}v_{4,1}\sum_{i\neq j}(\lambda_i-\lambda_j)^4 - \frac{a^2}{6}v_{2,2}\big[\sum_{i\neq j}(\lambda_i-\lambda_j)^2\big]^2 + ...\Big]. \end{aligned} \tag{6.3}$$

The logarithmic potential arises from the Vandermonde determinant, i.e. from diagonalization. The coefficients $v_{2,1}$, $v_{4,1}$ and $v_{2,2}$ are given by $v_{2,1} = +1$, $v_{4,1} = 0$, $v_{2,2} = 1/8$. Furthermore, it is not difficult to convince ourselves that the above action is a multitrace matrix model since it can be expressed in terms of various moments $m_n = TrM^n$ of the matrix M.

The original multitrace matrix model written down [187] comes with different values of v's and therefore, in the commutative limit $N \longrightarrow \infty$, it corresponds to a phi-four theory on the sphere modulo multi-integral terms.

Since these multitrace matrix models depend only on N independent eigenvalues their Monte Carlo sampling by means of the Metropolis algorithm does not suffer from any ergodic problem. The phase diagrams of these models obtained in Monte Carlo simulations will be reported elsewhere.

The remainder of this article is organized as follows:

1. Section 2: We describe our proposal for how fuzzy geometry can emerge in generic multitrace matrix models.

2. Section 3: We apply our proposal to an explicit example. We will show that if the multitrace matrix model under consideration does not sustain the uniform ordered phase then there is no emergent geometry. On the other hand, if the uniform ordered phase is sustained then there is an underlying or emergent geometry. In particular, we will show how

    - i) to determine the dimension from the critical exponents of the uniform-to-disordered (Ising) phase transition, and how
    - ii) to determine the metric (Laplacian, propagator) from the Wigner semicircle law behavior of the eigenvalues distribution of the matrix M.

3. Section 4: We conclude by giving a straightforward generalization to fuzzy $\mathbf{CP}^n$ and fuzzy $\mathbf{T}^n$.

6.2 The Proposal

We start with a general multitrace matrix model rewritten in terms of the moments TrM^n with generic parameters B, C, D, $B^{'}$, $C^{'}$, $D^{'}$, $A^{'}$,...as

$$V = BTrM^2 + CTrM^4 + D\Big[TrM^2\Big]^2 + B^{'}(TrM)^2 + C^{'}TrMTrM^3 + D^{'}(TrM)^4 + A^{'}TrM^2(TrM)^2 + ... \quad (6.4)$$

This action includes the noncommutative phi-four model on the fuzzy sphere (6.3) and the multitrace matrix model of [187] as special cases. It also includes as special cases the multitrace matrix models obtained by expanding the kinetic term on i) fuzzy $\mathbf{CP}^n$ [188, 197], on ii) Moyal-Weyl spaces with and without the harmonic oscillator term [193], and on iii) fuzzy tori [178].

The phase diagram of the action (6.4) will generically contain the matrix one-cut-to-two-cut transition line separating the two stables phases of disorder and non-uniform-order. However, the uniform ordered phase will typically be unstable as in the case of the real quartic matrix model

$$V = BTrM^2 + CTrM^4. \quad (6.5)$$

Our proposal goes as follows. We can check for a possible emergence of geometry in the multitrace matrix model (6.4) by following the three steps:

1. We compute the phase diagram of the model (6.4). If the uniform ordered phase remains unstable as in the case of the real quartic matrix model (6.5) then there is no geometry and the model is just a trivial deformation of (6.5). In the opposite case we claim that there is an underlying, i.e. emergent, geometry with a well defined dimension (step 2) and a well defined Laplacian/metric (step 3). This means that we can rewrite the multitrace matrix model, in the region of the phase diagram where the uniform ordered phase exists, in terms of a scalar function and a star product with a noncommutativity parameter θ by finding the appropriate Weyl map. As a consequence, a small noncommutativity parameter expansion can be performed and the the limit $\theta \longrightarrow 0$ can be taken. The disordered-to-uniform-ordered phase transition reduces therefore to the usual 2nd order Ising phase transition on the underlying geometry.

2. We compute the dimension of the underlying by computing the critical exponents of the disordered-to-uniform-ordered phase transition which, by universality, take specific values in each dimension.

3. We compute the Laplacian by computing the free behavior of the propagator. This is done explicitly by computing the eigenvalues distribution of the matrix M in the free regime, small values of C, and comparing with the Wigner semicircle law behavior which must hold with a specific radius depending crucially on the kinetic term.

6.3 Explicit Example: The Fuzzy Sphere

6.3.1 Phase Diagram

We consider as an example the multitrace matrix model of [187] which comes with the v values $v_{2,1} = -1$, $v_{4,1} = 3/2$, $v_{2,2} = 0$. The action is given explicitly by

$$V = BTrM^2 + CTrM^4 + D\big[TrM^2\big]^2 + B^{'}(TrM)^2 + C^{'}TrMTrM^3 \quad (6.6)$$

The parameters D, $B^{'}$ and $C^{'}$ are constrained as $D = 3N/4$, $B^{'} = \sqrt{N}/2$ and $C^{'} = -N$. The phase diagram of this model is computed by means of Monte Carlo elsewhere. The result is shown on figure (6.2). The details of the corresponding non-trivial lengthy Monte Carlo calculation will be reported elsewhere. As desired we have three stables phases in this particular model meeting at a triple point. In other words, we have established that this multitrace matrix model sustains the uniform ordered phase which is the first requirement.

6.3.2 Dimension from Critical Exponents

The uniform ordered phase is also called the Ising phase precisely because we believe that the corresponding transition to the disordered phase is characterized by the universal critical exponents of the Ising model in two dimensions derived from the Onsager solution. These critical exponents are defined as usual by the following behavior

$$\begin{aligned}
&m/N =<|TrM|>/N \sim (B_c - B)^{\beta} \sim N^{-\beta/\nu} \\
&C_v/N^2 \sim (B - B_c)^{-\alpha} \sim N^{\alpha/\nu} \\
&\chi =<|TrM|^2> - <|TrM|>^2 \sim (B - B_c)^{-\gamma} \sim N^{\gamma/\nu} \sim N^{2-\eta} \\
&\xi \sim |B - B_c|^{-\nu} \sim N.
\end{aligned} \tag{6.7}$$

There are in total six critical exponents, the above five plus the critical exponent δ which controls the equation of state, but only two are truly independent because of the so-called scaling laws. The Onsager solution of the Ising model in two dimensions gives the following celebrated values [198]

$$\nu = 1\ ,\ \beta = 1/8\ ,\ \gamma = 7/4\ ,\ \alpha = 0\ ,\ \eta = 1/4\ ,\ \delta = 15. \tag{6.8}$$

This fundamental result is very delicate to check explicitly in the Monte Carlo data. Since we must necessarily deal with the critical region we must face the two famous problems of finite size effects and critical slowing down. In this particular problem, the critical slowing down problem can be shown to start appearing in Monte Carlo simulations around $N > 60$ so we will keep below this value and employ very large statistics of the order of 2^{20} to avoid it. A more systematic solution to this problem is to employ the Wolf algorithm [199] which we do not attempt here. We simply employ here the ordinary Metropolis algorithm. The problem of finite size effects is also very serious for the measurement of the critical exponents since the above behavior (6.7) is supposed to hold only for large N. This problem can be avoided by not including values of N less than 20 and thus below we will quote for completeness $N = 10$ and $N = 15$ data but, in most cases, we will not take them into account in the fitting.

Since the Ising model appears from the Φ^4 theory for large values of the quartic coupling it is preferable to use values of $\tilde{C}$ as large as possible. However, we are limited from above by the appearance of the different physics of the transition between the disordered and non-uniform-ordered phases around $\tilde{C} = 1.5$. Thus, we choose $\tilde{C} = 1.0$ which is relatively large but well established to be within the Ising transition with an extrapolated critical point around $\tilde{B} = -3.07$ (see below). The critical behavior of the magnetization, susceptibility and specific heat around the critical value of $\tilde{B} = -3.10$ is shown on figure (6.4). We attach in table (6.1) some data relevant for the computation of the critical exponents ν, β, γ and α. The other critical exponents can be determined via scaling laws.

The measurements of the critical exponents ν, β, γ and α proceeds as follows:

- **Critical Point and The Critical Exponent ν:** By plotting the critical point $\tilde{B}_c$ obtained for each N versus N (first and second columns of table (6.1)) we get immediately both the $N = \infty$ critical point and the critical exponent ν. We obtain (see figure (6.7))

$$\tilde{B}_c = -1.061(168).N^{-0.926(83)} - 3.074(6) \Rightarrow \ , \ \nu = 0.926(83). \quad (6.9)$$

Also we obtain

$$\tilde{B}_* = -3.074(6). \quad (6.10)$$

This prediction for ν agrees reasonably well with the Onsager calculation. In the following we will assume for simplicity that $\nu = 1$. The above fit is the only instance in which we have included $N = 10$ and $N = 15$ and thus we believe that the obtained value of $-\tilde{B}_*$ is an underestimation of the true critical point.

- **Magnetization and The Critical Exponent β:** The magnetization and the zero power are defined by

$$m = < |TrM| > \ , \ \chi = < |TrM|^2 > - < |TrM| >^2 . \quad (6.11)$$

$$P_0 = < (\frac{1}{N} TrM)^2 > . \quad (6.12)$$

Measurements of the magnetization m/N were performed near the extrapolated critical point $\tilde{B} = -3.07$ for $\tilde{C} = 1.0$ but inside the uniform ordered phase. These are then used to compute the critical exponent β by searching for a power law behavior.

More precisely, we measure $\ln(m/N)$ versus $\ln N$ for each value of $\tilde{B}$ very near and around $\tilde{B} = -3.10$, fit to a straight line in the range $20 \leq N \leq 60$ and compute the slope β, then search for the flattest line, i.e. the smallest slope β. This value marks the transition from the Ising phase to the disordered phase. Deeep inside the Ising phase the slope should approach the mean field value $-1/4$ which can be shown from the scaling behavior of the dominant configuration. After determining the critical value we then consider the value of $\tilde{B}$ nearest to it but within the Ising phase and take the slope there to be the value of the critical exponent β. In our example here, the flattest line occurs at $\tilde{B} = -3.13$ with slope $-0.088(10)$ after which the slope becomes $-0.109(11)$ at $\tilde{B} = -3.14$. The slope goes fast to the mean field value -0.25 as we keep decreasing $\tilde{B}$. See figure (6.5). Our measured value of the critical point $\tilde{B}_*$ from the magnetization and of the critical exponent β are therefore

$$\tilde{B}_* = -3.13. \tag{6.13}$$

$$\ln \frac{m}{N} = -0.109(11). \ln N - 1.423(43) \Rightarrow \beta = -0.109(11). \tag{6.14}$$

- **Susceptibility and Zero Power and The Critical Exponent γ:** The measurement of the critical exponent γ is quite delicate and will be done indirectly as follows. We rewrite the susceptibility in terms of the zero power and magnetization as

$$\begin{aligned} \chi &= <|TrM|^2> - <|TrM|>^2 \\ &= N^2 P_0 - m^2. \end{aligned} \tag{6.15}$$

The critical exponent γ in terms of the critical exponent γ' of P_0 is then given by

$$\gamma = 2 + \gamma'. \tag{6.16}$$

By using the results shown on table (6.1) at $\tilde{B} = -3.14$, plotted on figure (6.7), we obtain the following exponents

$$\ln P_0 = -0.352(10). \ln N - 2.289(36) \Rightarrow \gamma' = -0.352(10). \tag{6.17}$$

Or equivalently

$$\ln N^2 P_0 = 1.648(10). \ln N - 2.289(36) \Rightarrow \gamma = 1.648(10). \tag{6.18}$$

For consistency we can check that the second term in the susceptibility behaves using the result (6.14) as

$$\ln m^2 = 1.782(22). \ln N - 2.846(86) \Rightarrow \gamma = 1.782(22). \tag{6.19}$$

Our two measurements of the critical exponent γ agree reasonably well with the Onsager values.

If we try to fit the values of the susceptibility at its maximum shown in third column of table (6.1), i.e. at the peak which keeps slowly moving with $\tilde{B}$, then we will obtain a very bad underestimate of the critical exponent γ given by

$$\ln \chi_{\max} = 0.515(08). \ln N - 0.652(30) \Rightarrow \gamma = 0.515(08). \quad (6.20)$$

This in our mind is due in part to the dependence of $\tilde{B}_c$ on N and in another part is an indication of the critical slowing down problem showing up in the measurement of this second moment, i.e. the size of the fluctuations is observed to grow with N at the critical point but not at the correct rate indicated by the independent measurements of the zero moment and the magnetization. See figure (6.7).

- **Specific Heat and The Critical Exponent α:** The sepcific heat is defined by

$$C_v = < S^2 > - < S >^2 . \quad (6.21)$$

The critical point $\tilde{B}_*$ as measured from the specific heat is identified by the intersection point of the various curves with different N shown on figure (6.4). We get

$$\tilde{B}_* = -3.08. \quad (6.22)$$

This measurement is contrasted very favorably with the independent measurement obtained from the extrapolated value of $\tilde{B}_c$ shown in equation (6.10) but should also be contrasted with the measurement obtained from the magnetization shown in equation (6.13).

By using the results shown on table (6.1) at the critical point $\tilde{B} = -3.08$, plotted on figure (6.7), we obtain the following exponent

$$\ln \frac{C_v}{N^2} = 0.024(9). \ln N - 0.623(31) \Rightarrow \alpha = 0.024(9). \quad (6.23)$$

N	$\tilde{B}_c, \tilde{B}_* = -3.07$	χ_c	$(C_v)_*, \tilde{B}_* = -3.08$	$\tilde{B} < \tilde{B}_* = -3.13$	$m_{<*}$	$10^3(P_0)_{<*}$
10	-3.20	1.704(2)	56.467(94)	-3.14	2.1776(12)	6.256(6)
15	-3.16	2.089(2)	129.111(217)	-3.14	2.7750(14)	4.315(4)
20	-3.14	2.436(3)	229.861(389)	-3.14	3.4423(15)	3.571(2)
25	-3.13	2.716(3)	365.183(621)	-3.14	4.1759(16)	3.220(2)
30	-3.12	3.017(4)	524.253(891)	-3.14	4.9772(16)	3.042(2)
36	-3.11	3.283(4)	749.099(1267)	-3.14	5.8878(15)	2.860(1)
40	-3.11	3.515(4)	941.139(1607)	-3.14	6.5134(14)	2.782(1)
50	-3.10	3.864(4)	1461.597(2479)	-3.14	7.9250(12)	2.576(1)
60	-3.10	4.301(5)	2144.929(3658)	-3.14	9.2021(11)	2.388(1)

Table 6.1: Measurements of the magnetization $(m/N)_{<*}$, the susceptibility $\chi_{<*}$, via the zero power $(P_0)_{<*}$, and the specific heat $(C_v/N^2)_*$ used to compute the critical exponents β, γ and α respectively. Here $\tilde{C} = 1.0$, the extrapolated critical point is $\tilde{B} = -3.07$, the critical point as intersection point of curves of specific heat is $\tilde{B} = -3.08$, and the critical point as the flattest line of decrease of magnetization is $\tilde{B} = -3.13$.

6.3.3 Free Propagator from Wigner Semicircle Law

We can also measure the emergent geometry by measuring the free propagator of the theory. This will give us information on both the dimension and the metric since the free propagator is the inverse of the Laplacian Δ which fully encodes the underlying geometry in the sense of [170, 171]. This goes as follows [127].

A noncommutative phi-four on a $d-$dimensional noncommutative Euclidean spacetime $\mathbf{R}^d_\theta$ reads in position representation

$$S = \int d^dx\big(\frac{1}{2}\partial_i\Phi\partial_i\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4}\Phi^4_*\big). \tag{6.24}$$

The first step is to regularize this theory in terms of a finite $\mathcal{N}-$dimensional matrix Φ and rewrite the theory in matrix representation. Then we diagonalize the matrix Φ. The measure becomes $\int\prod_i d\Phi_i\Delta^2(\Phi)\int dU$ where Φ_i are the eigenvalues, $\Delta^2(\Phi) = \prod_{i<j}(\Phi_i - \Phi_j)^2$ is the Vandermonde determinant and dU is the Haar measure. The effective probability distribution of the eigenvalues Φ_i can be determined uniquely from the behavior of the expectation values $< \int d^dx\Phi^{2n}_*(x) >$. These objects clearly depend only on the eigenvalues Φ_i and are computed using a sharp UV cutoff Λ. If we are only interested in the eigenvalues of the scalar matrix Φ then the free theory $\lambda = 0$ can be replaced by the effective matrix model [127]

$$S = \frac{2\mathcal{N}}{\alpha_0^2} Tr\Phi^2. \tag{6.25}$$

This result can be traced to the fact that planar diagrams dominates over the non-planar ones in the limit $\Lambda \longrightarrow \infty$. This means in particular that the eigenvalues Φ_i are distributed according to the famous Wigner semi-circle law with α_0 being the largest eigenvalue, viz

$$\rho(t) = \frac{2}{\pi\alpha_0^2}\sqrt{\alpha_0^2 - t^2} \ , \ \ -\alpha_0 \leq t \leq +\alpha_0. \tag{6.26}$$

In the most important cases of $d = 2$ and $d = 4$ dimensions we have explicitly

$$\alpha_0^2(m,\Lambda) = \frac{1}{4\pi^2}\big(\Lambda^2 - m^2 \ln(1 + \frac{\Lambda^2}{m^2})\big) \ , \ d = 4. \tag{6.27}$$

$$\alpha_0^2(m,\Lambda) = \frac{1}{\pi} \ln(1 + \frac{\Lambda^2}{m^2}) \ , \ d = 2. \tag{6.28}$$

Obviously, dimension four is eliminated by the results of the critical exponents. In two dimensions the regulator Λ originates in only one of two possible noncommutative spaces [127]:

1. **Fuzzy Torus:** As it turns the results on the fuzzy torus are different from those obtained using a sharp momentum cutoff due to the different behavior of the propagator for large momenta and as a consequence the resulting formula for α_0^2 is different from the above equation (6.28). We obtain instead

$$\alpha_0^2(m,\Lambda) = 4\int_0^{\pi} \frac{d^2r}{(2\pi)^2} \frac{1}{\sum_i (1 - \cos r_i) + m^2 l^2/2} \ , \ d = 2. \tag{6.29}$$

l here is the lattice spacing, the noncommutativity is quantized as $\theta = Nl^2/\pi$ and the cutoff is

$$\Lambda = \frac{\pi}{l} = \sqrt{\frac{N\pi}{\theta}}. \tag{6.30}$$

The above behavior can be easily excluded in our Monte Carlo data and by hindsight we know that this should be indeed so because the original multitrace approximation is relevant to the fuzzy sphere.

2. **Fuzzy Sphere:** The fuzzy sphere $\mathbf{S}^2_N = \mathbf{CP}^1_N$ is the simplest of fuzzy projective spaces $\mathbf{CP}^n_N$. In this case $\mathcal{N} = N + 1$ and the scalar field Φ becomes an $N \times N$ matrix ϕ given by $\phi = \sqrt{2\pi/Na}\Phi$. In this case the cutoff is given in terms of the matrix size N and the radius R of the sphere by

$$\Lambda = \frac{N}{R}. \tag{6.31}$$

Also, in this case the mass parameters B and m^2 are related by

$$m^2 = \frac{b}{aR^2}. \tag{6.32}$$

By using $\tilde{B} = B/N^{3/2}$ and choosing $a = 2\pi/N$, so that $\Phi = \phi$, we obtain

$$\frac{\Lambda^2}{m^2} = \frac{2\pi}{\sqrt{N}\tilde{B}}. \tag{6.33}$$

We get then

$$\alpha_0^2(m,\Lambda) = \frac{1}{\pi}\ln(1 + \frac{2\pi}{\sqrt{N}\tilde{B}}). \tag{6.34}$$

In the limit $B \longrightarrow \infty$ we get the one-cut $\delta^2 = 2N/B$ of the Gaussian matrix model $BTrM^2$, viz $B = 2\mathcal{N}/\alpha_0^2$. This can also be obtained by taking the limit $B \longrightarrow \infty$ of the one-cut (deformed Wigner semicircle law) solution

$$\rho(\lambda) \;=\; \frac{1}{N\pi}(2C\lambda^2 + B + C\delta^2)\sqrt{\delta^2 - \lambda^2}\ ,\ \delta^2 = \frac{1}{3C}(-B + \sqrt{B^2 + 12NC}) \tag{6.35}$$

of the quadratic matrix model $BTrM^2 + CTrM^4$.

This result was also generalized in [191]. The eigenvalues distribution of a free scalar field theory on the fuzzy sphere with an arbitrary kinetic term, viz $S = Tr(M\mathcal{K}M + BM^2)/2$, where $\mathcal{K}(0) = 0$ and $\mathcal{K}$ is diagonal in the basis of polarization tensors T^l_m, is always given by a Wigner semicircle law with a radius

$$R^2 = \delta^2 \;=\; \alpha_0^2 = \frac{4f(B)}{N}\ ,\ f(B) = \sum_{l=0}^{N-1}\frac{2l+1}{\mathcal{K}(l) + B}. \tag{6.36}$$

Some Monte Carlo results are shown on figures (6.9) and (6.10). These are obtained in Monte Carlo runs with 2^{20} thermalization steps and 2^{18} thermalized configurations where each two configurations are separated by 2^4 Monte Carlo steps in order to reduce auto-correlation effects. We consider $N = 20-40$, $\tilde{C} = 0.05-0.35$ and $\tilde{B} = 0-5$.

It is not difficult to convince ourselves that the mass parameter B is precisely the mass squared in this regime. For each value of $(N, \tilde{C}, \tilde{B})$ we compute the eigenvalues distribution $\rho(\lambda)$ and fit it to the Wigner semicircle law (6.26) (see figure (6.9)). We obtain thus a measurement of the radius of the Wigner semicircle law $\delta^2 = \alpha_0^2 = R^2$. We have checked carefully that in this regime the Wigner semicircle law is the appropriate behavior rather than the one-cut solution (6.35) as evidenced by the first graph in figure (6.9). The measurement of the radii δ^2 for various values of $\tilde{B}$ is then plotted and compared with the expected theoretical behaviors (6.34) as well as with the $B \longrightarrow \infty$ behavior $\delta^2 = 2N/B$ (see figure (6.10)). The agreement with (6.34) is very reasonable with some deviation for small values of $\tilde{B}$ as we approach the non-perturbative region where the uniform ordered phase appears at some $\tilde{B} < 0$. This discrepancy for small values of $\tilde{B}$ is already seen on figure (6.9) when we fit the distributions to the Wigner semicircle law. However, this effect is reduced as we decrease the value of $\tilde{C}$.

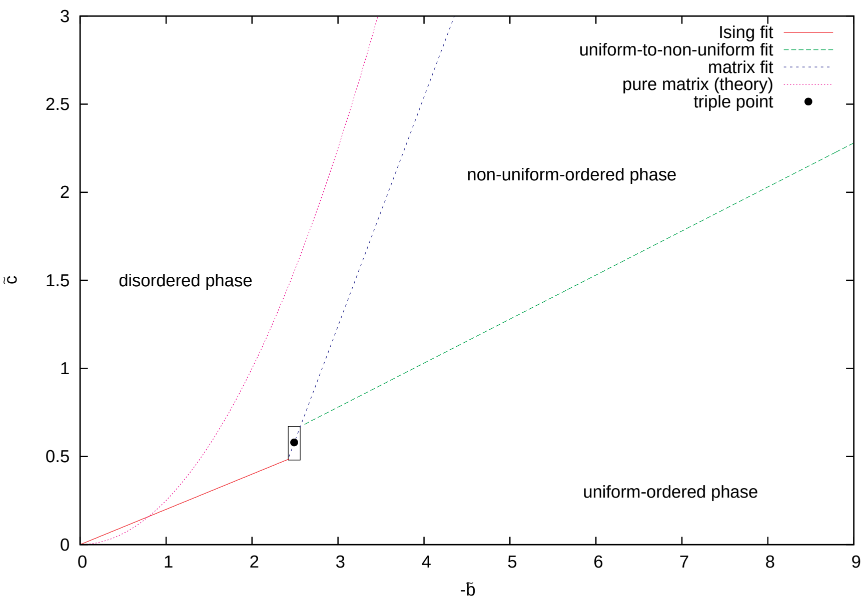
In summary we conclude that we are indeed dealing with the geometry of the fuzzy sphere and, given hindsight, we know that this should be true.

6.4 Generalization and Conclusion

The emergence of geometry in the very early universe is a problem of fundamental importance to our understanding of quantum gravity and cosmology. In this letter, we have proposed a novel scenario for the emergence of geometry in random multitrace matrix models which depend on a single hermitian matrix M with full unitary $U(N)$ invariance and without any kinetic term. Thus, the model under consideration has no geometry a priori precisely because of the absence of a kinetic term. On the other hand, previous proposals of emergent geometry required the input of several matrices with some rotational symmetry group besides the $U(N)$ gauge symmetry [140].

Our proposal consists in checking whether or not the uniform ordered phase is sustained by the multitrace matrix model under consideration. If yes, then the dimension of the underlying geometry, in the region of the phase diagram where the uniform ordered phase is stable, can be inferred from the values of the critical exponents of the Ising phase transition. Whereas, the metric/Laplacian of this geometry can be inferred from the behavior of the free propagator encoded in the Wigner semicircle law behavior of the eigenvalues distribution of the matrix M in the weakly coupled regime. An explicit example is given in which the geometry of the fuzzy sphere emerges, with all the correct properties, in the phase diagram of a particular multitrace matrix model containing multitrace terms depending on the moments $m_1 = TrM$, $m_2 = TrM^2$ and $m_3 = TrM^3$ in a particular way [187].

This idea can be generalized in a straightforward way to all higher fuzzy projective spaces $\mathbf{CP}^n$ and fuzzy tori $\mathbf{T}^n$ by tuning appropriately the coefficients of the multitrace matrix model and/or including higher moments in the multitrace matrix model .



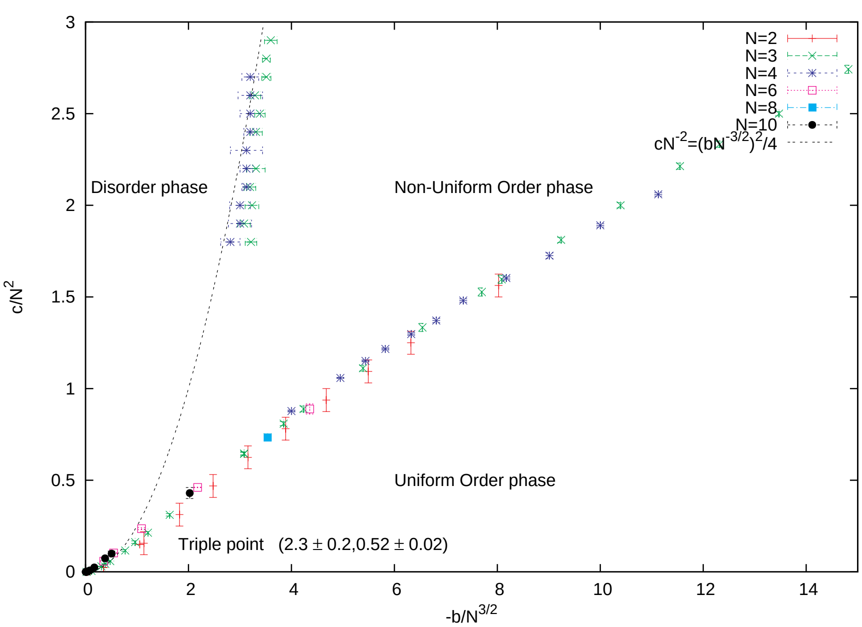


Figure 6.1: The phase diagram of noncommutative phi-four theory on the fuzzy sphere. In the first figure the fits are reproduced from actual Monte Carlo data [186]. Second figure reproduced from [180] with the gracious permission of D. O'Connor.

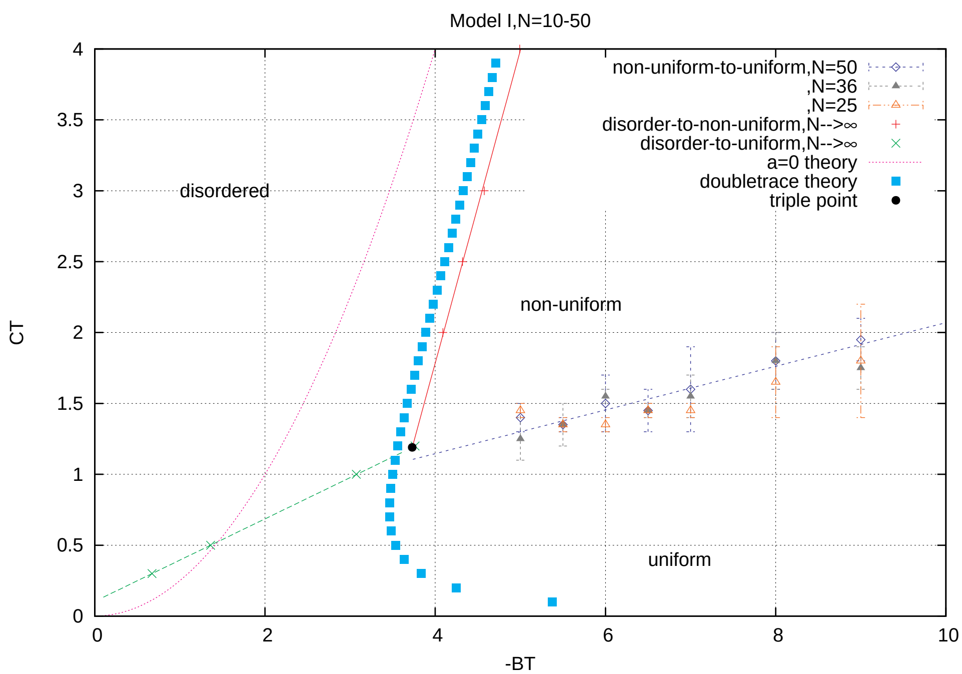


Figure 6.2: The phase diagram of the multitrace matrix model of [187]. The Ising and matrix transition data points are not shown but we only indicate their extrapolated fits whereas the $N = 25$, $N = 36$ and $N = 50$ stripe data points are included explicitly.

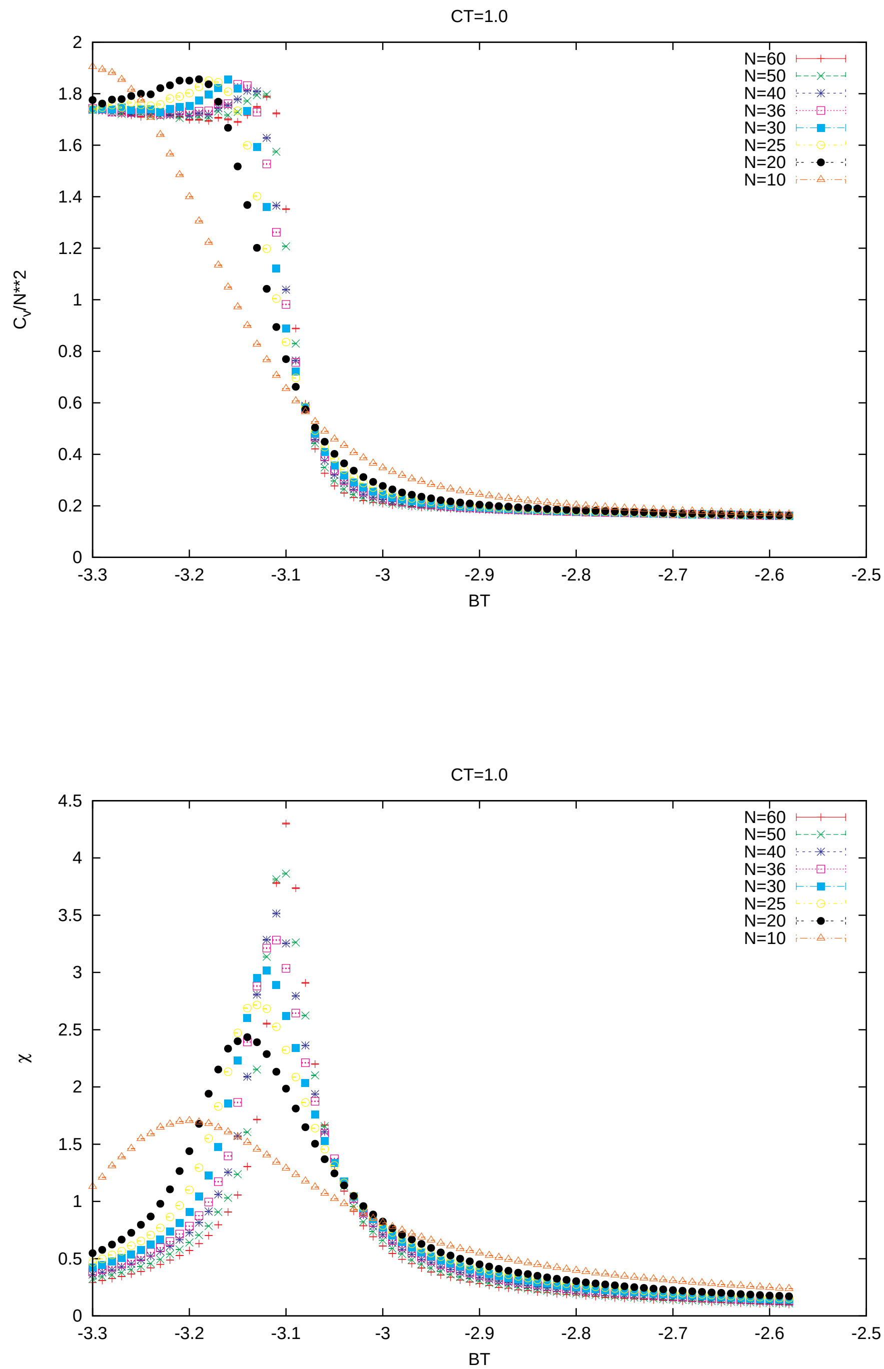


Figure 6.3: The Ising critical behavior.

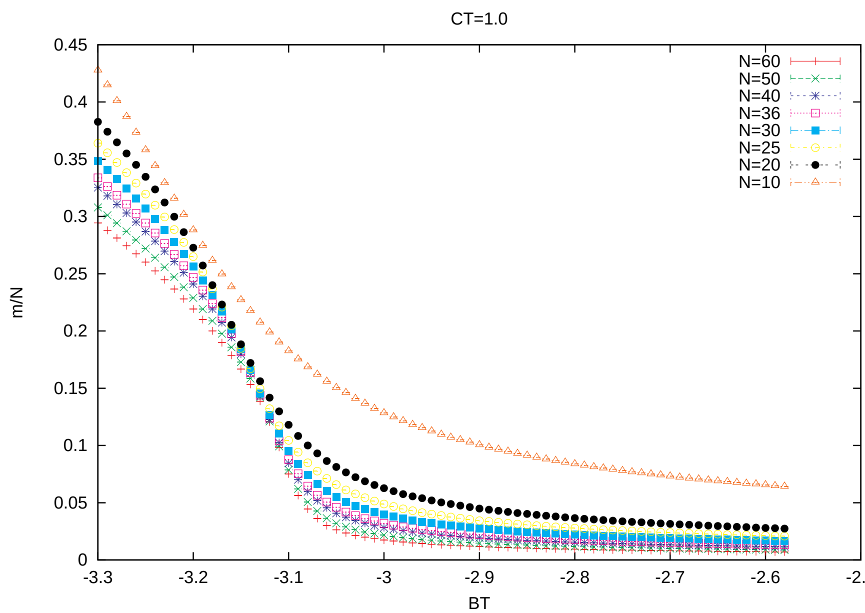


Figure 6.4: The Ising critical behavior.

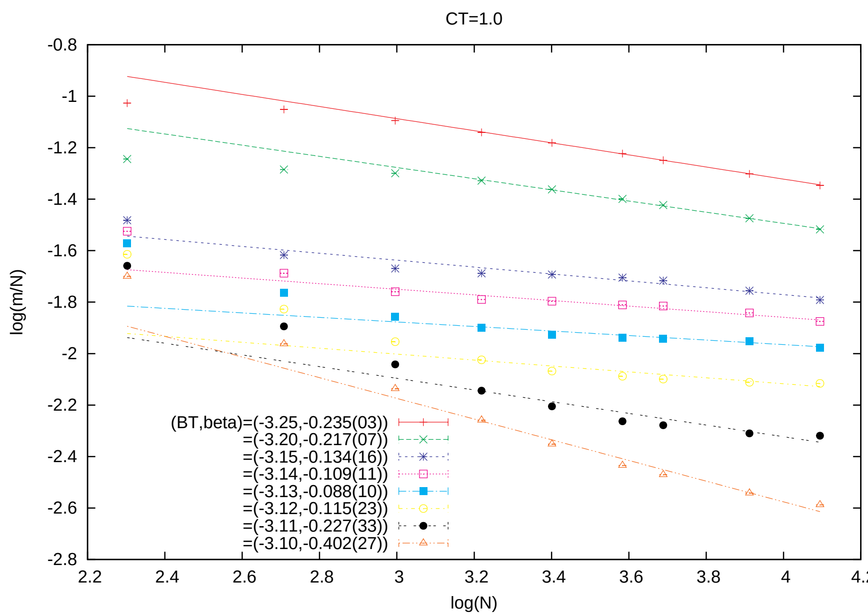
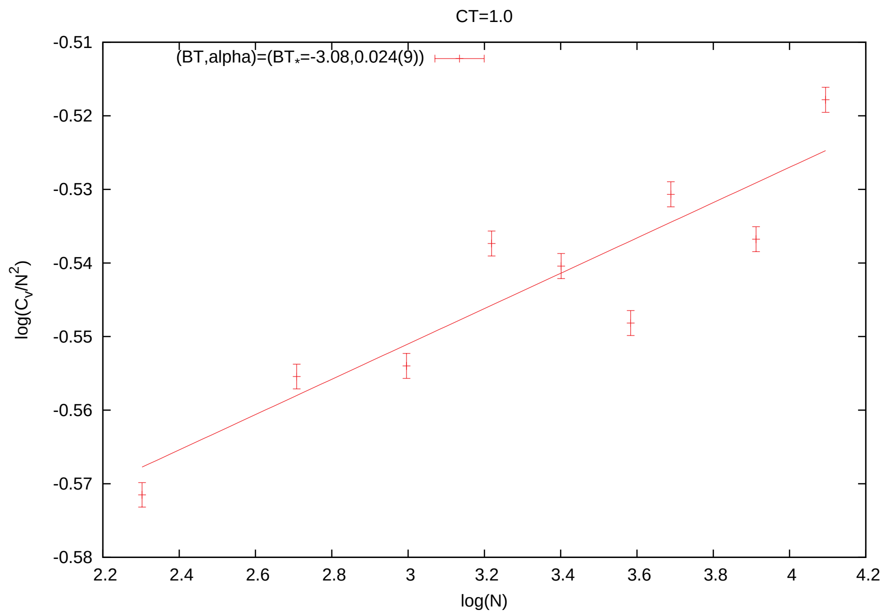


Figure 6.5: The critical exponent β.



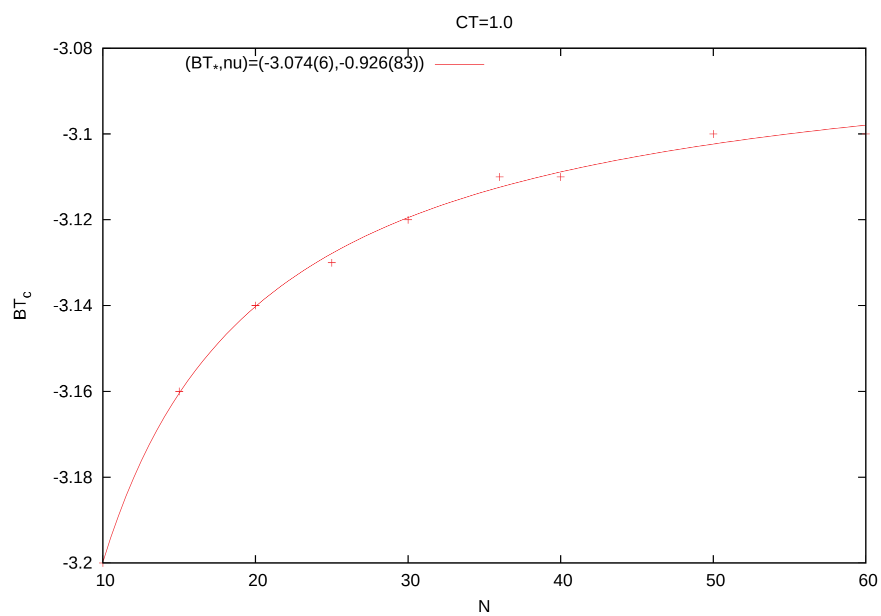


Figure 6.6: The critical exponents ν, γ and α.

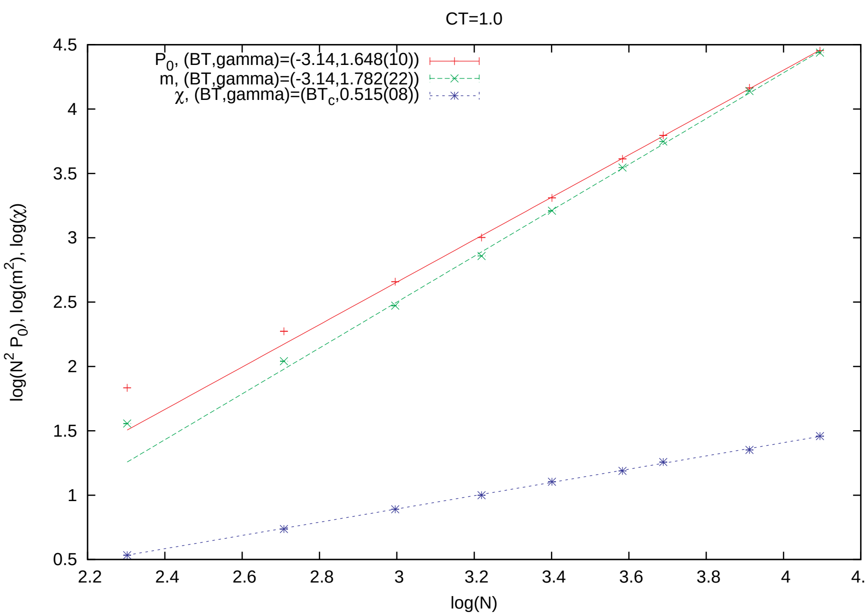
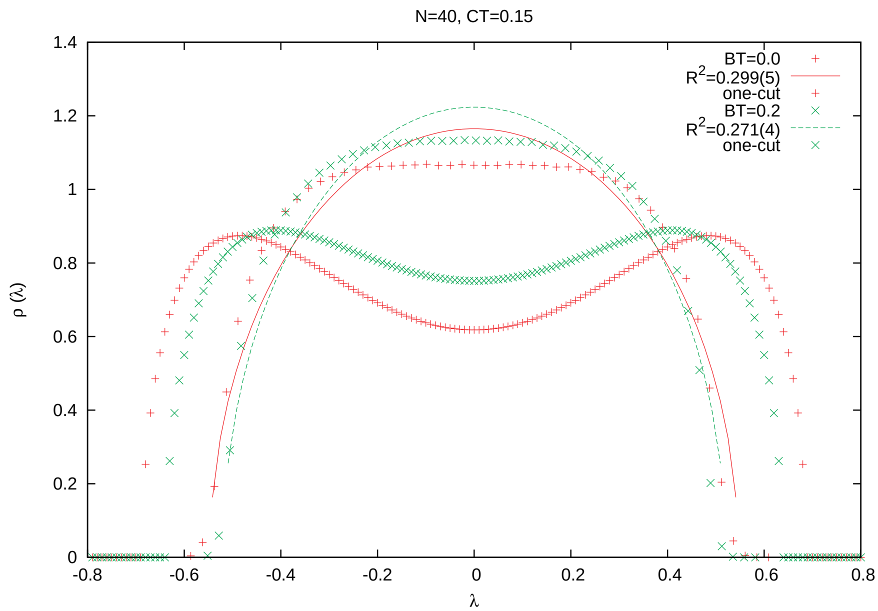


Figure 6.7: The critical exponents ν, γ and α.



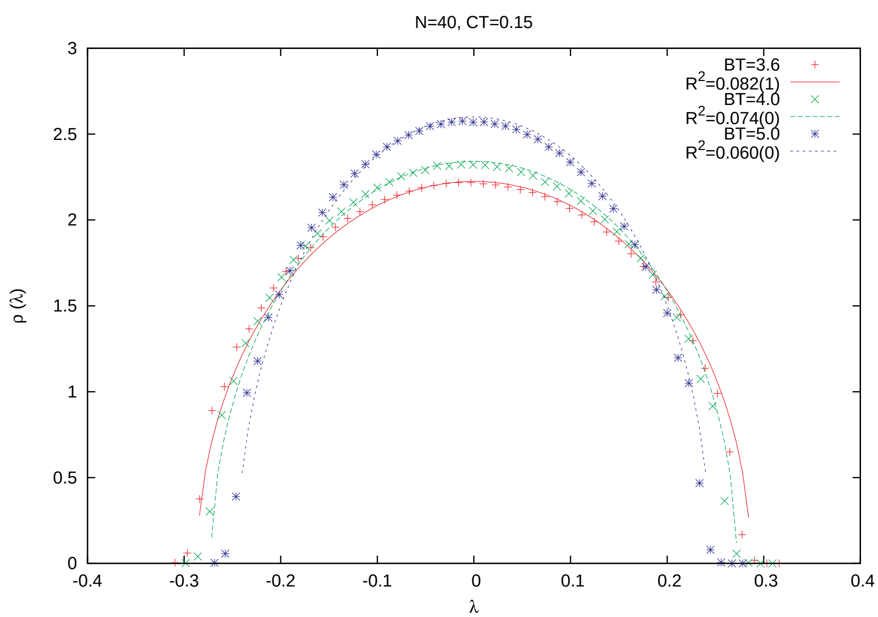
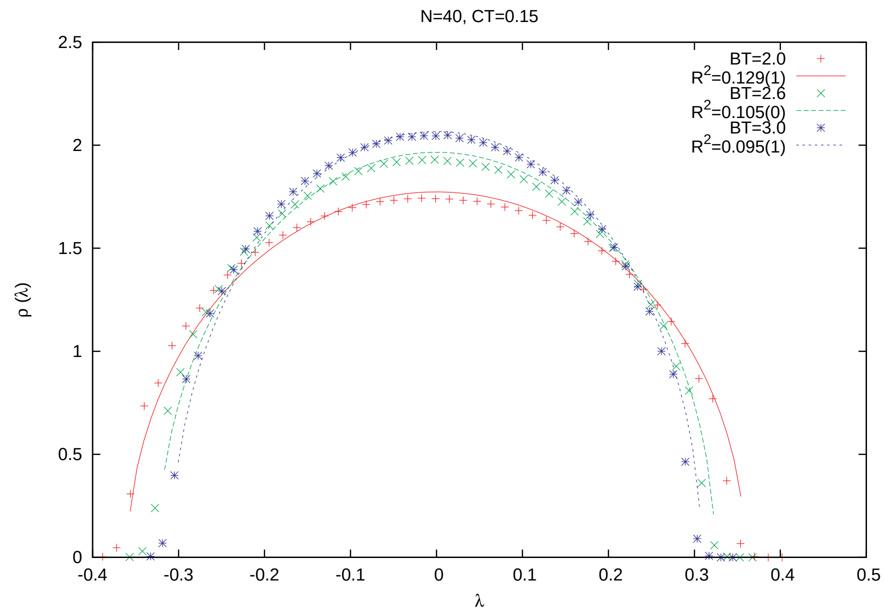


Figure 6.8: The semicircle law as a function of $\tilde{B}$.



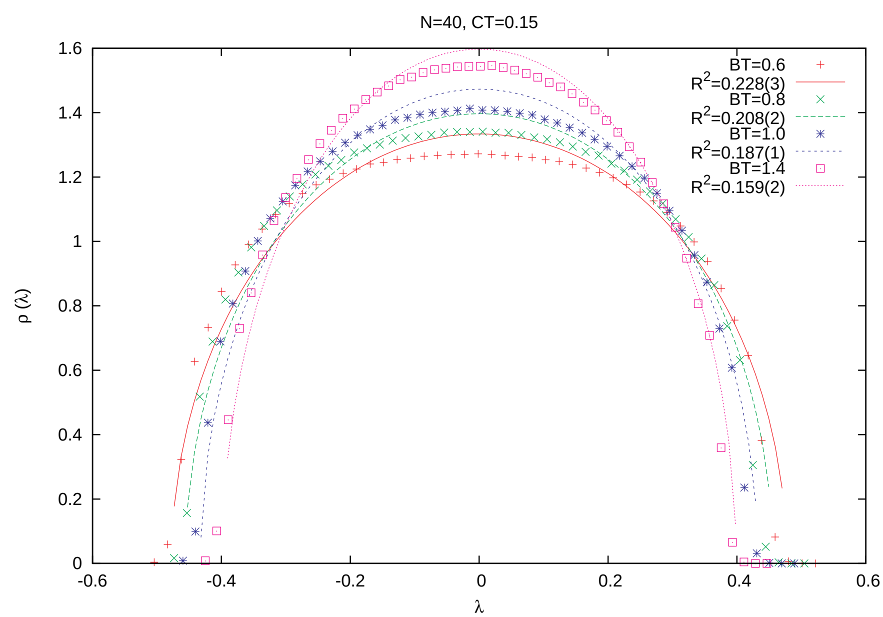
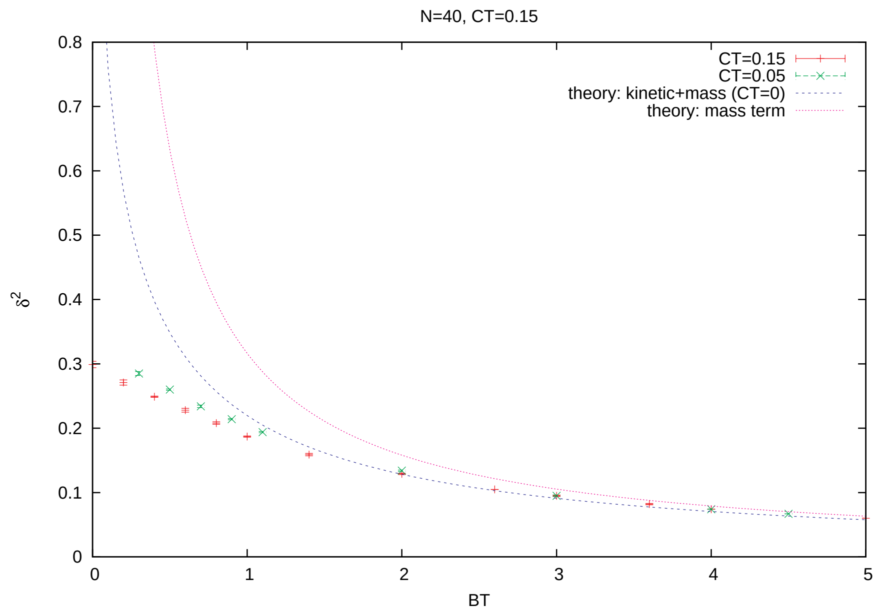


Figure 6.9: The semicircle law as a function of $\tilde{B}$.



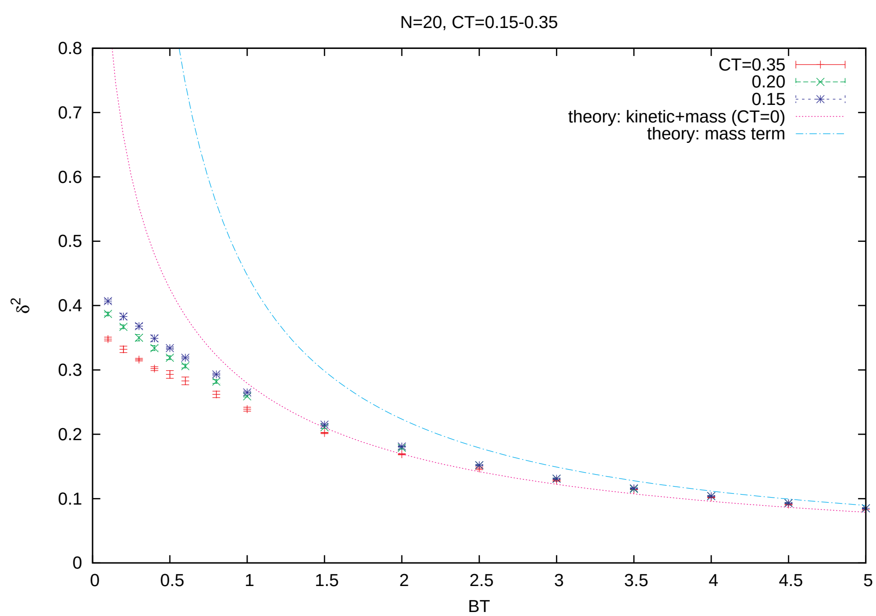


Figure 6.10: The behavior of the radius of the Wigner semicircle law $\delta^2 = \alpha_0^2$ as a function of $\tilde{B}$.

Chapter 7

Gauge Theory on The Noncommutative Torus

The study of massless field theories on a torus is of great interest in the noncommutative case because the compactness of the spacetime gives a natural infrared regularization of the theory. One may therefore analyse more carefully the ultraviolet behaviour and also the new light degrees of freedom which are responsible for the UV/IR mixing. From a more mathematical point of view, the noncommutative torus constitutes one of the original examples in noncommutative geometry [200–203] which captures the essential topological changes which occur when one deforms a compact space. It is perhaps the most basic example which still contains a rich geometrical structure. In this section we shall describe some basic aspects of the noncommutative torus with particular emphasis on the properties of vector bundles defined over them. From the study of the global properties of gauge theories defined on this space.

7.1 The Noncommutative Torus

Most of what we have said about noncommutative quantum field theory is true when $\mathcal{R}^D$ is replaced by a D dimensional torus $\mathbf{T}^D$, with only subtle changes that we shall now explain. Let $\sum_a^i$ be the $D \times D$ period matrix of $\mathbf{T}^D$ which is a vielbein for its metric, i.e.$\sum_a^i \delta^{ab} \sum_b^j = G^{ij}$. Here and in the following the indices $i, j, ...$ will label spacetime directions while $a, b, ...$ will denote indices in the frame bundle of $\mathbf{T}^D$. The matrices $\sum_a^i$ parametrize the moduli of D dimensional tori and they may be regarded as maps from the frame bundle to the tangent bundle of $\mathbf{T}^D$. They define the periods of the directions of $\mathbf{T}^D$,

$$x^i \sim x^i + \sum_a^i \quad , \quad a = 1, ..., D \tag{7.1}$$

for each $i = 1, ..., D$. When $\sum_i^a$ is not proportional to δ_i^a , the identifications (7.1) for $a \neq i$ describe how the torus is tilted in its parallelogram representation.

Smooth functions on the torus must be single-valued, which implies that the corre- sponding Fourier momenta $\vec{k}$ are quantized as

$$k_i = 2\pi \left(\sum^{-1} \right)_i^a m_a, \qquad m_a \in \mathcal{Z} \tag{7.2}$$

Therefore, to describe the deformation of the function algebra, one cannot use the unbounded operators $\hat{x}^i$ obeying (1.1).

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \tag{7.3}$$

Instead, one must restrict to the proper subalgebra of the algebra of noncommutative $\mathcal{R}^D$ that is generated by the $Weylbasis$ of unitary operators

$$\hat{Z}^a = e^{2\pi i (\sum^{-1})_i^a \hat{x}^i} \tag{7.4}$$

which generate the algebra

$$\hat{Z}^a \hat{Z}^b = e^{-2\pi i \Theta^{ab}} \hat{Z}^b \hat{Z}^a, \tag{7.5}$$

where

$$\Theta^{ab} = 2\pi \left(\sum^{-1} \right)_i^a \theta^{ij} \left(\sum^{-1} \right)_j^b \tag{7.6}$$

are the corresponding dimensionless noncommutativity parameters. The commutation relations (7.5) define the algebra of functions on the noncommutative torus. Formally, if $L \cong \mathcal{Z}^D$ is the lattice of rank$D$ (with bilinear form G_{ij}) which generates the torus as the quotient space $\mathbf{T}^D = \mathcal{R}^D/\mathcal{L}$, then the projective regular representations $\mathcal{L}_\Theta$ in (7.5) of the lattice group $\mathcal{L}$

are labelled by an element Θ^{ab} of the second Hochschild cohomology group $H^2(\mathcal{L}, U(1))$ This latter characterization can be generalized to describe other sorts of noncommutative compactifications of $\mathcal{R}^D$ [204].

Any function on $\mathbf{T}^D$ can be expanded as a Fourier series

$$f(x) = \sum_{\vec{m}\in\mathcal{Z}^D} f_{\vec{m}} e^{2\pi(\sum^{-1})^a_i m_a x^i} \tag{7.7}$$

The corresponding Weyl algebra is generated by the operators () and Weyl quantization takes the form of the map

$$\hat{\mathcal{W}}[f] = \int d^D x f(x)\hat{\Delta}(x), \tag{7.8}$$

where the integration is taken over $\mathbf{T}^D$ and

$$\hat{\Delta}(x) = \frac{1}{|\det \sum|} \sum_{\vec{m}\in\mathcal{Z}^D} \prod_{a=1}^{D} \left(\hat{Z}^a\right)^{m_a} \prod_{a<b} e^{-\pi i m_a \Theta^{ab} m_b} e^{-2\pi i(\sum^{-1})^a_i m_a x^i} \tag{7.9}$$

is a periodic field operator,

$$\hat{\Delta}(x + \sum_a^i \hat{i}) = \hat{\Delta}(x), \quad a = 1, ..., D, \tag{7.10}$$

with $\hat{i}$ a unit vector in the i-th direction of spacetime. Like on $\mathcal{R}^{\mathcal{D}}$, we may introduce anti-Hermitian, commuting linear derivations $\hat{\partial}_i$ which on the noncommutative torus are defined by their actions on the Weyl basis,

$$[\hat{\partial}_i, \hat{Z}^a] = 2\pi i \left(\sum^{-1}\right)^a_i \hat{Z}^a. \tag{7.11}$$

The basis (7.9) then has the requisite property

$$[\hat{\partial}_i, \hat{\Delta}(x)] = -\partial_i \hat{\Delta}(x) \tag{7.12}$$

7.2 Topological Quantum Numbers

A $U(N)$ noncommutative Yang-Mills theory on the torus $\mathbf{T}^D$ can be constructed in much the same way as we did in the previous section. If we restrict to gauge field configurations which are single-valued functions on $\mathbf{T}^D$, then everything we have said goes through with- out a hitch, with single-valued star-unitary functions $g(x)$ parametrizing the star-gauge transformations

$$A_i(x) \to g(x) \star A_i(x) \star g(x)^\dagger - ig(x) \star \partial_i g(x)^\dagger \tag{7.13}$$

The only difference which arises is that, like in the commutative case, there are extra observables associated with the non-trivial homotopy of the torus. The most general star-gauge invariant observable is still given by [67]

$$\mathcal{O}(C_v) \int d^D x \ \mathrm{tr}_N(\mathcal{U}(X; C_v)) \star e^{ik(v)x^i}, \tag{7.14}$$

but now there is a larger set of line momenta. Because the momenta are now quantized as in (7.2), the identification of the translation vector v in

$$e^{ik_i(v)x^i} \star g(x) \star e^{-ik_i(v)x^i} = g(x+v), \tag{7.15}$$

is ambiguous up to an integer translation of the periods of $\mathbf{T}^D$, and the relationship

$$k_i(v) = (\theta^{-1})_{ij} v^j, \tag{7.16}$$

is now modified to

$$v^i \theta^{ij} k_j(v,n) + \sum_a^i n^a \tag{7.17}$$

for arbitrary integer-valued vectors n^a . When $\theta = 0$, the relationship (7.17) reproduces the well-known result that the only open line observables in ordinary Yang-Mills theory are those which are associated with loops that wind n^a times around the a-th non-contractible cycle of the torus. Therefore, we obtain the analog of Polyakov lines in noncommutative Yang-Mills theory associated with the different homotopy classes of the torus [67].

More interesting things happen, however, when we consider gauge field configurations of non-vanishing topological charge on the noncommutative torus. An elegant way to keep track of the quantum numbers associated with topologically non-trivial gauge fields is through their Chern numbers. In the commutative case, these would be represented by the integers

$$\mu_{(n)} = \oint \mathrm{tr}_N F^n/(2\pi)^n \tag{7.18}$$

defined in terms of the curvature two-form F of some gauge connection of a $U(N)$ gauge bundle E over $\mathbf{T}^D$, and suitably integrated over cycles of the torus. For $n = 0$ they produce the rank N of the vector bundle E, for $n = 1$ they yield the fluxes Q ab of the gauge fields through the surface formed by the a-th and b-th cycles of $\mathbf{T}^D$, and for $n = 2$ they give the instanton number k of the bundle E when $D = 4$. We can collect these integers into the inhomogeneous Grassmann form

$$\mathrm{ch}_0(E) = N + \sum_{n=1}^{d} \frac{1}{n!}\mu(n)(E)_{a_1 \ldots a_{2n}} \rho^{a_1} \ldots \rho^{a_{2n}} \tag{7.19}$$

where here and in the following we will assume that the spacetime torus has even dimension $D = 2d$. We have introduced a set $\rho^a, a = 1, ..., D$, of anticommuting Grassmann variables,

$$\rho^a \rho^b = -\rho^b \rho^a, \tag{7.20}$$

which can be thought of as local generators of the cotangent bundle of $\mathbf{T}^D$. The quantity $ch_0(E)$ then defines an integer cohomology class of the ordinary torus $\mathbf{T}^D$. Given these integers which characterize the given bundle E, there is an elegant formula for the noncommutative Chern character

$$ch_\Theta(E) = \mathrm{Tr} \otimes \mathrm{tr}_N \mathrm{exp} \frac{\hat{\mathcal{W}}[F]}{2\pi} \tag{7.21}$$

which characterizes the corresponding gauge bundle over the noncommutative torus. Here F is the noncommutative curvature two-form of the bundle with local components

$$F_{ab} = \sum_{a}^{i} F_{ij} \sum_{b}^{j} \tag{7.22}$$

where F_{ij} is defined by

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i - i(A_i \star A_j - A_j \star A_i) \\ &= \partial_i A_j - \partial_j A_i - i[A_i, A_j] \\ &+ \frac{1}{2}\theta^{kl}(\partial_k A_i \partial_l A_j) - \partial_k A_j \partial_l A_i) + O(\theta^2) \end{aligned} \tag{7.23}$$

It can be regarded as an element of the ordinary cohomology ring $H^{even}(\mathbf{T}^D, \mathcal{R})$ of even degree differential forms on the torus. The quantity (7.21) can be written in terms of (7.19) through the Elliott formula [205]

$$ch_\Theta(E) = \exp\biggl(-\frac{1}{2}\Theta^{ab}\frac{\partial}{\partial\rho^a}\frac{\partial}{\partial\rho^b}\biggr) ch_0(E) \tag{7.24}$$

with Θ regarded as a two-cycle of the homology group $H_2(\mathbf{T}^D, \mathcal{R})$ [206,207]. The coefficients of $\rho^{a_1}...\rho^{a_{2n}}$ in the expansion of (7.24) define the $n-th$ noncommutative Chern numbers of the given noncommutative gauge theory. They represent the topological invariants of the corresponding deformation $E \to E_\Theta$ from a commutative to a noncommutative gauge bundle. In the commutative limit $\Theta = 0, ch_0(E)$ generates the ordinary integer-valued Chern numbers. But for $\Theta \neq 0$ they are non-integral in general.

For example, in two dimensions we find

$$ch_\Theta(E) = (N - Q\Theta) + Q\rho^1\rho^2, \tag{7.25}$$

where Q is the magnetic flux through $\mathbf{T}^2$

Conclusion

In this article we have studied IKKT Yang-Mills matrix models with mass deformations in three and six dimensions.

The $3-$dimensional IKKT matrix models considered here are very similar to the ones studied in [135, 138]. However, the dynamically emergent geometry, which is given by a fuzzy two-sphere $\mathbf{S}^2_N$, is found to be stable for all values of the deformation parameter M. This was anticipated previously for $M = 1/2$ in [141]. Indeed, the critical gauge coupling constant $\widetilde{\alpha}$ is found to scale as in equation (5.24), i.e. as $\widetilde{\alpha} \sim 1/\sqrt{N}$. The sphere-to-matrix transition line is pushed to 0 and only one phase survives.

In this case the fuzzy sphere acts then as a regulator of the commutative sphere, and as a consequence, fuzzy field theory and fuzzy physics, based on this emergent fuzzy sphere, makes full sense for all values of the gauge coupling constant.

We have also studied in this article $6-$dimensional IKKT matrix models, with global $SO(3) \times SO(3)$ symmetry, containing at most quartic powers of the matrices proposed in [146]. The value $M = 1/2$ of the deformation corresponds to the model of [147]. This theory exhibits a phase transition from a geometrical phase at low temperature, given by a fuzzy four-sphere $\mathbf{S}^2_N \times \mathbf{S}^2_N$ background, to a Yang-Mills matrix phase with no background geometrical structure at high temperature.

The geometry as well as an Abelian gauge field and two scalar fields are determined dynamically as the temperature is decreases and the fuzzy four-sphere condenses.

The transition is exotic in the sense that we observe, for small values of M, a discontinuous jump in the entropy, characteristic of a 1st order transition, yet with divergent critical fluctuations and a divergent specific heat with critical exponent $\alpha = 1/2$. The critical temperature is pushed upwards as the scalar field mass is increased. For small M, the system in the Yang-Mills phase is well approximated by 6 decoupled matrices with a joint eigenvalue distribution which is uniform inside a ball in $\mathbf{R}^6$. This gives what we call the $d = 6$ law given by equation (5.47). For large M, the transition from the four-sphere phase to the Yang-Mills matrix phase turns into a crossover and the eigenvalue distribution in the Yang-Mills matrix phase changes from the $d = 6$ law to a uniform distribution.

In the Yang-Mills matrix phase the specific heat is equal to 3/2 which coincides with the specific heat of 6 independent matrix models with quartic potential in the high temperature limit and is therefore consistent with this interpretation. Once the geometrical phase is well established the specific heat takes the value 5/2 with the gauge field contributing 3/2 and the two scalar fields contributing 1[1].

This should be contrasted with the case of $\mathbf{S}^2_N$ in which the specific heat in the Yang-Mills matrix phase is equal to 3/4, coinciding with the specific heat of 3 independent matrix models with quartic potential in the high temperature limit, while in the geometrical phase the specific heat takes the value 1 divided equally, i.e. with the gauge field contributing 1/2 and the two scalar fields contributing 1/2.

The counting on $\mathbf{S}^2_N$ is clear cut. In the sphere phase which coincides with the perturbative region of the theory the gauge field contributes 1/2 [167] and the scalar field contribute 1/2 (trivial to check in the quartic matrix model for very large positive values of the mass parameter [168, 169]).

The counting on $\mathbf{S}^2_N \times \mathbf{S}^2_N$ is more involved. There are here two scalar fields and one gauge field with 4 components. Again, it is very natural to suppose that each scalar field will contribute 1/2 (since they are free and dimension does not enter in the quartic matrix model). Therefore, the gauge field will contribute 3/2. This is the picture we also get, at least formally, by assuming that the gauge field is free and Abelian (which is true deep in the sphere phase in the large N limit) and then fixing the gauge in the axial gauge.

The $6-$dimensional IKKT Yang-Mills matrix models studied here present thus an appealing picture of a $4-$dimensional geometrical phase emerging as the system cools and suggests a scenario for the emergence of geometry in the early universe. See [148] and references therein.

[1]Recall that in the 3d Yang-Mills matrix model the specific heat takes the value 1 in the geometrical phase which is attributed in this case to the normal scalar field since there is no propagating gauge degrees of freedom in 2 dimensions.

The model presents thus an appealing picture of a geometrical phase emerging as the system cools and suggests a scenario for the emergence of geometry in the early universe.

Appendix A

A.1 Susceptibility and Specific Heat

A.1.1 Susceptibility

We consider Φ^4 on the fuzzy sphere coupled to a constant magnetic field H given by the action

$$S = Tr(a\Phi[L_a,[L_a,\Phi]]) + b\Phi^2 + c\Phi^4 + H\Phi \tag{A.1}$$

The magnetization and the susceptibility are defined by

$$\begin{aligned} \text{magnetization} &= \frac{1}{N} < Tr\Phi > \\ &= -\frac{1}{N}\frac{\partial}{\partial H}\ln Z \end{aligned} \tag{A.2}$$

$$\begin{aligned} \text{susceptibility} &= < (Tr\Phi)^2 >< Tr\Phi >^2 \\ &= \frac{\partial^2}{\partial H^2}\ln Z \\ &= -N\frac{\partial}{\partial H}\text{magnetization} \end{aligned} \tag{A.3}$$

On the fuzzy sphere we have

$$x_a = \frac{2R}{N}L_a,\ [x_a,x_b] = \frac{i\theta}{R}\epsilon_{abc}x_c,\ \theta\frac{2R^2}{N},\ Tr = \frac{N}{4\pi R^2}\int d^2x. \tag{A.4}$$

The regularized noncommutative plane is then defined by

$$x_3 = R,\ [x_1,x_2] = i\theta,\ \partial_i = -\frac{1}{R}\epsilon_{ij}L_j = -\frac{1}{\theta}\epsilon_{ij}x_j,\ \int d^2x = 2\pi\theta Tr. \tag{A.5}$$

We have $\epsilon_{12} = 1$. The above action becomes, including a rescaling of the field $\Phi \to \phi = \sqrt{Na/2\pi}\Phi$ given by the equation

$$S = 2\pi\theta Tr(\frac{1}{2}\phi\partial_i\partial_i\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + h\phi). \tag{A.6}$$

$$m^2 = \frac{b}{aR^2},\ \lambda = \frac{4\pi c}{Na^2R^2},\ h = \sqrt{\frac{N}{2\pi a}}\frac{H}{2r^2} \tag{A.7}$$

The commutative limit is $\theta \to 0$. By using a lattice in this limit we have

$$S = l^2\sum_n(\frac{1}{2}(\phi\partial_i\partial_i\phi)_{\text{lattice}} + \frac{1}{2}m^2\phi_n^2 + \frac{1}{4}\lambda\phi_n^4 + h\phi_n). \tag{A.8}$$

We compute in this limit on the lattice

$$\begin{aligned}\text{magnetization} &= \frac{1}{N} < Tr\phi > \\ &\to \frac{\mathcal{N}^2l^2}{4\pi R^2} < \frac{1}{\mathcal{N}^2}\sum_n \Phi_n > \end{aligned} \tag{A.9}$$

A.1.2 Specific Heat

The specific heat is defined by

$$\begin{aligned}C_v &= \frac{\partial^2}{\partial\beta^2}\ln z \\ &= < S^2 > - < S >^2 .\end{aligned} \tag{A.10}$$

The inverse temperature is introduced in the usual way as

$$Z = \int dM\exp(-\beta S[M]). \tag{A.11}$$

The calculation of the effective potential proceeds as before with the replacement $a \to a\beta$. The partition function in the quartic multitrace approximation is

$$Z = \int d\Lambda\Delta^2(\Lambda)\exp(-\beta V_0) + \beta\int d\Lambda\Delta^2(\Lambda)\exp(-\beta V_0)(-V_2) \tag{A.12}$$

$$+ \beta^2\int d\Lambda\Delta^2(\Lambda)\exp(-\beta V_0)(-V_4 + \frac{1}{2}V_2^2) \tag{A.13}$$

A straightforward calculation yields

$$C_v =< (V + V_4)^2 > - < (V + V_4) >^2 -2 < (V_4 + 2V_4^2 + 2V_2V_4)^2 > . \tag{A.14}$$

The last term could make this approximation of the specific heat negative. This actually happens in the approximation of [208].

A.2 Grosse-Wulkenhaar Model

The multitrace approach can also be applied to a regularized noncommutative Φ_2^4 on the Moyal-Weyl plane in the matrix basis [193] with action given by

$$S = Tr_N\Big[\frac{1}{2}m^2M^2 + \frac{u}{N}M^4 + a\Big(EM^2 + \sqrt{\omega}\Gamma^+M\Gamma M\Big)\Big]. \quad (A.15)$$

Two cases are of importance to us here:

1. The noncommutative theory without a harmonic oscillator term. In this case the effective action takes the form

   $$S_{\text{effe}} = bTr_NM^2 + cTr_NM^4 + d(Tr_NM^2)^2 + b_1(Tr_NM)^2 + c_1(Tr_NM)^4 + d_1Tr_NM^2(Tr_NM)^2 + eTr_NMTr_NM^3. \quad (A.16)$$

   The parameters are given by

   $$b = \frac{m^2}{2} + \frac{aN}{2}\ ,\ c = \frac{u}{N} - \frac{a^2N}{24}\ ,\ d = -\frac{a^2}{12}$$
   $$b_1 = -\frac{a}{2}\ ,\ c_1 = \frac{a^2}{24N^2}\ ,\ d_1 = -\frac{a^2}{12N}\ ,\ e = \frac{a^2}{6}. \quad (A.17)$$

   If we assume the symmetry $M \longrightarrow -M$ then all odd moments vanish identically and we end up with the action

   $$S_{\text{effe}} = bTr_NM^2 + cTr_NM^4 + d(Tr_NM^2)^2. \quad (A.18)$$

2. At the self-dual point we have $\Omega^2 = 1$, and thus $\sqrt{\omega} = 0$, and as a consequence the effective action reduces to the multitrace model

   $$S_{\text{effe}} = bTr_NM^2 + cTr_NM^4 + d(Tr_NM^2)^2. \quad (A.19)$$

   The parameters b, c and d are given by

   $$b = \frac{m^2}{2} + \frac{aN}{2}\ ,\ c = \frac{u}{N} - \frac{a^2N}{24}\ ,\ d = \frac{a^2}{24}. \quad (A.20)$$

Both the actions (A.18) and (A.19) do not contain odd moments and thus the corresponding phase diagrams are expected to not contain the uniform ordered phase with all matrix-like behavior as consequence.

Appendix B

B.1 Metropolis Algorithm for Yang-Mills Matrix Models

B.1.1 Metropolis Accept/Reject Step:

The basic Yang-Mills action of interest is

$$\begin{aligned} S_{YM} &= -\frac{N}{4}Tr[X_\mu, X_\nu]^2 \\ &= -N\sum_{\mu=1}^{d}\sum_{\nu=\mu+1}^{d}(X_\mu X_\nu X_\mu X_\nu - X_\mu^2 X_\mu 2). \end{aligned} \tag{B.1}$$

We perform the variation

$$X_\lambda \to X_\lambda^{'} = X_\lambda + \Delta X_\lambda \tag{B.2}$$

where

$$(X_\lambda)_{nm} = d\delta_{ni}\delta_{mj} + = d^*\delta_{nj}\delta_{mi} \tag{B.3}$$

The corresponding variation of the action is

$$\Delta S = S(X^{'}) - S(X) \tag{B.4}$$

The Metropolis accept/reject step is based on the probability distribution

$$P[X] = \min(1, \exp(-\Delta S)) \tag{B.5}$$

B.1.2 Auto-Correlation Time:

In any given ergodic process we obtain a sequence (Markov chain) of field/matrix configurations $\phi_1, \phi_2, ..., \phi_T$. We will assume that ϕ_i are thermalized configurations. Let f some (primary) observable with values $f_i \equiv f(\phi_i)$ in the configurations ϕ_i respectively. The average value $< f >$ of f and the statistical error δf are given by the usual formulas

$$< f > \;=\; \frac{1}{T}\sum_{i=1}^{T} f_i \tag{B.6}$$

$$\delta f \;=\; \frac{\sigma}{\sqrt{T}} \tag{B.7}$$

The standard deviation (the variance) is given by

$$\sigma^2 =< f^2 > - < f >^2 \tag{B.8}$$

The above theoretical estimate of the error is valid provided the thermalized configurations $\phi_1, \phi_2, .., \phi_T$ are statistically uncorrelated, i.e. independent. In real simulations, this is certainly not the case. In general, two consecutive configurations will be dependent, and the average number of configurations which separate two really uncorrelated configurations is called the auto-correlation time. The correct estimation of the error must depend on the auto-correlation time.

We define the auto-correlation function Γ_j and the normalized auto-correlation function ρ_j for the observable f by

$$\Gamma_j = \frac{1}{T-j}\sum_{i=1}^{T-j}(f_i- < f >)()f_{i+j}+ < f >). \tag{B.9}$$

$$\rho_j \;=\; \frac{\Gamma_j}{\Gamma_0} \tag{B.10}$$

These function vanish if there is no auto-correlation. Obviously Γ_0 is the variance σ^2 , viz $\Gamma_0 = \sigma^2$. In the generic case, where the auto-correlation function is not zero, the statistical error in the average $< f >$ will be given by

$$\delta_f = \frac{\sigma}{\sqrt{T}}\sqrt{2\tau_{\rm int}} \tag{B.11}$$

The so-called integrated auto-correlation time τ_{int} is given in terms of the normalized auto-correlation function ρ_j by

$$\tau_{\rm int} = \frac{1}{2} + \sum_{j=1}^{\infty} \rho_j \tag{B.12}$$

The auto-correlation function Γ_j , for large j, can not be precisely determined, and hence, one must truncate the sum over j in τ int at some cut-off M, in order to not increase the error $\delta\tau$ int in τ int by simply summing up noise. The integrated auto-correlation time τ int should then be defined by

$$\tau_{\rm int} = \frac{1}{2} + \sum_{j=1}^{M} \rho_j \tag{B.13}$$

The value M is chosen as the first integer between 1 and T such that

$$M \geq 4\tau_{\rm int} + 1 \tag{B.14}$$

The error $\delta\tau_{\rm int}$ in $\tau_{\rm int}$ is given by

$$\delta\tau_{\rm int}\sqrt{\frac{4M+2}{T}}\tau_{\rm int} \tag{B.15}$$

This formalism can be generalized to secondary observables F which are functions of n primary observables f^α , viz $F = F(f^1, f^2, ..., f^n)$.

B.1.3 Errors

We use the Jacknife method to estimate the errors. Given a set of $T = 2^P$ (with P some integer) data points $f(i)$ we proceed by removing z elements from the set in such a way that we end up with $n = T/z$ sets (or bins). The minimum number of data points we can remove is $z = 1$ and the maximum number is $z = T - 1$. The average of the elements of the ith bin is

$$< y(j) >_i = \frac{1}{T-z}\left(\sum_{j=1}^{T} f(j) - \sum_{j=1}^{z} f((i-1)z+j)\right) \ , \ i = 1, n. \tag{B.16}$$

For a fixed partition given by z the corresponding error is computed as follows

$$e(z) = \sqrt{\frac{n-1}{n}\sum_{i=1}^{n}(< y(j) >_i - < f >)^2} \ , \ \ < f >= \frac{1}{T}\sum_{j=1}^{T} f(j). \tag{B.17}$$

We start with $z = 1$ and we compute the error $e(1)$ then we go to $z = 2$ and compute the error $e(2)$. The true error is the largest value. Then we go to $z = 3$, compute $e(3)$, compare it with the previous error and again retain the largest value and so on until we reach $z = T - 1$.

B.2 The Hybrid Monte-Carlo Algorithm

The hybrid monte carlo algorithm is a combination of the molecular dynamics method and the metropolis algorithm. First we introduce a fictitious time τ and 3 bosonic matrices $P_a = P_a(\tau)$. These P_a play the role of conjugate momenta for the 3 matrices $X_a = X_a(\tau)$. We will have a classical dynamical system described by $(X_a(\tau), P_a(\tau))$ with Hamiltonian

$$H = \frac{1}{2}TrP_a^2 + S_{\rm eff}[X] \;\; = \;\; \frac{1}{2}TrP_a^2 + S_B[X] - TR\log\frac{\mathcal{D}[X]}{N}. \tag{B.18}$$

The partition function is given by

$$Z = \int [dX_a][dP_a]e^{-H[X,P]}. \tag{B.19}$$

By integrating out P_a we obatin the original partition function, viz

$$Z = \int [dX_a]e^{-S_{\rm eff}[X]}. \tag{B.20}$$

The Hamiltonian classical equations of motion are

$$\begin{aligned} \frac{d(P_a)_{ij}}{d\tau} = -\frac{\partial H}{\partial(X_a)_{ij}} \;\; &= \;\; -\frac{\partial S_{\rm eff}}{\partial(X_a)_{ij}} \\ &= \;\; -N\bigg(-[X_b,[X_a,X_b]] + 2i\alpha\epsilon_{abc}X_bX_c\bigg)_{ji} \\ &+ \;\; TR\mathcal{M}^{-1}\frac{\partial\mathcal{M}}{\partial(X_a)_{ij}}. \end{aligned} \tag{B.21}$$

$$\frac{d(X_a)_{ij}}{d\tau} = \frac{\partial H}{\partial(P_a)_{ij}} = (P_a)_{ji}. \tag{B.22}$$

Bibliography

[1] A. Connes, *Non-commutative differential geometry.*, Publications Mathmatiques de lIHES, 1985.

[2] A. Connes, *A short survey of noncommutative geometry.* Journal of Mathematical Physics **41:38323866**, (2000). doi: 10.1063/1.533329.

[3] A. Connes and M. Marcolli. *A walk in the noncommutative garden.* arXiv:math/0601054 .

[4] G. Landi. *An Introduction to Noncommutative Spaces and Their Geometries.* Springer-Verlag, 1997, arXiv:hep-th/9701078.

[5] M. Dubois-Violette. *Non-commutative differential geometry, quantum mechanics and gauge theory.* volume 375 of Lecture Notes in Physics. Springer-Verlag, 1991.

[6] A. Connes, *Noncommutative Geometry.* Academic Press, 1994.

[7] J. M. Gracia-Bonda, J.C.Vrilly, and H. Figueroa, *Elements of Noncommutative Geometry.*, Birkhaser Advanced Texts. Birkhaser, 2001.

[8] R. Coquereaux, *Connections, metrics, symmetries and scalar fields in commutative and noncommutative geometry.*, S41S53, 1992. ISSN 0264-9381. URL http://stacks.iop.org/0264-9381/9/41. Les Journes Relativistes (Cargse, 1991).

[9] M. Dubois-Violette, *Drivations et calcul differentiel non commutatif.* C.R. Acad. Sci. Paris, Srie I, 307:403408, 1988.

[10] M. Dubois-Violette, *Non-commutative differential geometry, quantum mechanics and gauge theory.* In C. Bartocci, U. Bruzzo, and R. Cianci,

editors, Differential Geometric Methods in Theoretical Physics, Proceedings of the 19th International Conference Held in Rapallo, Italy 1924 June 1990, volume 375 of Lecture Notes in Physics. Springer-Verlag, 1991.

[11] M. Dubois-Violette, *Lectures on graded differential algebras and noncommutative geometry.* In Y. Maeda and H. Moriyoshi, editors, Noncommutative Differential Geometry and Its Applications to Physics. Kluwer Academic Publishers, 2001.

[12] T. Masson, *Examples of derivation-based differential calculi related to noncommutative gauge theories.* International Journal of Geometric Methods in Modern Physics, 5(8):13151336, 2008. URL http://arxiv.org/abs/0810.4815.

[13] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory.* Perseus Books, 2008.

[14] Connes, A., Lott, J., Nucl.Phys. **18 B** (Proc.Suppl.) 29, (1990)

[15] H.Aoki, S.Iso, H.Kawai, Y.Kitazawa and T.Tada, *Space-Time Structures from IIB Matrix Model.* Prog.Theor.Phys.99 (1998)713, hep-th/9802085.

[16] H.Aoki, S.Iso, H.Kawai, Y.Kitazawa and T.Tada,and A.Tsuchiya, *IIB Matrix Model.* Prog.Theor.Phys.Suppl.134(1999)47, hep-th/9908038.

[17] H.Aoki, S.Iso, H.Kawai, Y.Kitazawa and T.Tada,and A.Tsuchiya, *Noncommutative Yang-Mills in IIB Matrix Model.* Nucl.Phys.B565(2000)176, hep-th/9908141.

[18] M.Li, *Strings from IIB Matrices*, Nucl.Phys.B499(1997)149, hep-th/9612222.

[19] I.Bars, D.Minic, *Non-Commutative Geometry on a Discrete Periodic Lattice and Gauge Theory*, hep-th/9910091.

[20] J.Ambjorn, Y.M.Makeenko, J.Nishimura and R.J.Szabo, *Finite N Matrix Models of Noncommutative Gauge Theory*, JHEP9911(1999)029, hep-th/9911041.

[21] N.Ishibashi, S.Iso, H.Kawai and Y.Kitazawa, *Wilson loops in Noncommutative Yang-Mills* , Nucl.Phys.B573(2000)573, hep-th/9910004.

[22] N.Seiberg and E.Witten,*String theory and Noncommutative Geometry*, JHEP9909(1999)032, hep-th/9908056.

[23] J.Hoppe,*Quantum Theory of A Massless Relativistic Surface and A Two-Dimensional Bound State Problem*, MIT Ph.D.Thesis,1982.

[24] N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, *A Large N reduced model as superstring,* Nucl.Phys.**498** (1997)467, hep-th/9612115

[25] D.Kabat and W.Taylor, *Spherical membranes in Matrix theory,* Adv.Theor.Math.Phys.2(1998)181, hep-th/9711078.

[26] S-J.Rey,*Gravitating M(atrix) Q-Balls,* hep-th/9711081.

[27] R.C.Myers, *Dielectric-Branes,* JHEP9912(1999)022, hep-th/9910053.

[28] A.Yu.Alekseev, A.Recknagel and V.Schomerus, *Non-commutative World-volume Geometries: Branes on SU(2) and Fuzzy Spheres,* JHEP9909(1999)023, hep-th/9908040.

[29] A.Yu.Alekseev, A.Recknagel and V.Schomerus, *Brane Dynamics in Background Fluxes and Non-commutative Geometry,* JHEP0005(2000)010, hep-th/0003187.

[30] J.Madore, *The Fuzzy Sphere*, Class.Quamtum.Grav.9(1992)69

[31] H.Grosse, C.Klimcik and P.Presnajder, *Towards Finite Quantum Field Theory in Non-Commutative Geometry*, Int.J.Theor.Phys.35(1996)231, hep-th/9505175.

[32] U.Carow-Watamura and S.Watamura, *Noncommutative Geometry and Gauge Theory on Fuzzy Sphere*, Comm.Math.Phys.212(2000)395, hep-th/9801195.

[33] C.Klimcik,*Gauge theories on the noncommutative sphere*, Comm.Math.Phys.199(1998)257, hep-th/9710153.

[34] H.Grosse and J.Madore,*A noncommutative version of the Schwinger model,* Phys.Lett.B283(1992)218.

[35] S.Baez, A.P.Balachandran, S.Vaidya and B.Ydri, *Monopoles and Solitons in Fuzzy Physics*, Comm.Math.Phys.208(2000)787, hep-th/9811169.

[36] I. Montvay and G. Munster, *Quantum Field on a Lattice*, Cambridge, 1997.

[37] S. Vaidya and B. Ydri, *On the Origin of the UV-IR Mixing in Noncommutative Matrix Geometry,* hep-th/0305201

[38] G. Alexanian, A.P. Balachandran, G. Immirzi and B. Ydri, J. of Geom. and Phys. **42** (2002) 28, hep-th/0103023.

[39] Denjoe OConnor and Julieta Medina, *Scalar Field Theory on Fuzzy* S^4, hep-th/0212170.

[40] Brian P. Dolan and Denjoe OConnor , *A Fuzzy Three Sphere and Fuzzy Tori*

[41] darajan and B. Ydri,*Fermion doubling problem and noncommutative geometry II,* hep-th/0006216

[42] H.J.Groenewold,*On the Principle of elementary quantum mechanics*, Physica 12,405 (1945).

[43] J.E.Moyal,*Quantum mechanics as a statistical theory*, Proc.Cambridge Phil.Soc.45,99 (1949).

[44] B.Jurco, S.Schraml, P.Schupp and J.Wess, *Noncommutative gauge theory for Poisson manifolds*, Nucl.Phys. **B84** 784 (2000), [hep-th/0005005].

[45] B.Jurco, L.Moller, S.Schraml, P.Schupp and J.Wess, *Construction of non-Abelian gauge theories on noncommutative spaces* , Eur.Phys. **C21** 383 (2001),[hep-th/0104153].

[46] Shiraz Minwalla, Mark Van Raamsdonk, Nathan Seiberg *Noncommutative Perturbative Dynamics* , JHEP **0002** 020 (2000),arXiv:hep-th/9912072v2.

[47] S. S. Gusber and S.L.Sondhi *Phase Structure of Non-commutative Scalar Field Theory* , Nucl. Phys **B605** 395 (2001),hep-th/0006119

[48] S. Brazovkii. *Phase Structure of an Isotropic System to a Nonuniform State*, Zh Eksp. Teo. Fiz **A15** 319 (1975)

[49] G. Chen and Y. Wu. *renormalization Group Equations and the Lifshitz Point in Noncommutative LandauGinsburg Theory*, Nucl. Phys **B622** 189 (2002),hep-th/0110134

[50] Frank Hofheinz *Field Theory on a NonCommutative Plane*, Thesis

[51] K. Matsubara. *Restrictions on Gauge Groups in Noncommutative Gauge Theory*, Phys.Lett **B482** 417 (2000),hep-th/0003294.

[52] A. Armoni *Comments on Perturbative Dynamics of Noncommutative YangMills Theory*, Nucl. Phys. **B593** 229 (2001), hep-th/0005208

[53] Ana Ros Camacho and Richard J. Szabo *Introduction to Noncommutative Field Theory*, at the 15th Saalburg Summer School, at Wolfersdorf Trockenborn, Thuringen, Germany

[54] David J. Gross, Akikazu Hashimoto, N. Itzhaki*Observables of Non-Commutative Gauge Theories*, Adv. Theor. Math. Phys. 4 (2000) 893-928, arXiv:hep-th/0008075v3.

[55] Nobuyuki Ishibashi, Satoshi Iso, Hikaru Kawai, Yoshihisa Kitazawa *Wilson Loops in Noncommutative Yang Mills*, Nucl. Phys. **B573** (2000) 573-593, arXiv:hep-th/9910004v2.

[56] Matti Tapoi Raasakka *NonCommutative Quantum field Theory and UV/IR Mixing*, Thesis

[57] M. Hayakawa *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on* $\mathcal{R}^4$, arXiv:hep-th/9912167v1.

[58] Alec Matusis, Leonard Susskind, Nicolaos Toumbas *The IR/UV Connection in the Non-Commutative Gauge Theories,*, JHEP **0012** (2000) 002, arXiv:hep-th/0002075v2.

[59] Nathan Seiberg, Edward Witten *String Theory and Noncommutative Geometry*, JHEP **9909** (1999) 032, arXiv:hep-th/9908142v3.

[60] Andreas Bichl, Jesper Grimstrup, Harald Grosse, Lukas Popp, Manfred Schweda, Raimar Wulkenhaar ,*Renormalization of the noncommutative photon self-energy to all orders via Seiberg-Witten map,* JHEP **0106** (2001) 013, arXiv:hep-th/0104097v3.

[61] Peter Schupp, Jiangyang You, *UV/IR mixing in noncommutative QED de- fined by Seiberg-Witten map,* JHEP **0808:107**,2008, arXiv:0807.4886v1 [hep-th].

[62] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys **73**,977 (2001),[hep-th/0106048].

[63] J.Madore, *The fuzzy sphere*, Class. Quant. **9** (1992) 69.

[64] D. J. Gross and N. A. Nekrasov, *Solitons in noncommutative gauge theory* ,, JHEP **03**,044 (2001),[hep-th/0010090].

[65] Wolfgang Behr *Noncommutative Gauge Theory beyond the Canonical Case*, Thesis

[66] Ofer Aharony et al. *The phase structure of low dimensional large n gauge theories on tori*. JHEP, 01:140, 2006.

[67] Alexander A. Migdal. *Recursion equations in gauge field theories*. Sov. Phys. JETP, 42:413, 1975.

[68] David J. Gross. *Two-dimensional qcd as a string theory*. Nucl. Phys., B400:161180,1993.

[69] David J. Gross and IV Taylor, Washington. *Two-dimensional qcd is a string theory*. Nucl. Phys., B400:181210, 1993.

[70] David J. Gross and IV Taylor, Washington. *Twists and wilson loops in the string theory of two- dimensional qcd*. Nucl. Phys., B403:395452, 1993.

[71] A. V. Smilga, Witten *Index Calculation In Supersymmetric Gauge Theory*, Nucl. Phys. **B266** (1986) 4557.

[72] P. Yi, Witten Index and Threshold Bound States of D-Branes, Nucl. Phys. **B505** (1997) 307, [hep-th/9704098].

[73] S. Sethi and M. Stern, D-Brane Bound States Redux, Commun. Math. Phys. **194** (1998) 675, [hep-th/9705046].

[74] T. Suyama and A. Tsuchiya, *Exact results in n(c)=2 iib matrix model,* Prog. Theor. Phys. **99** (1998) 321325, [hep-th/9711073].

[75] W. Krauth, J. Plefka, and M. Staudacher, *Yang-Mills Integrals,* Class. Quant. Grav. **17** (2000) 1171, [hep-th/9911170].

[76] W. Krauth, H. Nicolai, and M. Staudacher, *Monte Carlo Approach to M-theory,* Phys. Lett. **B431** (1998) 3141, [hep-th/9803117].

[77] W. Krauth and M. Staudacher,*Finite Yang-Mills Integrals*, Phys. Lett. **B435** (1998) 350, [hep-th/9804199].

[78] W. Krauth and M. Staudacher, *Yang-Mills Integrals for Orthogonal, Symplectic and Exceptional Groups,* Nucl. Phys. **B584** (2000) 641, [hep-th/0004076].

[79] T. Hotta, J. Nishimura, and A. Tsuchiya, *Dynamical aspects of large n reduced models,* Nucl. Phys. B545 (1999) 543575, [hep-th/9811220].

[80] S. Oda and F. Sugino, *Gaussian and mean field approximations for reduced yang- mills integrals,* JHEP **03** (2001) 026, [hep-th/0011175].

[81] F. Sugino, *Gaussian and mean field approximations for reduced 4d supersymmetric yang-mills integral,* hep-th/0105284.

[82] Edward Witten, *Bound states of strings and p-branes*, Nucl. Phys **B460**,335-350 (1996),[hep-th/9510135].

[83] Tom Banks, W. Fischler, S. H. Shenker, and Leonard Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev **D55**,5112-5128 (1997),[hep-th/9610043].

[84] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *A large-n reduced model as superstring*, Nucl. Phys **B498**,467-461 (1997),[hep-th/0404174].

[85] Iso Aoki, Kitazawa Kawai, and Tada Tsuchiya, *IIB matrix model*, Prog. Theor. Phys. Suppl **143**,47-83 (1999),[hep-th/9908038].

[86] Hajime Aoki, Satoshi Iso, Hikaru Kawai, Yoshihisa Kitazawa, and Tsukasa Tada, *Spacetime structures from iib matrix model*, Prog. Theor. Phys. Suppl **99**,713-746 (1998),[hep-th/9802085].

[87] H. Aoki et. al., *Iib matrix model*, Prog. Theor. Phys. Suppl. 134 (1999) **47**, [hep-th/9908038].

[88] W. Taylor, *M(atrix) theory: Matrix quantum mechanics as a fundamental theory*, hep-th/0101126.

[89] M. Fukuma, H. Kawai, Y. Kitazawa, and A. Tsuchiya,*String field theory from iib matrix model*, Nucl. Phys. **B510** (1998) 158174, [hep-th/9705128].

[90] A. Connes, *Noncommutative Geometry* , Academic Press. (1994).

[91] G. Landi, *An Introduction to Noncommutative Spaces and their Geometry* , Lecture Notes on Physics, Monographs m51, Spring-Verlag (1997), hep-th/9701078

[92] J. Madore, *The Fuzzy Sphere*, Class Quant. Grav. **9** (1992) 69.

[93] D. Karabali and V.P. Nair, *The effective action for edge states in higher dimensional quantum Hall systems*, Nucl. Phys **B 641** (2002) 533,hep-th/0203264.

[94] Yasuhiro abe, *Construction of fuzzy* S^4, arXiv: hep-th/0406135v3

[95] U. Carow-Watamura, H. Steinacker and S. Watamura, *Monopole Bundles over Fuzzy Complex Projective Spaces*, arXiv: hep-th/0404130.

[96] G. Alexanian, A.P. Balachandran, G. Immirzi and B. Ydri, *Fuzzy* CP^2, Nucl. Phys **42** (2002) 28,hep-th/0103023.

[97] H. Grosse, C. Klimcik, and P. Presnajder, *On Finite 4D Quantum Field Theory in Non-Commutative Geometry*, Commun. Math **180** (1996) 429,hep-th/9602115

[98] J. Medina and D. OConnor, *Scalar Field Theory on Fuzzy* S^4, JHEP **0311** (2003) 051,hep-th/0212170

[99] S. Ramgoolam, *On spherical harmonics for fuzzy spheres in diverse dimensions.*, Nucl. Phys. **B 610** (2001) 461,hep-th/0105006.

[100] P. Ho and M. Li, *Fuzzy Spheres in AdS/CFT Correspondence and Holography from Noncommutativity.*, Nucl. Phys. **B 693** (2001) 461,hep-th/0105006.

[101] R. Nakayama and Y. Shimono, *Matrix Configurations for Spherical 4-branes and Non-commutative Structures on S^4.*, Nucl. Phys. **B 693** (2004) 176,hep-th/0402010.

[102] Yasuhiro Abe, *Construction of Fuzzy Spaces and Their Applications to Matrix Models.*

[103] D. Karabali, V.P. Nair and S. Randjbar-Daemi, *Fuzzy spaces, the M(atrix) model and the quantum Hall effect, in From Fields to Strings.*,hep-th/0407007.

[104] D. Karabali, V.P. Nair, *Edge states for quantum Hall droplets in higher dimensions and a generalized WZW model.*, Nucl. Phys. **B697** (2004) 513, hep-th/0403111.

[105] R. Nakayama and Y. Shimono, *Matrix Configurations for Spherical 4-branes and Non-commutative Structures on S^4.*, Nucl. Phys. **B693** (2004) 176, hep-th/0402010.

[106] R. Penrose and M.A.H. MacCallum, *Spinors and Spacetime*, Phys. Rep. **6** (1972) 241; Penrose and W. Rindler,Volume 1, Cambridge University Press (1984). Volume 2, Cambridge University Press (1986). , Nucl. Phys. **B693** (2004) 176, hep-th/0402010.

[107] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ Super Yang Mills.*, JHEP **0204** (2002) 013, hep-th/0202021.

[108] Harold Steinacker, *Gauge theory on fuzzy spaces*, J. Phys.: Conf. Ser **53** (2006) 872-884

[109] H. Grosse, C. Klimčík, and P. Prešnajder,*Comm.Math.Phys.* **180**, 429 (1996),`hep-th/9602115`.

[110] H. Grosse, C. Klimčík, and P. Prešnajder,in *Les Houches Summer School on Theoretical Physics*, 1995, `hep-th/9603071` .

[111] P. Prešnajder,`hep-th/9912050` and *J.Math.Phys.*__41__(2000)2789-2804.

[112] H. Grosse, C. Klimčík and P. Prešnajder,*Commun.Math.Phys.* **178**,507-526 (1996); H. Grosse and P. Prešnajder,*Lett.Math.Phys.* **46**, 61 (1998) and ESI preprint,1999.

[113] J. Klauder and B.-S. Skagerstam , Coherent States : *Application in Physics and Mathematical Physics*, World Scientific(1985); A.M. Perelomov : *Generalized coherent states and their applications*, Springer-Verlag,(1986); M. Bordemann, M. Brischle, C. Emmrich and S. Waldmann,*J.Math.Phys.***37**,6311(1996); M. Bordemann, M. Brischle, C. Emmrich and S. Waldmann,*Lett.Math.Phys.***36**,357(1996); S. Waldmann,*Lett.Math.Phys.***44**,331(1998).

[114] A.P. Balachandran, B. Dolan, J. Lee, X. Martin and D. O'Connor (in preparation); G. Alexanian, S. Pinzul and A. Stern `hep-th/0010187` and *Nucl.Phys.***B600/3**,531(2001).

[115] E.Hawkins, `math.Q.A/9808116`.

[116] Fuzzy $\mathbf{CP}^2$ has already been investigated by H.Grosse and A.Strohmaier, *hep-th/9902138* and *Lett.Math.Phys.* **48**,163(1999).

[117] *Encyclopedic Dictionary of Mathematics*, edited by S.Iyanaga and Y.Kawada,The MIT Press (1977),275.J.

[118] U. Carow-Watamura and S. Watamura,`hep-th/9605003`,*Comm.Math.Phys.* **183**, 365 (1997).

[119] U. Carow-Watamura and S. Watamura,`hep-th/9801195`,*Comm.Math.Phys.* **212**,395(2000).

[120] A.P.Balachandran,T.R.Govindarajan,B.Ydri,`hep-th/9911087`;A. P. Balachandran, T. R. Govindarajan ,B. Ydri, `hep-th/0006216` and *Mod.Phys.Lett.***A15**, 1279 (2000);A. P. Balachandran and S. Vaidya, `hep-th/9910129` and *Int.J.Mod.Phys.***A16**,17(2001); A. P. Balachandran, X. Martin and D. O'Connor, `hep-th/0007030` and *Int.J.Mod.Phys.***A**(in press).

[121] F. Garcia Flores, X. Martin and D. OConnor, ıtSimulation of a scalar field on a fuzzy sphere, Int. J. Mod. Phys. A **24**, 3917 (2009 [arXiv:0903.1986 [hep-lat]].

[122] H. Grosse and P. Prešnajder,*Lett.Math.Phys.* **33**, 171 (1995) and references therein.

[123] A.U.Klimyk,*Rept.Math.Phys.***7**,153(1975)

[124] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model* , Nucl. Phys. **B679** , 66 (2004), [hep-th/0307075].

[125] H. Grosse and H. Steinacker, *Finite gauge theory on fuzzy CP^2* , Nucl. Phys. **B707** , 145 (2005), [hep-th/0407089].

[126] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model* , Nucl. Phys. **B679** , 66 (2004), [hep-th/0307075]

[127] H. Steinacker, JHEP **0503**, 075 (2005) [hep-th/0501174].

[128] A. Connes, "Gravity coupled with matter and foundation of non-commutative geometry," Commun. Math. Phys. **182**, 155 (1996) doi:10.1007/BF02506388 [hep-th/9603053].

[129] C. Sochichiu, "M[any] vacua of IIB," JHEP **0005**, 026 (2000) doi:10.1088/1126-6708/2000/05/026 [hep-th/0004062].

[130] J. Hoppe, Ph.D thesis,MIT,1982.

[131] J. Madore, Class. Quant. Grav. **9**, 69 (1992).

[132] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, "Noncommutative gauge theory on fuzzy sphere from matrix model," Nucl. Phys. B **604**, 121 (2001) [arXiv:hep-th/0101102].

[133] A. Y. Alekseev, A. Recknagel and V. Schomerus, "Brane dynamics in background fluxes and non-commutative geometry," JHEP **0005**, 010 (2000) [arXiv:hep-th/0003187].

[134] R. C. Myers, "Dielectric-branes," JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[135] P. Castro-Villarreal, R. Delgadillo-Blando and B. Ydri, "A gauge-invariant UV-IR mixing and the corresponding phase transition for U(1) fields on the fuzzy sphere," Nucl. Phys. B **704**, 111 (2005) [arXiv:hep-th/0405201].

[136] T. Azuma, K. Nagao and J. Nishimura, "Perturbative dynamics of fuzzy spheres at large N," JHEP **0506**, 081 (2005) [arXiv:hep-th/0410263].

[137] T. Azuma, S. Bal, K. Nagao and J. Nishimura, "Nonperturbative studies of fuzzy spheres in a matrix model with the Chern-Simons term," JHEP **0405**, 005 (2004) [arXiv:hep-th/0401038].

[138] D. O'Connor and B. Ydri, "Monte Carlo simulation of a NC gauge theory on the fuzzy sphere," JHEP **0611**, 016 (2006) [arXiv:hep-lat/0606013].

[139] R. Delgadillo-Blando, D. O'Connor and B. Ydri, "Matrix Models, Gauge Theory and Emergent Geometry," arXiv:0806.0558 [hep-th].

[140] R. Delgadillo-Blando, D. O'Connor and B. Ydri, "Geometry in transition: A model of emergent geometry," Phys. Rev. Lett. **100**, 201601 (2008) [arXiv:0712.3011 [hep-th]].

[141] H. Steinacker, "Quantized gauge theory on the fuzzy sphere as random matrix model," Nucl. Phys. B **679**, 66 (2004) [arXiv:hep-th/0307075].

[142] H. Steinacker and R. J. Szabo, "Nonabelian localization for gauge theory on the fuzzy sphere," J. Phys. Conf. Ser. **103**, 012017 (2008) [arXiv:0708.4365 [hep-th]].

[143] H. Steinacker and R. J. Szabo, "Localization for Yang-Mills Theory on the Fuzzy Sphere," Commun. Math. Phys. **278**, 193 (2008) [arXiv:hep-th/0701041].

[144] B. Ydri, "Quantum Equivalence of NC and YM Gauge Theories in 2 D and Matrix Theory," Phys. Rev. D **75**, 105008 (2007) [arXiv:hep-th/0701057].

[145] B. Ydri, "The one-plaquette model limit of NC gauge theory in 2D," Nucl. Phys. B **762**, 148 (2007) [arXiv:hep-th/0606206].

[146] R. Delgadillo-Blando and B. Ydri, "Towards Noncommutative Fuzzy QED," JHEP **0703**, 056 (2007) doi:10.1088/1126-6708/2007/03/056 [hep-th/0611177].

[147] W. Behr, F. Meyer and H. Steinacker, "Gauge theory on fuzzy S**2 x S**2 and regularization on noncommutative R**4," JHEP **0507**, 040 (2005) doi:10.1088/1126-6708/2005/07/040 [hep-th/0503041].

[148] B. Ydri, R. Khaled and R. Ahlam, "Geometry in transition in four dimensions: A model of emergent geometry in the early universe and dark energy,"

[149] R. Delgadillo-Blando and D. O'Connor, "Matrix geometries and Matrix Models," JHEP **1211**, 057 (2012) [arXiv:1203.6901 [hep-th]]. arXiv:1607.06761 [hep-th].

[150] P. Castro-Villarreal, R. Delgadillo-Blando and B. Ydri, "Quantum effective potential for U(1) fields on S**2(L) x S**2(L)," JHEP **0509**, 066 (2005) doi:10.1088/1126-6708/2005/09/066 [hep-th/0506044].

[151] D. E. Berenstein, M. Hanada and S. A. Hartnoll, "Multi-matrix models and emergent geometry," JHEP **0902**, 010 (2009) doi:10.1088/1126-6708/2009/02/010 [arXiv:0805.4658 [hep-th]].

[152] B. Ydri, "Impact of Supersymmetry on Emergent Geometry in Yang-Mills Matrix Models II," Int. J. Mod. Phys. A **27**, 1250088 (2012) doi:10.1142/S0217751X12500881 [arXiv:1206.6375 [hep-th]].

[153] A. P. Balachandran and G. Immirzi, "The Fuzzy Ginsparg-Wilson algebra: A Solution of the fermion doubling problem," Phys. Rev. D **68**, 065023 (2003) doi:10.1103/PhysRevD.68.065023 [hep-th/0301242].

[154] A. P. Balachandran and G. Immirzi, "Duality in fuzzy sigma models," Int. J. Mod. Phys. A **19**, 5237 (2004) doi:10.1142/S0217751X04020671 [hep-th/0408111].

[155] A. P. Balachandran and G. Immirzi, "Fuzzy Nambu-Goldstone physics," Int. J. Mod. Phys. A **18**, 5931 (2003) doi:10.1142/S0217751X03017440 [hep-th/0212133].

[156] D. Dou and B. Ydri, "Topology change from quantum instability of gauge theory on fuzzy CP**2," Nucl. Phys. B **771**, 167 (2007) doi:10.1016/j.nuclphysb.2007.02.010 [hep-th/0701160].

[157] D. O'Connor, B. P. Dolan and M. Vachovski, "Critical Behaviour of the Fuzzy Sphere," JHEP **1312**, 085 (2013) doi:10.1007/JHEP12(2013)085 [arXiv:1308.6512 [hep-th]].

[158] T. Imai and Y. Takayama, "Stability of fuzzy S**2 x S**2 geometry in IIB matrix model," Nucl. Phys. B **686**, 248 (2004) doi:10.1016/j.nuclphysb.2004.03.008 [hep-th/0312241].

[159] A.P. Balachandran, private communication.

[160] J. Medina and D. O'Connor, "Scalar field theory on fuzzy S**4," JHEP **0311**, 051 (2003) doi:10.1088/1126-6708/2003/11/051 [hep-th/0212170].

[161] J. Medina, I. Huet, D. O'Connor and B. P. Dolan, "Scalar and Spinor Field Actions on Fuzzy S^4: fuzzy CP^3 as a S^2_F bundle over S^4_F," JHEP **1208**, 070 (2012) doi:10.1007/JHEP08(2012)070 [arXiv:1208.0348 [hep-th]].

[162] H. C. Steinacker, "One-loop stabilization of the fuzzy four-sphere via softly broken SUSY," JHEP **1512**, 115 (2015) doi:10.1007/JHEP12(2015)115 [arXiv:1510.05779 [hep-th]].

[163] G. Alexanian, A. P. Balachandran, G. Immirzi and B. Ydri, "Fuzzy CP**2," J. Geom. Phys. **42**, 28 (2002) doi:10.1016/S0393-0440(01)00070-5 [hep-th/0103023].

[164] H. Grosse and A. Strohmaier, "Towards a nonperturbative covariant regularization in 4-D quantum field theory," Lett. Math. Phys. **48**, 163 (1999) doi:10.1023/A:1007518622795 [hep-th/9902138].

[165] T. Azuma, S. Bal, K. Nagao and J. Nishimura, "Dynamical aspects of the fuzzy CP**2 in the large N reduced model with a cubic term," JHEP **0605**, 061 (2006) doi:10.1088/1126-6708/2006/05/061 [hep-th/0405277].

[166] H. Grosse and H. Steinacker, "Finite gauge theory on fuzzy CP**2," Nucl. Phys. B **707**, 145 (2005) doi:10.1016/j.nuclphysb.2004.11.058 [hep-th/0407089].

[167] D. J. Gross and E. Witten, "Possible Third Order Phase Transition in the Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446 (1980).

[168] Y. Shimamune, Phys. Lett. B **108**, 407 (1982).

[169] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[170] A. Connes, Academic Press,London, 1994.

[171] J. Frohlich and K. Gawedzki, arXiv:hep-th/9310187.

[172] B. Eynard, Cours de Physique Theorique de Saclay.

[173] N. Kawahara, J. Nishimura and A. Yamaguchi, JHEP **0706**, 076 (2007) [hep-th/0703209].

[174] S. S. Gubser and S. L. Sondhi, Nucl. Phys. B **605**, 395 (2001) [hep-th/0006119].

[175] J. Ambjorn and S. Catterall, Phys. Lett. B **549**, 253 (2002) [hep-lat/0209106].

[176] S. A. Brazovkii, Zh. Eksp. Teor. Fiz **68**, (1975) 175-185.

[177] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP **0002**, 020 (2000) [arXiv:hep-th/9912072].

[178] J. Ambjorn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, JHEP **0005**, 023 (2000) [arXiv:hep-th/0004147].

[179] H. Grosse and R. Wulkenhaar, JHEP **0312**, 019 (2003) [arXiv:hep-th/0307017].

[180] F. Garcia Flores, X. Martin and D. O'Connor, Int. J. Mod. Phys. A **24**, 3917 (2009) [arXiv:0903.1986 [hep-lat]].

[181] F. Garcia Flores, D. O'Connor and X. Martin, PoS LAT **2005**, 262 (2006) [hep-lat/0601012].

[182] X. Martin, JHEP **0404**, 077 (2004) [hep-th/0402230].

[183] M. Panero, JHEP **0705**, 082 (2007) [hep-th/0608202].

[184] J. Medina, W. Bietenholz and D. O'Connor, JHEP **0804**, 041 (2008) [arXiv:0712.3366 [hep-th]].

[185] C. R. Das, S. Digal and T. R. Govindarajan, Mod. Phys. Lett. A **23**, 1781 (2008) [arXiv:0706.0695 [hep-th]].

[186] B. Ydri, JHEP **1403**, 065 (2014) [arXiv:1401.1529 [hep-th]].

[187] D. O'Connor and C. Saemann, JHEP **0708**, 066 (2007) [arXiv:0706.2493 [hep-th]].

[188] C. Saemann, SIGMA **6**, 050 (2010) [arXiv:1003.4683 [hep-th]].

[189] A. P. Polychronakos, arXiv:1306.6645 [hep-th].

[190] J. Tekel, arXiv:1407.4061 [hep-th].

[191] V. P. Nair, A. P. Polychronakos and J. Tekel, Phys. Rev. D **85**, 045021 (2012) [arXiv:1109.3349 [hep-th]].

[192] J. Tekel, Phys. Rev. D **87**, no. 8, 085015 (2013) [arXiv:1301.2154 [hep-th]].

[193] B. Ydri, arXiv:1410.4881 [hep-th].

[194] W. Bietenholz, F. Hofheinz and J. Nishimura, JHEP **0406**, 042 (2004) [hep-th/0404020].

[195] F. Lizzi and B. Spisso, Int. J. Mod. Phys. A **27**, 1250137 (2012) [arXiv:1207.4998 [hep-th]].

[196] H. Meja-Daz, W. Bietenholz and M. Panero, arXiv:1403.3318 [hep-lat].

[197] A. P. Balachandran, B. P. Dolan, J. H. Lee, X. Martin and D. O'Connor, J. Geom. Phys. **43**, 184 (2002) [hep-th/0107099].

[198] L. Onsager, Phys. Rev. **65**, 117 (1944).

[199] U. Wolff, Phys. Rev. Lett. **62**, 361 (1989).

[200] A. Connes, Noncommutative Geometry (Academic Press, 1994).

[201] G. Landi, An Introduction to Noncommutative Spaces and their Geometries (Springer-Verlag, 1997).

[202] J. Madore, An Introduction to Noncommutative Geometry and its Physical Applications (Cambridge University Press, 1999).

[203] J.M. Gracia-Bondia, J.C. Varilly and H. Figueroa, Elements of Noncommutative Geometry (Birkhauser, 2000).

[204] P.-M. Ho and Y.-S. Wu, Noncommutative Gauge Theories in Matrix Theory, Phys. Rev. **D58** (1998) 066003 [hep-th/9801147].

[205] G.A. Elliott, *On the K-Theory of the C^* Algebra Generated by a Projective Representation of a Torsion-Free Discrete Abelian Group, in: Operator Algebras and Group Representations 1*, eds. G. Arsene et al. (Pitman, London, 1984), p. **159.**

[206] A. Astashkevich, N.A. Nekrasov and A. Schwarz, *On Noncommutative Nahm Transform,* Commun. Math. Phys. **211** (2000) 167 [hep-th/9810147]

[207] A. Konechny and A. Schwarz, *BPS States on Noncommutative Tori and Duality,* Nucl. Phys. **B550** (1999) 561 [hep-th/9811159].

[208] D. OConnor and C. Saemann, *Fuzzy Scalar Field Theory as a Multitrace Matrix Model,* JHEP 0708, **066** (2007) [arXiv:0706.2493 [hep-th]].