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وزارة التعليم العالي و البحث العلمي جـامعة
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Département de Génie Civil

Polycopié pédagogique

Titre : DYNAMIC of STRCUTURES
(DDS 2)

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Cours destiné aux étudiants de

Master (spécialité et niveau) : Master 1 Structures

Année : 2025

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Chapter 1

Free vibration of MDOF systems

1. Introduction

Vibration in multi-degree-of-freedom (MDOF) systems is a fundamental concept in structural dynamics that plays a crucial role in the analysis and design of complex structures. In real-life situations, structures are inherently complex, containing distributed mass and stiffness throughout their components. To accurately analyze these structures, it is more appropriate to discretize them into a series of interconnected masses and stiffnesses. This discretization process typically involves lumping masses at the center of gravity of the discretized elements, resulting in a model that requires multiple displacement coordinates to define the structure's deformed position at any given time.

This approach leads to the concept of multiple degrees of freedom, where each degree represents an independent way in which the system can move or deform. Unlike single-degree-of-freedom systems, MDOF systems require multiple coordinates to describe their motion, making them more representative of real-world structures such as multi-story buildings, bridges, and complex mechanical systems. These systems are characterized by their ability to vibrate in multiple modes, each with its own natural frequency and mode shape.

The analysis of MDOF systems is essential for civil engineers, as it provides insights into how structures respond to various dynamic loads, including earthquakes, wind, and machinery-induced vibrations. Understanding MDOF vibrations allows engineers to predict and mitigate potential resonance phenomena, optimize structural designs, and ensure the safety and serviceability of structures under dynamic conditions. The complexity of MDOF systems necessitates the use of advanced mathematical techniques, including matrix methods, eigenvalue analysis, and modal decomposition, to solve the coupled equations of motion that govern their behavior.

As structures become increasingly complex and performance requirements more stringent, mastering the principles of MDOF vibration becomes indispensable for civil engineering students pursuing advanced degrees. This knowledge forms the foundation for more advanced topics in structural dynamics, such as seismic analysis, vibration control, and structural health monitoring. The study of MDOF systems provides a comprehensive framework for understanding and analyzing the dynamic behavior of complex structures, enabling engineers to design safer, more efficient, and more resilient infrastructure.

2. Two Degree of Freedom System

In general, the dynamic response of a structure cannot be described adequately by a SDOF system model because the response includes time variations of the displacement shape as well as its amplitude. Such behaviour can be described only in terms of more than one displacement coordinate; the motion must be represented by more than one degree of freedom. The number of degrees of freedom, that is, displacement components, to be considered is left to the judgment of the analyst. However, large numbers provide better approximations of the true dynamic behaviour, but in many cases excellent results can be obtained with only two or three degrees of freedom. Therefore a complex structure has to be idealized into a number of masses and springs and assumed interconnected together. Although a large number of degrees of freedom are usually associated with a complex structural system, acceptable results may be obtained from the analysis of the response of only a few degrees of freedom. A number of masses and springs interconnected together in a system constitute a multi degree of freedom (MDOF) system. In such an arrangement each mass displaces in its own way independent of the other masses while the entire system vibrates. A force can be applied externally on any of the masses independently and excite the system. Different forces can be applied on different masses.

Derivation of equation of motion for MDOF system straight away is cumbersome in terms of comprehending its dynamic behavior. Therefore for quick and better understanding of the dynamic behavior we consider here a simple two mass system with two coordinates for the derivation of the equation of motion. This two mass system is a special case of the MDOF system. In this we assume that each mass is constrained to move only in the horizontal plane. The associated displacements, therefore, represent the two independent co-ordinates or degrees of freedom which will be used to define the configuration of the system.

This chapter deals with the dynamic analysis of Two DOF systems for which the matrix method is followed. Initially a formal analysis is presented in which it will be demonstrated that the dynamic analysis of a structural system is a form of the classical eigenvalue problem of matrix algebra.

2.1.Equation of Motion for a System with TDOF

We assume that the masses are excited by forces $P_1(t)$ and $P_2(t)$ as in Fig. 1.1. The equation of mass is obtained by considering the dynamic equilibrium of each mass in turn shown in Fig. 1.1(a) and Fig. 1.1(b).

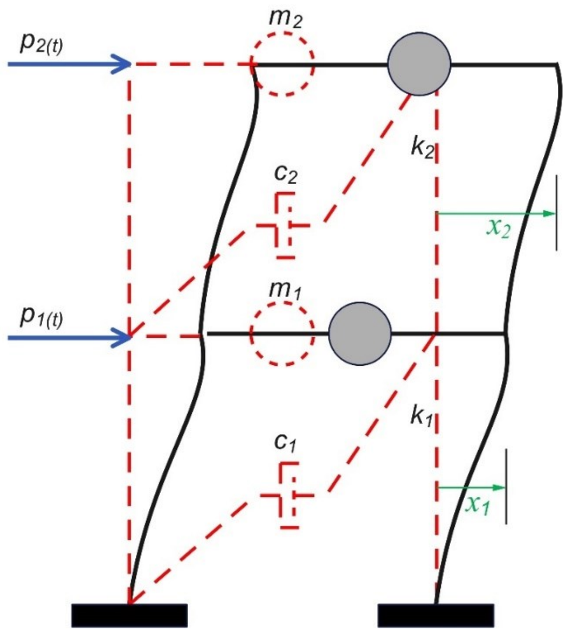


Fig. 1.1 Two-story shear frame system

In general, four types of forces act on each mass, namely, inertia, damping, elastic and applied forces, respectively. The elastic force acting on a mass depends not only upon the displacement of the mass under consideration, but also upon the displacement of the adjacent mass. Similarly, the damping force depends on the velocity of the mass as well as on the adjacent mass too. The equation of dynamic equilibrium for each mass may now be written as:

For mass 1

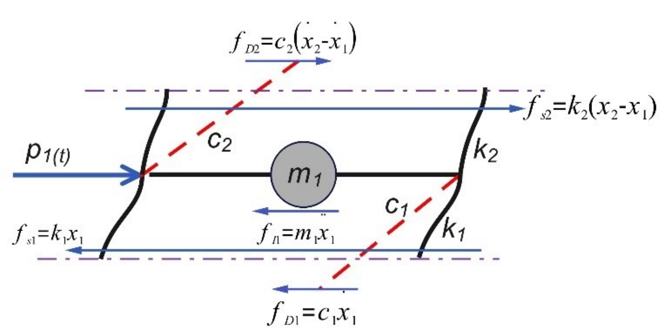


Fig. 1.1(a) Dynamic equilibrium of mass 1

$$m_1\ddot{x}_1 + c_1\dot{x}_1 - c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_1x_1 - k_2\left(x_2 - x_1\right) - p_1\left(t\right) = 0$$

$$m_1\ddot{x}_1 + c_1\dot{x}_1 - c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_1x_1 - k_2\left(x_2 - x_1\right) - p_1\left(t\right) = 0$$

$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = p_1\left(t\right)$$

$$\Rightarrow m_1\ddot{x}_1 + 0\ddot{x}_2 + \left(c_1 - c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = p_1\left(t\right) \tag{1.1}$$

For mass 2:

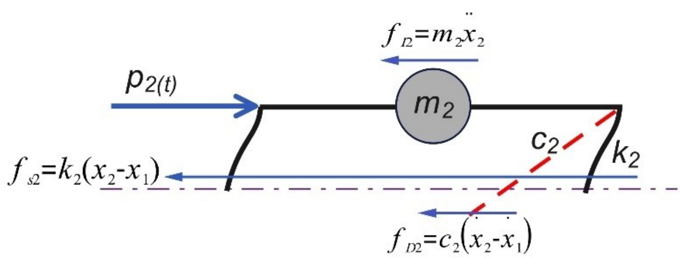


Fig. 1.1(b) Dynamic equilibrium of mass 2

$$m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) - p_2\left(t\right) = 0$$

$$m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) - p_2\left(t\right) = 0$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 - c_2\dot{x}_1 + k_2x_2 - k_2x_1 = p_2\left(t\right)$$

$$\Rightarrow O\ddot{x}_1 + m_2\ddot{x}_2 - c_2\dot{x}_1 + c_2\dot{x}_2 - k_2x_1 + k_2x_2 = p_2\left(t\right)$$

$$\Rightarrow m_1\ddot{x}_1 + o\ddot{x}_2 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = p_1\left(t\right) \tag{1.2}$$

Equations (1.1) and (1.2) are interconnected and hence are called coupled. They can be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & (c_1 + c_2) \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 & -K_1 \\ -K_1 & (K_1 + K_2) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} P_1(t) \\ P_2(t) \end{Bmatrix} \tag{1.3}$$

In compact form Eq. (1.3) can be written as

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [K]\{x\} = \{P(t)\} \tag{1.4}$$

Equation (1.4) can also be rewritten as

$$[m]\{\ddot{x}\} + \{f_D\} + \{f_s\} = \{P(t)\} \tag{1.5}$$

where

$$\{f_D\} = \begin{Bmatrix} f_{D1} \\ f_{D2} \end{Bmatrix} = [c]\{\dot{x}\}\begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \tag{1.6}$$

$$\{f_s\} = \begin{Bmatrix} f_{s1} \\ f_{s2} \end{Bmatrix} = [K]\{x\} = \begin{bmatrix} K_1 & -K_1 \\ K_1 & K_1 + K_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{1.7}$$

Similar to SDOF system, the equation of motion for Two DOF is expressed generically as:

$$\overline{\mathrm{m}}\tilde{\ddot{\mathrm{x}}} + \overline{\mathrm{c}}\tilde{\dot{\mathrm{x}}} + \overline{\mathrm{K}}\,\tilde{\mathrm{x}} = \tilde{\mathrm{P}}(\mathrm{t}) \tag{1.8}$$

In Eq. (1.8), $\overline{m}$ - mass matrix; $\overline{c}$ - damping matrix; $\overline{K}$ - stiffness matrix and $\overline{P}$ - force matrix and $\tilde{\ddot{x}}$, $\tilde{\dot{x}}$, and $\tilde{x}$ are, respectively, acceleration, velocity and displacement vectors.

2.2. Free Vibration of Two Degree System without Damping

Free vibration occurs when no external force or support motion acts on the system. It begins when the system is disturbed from its equilibrium position, either through initial displacements, initial velocities, or both. As discussed earlier, the vibration of a two-degree-of-freedom system is generally described by Eq. (1.8).

In the case of free vibration, $\overline{P}(t) = 0$ because no dynamic force is applied. For a system without damping, $\overline{c} = 0$. Therefore an undamped system under free vibration is governed by

$$\overline{m}\tilde{\ddot{x}} + \overline{K}\tilde{x} = 0 \tag{1.9}$$

For a two-degree-of-freedom system, Equation (1.9) consists of two coupled homogeneous differential equations. These equations are linked through either the mass matrix, the stiffness matrix, or both. The term "two" here refers to the number of degrees of freedom (DOFs) in the system. The goal is to determine a solution, x(t), for Equation (1.9) that fulfills the given initial conditions.

$$\tilde{x} = \tilde{x}(0) \text{ and } \tilde{\dot{x}} = \tilde{\dot{x}}(0) \tag{1.10}$$

2.2.1. Normal Modes and Natural Frequencies

Before determining the dynamic response *x(t)* of the structure, it is essential to first evaluate the frequencies of the two-degree-of-freedom (DOF) system, similar to what was done for a single-degree-of-freedom (SDOF) system. In the SDOF system, there is only one frequency of vibration, which depends on the system's mass and stiffness, as it involves a single mass. However, in a two DOF system, there are two masses, resulting in two distinct frequencies of vibration. While the SDOF system's response involved a single displacement with an easily identifiable deflected shape, the two DOF system allows its masses to displace in various ways depending on the vibration frequency. This leads to characteristic deformed shapes under different frequencies, known as modes. The natural period of vibration T_{nm} for a two DOF system represents the time taken for one complete cycle of simple harmonic motion in one of these natural modes. The corresponding natural circular frequency of vibration is ω_{nm}, and the natural cyclic frequency of vibration is f_{nm} where:

$$T_{nm} = \frac{2\pi}{\omega_{nm}}; \qquad f_{nm} = \frac{1}{T_{nm}} \tag{1.11}$$

Here subscript m refers to modes, (m = 1, 2). Figure 5.12 and 5.13 show two natural periods T_{nm} and natural frequency ω_{nm} (m = 1, 2) of the two-storey building vibrating in its natural modes $\phi_{im} = (\phi_{1m}\ \phi_{2m})^T$. Here subscript i refers to masses (i = 1, 2). The lower of the two natural vibration frequencies is referred to as ω_{n1}, while the higher frequency is labeled as ω_{n2}. Similarly, the longer natural vibration period is designated as T_{n1}, and the shorter period is identified as T_{n2}.

A system with two degrees of freedom (DOF) is found to have two natural frequencies. The smaller of these is referred to as the fundamental frequency or the first mode, while the larger one is known as the second mode. By applying specific initial conditions, it is possible to make the system vibrate entirely at one of these natural frequencies. In such cases, both masses will simultaneously pass through their equilibrium or mean position and also reach their maximum displacement at the same time. This type of vibration pattern is called the principal mode of vibration. During this principal mode, the amplitude of vibration for any one of the masses is referred to as the normal mode of vibration, which describes the displacement configuration. This normal mode depends solely on how the mass and stiffness are distributed within the system.

2.2.2. Mode Shape

A two-story shear frame is illustrated in Fig. 1.2 which depicts the first natural modes of vibration of this frame at various time points: a, b, c, and d. In a two-degree-of-freedom (DOF) or multi-degree-of-freedom (MDOF) system, a natural mode of vibration represents its characteristic deflected shape when free vibration occurs due to specific distributions of displacements across different DOFs. If the two-DOF system is displaced into the shapes shown in these figures and then released, it will undergo simple harmonic motion, consistently maintaining the initial deflected shape. Notably, during this motion, the displacements of both floors move in the same direction.

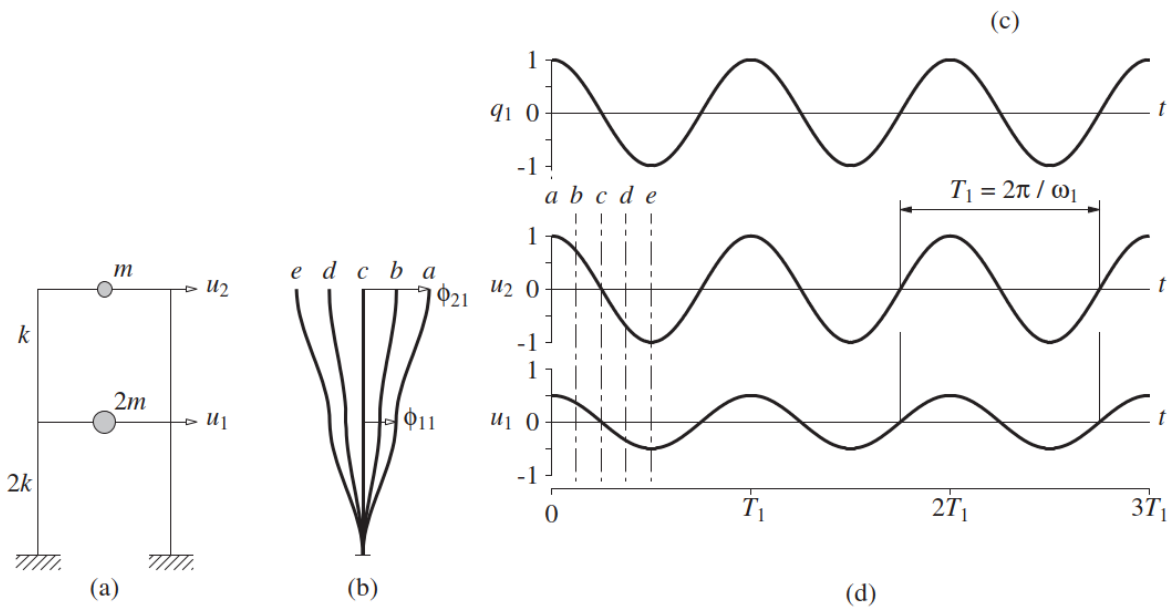


Fig. 1.2 Free vibration of an undamped system in its first natural mode of vibration

The two-story frame oscillates naturally in the mode shape illustrated in Fig. 1.2(b), at a frequency denoted as ω_{n1}. The natural period of vibration is $T_{n1} = \dfrac{2\pi}{\omega_{n1}}$. The mode shape is described by $\phi_{i1} = (\phi_{11} \;\; \phi_{21})^T$. The mode shape visually illustrates the relative amplitudes of two coordinates and how their phase angles are related. At a vibrating frequency of ω_{n1}, the displacement of the topmost mass in the mode shape is referred to as the modal coordinate or normal coordinate, denoted as q_1, which is a scalar value. Figure 1.2(c) depicts how this modal coordinate changes over time, while Figure 1.2(d) shows the displacement history of this degree of freedom (DOF).

The second natural mode of vibration of the same two-storey frame is shown in Fig. 1.3(b) . The frequency of vibration of this second mode is ω_{n2} and period $T_{n2} = \frac{2\pi}{\omega_{n2}}$.

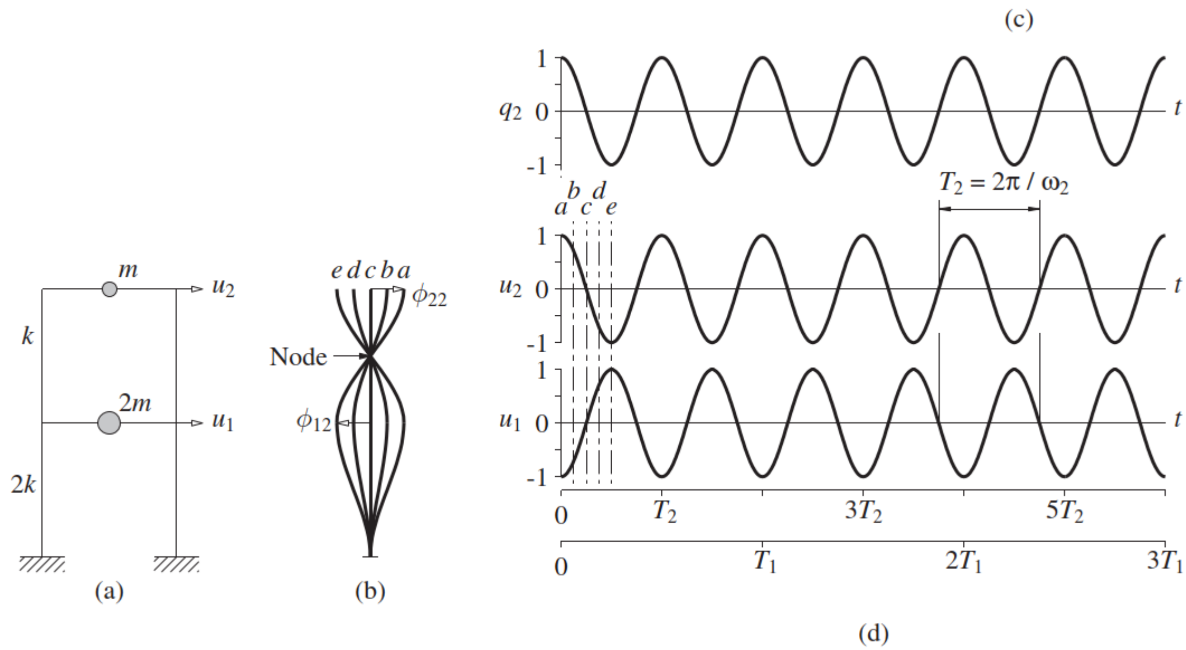


Fig. 1.3 Free vibration of an undamped system in its second natural mode of vibration

In this mode, the displacements of the two floors move in opposite directions, with a point of zero displacement known as a node. A node is a stationary point where the amplitude shifts between positive and negative or vice versa. The fundamental mode of vibration includes the minimum possible modes, including zero. As the number of modes increases, the number of nodes rises accordingly. Mode shapes help identify these nodal points within the system. The time variation of the modal coordinate q_2 is illustrated in Fig. 1.3(c), while Fig. 1.3(d) shows the displacement history of both degrees of freedom (DOFs) corresponding to this second mode of vibration. The second mode shape is denoted by $\phi_{i2} = (\phi_{12}\ \ \phi_{22})^T$.

If the two masses are initially displaced equally in the same direction, the system will vibrate at its first natural frequency. Conversely, if the initial displacement is equal but in opposite directions, the system will vibrate in its second principal mode at the second natural frequency. However, if the masses are given unequal initial displacements in any direction, their motion will be the superposition of two harmonic motions associated with the two natural frequencies.

The conclusions from the above analysis may be summarised as follows:

1. The normal or natural modes are the free, undamped periodic oscillations within which linear combinations represent the position of the system at every moment.

2. For every such normal mode all the masses of the system oscillate in phase, that is, at every moment the ratio of the displacements of the damped masses remains constant. As a result, all masses go through rest position and reach maximum amplitude simultaneously.

3. The number n of normal modes is equal to the number of degrees of freedom. Every normal mode is related to a natural frequency or period of vibration known as the natural period. The normal mode with the longest natural period is by definition the first or fundamental normal mode.

2.2.3. Eigen Value Problem

This section presents the eigenvalue problem, which provides the natural frequencies and vibration modes of a system. The free vibration of an undamped system, occurring in one of its natural modes, is visually depicted in Figures 5.12 and 5.13 for a two-degree-of-freedom (DOF) system and can be described mathematically as:

$$\tilde{x}(t) = q_m(t)\phi_m \tag{1.12}$$

In this context, q_m represents the modal coordinates, while ϕ_m denotes the deflected shape, which remains constant over time. To describe how the displacements of the masses vary with time, we use a simple harmonic function, as illustrated in Figures 1.2(d) and 1.3(d).

$$q_m(t) = A_m \cos \omega_{nm} t + B_m \sin \omega_{nm} t \tag{1.13}$$

where A_m and B_m are integration constants. The constants are determined by applying the initial conditions, which play a crucial role in initiating the vibratory motion. We now substitute Eq. (1.13) into Eq. (1.12) and get

$$\tilde{x}(t) = \phi_m A_m \cos \omega_{nm} t + B_m \sin \omega_{nm} t \tag{1.14}$$

where ω_{nm} and ϕ_m are unknowns. Now Eq. (1.14) is substituted in Eq. (1.9) to get

$$\left[-\omega_{nm}{}^2 \overline{m}\, \phi_m + \overline{K} \phi_m \right] q_m(t) = \overline{0} \tag{1.15}$$

We know that m = 1 refers to first frequency ω_{n1} and the corresponding mode shape ϕ_1 which is given by

$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix}$$

and m = 2 refers to second frequency ω_{n2} and the corresponding mode shape ϕ_2 which is given by

$$\phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix}$$

In expanded form Eq. (1.15) can be written as

$$-\omega_{nm}^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.16}$$

Two possibilities are there as solutions to Eq. (1.15). First $q_m(t) = 0$. This means $\tilde{x}(t) = \tilde{0}$ indicating that the system cannot exhibit any motion. This solution is considered trivial. Alternatively, another possibility is...

$$-\omega_{nm}^2 \overline{m}\,\phi_m + K\phi_m = 0 \tag{1.17}$$

From Eq. (1.17), we get

$$\overline{K}\phi_m = \omega_{nm}^2 \overline{m}\,\phi_m \tag{1.18}$$

Equation (1.18) offers a valuable condition and represents an algebraic equation, commonly referred to as the matrix eigenvalue problem. In this context, the stiffness matrix $\overline{K}$ and the mass matrix $\overline{m}$ are given. Therefore, the task is reduced to finding the scalar ω_{nm}^2 and the vector ϕ_m. We can now rewrite Eq. (1.18) as follows to indicate its formal solution

$$\left\lfloor \overline{K} - \omega_{nm}^2 \overline{m} \right\rfloor \phi_m = \overline{0} \tag{1.19}$$

The given expression represents a set of m homogeneous algebraic equations for the m elements ϕ_{im} (i = 1, 2). The trivial solution to this equation set is $\phi_m = \overline{0}$. However, this solution is unacceptable as it implies that the system cannot undergo any motion. The non-trivial solution of Eq. (1.19) is given by

$$\left|\overline{K} - \omega_{nm}{}^2\overline{m}\right| = \overline{0} \tag{1.20}$$

Equation (1.20), referred to as the characteristic or frequency equation, is expressed as a determinant. When expanded, this determinant produces a polynomial of degree m in terms of $\omega_{nm}{}^2$. Since the structural mass matrix $\overline{m}$ and stiffness matrix $\overline{K}$ are symmetric and represent physical quantities, they are positive definite. This ensures that the roots of Equation (1.20) for $\omega_{nm}{}^2$ are both real and positive. For structures where the support conditions eliminate rigid body motion, the stiffness matrix $\overline{K}$ retains its positive definite property. This scenario is particularly relevant to civil engineering structures, which are typically restrained and meet this condition. Additionally, in the mass matrix $\overline{m}$, the diagonal elements represent lumped masses at all degrees of freedom (DOFs), and these values are always non-zero, further guaranteeing its positive definite nature.

The 2 roots of Eq. (1.20) result in 2 natural frequencies ω_{n1} and ω_{n2} of vibration. These 2 roots of the characteristic equation are called *eigenvalues*, *characteristic values*, or *normal values*.

When the natural frequencies, denoted as ω_{n1} and ω_{n2}, are determined by solving Eq. (1.20), the corresponding vectors, ϕ_1 and ϕ_2, can then be found by solving Eq. (1.19). However, the eigenvalue problem does not define the exact amplitudes of these vectors. Instead, it provides only the relative values of the displacements, represented as ϕ_{im}, where i = 1 or 2 for each mode. For a two-degree-of-freedom (TDOF) system, two independent vectors, ϕ_1 and ϕ_2, are obtained, each associated with one of the natural vibration frequencies, ω_{n1} or ω_{n2}. These vectors represent the natural modes or mode shapes of vibration and are also referred to as eigenvectors, characteristic vectors, or normal modes.

In general, if a vibrating system has m degrees of freedom (DOFs), it will exhibit m natural frequencies of vibration, denoted as ω_{np}, where p = 1, 2, ..., m, representing the mode number. The first mode (p = 1) is referred to as the fundamental mode. These m natural frequencies are typically listed in ascending order, such that $\omega_1 < \omega_2 < \omega_3 < \ldots < \omega_m$. Each frequency has an associated natural period (T_{np}) and a corresponding natural mode (ϕ_p). The term "natural" is used to emphasize that these vibration characteristics are inherent to the structure during free vibration. They arise naturally from the system's properties and depend solely on the structure's mass and stiffness.

Example 1.1

An undamped two DOF system is shown in Figure (I). Determine its frequencies and mode shapes.

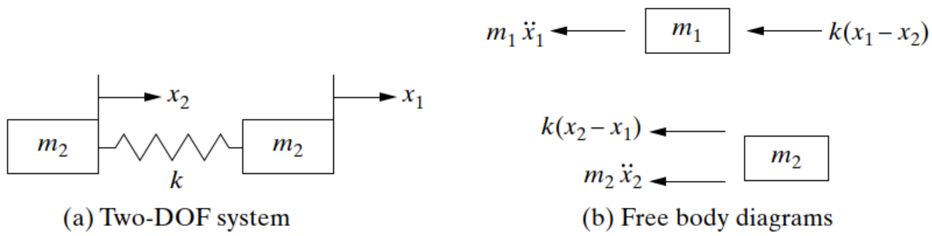


Figure 1 Two-DOF system

Solution 1.1

The free body diagrams of the mass m_1 and m_2 are shown in the same Figure. The equations of motion can be written as:

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0$$

And $$m_2\ddot{x}_2 + k(x_2 - x_1) = 0 \quad \text{(i)}$$

Let $$x_1 = x_{10}\sin\omega t, \quad x_2 = x_{20}\sin\omega t \quad \text{(ii)}$$

Substituting these values in Equation (i) gives,

$$(-)\begin{bmatrix} m_1\omega^2 & 0 \\ 0 & m_2\omega^2 \end{bmatrix}\begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or,

$$\begin{bmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{bmatrix}\begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(iii)}$$

It can be written in short as follows:

$$[[K] - [M]\omega^2][X_0] = \{0\} \quad \text{(iv)}$$

where [K] and [M] are stiffness and mass matrices of the system. Its solution is given by setting the determinant to zero,

$$\det[K - M\omega^2] = 0 \quad \text{(v)}$$

This is referred to as the characteristic equation. Equations (iv) and (v) are very important for eigenvalue analysis.

$$m_1 m_2 \omega^4 - (m_1 + m_2) k \omega^2 = 0$$

Its roots are given by: $\omega_1^{*2} = 0 \quad \text{or} \quad \omega_2^{*2} = \frac{(m_1 + m_2)k}{m_1 m_2}$

or, the frequencies are given by: $\omega_1^{*} = 0 \quad \text{and} \quad \omega_2^{*} = \sqrt{\frac{(m_1 + m_2)}{m_1 m_2} k}$

The mode shapes can be obtained by substituting the value of ω in Equation (iii). Substituting $\omega^2 = 0$ in Equation (iii) gives $x_{10} = x_{20} = 1$. Similarly, substituting the other value,

$$\omega^2 = \frac{(m_1 + m_2)k}{m_1 m_2}$$

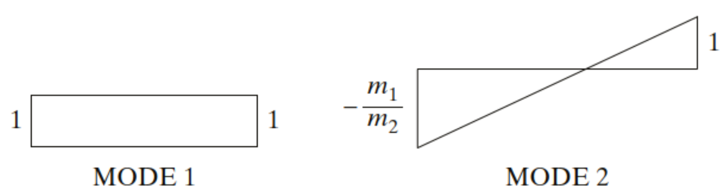
$$(k - m_1 \omega^2) x_{10} - k x_{20} = 0$$

Solving for mode shape ratio,

$$\frac{x_{20}}{x_{10}} = 1 - \frac{m_1}{k}\omega^2 = 1 - \frac{m_1 + m_2}{m_2} = -\frac{m_1}{m_2}$$

Thus if, $x_{10} = 1, x_{20} = -\frac{m_1}{m_2}.$

Now the mode shapes can be plotted as shown in Figure below. It can be seen that the spring remain undeformed in Mode 1, that is, Mode 1 is a rigid body displacement mode.



3. Multi-degree of freedom System (MDOF)

Let's examine a three-story shear frame, as illustrated in Fig. 1.4. When subjected to external forces $P_1(t)$, $P_2(t)$ and $P_3(t)$, the system's state at any given moment is described by its displacements x_i, velocities $\dot{x}_i$, and accelerations $\ddot{x}_i$.

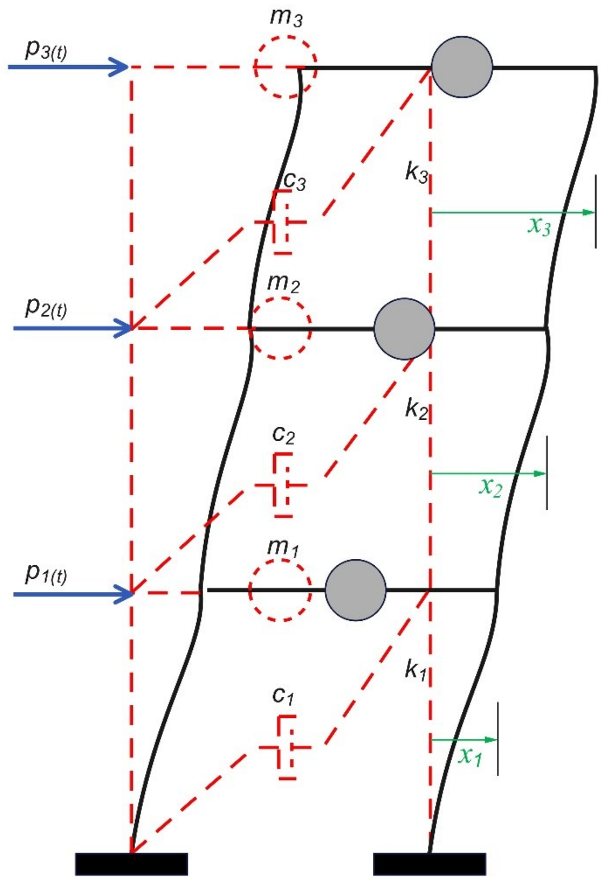


Fig. 1.4 A three-story shear frame system

The relationship between the elastic forces f_{si} in the stiffness component and the displacements x_i is described by Eq. (1.7). Similarly, the damping forces f_{Di} in the damping component are related to velocities $\dot{x}_i$ through Eq. (1.6). Likewise, the inertia forces f_{Ii} in the mass component are tied to accelerations $\ddot{x}_i$ by $\widetilde{f}_I = \overline{m}\ddot{\widetilde{x}}$. Thus, we can interpret the external forces $\widetilde{P}(t)$ as being distributed across these three structural components. Thus, $\bar{f}_s + \bar{f}_D + \bar{f}_I$ must be equal to applied external forces $\widetilde{P}(t)$ leading to Eq. (1.5).

3.1. Elastic Forces (Stiffness Component)

In this section, we will use the method of superposition and the concept of stiffness influence coefficients to establish a relationship between the external elastic forces, denoted as f_{sp}, acting on the stiffness components of a structure, and the resulting displacements, x_q. Considering an 8-floor shear frame system, we begin by applying a unit displacement along a specific degree of freedom (DOF) p, while keeping all other displacements fixed at zero. For instance, a unit displacement $x_1 = 1$ is applied, causing all other displacements (x_2, x_3, x_4, x_5, x_6, x_7, and x_8) to remain zero. The resulting deflected shape of the frame due to this unit displacement. To maintain this deflected configuration, forces must be applied along all DOFs. These forces are represented as K_{11}, K_{21}, ……, K_{81} and are known as stiffness influence coefficients (K_{pq}). These coefficients represent the forces induced at DOF p when a unit displacement is applied at DOF q, where p = 1, 2, ……, 8 and q = 1, 2, ……, 8. Similarly, a unit rotation ($x_6 = 1$) is applied while keeping all other DOFs (x_1, x_2, x_3, x_4, x_5, x_7, and x_8) fixed at zero. The corresponding deflected profile and stiffness influence coefficients (K_{16}, K_{26}, ……, K_{86}. Here, positive values are assumed for anticlockwise moments and translations along the x-direction; however, some forces may act in reverse directions and should be treated as negative to align with the imposed deformations. These principles can be extended to a system with N degrees of freedom (DOFs), where the forces f_{sp} at DOF p, associated with displacements x_q for q = 1 to N, can be determined through superposition as:

$$f_{sp} = K_{p1}x_1 + K_{p2}x_2 + K_{p3}x_3 + \ldots\ldots\ldots\ldots\ldots + K_{pN}x_N \tag{1.21}$$

For each value of p = 1 to N, there is one such equation. We can now write in matrix form the set of N equations as:

$$\begin{Bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \\ . \\ . \\ f_{sN} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & . & . & K1N \\ K_{21} & K_{22} & K_{23} & .. & .. & K_{2N} \\ K_{31} & K_{32} & K_{33} & .. & .. & K_{3N} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ K_{N1} & K_{N2} & K_{N3} & .. & .. & K_{NN} \end{bmatrix} . \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_N \end{Bmatrix} \tag{1.22}$$

In compact form Eq. (1.22) can be given as

$$\tilde{f}_s = \overline{K}\tilde{x} \tag{1.23}$$

where $\overline{K}$ is a *stiffness matrix* and also symmetric, i.e., $K_{pq} = K_{qp}$.

The stiffness matrix $\overline{K}$ for a discretized system can be constructed using any of the structural analysis methods that readers may already know. To determine the *q*th column of $\overline{K}$, we calculate the forces K_{pq} (where p = 1, 2, 3, ..., N) required to produce a displacement of $x_q = 1$, while keeping all other displacements ($x_q = 0$) fixed. This process of determining stiffness influence coefficients is referred to as the system approach. A widely used alternative is the direct stiffness method, where the stiffness matrices of individual elements are assembled together to form the overall structural stiffness matrix.

3.2. Damping Forces (Damping Component)

In Chapter 2, we explored how damping serves as a mechanism for energy dissipation in structures. It can be simplified and represented as equivalent viscous damping since various damping mechanisms may exist within a structure. Building on this concept, we will now establish a relationship between the external forces, denoted as f_{Dp}, acting on the damping component of the structure, and the velocity $\dot{x}_q$. To do this, we apply a unit velocity along a specific degree of freedom (DOF) q, while ensuring that velocities at all other DOFs remain zero. This action generates internal damping forces that resist the applied velocity, requiring external forces to maintain equilibrium. The damping influence coefficient c_{pq} represents the external force at DOF p caused by a unit velocity at DOF q. Consequently, for a system with N degrees of freedom, the resulting force f_{Dp} is determined by the velocities across all DOFs, from q = 1 to N. The forces f_{D1}, f_{D2}, f_{D3}, f_{D4}, f_{D5}, f_{D6}, f_{D7}, and f_{D8} act at the six nodes in 8 degrees of freedom are obtained by superposition. Therefore:

$$f_{Dp} = c_{p1}\dot{x}_1 + c_{p2}\dot{x}2 + c_{p3}\dot{x}_3 + ...+ c_{pN}\dot{x}_N \tag{1.24}$$

Using all influence coefficients for p = 1 to N and expressing them in matrix form, we get

$$\begin{Bmatrix} f_{D1} \\ f_{D2} \\ f_{D3} \\ . \\ . \\ f_{DN} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & . & . & c1N \\ c_{21} & c_{22} & c_{23} & .. & .. & c_{2N} \\ c_{31} & c_{32} & c_{33} & .. & .. & c_{3N} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ c_{N1} & c_{N2} & c_{N3} & .. & .. & c_{NN} \end{bmatrix} . \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ . \\ . \\ \dot{x}_N \end{Bmatrix} \tag{1.25}$$

In compact form we can write Eq. (1.25) as:

$$\bar{f}_D = \bar{c}\tilde{\dot{x}} \tag{1.26}$$

where $\bar{c}$ is the damping matrix of the structure.

It is typically neither practical nor feasible to directly determine the coefficients c_{pq} of the damping matrix based solely on the structure's geometry. Instead, a common approach is to assign damping ratios to a multi-degree-of-freedom (MDOF) system, similar to how it is done for single-degree-of-freedom (SDOF) systems. These damping ratios are usually derived from experimental data collected from structures with similar characteristics.

3.3. Inertia Forces (Mass component)

In this section, we aim to establish a connection between the external forces f_{Ip} acting on the mass components of a structure and the resulting acceleration $\ddot{x}_q$. To achieve this, we apply a unit acceleration along a specific degree of freedom (DOF) q, while ensuring that all other DOFs maintain zero acceleration. Based on D'Alembert's principle, the imaginary inertia forces generated by this acceleration act in opposition to it at the nodes. To maintain equilibrium in the structure, external forces must be applied at these nodes. The inertia force at any node is determined by multiplying the mass influence coefficient with the unit acceleration along the corresponding DOF. The mass influence coefficient, m_{pq}, represents the external force in DOF p caused by a unit acceleration along DOF q. For instance, we illustrate the external forces f_{I1}, f_{I2}, f_{I3}, f_{I4}, f_{I5}, f_{I6}, f_{I7}, and f_{I8} acting on various nodes. Correspondingly, the mass influence coefficients m_{11}, m_{21}, m_{31}, m_{41}, m_{51}, m_{61}, m_{71}, and m_{81} at these nodes are due to unit acceleration applied along DOF 1. In general, for a system with N degrees of freedom, the external force f_{Ip} resulting from unit acceleration $\ddot{x}_q$ across all DOFs (q = 1 to N) can be determined using superposition principles.

$$f_{Ip} = m_{p1}\ddot{x}_1 + m_{p2}\ddot{x}_2 + m_{p3}\ddot{x}_3 + \dots\dots\dots + m_{pN}\ddot{x}_N \tag{1.27}$$

We can vary p = 1 to N in Eq. (1.27). For each value of p we have one equation. We can arrange the set of N equations in matrix form as

$$\begin{Bmatrix} f_{I1} \\ f_{I2} \\ f_{I3} \\ . \\ . \\ f_{IN} \end{Bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & .. & .. & m_{1N} \\ m_{21} & m_{22} & m_{23} & .. & .. & m_{2N} \\ m_{31} & m_{32} & m_{33} & .. & .. & m_{3N} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ m_{N1} & m_{N2} & m_{N3} & .. & .. & m_{NN} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ . \\ . \\ \ddot{x}_N \end{Bmatrix} \tag{1.28}$$

We can write Eq. (1.28) in compact form as

$$\tilde{f}_I = \overline{m}\ddot{x} \tag{1.29}$$

where $\overline{m}$ is the *mass matrix.* The mass matrix is symmetric similar to the stiffness matrix. That is $m_{ij} = m_{ji}$.

In most cases, the mass of a structure is spread out across its entirety. However, for the purpose of dynamic analysis, we can simplify this by treating the mass as if it is concentrated or "lumped" at specific points, known as the nodes, within the discretized structure. This approach is generally considered to provide accurate results. Each structural element connects two nodes, and its mass is evenly divided between them—half is assigned to one node and the other half to the second node. When multiple elements converge at a single node, the contributions from all these elements are added together at that point.

3.4. Equations of Motion for MDOF System

The governing equation of motion for a multi-degree-of-freedom (MDOF) system subjected to an external dynamic force, $P_p(t)$, where p = 1 to N, can be constructed by combining the three key components previously discussed. The structure's dynamic response to this external force is characterized by its displacements, $x_p(t)$; velocities, $\dot{x}_p(t)$; and accelerations, $\ddot{x}_p(t)$, for each degree of freedom (p = 1 to N). The applied dynamic force, $P_p(t)$, can be understood as being distributed among three structural components: the stiffness component, represented by the force $f_s(t)$; the damping component, represented by the force $f_D(t)$; and the mass component, represented by the force $f_I(t)$. Thus

$$\tilde{f}_s + \tilde{f}_D + \tilde{f}_I = P(t) \tag{1.30}$$

We can now substitute Eqs (1.23), (1.26) and (1.29) in Eq. (1.30) and rewrite it as

$$\overline{m}\ddot{\tilde{x}} + \overline{c}\dot{\tilde{x}} + \overline{K}\tilde{x} = \tilde{P}(t) \tag{1.31}$$

Equation (1.31) represents a system of N ordinary differential equations that describe the displacements x(t) caused by the applied forces $\tilde{P}(t)$. This equation serves as the equation of motion for a multi-degree-of-freedom (MDOF) system, analogous to Equation (1.3) for a single-degree-of-freedom (SDOF) system. However, in the MDOF case, the scalar quantities in the SDOF system are replaced by vectors or matrices of size N, corresponding to the number of degrees of freedom in the MDOF system. The off-diagonal elements in the coefficient matrices $\overline{m}$, $\overline{c}$, and $\overline{K}$ are grouped together and referred to as coupling terms. Typically, these equations exhibit coupling in mass, damping, and stiffness, which depend on how the degrees of freedom are selected to describe the motion of the system.

4. Free Vibration of MDOF Systems without Damping

In Section 5.3, we thoroughly examined the methodology for determining normal modes and natural frequencies through free vibration analysis of a two-degree system. This approach can be generalized to multi-degree-of-freedom (MDOF) systems with N degrees of freedom. The free vibration behavior of linear MDOF systems is described by Equation (1.31), which serves as the governing equation for such systems with $\overline{P}(t) = \tilde{0}$ which for undamped systems becomes

$$\overline{m}\ddot{\tilde{x}} + \overline{K}\tilde{x} = \tilde{0} \tag{1.32}$$

The system of N homogeneous differential equations, represented by Equations (1.32), exhibits coupling through either the mass matrix, the stiffness matrix, or both. These equations correspond to the number of degrees of freedom (DOFs) in the system.

4.1. Normal Modes and Natural Frequencies for MDOF Systems

In this section, we extend the eigenvalue problem previously examined for a two degrees of freedom (DOF) system to a multi-degree of freedom (MDOF) system. The solution of the eigen equation, which yielded natural frequencies and modes for the two DOF system, can be generalized to MDOF systems. The free vibration of an undamped system in one of its natural vibration modes, as depicted graphically for a two DOF system, can be mathematically expressed for the pth mode shape in an MDOF system as

$$x(t) = q_p(t)\phi_p \tag{1.33}$$

where the deflected shape ϕ_p remains invariant with time. Therefore, we can describe the variation of displacements with time by the simple harmonic function as

$$q_p(t) = A_p \cos\omega_{np} + B_p \sin\omega_{np}t \tag{1.34}$$

The constants of integration, denoted as A_p and B_p, can be determined using the initial conditions. By combining Equations (1.32) and (1.33), we arrive at the subsequent expression.

$$x(t) = \phi_p(A_p \cos\omega_{np}t + B_p \sin\omega_{np}t) \tag{1.35}$$

in which ω_{np} and ϕ_p are not known. Substituting Eq. (1.35) in Eq. (1.32) and simplifying we get

$$\left\lfloor -\omega_{np}{}^2\overline{m}\phi_p + \overline{K}\phi_p \right\rfloor q_p(t) = \widetilde{0} \tag{1.36}$$

In Eq. (1.36), either $q_p(t) = 0$ or $\left\lfloor -\omega_{np}{}^2\overline{m}\phi_p + \overline{K}\phi_p \right\rfloor = 0$. If $q_p(t) = 0$, then $x(t) = 0$ which means the system does not vibrate. Hence, this is a trivial solution. If $\left\lfloor -\omega_{np}{}^2\overline{m}\phi_p + \overline{K}\phi_p \right\rfloor = 0$, then ω_{np} and ϕ_{np} satisfy the following algebraic equation,

$$\overline{K}\phi_p = \omega_{np}{}^2\overline{m}\phi_p \tag{1.37}$$

The matrix eigenvalue problem, as defined in Equation (1.37), presents a valuable criterion for analysis. Within this equation, both the stiffness matrix $\overline{K}$ and mass matrix $\overline{m}$ are given quantities, specifically derivable through the influence coefficients method outlined in the preceding chapter. Consequently, the unknown elements to be determined are the scalar ω_{np}^2 and vector ϕ_p. By rearranging Equation (1.37), we can arrive at a formal solution to this problem. Therefore,

$$\left[\overline{K} - \omega np 2\overline{m}\right]\phi p = 0 \tag{1.38}$$

The expression (1.38) represents a set of N homogeneous algebraic equations corresponding to N vectors ϕ_{qp} (where q ranges from 1 to N). While $\phi_p = 0$ consistently provides a trivial solution which is not useful because it means the system does not execute any motion. To achieve a meaningful outcome, we must seek a non-trivial solution, which is attainable only under specific condition:

$$\left|\overline{K} - \omega_{np}{}^{2}\overline{m}\right| = 0 \tag{1.39}$$

The characteristic equation (1.39) for an MDOF system yields N natural frequencies ω_{np} (p = 1, 2, ..., N), which are the system's eigenvalues or characteristic values. These frequencies, when applied to Eq. (1.38), determine ϕ_p, though only relative displacement values ϕ_{qp} (q = 1, 2, ..., N) can be obtained, defining the vector's shape. Each natural frequency corresponds to an independent vector ϕ_p, resulting in N eigenvectors known as natural modes or mode shapes of vibration. These vectors are also referred to as characteristic vectors or normal modes. The mode number is denoted by p, with p = 1 representing the fundamental mode. This equation is crucial in understanding the vibrational behavior of multi-degree-of-freedom systems, providing insights into their natural frequencies and corresponding mode shapes, which are essential for analyzing and predicting system responses to various excitations.

4.2. Properties of Modes

In this phase, it's advantageous to present certain characteristics of the free vibration mode shapes, which will prove invaluable in future dynamic examinations. These attributes are known as the orthogonality relations. We'll now explore modal properties that facilitate the decoupling of motion equations in multi-degree-of-freedom (MDOF) systems. Our primary focus is on elucidating the orthogonality features of modes and their implications. Additionally, we'll delve into mode normalization, addressing the relevance of spectral and modal matrices during this process. By resolving Equations (1.38) and (1.39), we can derive N eigenvalues and N natural modes, which can be consolidated into concise matrices.

4.3. Spectral and Modal Matrices

As a solution of Eq. (1.39) we get N eigenvalues ω_{np}^2. We can assemble these eigenvalues ω_{np}^2 into a diagonal matrix Ω^2. This diagonal matrix Ω^2 is called *spectral matrix* of the eigenvalue problem given by Eq. (1.37). Now we can write the spectral matrix as

$$\overline{\Omega}^2 = \begin{bmatrix} \omega_{n1}{}^2 & & & & & \\ & \omega_{n2}{}^2 & & & & \\ & & \omega_{n3}{}^2 & & & \\ & & & . & & \\ & & & & . & \\ & & & & & \omega_{nN}{}^2 \end{bmatrix} \tag{1.40}$$

We denote here ϕ_{pq} to represent the components of a vector associated with the ϕ_p natural mode and its corresponding natural frequency ω_{np}. Here, q ranges from 1 to N, indicating the degrees of freedom (DOFs) in the system. We can then neatly organize these N eigenvectors into a single square matrix, where each column stands for a distinct natural mode. This arrangement provides a compact and insightful representation of the system's vibrational characteristics.

$$\overline{\Phi} = \left[\phi qp\right] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & . & . & \phi_{1N} \\ \phi_{21} & \phi_{22} & \phi_{23} & . & . & \phi_{2N} \\ \phi_{31} & \phi_{32} & \phi_{33} & . & . & \phi_{3N} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \phi_{N1} & \phi_{N2} & \phi_{N3} & . & . & \phi_{NN} \end{bmatrix} \tag{1.41}$$

The modal matrix, denoted as $\overline{\Phi}$, plays a crucial role in the eigenvalue problem outlined in Equation (1.37). This matrix is composed of eigenvectors, each paired with its corresponding eigenvalue. Together, these eigenvalue-eigenvector pairs fulfill the conditions set forth in Eq. (1.37). We can express this relationship in an alternative form, providing a different perspective on the problem.

$$\overline{K}\phi_p = \overline{m}\,\phi_p \omega_{np}{}^2 \tag{1.42}$$

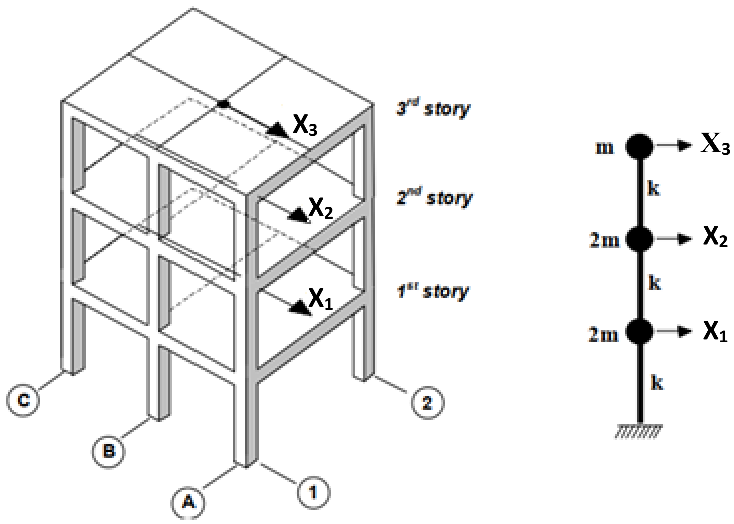
Using the modal and spectral matrices, all these relationships (p = 1, 2, 3, …, N) can be combined into one unified matrix equation.

$$\overline{K\Phi} = \overline{m}\,\overline{\Phi}\vec{\Omega}^2 \tag{1.43}$$

Equation (1.43) provides a compact presentation of the equations relating all eigenvalues and eigenvectors.

Example 1.2

For the building depicted in Fig. 1, we are analyzing the response along the numerical reference axes. Each story has identical lateral stiffness, denoted as K. The mass of the two lower stories is twice that of the roof, with the roof mass represented as M.



Solution 1.2

The mass matrix of the structure is:

$$[M]=\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

The stiffness matrix, obtained from equilibrium of each mass is:

$$[K]=\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

The dynamic equilibrium equations are then:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now proceed to find the solution of the free vibration response of the system for different initial conditions: $\Delta = [[K] - \omega_i^2 [M]] = 0$. After replacing [K] and [M] we obtain the following determinant:

$$\Delta = \begin{vmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 2m & -k \\ 0 & -k & 2k - \omega^2 2m \end{vmatrix} = 0$$

Expanding this determinant, we obtain the following characteristic equation:

$$\Delta = 4m^3\omega^6 - 12km^2\omega^4 + 9k^2m\omega^2 - k^3 = 0$$

After dividing all terms of the characteristic equation by $4m^3$ we obtain:

$$\omega^6 - 3\frac{k}{m}\omega^4 + \frac{9}{4}\frac{k^2}{m^2}\omega^2 - \frac{k^3}{4m^3} = 0$$

A simple inspection of the equation tell us that $\omega^2 = k/m$ is a root, and by using synthetic division, we transform the characteristic equation into:

$$(\omega^2 - \frac{k}{m})(4m\omega^4 - 8km\omega^2 + k^2m) = 0$$

Solving the second-degree equation contained in the second term of the previous equation, we obtain:

$$\omega^2 = \frac{8km}{8m} \pm \sqrt{\frac{64k^2m^2 - 16k^2m^2}{8m}} = \frac{k}{m}\left[1 \pm \frac{\sqrt{3}}{2}\right] = \begin{matrix} 1.866\frac{k}{m} \\ 0.134\frac{k}{m} \end{matrix}$$

Then, the natural frequencies of the building — properly ordered — are:

$$\omega_1^2 = 0.134\frac{k}{m}, \quad \omega_2^2 = \frac{k}{m}, \quad \omega_3^2 = 1.866\frac{k}{m}$$

Now, by using Eq. (1.38) we can obtain the vibration modes by going back to the characteristic determinant:

$$[[K]-\omega_r^2[M]]\{\phi^{(r)}\}=\{0\}, \quad r=1,2,3$$

Replacing here the mass and stiffness matrices, we obtain the following set of homogeneous simultaneous equations:

$$\begin{bmatrix} k-\omega_r^2 m & -k & 0 \\ -k & 2k-\omega_r^2 2m & -k \\ 0 & -k & 2k-\omega_r^2 2m \end{bmatrix} \begin{bmatrix} \phi_1^{(r)} \\ \phi_2^{(r)} \\ \phi_3^{(r)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding the product, we see the system in the classical simultaneous equation format:

$$\begin{aligned} &(k-\omega_r^2 m)\phi_3^{(r)} - k\phi_2^{(r)} = 0 \\ &-k\phi_3^{(r)} + (2k-\omega_r^2 2m)\phi_2^{(r)} - k\phi_1^{(r)} = 0 \\ &-k\phi_2^{(r)} + (2k-\omega_r^2 2m)\phi_1^{(r)} = 0 \end{aligned}$$

From the third equation, we can see that, in this case, the ratio between the second unknown and the first unknown is:

$$\frac{\phi_2^{(r)}}{\phi_1^{(r)}} = \frac{2k-\omega_r^2 2m}{k}$$

Now replacing the third equation into the second, we obtain the following ratio between the third unknown and the first unknown:

$$\frac{\phi_3^{(r)}}{\phi_1^{(r)}} = \left(\frac{2k-\omega_r^2 2m}{k}\right)^2 - 1$$

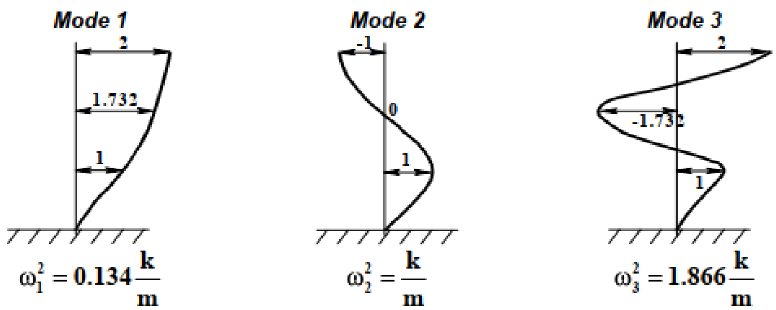
These two ratios are fixed for any value of ω_i^2 . We now replace the values of ω_i^2 obtained previously and the values of the unknowns are found for each case:

	ω_1^2	ω_2^2	ω_3^2
ϕ_2/ϕ_1	1.732	0	−1.732
ϕ_3/ϕ_1	2	−1	2

We may assign any arbitrary value to the φ_1 term and thus from the obtained ratios compute the other two values of the terms of the mode. We choose, arbitrarily again, a value of one for φ_1. By doing so, the modes are defined as:

$$\{\phi^{(1)}\} = \begin{bmatrix} 2 \\ 1.732 \\ 1 \end{bmatrix}, \quad \{\phi^{(2)}\} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \{\phi^{(3)}\} = \begin{bmatrix} 2 \\ -1.732 \\ 1 \end{bmatrix}$$

Corresponding, graphically, to:



4.4. Orthogonality of Modes

The free vibration mode shapes (ϕ_p) possess unique properties that are invaluable in structural dynamics analysis. This crucial characteristic of modes is known as the orthogonality property or orthogonality relationships. It plays a fundamental role in solving dynamic problems using the Modal Superposition Method for MDOF systems. The mode shapes or eigenvectors exhibit mutual orthogonality with respect to the mass and stiffness matrices. Orthogonality is a key attribute of normal modes or eigenvectors, utilized to decouple the modal mass and stiffness matrices. We can demonstrate that the natural modes corresponding to distinct natural frequencies satisfy specific orthogonality conditions.

When $\omega_{np} \neq \omega_{nr}$,

$$\tilde{\phi}_p^{\,T} \overline{K} \tilde{\phi}_r = 0 \tag{1.44a}$$

$$\tilde{\phi}_p^{\,T} \overline{m} \tilde{\phi}_r = 0 \tag{1.44b}$$

The subsequent discourse will demonstrate these crucial characteristics. The *p*th natural frequency and mode satisfy Eq. (1.37). We now pre-multiply Eq. (1.37) by the transpose of ϕ_r, i.e., ϕ_r^T. Thus we get

$$\tilde{\phi}_r^T \overline{K} \tilde{\phi}_p = \omega_{np}^2 \tilde{\phi}_r^T \overline{m} \phi_p \tag{1.45}$$

Likewise, the rth natural frequency and mode too satisfy Eq. (1.37). Thus we get

$$\overline{K} \tilde{\phi}_r = \omega_{nr}^2 \overline{m} \tilde{\phi}_r \tag{1.46}$$

Now pre-multiplying Eq. (1.46) by ϕ_p^T we get,

$$\tilde{\phi}_p^T \overline{K} \phi_r = \omega_{nr}^2 \phi_p^T \overline{m} \tilde{\phi}_r \tag{1.47}$$

The transpose of the matrix on the LHS of Eq. (1.45) will be equal to the transpose of the matrix on the RHS of the equation. Therefore

$$\tilde{\phi}_p^T \overline{K} \tilde{\phi}_r = \omega_{np}^2 \tilde{\phi}_p^T \overline{m} \tilde{\phi}_r \tag{1.48}$$

In deriving Eq. (1.48), we leveraged the symmetric nature of both mass and stiffness matrices. To proceed, we subtract Eq. (1.47) from Eq. (1.48), yielding the following result.

$$(\omega_{np}^2 - \omega_{nr}^2) \tilde{\phi}_p^T \overline{m} \tilde{\phi}_r = 0 \tag{1.49}$$

In Eq. (1.49), if $\omega_{np}^2 \neq \omega_{nr}^2$, then it implies that $\tilde{\phi}_p^T \overline{m} \tilde{\phi}_r = 0$. This establishes that Eq. (1.44b) is true because for systems with positive natural frequencies, $\omega_{np} \neq \omega_{nr}$.

Now we substitute Eq. (1.44b) into Eq. (1.47) we get $\tilde{\phi}_p \overline{K} \tilde{\phi}_r = 0$. This clearly shows that Eq. (1.44a) is true when $\omega_{np} \neq \omega_{nr}$ Therefore we have completed the proof of the orthogonality conditions prescribed in Eq. (1.44).

For systems in which no two modes share the same frequency, orthogonality conditions hold true for any pair of distinct modes, as shown in Equation (1.44). However, these conditions don't apply when two modes have identical frequencies. This concept is crucial for understanding how different vibration modes interact within a system.

The natural modes' orthogonality indicates that the corresponding square matrices are diagonal:

$$K^* \equiv \overline{\Phi}^T \overline{K} \overline{\Phi} \tag{1.50a}$$

Which denotes as modal stiffness matrix

$$M^* \equiv \overline{\Phi}^T \overline{m} \overline{\Phi} \tag{1.50b}$$

Which denotes as modal mass matrix in which the diagonal elements are

$$K_p = \bar{\phi}_p{}^T \bar{K} \bar{\phi}_p \tag{1.51a}$$

$$M_p = \bar{\phi}_p \bar{m} \bar{\phi}_p \tag{1.51b}$$

As $\bar{m}$ and $\bar{K}$ are positive definite, the diagonal elements of K^* and M^* are positive. They are related by

$$K_p = \omega_{np}{}^2 M_P \tag{1.52}$$

This can be demonstrated from the definitions of K_p and M_p as follows. Substituting Eq. (1.37) in Eq. (1.51a)

$$K_p = \varphi_p{}^T (\omega_p{}^2 \bar{m} \varphi_p) = \omega_p{}^2 (\varphi_p \bar{m} \varphi_p) = \omega_n{}^2 M_p \tag{1.53}$$

As summury:

$$\Phi = [\phi_1 \quad \phi_2 \quad \phi_3]$$

Generalized mass

$$\Phi^T M \Phi = \begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix}$$

Generalized stiffness

$$\Phi^T K \Phi = \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix}$$

Generalized damping

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Generalized force

$$\Phi^T F(t) = \begin{Bmatrix} f_1^*(t) \\ f_2^*(t) \\ f_3^*(t) \end{Bmatrix}$$

4.5. Interpretation of Orthogonality of Modes

The physical interpretation of the orthogonality property of natural modes will now be discussed. Modal orthogonality implies that the work performed by the inertia force associated with the pth mode during the displacement in the rth mode is zero. To clarify this concept, consider a structure vibrating in the pth mode with corresponding displacements.

$$\tilde{x}_p(t) = \phi_p q_p(t) \tag{1.54}$$

The accelerations corresponding to Eq. (1.54) are $\ddot{x}_p(t) = \phi_p \ddot{q}_p(t)$. The inertia forces corresponding to these accelerations are

$$\left(\bar{f}_I\right)_p = -\overline{m}\ddot{x}_p(t) = -\overline{m}\,\phi_p\ddot{q}_p(t) \tag{1.55}$$

We now consider the *r*th natural mode of displacements of the structure,

$$\widetilde{x}_r(t) = \phi_r\widetilde{q}_r(t) \tag{1.56}$$

The work performed by the inertia force, as defined in Eq. (1.55), on the displacement outlined in Eq. (1.56) can be represented as:

$$\left(\bar{f}_I\right)_p^T \widetilde{x}_r = -\left(\phi_p^T\overline{m}\,\phi_r\right)\ddot{q}(t)\ddot{q}_r(t) \tag{1.57}$$

This quantity stated in Eq. (1.57) equals zero due to the orthogonality condition outlined in Eq. (1.44b), thereby confirming the validity of the statement above.

The orthogonality property can be interpreted physically in a different way: when the displacements of the *p*th mode act upon the displacements of the *r*th mode, the work performed by the equivalent static forces equals zero. To clarify, the equivalent static forces in the *p*th mode are represented as follows:

$$\left(\bar{f}_s\right)_p = \overline{K}\widetilde{x}_p(t) = \overline{K}\phi_p q_p(t) \tag{1.58}$$

The displacements in the *r*th mode are given by Eq. (1.54). The work done of the static forces in Eq. (1.58) on the displacements given in Eq. (1.54) is given as

$$\left(\bar{f}_s\right)_p^T \widetilde{x}_r = \left(\phi_p^T\overline{K}\phi_r\right)q_p(t)q_r(t) \tag{1.59}$$

Due to the orthogonality property described in Eq. (1.44a), Eq. (1.59) equals zero, thereby confirming the second physical interpretation as well.

4.6. Normalization of Modes

The eigenvalue problem in Eq. (1.37) yields relative, not absolute, natural vibration modes. These modes are arbitrary in amplitude, as any scalar multiple of the eigenvector ϕ_p satisfies the equation, with only the mode shapes being uniquely defined. The normalization of Two DOF system normal modes involved setting one amplitude to unity and determining the other relative to this reference. This process, known as mode shape normalization, standardizes elements associated with various degrees of freedom (DOFs). Common normalization methods include setting the largest element to unity or normalizing a specific DOF element, such as the top floor in a multistory building analysis. Notably, in dynamic analysis and computational applications, modes are frequently normalized to achieve unit modal mass (Mn). In such a case we have:

$$M_n = \phi_p^{\;T} \overline{m} \phi_p = 1 \qquad\qquad \overline{\Phi}^T \overline{m} \overline{\Phi} = \bar{I} \tag{1.60}$$

The equation $\bar{I}$ demonstrates the orthonormality of the natural modes with respect to the mass matrix $\overline{m}$. Here, $\bar{I}$ identity matrix, characterized by unit values along its principal diagonal, while $\overline{\Phi}$ represents the complete set of N normalized mode shapes, also referred to as the modal matrix. This relationship signifies that the natural modes are not only orthogonal but also normalized with respect to the mass matrix $\overline{m}$, thus constituting a mass orthonormal set. When the modes are normalized in this manner, Eq. (1.51a) becomes

$$K_p = \phi_p^{\;T} \overline{m} \phi_p = \omega_{np}^{\;2} M_p = \omega_{np}^{\;2} \tag{1.61a}$$

and Eq. (1.50a),

$$\overline{K}_{Diag} = \overline{\Phi}^T \overline{K\Phi} = \overline{\Omega}^2 \tag{1.61b}$$

Example 1.3

Uncouple the dynamic system of Example 1.2

Solution 1.3

We change the normalization of the modes in such a way that they comply with Eq. (15) to obtain orthonormal modes: $\{\phi^{(r)}\}^T [M] \{\phi^{(r)}\} = 1$

<u>Mode 1</u>

$$\{2|\sqrt{3}|1\} \cdot \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \cdot \begin{bmatrix} 2 \\ \sqrt{3} \\ 1 \end{bmatrix} = 12m \quad \Rightarrow \{\phi^{(1)}\} = \frac{1}{\sqrt{12m}} \begin{bmatrix} 2 \\ \sqrt{3} \\ 1 \end{bmatrix} = \begin{vmatrix} 0.5774/\sqrt{m} \\ 0.5000/\sqrt{m} \\ 0.2887/\sqrt{m} \end{vmatrix}$$

<u>Mode 2</u>

$$\{-1|0|1\}\cdot\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 3m \quad \Rightarrow \{\phi^{(2)}\} = \frac{1}{\sqrt{3m}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5774/\sqrt{m} \\ 0.0000/\sqrt{m} \\ 0.5774/\sqrt{m} \end{bmatrix}$$

<u>Mode 3</u>

$$\{2|-\sqrt{3}|1\}\cdot\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\cdot\begin{bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{bmatrix} = 12m \quad \Rightarrow \{\phi^{(3)}\} = \frac{1}{\sqrt{12m}}\begin{bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5774/\sqrt{m} \\ -0.5000/\sqrt{m} \\ 0.2887/\sqrt{m} \end{bmatrix}$$

The modal matrix is then:

$$[\Phi] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5000 & 0.0000 & -0.5000 \\ 0.2887 & 0.5774 & 0.2887 \end{bmatrix}$$

In order to uncouple the system, the following operations are performed:

$$[\Phi]^T[M][\Phi] = \frac{1}{m}\begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0.0000 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5000 & 0.0000 & -0.5000 \\ 0.2887 & 0.5774 & 0.2887 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And

$$[\Phi]^T[K][\Phi] = \frac{k}{m}\begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0.0000 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix}\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5000 & 0.0000 & -0.5000 \\ 0.2887 & 0.5774 & 0.2887 \end{bmatrix}$$

$$= \frac{k}{m}\begin{bmatrix} \cdot\, 0.134 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.866 \end{bmatrix}$$

The uncoupled equations are:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \frac{k}{m}\begin{bmatrix} 0.134 & 0 & 0 \\ 0 & 1.000 & 0 \\ 0 & 0 & 1.866 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4.7. Modal Analysis

In earlier discussions of MDOF systems in Chapters 6 and 7, the displacement was described using the N components of the displacement vector $\tilde{x}$. However, when analyzing the dynamic response of linear systems, it is often more practical to represent displacements using free vibration mode shapes. These mode shapes are N independent displacement patterns, and their amplitudes can act as generalized coordinates to describe any displacement configuration. Similar to how trigonometric functions are used in a Fourier series, mode shapes are beneficial due to their orthogonality properties and their ability to efficiently represent displacements, allowing accurate approximations with only a few terms.

4.7.1. Model Expansion of Displacements

Any vector of order N can be represented using a set of N independent vectors. In this context, we will use the natural modes as the basis for our discussion. This leads us to express any displacement through a modal expansion, utilizing the normal modes as :

$$\tilde{x} = \sum_{p=1}^{N} \phi_p q_p = \overline{\Phi}\tilde{q} \tag{1.62}$$

where q_p are scalar multipliers called *generalized coordinates* or *modal coordinates* or *normal coordinates* and

$$\tilde{q} = \begin{Bmatrix} q_1 \\ q_2 \\ . \\ . \\ q_N \end{Bmatrix} \tag{1.63}$$

If we know ϕ_p for a given $\tilde{x}$ then we can determine the q_p by multiplying both sides of Eq. (1.62) by ϕr^T as

$$\phi_r^T \overline{m}\tilde{x} = \sum_{p=1}^{N} \left(\phi_r^T \overline{m}\phi_p\right) q_p \tag{1.64}$$

Due to the orthogonality relationship described in Eq. (1.44b), all the terms within the summation in Eq. (1.64) are eliminated, except for the term where p = r. Consequently,

$$\phi_r^T \overline{m}\widetilde{x} = \left(\phi_r^T \overline{m}\phi_r\right) q_p \tag{1.65}$$

The matrix products of both sides of Eq. (1.65) are scalars. So, we can rewrite Eq. (1.65) as

$$q_p = \frac{\phi_p^T \overline{m}\widetilde{x}}{\phi_p^T \overline{m}\phi_p} = \frac{\phi_p^T \overline{m}\widetilde{x}}{M_p} \tag{1.66}$$

Each normal coordinate is expressed as shown in Eq. (1.66). The modal expansion of the displacement vector $\widetilde{x}$, described in Eq. (1.62), is utilized to derive solutions for the free vibration response of undamped systems. Additionally, it is crucial for analyzing the response of systems subjected to forced vibrations and earthquake excitations in multi-degree-of-freedom (MDOF) systems.

4.7.2. Concept of Mode Superposition

Let us now examine a cantilever column, as illustrated in Fig. 1.5. The deflected shape of this column is described by translational displacement coordinates at three distinct levels.

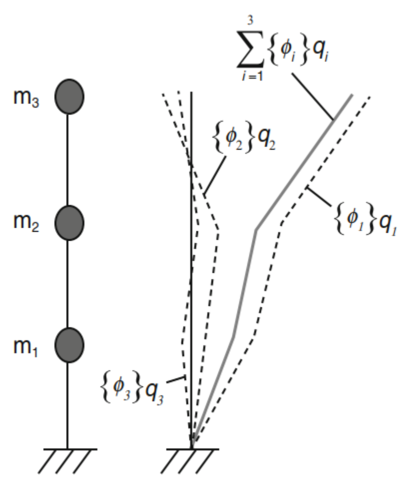


Fig. 1.5 Modal analysis of cantilever column

Any displacement vector $\widetilde{x}$ associated with this column can be constructed by combining appropriate amplitudes of the three vibration modes depicted in Fig. 1.5. For a specific modal component, denoted as x_p, the displacements are expressed as the product of the mode shape vector

ϕ_p and the modal amplitude q_p, as outlined in Eq. (1.62). The overall displacement is then determined by summing up all the individual modal components.

$$\tilde{x} = \phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3 + ...+ \phi_N q_N = \sum_{p=1}^{N} \phi_p q_p \tag{1.67}$$

Equation (1.67) can be expressed in matrix form as shown in Equation (1.62), where $\tilde{x} = \overline{\Phi}\tilde{q}$. This equation highlights that the mode shape matrix, $\overline{\Phi}$, acts as a transformation tool, converting the normal coordinates into geometric coordinates, $\tilde{x}$. These generalized coordinates, which represent the amplitudes of the modes, are referred to as the normal coordinates of the structure. The mode shape matrix $\overline{\Phi}$ for a system with N degrees of freedom is made up of N independent modal vectors. Because of this, the matrix is non-singular, meaning it can be inverted. This allows us to directly solve Equation (1.62) to find the normal coordinate amplitudes $\tilde{q}$ corresponding to any given displacement vector $\tilde{x}$. However, thanks to the orthogonality property of the mode shapes, there's no need to solve simultaneous equations to determine $\tilde{q}$, as explained in Section 8.1.1.

4.8. Response of Undamped Free Vibration

The equation of motion that governs the free vibration of a multi-degree-of-freedom (MDOF) system is represented by Eq. (7.1). To solve this equation, specific initial conditions must be applied:

$$\tilde{x} = \tilde{x}(0) \text{ and } \dot{\tilde{x}} = \dot{\tilde{x}}(0) \tag{1.68}$$

In Chapter 1, we explored how solving the differential equation leads to the matrix eigenvalue problem described in Eq. (1.37). Once the eigenvalue problem is resolved and the natural frequencies and modes are identified, the general solution can be represented as a superposition of different modes, similar to what is shown in Eq. (1.67). Mathematically, this is expressed in Eq. (1.62), which breaks down the response into contributions from individual modes. As a result, the overall response can be written as the sum of these individual mode responses as following:

$$\tilde{x}(t) = \sum_{p=1}^{N} \phi_p \left(A_p \cos \omega_{np} t + B_p \sin \omega_{np} t \right) \tag{1.69}$$

where A_p and B_p are the constants of integration each containing N constants. By differentiating Eq. (1.69) we can get the velocity vector as

$$\dot{\tilde{x}}(t) = \sum_{p=1}^{N} \phi_p \omega_{np} \left(-A_p \sin \omega_{np} t + B_p \cos \omega_{np} t \right) \tag{1.70}$$

Now we set t = 0 in Eqs. (1.69) and (1.70) and get

$$\tilde{x}(0) = \sum_{p=1}^{N} \phi_p A_p \qquad \dot{\tilde{x}}(0) = \sum_{p=1}^{N} \phi_p \omega_p B_p \tag{1.71}$$

Given the initial displacements, $\tilde{x}(0)$, and initial velocity, $\dot{\tilde{x}}(0)$, as defined in Equation (1.68), each equation in (1.71) forms a set of N algebraic equations involving the unknowns A_p and B_p. However, it is not necessary to solve these equations simultaneously. Instead, they can be addressed as a modal expansion of the vectors $\tilde{x}(0)$ and $\dot{\tilde{x}}(0)$, as explained earlier in Section 8.1.1. By following the approach outlined in Equation (1.62), we can reformulate these equations using normal coordinates for simplicity and clarity as:

$$\tilde{x}(0) = \sum_{p=1}^{N} \phi_p q_p(0) \qquad \dot{\tilde{x}}(0) = \sum_{p=1}^{N} \phi_p \dot{q}_p(0) \tag{1.72}$$

Following the analogy given in Eq. (1.66) we can express $q_p(0)$ and $\dot{q}_p(0)$ as

$$q_p(0) = \frac{\phi_p \overline{m} \tilde{x}(0)}{M_p} \qquad \dot{q}_p(0) = \frac{\phi_p^T \overline{m} \dot{\tilde{x}}(0)}{M_p} \tag{1.73}$$

Equations (1.71) and (1.72) are equivalent, indicating that $A_p = q_p(0)$ and $B_p = \left(\frac{q_p(0)}{\omega_p} \right)$. We now substitute these in Eq. (1.69) and obtain

$$\tilde{x}(t) = \sum_{p=1}^{N} \phi_p \left[q_p(0) \cos \omega_{np} t + \frac{\dot{q}(0)}{\omega_{np}} \sin \omega_{np} t \right] \tag{1.74}$$

We can also express Eq. (1.74) alternatively as

$$\tilde{x}(t) = \sum_{p=1}^{N} \phi_p q_p(t) \tag{1.75}$$

where

$$q_p(t) = q_p(0)\cos\omega_{np}(t) + \frac{\dot{q}_p(0)}{\omega_{np}}\sin\omega_{np}(t) \tag{1.76}$$

The time variation of modal coordinates is described by Equation (1.76), which bears a resemblance to Equation (3.1) for an SDOF system. For an undamped free vibration in an MDOF system, the solution is expressed through Equation (1.74). This equation determines the displacements, $\tilde{x}(t)$, over time based on the initial displacement, $\tilde{x}(0)$, and initial velocity, $\dot{\tilde{x}}(0)$. By assuming that the natural frequencies ω_{np} and mode shapes ϕ_p are known, and by defining the normal coordinates q_N and $\dot{q}_N$ as per Equation (1.73), the right-hand side of Equation (1.74) can be computed. Consequently, the complete response of the MDOF system undergoing undamped free vibration can be fully determined.

Example 1.4

For the building in Example 1.1, find the free vibration response for different cases of initial displacement conditions.

- Case (a) - Suppose a unit displacement at each story of the building at time = 0, without any initial velocity.
- Case (b) - Suppose an initial displacement condition in the shape of the first mode $2 \mid \sqrt{3} \mid 1$, without initial velocity.
- Case (c) - Suppose an initial displacement condition in the shape of the second mode $-1 \mid 0 \mid 1$, without initial velocity.
- Case (d) - Suppose an initial displacement condition in the shape of the third mode $2 \mid -\sqrt{3} \mid 1$, without initial velocity.

Solution 1.4

<u>Case (a) :</u> The initial displacement vector is:

$$\{X_0\} = \begin{bmatrix} x_3(0) \\ x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Constants bi are obtained from:

$$\{B\} = [\Phi]^T [M] \{X_0\}$$

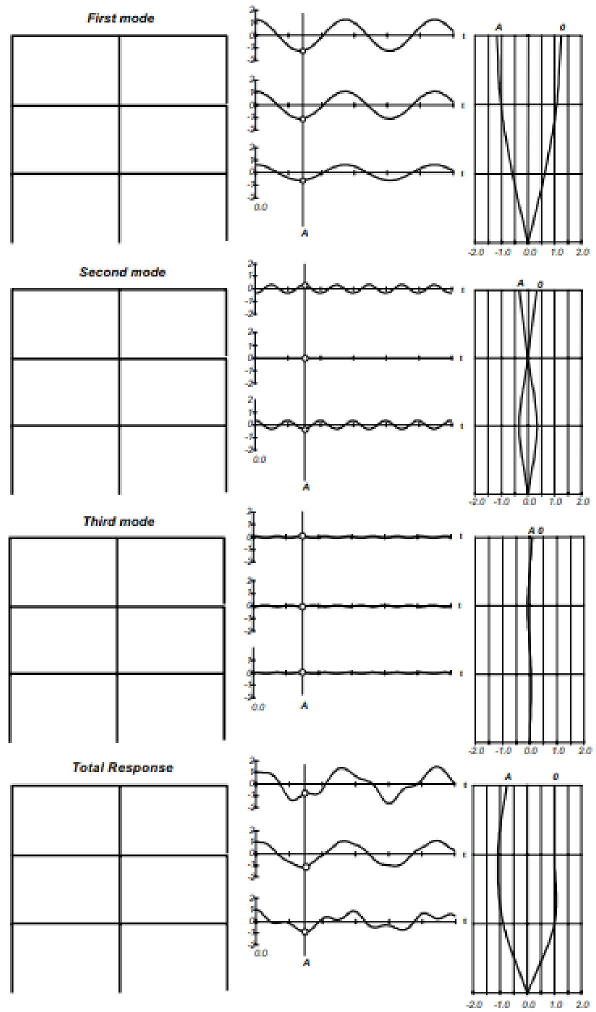
$$\{B\} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0.0000 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix} \cdot \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 2.1547 \\ 0.0000 \\ 0.1548 \end{bmatrix}$$

Then, the response of the system is described by the following equation:

$$\begin{bmatrix} U_3 \\ U_2 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0.5774 \\ 0.5000 \\ 0.2887 \end{bmatrix} 2.1547 \cos \omega_1 t + \begin{bmatrix} -0.5774 \\ 0.0000 \\ 0.5774 \end{bmatrix} 0 \cos \omega_2 t + \begin{bmatrix} 0.5774 \\ -0.5000 \\ 0.2887 \end{bmatrix} 0.1547 \cos \omega_3 t$$

$$= \begin{bmatrix} 1.2441 \\ 1.0774 \\ 0.6221 \end{bmatrix} \cos \omega_1 t + \begin{bmatrix} -0.3333 \\ -0.3333 \\ -0.3333 \end{bmatrix} \cos \omega_2 t + \begin{bmatrix} 0.0893 \\ -0.0774 \\ 0.0447 \end{bmatrix} \cos \omega_3 t$$

It is evident that the response of the system corresponds to the superposition of the individual responses from each mode. Fig. 2 shows the response for each mode and the total response of the building. Supposing that at some instant in time the three responses are in phase, 62.2% would be contributed by the first mode, 33.3% by the second, and 4.5% by the third.



<u>Case (b):</u> The initial displacement vector is:

$$\{X_0\} = \begin{bmatrix} x_3(0) \\ x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{3} \\ 1 \end{bmatrix}$$

Constants bi are obtained from:

$$\{B\} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0.0000 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} 2/\sqrt{3} \\ 1 \\ 1 \end{bmatrix} = \sqrt{m} \begin{bmatrix} 2\sqrt{3} \\ 0 \\ 0 \end{bmatrix}$$

The response would be described by:

$$\begin{bmatrix} U_3 \\ U_2 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0.5774 \\ 0.5000 \\ 0.2887 \end{bmatrix} \cdot 2\sqrt{3}\cos\omega_1 t = \begin{bmatrix} 2/\sqrt{3} \\ \sqrt{3} \\ 1 \end{bmatrix} \cos\omega_1 t$$

100% of the response is contributed by the first mode alone. The other modes don't contribute.

<u>Case (c):</u> The initial displacement vector is:

$$\{X_0\} = \begin{bmatrix} x_3(0) \\ x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Constants bi are obtained from:

$$\{B\} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0.0000 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{m} \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

The response of the system is described by the following equation:

$$\begin{bmatrix} U_3 \\ U_2 \\ U_1 \end{bmatrix} = \begin{bmatrix} -0.5774 \\ 0 \\ 0.5774 \end{bmatrix} \sqrt{3}\cos\omega_2 t = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cos\omega_2 t$$

Only the second mode contributes with a 100% of the response.

<u>Case (d):</u> The initial displacement vector is:

$$\{X_0\} = \begin{bmatrix} x_3(0) \\ x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{bmatrix}$$

Constants bi are obtained from:

$$\{B\} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5774 & 0.5000 & 0.2887 \\ -0.5774 & 0.0000 & 0.5774 \\ 0.5774 & -0.5000 & 0.2887 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{bmatrix} = \sqrt{m} \begin{bmatrix} 0 \\ 0 \\ 2\sqrt{3} \end{bmatrix}$$

Response is described by the following equation:

$$\begin{bmatrix} U_3 \\ U_2 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0.5774 \\ -0.5000 \\ 0.2887 \end{bmatrix} 2\sqrt{3}\cos\omega_3 t = \begin{bmatrix} 2 \\ -\sqrt{3} \\ 1 \end{bmatrix} \cos\omega_3 t$$

Only the third mode contributes with a 100% of the response.

5. Response of Damped Free Vibration with Classically Damped System

When damping is considered in the free vibration of a multi-degree-of-freedom (MDOF) system, the equation of motion can be derived by modifying Eq. (1.31). This is done by setting $\widetilde{P}(t) = 0$, which simplifies the equation to its reduced form:

$$\overline{m}\ddot{\widetilde{x}} + \overline{c}\dot{\widetilde{x}} + \overline{K}\widetilde{x} = \widetilde{0} \tag{1.77}$$

To solve Eq. (1.77) for $\widetilde{x}(t)$, we use the initial conditions provided in Eq. (1.68) at t = 0. The main challenge lies in modeling the type of damping and determining whether it can be addressed theoretically. To overcome this, we focus on presenting the solution for a specific system in a graphical format, making it easier to intuitively understand how damping influences the free vibration of multi-degree-of-freedom (MDOF) systems. This is achieved by expressing the displacement $\widetilde{x}$ in terms of the system's natural modes without damping, as outlined in Eq. (1.62), and substituting this expression into Eq. (1.77). This approach simplifies the process and provides clarity on the effects of damping.

$$\overline{m}\overline{\Phi}\ddot{\widetilde{q}} + \overline{c}\overline{\Phi}\dot{\widetilde{q}} + \overline{K}\widetilde{q} = \widetilde{0} \tag{1.78}$$

Premultiplying Eq. (1.78) by $\overline{\Phi}^T$ and using the identity in Eq. (1.50) we can obtain,

$$\overline{M}\ddot{\widetilde{q}} + \overline{C}\dot{\widetilde{q}} + \overline{K}_{Diag}\widetilde{q} = \widetilde{0} \tag{1.79}$$

While the diagonal matrices $\overline{M}$ and $\overline{K}_{Diag}$ have been defined in Eq. (1.50) and now

$$\overline{C} = \overline{\Phi}^T \overline{c}\,\overline{\Phi} \tag{1.80}$$

The nature of the damping distribution in a system determines whether the square matrix $\overline{C}$ in Eq. (1.80) is diagonal or not. If $\overline{C}$ is diagonal, it represents N independent differential equations in modal coordinates (q_p), and such systems are described as having classical damping. In these cases, classical modal analysis can be applied, and the natural modes of the system remain identical to those of the undamped system. On the other hand, if $\overline{C}$ is not diagonal, the system is said to exhibit non-classical damping. These systems cannot be analyzed using classical modal analysis because their natural modes differ from those of the undamped system.

This section provides a structured approach to solving free vibration problems in systems with classical damping, caused by initial displacements and/or velocities. In classically damped systems, damping does not alter the natural modes. Therefore, the natural frequencies and modes are initially determined for the undamped system. Afterward, the influence of damping on the natural frequencies is analyzed, similar to how it is done for a single-degree-of-freedom (SDOF) system.

In a multi-degree-of-freedom (MDOF) system with classical damping, the motion in modal coordinates can be described by a set of N differential equations. Each equation corresponds to one mode of vibration:

$$M_p \ddot{q}_p + C_p \dot{q}_p + K_p q_p = 0 \tag{1.81}$$

where M_p and K_p are defined in Eq. (1.51). Now,

$$C_n = \phi_n^T \bar{c} \phi_n \tag{1.82}$$

Equation (1.81) is similar to Eq. (2.17) of a SDOF system with damping. Therefore the damping ratio can be defined for each mode in the same manner for a SDOF system presented in Eq. (3.13)

$$\xi_p = \frac{C_p}{2M_p \omega_{np}} \tag{1.83}$$

We now divide Eq. (1.81) by M_p and combine with Eq. (1.83) to obtain

$$\ddot{q}_p + 2\xi_p \omega_{np} \dot{q}_p + \omega_{np}^2 q_p = 0 \tag{1.84}$$

Equation (1.84) closely resembles Equation (3.15), which describes the free vibration of a single-degree-of-freedom (SDOF) system with damping. As we know, the solution for the free vibration of an SDOF system with damping is provided in Equation (3.16). By extending some of the principles and results from the SDOF system, we can apply them to a multi-degree-of-freedom (MDOF) system. Consequently, the solution to Equation (1.84) can be derived using this approach.

$$q_p(t) = e^{-\xi\omega_{np}t}\left[q_p(0)\cos\omega_{Dp}t + \frac{\dot{q}_p(0) + \xi_p\omega_{np}q_p(0)}{\omega_{Dp}}\sin\omega_{Dp}t\right] \tag{1.85}$$

where p[th] natural frequency with damping is

$$\omega_{Dp} = \omega_{np}\sqrt{(1-\xi_p{}^2)} \tag{1.86}$$

The displacement response of the system is then obtained by substituting Eq. (1.85) for q(t) in Eq. (1.62):

$$\tilde{x}(t) = \sum_{p=1}^{N}\phi_p e^{-\xi_p\omega_{np}t}\left[q_p(0)\cos\omega_{Dp}t + \frac{\dot{q}_p(0) + \xi_p\omega_{np}q_p(0)}{\omega_{Dp}}\sin\omega_{Dp}t\right] \tag{1.87}$$

Equation (1.87) represents the solution to the free vibration problem for a multi-degree-of-freedom (MDOF) system with classical damping. This equation describes how displacement, $\tilde{x}$, changes over time due to initial conditions, specifically the initial displacement $\tilde{x}(0)$ and initial velocity $\dot{\tilde{x}}(0)$. Once the system's natural frequencies ω_{np} and mode shapes ϕ_p are determined for the undamped case, along with the modal damping ratios ξ_{np}, the right-hand side of Equation (1.87) becomes fully defined. These terms depend on the initial modal coordinates $q_p(0)$ and $\dot{q}_p(0)$, which are described by Equation (1.73).

Chapter 2

Forced Vibration Response of MDOF System

1. Introduction

In Chapter 7, we explained how the free motion of a multi-degree-of-freedom (MDOF) system can be described using its normal modes of vibration. Here, we extend that concept to show that the forced motion of an MDOF system can also be expressed in terms of these normal modes. The total response can be determined by superimposing the solutions of independent modal equations. Essentially, this means that normal modes can be used to transform a system of coupled differential equations into a set of uncoupled equations, where each equation involves only one dependent variable. This approach, known as the modal superposition method, simplifies the process by reducing the problem of analyzing the response of a forced MDOF system to evaluating the response of multiple forced single-degree-of-freedom (SDOF) systems.

2. Modal Equations for Forced Undamped Systems

In Chapter 6, we derived the general equation of motion for a damped MDOF system under forced vibration, and it is reiterated here for reference.

$$\overline{m}\ddot{\tilde{x}} + \overline{c}\dot{\tilde{x}} + \overline{K}\tilde{x} = \tilde{P}(t) \tag{1.31}$$

Assuming the system is undamped, then $\overline{c} = 0$, the equation simplifies to:

$$\overline{m}\ddot{x} + \overline{K}\tilde{x} = \tilde{P}(t) \tag{2.1}$$

Equation (2.1) is coupled, meaning the equations of motion are interdependent. While we previously demonstrated in Section 5.5 how to solve such coupled equations for a two-DOF system under harmonic excitation, this approach becomes inefficient for systems with more degrees of freedom or when subjected to other types of dynamic forces. To address this, it is beneficial to convert these equations into modal coordinates. By doing so, we aim to transform the coupled system of differential equations into a set of independent or uncoupled equations, where each equation involves only one mode, scaled by factors representing the contributions of each mode. This process is explained further in the following section.

The displacement $\widetilde{x}$ of a MDOF system can be represented as a combination of modal contributions, as explained in Section 8.1.1. Based on this, the dynamic response of a MDOF system can be described as:

$$\widetilde{x}(t) = \sum_{p=1}^{N} \phi_p q_p(t) = \overline{\Phi}\widetilde{q}(t) \tag{2.2}$$

By applying Eq. (2.2) to the general coupled Eq. (2.1) expressed in $x_i(t)$, the system can be reformulated into a set of uncoupled equations using modal coordinates $q_m(t)$ as the unknowns. Substituting Eq. (2.2) into Eq. (2.1) yields this transformation:

$$\sum_{p=1}^{N} \overline{m}\phi_p \ddot{q}_p(t) + \sum_{p=1}^{N} \overline{K}\phi_p q_p(t) = \widetilde{P}(t) \tag{2.3}$$

Premultiplying each term in Eq. (2.3) by $\phi_r^{\,T}$ gives

$$\sum_{p=1}^{N} \phi_r^{\,T} \overline{m}\phi_p \ddot{q}_p(t) + \sum_{p=1}^{N} \phi_r^{\,T} \overline{K}\phi_p q_p(t) = \phi_r \widetilde{P}(t) \tag{2.4}$$

According to the orthogonality conditions outlined in Eq. (1.51), all terms in the summations are eliminated except for the term where p = r. Under this condition, Eq. (2.4) simplifies to:

$$(\phi_p^{\,T} \overline{m}\phi_p)\ddot{q}_p(t) + (\phi_p^{\,T} \overline{K}\phi_p) q_p(t) = \phi_p^{\,T} \widetilde{P}(t) \tag{2.5}$$

or

$$M_p \ddot{q}_p(t) + K_p q_p(t) = P_p(t) \tag{2.6}$$

where

$$M_p = \phi_p^{\,T} \overline{m}\phi_p \qquad K_p = \phi_p^{\,T} \overline{K}\phi_p \qquad P_p(t) = \phi_p^{\,T} \widetilde{P}(t) \tag{2.7}$$

Equation (2.6) represents the motion of a single degree of freedom (SDOF) system in terms of its response, qp(t). This system is characterized by a mass M_p, stiffness K_p, and dynamic force $P_p(t)$, where M_p is referred to as the generalized mass, K_p as the generalized stiffness, and $P_p(t)$ as the generalized force for the *p*th mode. These parameters are determined solely by the *p*th mode shape, ϕ_p. Consequently, if the pth mode is known, we can formulate and solve the equation for q_p without requiring information about other modes. By dividing Eq. (2.6) by M_p and applying Eq. (1.52), the resulting expression can be rewritten accordingly.

$$\ddot{q}_p + \omega_{np}^{\;2} q_p = \frac{P_p(t)}{M_p} \tag{2.8}$$

Equation (2.6) or Eq. (2.8) represents the governing equation of motion, with the sole unknown being $q_p(t)$, the normal coordinate of the pth mode. Similarly, there is a corresponding equation for each mode, resulting in a total of N equations for a multi-degree-of-freedom (MDOF) system.

A set of N coupled differential equations (2.1) expressed in terms of displacements $x_i(t)$, where i ranges from 1 to N, has been converted into a set of N independent equations (2.6) using modal coordinates $q_m(t)$, with m = 1, 2, ….., N. These uncoupled equations can be represented in matrix form as:

$$M^* \tilde{\ddot{q}} + K^* \tilde{q} = \tilde{P}(t) \tag{2.9}$$

Here, M^* represents a diagonal matrix containing the generalized modal masses M_p, while K^* is a diagonal matrix of the generalized modal stiffnesses K_p. Additionally, $\tilde{P}(t)$ is a column vector comprising the generalized modal forces $P_p(t)$. The definitions of M^* and K^* were previously provided in Equation (1.50).

3. Modal Equations for Forced Damped Systems

We know already that the governing equations of motion for a forced damped system is expressed as

$$\overline{m}\tilde{\ddot{x}} + \overline{c}\tilde{\dot{x}} + \overline{K}\tilde{x} = \tilde{P}(t) \tag{1.31}$$

By applying Eq. (2.2), which represents the natural modes ϕ_p of the system without damping, we can reformulate Eq. (1.31) to express it in terms of modal coordinates. For undamped systems, as explained in Section 9.2, these equations are independent and uncoupled. However, when dealing with damped systems, the modal equations may be coupled due to the influence of damping. If we substitute Eq. (2.2) in Eq. (1.31) we get :

$$\sum_{p=1}^{N} \overline{m}\,\phi_p \ddot{q}_p(t) + \sum_{p=1}^{N} \overline{c}\,\phi_p \dot{q}_p(t) + \sum_{p=1}^{N} \overline{K}\phi_p q_p(t) = \tilde{P}(t) \tag{2.10}$$

In Eq. (2.10) if each term is premultiplied by ϕ_r^T then we get

$$\sum_{p=1}^{N}\phi_r^T\overline{m}\phi_p\ddot{q}_p(t)+\sum_{p=1}^{N}\phi_r^T\overline{c}\phi_p\dot{q}_p(t)+\sum_{p=1}^{N}\phi_r^T\overline{K}\phi_p q_p(t)=\phi_r\widetilde{P}(t) \tag{2.11}$$

Making use of Eq. (2.7), Eq. (2.11) can be written as

$$M_p\ddot{q}_p(t)+C_{rp}\dot{q}(t)+K_p q_p(t)=P_p(t) \tag{2.12}$$

We have defined M_p, K_p, and P_p in Eq. (2.7). Now we define

$$C_{rp}=\phi_p^T\overline{c}\phi_p \tag{2.13}$$

For every value of p ranging from 1 to N, Equation (2.12) holds true. As a result, these N equations can collectively be expressed in matrix form as :

$$\overline{M}\ddot{\widetilde{q}}+\overline{C}\dot{\widetilde{q}}+\overline{K}_{Diag}\widetilde{q}=\widetilde{P}(t) \tag{2.14}$$

In Equation (2.14), the matrices $\overline{M}$, $\overline{K}_{Diag}$, and $\widetilde{P}(t)$ were previously defined in Equation (2.9). The matrix $\overline{C}$, introduced here, is a non-diagonal matrix containing the damping coefficients C_p. This equation represents a system of N equations expressed in terms of the modal coordinates $q_p(t)$. These equations are interconnected due to the damping terms, as Equation (2.12) includes multiple modal velocity components, leading to coupling between the modes.

If the system has classical damping, then the modal equations are uncoupled. For such type of systems $C_{rp} = 0$ if $r \neq p$. Therefore Eq. (2.12) is reduced to

$$M_p\ddot{q}_p(t)+C_p\dot{q}(t)+K_p q_p(t)=P_p(t) \tag{2.15}$$

The generalized damping is expressed by Eq. (1.82), which describes the behavior of the single-degree-of-freedom (SDOF) system. To simplify Eq. (2.15), we can divide it by M_p:

$$\ddot{q}_p+2\xi_p\omega_{np}q_p+\omega_{np}^2 q_p=\frac{P_p(t)}{M_p} \tag{2.16}$$

The damping ratio for the *p*th mode (ξ_p) is typically not calculated using Eq. (1.83); instead, it is estimated based on experimental data from structures similar to the one under analysis. Equation

(2.15) describes the pth modal coordinate $q_p(t)$, with parameters M_p, K_p, C_p, and $P_p(t)$ depending solely on the *p*th mode ϕ_p and are independent of other modes. Consequently, there are N uncoupled equations, each resembling Eq. (2.15), corresponding to each natural mode. In summary, the original set of N coupled differential equations (1.31) expressed in terms of nodal displacements $x_i(t)$ has been transformed into a set of uncoupled equations (2.15) in modal coordinates $q_p(t)$ through the application of the modal superposition method.

4. **Determination of Total Response**

For a multi-degree-of-freedom (MDOF) system subjected to known excitation forces, the dynamic response can be determined by solving either Eq. (2.15) or Eq. (2.16) in terms of the modal coordinates $q_p(t)$. Each modal equation has the same structure as the equation of motion for a single-degree-of-freedom (SDOF) system. Therefore, the solution methods and results used for SDOF systems can also be applied to solve for $q_p(t)$ in the modal equations. Once the modal coordinates $q_p(t)$ are obtained, Eq. (2.2) can then be used to calculate the contribution of the *p*th mode to the nodal displacement $\tilde{x}(t)$ as:

$$\tilde{x}(t) = \phi_p q_p(t) \tag{2.17}$$

We combine these modal contributions to get the total displacement response:

$$\tilde{x}(t) = \sum_{p=1}^{N} \tilde{x}_p(t) = \sum_{p=1}^{N} \phi_p q_p(t) \tag{2.18}$$

The process of determining the total response of a multi-degree-of-freedom (MDOF) system by combining the contributions of various modes is known as classical modal analysis or the classical mode superposition method. This approach involves solving individual uncoupled modal equations to find the modal coordinates $q_p(t)$ and modal responses $\tilde{x}_p(t)$. These modal responses are then combined to derive the total response $\tilde{x}(t)$. More specifically, this method is referred to as the classical mode displacement superposition method because it relies on the superposition of modal displacements. Commonly abbreviated as modal analysis, this technique is applicable only to linear systems with classical damping. The system's linearity is essential, as it allows the use of the principle of superposition, as expressed in Eq. (2.2). Additionally, damping must be in classical form to ensure that the modal equations remain uncoupled, which is a fundamental aspect of modal analysis.

5. Seismic Excitation

Dynamic analysis of multi-degree-of-freedom (MDOF) systems involves two slightly different analytical approaches. The first approach, time-history analysis, calculates how a structure responds over time when subjected to a base acceleration. This method uses either normal mode superposition or direct numerical integration of motion equations. The total system response is determined incrementally at very small time steps, with each step using the previous step's results as initial conditions for the next. This stepwise process continues until the full response is obtained. The second approach, called modal response spectrum analysis, estimates the maximum response parameters (such as displacements or bending moments) by combining the peak responses from individual modes. Each mode behaves like an independent single-degree-of-freedom (SDOF) system with its own natural period. The maximum response for a specific mode is derived from the corresponding SDOF system's spectrum. Since the peak responses of different modes do not occur simultaneously, various methods are used to combine these modal contributions. The most common method is the square root of the sum of the squares (SRSS), which assumes the modal maxima are random quantities. However, when natural periods of modes are very close (closely coupled modes), SRSS can underestimate the actual response, necessitating more accurate combination techniques, which will be discussed later. Typically, only the first few modes are considered, as they contribute most significantly to the overall response.

Modal response spectrum analysis is widely used in structural design and serves as a reference method for analyzing buildings such as RPA2024. It relies on normalized response spectra derived from multiple seismic records scaled to standard intensity levels. These spectra offer a straightforward way to study how structures respond to varying seismic inputs. In subsequent chapters, normalized response spectra will be frequently referenced.

5.1. Equation of motion

As shown in the previous section, the equation of motion of a system subjected to a base excitation is:

$$M\ddot{x}_a + C\dot{x} + Kx = 0$$

Where $\ddot{x}_a$ is vector of the absolute accelerations of the DoFs of the system while $\dot{x}$ and x are the vectors of the relative velocities and of the relative displacements of the DoFs of the system, respectively. The absolute displacement x_a of the system can be expressed as:

$$x_a = x_s + x$$

where is displacement x_s of the DoFs due to the static application (i.e. very slow so that no inertia and damping forces are generated) of the ground motion, and x is again the vector of the relative displacements of the DoFs of the system. The "static displacements" $x_s(t)$ can now be expressed in function of the ground displacement as follows:

$$x_s(t) = \iota . x_g(t)$$

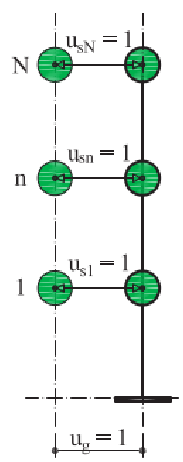
Where ι is the so-called influence vector. Equation (12.6) can now be rewritten as:

$$M(\iota \ddot{x}_g + \ddot{x}) + C\dot{x} + Kx = 0$$

$$M\ddot{x} + C\dot{x} + Kx = -M\iota\ddot{x}_g(t) \qquad (2.19)$$

An example of the influence vector for some typical cases is presented here

- Case 1 : Planar system with translational ground motion: In this case all DoFs of the system undergo static displacements which are equal to the ground displacement , hence:



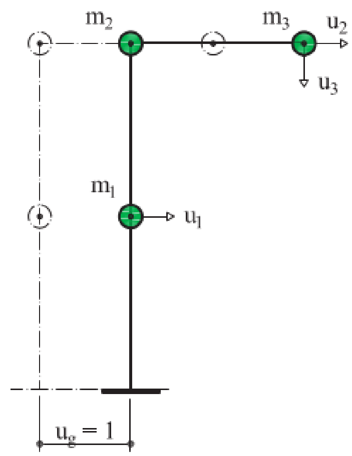
$$\boldsymbol{\iota} = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} = \mathbf{1}$$

Fig. 2.1 The influence vector for case 1

Where 1 is a vector of order N, i.e. the number of DoFs, with all elements equal to 1.

- Case 2 : Planar system with translational ground motion: The axial flexibility of the elements of the depicted system can be neglected, hence 3 DoFs are defined. In this case

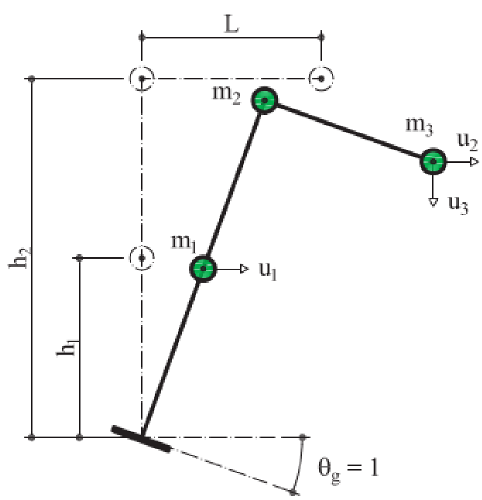
DoFs 1 and 2 undergo static displacements which are equal to the ground displacement, while the static displacement of DoF 3 is equal to 0, i.e.:



$$\iota = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Fig. 2.2 The influence vector for case 2

- Case 3: Planar system with rotational ground motion: The depicted system is subjected to a rotational ground motion θ_g which generates the following static displacements of the DoFs



$$x_s(t) = \begin{bmatrix} h_1 \\ h_2 \\ L \end{bmatrix} \theta_g(t) \text{ hence } \iota = \begin{bmatrix} h_1 \\ h_2 \\ L \end{bmatrix}$$

Fig. 2.3 The influence vector for case 3

5.2. Time-history of the response of elastic systems

As discussed in the previous sections, the equation of motion of a MDoF system under base excitation is:

$$M\ddot{x} + C\dot{x} + Kx = -M\iota\ddot{x}_g(t)$$

As already stated above, the dynamic response of the MDoF system can be written as:

$$x(t) = \sum_{n=1}^{N} \phi_n q_n(t) \tag{1.62}$$

If the damping of the MDoF system is classical, Equation (2.19) can be written in the form of decoupled modal equations, where is the number of modes of the system. The modal equations are of the following form:

$$m_n^* \ddot{q}_n + c_n^* \dot{q}_n + k_n^* q_n = -\phi_n^T M \ddot{x}_g \tag{2.20}$$

Where m_n^* and k_n^* are the modal mass and the modal stiffness respectively of the *n*th mode. These parameters are defined previously as follows:

$$m_n^* = \phi_n^T \cdot M \cdot \phi_n$$

$$k_n^* = \phi_n^T \cdot K \cdot \phi_n = \omega_n^2 \cdot m_n^*$$

ω_n : *n*th modal circular frequency of the MDOF system

The equation (2.20) can be rewritten as :

$$\ddot{q}_n + 2\zeta_n^* \omega_n \dot{q}_n + \omega_n^2 q_n = -\frac{\phi_n^T M \mathbf{1}}{\phi_n^T M \phi_n} \ddot{x}_g \tag{2.21}$$

The modal participation factor is a measure for the contribution of the n-th mode to the total response of the system. It is defined as follows:

$$\Gamma_n = \frac{\phi_n^T M \mathbf{1}}{\phi_n^T M \phi_n} \tag{2.22}$$

In addition the so-called effective modal mass of the *n*th mode is defined as:

$$m_{n,eff}^* = \Gamma_n^2 . m_n^* \tag{2.23}$$

Unlike the modal mass m_n^* and the modal participation factor Γ_n, the effective modal mass $m_{n,eff}^*$ is independent of the normalization of the eigenvectors. The following equation holds:

$$\sum_{n=1}^{N} m_{n,\text{eff}}^* = \sum_{n=1}^{N} m_n = m_{\text{tot}} \tag{2.24}$$

where m_{tot} is the total mass of the dynamic system. The effective modal height h_n^* of the *n*th mode is:

$$h_n^* = \frac{L_n^\theta}{L_n} \quad \text{with} \quad L_n^\theta = \sum_{j=1}^{N} h_j \cdot m_j \cdot \phi_{j,n} \quad \text{and} \quad L_n = \phi_n^T \cdot M\iota \tag{2.25}$$

- Significance of the effective modal mass $m_{n,eff}^*$: The effective modal mass is the lumped mass of a single-storey substitute system which is subjected to a base shear force V_{bn} equal to the *n*th modal base shear force of a multi-storey system. If in addition the height of the single storey substitute system with the lumped mass $m_{n,eff}^*$ equals the modal height h_n^*, the single-storey system is subjected to a base moment M_{bn} which is equal to the nth modal base moment of the multi-storey system. The following holds:

$$V_{bn} = m_{n,\text{eff}}^* \cdot S_{\text{pa},n} = \sum_{j=1}^{N} f_{jn} \tag{2.26}$$

$$M_{bn} = m_{n,\text{eff}}^* \cdot S_{\text{pa},n} \cdot h_n^* = \sum_{j=1}^{N} f_{jn} \cdot h_j \tag{2.27}$$

Where $S_{pa,n}$ is the pseudo-acceleration of the nth mode.

- Distribution of the internal forces: If the internal forces of the entire system are to be determined, the modal equivalent static forces should be computed first:

$$\mathbf{f}_n = s_n \cdot S_{\text{pa},n} \tag{2.28}$$

Where : $\mathbf{f}_n = \left[f_{1n} \quad f_{2n} \quad \cdots \quad f_{nn} \right]$

The excitation vector is defined according to equation (12.66) and specifies the distribution of the inertia forces due to excitation of the *n*th mode:

$$s_n = \Gamma_n M \phi_n \tag{2.29}$$

s_n is independent of the normalization of the eigenvector ϕ_n and we have that:

$$\sum_{n=1}^{N} s \;= M\iota \tag{2.30}$$

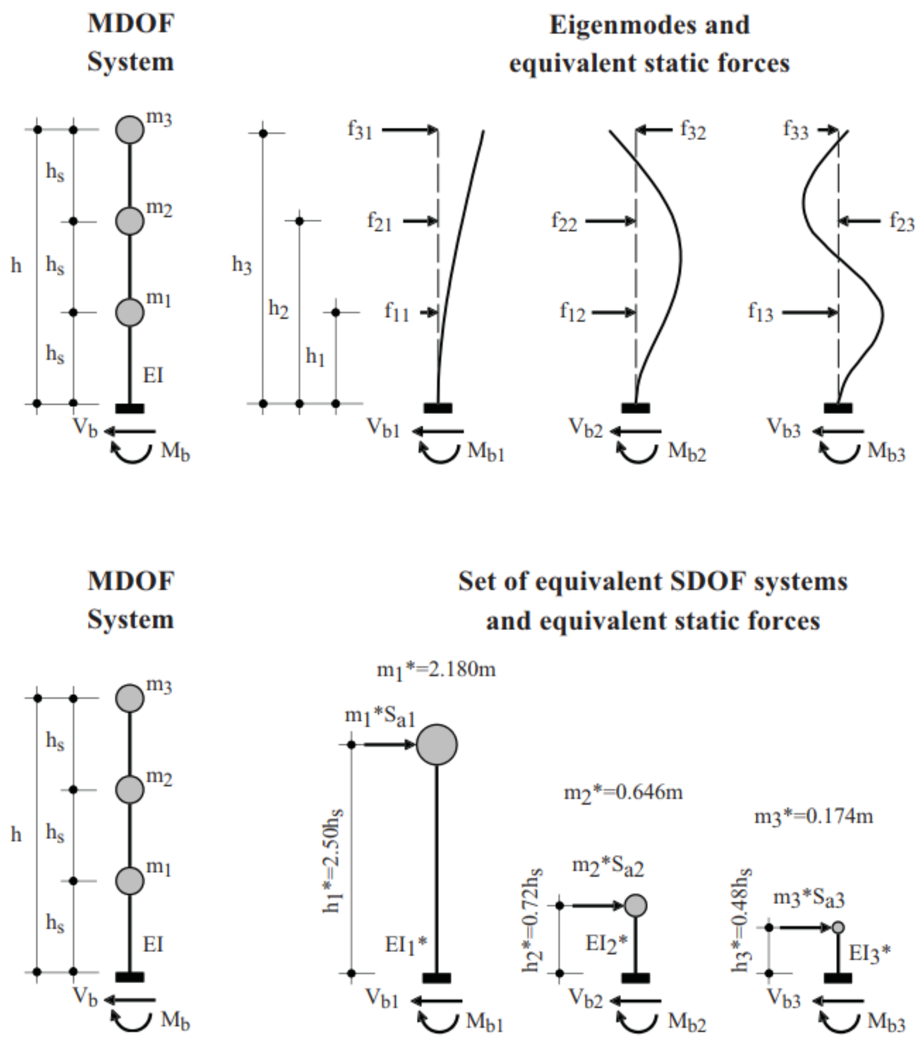


Fig. 2.3 MDoF system with eigenmodes and equivalent SDoF systems

5.3. Response spectrum method

If the maximum response only and not the response to the entire time history according to Equation (1.62) is of interest, the response spectrum method can be applied. The response spectrum can be computed for the considered seismic excitation and the maximum value of the modal coordinate $q_{n,max}$ can be determined as follows:

$$q_{n,\max} = \Gamma_n \cdot \; S_d(\omega_n, \zeta_n^*) = \Gamma_n \cdot \; \frac{1}{\omega_n^2} \cdot \; S_{pa}(\omega_n, \zeta_n^*) \tag{2.31}$$

where:

Γ_n: modal participation factor of the n-th mode

$S_d(\omega_n, \zeta_n^*)$: Spectral displacement for the circular eigenfrequency ω_n and the modal damping rate ζ_n^*.

$S_{pa}(\omega_n, \zeta_n^*)$: Spectral pseudo-acceleration for the circular eigenfrequency and the modal damping rate ζ_n^*.

The contribution of the *n*th mode to the total displacement is:

$$x_{n,\max} = \phi_n q_{n,\max} \tag{2.32}$$

5.3.1. Modal Seismic Response of Building

For the modal analysis of buildings, the participation factor Γn given by Eq. (2.22) can be rewritten with respect to weight as:

$$\Gamma_n = \frac{\sum_{i=1}^{N} W_i \phi_{in}}{\sum_{i=1}^{N} W_i \phi_{in}^2} \tag{2.33}$$

For normalized eigenvectors, the participation factor reduces to:

$$\Gamma_n = \frac{1}{g} \sum_{i=1}^{N} W_i \phi_{in} \tag{2.34}$$

because for normalized eigenvectors, $\sum_{i=1}^{N} W_i \phi_{in}^2 = 1$ where g is the acceleration due to gravity.

For convenience Eq. (2.61), can be written with omission of the participation factor as

$$\ddot{q}_n + 2\zeta_n^* \omega_n \dot{q}_n + \omega_n^2 q_n = \ddot{x}_g(t) \tag{2.35}$$

With the substitution

$$x_n = -\Gamma_n q_n \tag{2.36}$$

5.3.1.1. Modal Shear Force

The value of the maximum response for the modal spectral acceleration, S_{an}, is found from an appropriate response spectral chart.

From Eqs. (1.62) and (2.36), the maximum acceleration a_{zn} of the *n*th mode at the level z of the building is given by

$$a_{zn} = \Gamma_n \phi_{zn} S_{zn} \tag{2.37}$$

in which Sam and a_{zn} are usually expressed in units of the gravitational acceleration g. As stated, the modal values of the spectral acceleration Sa_n, the spectral velocity Sv_n, and the spectral displacement Sd_n are related by an apparent harmonic relationship:

$$Sa_n = \omega_n Sv_n = \omega_n^2 Sd_n \tag{2.38}$$

or in terms of the modal period $T_n = 2\pi/\omega_n$ by:

$$Sa_n = \frac{2\pi}{T_n} Sv_n = \left(\frac{2\pi}{T_n}\right)^2 Sd_n \tag{2.39}$$

On the basis of these relations, the modal spectral acceleration Sam in Eq. (2.37) may be replaced by the spectral displacement Sd_n times ωm2 or by the spectral velocity Sv_n times ω_n. The modal lateral force F_{zn} at the level z of the building is then given by Newton's Law as:

$$F_{zn} = a_{zn} W_z \quad \text{or} \quad F_{zn} = \Gamma_n \phi_{zn} Sa_n W_z \tag{2.40}$$

in which San is the modal spectral acceleration in g units (Note: RPA2024 requires to scale Sa_n multiplied by the importance factor, I and divided by response modifications coefficient, R and W_z is the weight attributed to the level z of the building.

The modal shear force V_{zn} at the level z of the building is equal to the sum of the seismic forces F_{zn} above that level, namely,

$$V_{zn} = \sum_{i=z}^{N} F_{iz} \tag{2.41}$$

The total modal shear force Vn at the base of the building is then calculated as:

$$V_n = \sum_{i=1}^{N} F_{iz} \tag{2.42}$$

or using Eq. (2.40)

$$V_n = \sum_{i=1}^{N} \Gamma_n \phi_{in} W_i Sa_n \tag{2.43}$$

5.3.1.2. Effective Modal Weight

The effective modal weight Wm is defined by the equation

$$V_n = W_n Sa_n \tag{2.41}$$

Then, from Eq. (2.43), the modal weight is

$$W_n = \Gamma_n \sum_{i=1}^{N} \phi_{in} W_i \tag{2.42}$$

Combining Eqs. (2.33) and (2.42) results in the following important expression for the effective modal weight:

$$W_n = \frac{\left[\sum_{i=1}^{N} \phi_{in} W_i \right]^2}{\sum_{i=1}^{N} \phi_{in}^2 W_i} \tag{2.43}$$

It can be proven analytically that the sum of the effective modal weights for all the modes of the building is equal to the total design weight of the building, that is:

$$\sum_{n=1}^{N} W_n = \sum_{i=1}^{N} W_i \tag{2.44}$$

Equation (2.44) is most convenient in assessing the number of significant modes of vibration to consider in the design. Specifically, the RPA2024 requires that, in applying the dynamic method of analysis, a sufficient number of modes are needed to estimate a combined modal mass participation of 100% of the structure's mass. Alternatively, this requirement can be satisfied by including a sufficient number of modes such that their total effective modal weight is at least 90% of the total design weight of the building. Thus, this requirement can be satisfied by simply adding a sufficient number of effective modal weights [Eq. (2.43)] until their total weight is 90% or more of the seismic design weight of the building.

5.3.1.3. Modal Lateral Forces

By combining Eq. (2.40) with Eqs. (2.41) and (2.42), we may express the modal lateral force F_{zn} as:

$$F_{zn} = C_{zn} V_n \tag{2.45}$$

Where the modal seismic coefficient C_{zn} at level x is given by:

$$C_{zn} = \frac{\phi_{zn} W_z}{\sum_{i=1}^{N} \phi_{in} W_i} \tag{2.46}$$

5.3.1.4. Modal Displacements

The modal displacement δ_{zn} at the level x of the building may be expressed as:

$$\delta_{zn} = \Gamma_n \phi_{zn} S d_n \tag{2.47}$$

where Γ_n is the participation factor for the nth mode, ϕ_{zn} is the component of the modal shape at level x of the building, and Sd_n is the spectral displacement for that mode. Alternatively, the modal displacement δ_{zn} may be calculated from Newton's Law of Motion in the form

$$F_{zn} = \frac{W_z}{g}\omega_n^2\delta_{zn} \tag{2.48}$$

because the magnitude of the modal acceleration corresponding to the modal displacement δ_{zn} is $\omega_n^2\ \delta_{zn}$. Hence, from Eq.

$$\delta_{zn} = \frac{g}{\omega_n^2}\frac{F_{zn}}{W_z} \tag{2.49}$$

or substituting $\omega_n = 2\pi/T_n$

$$\delta_{zn} = \frac{g}{4\pi^2}\cdot\frac{T_n^2 F_{zn}}{W_z} \tag{2.50}$$

where T_n is the *n*th natural period.

5.3.1.5. Modal Drift

The modal drift Δ_{zn} for the *z*th story of the building, defined as the relative displacement of two consecutive levels, is given by

$$\Delta_{zn} = \delta_{zn} - \delta_{(z-1)n} = \Delta a \tag{2.51}$$

5.3.1.6. Modal Overturning Moment

The modal overturning moment M_{zn} at the level x of the building which is calculated as the sum of the moments of the seismic forces F_{zn} above that level is given by:

$$M_{zn} = \sum_{i=z+1}^{N} F_{in}(h_i - h_z) \tag{2.52}$$

where hi and h_z are, respectively, the height of levels i and z. The modal overturning moment Mm at the base of the building then is given by:

$$M_n = \sum_{i=z+1}^{N} F_{in}h_i \tag{2.53}$$

5.3.1.7. Modal Torsional Moment

The modal torsional moment Mt_{nz} at level z, which is due to eccentricity e_z between the center of the above mass and the center of stiffness at that level (measured normal to the direction considered), is calculated as:

$$Mt_{nz} = e_z V_{zn} \tag{2.54}$$

where V_{zn} is the modal shear force at level z.

The RPA2024 requires that an accidental torsional moment be added to the torsional moment existent at each level. The recommended way to add the accidental torsion is to offset the center of mass at each level by 5% of the dimension of the building normal to the direction under consideration.

5.3.2. Modal combination

The maxima of different modes do not occur at the same instant. An exact computation of the total maximum response on the basis of the maximum modal responses is hence impossible. Different methods have been developed to estimate the total maximum response from the maximum modal responses.

5.3.2.1. Combination rule: "Absolute Sum (ABSSUM)"

$$x_{i,\max} \leq \sum_{j=1}^{n} \phi_{ij} q_{j,\max}$$

The assumption that all maxima occur at the same instant and in the same direction yields an upper bound value for the response quantity. This assumption is commonly too conservative.

5.3.2.2. Combination Rule: "Square-Root-of Sum-of-Squares (SRSS)"

$$x_{i,\max} = \sqrt{\sum_{j=1}^{n} \left(\phi_{ij} q_{j,\max}\right)^2}$$

This rule is often used as the standard combination method and yields very good estimates of the total maximum response if the modes of the system are well separated. If the system has several modes with similar frequencies the SRSS rule might yield estimates which are significantly lower than the actual total maximum response.

5.3.2.3. Combination Rule: "Complete Quadratic Combination (CQC)"

$$x_{i,\max} = \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} x_{i,\max}^{(j)} x_{i,\max}^{(k)} \rho_{jk}}$$

Where $x_{i,\max}^{(j)}$ and $x_{i,\max}^{(k)}$ are the max modal responce of mode j and mode k, and ρ_{jk} is the modal correlation coefficient between mode j and mode k

$$\rho_{jk} = \frac{8\sqrt{\zeta_j \zeta_k}(\zeta_j + r\zeta_k) r^{3/2}}{(1-r^2)^2 + 4\zeta_j \zeta_k r(1+r^2) + 4(\zeta_j^2 + \zeta_k^2) r^2} \quad \text{with } r = \frac{\omega_k}{\omega_j}$$

5.3.3. Number of modes to be considered

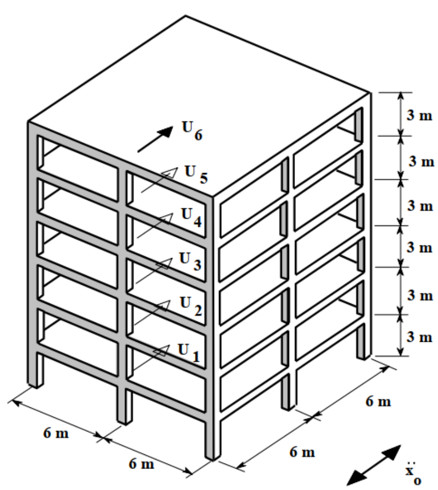
A comprehensive dynamic analysis should consider all contributing modes. However, practical applications often focus on modes exceeding a specific contribution threshold. Notably, the number of modes required for accurate results may vary depending on the response measure (e.g., displacements, shear forces, bending moments), necessitating a tailored approach to modal selection for each parameter of interest.

For a regular building the top displacement can be estimated fairly well based on the fundamental mode only. To estimate the internal forces, however, higher modes need to be considered too.

According to RPA2024, all modes should be considered (starting from the lowest) until the sum of the effective modal masses of all considered modes corresponds to at least 90% of the total mass.

Example 2.1

Figure below shows a building that is part of an industrial facility. We want to study the response of the building to the N-S component of the recorded accelerations at El Centro, California, in Mayo 18 of 1940. We are interested in the response in the direction shown in the figure. Damping for the system was estimated in $\xi = 5\%$ of critical. All girders of the structure have width b = 0.40 m and depth h = 0.50 m. All columns have square section with a cross-section dimension h = 0.50 m. The material of the structure has a modulus of elasticity E = 25 GPa. The self-weight of structure plus additional dead load is 780 kg/m^2 and the industrial machinery, which is firmly connected to the building slabs, increases the mass per unit area by 1000 kg/m^2, for a total mass per unit area of 1780 kg/m^2.



Solution 2.1

The area of each floor slab is 12 m · 12 m = 144 m^2. The total translational mass of each story is m = 144 m^2· 1780 kg/m^2 = 256 Mg. The mass matrix of the buildings is:

$$[K_E]=10^3\times\begin{bmatrix} 216.76 & -306.77 & 105.49 & -19.561 & 4.2822 & -0.51088 \\ -306.77 & 668.24 & -475.14 & 137.94 & -29.375 & 5.3857 \\ 105.49 & -475.14 & 731.37 & -493.23 & 159.60 & -29.327 \\ -19.561 & 137.94 & -493.23 & 749.02 & -494.47 & 145.71 \\ 4.2822 & -29.375 & 159.60 & -494.47 & 738.11 & -515.90 \\ -0.51088 & 5.3857 & -29.327 & 145.71 & -515.90 & 889.94 \end{bmatrix}$$

$$[M]=\begin{bmatrix} 256 & 0 & 0 & 0 & 0 & 0 \\ 0 & 256 & 0 & 0 & 0 & 0 \\ 0 & 0 & 256 & 0 & 0 & 0 \\ 0 & 0 & 0 & 256 & 0 & 0 \\ 0 & 0 & 0 & 0 & 256 & 0 \\ 0 & 0 & 0 & 0 & 0 & 256 \end{bmatrix}$$

Matrix$\left[\iota\right]$ is in this case a single column vector having one in all rows, because all the lateral degrees of freedom of the structure are parallel to the ground motion acceleration. The dynamic equilibrium equations are:

$$\begin{bmatrix} 256 & 0 & 0 & 0 & 0 & 0 \\ 0 & 256 & 0 & 0 & 0 & 0 \\ 0 & 0 & 256 & 0 & 0 & 0 \\ 0 & 0 & 0 & 256 & 0 & 0 \\ 0 & 0 & 0 & 0 & 256 & 0 \\ 0 & 0 & 0 & 0 & 0 & 256 \end{bmatrix}\begin{bmatrix} \ddot{x}_6 \\ \ddot{x}_5 \\ \ddot{x}_4 \\ \ddot{x}_3 \\ \ddot{x}_2 \\ \ddot{x}_1 \end{bmatrix}$$

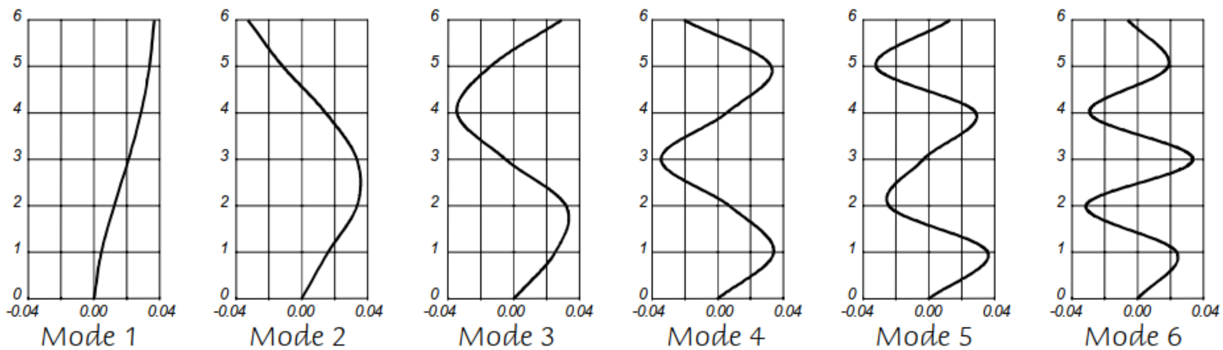
$$+10^3\times\begin{bmatrix} 216.76 & -306.77 & 105.49 & -19.561 & 4.2822 & -0.51088 \\ -306.77 & 668.24 & -475.14 & 137.94 & -29.375 & 5.3857 \\ 105.49 & -475.14 & 731.37 & -493.23 & 159.60 & -29.327 \\ -19.561 & 137.94 & -493.23 & 749.02 & -494.47 & 145.71 \\ 4.2822 & -29.375 & 159.60 & -494.47 & 738.11 & -515.90 \\ -0.51088 & 5.3857 & -29.327 & 145.71 & -515.90 & 889.94 \end{bmatrix}\begin{bmatrix} x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}=-[M]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\ddot{x}_0$$

After solving the eigenvalues problem for this system, we find:

Mode	ω^2 (rad/s)²	ω (rad/s)	f (Hertz)	T (s)
1	29.108	5.39	0.859	1.16
2	301.81	17.4	2.76	0.36
3	973.78	31.2	4.97	0.2
4	2494.3	49.9	7.95	0.13
5	4686.5	68.5	10.9	0.092
6	7113.8	84.3	13.4	0.075

The corresponding vibration modes are:

$$[\Phi]=\begin{bmatrix} 0.036721 & -0.032775 & 0.029168 & -0.020667 & 0.013049 & -0.005955 \\ 0.033690 & -0.011592 & -0.014245 & 0.032483 & -0.032188 & 0.018512 \\ 0.028524 & 0.014524 & -0.034529 & 0.005317 & 0.028533 & -0.029103 \\ 0.020961 & 0.033322 & -0.005049 & -0.034504 & -0.003317 & 0.033609 \\ 0.012243 & 0.033525 & 0.031633 & 0.006893 & -0.024392 & -0.031454 \\ 0.004460 & 0.015888 & 0.025184 & 0.034025 & 0.035774 & 0.023711 \end{bmatrix}$$



(T_1 = 1.16 s) (T_2 = 0.36 s) (T_3 = 0.20 s) (T_4 = 0.13 s) (T_5 = 0.092 s) (T_6 = 0.075 s)

The modal participation factors are obtained from:

$$\{\Gamma\}=[\Phi]^T[M][\iota]=\begin{Bmatrix} 34.970 \\ 13.540 \\ 8.2331 \\ 6.0279 \\ 4.4695 \\ 2.3861 \end{Bmatrix}$$

The total effective mass is computed as αi2

Mode	Γ_i	Γ_i^2	%M_{tot}	%M_{tot} accumulated
1	34.97	1222.901	79.62%	79.62%
2	13.54	183.332	11.93%	91.55%
3	8.2331	67.784	4.41%	95.96%
4	6.0279	36.336	2.37%	98.33%
5	4.4695	19.976	1.30%	99.63%
6	2.3861	5.693	0.37%	100.00%

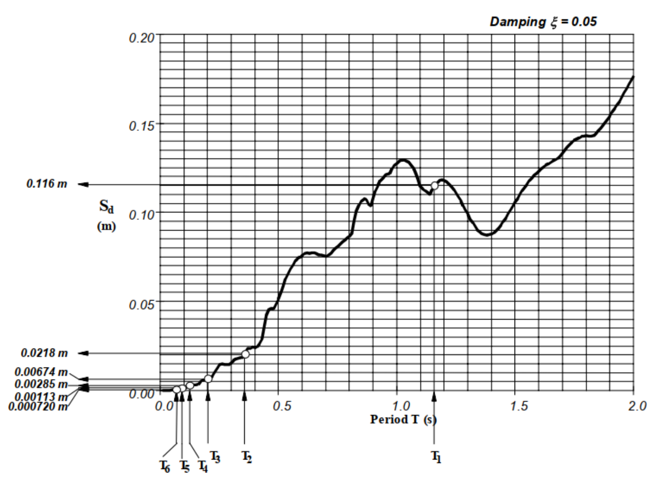
Now we modify the dynamic equilibrium equations by pre-multiplying by $[\Phi]^T$ and using the following coordinate transformations:

$$\{X\} = [\Phi]\{q\} \quad \text{and} \quad \{\ddot{X}\} = [\Phi]\{\ddot{q}\}$$

The uncoupled vibration equations are:

$$\begin{aligned}
\ddot{q}_1 + 2\xi_1\omega_1\dot{q}_1 + \omega_1^2 q_1 &= -34.970\ddot{x}_0 \\
\ddot{q}_2 + 2\xi_2\omega_2\dot{q}_2 + \omega_2^2 q_2 &= -13.540\ddot{x}_0 \\
\ddot{q}_3 + 2\xi_3\omega_3\dot{q}_3 + \omega_3^2 q_3 &= -8.2331\ddot{x}_0 \\
\ddot{q}_4 + 2\xi_4\omega_4\dot{q}_4 + \omega_4^2 q_4 &= -6.0279\ddot{x}_0 \\
\ddot{q}_5 + 2\xi_5\omega_5\dot{q}_5 + \omega_5^2 q_5 &= -4.4695\ddot{x}_0 \\
\ddot{q}_6 + 2\xi_6\omega_6\dot{q}_6 + \omega_6^2 q_6 &= -2.3861\ddot{x}_0
\end{aligned}$$

In all six equations $\xi_i = 0.05$. The response for each of the uncoupled equations is obtained using the displacement response spectra for the N-S component of the El Centro record. The figure below shows the spectrum and period for each mode and the displacement read from the spectrum for each period.



Mode	T_i (s)	$S_d(T_i,\xi_i)$ (m)
1	1.16	0.116
2	0.36	0.0218
3	0.2	0.00674
4	0.13	0.00285
5	0.092	0.00113
6	0.075	0.00072

With this information, it is possible to compute the maximum displacement that the uncoupled degrees of freedom can attain:

Mode	Γ_i	$S_d(T_i,\xi_i)$ (m)	$(q_i)_{max} = \Gamma_i x\, S_d(T_i,\xi_i)$ (m)
1	34.97	0.116	4.0495
2	13.54	0.0218	0.29571
3	8.233	0.00674	0.055458
4	6.028	0.00285	0.017155
5	4.469	0.00113	0.005064
6	2.386	0.00071	0.001717

Maximum modal displacements (m)

The maximum displacements for each mode are obtained from:

$$\{X_{\text{mod}}^{(i)}\} = \{\phi^{(i)}\}(q_i)_{\max}$$

These results can be computed for all the modes at the same time by introducing the values of (ηi)max in the diagonal of a square matrix [H_{mod}] y and performing the operation:

$$[X_{\text{mod}}] = [\Phi][H_{\text{mod}}] = \left\lfloor \{X_{\text{mod}}^{(1)}\} \quad \{X_{\text{mod}}^{(2)}\} \quad \cdots \quad \{X_{\text{mod}}^{(6)}\} \right\rfloor$$

In present case matrix [H_{mod}] has the following form:

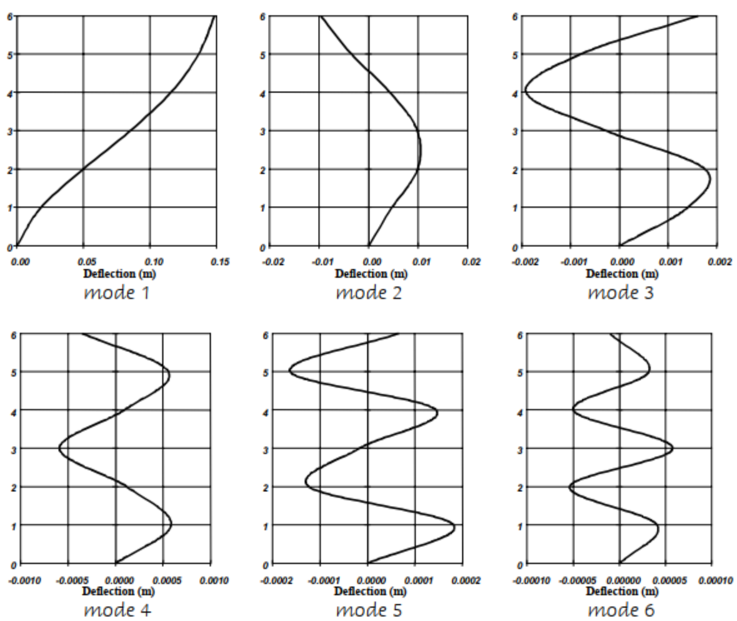
$$[H_{\text{mod}}] = \begin{bmatrix} (q_1)_{\max} & 0 & 0 & 0 & 0 & 0 \\ 0 & (q_2)_{\max} & 0 & 0 & 0 & 0 \\ 0 & 0 & (q_3)_{\max} & 0 & 0 & 0 \\ 0 & 0 & 0 & (q_4)_{\max} & 0 & 0 \\ 0 & 0 & 0 & 0 & (q_5)_{\max} & 0 \\ 0 & 0 & 0 & 0 & 0 & (q_6)_{\max} \end{bmatrix}$$

And replacing the appropriate values from Table 2:

$$[H_{\text{mod}}]=\begin{bmatrix} 4.0495 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.29571 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0055458 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.015155 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0050639 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0017177 \end{bmatrix}$$

The values for [U_{mod}] are:

$$[X_{\text{mod}}]=[\Phi][H_{\text{mod}}]=\begin{bmatrix} 0.148703 & -0.009692 & 0.001618 & -0.000355 & 0.000066 & -0.000010 \\ 0.136429 & -0.003428 & -0.000790 & 0.000557 & -0.000163 & 0.000032 \\ 0.115519 & 0.004295 & -0.001915 & 0.000091 & 0.000144 & -0.000050 \\ 0.084882 & 0.009854 & -0.000280 & -0.000592 & -0.000017 & 0.000058 \\ 0.049588 & 0.009914 & 0.001754 & -0.000118 & -0.000124 & -0.000054 \\ 0.018061 & 0.004698 & 0.001397 & 0.000584 & 0.000181 & 0.000041 \end{bmatrix}$$

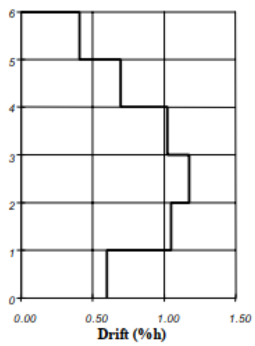


<u>Maximum story drift as a percentage of story height (%h)</u>

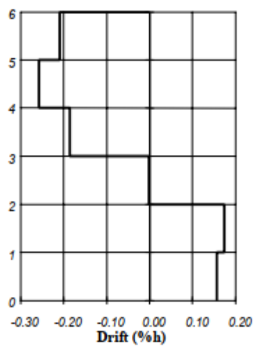
Using the displacements just computed the story drift for each story and mode could be computed as the algebraic difference of the displacement of two consecutive stories. Drift is usually expressed as percentage of the inter-story height.

Story	mode 1	mode 2	mode 3	mode 4	mode 5	mode 6
6	0.409%	-0.209%	0.080%	-0.030%	0.008%	-0.001%
5	0.697%	-0.257%	0.037%	0.016%	-0.010%	0.003%
4	1.021%	-0.185%	-0.054%	0.023%	0.005%	-0.004%
3	1.177%	-0.002%	-0.068%	-0.024%	0.004%	0.004%
2	1.051%	0.174%	0.012%	-0.016%	-0.010%	-0.003%
1	0.602%	0.157%	0.047%	0.019%	0.006%	0.001%

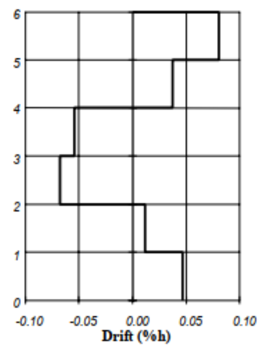
Next figure shows the story drifts for each mode:



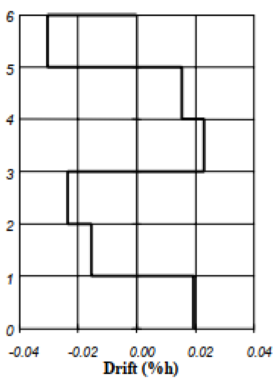
mode 1



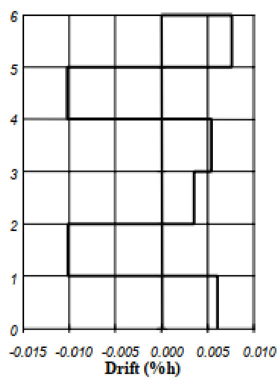
mode 2



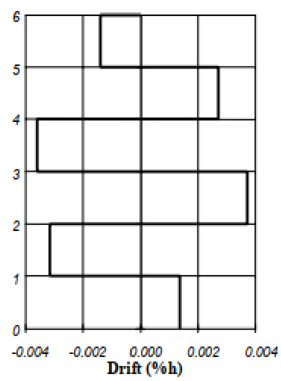
mode 3



mode 4



mode 5



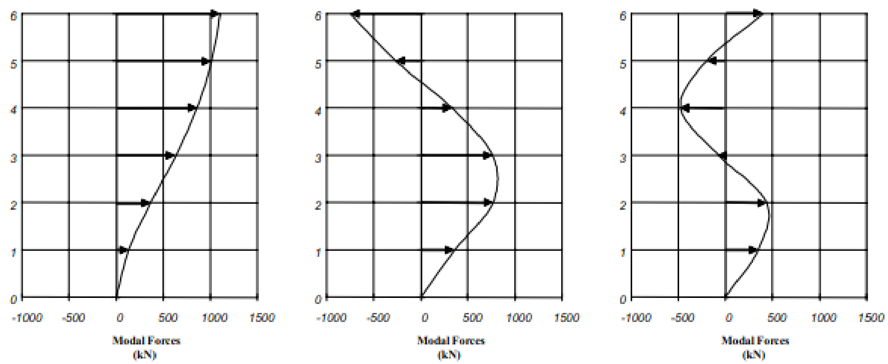
mode 6

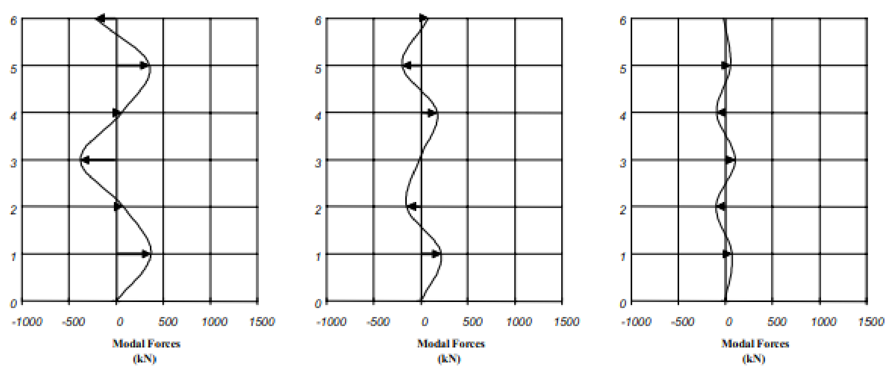
Maximum modal lateral forces (kN)

To obtain the maximum modal lateral forces imposed on the structure by the ground motions the stiffness matrix of the structure is multiplied by the modal lateral displacements. Results are obtained in kN.

$$[\mathbf{F}_{\text{mod}}] = [\mathbf{K}_E]\begin{bmatrix} X_{\text{mod}}^{(1)} & X_{\text{mod}}^{(2)} & \cdots & X_{\text{mod}}^{(6)} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\text{mod}}^{(1)} & \mathbf{F}_{\text{mod}}^{(2)} & \cdots & \mathbf{F}_{\text{mod}}^{(6)} \end{bmatrix}$$

$$[F_{\text{mod}}]=[K_E][X_{\text{mod}}]=\begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$



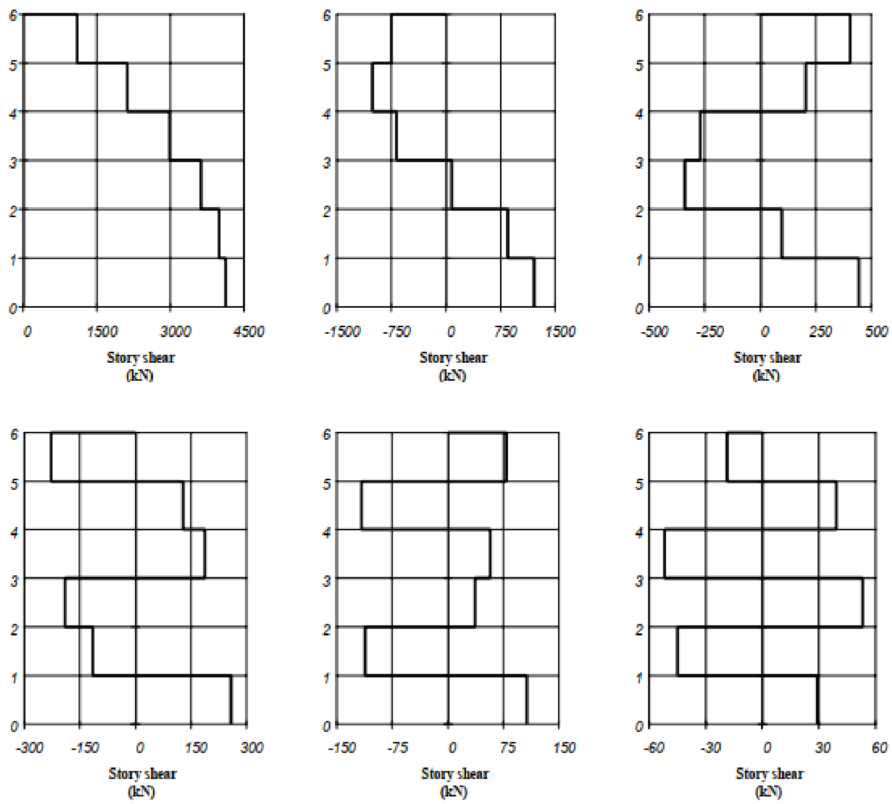


mode 4 mode 5 mode 6

Maximum modal story shear (kN)

The maximum modal story shear is obtained from $V_j^{(i)}=\sum_{k=j}^{n}F_k^{(i)}$

Story	**$V^{(1)}_{mod}$ (kN)**	**$V^{(2)}_{mod}$ (kN)**	**$V^{(3)}_{mod}$ (kN)**	**$V^{(4)}_{mod}$ (kN)**	**$V^{(5)}_{mod}$ (kN)**	**$V^{(6)}_{mod}$ (kN)**
6	1108.3	-748.9	403.3	-226.4	79.3	-18.6
5	2124.6	-1013.7	206.3	129.4	-116.3	39.3
4	2984.8	-681.9	-271	187.7	57.1	-51.7
3	3617.6	79.6	-340.9	-190.3	36.9	53.4
2	3987	845.5	96.5	-114.8	-111.3	-45
1	4122.1	1208.5	444.6	257.9	106.1	29.1
0	4122.1	1208.5	444.6	257.9	106.1	29.1



mode 4 mode 5 mode 6

Base shear (kN)

The base shear in kN for each mode is obtained from

$$\{V_{\text{mod}}\} = \{1\}^T[F_{\text{mod}}] = \{1|1|1|1|1|1\}\begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

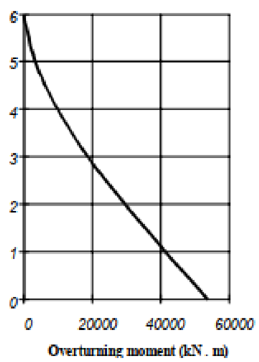
$$= \{4122.1|1208.5|444.6|257.9|106.1|29.1\}$$

$$= \{V_{\text{mod}}^{(1)} \quad |V_{\text{mod}}^{(1)}| \quad V_{\text{mod}}^{(1)}|V_{\text{mod}}^{(1)} \quad |V_{\text{mod}}^{(1)}|V_{\text{mod}}^{(1)}\}$$

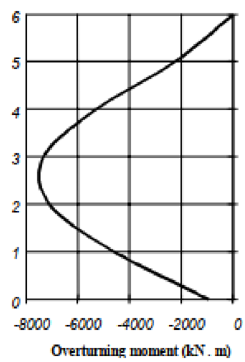
Overturning moment (kN · m)

The overturning moment for each story is obtained from $M_j^{(i)} = \sum_{k=j+1}^{n} \left(h_k - h_j\right) \cdot F_j^{(i)}$

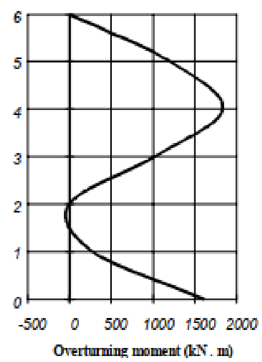
Story	$M^{(1)}_{mod}$ (kN . m)	$M^{(2)}_{mod}$ (kN . m)	$M^{(3)}_{mod}$ (kN . m)	$M^{(4)}_{mod}$ (kN . m)	$M^{(5)}_{mod}$ (kN . m)	$M^{(6)}_{mod}$ (kN . m)
6	0	0	0	0	0	0
5	3324.9	-2246.7	1209.8	-679.2	237.8	-55.9
4	9698.6	-5287.8	1828.7	-290.9	-111	61.9
3	18652.9	-7333.6	1015.6	272.2	60.2	-93.3
2	29505.8	-7094.7	-6.9	-298.7	170.9	66.8
1	41466.8	-4558.2	282.4	-643.1	-162.9	-68.2
0	53833.1	-932.7	1616.3	130.7	155.3	19.2



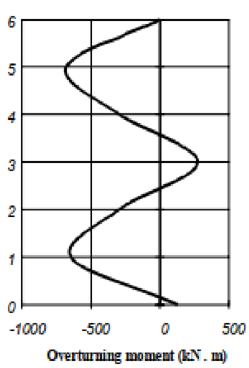
mode 1



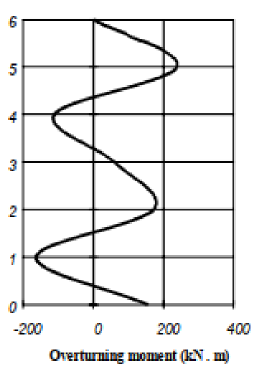
mode 2



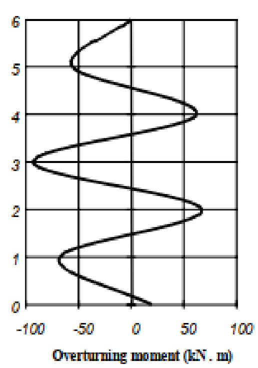
mode 3



mode 4



mode 5



mode 6

The maximum overturning moment at the base, in kN·m, contributed by each mode can be obtained from:

$$\{\mathbf{M}_{\text{mod}}\} = \{\mathbf{h}\}^T[\mathbf{F}_{\text{mod}}] = \{18|15|12|9|6|3\}\begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

$$= \{53833|-933|1616|131|155|19\}$$
$$= \{M_{\text{mod}}^{(1)}|M_{\text{mod}}^{(2)}|M_{\text{mod}}^{(3)}|M_{\text{mod}}^{(4)}|M_{\text{mod}}^{(5)}|M_{\text{mod}}^{(6)}\}$$

<u>Maximum credible lateral displacements (m)</u>

The maximum modal displacements were obtained from: $\{X_{\text{mod}}^{(i)}\} = \{\phi^{(i)}\} \cdot (q_i)_{\max}$

$$[X_{\text{mod}}] = [\Phi][H_{\text{mod}}] = \begin{bmatrix} 0.148703 & -0.009692 & 0.001618 & -0.000355 & 0.000066 & -0.000010 \\ 0.136429 & -0.003428 & -0.000790 & 0.000557 & -0.000163 & 0.000032 \\ 0.115519 & 0.004295 & -0.001915 & 0.000091 & 0.000144 & -0.000050 \\ 0.084882 & 0.009854 & -0.000280 & -0.000592 & -0.000017 & 0.000058 \\ 0.049588 & 0.009914 & 0.001754 & -0.000118 & -0.000124 & -0.000054 \\ 0.018061 & 0.004698 & 0.001397 & 0.000584 & 0.000181 & 0.000041 \end{bmatrix}$$

We now apply the SRSS procedure to each of the row of previous matrix. For example, for the roof (6^{th} story):

$$X_6^{\max} = \sqrt{(0.148703)^2 + (-0.009692)^2 + (0.001618)^2 + (-0.000355)^2 + (0.000066)^2 + (-0.000010)^2}$$
$$= 0.14903\text{m}$$

<u>Maximum credible story drift</u>

The modal spectral story drifts are computed from the values shown in [X_{mod}] Using Eq. (2.51) the following result are obtained:

$$[\Delta_{\text{mod}}] = \begin{bmatrix} 0.012274 & -0.006264 & 0.002408 & -0.000912 & 0.000229 & -0.000042 \\ 0.020920 & -0.007723 & 0.001125 & 0.000466 & -0.000307 & 0.000082 \\ 0.030627 & -0.005559 & -0.001635 & 0.000683 & 0.000161 & -0.000108 \\ 0.035304 & -0.000060 & -0.002034 & -0.000710 & 0.000107 & 0.000112 \\ 0.031517 & 0.005216 & 0.000358 & -0.000465 & -0.000305 & -0.000095 \\ 0.018061 & 0.004698 & 0.001397 & 0.000584 & 0.000181 & 0.000041 \end{bmatrix}$$

As an example, we now apply the SRSS procedure to the third story:

$$\Delta_3^{SRSS} = \sqrt{(0.035304)^2 + (-0.000060)^2 + (-0.002034)^2 + (-0.000710)^2 + (0.000107)^2 + (0.000112)^2}$$
$$= 0.03537\text{m}$$

And for all stories:

$$\{\Delta_{SRSS}\} = \begin{bmatrix} 0.0140 \\ 0.0223 \\ 0.0312 \\ 0.0354 \\ 0.0320 \\ 0.0188 \end{bmatrix}, \quad m = \begin{bmatrix} 0.47\%/h \\ 0.74\%/h \\ 1.04\%/h \\ 1.18\%/h \\ 1.07\%/h \\ 0.62\%/h \end{bmatrix}$$

Maximum credible story forces (kN)

The maximum modal spectral forces were obtained for each mode multiplying the stiffness matrix by the modal spectral displacements of each mode, obtaining there the following forces in

$$\text{kN: } [\mathbf{F}_{\text{mod}}] = [\mathbf{K}_E]\begin{bmatrix} X_{\text{mod}}^{(1)} & X_{\text{mod}}^{(2)} & \cdots & X_{\text{mod}}^{(6)} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\text{mod}}^{(1)} & \mathbf{F}_{\text{mod}}^{(2)} & \cdots & \mathbf{F}_{\text{mod}}^{(6)} \end{bmatrix}$$

$$[F_{\text{mod}}] = [K_E][X_{\text{mod}}] = \begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

Maximum credible story shear (kN)

The maximum credible modal spectral story shear may be obtained from Eq.(2.41) $V_j^{(i)} = \sum_{k=j}^{p} F_k^{(i)}$

Story	**$V^{(1)}_{mod}$ (kN)**	**$V^{(2)}_{mod}$ (kN)**	**$V^{(3)}_{mod}$ (kN)**	**$V^{(4)}_{mod}$ (kN)**	**$V^{(5)}_{mod}$ (kN)**	**$V^{(6)}_{mod}$ (kN)**
6	1108.3	-748.9	403.3	-226.4	79.3	-18.6
5	2124.6	-1013.7	206.3	129.4	-116.3	39.3
4	2984.8	-681.9	-271	187.7	57.1	-51.7
3	3617.6	79.6	-340.9	-190.3	36.9	53.4
2	3987	845.5	96.5	-114.8	-111.3	-45
1	4122.1	1208.5	444.6	257.9	106.1	29.1

Applying, for example, the SRSS procedure to the second story, we obtain:

$$V_2^{\text{SRSS}} = \sqrt{(3987.0)^2 + (845.5)^2 + (96.5)^2 + (-114.8)^2 + (-111.3)^2 + (-45.0)^2} = 4080.2\text{kN}$$

The result, in kN, for all stories is

$$\{V_{\text{SRSS}}\} = \pm \begin{bmatrix} 1417.6 \\ 2369.8 \\ 3080.3 \\ 3640.1 \\ 4080.2 \\ 4327.6 \end{bmatrix}$$

Maximum credible base shear

The base shear, in kN, was obtained in Example 6 for each mode as:

$$\{\mathbf{V}_{\text{mod}}\} = \{1\}^T [\mathbf{F}_{\text{mod}}] = \{1|1|1|1|1|1\} \begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

$$= \{4122.1|1208.5|444.6|257.9|106.1|29.1\}$$

Applying the SRSS procedure:

$$V^{\text{SRSS}} = \sqrt{(4122.1)^2 + (1208.5)^2 + (444.6)^2 + (257.9)^2 + (106.1)^2 + (29.1)^2} = 4327.6\text{kN}$$

Maximum credible overturning moment

The overturning moment for each story and mode is obtained using Eq. (2.52):

Story	$M^{(1)}_{mod}$ (kN . m)	$M^{(2)}_{mod}$ (kN . m)	$M^{(3)}_{mod}$ (kN . m)	$M^{(4)}_{mod}$ (kN . m)	$M^{(5)}_{mod}$ (kN . m)	$M^{(6)}_{mod}$ (kN . m)
6	0	0	0	0	0	0
5	3324.9	-2246.7	1209.8	-679.2	237.8	-55.9
4	9698.6	-5287.8	1828.7	-290.9	-111	61.9
3	18652.9	-7333.6	1015.6	272.2	60.2	-93.3
2	29505.8	-7094.7	-6.9	-298.7	170.9	66.8
1	41466.8	-4558.2	282.4	-643.1	-162.9	-68.2
0	53833.1	-932.7	1616.3	130.7	155.3	19.2

Now using the SRSS procedure for example to the fourth story:

$$M_4^{\text{SRSS}} = \sqrt{(9698.6)^2 + (-5287.7)^2 + (1828.7)^2 + (-290.9)^2 + (-111.0)^2 + (61.9)^2} = 4080.2\text{kN}$$

The result, in kN·m, for all stories is:

$$\{\mathbf{M}^{\text{SRSS}}\} = \begin{Bmatrix} 0.0 \\ 4252.9 \\ 11201.3 \\ 20070.6 \\ 30348.8 \\ 41722.9 \\ 53865.8 \end{Bmatrix}$$

Maximum credible base overturning moment

Base overturning moment contributed by each mode can be computed from:

$$\{\mathbf{M}_{\text{mod}}\} = \{\mathbf{h}\}^T[\mathbf{F}_{\text{mod}}] = \{18|15|12|9|6|3\}\begin{bmatrix} 1108.3 & -748.9 & 403.3 & -226.4 & 79.3 & -18.6 \\ 1016.2 & -264.8 & -196.9 & 355.8 & -195.6 & 57.9 \\ 860.2 & 331.8 & -477.4 & 58.2 & 173.4 & -91.0 \\ 632.9 & 761.5 & -69.8 & -378.0 & -20.2 & 105.1 \\ 369.4 & 765.9 & 437.3 & 75.5 & -148.2 & -98.4 \\ 135.1 & 363.0 & 348.2 & 372.7 & 217.3 & 74.1 \end{bmatrix}$$

$$= \{53833|-933|1616|131|155|19\}$$

And

$$M^{\text{SRSS}} = \sqrt{(53833.1)^2 + (-932.7)^2 + (1616.3)^2 + (130.7)^2 + (155.3)^2 + (19.2)^2} = 53865.8 \quad \text{kN.m}$$

Chapter 3

Progressive Pushover Method

1. Introduction

Pushover analysis is a nonlinear static analysis method widely used in structural engineering to evaluate the seismic performance of buildings and other structures. This method involves applying progressively increasing lateral loads to a structural model until it reaches its ultimate capacity or a predefined target displacement. The procedure accounts for the redistribution of forces as structural elements yield, enabling engineers to simulate the inelastic behavior of a structure under seismic loading. By plotting the relationship between base shear and roof displacement, known as the capacity curve, pushover analysis provides valuable insights into the structure's strength, deformation capacity, and potential failure mechanisms.

The primary objective of pushover analysis is to identify weak points in the structure and predict how it will behave during an earthquake. It helps engineers assess critical parameters such as plastic hinge formation, interstory drifts, and force demands on individual members. This method is particularly useful for performance-based seismic design and retrofitting of existing buildings, offering a practical alternative to more complex nonlinear dynamic analyses. However, it is important to note that pushover analysis relies on simplified assumptions about load patterns and may not fully capture dynamic effects or higher-mode contributions during an earthquake. Despite these limitations, it remains a powerful tool for understanding and enhancing the seismic resilience of structures.

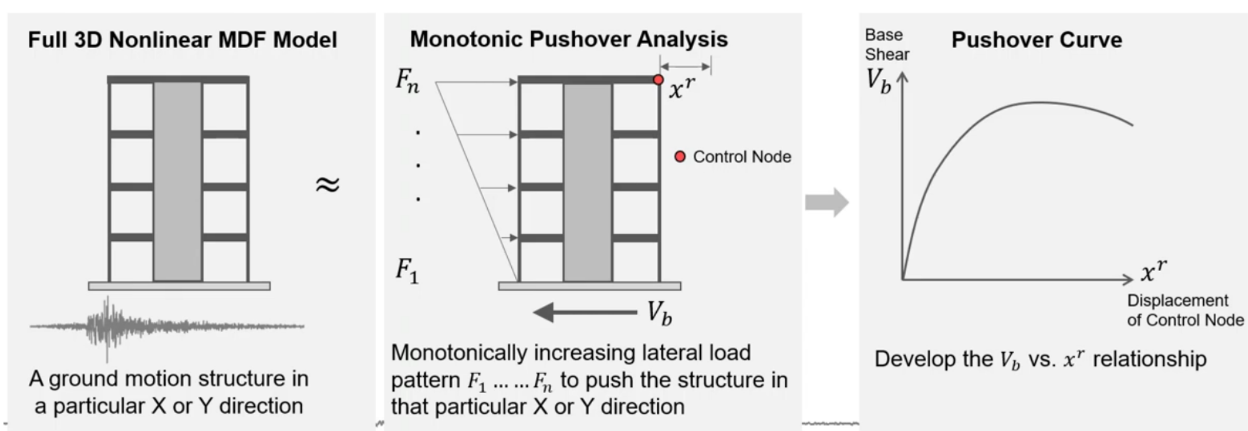


Fig 3.1 The basic idea of Pushover analysis

2. Definition of the structure and behavior laws of plastic nodes

In the context of pushover analysis, the definition of the structure and behavior laws of plastic nodes revolves around the modeling of structural elements and their transition from elastic to plastic behavior under increasing lateral loads. Structural elements are typically modeled using linear elastic behavior up to a certain threshold, defined by their yield point. Beyond this yield point, plastic hinges are introduced at specific locations, such as the ends of beams or columns, to simulate the inelastic behavior of the structure. These plastic hinges represent localized zones where plastic deformations occur, allowing for redistribution of forces within the structure.

The behavior of plastic nodes is governed by moment-rotation relationships that describe their response under loading. Initially, the structure behaves elastically with high stiffness (zone AB in a typical moment-rotation curve). Upon reaching the yield point (point B), the stiffness decreases as the structure enters an inelastic phase (zone BC), where deformations increase without significant additional resistance. The post-yield behavior can be idealized as elastic-perfectly plastic or include strain hardening or softening effects depending on material properties and design assumptions. In advanced models, these hinges are characterized by acceptance criteria such as Immediate Occupancy, Life Safety, and Collapse Prevention states, which correspond to increasing levels of deformation and damage.

The placement and properties of these plastic nodes are critical for accurately predicting the nonlinear response of structures during seismic events or other extreme loading scenarios. By tracking the formation and progression of plastic hinges during pushover analysis, engineers can evaluate structural performance, identify failure mechanisms, and ensure compliance with safety standards.

3. Lateral force distribution

The lateral force distribution refers to the manner in which horizontal forces are applied across the height of a structure during analysis. This distribution is a critical factor influencing the accuracy of the results, as it determines how seismic demands are represented and how structural responses, such as inter-story drifts and member deformations, are captured.

The Nonlinear Static Pushover Procedure outlined in RPA2024 is based on the N2 method introduced by Fajfar in 1999. This approach involves applying fixed load patterns to a building model, which simulate the lateral forces caused by ground motion. The intensity of these loads is gradually increased in a pseudo-static manner. Depending on the building's structural characteristics, the model can be either planar (2D) or spatial (3D). However, the load pattern is always applied in a single direction. For scenarios where ground motion input occurs in multiple directions, such as both x and y axes, RPA2024 provides specific combination rules for analysis.

This nonlinear pushover analysis involves incrementally increasing constant-shape lateral load distributions on the structure being studied. The structural model can be either 2D or 3D depending on the building's plan regularity. Generally, buildings with regular plans can be analyzed using a 2D single-plane frame model, while those with irregular plans require a full 3D model. Since nonlinear methods are particularly useful for existing buildings—which are often irregular—a 3D model is typically necessary in most cases.

The N2 method was originally developed using a shear building model, meaning it assumes a frame structure with floors that are rigid within their planes. Vertical displacements are generally ignored in this method, focusing instead on the two horizontal components of ground motion (x and y directions). Extending this method to more complex cases involving fully deformable frames is relatively straightforward. The N2 method applies two distinct load distributions to the frame for analysis.

Typically, lateral forces in pushover analysis are applied using predefined patterns that approximate the effects of seismic loads. Commonly used distributions include:

- Uniform Distribution: Forces are proportional to the mass at each floor level, assuming a uniform response across the structure.
- Mode Shape-Based Distribution: Forces are distributed according to the fundamental mode shape of vibration, which is suitable for regular buildings with fundamental periods up to approximately one second.

The chosen lateral force distribution significantly impacts the resulting pushover curve (base shear vs. roof displacement) and structural performance predictions. While simpler patterns may suffice for regular structures, more complex patterns are essential for accurately assessing irregular or flexible buildings.

In the N2 method, the mode shape ϕ_1 is scaled such that the displacement at the top floor equals 1, denoted as $\phi_{1,n} = 1$. The two types of load distributions are illustrated schematically in the figure below. These lateral load distributions are gradually increased, and the resulting response is represented as a plot of base shear (Vb) against the top floor displacement (D), typically measured at the center of mass of the top floor. This plot is commonly referred to as the pushover curve or capacity curve.

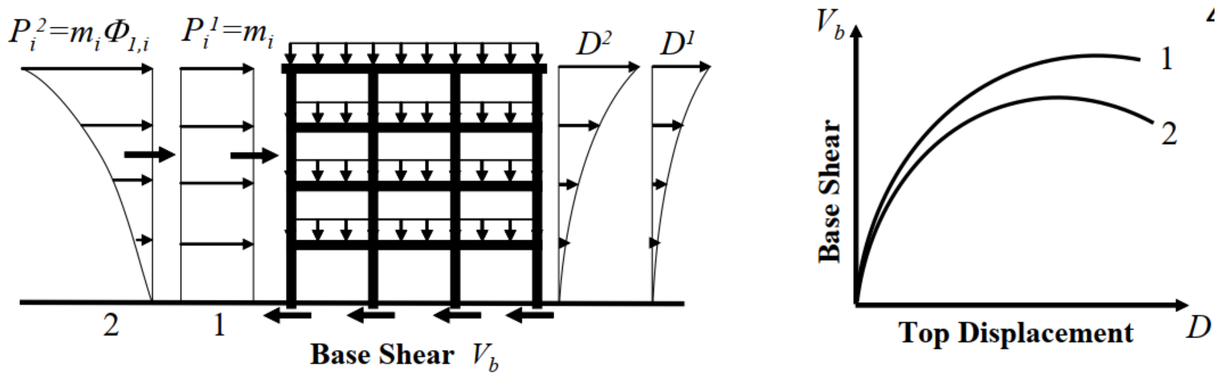


Fig. 3.2 Lateral force distribution in Pushover analysis

4. Capacity curve

4.1. Equivalent SDOF systems

According to Fajfar (1999), it is assumed that the building behaves as a shear frame, meaning the floors are considered rigid within their own plane. When vertical displacements of the building are disregarded, the floor movements can be described using three degrees of freedom, as illustrated in Window 3-2. These degrees of freedom are generally defined at the center of mass. It is important to note that the beams are capable of deforming outside the plane of the floor, allowing the nodes to exhibit rotational degrees of freedom beyond the floor plane.

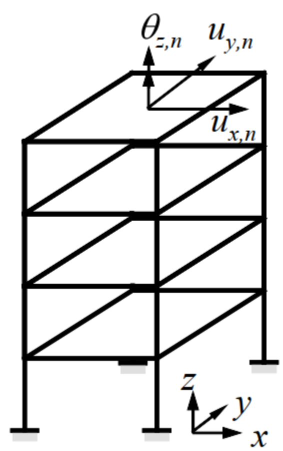


Fig 3.3 Typically shear frame for pushover analysis

In modern earthquake engineering, seismic input is typically characterized using design spectra or accelerograms. However, in codified design, this input is always defined or at least linked to spectra, which provide critical data about the acceleration and/or displacement of single-degree-of-freedom (SDOF) systems. As a result, it becomes crucial to connect the outcomes of pushover analysis—applied to multi-degree-of-freedom (MDOF) systems—with the characteristics of an equivalent SDOF system. This process must account for the nonlinear response in terms of both forces (such as base shear) and deformations. To address this, an equivalent SDOF oscillator is employed. A specific approach to this is outlined in the following two paragraphs: the first focuses on the standard translational case, while the second extends to scenarios involving both translational and rotational (torsional) behavior.

Various methods for defining an equivalent Single Degree of Freedom (SDOF) oscillator can be found in the literature. However, they all share a common starting point: the assumption that the deformation of a Multi-Degree of Freedom (MDOF) system can be represented by a deformation vector [Φ], which remains unchanged throughout the duration of the loading time-history, regardless of the magnitude of the applied deformation. This section introduces a widely recognized approach for defining an SDOF oscillator specifically for the translational behavior of spatial (3-D) structures. The equation governing the dynamic elastic response of the system to external excitation, as illustrated in Figure 5.29, is presented in vector form.

$$[M][\ddot{x}(t)] + [C][\dot{x}(t)] + [P(t)] = -[M][1]\ddot{x}_0(t) \qquad 3.1$$

By eliminating the damping terms from Equation 3.1, [C]$[\dot{x}(0)]$, Equation 3.2 results:

$$[M][\ddot{x}(t)] + [P(t)] = -[M][1]\ddot{x}_0(t) \qquad 3.2$$

It is assumed that the displacement vector [x] and the restoring force vector [P] of the elastic multi-degree-of-freedom (MDOF) system can be related to the corresponding parameters of an equivalent single-degree-of-freedom (SDOF) nonlinear oscillator $u_n(t)$ and $P_n(t)$, using two vectors, [Φ] and [Ψ]:

$$[x(t)] = [\Phi]x_n(t) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} x_n(t) \qquad 3.3$$

$$[P] = [\Psi]P_n(t) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} P_n(t) \qquad 3.4$$

Therefore, using these transformations, the equation of vibration of the MDOF system becomes, in vector form:

$$[M][\Phi]\ddot{x}_n(t) + [\Psi]P_n(t) = -[M][1]\ddot{x}_0(t) \qquad 3.5a$$

And in algebraic form:

$$\left.\begin{aligned} m_1\phi_1\ddot{x}_n(t) + \psi_1 P_n &= -m_1\ddot{x}_0(t) \\ m_2\phi_2\ddot{x}_n(t) + \psi_2 P_n &= -m_2\ddot{x}_0(t) \\ &\vdots \\ m_n\phi_n\ddot{x}_n(t) + \psi_n P_n &= -m_n\ddot{x}_0(t) \end{aligned}\right\} \qquad 3.5b$$

By multiplying Equation (3.5a) times $[\Phi]^T$:

$$[\Phi]^T[M][\Phi]\ddot{x}_n(t) + [\Phi]^T[\Psi]P_n(t) = -[\Phi]^T[M][1]\ddot{x}_0(t) \qquad 3.8$$

and by transforming the first term:

$$[\Phi]^T[M][\Phi]\left(\frac{[\Phi]^T[M][1]}{[\Phi]^T[M][1]}\right)\ddot{x}_n(t) + [\Phi]^T[\Psi]P_n(t) = -[\Phi]^T[M][1]\ddot{x}_0(t) \qquad 3.8a$$

or

$$[\Phi]^T[M][1]\left(\frac{[\Phi]^T[M][\Phi]}{[\Phi]^T[M][1]}\right)\ddot{x}_n(t) + [\Phi]^T[\Psi]P_n(t) = -[\Phi]^T[M][1]\ddot{x}_0(t) \qquad 3.8b$$

We define

$$x^* = \frac{[\Phi]^T[M][\Phi]}{[\Phi]^T[M][1]} x_n(t) \tag{3.9a}$$

$$m^* = [\Phi]^T[M][1] \tag{3.9b}$$

so, Equation 3.5a becomes in vector form:

$$m^* \ddot{x}_n^*(t) + [\Phi]^T[\Psi]P_n(t) = -m^* \ddot{x}_0(t) \tag{3.10}$$

While Equations 3.9a and 3.9b are in algebraic form:

$$m^* = m_1\phi_1 + m_2\phi_2 + \cdots + m_n\phi_n = \sum_{i=1}^{n} m_i\phi_i$$

$$[\Phi]^T[M][\Phi] = m_1\phi_1^2 + m_2\phi_2^2 + \cdots + m_n\phi_n^2 = \sum_{i=1}^{n} m_i\phi_i^2$$

$$[\Phi]^T[\Psi] = \sum_{i=1}^{n} \psi_i\phi_i$$

meaning

$$m^* = \sum_{i=1}^{n} m_i\phi_i \tag{3.11a}$$

$$x_n^*(t) = \frac{\sum_{i=1}^{n} m_i\phi_i^2}{\sum_{i=1}^{n} m_i\phi_i} x_n(t) \tag{3.11b}$$

$$[\Phi]^T[\Psi] = \sum_{i=1}^{n} \psi_i\phi_i \tag{3.11c}$$

Equation 3.10 by introducing Equations 3.11a through 3.11c is transformed to:

$$\left[\sum_{i=1}^{n} m_i\phi_i\right]\ddot{x}_n(t) + \left[\sum_{i=1}^{n} m_i\psi_i\right]P_n(t) = -\left[\sum_{i=1}^{n} m_i\phi_i\right]\ddot{x}_0(t) \tag{3.12}$$

Taking into account that:

$$\left[\sum_{i=1}^{n} m_i \psi_i\right] P_n(t) = V(t) \quad \text{or} \quad P_n(t) = \frac{V(t)}{\sum_{i=1}^{n} m_i \psi_i} \qquad 3.13$$

Where V is the base shear of the MDOF excited system, Equation 3.12 may be re-written as:

$$m^* \ddot{x}_n^*(t) + \frac{\sum_{i=1}^{n} \phi_i \psi_i}{\sum_{i=1}^{n} m_i \psi_i} V(t) = -m^* \ddot{x}_0(t) \qquad 3.14$$

This corresponds to an SDOF system, which has defined the following properties:

$$m^* = [\Phi]^T [M][1] = \sum_{1}^{n} m_i \phi_i$$

See equation 3.8a

$$V^*(t) = \frac{[\Phi]^T [\Psi]}{[1]^T [\Psi]} V(t) = \frac{\sum_{i=1}^{n} \phi_i \psi_i}{\sum_{i=1}^{n} m_i \psi_i} V(t) \qquad 3.15$$

Where m^*, $u^*(t)$, V^* are the mass, displacement, and base shear of the equivalent SDOF oscillator. The equation of vibration under excitation for this SDOF system is:

$$m^* \ddot{x}_n^*(t) + V^*(t) = -m^* \ddot{x}_0(t) \qquad 3.16$$

Having

$$x^*(t) = \Gamma_1 x_n(t) \qquad 3.17a$$

$$V^*(t) = \Gamma_2 V(t) \qquad 3.17b$$

$$\Gamma_1 = \frac{[\Phi]^T[M][\Phi]}{[\Phi]^T[M][1]} = \frac{\sum_{i=1}^{n} m_i\phi_i^2}{\sum_{i=1}^{n} m_i\phi_i} \qquad 3.17c$$

$$\Gamma_2 = \frac{[\Phi]^T[\Psi]}{[1]^T[\Psi]} = \frac{\sum_{i=1}^{n} \phi_i\psi_i}{\sum_{i=1}^{n} \psi_i} \qquad 3.17d$$

By introducing $\psi_i=\phi_i$, one gets:

$$\Gamma_2 = \frac{\sum_{i=1}^{n} \phi_i^2}{\sum_{i=1}^{n} \phi_i} \qquad 3.18a$$

And for $m_i=m=$constant:

$$\Gamma_1 = \frac{\sum_{i=1}^{n} m\phi_i^2}{\sum_{i=1}^{n} m\phi_i} = \frac{m\sum_{i=1}^{n} \phi_i^2}{m\sum_{i=1}^{n} \phi_i} = \Gamma_2 \qquad 3.18b$$

So, in the end, one factor Γ is used, with:

$$\Gamma = \frac{\sum_{i=1}^{n} \phi_i^2}{\sum_{i=1}^{n} \phi_i} \qquad 3.19$$

This transformation factor is denoted as Γ in the Annex B of EC8-1/2004, Equation B3:

$$\Gamma = \frac{m^*}{\sum_{i=1}^{n} m_i\phi_i^2} = \frac{\sum_{i=1}^{n} m_i\phi_i}{\sum_{i=1}^{n} m_i\phi_i^2} \qquad 3.20$$

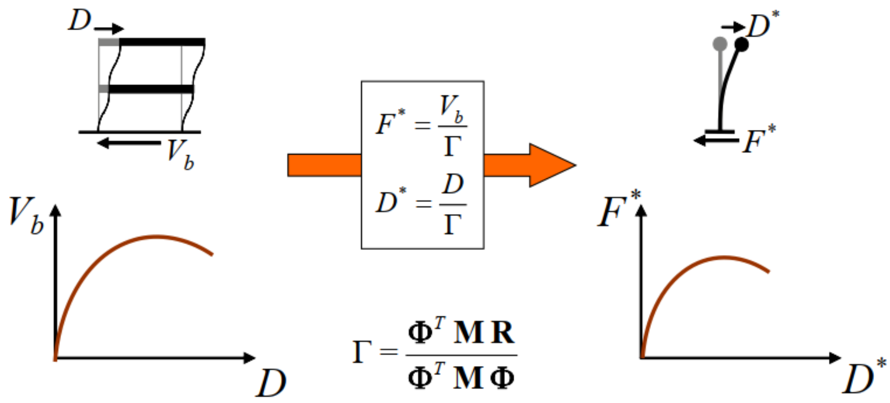


Fig 3.4 Summary of equivalent SDOF system

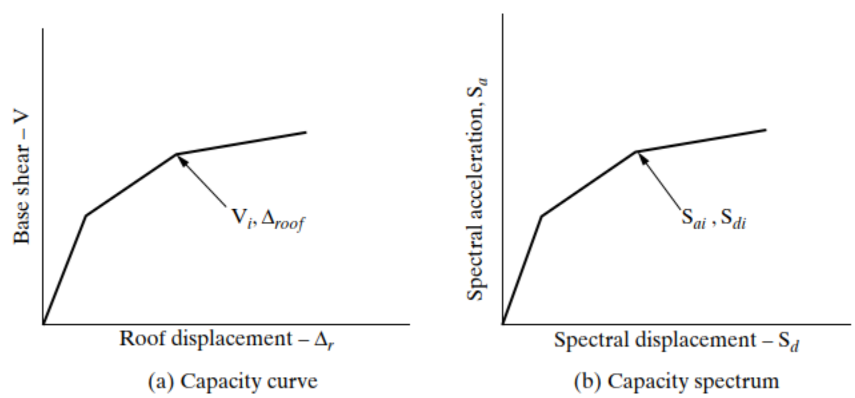


Fig. 3.5 Conversion of pushover curve to spectral form

4.2. Linearization of the capacity curve

To compare the capacity curve with the demand curve provided by the design spectrum, the nonlinear pushover curves of the single-degree-of-freedom (SDOF) system are simplified into elastic-perfectly plastic (or bilinear) representations. As outlined in Annex J of RAP2024, this transformation relies on the equal energy principle. A target displacement is defined, and it is assumed that the energy remains equal between the bilinear and nonlinear pushover curves. This straightforward method is demonstrated in the figure below.

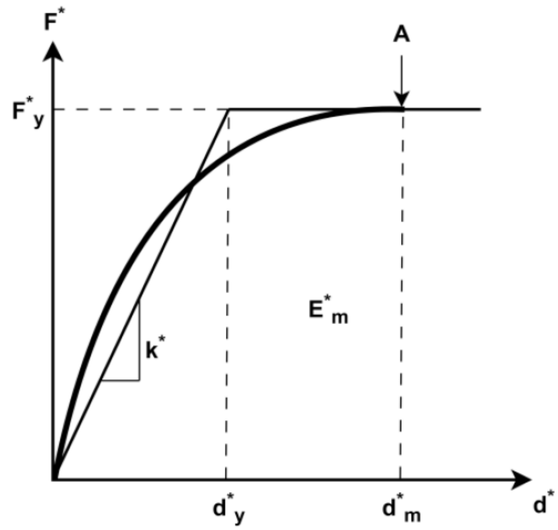


Fig. 3.6 Linearization of the capacity curve

The bilinearization of figure gives the yield force and the yield displacement

$$d_y^* = 2\left(d_m^* - \frac{E_m^*}{F_y^*}\right) \qquad 3.21$$

which allow the initial elastic period to be computed as:

$$T^* = 2\pi\sqrt{\frac{m^*}{k^*}} \qquad 3.22$$

Secondly, the capacity curve is transformed into capacity spectrum by normalizing the force with respect to the SDOF weight. The resulting capacity spectrum is shown in :

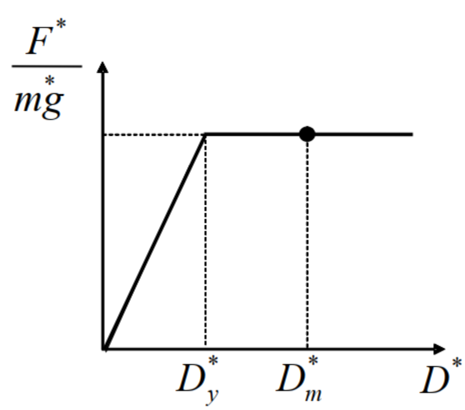


Fig 3.7 The bilinear capacity curve

5. **Seismic demand**

The building's demand is determined using the design spectrum outlined in the design code (RPA2024).

$$\frac{S_a(T)}{g} = \begin{cases} A.I.S.\left[\frac{2}{3} + \frac{T}{T_1}\cdot\left(2.5\frac{Q_F}{R} - \frac{2}{3}\right)\right] & \text{if } 0 \le T < T_1 \\ A.I.S.\left[2.5\frac{Q_F}{R}\right] & \text{if } T_1 \le T < T_2 \\ A.I.S.\left[2.5\frac{Q_F}{R}\cdot\frac{T_2}{T}\right] & \text{if } T_2 \le T < T_3 \\ A.I.S.\left[2.5\frac{Q_F}{R}\cdot\frac{T_2 T_3}{T^2}\right] & \text{if } T_3 \le T < 4s \end{cases} \qquad 3.23$$

To effectively compare the building's capacity with this demand, the initial step involves converting the design spectrum from its traditional format, which plots Acceleration (A) against Period (T), into the ADRS format, where Acceleration (A) is plotted against Displacement (D). This transformation is straightforward since there is a direct relationship between Acceleration and Displacement.

$$S_D = \left(\frac{T}{2\pi}\right)^2 S_A \qquad 3.24$$

The transformation to the ADRS spectrum is shown in figure. Lines from the origin represent constant periods.

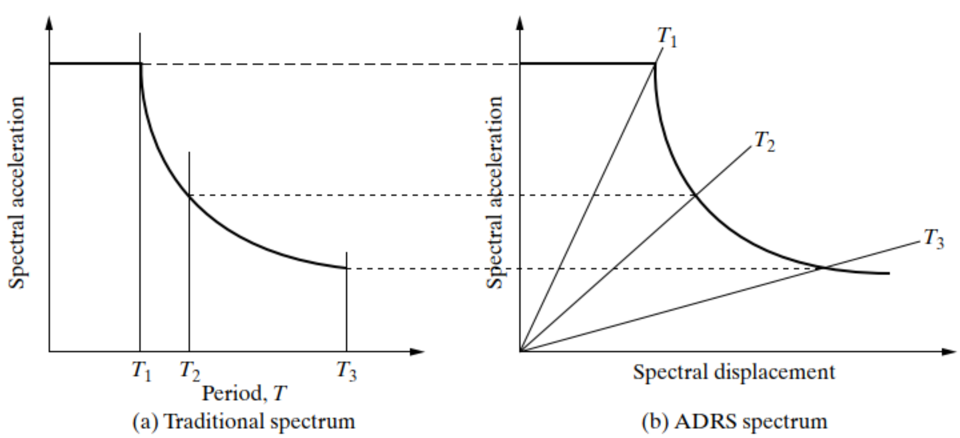


Fig. 3.8 Transformation to ADRS linear spectrum

The capacity spectrum shown in Fig. 3.7 is compared to the ADRS demand spectrum depicted in Fig. 3.8. However, this comparison is not straightforward because the capacity spectrum exhibits nonlinear behavior, whereas the ADRS spectrum provided by design codes is linear. For a single-degree-of-freedom (SDOF) system with bilinear plastic behavior, the acceleration spectrum (SA) and displacement spectrum (SD) can be calculated as follows:

$$S_a = \frac{S_{ae}}{R_\mu}, \quad S_d = \frac{\mu}{R_\mu} S_{de} \qquad 3.25$$

$$S_a(T,\xi) = \frac{S_{ae}(T,\xi)}{R_\mu} \qquad 3.26$$

$$S_{de}(T,\xi) = \frac{T^2}{4\pi^2} S_{ae}(T,\xi) \qquad 3.27$$

Where

ξ : viscous damping ratio fixed at 5%

$S_{ae}(T,\xi)$: acceleration in the elastic spectrum corresponding to periods T et $\xi = 5\%$

$S_{de}(T,\xi)$: displacement in the elastic spectrum corresponding to periods T et $\xi = 5\%$

$$S_d(T,\xi) = \frac{\mu}{R_\mu} S_{de}(T,\xi) = \frac{\mu}{R_\mu} \frac{T^2}{4\pi^2} S_{ae}(T,\xi) = \mu \frac{T^2}{4\pi^2} S_a(T,\xi) \qquad 3.28$$

$S_a(T,\xi)$: inelastic acceleration

$S_d(T,\xi)$: inelastic displacement

μ: ductility factor

Rμ: reduction factor given by:

$$R_\mu = \begin{cases} (\mu - 1)\frac{T}{T_2} + 1 & : T < T_2 \\ \mu & : T \geq T_2 \end{cases} \qquad 3.29$$

With

T: vibration period of a linear single-degree-of-freedom system

T_2: upper limit of periods corresponding to the constant spectral acceleration plateau

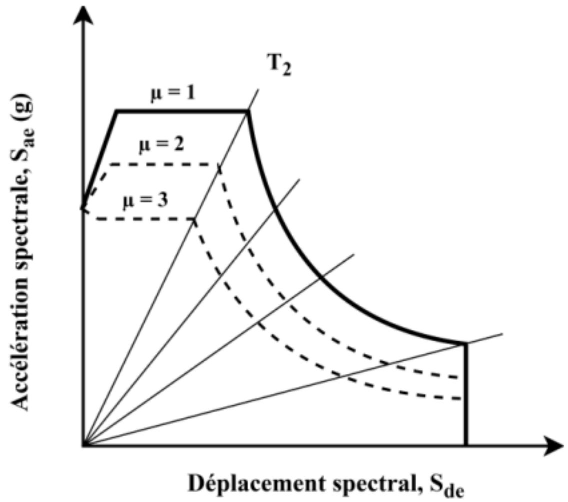


Fig. 3.9 The reduced seismic demand curve

6. **Determination of the target displacement**

6.1. **For the SDOF equivalent system**

From a theoretical perspective, the target displacement (D_t*) is identified by locating the point where the inelastic demand spectrum, defined by a ductility (μ) value, intersects with the bilinear capacity spectrum at a corresponding capacity ductility (μ). Essentially, this means that the design point is determined where the demand and capacity ductility are equal. The target displacement of the structure with period T ∗ and unlimited elastic behavior is given by

$$d_{et}^{*} = S_e(T^{*})\left[\frac{T^{*}}{2\pi}\right]^{2} \qquad 3.30$$

For the determination of the target displacement dt∗ of structures in the short period domain and of structures in the medium and long period domains, different expressions should be used, as shown below. The boundary period between the short period and medium period domains is T_2

- Short period domain

1- If $\frac{F_y^*}{m^*} \geq S_e(T^*)$ the response is elastic, so: $d_l^* = d_{et}^*$

2- If $\frac{F_y^*}{m^*} < S_e(T^*)$ the response is nonlinear, and: $d_l^* = \frac{d_{et}^*}{R_\mu} \cdot \left[1 + (R_\mu - 1)\frac{T_2}{T^*} \right] \geq d_{et}^*$

Where Rμ is the ratio between the acceleration $Se(T^*)$ in a structure with unlimited elastic behavior and the acceleration F_y^* / m^* in a structure with limited resistance, i.e.,

$$R_\mu = \frac{S_e(T^*) \cdot \ m^*}{F_y^*} \qquad 3.31$$

- Medium and long period domain:

$$d_l^* = d_{et}^* \qquad 3.32$$

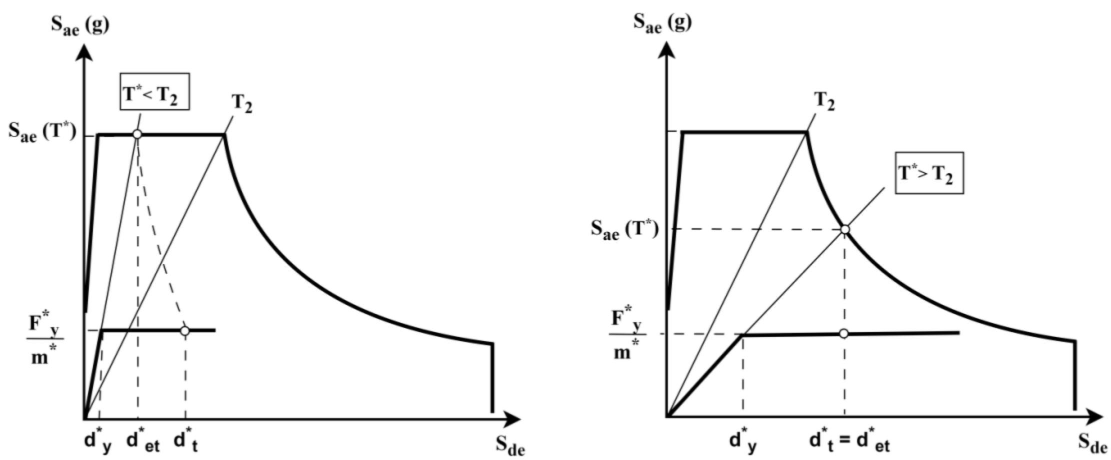


Fig. 3.10 Determination of the target displacement

6.2. For the MDOF system

The target move corresponds to the control node.

$$d_t = \Gamma . d_t^* \qquad 3.33$$

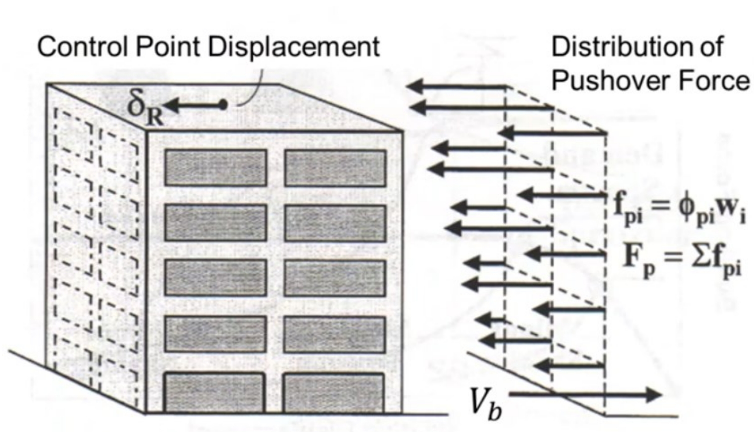


Fig. 3.11 The target displacement for MDOF system

A schematic representation of nonlinear static procedure – pushover analysis is shown in Fig. 3.12.

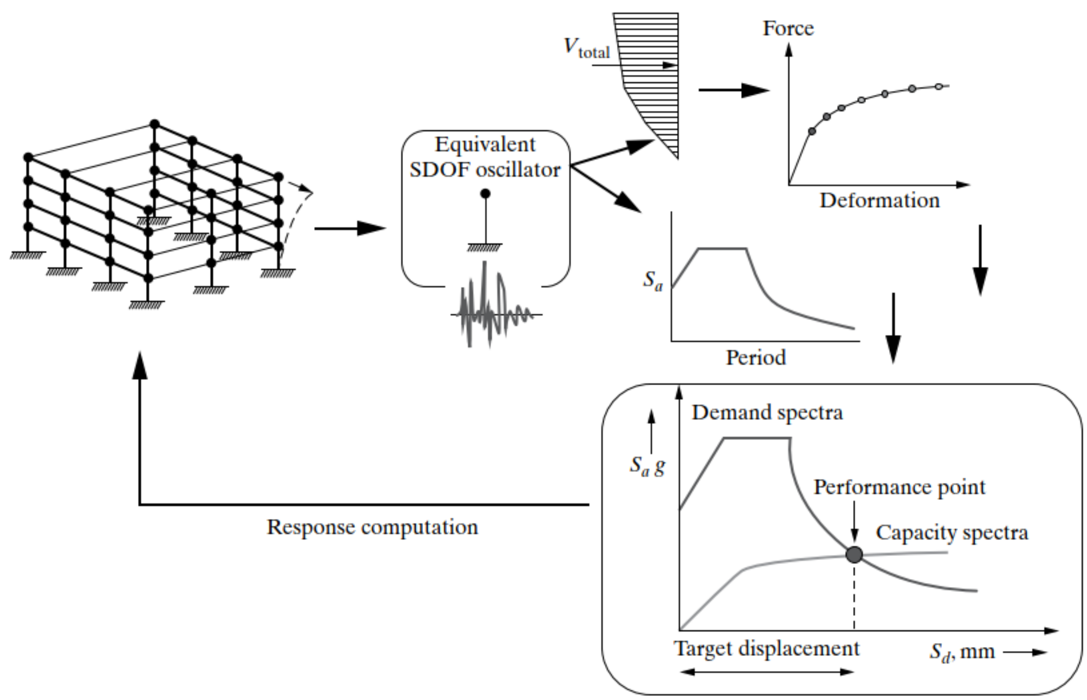


Fig. 3.12 Schematic representation of nonlinear static procedure – pushover analysis.

References

Anil K. Chopra. « Dynamics of structures - Theory and Applications to Earthquake Engineering », Fifth Edition in SI Units, University of California at Berkeley

Alain PECKER. « Dynamique des structures et des ouvrages », Edition 2006

Ashok K. Jain. « Dynamics of structures with matlab applications », ISBN 978-93-325-5855-7

Ahmed Elsayed Ahmed Ali. « Structural Dynamics and Earthquake Engineering by using Autodesk Robot & Matlab »

Akhtar Naeem Khan, Mohammad Javed. « CE-412: Introduction to Structural Dynamics and Earthquake Engineering », University of Engineering and Technology, Peshawar, Pakistan

A. Dazio. « Course Fundamentals of Structural Dynamics », An-Najah National University

Abdelhafid OUNIS. « Cours de Dynamique des structures », Université de Biskra

DaniEl J. inman. « Engineering Vibration », Fourth Edition, University of Michigan

Fawad A. Najam, Pennung Warnitchai. « CE 809 - Structural Dynamic », Department of Structural Engineering. H-12 Islamabad, Pakistan

John T. Katsikadelis. « Dynamic Analysis of Structures », Academic Press is an imprint of Elsevier, ISBN 978-0-12-818643-5

Luis E. García and Mete A. Sozen, Purdue University CE571 - Earthquake Engineering, Multiple degrees of freedom structural dynamics. Spring 2003

Mario Paz · Young Hoon Kim. « Structural Dynamics - Theory and Computation », Sixth Edition. ISBN 978-3-319-94743-3 (eBook). https://doi.org/10.1007/978-3-319-94743-3

Pelin Gundes Bakir. « Vibration of single degree of freedom systems ».

Ray W. Clough, Joseph Penzien. « Dynamics of structures », Computers & Structures, Inc. 1995 University Ave. Berkeley, CA 94704, USA

Rabee AlSafadi. « Dynamique of structures”, Arab International University (AIU)

Singiresu S. Rao. « Mechanical Vibrations », Fifth Edition, University of Miami

Thandavamoorthy Thumati. “course Dynamics of structures”, Adhiparasakthi Engineering College

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