الجمهوريـة الجزائـريـة الديمقراطيـة الشعبيـة
People's Democratic Republic of Algeria
وزارة التعليـم العـالـي والبحـث العلمـي
Ministry of Higher Education and Scientific Research

Faculty of Science كلية العلوم – قسم الرياضيات
Department of Mathematics

Badji Mokhtar-Annaba University
جـامعة بـاجـي مختـار–عنـابـة -

THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat en Sciences

Option : **Mathématiques Appliquées**

ÉTUDE QUALITATIVE SUR QUELQUES SYSTÈMES COUPLÉS DE MULTI-PHYSIQUES POUR DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES DE TYPE HYPERBOLIQUE

Par :
Houma Nabil

Sous la direction de

DIRECTEUR DE THÈSE :	Zennir Khaled	M.C.A.	Qassim Univ. KSA
CO-DIRECTEUR DE THÈSE :	Djebabla Abdelhak	Prof.	Univ. Annaba

Devant le jury

PRESIDENT :	Chorfi Lahcène	Prof.	Univ. Annaba
EXAMINATEUR :	Kouche Mahiéddine	Prof.	Univ. Annaba
EXAMINATEUR :	Benrabah Abderafik	M.C.A.	Univ. Guelma
EXAMINATEUR :	Zitouni Salah	M.C.A.	Univ. Souk Ahras

Année : 2023

Acknowledgements

First of all, I want to thank Allah for giving me strength, courage and above all knowledge.
I would like to express my deep gratitude to Dr Khaled Zennir, my thesis supervisor, for his patience, motivation , enthusiastic encouragements and guidance.
My warmest thanks go also to Pr Abdelhak DJEBABLA, for his guidance, encouragements and continuous advice and support throughout this research thesis. I am very grateful to him.
My thanks go also to Pr Chorfi Lahcène for accepting to chair the defence commite and also to the following members of the defence commite Pr Kouche Mahiéddine, Dr Benrabah Abderafik and Dr Zitouni Salah for having accepted to be part of my jury. I thank them for their interest in my work.
Last but no least, i want to thank my family and especially my parents, my friends for their encouragements and support during my studies.

Abstract

In this thesis, we consider the theoretical study of some problems of hyperbolic type (equations and systems of equations) with viscoelastic term under some assumptions on initial and boundary conditions, conditions on dissipation terms, source terms. We have studied the existence and the asymptotic behavior of the energy of the solutions.

Keywords: Nonlinear wave equations, Delay term, Infinite memory, Multiplier method, polynomial decay, exponential stability.

Mathematics Subject Classification: 2010, 35L05, 35L20, 35L70, 35L71, 37B25, 35B35, 93D15, 93D20, 93C20.

Résumé

Dans cette thèse, on considère l'étude théorique de quelques problèmes de type hyperbolique (équations et systèmes d'équations) à terme viscoélastique sous quelques hypothèses sur les conditions initiale et au bord, des conditions sur les termes de dissipation et des termes sources. Nous avons étudié l'existence et le comportement asymptotique de l'énergie associée à la solution.

Mots-clés: Equations des ondes non lineaires, Terme de retard, Mémoire infinie, Méthode des Multiplicateurs, décroissance polynomiale, stabilité exponentielle.

Mathematics Subject Classification: 2010, 35L05, 35L20, 35L70, 35L71, 37B25, 35B35, 93D15, 93D20, 93C20.

ملخص
في هذه الأطروحة، نهتم بالدراسة النظرية لبعض مسائل النوع الزائدي (المعادلات وأنظمة المعادلات) مع حدود المرونة اللزجة تحت بعض افتراضات بشأن الشروط الأولية والحدودية ، شروط التبديد ، وشروط المصدر. لقد درسنا مسألة الوجود وسلوك تقارب طاقة الحلول.

الكلمات المفتاحية: معادلات الموجة غير الخطية ، مصطلح التأخير ، الذاكرة اللانهائية ، طريقة Multiplier ، اضمحلال كثير الحدود ، الاستقرار الأسي.
تصنيف مادة الرياضيات: 2010 ، 35L05 ، 35L20 ، 35L70 ، 35L71 ، 37B25 ، 35B35 ، 93D15 ، 93D20 ، 93C20 .

Contents

Introduction

The problem of stabilization and control of PDEs plays an essential role in current fundamental sciences. Evolution equations, i.e. partial differential equations in which time t is one of the independent variables, appear not only in many fields of mathematics, but also in other branches of science such as physics, mechanics and materials. For example, Navier-Stokes and Euler equations of acid mechanics, nonlinear reaction-diffusion equations of heat transfer and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrodinger equations of quantum mechanics and the Cahn-Hilliard equations of materials science.

In recent years, wave equation control with delay effects has become an active area of research. It is well known that p-Laplace equations are degenerate equations in divergence form. It has been the subject of many studies in recent years and their results are now quite developed, especially with delays. In the classical theory of evolution equations, several main parts of mathematics are fruitfully connected, it is very remarkable that the p-Laplace wave equation occupies a similar position with respect to nonlinear problems.

The primary objective of this thesis is to delve into the qualitative aspects of a range of coupled multi-physics systems governed by hyperbolic partial differential equations. The term "qualitative study" emphasizes the intention to explore and comprehend the nature, characteristics, and behaviors of these systems without relying solely on precise numerical measurements or quantitative analysis. By taking this approach, we aim to gain a deeper understanding of the intricate dynamics and interactions within these interconnected systems.

The reference to "coupled multi-physics systems" highlights the interconnected nature of the physical processes under investigation. These systems involve various physical phenomena, such as fluid dynamics, heat transfer, electromagnetics, and potentially other fields, which interact or couple with one another. By examining these coupled systems, we aim to unravel the intricate interplay between different physical phenomena, contributing to a comprehensive understanding of their behavior and functioning.

Central to this study are hyperbolic partial differential equations. These mathematical equations incorporate partial derivatives and exhibit a hyperbolic character. Hyperbolic equations typically govern wave-like behavior or propagation phenomena, such as wave equations, advection equations, or the transport of signals. Therefore, the inclusion of hyperbolic partial differential equations in the title signifies their fundamental role in characterizing and describing the dynamics of the coupled multi-physics systems under investigation.

The thesis has 4 chapters. The first chapter summarizes some concepts, definitions and results that mainly concern the undergraduate program and are assumed to be fundamentally known or to have specific bases in rather isolated areas and have a rather auxiliary character with respect to the purpose of the present study. In the next three chapters, we develop our main results for nonlinear evolution problems of the hyperbolic type.

In chapter two, we investigate an initial boundary value problem with weak and strong damping and a logarithmic type source. We analyze the stability of the unique solution under certain conditions on the dissipations using the Lyapunov method.

In the third chapter, we consider a plate equation with time-varying delay and viscoelasticity in $\mathbb{R}^n$. Under appropriate assumptions on the relaxation function and the source term, we establish energy stability using an appropriate Lyapunov function.

In chapter four, our main contributions are to demonstrate the existence and lack of stability for a Love equation with infinite memory. This shows that, for these types of materials, the dissipation produced by the memories at the end of the time is not strong enough to ensure solution stability under usual or non-usual conditions regarding the growth of the relaxation functions.

The results of this thesis have been the subject of the following publication:

Nabil Houma, Khaled Zennir, Abderrahmane Beniani, Abdelhak Djebabela, *New Stability Estimates of Solutions to Strong Damped Wave Equation with Logarithmic External Forces*, Discontinuity, Nonlinearity, and Complexity 10(4) (2021) 625-634.

Chapter 1

Preliminary

1- Continuous function spaces

2- L^p Spaces

3- Sobolev Spaces

4- Banach-Alaoglu theorem

5- P-Laplace operator

1.1 Continuous function spaces

We start this work by giving some useful notations and conventions.

Let $x = (x_1, x_2, ..., x_n)$ denote the generic point of an open set Ω of $\mathbb{R}^n$. Let u be a function defined from Ω to $\mathbb{R}^n$, we designate by $D_i u(x) = \partial_i u(x) = \dfrac{\partial u(x)}{\partial x_i}$ the partial derivative of u with respect to $x_i\,(1 \le i \le n)$. Let's also define the gradient and the p-Laplacian from u, respectively as following

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n})^{\,T} \quad \text{and} \quad |\nabla u|^2 = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2$$

$$\Delta_p u(x) = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right)(x).$$

Note by $C(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$, $C(\Omega, \mathbb{R}^m)$ the space of continuous functions from Ω to $\mathbb{R}^m$ and $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$, it is equipped with the norm $\|.\|_\infty$;

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$$

For $k \ge 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times derivable and whose derivation of order k is continuous on Ω. $C_c^k(\Omega)$ is the set of functions of $C^k(\Omega)$ whose support is compact and contained in Ω.

We also define $C^k\left(\overline{\Omega}\right)$ as the set of restrictions to $\overline{\Omega}$ of elements from $C^k(\mathbb{R}^n)$ or as being the set of functions of $C^k(\Omega)$, such that for all $0 \le j \le k$, and for all $x_0 \in \partial\Omega$, the limit $\lim_{x \to x_0} D_j u(x)$ exists and depends only on x_0.

$C_0^\infty(\Omega)$ or $\mathfrak{D}(\Omega)$, is the space of the infinitely differentiable functions, with compact supports called test function space.

The Hölder space $C^{k,\alpha}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ and $k \ge 0$ an integer, $0 < \alpha \le 1$, consists of those real or complex-valued k-times continuously differentiable functions f on Ω

verifying

$$|f^{(\beta)}(x) - f^{(\beta)}(y)| \leq C\|x-y\|^{\alpha}$$

where $C > 0$, $|\beta| \leq k$.

1.2 L^p Spaces

Let Ω be an open set of $\mathbb{R}^n$, equipped with the Lebesgue measure dx. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values in $\mathbb{R}$, it is provided with the norm

$$\|u\|_{L^1} = \int_{\Omega} |u(x)|\, dx.$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } \int_{\Omega} |f(x)|^p\, dx < +\infty \right\}$$

equipped with norm

$$\|u\|_{L^p} = \left(\int_{\Omega} |u(x)|^p\, dx \right)^{\frac{1}{p}}.$$

We also define the space $L^{\infty}(\Omega)$

$$L^{\infty}(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ measurable},\ \exists c > 0, \text{ so that } |f(x)| \leq c \quad \text{a.e. on } \Omega \},$$

it will be equipped with the essential-sup norm

$$\|u\|_{L^{\infty}} = ess \sup_{x \in \Omega} |u(x)| = \inf \{ c\ ;\ \ |u(x)| \leq c \quad \text{a.e. on } \Omega \}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $\mathbf{1}_K f \in L^p(\Omega)$ for any compact $K \subset \Omega$.

Theorem 1. *(Dominated convergence theorem) [1]*

Let $\{f_n\}_{n\geq 1}$ be a series of functions of $L^1(\Omega)$ converging almost everywhere to a measurable *function f. It is assumed that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we get*

$$|f_n| \leq g \quad a.e \text{ on } \Omega.$$

Then $f \in L^1(\Omega)$ *and*

$$\lim_{n\to+\infty} \|f_n - f\|_{L^1} = 0, \text{ and } \int_\Omega f(x)\,dx = \lim_{n\to+\infty} \int_\Omega f_n(x)\,dx.$$

1.3 Sobolev spaces

1.3.1 Weak derivative

Definition 1. *Let Ω be an open set of $\mathbb{R}$, and $1 \leq i \leq n$. A function $u \in L^1_{loc}(\Omega)$ has an i^{th} weak derivative in $L^1_{loc}(\Omega)$ if there exists $f_i \in L^1_{loc}(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u(x)\,\partial_i\varphi(x)\,dx = -\int_\Omega f_i(x)\,\varphi(x)\,dx.$$

This leads to say that the i^{th} derivative within the meaning of distributions of u belongs to $L^1_{loc}(\Omega)$, we write

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i$$

1.3.2 $W^{1,p}(\Omega)$ spaces

Let Ω be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \text{such that } \partial_i u \in L^p(\Omega), 1 \leq i \leq n\}$$

where $\partial_i u$ is the i^{th} weak derivative of $u \in L^1_{loc}(\Omega)$.

For $1 \leq p < +\infty$ we define the space $W_0^{1,p}(\Omega)$ as being the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$, and we write

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}.$$

Theorem 2. *(Poincaré's inequality) [1]*

Assume Ω is a bounded open subset of $\mathbb{R}^n$, $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$. Then we have the estimate

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

for each $q \in [1, p^*]$, where $p^* = \dfrac{np}{n-p}$ and the constant C depends only on q, p, n and Ω.

Remark 1. *In view of this Poincaré's inequality, if Ω is bounded, then on $W_0^{1,p}(\Omega)$ the norm $\|u\|_{W^{1,p}(\Omega)}$ is equivalent to $\|\nabla u\|_{L^p(\Omega)}$.*

Young's inequality

Let a and b be strictly positive realities p and q such as, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$, we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

A simple case of Young's inequality is the inequality for $p = q = 2$:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

which also gives Young's inequality for all $\delta > 0$:

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2.$$

Theorem 3. *(Rellich-Kondrachov compactness theorem) [1]*

Assume Ω is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary, and $1 \leq p < n$. Then

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \leq q < p^*$.

1.3.3 $W^{m,p}(\Omega)$ Spaces

Let Ω be an open set of $\mathbb{R}^n, m \geq 2$ integer number and p real number such that $1 \leq p \leq +\infty$, we define the space $W^{m,p}(\Omega)$ as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \text{ such that } \partial^\alpha u \in L^p(\Omega), \ \forall \alpha, |\alpha| \leq m\}$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ the length of α and $\partial^\alpha u = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$ in the sense of definition (1).

The space $W^{m,p}(\Omega)$ is equiped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial^\alpha u\|_{L^p}.$$

For $p = 2$, the space $W^{m,2}(\Omega)$ is noted $H^m(\Omega)$.

1.4 Banach-Alaoglu theorem

The next result from functional analysis will be useful later in the following chapters.

Theorem 4. *[2]*
Let E be a normed space, and E^* its dual which is also a normed space (with the operator norm). The closed unit ball of E^* is compact with respect to the weak-* topology.

Corollary 1. *In case where E is a Hilbert space, every bounded and closed set is weakly relatively compact, i.e. every bounded sequence has a weakly convergent subsequence (since Hilbert spaces are reflexive).*

1.5 P-Laplace operator

The study of eigenvalue problems is an important object of research in functional analysis. It is known that in the framework of the Ljusternik-Schnirelman theory one can find estimates for the number of critical points of functionals from which some results on eigensolutions for nonlinear differential equations are deduced.

A nonlinear operator equation can be formulated of the form

$$Au = \lambda Bu.$$

In the case of p-Laplace operator, the following nonlinear eigenvalue problem has been extensively investigated in the past thirty years

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5.1}$$

See for exemple [4], [6], [5] and [7], from which we are going to mention the following definition and some famous results.

Definition 2. *We say that $u \in W_0^{1,p}(\Omega), u \neq 0$, is an eigenfunction of the operator $-\triangle_p u$ if:*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx = \lambda \int_\Omega |u|^{p-2} u.\varphi \; dx \tag{1.5.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. The corresponding real number λ is called eigenvalue.

Let λ_1 defined by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{1.5.3}$$

equivalent to

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p \, dx; \; \int_\Omega |u|^p \, dx = 1, u \in W_0^{1,p}(\Omega) \right\}.$$

λ_1 is the first eigenvalue of the p-Laplacian operator with null Dirichlet conditions at the edge.

Lemma 1. *λ_1 is isolated, i.e : there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$, there is no other eigenvalues of (1.5.2).*

Lemma 2. *The first eigenvalue λ_1 is simple, i.e : if u, v are two eigenfunctions associated with λ_1, then, there exists k such that $u = kv$.*

Lemma 3. *Let u be an eigenfunction associated with the eigenvalue λ_1, then u does not change sign on Ω. Further if $u \in C^{1,\alpha}(\Omega)$, then $u(x) \neq 0$, $\forall x \in \overline{\Omega}$.*

Definition 3. *[3]*

Let ω be a part of a Banach space X and $F : \omega \to \mathbb{R}$. If $u \in \omega$, we say that F is Gâteaux differentiable (or G-differentiable) at u, if there exists $l \in X'$ such that in each direction $z \in X$ where $F(u+tz)$ exists for $t > 0$ small enough, the directional derivative $F'_z(u)$ exists and we have

$$\lim_{t \to 0^+} \frac{F(u+tz) - F(u)}{t} = \langle l, z \rangle .$$

We write $F'(u) = l$.

Theorem 5. *[3]*

Let $\Omega \subset \mathbb{R}^n$ an open set, $n \geq 3$. For $p \in (1, +\infty)$, we define a functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega |\nabla u|^p \, dx$$

then J is differentiable in $W_0^{1,p}(\Omega)$ and

$$J'(u)(v) = p \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx, \forall v \in W_0^{1,p}(\Omega) .$$

Proof. We consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, defined by $\varphi(x) = |x|^p$, it is a function of class C^1, and $\nabla \varphi = p|x|^{p-2} x$.

Then for all $x, y \in \mathbb{R}^n$,

$$\lim_{t \to 0} \frac{\varphi(x+ty) - \varphi(x)}{t} = p|x|^{p-2} x.y$$

as a consequence

$$\lim_{t \to 0} \frac{|\nabla u(x) + t\nabla v(x)|^p - |\nabla u(x)|^p}{t} = p|\nabla u(x)|^{p-2} \nabla u(x) . \nabla v(x) .$$

By Mean value theorem, for almost every $x \in \Omega$ and for $t > 0$, there exists a function θ that takes its values in $]0,1[$ and we can write

$$\begin{aligned} & \left|\nabla u(x) + t\nabla v(x)\right|^p - \left|\nabla u(x)\right|^p - tp\left|\nabla u(x)\right|^{p-2}\nabla u(x).\nabla v(x) \\ = \; & tp\left|\nabla u(x) + \theta(t,x)t\nabla v(x)\right|^{p-2}\left(\nabla u(x) + \theta(t,x)t\nabla v(x)\right).\nabla v(x) \\ & -tp\left|\nabla u(x)\right|^{p-2}\nabla u(x).\nabla v(x). \end{aligned} \tag{1.5.4}$$

Dividing by t, we get for almost every x

$$\lim_{t\to 0}\frac{\left|\nabla(u+tv)(x)\right|^p - \left|\nabla u(x)\right|^p - tp\left|\nabla u(x)\right|^{p-2}\nabla u(x).\nabla v(x)}{t} = 0.$$

On the other hand, one can see that the second member of the equality (1.5.4) devided by t is bounded by

$$h(x) = 2\left|\nabla v(x)\right|\left(\left|\nabla u(x)\right| + \left|\nabla v(x)\right|\right)^{p-1}.$$

Then using the Holder inequality we have

$$|h| \leq C\left\|\nabla v\right\|_p\left(\left\|\nabla u\right\|_p^{p-1} + \left\|\nabla v\right\|_p^{p-1}\right).$$

One can apply the Dominated convergence theorem and conclude

$$J'(u)(v) = p\int_\Omega \left|\nabla u\right|^{p-2}\nabla u.\nabla v dx, \forall v \in W_0^{1,p}(\Omega),$$

then J is Gâteaux differentiable. □

Lemma 4. (*Comparison lemma*) *[8]*
Let $u, v \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega \left|\nabla u\right|^{p-2}\nabla u.\nabla\varphi dx \leq \int_\Omega \left|\nabla v\right|^{p-2}\nabla v.\nabla\varphi dx \tag{1.5.5}$$

for all $\varphi \in W_0^{1,p}(\Omega), \varphi \geq 0$, then $u \leq v$ a.e in Ω.

Proof. This proof is based on the arguments presented in [9] and [10]. We start by defining the function $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by the formula

$$J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx. \tag{1.5.6}$$

It is clear that the functional J is Gâteaux differentiable and continuous and its derivative at $u \in W_0^{1,p}(\Omega)$ is the function $J'(u) \in W_0^{-1,p}(\Omega)$, given by

$$J'(u)(\varphi) = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx, \ \forall \varphi \in W_0^{1,p}(\Omega). \tag{1.5.7}$$

$J'(u)$ is continuous and bounded. We will show that $J'(u)$ is strictly monotonic in $W_0^{1,p}(\Omega)$. Indeed, for all $u, v \in W_0^{1,p}(\Omega), u \neq v$ without loss of generality, we can suppose that

$$\int_\Omega |\nabla u|^p \, dx \geq \int_\Omega |\nabla v|^p \, dx.$$

Using the Cauchy inequality we have

$$\nabla u.\nabla v \leq |\nabla u| |\nabla v| \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right). \tag{1.5.8}$$

From formula (1.5.8) we deduce

$$\int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla v dx \geq \frac{1}{2}\int_\Omega |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx \tag{1.5.9}$$

$$\int_\Omega |\nabla v|^p \, dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla u dx \geq \frac{1}{2}\int_\Omega |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right) dx. \tag{1.5.10}$$

If $|\nabla u| \geq |\nabla v|$, by using (1.5.6)-(1.5.8) we get

$$
\begin{aligned}
I_1(u) &= J'(u)(u) - J'(u)(v) - J'(v)(u) + J'(v)(v) \\
&= \left(\int_\Omega |\nabla u|^p\, dx - \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx\right) - \left(\int_\Omega |\nabla v|^{p-2} \nabla v . \nabla u dx - \int_\Omega |\nabla v|^p\, dx\right) \\
&\geq \int_\Omega \tfrac{1}{2} |\nabla u|^{p-2} \left(|\nabla u|^2 - |\nabla v|^2\right) dx \\
&-\tfrac{1}{2} \int_\Omega |\nabla v|^{p-2} \left(|\nabla u|^2 - |\nabla v|^2\right) dx \\
&\geq \tfrac{1}{2} \int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right) \left(|\nabla u|^2 - |\nabla v|^2\right) dx \\
&\geq \tfrac{1}{2} \int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right) \left(|\nabla u|^2 - |\nabla v|^2\right) dx.
\end{aligned} \tag{1.5.11}
$$

If $|\nabla v| \geq |\nabla u|$, by changing the role of u and v in (1.5.6)-(1.5.8) we have

$$
\begin{aligned}
I_2(v) &= J'(v)(v) - J'(v)(u) - J'(u)(v) + J'(u)(u) \\
&= \left(\int_\Omega |\nabla v|^p\, dx - \int_\Omega |\nabla v|^{p-2} \nabla v . \nabla u dx\right) \\
&- \left(\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx - \int_\Omega |\nabla u|^p\, dx\right) \\
&\geq \tfrac{1}{2} \int_\Omega |\nabla v|^{p-2} \left(|\nabla v|^2 - |\nabla u|^2\right) dx \\
&-\tfrac{1}{2} \int_\Omega |\nabla u|^{p-2} \left(|\nabla v|^2 - |\nabla u|^2\right) dx
\end{aligned}
$$

We obtain,

$$\begin{aligned} I_2(v) &= \frac{1}{2}\int_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2}\right)\left(|\nabla v|^2 - |\nabla u|^2\right)dx \\ &\geq \frac{1}{2}\int_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2}\right)\left(|\nabla v|^2 - |\nabla u|^2\right)dx. \end{aligned} \tag{1.5.12}$$

From (1.5.9)-(1.5.10), we have

$$(J'(u) - J'(v))(u-v) = I_1 = I_2 \geq 0, \forall u, v \in W_0^{1,p}(\Omega).$$

In addition, if $u \neq v$ and $(J'(u) - J'(v))(u-v) = 0$, then we have

$$\int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx = 0.$$

If $|\nabla u| = |\nabla v|$ in Ω, we deduce that

$$\begin{aligned} (J'(u) - J'(v))(u-v) &= J'(u)(u-v) - J'(v)(u-v) \\ &= \int_\Omega |\nabla u|^{p-2}|\nabla u - \nabla v|^2 dx = 0, \end{aligned} \tag{1.5.13}$$

i.e. $u - v$ is a constant. Given $u = v = 0$ on $\partial\Omega$ we are getting $u = v$, which is contrary with $u \neq v$. Then $(J'(u) - J'(v))(u-v) > 0$ and $J'(u)$ is strictly monotonic in $W_0^{-1,p}(\Omega)$. Let u, v two functions such that (1.5.7) is satisfied, let's take $\varphi = (u-v)^+$ the positive part of $u - v$ as a test function in (1.5.7), we get

$$(J'(u) - J'(v))(\varphi) = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi dx - \int_\Omega |\nabla v|^{p-2}\nabla v.\nabla\varphi dx \leq 0. \tag{1.5.14}$$

Relationships (1.5.11) and (1.5.12) imply that $u \leq v$. □

Chapter 2

New stability estimates of solutions to strong damped wave equation with logarithmic external forces

2.1 Introduction

In this chapter, we consider an initial boundary value problem with weak and strong damping terms and logarithmic source

$$\begin{cases} \rho(x)v_{tt} + a\rho(x)v_t = \Delta v - \omega\Delta v_t + \int_0^t \varpi(t-p)\Delta v(p)\,dp + k\rho(x)v\ln|v|, & x \in \mathbb{R}^n,\ t>0 \\ v(x,0) = v_0(x), & x \in \mathbb{R}^n \\ v_t(x,0) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1.1}$$

where $a \in \mathbb{R}$, $n \geq 3$, and k is a small positive real number. The density function $\rho(x) > 0$, for all $x \in \mathbb{R}^n$

$$\omega \geq 0, \quad a > -\omega\lambda_1, \tag{2.1.2}$$

λ_1 being the first eigenvalue of the operator $-\phi(x)\Delta$, where $(\phi(x))^{-1} = 1/\phi(x) \equiv \rho(x)$, under homogeneous Drichlet boundary conditions.

A related initial boundary value problem was considered by Han in [13]

$$\begin{cases} u_{tt} + u_t - \Delta u + u + |u|^2 u = u\ln|u|^2, & x \in \Omega,\ t \in [0,T) \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in [0,T). \end{cases} \tag{2.1.3}$$

and the global existence of weak solutions was proved, for all $(u_0, u_1) \in H_0^1 \times L^2$ in $\mathbb{R}^3$. The weak and strong damping terms in logarithmic wave equation

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega\Delta u_t = u\ln|u|, & x \in \Omega,\ t \in (0,\infty) \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,\infty). \end{cases} \tag{2.1.4}$$

was introduced by Lian and Xu [16]. The global existence, asymptotic behavior and blowup at three different initial energy levels was proved, i.e., with subcritical energy $E(0) < d$, critical initial energy $E(0) = d$ and the arbitrary high initial energy $E(0) > 0 (\omega = 0)$. In [15], Al-

Gharabli established an explicit and general energy decay results of the problem

$$\begin{cases} u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u ds = ku \ln |u|, & x \in \Omega,\ t \in (0,\infty) \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), & x \in \Omega \\ u(x,t) = \frac{\partial u}{\partial v} = 0, & x \in \partial\Omega,\ t \in (0,\infty). \end{cases} \tag{2.1.5}$$

When the density $\phi(x) \neq 1$, Karachalios and Stavrakakis [23] considered the following semilinear hyperbolic initial value problem

$$u_{tt} + \phi(x)\Delta u + \delta u_t + \lambda f(u) = \eta(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

The authors proved local existence of solutions and established the existence of a global attractor in energy space $\mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$ where $(\phi(x))^{-1} := g(x)$. Miyasita and Zennir [17] proved the global existence of the following viscoelastic wave equation

$$\begin{cases} u_{tt} + au_t - \phi(x)\left(\Delta u + \omega\Delta u_t - \int_0^t g(t-s)\Delta u(s)\, ds\right) = u\,|u|^{p-1}, & x \in \mathbb{R}^n \quad t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \\ u_t(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.1.6}$$

The novelty of our work lies primarily in the use of a new condition between the weights of weak and strong damping in (2.1.2), which is useful in the calculation, where we outlined the effects of damping terms. The constant λ_1 being the first eigenvalue of the operator $-\phi(x)\Delta$. We also proposed a logarithmic nonlinearities in sources and used a classical arguments to estimate them. This nonlinearities make the problem very interesting in the application point of view. In order to compensate the lack of classical Poincaré's inequality in $\mathbb{R}^n$, we used the weighted function to use the generalized poincaré's one. The main contribution located in Theorem 9, where we obtained a decay estimates with positive initial energy under a general assumption on the kernal. The outline of this chapter is as follows. In Section 2, we give some preliminaries and our main results. In Section 3, we will prove the general decay of energy to the problem.

2.2 Preliminaries and main results

We give some assumptions used in this chapter. With respect to the relaxation function ϖ, we assume

(H1) $\varpi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying for any $t \geq 0$,

$$\varpi(0) > 0, \quad \int_0^\infty \varpi(p)dp = l_0 < \infty, \; 1 - \int_0^t \varpi(p)dp = l > 0. \tag{2.2.1}$$

(H2) There exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\xi(t) > 0, \quad \varpi'(t) \leq -\xi(t)\varpi(t) \quad \text{for } t \geq 0. \tag{2.2.2}$$

(H3) The function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.

Definition 4. *[23] We define the function spaces of our problem and its norm as follows:*

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^n)\}.$$

and the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|v\|_{\mathcal{H}} = (v,v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v,w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_\rho(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\rho} = (v,v)_{L^2_\rho}^{1/2}$ for

$$(v,w)_{L^2_\rho} = \int_{\mathbb{R}^n} \rho v w \, dx,$$

respectively.

For general $q \in [1, +\infty)$, $L^q_\rho(\mathbb{R}^n)$ is the weighted L^q space under a weighted norm

$$\|v\|_{L^q_\rho} = \left(\int_{\mathbb{R}^n} \rho \, |v|^q \, dx \right)^{\frac{1}{q}}.$$

To distinguish the usual L^q space from the weighted one, we denote the standard L^q norm by

$$\|v\|_q = \left(\int_{\mathbb{R}^n} |v|^q \, dx. \right)^{\frac{1}{q}}.$$

We denote an eigenpair $\{(\lambda_j, w_j)\}_{j\in\mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ of

$$-\phi(x)\Delta w_j = \lambda_j w_j \quad x \in \mathbb{R}^n,$$

for any $j \in \mathbb{N}$. Then according to [23],

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots \uparrow +\infty,$$

holds and $\{w_j\}$ is a complete orthonormal system in $\mathcal{H}$.

First, we introduce Sobolev embedding and generalized Poincaré inequalities.

Lemma 5. *Let ρ satisfy (H3). Then there are positive constants $C_S > 0$ and $C_P > 0$ which depends only on n and ρ such that*

$$\|v\|_{\frac{2n}{n-2}} \le C_S \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_\rho} \le C_P \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 6 (Lemma 2.2 in [20])**.** *Let ρ satisfy (H3). Then we have*

$$\|v\|_{L^q_\rho} \le C_q \|v\|_{\mathcal{H}} \quad \text{and} \quad C_q = C_S \|\rho\|_s^{\frac{1}{q}},$$

for $v \in \mathcal{H}$, where $s = 2n/(2n - qn + 2q)$ for $1 \le q \le 2n/(n-2)$.

The energy functional associated to problem (2.1.1) by

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\|v_t(t)\|^2_{L^2_\rho} + \frac{1}{2}\left(1 - \int_0^t \varpi(p)dp\right)\|\nabla v(t)\|^2 \\ &+\frac{1}{2}(\varpi \circ \nabla v)(t) - \frac{k}{2}\int_{\mathbb{R}^n} \rho(x)v^2 \ln|v|dx + \frac{k}{4}\|v\|^2_{L^2_\rho}, \end{aligned} \tag{2.2.3}$$

where

$$(\varpi \circ v) = \int_0^t \varpi(t-p)\|v(t) - v(p)\|_2^2 dp.$$

Direct differentiation of (2.2.3), using (2.1.1), we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\left(a\|\nabla v_t\|^2_{L^2_\rho} + \omega\|v_t\|^2_{\mathcal{H}} + \frac{1}{2}\varpi(t)\|v\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi' \circ \nabla v)\right) \leq 0. \tag{2.2.4}$$

Lemma 7. *(Logarithmic Sobolev inequality) [14]*

Lets u be any function in $H^1_0(\Omega)$ and $a > 0$ be any number. Then

$$\int_\Omega v^2 \ln|v|dx \leq \frac{1}{2}\|v\|_2^2 \ln\|v\|_2^2 + \frac{a^2}{2\pi}\|\nabla v\|_2^2 - (1 + \ln a)\|v\|_2^2. \tag{2.2.5}$$

Lemma 8. *(Logarithmic Gronwall inequality) [11]*

Let $c > 0$, $\gamma \in L^1(0,T;\mathbb{R}^+)$, and assume that the function $\omega : [0,T] \to [1,\infty)$ satisfies

$$\omega(t) \leq c\left(1 + \int_0^t \gamma(p)\omega(p)\ln\omega(p)dp\right), \quad 0 \leq t \leq T, \tag{2.2.6}$$

then

$$\omega(t) \leq c\exp\left(c\int_0^t \gamma(p)dp\right), \quad 0 \leq t \leq T. \tag{2.2.7}$$

We define the following functionals

$$J(v) = \frac{1}{2}\left(1 - \int_0^t \varpi(p)dp\right)\|\nabla v(t)\|^2 + \frac{1}{2}(\varpi \circ \nabla v)(t) - \frac{k}{2}\int_{\mathbb{R}^n}\rho(x)v^2\ln|v|dx + \frac{k}{4}\|v\|^2_{L^2_\rho} \tag{2.2.8}$$

$$I(v) = \frac{1}{2}\left(1 - \int_0^t \varpi(p)dp\right)\|\nabla v(t)\|^2 + \frac{1}{2}(\varpi \circ \nabla v)(t) - \frac{k}{2}\int_{\mathbb{R}^n}\rho(x)v^2\ln|v|dx. \tag{2.2.9}$$

Then, we introduce

$$W = \{v : v \in \mathcal{H}/I(v) > 0, J(v) < d\} \cup \{0\}. \tag{2.2.10}$$

Lemma 9. *Let $(v_0, v_1) \in \mathcal{H} \times L^2_\rho(\mathbb{R}^n)$ such that $0 < \mathcal{E}(0) < d$ and $I(v_0) > 0$. Then we have*

$$v(t) \in W \quad \textit{and} \quad \|v\|^2 < 4d \textit{ for all } t \in [0, T). \tag{2.2.11}$$

Theorem 6. *[17] Let $(v_0, v_1) \in \mathcal{H} \times L^2_\rho(\mathbb{R}^n)$. Under the assumptions $(H1) - (H3)$ and (2.1.2). Then problem (2.1.1) has a global weak solution u in the space*

$$v \in C([0, +\infty); \mathcal{H}) \cap C^1\left([0, +\infty); L^2_\rho(\mathbb{R}^n)\right).$$

Then the main result in this chapter is the general decay of energy to problem (2.1.1) which is given in the following theorem.

Theorem 7. *Assume the assumptions $(H1) - (H3)$ hold and $0 < \mathcal{E}(0) < d$. Let (v, v_t) be the weak solutions of problem (2.1.1) with the initial data $(v_0, v_1) \in \mathcal{H}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$. Then there exist constant $\beta > 0$ such that the energy $\mathcal{E}(t)$ defined by (2.2.3) satisfies for all $t > 0$,*

$$\mathcal{E}(t) \leq \beta \left(1 + \int_{t_0}^{t} \xi^{\varepsilon_0 + 1}(p)dp\right)^{\frac{-1}{\varepsilon_0}}. \tag{2.2.12}$$

2.3 Asymptotic behavior for $\mathcal{E}(0) < d$

In this section, we shall establish the general decay of energy to problem (2.1.1). We need the following technical lemmas.

Lemma 10. *Under the assumptions in Theorem 7, then the functional $\Phi(t)$ defined by*

$$\Phi(t) = \int_{\mathbb{R}^n} \rho(x)v(t)v_t(t)dx + \frac{\omega}{2}\int_{\mathbb{R}^n} |\nabla v(t)|^2 dx, \tag{2.3.1}$$

satisfies the property: there exist positive constants C_1, C_2 *and* C_3 *such that for any* $t \geq 0$,

$$\begin{aligned} \Phi'(t) \leq & \left(1 - \frac{a}{4\varepsilon}\right) \|v_t(t)\|_{L^2_\rho}^2 - \frac{l}{2}\|\nabla v(t)\|^2 + \frac{1-l}{4\varepsilon}(\varpi \circ \nabla v)(t) \\ & -\omega(1-\varepsilon)\|\nabla v_t(t)\|_{L^2_\rho}^2 + k\int_{\mathbb{R}^n} \rho(x)v^2 \ln|v|dx. \end{aligned} \tag{2.3.2}$$

Proof. We differentiate $\Phi(t)$, using (2.1.1), we can get

$$\begin{aligned} \Phi'(t) = & \|v_t\|_{L^2_\rho}^2 - \|\nabla v\|^2 + \int_{\mathbb{R}^n} \nabla v(t) \cdot \int_0^t \varpi(t-p)\nabla v(p)dpdx \\ & -a\int_{\mathbb{R}^n} \rho(x)v_t v dx + \int_{\mathbb{R}^n} \rho(x)v^2 \ln|v|dx - \omega\|\nabla v_t(t)\|_{L^2_\rho}^2. \end{aligned} \tag{2.3.3}$$

It follows from Young and Poincaré's inequality that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} \nabla v(t) \cdot \int_0^t \varpi(t-p)\nabla v(p)dpdx \tag{2.3.4}$$

$$\begin{aligned} = & \int_{\mathbb{R}^n} \nabla v(t) \cdot \int_0^t \varpi(t-p)(\nabla v(p) - \nabla v(t))dpdx + \int_0^t \varpi(p)dp\|\nabla v(t)\|_2^2 \\ \leq & (1-l)\|\nabla v\|^2 + \varepsilon\|\nabla v\|_2^2 + \frac{1}{4\varepsilon}\int_{\mathbb{R}^n}\left(\int_0^t \varpi(t-p)(\nabla v(p) - \nabla v(t))dp\right)^2 dx \\ \leq & (1-l+\varepsilon)\|\nabla v\|_2^2 + \frac{1-l}{4\varepsilon}(\varpi \circ \nabla v)(t). \end{aligned} \tag{2.3.5}$$

Exploit Young's inequality and Poincaré's inequality to estimate

$$\int_{\mathbb{R}^n} \rho(x)v_t v dx \leq \varepsilon c_* \|\nabla v\|_{L^2_\rho}^2 + \frac{1}{4\varepsilon}\int_{\mathbb{R}^n} \rho(x)v_t^2 dx, \tag{2.3.6}$$

and

$$\int_{\mathbb{R}^n} \rho(x)\nabla v_t \nabla v dx \leq \varepsilon c_* \|\nabla v\|_{L^2_\rho}^2 + \frac{1}{4\varepsilon}\|\nabla v_t\|_{L^2_\rho}^2. \tag{2.3.7}$$

Inserting (2.3.4)-(2.3.7) into (2.3.3), we shall see that for any $\varepsilon > 0$,

$$\begin{aligned}\Phi'(t) \;\leq\; & \left(1-\frac{a}{4\varepsilon}\right)\|v_t(t)\|^2_{L^2_\rho} - (l-\varepsilon-\varepsilon c_*(a-\omega))\,\|\nabla v(t)\|^2 + \frac{1-l}{4\varepsilon}(\varpi\circ\nabla v)(t) \\ & -\omega(1-\varepsilon)\|\nabla v_t(t)\|^2_{L^2_\rho} + k\int_{\mathbb{R}^n}\rho(x)v^2\ln|v|dx. \end{aligned} \tag{2.3.8}$$

Taking $\varepsilon > 0$ small enough in (2.3.8) such that

$$l-\varepsilon-\varepsilon c_*(a-\omega) > \frac{l}{2}.$$

The proof is hence complete.

Lemma 11. *Under the assumptions in Theorem 7, then the functional $\psi(t)$ defined by*

$$\psi(t) = -\int_{\mathbb{R}^n}\rho(x)v(t)\int_0^t \varpi(t-p)(v(t)-v(p))dpdx, \tag{2.3.9}$$

satisfies the property: there exist a positive constant C_4 such that for any $\delta > 0$,

$$\begin{aligned}\psi'(t) \;\leq\; & \delta\Big[(1-l)^2+1+c_*\Big]\|\nabla v(t)\|^2 - \left[\left(\int_0^t\varpi(s)dp\right)-2\delta\right]\|v_t(t)\|^2_{L^2_\rho} \\ & +\omega\|\nabla v_t\|^2_{L^2_\rho} + C_4\left(\int_0^t\varpi(s)dp\right)(\varpi\circ\nabla v)(t) \\ & -\frac{\varpi(0)c_*}{4\delta}(\varpi'\circ\Delta v)(t) + c_{\varepsilon_0}\left(\varpi\circ\nabla v\right)^{\frac{1}{1+\varepsilon_0}}. \end{aligned} \tag{2.3.10}$$

Proof. Taking the derivative of $\psi(t)$ and using (2.1.1), we conclude that

$$\begin{aligned}
\psi'(t) &= \int_{\mathbb{R}^n} \nabla v(t) \int_0^t \varpi(t-p)(\nabla v(t) - \nabla v(p))dpdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t \varpi(t-p) \nabla v(p) dp \right) \left(\int_0^t \varpi(t-p)(\nabla v(t) - \nabla v(p)) dp \right) dx \\
&+ a \int_{\mathbb{R}^n} \rho(x) v_t \int_0^t \varpi(t-p)(v(t) - v(p))dpdx \\
&- k \int_{\mathbb{R}^n} \rho(x) v \ln |v| \int_0^t \varpi(t-p)(v(t) - v(p))dpdx \\
&+ \omega \int_{\mathbb{R}^n} \nabla v_t(t) \int_0^t \varpi(t-p)(\nabla v(t) - \nabla v(p))dpdx \\
&- \int_0^t \varpi(p)dp \|v_t\|^2_{L^2_\rho} - \int_{\mathbb{R}^n} \rho(x) v_t \int_0^t \varpi'(t-p)(v(t) - v(p))dpdx. \qquad (*)
\end{aligned}$$

Then we use Young's inequality and Poincaré's inequality in the first term of (*), to get for any $\delta > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla v(t) \int_0^t \varpi(t-p)(\nabla v(t) - \nabla v(p))dpdx \\
\leq\ & \delta \|\nabla v\|^2 + \frac{1}{4\delta} \left(\int_0^t \varpi(p)dp \right) (\varpi \circ \nabla v)(t). \qquad (2.3.11)
\end{aligned}$$

From the second term, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\int_0^t \varpi(t-p) \nabla v(p) dp \right) \left(\int_0^t \varpi(t-p)(\nabla v(t) - \nabla v(p)) dp \right) dx \\
\leq\ & \delta (1-l)^2 \|\nabla v\|^2 + \left(1 + \frac{1}{4\delta} \right) \left(\int_0^t \varpi(p)dp \right) (\varpi \circ \nabla v)(t). \qquad (2.3.12)
\end{aligned}$$

For the third term

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho(x) v_t \int_0^t \varpi(t-p)(v(t) - v(p))dpdx \\
\leq\ & \delta \|v_t\|^2_{L^2_\rho} + \frac{c_*}{4\delta} \left(\int_0^t \varpi(p)dp \right) (\varpi \circ \nabla v)(t). \qquad (2.3.13)
\end{aligned}$$

For the fourth term

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla v_t(t) \int_0^t \varpi(t-p)(\nabla v(t) - \nabla v(p))dpdx \\ \leq\ & \delta\|\nabla v_t\|_2^2 + \frac{1}{4\delta}\left(\int_0^t \varpi(p)dp\right)(\varpi \circ \nabla v)(t). \end{aligned} \tag{2.3.14}$$

For the last term of (*)

$$\begin{aligned} &-\int_{\mathbb{R}^n} \rho(x) v_t \int_0^t \varpi'(t-p)(v(t)-v(p))dpdx \\ \leq\ & \delta\|v_t\|_{L^2_\rho}^2 + \frac{c_*}{4\delta}\left(\int_0^t -\varpi'(p)dp\right)(\varpi' \circ \nabla v)(t) \\ \leq\ & \delta\|v_t\|_{L^2_\rho}^2 - \frac{\varpi(0)c_*}{4\delta}(\varpi' \circ \nabla v)(t). \end{aligned}$$

Combining (2.3.11)-(2.3.14) with (2.3.9), gives us (2.3.10) with

$$C_4 = \frac{\omega + ac_* + 2}{4\delta} + 2\delta.$$

Let $\varepsilon_0 \in (0,1)$ and $g(s) = s^{\varepsilon_0}(|\ln s| - s)$. Notice that g is continous on $(0,\infty)$, its limit at 0 is 0 and its limit at ∞ is $-\infty$.Then g has a maximum m_{ε_0} on $[0,\infty)$, so the following inequality holds

$$s|\ln s| \leq s^2 + m_{\varepsilon_0} s^{1-\varepsilon_0}, \quad \text{for} \quad \text{all } s > 0. \tag{2.3.15}$$

Using the Cauchy-Schwartz inequality and applying (2.3.15), we get, for any $\delta > 0$

$$\begin{aligned} & k\int_{\mathbb{R}^n} \rho(x) v \ln|v| \int_0^t \varpi(t-p)(v(t)-v(p))dpdx \\ \leq\ & k\int_{\mathbb{R}^n} \rho(x)\left(v^2 + m_{\varepsilon_0}|v|^{1-\varepsilon_0}\right)\left|\int_0^t \varpi(t-p)(v(t)-v(p))dpdx\right| \\ \leq\ & c\int_{\mathbb{R}^n} \rho(x) v^2 \left|\int_0^t \varpi(t-p)(v(t)-v(p))dpdx\right| \\ +\ & \delta\|v\|_{L^2_\rho}^2 + \int_{\mathbb{R}^n}\left|\int_0^t \varpi(t-p)(v(t)-v(p))dpdx\right|^{\frac{2}{1+\varepsilon_0}} \\ \leq\ & \delta c_*\|\nabla v\|_2^2 + \frac{1}{4\delta}(\varpi \circ \nabla v)(t) + c_{\varepsilon_0}(\varpi \circ \nabla v)^{\frac{1}{1+\varepsilon_0}}. \end{aligned} \tag{2.3.16}$$

Therefore the proof is complete. □

Now we define a Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = M\mathcal{E}(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \psi(t), \tag{2.3.17}$$

where M, ε_1 and ε_2 are positive constants will be taken later.

It is easy to see that $\mathcal{L}(t)$ and $\mathcal{E}(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 such that

$$\beta_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \beta_2 \mathcal{E}(t). \tag{2.3.18}$$

Remark 2. *[15] Since ξ is nonincreasing, we have*

$$\xi(t) \left(\varpi \circ \nabla v\right)\right)^{\frac{1}{1+\varepsilon_0}} \leq C \left(-\mathcal{E}'(t)\right)^{\frac{1}{1+\varepsilon_0}} . \tag{2.3.19}$$

Proof of Theorem 7. First for any fixed $t_0 > 0$, we have for any $t \geq t_0$,

$$\int_0^t \varpi(p)dp \geq \int_0^{t_0} \varpi(p)dp := \varpi_0.$$

It follows from (2.3.10), (2.3.2) and (2.2.4) that for any $t > t_0$,

$$\begin{aligned}
\mathcal{L}'(t) &= M\mathcal{E}'(t) + \varepsilon_1 \Phi'(t) + \psi'(t) \\
&\leq -\Big(\varpi_0 + M\omega - (1+a)\,\delta - \varepsilon_1 \Big(1 - \frac{a}{4\varepsilon}\Big)\Big) \|v_t(t)\|^2_{L^2_\rho} \\
&\quad - \left[\frac{l}{2}\varepsilon_1 - \delta\Big((1-l)^2 + 1 + c_*\Big)\right] \|\nabla v(t)\|_2^2 + \left[C_1\varepsilon_1 + C_4 l_0\right] (\varpi \circ \nabla v)(t) \\
&\quad - \left(Ma + \varepsilon_1 \omega(1-\varepsilon) - \omega\right) \|\nabla v_t(t)\|^2_{L^2_\rho} - \frac{M}{2}\varpi(t)\|v(t)\|^2 \\
&\quad + \varepsilon_1 k \int_{\mathbb{R}^n} \rho(x) v^2 \ln|v| dx + \varepsilon_1 c_{\varepsilon_0} \left(\varpi \circ \nabla v\right)\right)^{\frac{1}{1+\varepsilon_0}} + C_3 (\varpi' \circ \nabla v)(t).
\end{aligned}$$

Using the Logarithmic Sobolev inequality, we have

$$\begin{aligned}\mathcal{L}'(t) \;\leq\; & -\left(\varpi_0 + M\omega - (1+a)\,\delta - \varepsilon_1\left(1 - \frac{a}{4\varepsilon}\right)\right)\|v_t(t)\|_{L^2_\rho}^2 + C_3(\varpi' \circ \nabla v)(t) \\ & - \left[\frac{l}{2}\varepsilon_1 - \delta\Big((1-l)^2 + 1 - \varepsilon_1 k\frac{\alpha^2}{2\pi}\Big)\right]\|\nabla v(t)\|_2^2 + \left[C_1\varepsilon_1 + C_4 l_0\right](\varpi \circ \nabla v)(t) \\ & - \left(Ma + \varepsilon_1\omega(1-\varepsilon) - \omega\right)\|\nabla v_t(t)\|_{L^2_\rho}^2 - \frac{M}{2}\varpi(t)\|v(t)\|^2 \\ & + \varepsilon_1 k\frac{1}{2}\|v\|_2^2 \ln\|v\|_2^2 - \varepsilon_1 k(1+\ln\alpha)\|v\|_2^2 + \varepsilon_1 c_{\varepsilon_0}\left(\varpi \circ \nabla v\right)^{\frac{1}{1+\varepsilon_0}}.\end{aligned}$$

Recalling (2.2.3) and $\mathcal{E}(t) \leq \mathcal{E}(0) < d$, we get

$$\ln\|v\|_2^2 < \ln\left(\frac{4}{k}\mathcal{E}(t)\right) < \ln\left(\frac{4}{k}\mathcal{E}(0)\right) < \ln\left(\frac{4}{k}d\right). \tag{2.3.20}$$

Now, we take $\varepsilon_1 > 0$ small enough so that

$$\left(\varpi_0 - (1+a)\,\delta - \varepsilon_1\left(1 - \frac{a}{4\varepsilon}\right)\right) > 0.$$

For any fixed $\varepsilon_1 > 0$, we pick $\delta > 0$ so small that

$$\frac{l}{2}\varepsilon_1 - \delta\Big((1-l)^2 + 1\Big) > \frac{l}{4}\varepsilon_1.$$

At last we choose $M > 0$ large enough so that (2.3.18) hold, and further

$$C_3 = \frac{M}{2} - \frac{\varpi(0)}{4\delta} > 0.$$

We can conclude that there exist two positive constant m and C' such that for any $t \geq t_0$,

$$\mathcal{L}'(t) \leq -m\mathcal{E}(t) + C'(\varpi \circ \nabla v)(t) + \varepsilon_1 c_{\varepsilon_0}\left(\varpi \circ \nabla v\right)^{\frac{1}{1+\varepsilon_0}}. \tag{2.3.21}$$

Multiplying (2.3.21) by $\xi(t)$, $(H2)$ and use $(\varpi \circ \nabla v)(t) \leq\; c\,(\varpi \circ \nabla v)^{\frac{1}{1+\varepsilon_0}}(t)$, and (2.3.19), we get

$$\xi(t)\mathcal{L}'(t) \quad \leq \quad -m\xi(t)\mathcal{E}(t) + c\left(-\mathcal{E}'(t)\right)^{\frac{1}{1+\varepsilon_0}}. \tag{2.3.22}$$

Multiply (2.3.22) by $\xi^{\varepsilon_0}(t)\mathcal{E}^{\varepsilon_0}(t)$ and recall that $\xi'(t) \leq 0$ to obtain

$$\xi^{\varepsilon_0+1}(t)\mathcal{E}^{\varepsilon_0}(t)\mathcal{L}'(t) \leq -m\xi^{\varepsilon_0}(t)\mathcal{E}^{\varepsilon_0+1}(t) + c\left(\xi\mathcal{E}\right)^{\varepsilon_0}(t)\left(-\mathcal{E}'(t)\right)^{\frac{1}{\varepsilon_0+1}}.$$

Using Young's inequality, for any $\delta > 0$,

$$\begin{aligned} \xi^{\varepsilon_0+1}(t)\mathcal{E}^{\varepsilon_0}(t)\mathcal{L}'(t) &\leq -m\xi^{\varepsilon_0+1}(t)\mathcal{E}^{\varepsilon_0+1}(t) + c\left(\delta\xi^{\varepsilon_0+1}(t)\mathcal{E}^{\varepsilon_0+1}(t) - c_\delta\mathcal{E}'(t)\right) \\ &\leq -\left(m - \delta c\right)\xi^{\varepsilon_0+1}(t)\mathcal{E}^{\varepsilon_0+1}(t) - c\mathcal{E}'(t), \end{aligned} \tag{2.3.23}$$

which implies

$$\left(\xi^{\varepsilon_0+1}\mathcal{E}^{\varepsilon_0}\mathcal{L} + c\mathcal{E}\right)'(t) \quad \leq \quad -\left(m - \delta c\right)\xi^{\varepsilon_0+1}(t)\mathcal{E}^{\varepsilon_0+1}(t). \tag{2.3.24}$$

It is clear that to get

$$\mathcal{L}_1(t) = \left(\xi^{\varepsilon_0+1}\mathcal{E}^{\varepsilon_0}\mathcal{L} + c\mathcal{E}\right) \sim \mathcal{E}(t). \tag{2.3.25}$$

By using (2.3.24) and $\xi'(t) \leq 0$,we arrive at

$$\mathcal{L}_1'(t) = \left(\xi^{\varepsilon_0+1}\mathcal{E}^{\varepsilon_0}\mathcal{L} + c\mathcal{E}\right)' \quad \leq \quad -m'\xi^{\varepsilon_0}(t)\mathcal{E}^{\varepsilon_0+1}(t). \tag{2.3.26}$$

Integration over (t_0, t) leads to, for some constant $m' > 0$ such that

$$\mathcal{L}_1(t) \leq m'\left(1 + \int_{t_0}^{t} \xi^{\varepsilon_0+1}(p)dp\right)^{\frac{-1}{\varepsilon_0}}.$$

The equivalence of $\mathcal{L}_1(t)$ and $\mathcal{E}$ completes the proof of Theorem 7. □

Chapter 3

Stabilization for solutions of plate equation with time-varying delay and weak-viscoelasticity in $\mathbb{R}^n$

3.1 Introduction, related results and position of problem

The phenomena of delay appear naturally in the modeling of many real processes. Physic, biology, ecology, engineering sciences and telecommunications are areas in which differential equations are involved, the evolution of which depends not only on the value of their variables at present, but also part of their "history", that is, values at a time $t_1 < t$. These problems are thus said to be "delayed". The analysis of stability for constant or variable delays, single or coupled systems of viscoelastic wave equations in bounded domain has attracted a lot of attention in recent decades. The stabilization of solutions for evolution systems with different dissipations have been considered by several authors and different stability/instability results have been obtained (See [25], [26], [31], [32], [35], [37]-[39], [40]-[42], [45], ...). The first studies on the stability of delayed systems mainly concerned constant delays. Since the constancy of delay is a hypothesis rarely tested in real life, the case of variable delays (known or unknown) has also been the subject of much research. Define the major delays for which there is a known reality $\nu_1 > 0$, such that

$$0 \leq \nu(t) \leq \nu_1$$

A large part of the existing results assume that the delays vary in an interval $[0, \nu_1]$. Delays in real processes are most often due to phenomena of information or material transfer. To allow the delay and take the value 0, it is to assume that at some point this transfer is instantaneous. In this thesis and provided of course that this leads to less restrictive criteria, it seems worthwhile to set a lower bound for the delay and then to provide the means to measure its impact on stability of the system. We then define the bounded delays $\nu(t)$ for which there exist two real ν_0 and ν_1 such that

$$0 < \nu_0 \leq \nu(t) \leq \nu_1, \quad \forall\, t > 0. \tag{3.1.1}$$

As we have said, delayed systems are dynamic systems governed by differential equations dealing with both present and past values. To begin with, we consider the following plate equation

$$\begin{cases} u'' + \phi(x)\Delta_x^2[u - \vartheta(t)\int_0^t \mu(t-\tau)u(\tau)d\tau] + \mu_1 u' + \mu_2 u'(t-\nu(t)) + h(u) = 0, \\ u(0,x) = u_0(x) \in \mathcal{D}^{2,2}(\mathbb{R}^n), \quad u'(0,x) = u_1(x) \in L_g^2(\mathbb{R}^n), \\ u'(x,t) = h_0(x,t),\ x \in \mathbb{R}^n,\ t \in [-\nu(0), 0), \end{cases} \tag{3.1.2}$$

where $u_0(x), u_1(x)$ and $h_0(x, t-\nu(0))$ are given. The function $(\phi(x))^{-1} = g(x)$ is the density. The constants μ_1 and μ_2 are two real numbers. The time-varying delay is given by the function $\nu(t)$. The function $h(u)$ is considered as source term.

In any space dimension without delay when $\phi(x) \neq 1$, Karachalios and Stavrakakis [36] considered the system

$$u'' + \phi(x)\Delta_x u + \delta u' + \lambda f(u) = \eta(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

The authors proved local existence of solutions and established the existence of a global attractor in energy space $\mathcal{D}^{1,2}(\mathbb{R}^n) \times L_g^2(\mathbb{R}^n)$. To compensate for Poincaré's lack of inequality in $\mathbb{R}^n$ and for a wider class of relaxation functions, Zennir used in [47] a weighted spaces to establish very general decay rate of solutions of Kirchhoff type viscoelastic wave equations

$$\rho(x)\left(|u'|^{q-2}u'\right)' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t g(t-\tau)\Delta_x u(\tau)d\tau = 0, x \in \mathbb{R}^n, t > 0, \tag{3.1.3}$$

where the density function satisfies

$$\rho : \mathbb{R}^n \to \mathbb{R}_+^*, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \tag{3.1.4}$$

with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.

The earliest result was established by [27]. In this work, the authors discussed question of an algebraic decay rate in $\mathbb{R}^n$. The main contribution was to show that the system can not be exponentially stable even the kernel is sub-exponential and the problem still dissipative.

Concerning the plate equation with delay, Park in [43] considered the following problem

$$u'' + \Delta_x^2 u - M(\|\nabla_x u\|^2)\Delta_x u + \sigma(t)\int_0^t \mu(t-\tau)\Delta_x u(\tau)d\tau + a_0 u' + a_1 u'(t-\nu(t)) = 0,$$

and obtained a general decay of solution under an appropriate assumption $|a_1| < \sqrt{1-d}a_0$. Next, in [34], the global existence of solutions with $|\mu_2| \leq \mu_1$ was proved and general decay of energy was showed under the assumption $|\mu_2| < \mu_1$ of the viscoelastic plate equation

$$u'' + \Delta_x^2 u - M(\|\nabla_x u\|^2)\Delta_x u - \int_0^t g(t-\tau)\Delta_x u(\tau)d\tau + \mu_1 u' + \mu_2 u'(t-\nu) = 0.$$

On the other hand, Yang [46] considered

$$u'' + \Delta_x^2 u - \int_0^t g(t-\tau)\Delta_x^2 u(\tau)d\tau + \mu_1 u' + \mu_2 u'(t-\nu) = 0,$$

and obtained a global existence without any restriction on real numbers μ_1, μ_2. But the exponential stability of energy have been obtained under the following assumption $0 < |\mu_2| < \mu_1$. For the non-linear delay, Benaissa et *all* [28] proved global existence of solutions for

$$u'' - \Delta_x u + \mu_1\sigma(t)g_1(u') + \mu_2\sigma(t)g_2(u(t-\nu(t))) = 0,$$

under some conditions on delay. Many authors established further the energy decay of energy. (For more, please see [33], [37], [38], [42]).

3.2 Preliminaries

We list here some useful mathematical tools and assumption.

Definition 5. *A system (3.1.2) is said to be exponentially stable of degree $\kappa(t)$, if there exists two constants $W, w > 0$ and function $\kappa \in C(\mathbb{R}^+, \mathbb{R}^+)$, such that solution $u(x,t)$, for all initial conditions, satisfies*

$$\|u(x,t)\| \leq W \exp\left(-w \int_0^t \kappa(\tau)d\tau\right), \ \forall t > 0. \tag{3.2.1}$$

The function $h(u)$ is a nonlinear such that $h(0) = 0$ and

$$|h(x) - h(y)| \leq c_h(1 + |x|^p + |y|^p)|x - y|, \ \ \forall \ x, y \in \mathbb{R}, \tag{3.2.2}$$

where $c_h > 0$ and

$$0 < p \leq \frac{4}{n-4} \text{ if } n \geq 5 \ \text{ and } \ p > 0 \text{ if } 1 \leq n \leq 4. \tag{3.2.3}$$

We assume further that

$$0 \leq H(u) \leq h(u)u, \ \ \forall \ u \in \mathbb{R}, \tag{3.2.4}$$

where $H(u) = \int_0^u h(z)dz$. Statement (3.2.2) and (3.2.4) include the following nonlinear type

$$h(u) \approx |u|^p u + |u|^\alpha u, \quad 0 < \alpha < p.$$

With respect to the relaxation function μ and the potential function ϑ, we assume

(A1) $\mu, \vartheta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ verifying $\forall t \geq 0$,

$$\mu(0) > 0, \quad \int_0^\infty \mu(\tau)d\tau = l_0 < \infty, \quad \vartheta(t) > 0, \ 1 - \vartheta(t)\int_0^t \mu(\tau)d\tau \geq l > 0. \tag{3.2.5}$$

(A2) There exists a function $\zeta \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\zeta(t) > 0, \ \ \mu'(t) \leq -\zeta(t)\mu(t) \ \ \text{ for } t \geq 0, \quad \lim_{t\to\infty} \frac{-\vartheta'(t)}{\zeta(t)\vartheta(t)} = 0. \tag{3.2.6}$$

(A3) The density function $g : \mathbb{R}^n \to \mathbb{R}^*_+, g(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$,

and $g \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}, q \geq 2$.

(A4) We assume that there exist two constants $\nu_0, \nu_1 > 0$ such that (3.1.1) satisfied.

(A5) We assume that there is a real d such that

$$\forall\, T, t > 0, \nu(t) \in W^{2,\infty}(0,T) \text{ and } \; 0 < \nu'(t) \leq d < 1, \tag{3.2.7}$$

and for some $\alpha_1, \alpha_2 > 0$, we have

$$\mu_2 \alpha_2 (1 - \alpha_1 d) < \mu_1 \alpha_1 (1 - d). \tag{3.2.8}$$

Remark 3. *If we look at the function $f(t) = t - \nu(t)$, the condition (3.2.7) implies that f is a strictly increasing function. This means that the delayed information arrives in a chronological order.*

As in [40], we introduce a new variable

$$z(x, \varrho, t) = u'(x, t - \nu(t)\varrho), \quad x \in \mathbb{R}^n, \quad \varrho \in (0,1), \quad t > 0, \tag{3.2.9}$$

then

$$\nu(t) z_t(x, \varrho, t) + (1 - \nu'(t)\varrho) z_\varrho(x, \varrho, t) = 0, \quad \text{in } \mathbb{R}^n \times (0,1) \times (0, \infty). \tag{3.2.10}$$

The original system becomes

$$\begin{cases} u'' + \phi(x)\Delta_x^2 u - \vartheta(t) \int_0^t \mu(t-\tau)\phi(x)\Delta_x^2 u(\tau) d\tau \\ \qquad\qquad\qquad\qquad + \mu_1 u' + \mu_2 z(x,1,t) + h(u) = 0, \\ \nu(t) z_t(x, \varrho, t) + (1 - \nu'(t)\varrho) z_\varrho(x, \varrho, t) = 0, \end{cases} \tag{3.2.11}$$

here $x \in \mathbb{R}^n$, $\varrho \in (0,1)$ and $t > 0$. The boundary and initial conditions are given as

$$\begin{cases} u(x,0) = u_0, \; u'(x,0) = u_1, \quad x \in \mathbb{R}^n \\ z(x, 0, \varrho) = h_0(x, -\varrho\nu(0)), \quad (x, \varrho) \in \mathbb{R}^n \times (0,1), \\ z(x, 0, t) = u'(x,t) \quad x \in \mathbb{R}^n \times \mathbb{R}^+. \end{cases} \tag{3.2.12}$$

Definition 6. *The function spaces of our problem and its norm are defined as*

$$\mathcal{D}^{2,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-4)}(\mathbb{R}^n) : \Delta_x f \in L^2(\mathbb{R}^n) \right\}. \tag{3.2.13}$$

We use the spaces $L^2_g(\mathbb{R}^n)$ defined with the inner product

$$(f,h)_{L^2_g(\mathbb{R}^n)} = \int_{\mathbb{R}^n} gfh dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_g(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} g|f|^q dx \right)^{1/q}, \tag{3.2.14}$$

and

$$\|f\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^q dx \right)^{1/q}. \tag{3.2.15}$$

Then $\mathcal{D}^{2,2}(\mathbb{R}^n)$ can be embedded continuously in $L^{2n/(n-4)}(\mathbb{R}^n)$, i.e there exists $k > 0$ such that

$$\|u\|_{L^{2n/(n-4)}(\mathbb{R}^n)} \leq k \|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)}. \tag{3.2.16}$$

The generalized Poincaré's inequality will be used

$$\int_{\mathbb{R}^n} |\Delta_x u|^2 dx \geq \gamma \int_{\mathbb{R}^n} g u^2 dx, \tag{3.2.17}$$

for all $u \in C_0^\infty$ and $g \in L^{n/4}(\mathbb{R}^n)$, where $\gamma =: k^{-2} \|g\|^{-1}_{L^{n/4}(\mathbb{R}^n)}$.
The separable Hilbert spaces $L^2_g(\mathbb{R}^n)$ with

$$(f,f)_{L^2_g(\mathbb{R}^n)} = \|f\|^2_{L^2_g(\mathbb{R}^n)},$$

consist of all f for which $\|f\|_{L^q_g(\mathbb{R}^n)} < \infty$ and $1 < q < +\infty$.
Let $g \in L^{n/4}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the embedding $\mathcal{D}^{2,2}(\mathbb{R}^n) \subset L^2_g(\mathbb{R}^n)$ is compact.
For any $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$

$$\|u\|_{L^2_g(\mathbb{R}^n)} \leq \|g\|_{L^{n/4}(\mathbb{R}^n)} \|\Delta_x u\|_{L^2(\mathbb{R}^n)}, \tag{3.2.18}$$

where $\|g\|_{L^s(\mathbb{R}^n)} = c_* > 0$, then

$$\|u\|_{L^2_g(\mathbb{R}^n)} \leq c_* \|\Delta_x u\|_{L^2(\mathbb{R}^n)}. \tag{3.2.19}$$

The weak solution to our system is introduced in the following

Definition 7. *A pair $(u, u') \in C(\mathbb{R}^+, \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n))$ is said to be weak solution of problem* (3.1.2)*, for given initial data $(u_0, u_1) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$ and $h_0 \in L^2_g(\mathbb{R}^n \times (0,1))$, if satisfies*

$$\int_{\mathbb{R}^n} u'' \omega dx + \int_{\mathbb{R}^n} \phi(x) \Delta_x u \Delta_x \omega dx - \vartheta(t) \int_0^t \mu(t-\tau) \int_{\mathbb{R}^n} \phi(x) \Delta_x u(\tau) \Delta_x \omega dx d\tau$$
$$+\mu_1 \int_{\mathbb{R}^n} u' \omega dx + \mu_2 \int_{\mathbb{R}^n} u'(t - \nu(t)) \omega dx + \int_{\mathbb{R}^n} h(u) \omega dx = 0,$$

for all test function $\omega \in \mathcal{D}^{2,2}(\mathbb{R}^n)$, for almost all $t \in [0, T]$.

We can follow the steps in [30, 34] to prove the following Theorem.

Theorem 8. *Let* (3.2.2)-(3.2.8) *hold. If $(u_0, u_1, h_0) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n \times (0,1))$ and the compatibility condition $h_0(\cdot, 0) = u_1$ is satisfied, then problem (3.2.11)-(3.2.12) has weak solution $(u, u') \in C(0, T, \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n))$ such that for any $T > 0$,*

$$u \in L^\infty(0, T, \mathcal{D}^{2,2}(\mathbb{R}^n)), \quad u' \in L^\infty(0, T, L^2_g(\mathbb{R}^n)).$$

We mean by, Δ_x^κ, the κth-order of the derivatives for the dependent variable (κ is degree of the Laplace operator) and we mention here that

$$|\nabla^\kappa u|^2 = (\Delta^{\kappa/2} u)^2, \text{for par value of } \kappa$$

and

$$|\nabla^\kappa u|^2 = |\nabla(\Delta^{(\kappa-1)/2} u)|^2, \text{for odd } \kappa$$

where

$$|\nabla u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2, \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}.$$

With the notation

$$(\mu \circ w) = \int_0^t \mu(t-\tau)\|w(t) - w(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau,$$

we have the following technical Lemma

Lemma 12. *For any* $v \in C^1(0, T, H^\kappa(\mathbb{R}^n)), \kappa \geq 1$ *we have*

$$\begin{aligned}
& \int_{\mathbb{R}^n} \vartheta(t) \int_0^t \mu(t-\tau)\Delta_x^\kappa v(\tau) v'(t) d\tau dx \\
= & \ \frac{1}{2}\frac{d}{dt}\vartheta(t)\left(\mu \circ \nabla_x^\kappa v\right)(t) \\
- & \ \frac{1}{2}\frac{d}{dt}\left[\vartheta(t)\int_0^t \mu(\tau)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx d\tau\right] \\
& -\frac{1}{2}\vartheta(t)\left(\mu' \circ \nabla_x^\kappa v\right)(t) + \frac{1}{2}\vartheta(t)\mu(t)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx d\tau \\
& -\frac{1}{2}\vartheta'(t)\left(\mu \circ \nabla_x^\kappa v\right)(t) + \frac{1}{2}\vartheta'(t)\int_0^t v(\tau) d\tau \int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx d\tau.
\end{aligned}$$

Proof. It's easy to see that

$$\begin{aligned}
& \int_{\mathbb{R}^n} \vartheta(t) \int_0^t \mu(t-\tau)\Delta_x^\kappa v(\tau) v'(t) d\tau dx \\
= & -\vartheta(t)\int_0^t \mu(t-\tau)\int_{\mathbb{R}^n} \nabla_x^\kappa v'(t)\nabla_x^\kappa v(\tau) dx d\tau \\
= & -\vartheta(t)\int_0^t \mu(t-\tau)\int_{\mathbb{R}^n} \nabla_x^\kappa v'(t)\left[\nabla_x^\kappa v(\tau) - \nabla_x^\kappa v(t)\right] dx d\tau \\
& -\vartheta(t)\int_0^t \mu(t-\tau)\int_{\mathbb{R}^n} \nabla_x^\kappa v'(t)\nabla_x^\kappa v(t) dx d\tau.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{\mathbb{R}^n} \vartheta(t) \int_0^t \mu(t-\tau)\Delta_x^\kappa v(\tau) v'(t) d\tau dx \\
= & \ \frac{1}{2}\vartheta(t)\int_0^t \mu(t-\tau)\frac{d}{dt}\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(\tau) - \nabla_x^\kappa v(t)\right|^2 dx d\tau \\
& -\vartheta(t)\int_0^t \mu(\tau)\left(\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx\right) d\tau
\end{aligned}$$

which implies,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \vartheta(t) \int_0^t \mu(t-\tau)\Delta_x^\kappa v(\tau) v'(t) d\tau dx \\
= &\frac{1}{2}\frac{d}{dt}\left[\vartheta(t)\int_0^t \mu(t-\tau)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(\tau) - \nabla_x^\kappa v(t)\right|^2 dx d\tau\right] \\
&-\frac{1}{2}\frac{d}{dt}\left[\vartheta(t)\int_0^t v(\tau)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx d\tau\right] \\
&-\frac{1}{2}\vartheta(t)\int_0^t \mu'(t-\tau)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(\tau) - \nabla_x^\kappa v(t)\right|^2 dx d\tau \\
&+\frac{1}{2}\vartheta(t)\mu(t)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx d\tau. \\
&-\frac{1}{2}\vartheta'(t)\int_0^t \mu(t-\tau)\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(\tau) - \nabla_x^\kappa v(t)\right|^2 dx d\tau \\
&+\frac{1}{2}\vartheta'(t)\int_0^\tau \mu(\tau) d\tau\int_{\mathbb{R}^n} \left|\nabla_x^\kappa v(t)\right|^2 dx d\tau.
\end{aligned}
$$

This completes the proof. □

For $\kappa = 2$ in Lemma 12, the modified energy of problem (3.2.11)-(3.2.12) is given by the functional

$$
\begin{aligned}
E(t) \;=\; &\frac{1}{2}\|u'(t)\|^2_{L^2_g(\mathbb{R}^n)} + \frac{1}{2}\left(1 - \vartheta(t)\int_0^t \mu(\tau) d\tau\right)\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} \\
&+\frac{1}{2}\vartheta(t)(\mu \circ \Delta_x u)(t) + \frac{1}{2}\xi\nu(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z^2(x,\varrho,t) d\varrho dx + \int_{\mathbb{R}^n} g(x) H(u) dx,
\end{aligned}
\tag{3.2.20}
$$

where $\xi > 0$ satisfies

$$
\frac{\mu_2(1-\alpha_1)}{(1-d)\alpha_1} < \xi < \frac{\mu_1 - \mu_2\alpha_2}{\alpha_2}, \tag{3.2.21}
$$

3.3 Stability result and proofs

Our main result regarding the general decay is given in the next theorem.

Theorem 9. *Let* (3.2.2)-(3.2.8) *hold. If* (u, u') *is weak solution of problem* (3.2.11)-(3.2.12) *with the initial data* $(u_0, u_1) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$, $h_0 \in L^2_g(\mathbb{R}^n \times (0,1))$, *then the solution is* $\kappa(t)-$*stable in the sense of Definition 5.*

We need the following Lemma

Lemma 13. *Under the same statement of Theorem 9, we have*

$$\begin{aligned}
E'(t) \leq & -(\mu_1 - \xi\alpha_2 - \mu_2\alpha_2)\int_{\mathbb{R}^n} g(x)u'^2 dx \\
& -\Big[\xi(1-d)\alpha_1 - \frac{\mu_2}{2} + \mu_2\alpha_1\Big]\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx \\
& -\frac{1}{2}\vartheta'(t)\int_0^t \mu(\tau)d\tau \|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2}\vartheta(t)\mu(t)\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} \\
& +\frac{1}{2}\vartheta'(t)(\mu\circ\Delta_x u)(t) + \frac{1}{2}\vartheta(t)(\mu'\circ\Delta_x u)(t). \qquad (3.3.1)
\end{aligned}$$

Proof. Differentiating (3.2.20) and by (3.2.11), we have

$$\begin{aligned}
E'(t) = & \int_{\mathbb{R}^n} u'\Big(-\Delta_x^2 u + \vartheta(t)\int_0^t \mu(t-\tau)\Delta_x^2 u(\tau)d\tau - \mu_1 g(x)u' - \mu_2 g(x)z(x,1,t) - g(x)h(u)\Big)dx \\
& -\frac{1}{2}\vartheta'(t)\int_0^t \mu(\tau)d\tau\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2}\vartheta(t)\mu(t)\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\vartheta'(t)(\mu\circ\Delta_x u) \\
& +\frac{1}{2}\vartheta(t)(\mu\circ\Delta_x u)' + \Big(1-\vartheta(t)\int_0^t \mu(\tau)d\tau\Big)\int_{\mathbb{R}^n}\Delta_x u\cdot\Delta_x u' dx \\
& +\Big(\frac{1}{2}\xi\nu(t)\int_{\mathbb{R}^n}\int_0^1 g(x)z^2(x,\varrho,t)d\varrho dx\Big)' \\
= & -\mu_1\int_{\mathbb{R}^n} g(x)u'^2 dx - \mu_2\int_{\mathbb{R}^n} g(x)z(x,1,t)u' dx - \frac{1}{2}\vartheta'(t)\int_0^t \mu(\tau)d\tau\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} \\
& -\frac{1}{2}\vartheta(t)\mu(t)\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\vartheta'(t)(\mu\circ\Delta_x u) + \frac{1}{2}\vartheta(t)(\mu'\circ\Delta_x u) \\
& +\frac{d}{dt}\Big(\frac{1}{2}\xi\nu(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z^2(x,\varrho,t)d\varrho dx\Big). \qquad (3.3.2)
\end{aligned}$$

A multiplication of (3.2.11) by $\xi g(x)z(x,\varrho,t)$ and integration over $\mathbb{R}^n\times(0,1)$, yields

$$\begin{aligned}
& \xi\nu(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z_t(x,\varrho,t).z(x,\varrho,t)d\varrho dx \\
& = -\xi\int_{\mathbb{R}^n} g(x)\int_0^1 (1-\nu'(t)\varrho)z_\varrho(x,\varrho,t)z(x,\varrho,t)d\varrho dx \\
& = -\frac{\xi}{2}\int_{\mathbb{R}^n} g(x)\int_0^1 \frac{\partial}{\partial\varrho}\Big[(1-\nu'(t)\varrho)z^2(x,\varrho,t)\Big]d\varrho dx - \xi\nu'(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z^2(x,\varrho,t)d\varrho dx \\
& = -\frac{\xi}{2}\int_{\mathbb{R}^n} g(x)\Big[(1-\nu'(t))z^2(x,1,t) - u'^2(t)\Big]dx - \frac{1}{2}\xi\nu'(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z^2(x,\varrho,t)d\varrho dx,
\end{aligned}$$

then

$$\begin{aligned}
&\left(\frac{1}{2}\xi\nu(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z^2(x,\varrho,t)d\varrho dx\right)' \\
&= \frac{1}{2}\xi\nu'(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z^2(x,\varrho,t)d\varrho dx + \xi\nu(t)\int_{\mathbb{R}^n} g(x)\int_0^1 z_t(x,\varrho,t)z(x,\varrho,t)d\varrho dx \\
&= -\frac{\xi}{2}\int_{\mathbb{R}^n} g(x)(1-\nu'(t))z(x,1,t)dx + \frac{\xi}{2}\int_{\mathbb{R}^n} g(x)u'^2(t)dx. \qquad (3.3.3)
\end{aligned}$$

From (3.3.2), (3.3.3), we have

$$\begin{aligned}
E'(t) \leq\ & -(\mu_1-\xi\alpha_2)\int_{\mathbb{R}^n} g(x)u'^2dx - \mu_2\int_{\mathbb{R}^n} g(x)z(x,1,t)u'dx \\
& -\frac{\xi}{2}\int_{\mathbb{R}^n} g(x)(1-\nu'(t))z^2(x,1,t)dx - \frac{1}{2}\vartheta'(t)\int_0^t \mu(\tau)d\tau\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} \\
& -\frac{1}{2}\vartheta(t)\mu(t)\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\vartheta'(t)(\mu\circ\Delta_x u) + \frac{1}{2}\vartheta(t)(\mu'\circ\Delta_x u). \qquad (3.3.4)
\end{aligned}$$

Together with Young's inequality and using (3.2.7), we have

$$\begin{aligned}
E'(t) \leq\ & -(\mu_1-\xi\alpha_2-\mu_2)\int_{\mathbb{R}^n} g(x)u'^2dx + \Big[\mu_2-\xi(1-d)\alpha_1\Big]\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx \\
& -\frac{\mu_2}{2}\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx - \frac{1}{2}\vartheta'(t)\int_0^t \mu(\tau)d\tau\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} \\
& -\frac{1}{2}\vartheta(t)\mu(t)\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\vartheta'(t)(\mu\circ\Delta_x u) + \frac{1}{2}\vartheta(t)(\mu'\circ\Delta_x u). \qquad (3.3.5)
\end{aligned}$$

Then the proof is completed. □

Lemma 14. *Under the same statement in Theorem 9, the functional* $\Phi(t)$ *defined by*

$$\Phi(t) = \int_{\mathbb{R}^n} g(x)u(t)u'(t)dx, \qquad (3.3.6)$$

satisfies

$$\begin{aligned}
\Phi'(t) \leq\ & C_2\|u'(t)\|^2_{L^2_g(\mathbb{R}^n)} - \frac{l}{2}\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} + C_1\vartheta(t)(\mu\circ\Delta_x u)(t) \\
& + C_3\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx, \qquad (3.3.7)
\end{aligned}$$

for $C_1, C_2, C_3 > 0$ *and for any* $t \geq 0$.

Proof. Using (3.2.11) to get

$$\begin{aligned}
\Phi'(t) &= \|u'\|^2_{L^2_g(\mathbb{R}^n)} - \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \vartheta(t)\int_{\mathbb{R}^n} \Delta_x u(t) \cdot \int_0^t \mu(t-\tau)\Delta_x u(\tau)d\tau dx - \mu_1 \int_{\mathbb{R}^n} g(x)u'udx \\
&\quad -\mu_2 \int_{\mathbb{R}^n} g(x)z(x,1,t)u - \int_{\mathbb{R}^n} g(x)h(u)udx \\
&\leq \|u'\|^2_{L^2_g(\mathbb{R}^n)} - \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \underbrace{\vartheta(t)\int_{\mathbb{R}^n} \Delta_x u(t) \cdot \int_0^t \mu(t-\tau)\Delta_x u(\tau)d\tau dx}_{:=I_1} \\
&\quad \underbrace{-\mu_1 \int_{\mathbb{R}^n} g(x)u'udx}_{:=I_2} \underbrace{-\mu_2 \int_{\mathbb{R}^n} g(x)z(x,1,t)udx}_{:=I_3}. \qquad (3.3.8)
\end{aligned}$$

Thanks to generalized Poincaré and Young's inequalities, $\forall \varepsilon > 0$, we have

$$\begin{aligned}
I_1 &= \vartheta(t)\int_{\mathbb{R}^n} \Delta_x u(t) \cdot \int_0^t \mu(t-\tau)(\Delta_x u(\tau) - \Delta_x u(t))d\tau dx + \vartheta(t)\int_0^t \mu(\tau)d\tau \|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} \\
&\leq (1-l)\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \varepsilon\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{\vartheta^2(t)}{4\varepsilon}\int_{\mathbb{R}^n}\left(\int_0^t \mu(t-\tau)(\Delta_x u(\tau) - \Delta_x u(t))d\tau\right)^2 dx \\
&\leq (1-l+\varepsilon)\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1-l}{4\varepsilon}\vartheta(t)(\mu \circ \Delta_x u)(t), \qquad (3.3.9)
\end{aligned}$$

and

$$I_2 \leq \frac{\varepsilon c_*}{\lambda_1}\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{\mu_1^2}{4\varepsilon}\int_{\mathbb{R}^n} g(x)u'^2 dx, \qquad (3.3.10)$$

and

$$I_3 \leq \frac{\varepsilon c_*}{\lambda_1}\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{\mu_2^2}{4\varepsilon}\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx. \qquad (3.3.11)$$

For any $\varepsilon > 0$, inserting (3.3.9)-(3.3.11) into (3.3.8), to get

$$\begin{aligned}
\Phi'(t) &\leq C_2\|u'(t)\|^2_{L^2_g(\mathbb{R}^n)} - \left(l - \varepsilon - \frac{2\varepsilon c_*}{\lambda_1}\right)\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{1-l}{4\varepsilon}(\mu \circ \Delta_x u)(t) \\
&\quad + \frac{\mu_2^2}{4\varepsilon}\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx. \qquad (3.3.12)
\end{aligned}$$

Choosing $\varepsilon > 0$ small enough such that

$$l - \varepsilon - \frac{2\varepsilon c_*}{\lambda_1} > \frac{l}{2},$$

then, by (3.3.7) and for

$$C_1 = \frac{1-l}{4\varepsilon},$$

$$C_2 = \frac{\mu_1^2}{4\varepsilon} + 1,$$

$$C_3 = \frac{\mu_2^2}{4\varepsilon}.$$

The proof is completed. □

Lemma 15. *Under the same statement in Theorem 9, then the functional $\psi(t)$ defined by*

$$\psi(t) = -\int_{\mathbb{R}^n} g(x)u(t)\int_0^t \mu(t-\tau)(u(t)-u(\tau))d\tau dx, \tag{3.3.13}$$

satisfies

$$\begin{aligned}\psi'(t) \;\; \leq \;\; & \left[\delta(1-l)^2 + 3\delta\right]\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} - \left[\left(\int_0^t \mu(\tau)d\tau\right) - 2\delta\right]\|u'(t)\|^2_{L^2_g(\mathbb{R}^n)} \\ & +\delta\|z(x,1,t)\|^2_{L^2_g(\mathbb{R}^n)} + C_4\left(\int_0^t \mu(\tau)d\tau\right)(\mu\circ\Delta_x u)(t) - \frac{\mu(0)c_*}{4\delta\lambda_1}(\mu'\circ\Delta_x u)(t).\end{aligned} \tag{3.3.14}$$

for $C_4 > 0$ and for any $\delta > 0$,

Proof. Using (3.2.11), we have

$$\begin{aligned}\psi'(t) \;\; = \;\; & \int_{\mathbb{R}^n}\left(\Delta_x^2 u - \vartheta(t)\int_0^t \mu(t-\tau)\Delta_x^2 u(\tau)d\tau\right. \\ & \left.+\mu_1 g(x)u' + \mu_2 g(x)z(x,1,t) + g(x)h(u)\right)\int_0^t \mu(t-\tau)(u(t)-u(\tau))d\tau \\ & -\int_{\mathbb{R}^n} g(x)u'\int_0^t \mu'(t-\tau)(u(t)-u(\tau))d\tau dx - \int_0^t \mu(\tau)d\tau\|u'\|^2_{L^2_g}\end{aligned}$$

Then,

$$\begin{aligned}
\psi'(t) &= \int_{\mathbb{R}^n} \Delta_x u(t) \int_0^t \mu(t-\tau)(\Delta_x u(t) - \Delta_x u(\tau)) d\tau dx \\
&\quad -\vartheta(t) \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-\tau)\Delta_x u(\tau) d\tau \right) \left(\int_0^t \mu(t-\tau)(\Delta_x u(t) - \Delta_x u(\tau)) d\tau \right) dx \\
&\quad +\mu_1 \int_{\mathbb{R}^n} g(x) u' \int_0^t \mu(t-\tau)(u(t)-u(\tau)) d\tau dx \\
&\quad +\mu_2 \int_{\mathbb{R}^n} g(x) z(x,1,t) \int_0^t \mu(t-\tau)(u(t)-u(\tau)) d\tau dx \\
&\quad + \int_{\mathbb{R}^n} g(x) h(u) \int_0^t \mu(t-\tau)(u(t)-u(\tau)) d\tau dx \\
&\quad - \int_{\mathbb{R}^n} g(x) u' \int_0^t \mu'(t-\tau)(u(t)-u(\tau)) d\tau dx - \int_0^t \mu(\tau) d\tau \|u'\|^2_{L^2_g(\mathbb{R}^n)} \\
&= \sum_{i=1}^{7} J_i - \int_0^t \mu(\tau) d\tau \|u'\|^2_{L^2_g(\mathbb{R}^n)}. \qquad (3.3.15)
\end{aligned}$$

Thanks to generalized Poincaré and Young's inequalities, $\forall \delta > 0$, we have

$$J_1 \le \delta \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{4\delta} \left(\int_0^t \mu(\tau) d\tau \right) (\mu \circ \Delta_x u)(t), \qquad (3.3.16)$$

$$J_2 \le \delta \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{C'}{4\lambda_2 \delta} \left(\int_0^t \mu(\tau) d\tau \right) (\mu \circ \Delta_x u)(t), \qquad (3.3.17)$$

$$\begin{aligned}
J_3 &= \vartheta(t) \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-\tau)(\Delta_x u(t) - \Delta_x u(\tau)) d\tau \right)^2 dx \\
&\quad -\vartheta(t) \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-\tau)\Delta_x u(t) d\tau \right) \left(\int_0^t \mu(t-\tau)(\Delta_x u(t) - \Delta_x u(\tau)) d\tau \right) dx \\
&\le \vartheta(t) \left(\int_0^t \mu(\tau) d\tau \right) (\mu \circ \Delta_x u) + \delta \vartheta^2(t) \left(\int_0^t \mu(\tau) d\tau \right)^2 \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} \\
&\quad + \frac{1}{4\delta} \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-\tau)(\Delta_x u(t) - \Delta_x u(\tau)) d\tau \right)^2 dx \\
&\le \delta (1-l)^2 \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \left(\vartheta(t) + \frac{1}{4\delta} \right) \left(\int_0^t \mu(\tau) d\tau \right) (\mu \circ \Delta_x u)(t), \qquad (3.3.18)
\end{aligned}$$

$$J_4 \leq \delta \|u'\|^2_{L^2_g(\mathbb{R}^n)} + \frac{\mu_1^2 c_*}{4\delta\lambda_1}\left(\int_0^t \mu(\tau)d\tau\right)(\mu \circ \Delta_x u)(t), \tag{3.3.19}$$

$$J_5 \leq \delta \|z(x,1,t)\|^2_{L^2_g(\mathbb{R}^n)} + \frac{\mu_2^2 c_*}{4\delta\lambda_1}\left(\int_0^t \mu(\tau)d\tau\right)(\mu \circ \Delta_x u)(t), \tag{3.3.20}$$

$$J_6 \leq \delta \|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{4\delta\lambda_1}\left(\int_0^t \mu(\tau)d\tau\right)(\mu \circ \Delta_x u)(t), \tag{3.3.21}$$

$$\begin{aligned} J_7 &\leq \delta \|u'\|^2_{L^2_g(\mathbb{R}^n)} + \frac{c_*}{4\delta\lambda_1}\left(\int_0^t \mu'(\tau)d\tau\right)(\mu' \circ \Delta_x u)(t) \\ &\leq \delta \|u'\|^2_{L^2_g(\mathbb{R}^n)} - \frac{\mu(0)c_*}{4\delta\lambda_1}(\mu' \circ \Delta_x u)(t). \end{aligned} \tag{3.3.22}$$

By the fact $\vartheta(t) < \vartheta(0)$ and for

$$C_4 = \frac{1}{2\delta} + \frac{C'}{4\lambda_2\delta} + \vartheta(0) + \frac{\mu_1^2 c_*}{4\delta\lambda_1} + \frac{\mu_2^2 c_*}{4\delta\lambda_1} + \frac{1}{4\delta\lambda_1}.$$

Replacing (3.3.16)-(3.3.22) into (3.3.15). Which completes the proof. □

Lemma 16. *Under the same statement in Theorem 9, we define the functional $I(t)$ by*

$$I(t) = \int_{\mathbb{R}^n}\int_0^1 g(x)e^{-2\nu(t)\varrho} z(x,\varrho,t)d\varrho dx. \tag{3.3.23}$$

Then we have

$$\begin{aligned} I'(t) &\leq -2(1-d)I(t) - \frac{1-d}{\nu_1}e^{-2\nu_1}\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx \\ &+ \frac{1}{\nu_0}\int_{\mathbb{R}^n} g(x)u'^2(t)dx. \end{aligned} \tag{3.3.24}$$

Proof. We have $\forall t > 0$,

$$I'(t) = -\varrho\nu'(t)\int_{\mathbb{R}^n}\int_0^1 g(x)e^{-2\nu(t)\varrho}z^2(x,\varrho,t)d\varrho dx + \int_{\mathbb{R}^n}\int_0^1 g(x)e^{-2\nu(t)\varrho}z(x,\varrho,t)z_t(x,\varrho,t)d\varrho dx$$

Then,

$$\begin{aligned}
I'(t) &= -\varrho\nu'(t)I(t) - \frac{1}{\nu(t)}\int_{\mathbb{R}^n}\int_0^1 g(x)e^{-2\nu(t)\varrho}z(x,\varrho,t)z_\varrho(x,\varrho,t)(1-\nu'(t)\varrho)d\varrho dx \\
&= -\varrho\nu'(t)I(t) - \frac{1}{\nu(t)}\int_{\mathbb{R}^n}\int_0^1 g(x)\Big[\frac{\partial}{\partial\varrho}\Big(e^{-2\nu(t)\varrho}z^2(x,\varrho,t)\Big) \\
&\quad +\nu(t)z^2(x,\varrho,t)e^{-2\nu(t)\varrho}\Big](1-\nu'(t)\varrho)d\varrho dx \\
&= -\varrho\nu'(t)I(t) - \frac{1}{\nu(t)}\Big[\int_{\mathbb{R}^n} g(x)\Big(e^{-2\nu(t)\varrho}z^2(x,\varrho,t)(1-\nu'(t)\varrho)\Big|_0^1 \\
&\quad +\nu'(t)\int_0^1 e^{-2\nu(t)\varrho}z^2(x,\varrho,t)d\varrho\Big)dx\Big] \\
&\quad -(1-\nu'(t)\varrho)\int_{\mathbb{R}^n}\int_0^1 e^{-2\nu(t)\varrho}z^2(x,\varrho,t)d\varrho dx \\
&= -\varrho\nu'(t)I(t) - \frac{1-\nu'(t)}{\nu(t)}\int_{\mathbb{R}^n} g(x)e^{-2\nu(t)}z^2(x,1,t)dx + \frac{1}{\nu(t)}\int_{\mathbb{R}^n} g(x)u'^2(t)dx \\
&\quad -\frac{\nu'(t)}{\nu(t)}I(t) - (1-\nu'(t)\varrho)\int_{\mathbb{R}^n}\int_0^1 e^{-2\nu(t)\varrho}g(x)z^2(x,\varrho,t)d\varrho dx \\
&\leq \Big[-\varrho\nu'(t) - \frac{\nu'(t)}{\nu(t)} - (1-d)\Big]I(t) - \frac{1-d}{\nu_1}e^{-2\nu_1}\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx \\
&\quad +\frac{1}{\nu_0}\int_{\mathbb{R}^n} g(x)u'^2(t)dx \\
&\leq -(1-d)I(t) - \frac{1-d}{\nu_1}e^{-2\nu_1}\int_{\mathbb{R}^n} g(x)z^2(x,1,t)dx + \frac{1}{\nu_0}\int_{\mathbb{R}^n} g(x)u'^2(t)dx.
\end{aligned}$$

This is completes the proof. □

We introduce now a functional $\mathcal{L}(t)$ of Lyapunov type, by

$$\mathcal{L}(t) = ME(t) + \varepsilon_1\vartheta(t)\Phi(t) + \varepsilon_2\vartheta(t)I(t) + \vartheta(t)\psi(t), \tag{3.3.25}$$

where $M, \varepsilon_1, \varepsilon_2 > 0$ will be taken later, then we can get the following lemma.

Lemma 17. *Let M be large enough, then there exist $\beta_1, \beta_2 > 0$ such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t). \tag{3.3.26}$$

Proof. For any $t > 0$, we have

$$\begin{aligned}|\mathcal{L}(t) - ME(t)| &= |\varepsilon_1\vartheta(t)\Phi(t) + \varepsilon_2\vartheta(t)I(t) + \vartheta(t)\psi(t)| \\ &\le \frac{\varepsilon_1 c_*}{2\lambda_1}\|\Delta_x u\|^2_{L^2(\mathbb{R}^n)} + \left(\frac{\varepsilon_1}{2}\vartheta(0) + \frac{\vartheta(0)}{2}\right)\|u'\|^2_{L^2_g(\mathbb{R}^n)} \\ &\quad + \varepsilon_2\vartheta(0)\int_{\mathbb{R}^n}\int_0^1 g(x)z^2(x,\varrho,t)d\varrho dx + \frac{(1-l)c_*}{2}(\mu\circ\Delta_x u),\end{aligned}$$

this implies

$$|\mathcal{L}(t) - KE(t)| \le cE(t), c > 0.$$

For M large with $\beta_1 = K - c$ and $\beta_2 = K + c$.

This is completes the proof. □

Proof. (Of Theorem 9.) Let $t_0 > 0$ fixed, for any $t \ge t_0$, we have

$$\mu_0 = \int_0^{t_0}\mu(\tau)d\tau \le \int_0^t \mu(\tau)d\tau.$$

Thanks to generalized Poincaré and Young's inequalities, we get

$$\begin{aligned}&\varepsilon_1\vartheta'(t)\Phi(t) + \varepsilon_2\vartheta'(t)I(t) + \vartheta'(t)\psi(t) \\ &\le \frac{1+\varepsilon_1}{2}\vartheta'(t)\|u'(t)\|^2_{L^2_g(\mathbb{R}^n)} + \frac{\varepsilon_1 c_*}{2\lambda_1}\vartheta'(t)\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{c_*\vartheta'(t)}{2\lambda_1}\left(\int_0^t\mu(\tau)d\tau\right)(\mu\circ\Delta_x u)(t) \\ &\quad + \varepsilon_2\vartheta'(t)\int_{\mathbb{R}^n}\int_0^1 g(x)z^2(x,\varrho,t)d\varrho dx. \qquad (3.3.27)\end{aligned}$$

By (3.3.1), (3.3.7), (3.3.14), (3.3.24), (3.3.27) and for any $t > t_0$, we have

$$\begin{aligned}\mathcal{L}'(t) &= KE'(t) + \varepsilon_1\vartheta(t)\Phi'(t) + \varepsilon_2\vartheta(t)I'(t) + \vartheta(t)\psi'(t) \\ &\quad + \varepsilon_1\vartheta'(t)\Phi(t) + \varepsilon_2\vartheta'(t)I(t) + \vartheta'(t)\psi(t) \\ &\le -\vartheta(t)\left(\mu_0 - \delta - \varepsilon_1 + \frac{1+\varepsilon_1}{2}\frac{\vartheta'(t)}{\vartheta(t)}\right)\|u'(t)\|^2_{L^2_g(\mathbb{R}^n)} + \vartheta(t)\left(\frac{K}{2} - \frac{\mu(0)c_*}{4\delta\lambda_1}\right)(\mu'\circ\Delta_x u)(t) \\ &\quad - \vartheta(t)\left[-\varepsilon_1\frac{\vartheta'(t)c_*}{2\lambda_1\vartheta(t)} + \frac{l}{2}\varepsilon_1 - \left(\delta(1-l)^2 + 3\delta\right)\right]\|\Delta_x u(t)\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + \vartheta(t)\left[C_1\varepsilon_1\vartheta(0) + C_4 l_0 + \frac{\vartheta'(t)c_*}{2\lambda_1\vartheta(t)}l_0\right](\mu\circ\Delta_x u)(t) - \varepsilon_2(1-d)\vartheta(t)I(t)\end{aligned}$$

$$
\begin{aligned}
&-\left[Kc+\frac{(1-d)\varepsilon_2}{\nu_1}e^{-2\nu_1}\alpha_1-\vartheta(0)(\varepsilon_1C_3+\delta)\right]\int_{\mathbb{R}^n}g(x)z^2(x,1,t)dx\\
&-\left(Kc-\frac{\varepsilon_2\alpha_2}{\nu_0}-\vartheta(t)(\varepsilon_1C_2+\delta)\right)\int_{\mathbb{R}^n}g(x)u'^2(t)dx\\
&+\vartheta(t)\left(\varepsilon_2\frac{\vartheta'(t)}{2\vartheta(t)}\right)\int_{\mathbb{R}^n}\int_0^1 g(x)z^2(x,\varrho,t)d\varrho dx. \qquad (3.3.28)
\end{aligned}
$$

Choosing $\varepsilon_1>0$ small enough so that

$$\varepsilon_1<\frac{1}{2}.$$

Picking $\delta>0$ for any fixed $\varepsilon_1>0$ such that

$$\frac{l}{2}\varepsilon_1-\left(\delta(1-l)^2+3\delta\right)>\frac{l}{4}\varepsilon_1,$$

and

$$\mu_0-\delta-\varepsilon_1>\frac{1}{4}\mu_0.$$

We can now choose $K>0$ so that (3.3.26) hold, where

$$\frac{K}{2}-\frac{\mu(0)}{4\delta\lambda_1}>0,\quad Kc-\frac{\varepsilon_2\alpha_2}{\nu_0}>0,$$

and

$$Kc+\frac{(1-d)\varepsilon_2}{\nu_1}e^{-2\nu_1}\alpha_1-\vartheta(0)(\varepsilon_1C_3+\delta)>0.$$

Then from (3.3.28) and by the fact that

$$\lim_{t\to+\infty}\frac{\vartheta'(t)}{\xi(t)\vartheta(t)}=0,$$

there exist two positive constant C_5 and C_6 such that for any $t\geq t_0$,

$$\mathcal{L}'(t)\leq -C_5\vartheta(t)E(t)+C_6\vartheta(t)(\mu\circ\Delta_x u)(t). \qquad (3.3.29)$$

Multiplying (3.3.29) by $\xi(t)$ and using

$$\xi(t)(\mu \circ \Delta_x u) \leq -\mu' \circ \Delta_x u) \leq -2E'(t),$$

to get

$$\begin{aligned} \xi(t)\mathcal{L}'(t) &\leq -C_5\vartheta(t)\xi(t)E(t) + C_6\vartheta(t)\xi(t)(\mu \circ \Delta_x u)(t) \\ &\leq -C_5\vartheta(t)\xi(t)E(t) - 2C_6\vartheta(t)\xi(t)E'(t). \end{aligned} \tag{3.3.30}$$

this implies

$$\xi(t)\mathcal{L}'(t) + 2C_6\vartheta(t)\xi(t)E'(t) \leq -C_5\vartheta(t)\xi(t)E(t). \tag{3.3.31}$$

Using again (3.3.26), to get

$$\mathcal{E}(t) = \xi(t)\mathcal{L}(t) + 2C_6\vartheta(t)\xi(t)E(t), \quad \mathcal{E}(t) \sim E, \tag{3.3.32}$$

By using (3.3.31) and the fact that $\xi'(t) \leq 0$, we get for any $t \geq t_0$, that

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)Exp\left(-C_5\int_{t_0}^{t}\vartheta(\tau)\xi(\tau)d\tau\right). \tag{3.3.33}$$

This completes the proof. □

Chapter 4

Local existence and Global nonexistence of solution for Love-equation with infinite memory

4.1 Introduction

Many interesting physical phenomena in which delay effects occur (e.g., population dynamics) can be modeled by partial differential equations with finite or infinite visco-elastic memory which provides a typical damping mechanism in nature. The well-posedness and stability for elasticity and visco-elasticity systems attracted lots of interests in recent years, where different types of dissipative mechanisms have been introduced to obtain diverse results. In general, the stability properties of visco-elastic system are in dependence on the form of the convolution kernel (see in this direction results in, [27], [25], [47], [49], [64], [65], ...). The blow up is an essential and very important phenomena to be studied in the evolution PDEs, there is a difference between global nonexistence which means that the local solution can't be continued to exist in time i.e. there exists a finite time blow up which is our case in this thesis and the blowing up $\forall t > 0$, that is the solution goes to infinity for all $t > 0$. An effect of blow up occurs, for example, when a sea wave tumbles to the shore, when a computer breaks down as a result of electrical breakdown, when a nuclear bomb explodes and in a number of other interesting physical phenomena (see [48], [50], [55], [62], [63], [64], [66], [68], ...).

4.1.1 Formulation of problem

Denote $u = u(x,t)$, $u' = u_t = \frac{\partial u}{\partial t}(x,t)$, $u'' = u_{tt} = \frac{\partial^2 u}{\partial t^2}(x,t)$, $\nabla u = u_x = \frac{\partial u}{\partial x}(x,t)$ and $\Delta u = u_{xx} = \frac{\partial^2 u}{\partial x^2}(x,t)$. In this chapter, we consider the Love-equation in the form

$$\begin{aligned} &u'' - \big(\lambda_0 u_x + \lambda_1 u'_x + u''_x\big)_x + \lambda \int_{-\infty}^{t} g(t-s) u_{xx}(s)\, ds \\ &= F_1[u] - \Big(F_2[u]\Big)_x + f(x,t), \quad x \in \Omega \equiv (0,1),\ 0 < t < T, \end{aligned} \tag{4.1.1}$$

where

$$F_k[u] = F_k\Big(x, t, u, u_x, u', u'_x\Big) \in C^1\Big([0,1] \times \mathbb{R}^+ \times \mathbb{R}^4\Big) \tag{4.1.2}$$

for $k = 1, 2$ and λ, λ_0, $\lambda_1 > 0$ are constants. The given functions g, f are specified later. With $F_k = F_k(x, t, y_1, \dots, y_4)$, we put $D_1 F_k = \frac{\partial F_k}{\partial x}$, $D_2 F_k = \frac{\partial F_k}{\partial t}$ and $D_{i+2} F_k = \frac{\partial F_k}{\partial y_i}$ with $i = 1, \dots, 4$

and $k = 1, 2$. Equation (4.1.1) satisfies the homogeneous Dirichlet boundary conditions:

$$u(0,t) = u(1,t) = 0, \qquad 0 < t < T, \tag{4.1.3}$$

and the following initial conditions

$$u(x,-t) = u_0(x,t), \quad u'(x,0) = u_1(x), \qquad t > 0. \tag{4.1.4}$$

To deal with a wave equation with infinite history, we assume that the kernel function g satisfies the following hypothesis:
(Hyp1:) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing C^1 function such that

$$\lambda_0 - \lambda \int_0^\infty g(s)\, ds = l > 0, \quad g(0) > 0. \tag{4.1.5}$$

We need the following assumptions on source forces:

(Hyp2:) $u_0(0), u_1 \in H_0^1 \cap H^2$;

(Hyp3:) $f \in H^1((0,1) \times (0,T))$;

(Hyp4:) $F_k \in C^1\Big([0,1] \times [0,T] \times \mathbb{R}^4\Big)$ such that

$$F_k(0,t,0,y_2,0,y_4) = F_k(1,t,0,y_2,0,y_4) = 0$$

for all $k = 1, 2$, $t \in [0,T]$ and $y_2, y_4 \in \mathbb{R}$.

4.1.2 Bibliographical notes

Now, we start our literature review concerning visco-elastic problems with the pioneer work of Dafermos [53], where the author considered a one-dimensional visco-elastic problem

$$\rho u'' = c u_{xx} - \int_{-\infty}^{t} g(t-s) u_{xx}\, ds,$$

established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as t goes to infinity. In [57], Hrussa considered a one-dimensional nonlinear visco-elastic equation

$$u'' = cu_{xx} - \int_0^t m(t-s)(\phi(u_x))_x \, ds + f$$

and proved several global existence results for large data. Here, the author also obtained a decay rate of solution.

Concerning problems with infinite history, we mention the work [54] in which considered the following semi-linear hyperbolic equation, in a bounded domain of $\mathbb{R}^3$,

$$u'' - K(0)\Delta u - \int_0^\infty K'(s)\Delta u(t-s)ds + g(u) = f$$

with $K(0)$, $K(\infty) > 0$, $K' \leq 0$ and gave the existence of global attractors for the problem. Next in [64], the authors proved that the solutions of a system of wave equations with visco-elastic term, degenerate damping and strong nonlinear sources acting in both equations at the same time are globally non-existing provided that the initial data are sufficiently large in a bounded domain, the initial energy is positive and the strongly nonlinear functions f_1 and f_2 located in the sources satisfy an appropriate conditions. The authors concentrate their studies on the role of the nonlinearities of sources. After that, in [59], the authors considered a fourth-order suspension bridge equation with nonlinear damping term and source term

$$u'' + \Delta^2 u + au + |u_t|^{m-2}u_t = |u|^{p-2}u$$

The authors gave necessary and sufficient condition for global existence and energy decay results without considering the relation between m and p. Moreover, when $p > m$, they gave sufficient condition for finite time blow-up of solutions. The lower bound of the blow-up time is also established.

Recently, in [66], the authors studied a three-dimensional (3D) visco-elastic wave equation with nonlinear weak damping, supercritical sources and prescribed past history for $t \leq 0$ in

$$u'' - k(0)\Delta u - \int_0^\infty k'(s)\Delta u(t-s)\, ds + |u'|^{m-1}u' = |u|^{p-1}u,$$

where the relaxation function k is monotone decreasing with $k(+\infty) = 1$, $m \geq 1$ and $1 \leq p < 6$. When the source is stronger than dissipations, i.e. $p > \max\{m, \sqrt{k(0)}\}$, they obtained some finite time blow-up results with positive initial energy. In particular, they obtained the existence of certain solutions which blow up in finite time for initial data at arbitrary energy level. In [56], the abstract thermo-elastic system is considered,

$$\begin{cases} u'' + Au + Bu' - \int_0^\infty g(s)u_{xx}(t-s)\, ds - A^\alpha\theta = 0, \\ \theta' + kA^\beta\theta + A^\alpha u' = 0, \\ u(-t) = u_0(t), u'(0) = u_1, \quad \theta(0) = \theta_0 \end{cases}$$

in which u is the displacement vector, θ is the temperature difference, $\alpha \in [0,1)$ and $\beta \in (0,1]$ are constants. H is a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the related norm $\|\cdot\|$. The operators $A : D(A) \to H$ and $B : D(B) \to H$ are self-adjoint linear positive definite operators. Under suitable conditions on the order of the coupling, the memory kernel function and the initial values, the well-posedness and the general decay rate of solution are given by semigroup theory and perturbed energy functional technique to allow a wider thermo-elastic systems.

Without infinite memory term, when $\lambda = 0$ in (4.1.1), Triet and his collaborators in [67] considered an IBVP for a nonlinear Kirchhoff-Love equation

$$\begin{aligned} & u_{tt} - \frac{\partial}{\partial x}\left[B\big(x,t,u,\|u(t)\|^2,\|u_x(t)\|^2,\|u_t(t)\|^2,\|u_{xt}(t)\|^2\big)\big(u_x + \lambda_1 u_{xt} + u_{xtt}\big)\right] \\ & + \lambda u_t = F\big(x,t,u,u_x,u_t,u_{xt},\|u(t)\|^2,\|u_x(t)\|^2,\|u_t(t)\|^2,\|u_{xt}(t)\|^2\big) \\ & \quad - \frac{\partial}{\partial x}\left[G\big(x,t,u,u_x,u_t,u_{xt},\|u(t)\|^2,\|u_x(t)\|^2,\|u_t(t)\|^2,\|u_{xt}(t)\|^2\big)\right] \\ & \quad + f(x,t), \quad x \in \Omega = (0,1),\ 0 < t < T, \end{aligned}$$

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x),$$

where $\lambda > 0$, $\lambda_1 > 0$ are constants and $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$, f, F and G are given functions. By applying the Faedo-Galerkin method, the authors proved existence and uniqueness of a solution and by constructing Lyapunov functional, they proved a blow-up of the solution with a negative initial energy and established a sufficient condition for the exponential decay of weak solutions. This chapter is organized as follows: In the second section, owing to the nonlinearities, we combine three techniques to prove the local existence of unique weak solution in Theorem 10. In the third section, the blow up result with negative initial energy is obtained in Theorem 11 under certain conditions on the sources and the function g. It is not surprising that this work is inspired from [60], [61], [64] and [67].

4.2 Existence of weak solution

The weak formulation

We define in the following, the weak solution of (4.1.1)–(4.1.4).

Definition 8. *A function u is said to be weak solution of (4.1.1)–(4.1.4) on $[0,T]$ if there exists*

$$u, u', u'' \in L^\infty(0,T; H_0^1 \cap H^2)$$

satisfying the variational equation

$$\begin{aligned}
&\int_0^1 u'' w \, dx + \int_0^1 (\lambda_0 u_x + \lambda_1 u'_x + u''_x) w_x \, dx \\
&- \lambda \int_0^1 \int_0^\infty g(s) u_x(t-s) \, ds w_x \, dx \\
&= \int_0^1 f w \, dx + \int_0^1 F_1[u] w \, dx + \int_0^1 F_2[u] w_x \, dx
\end{aligned}$$

for all test functions $w \in H_0^1$ and for almost all $t \in (0,T)$.

The following famous and widely used technical Lemma will play an important role in the sequel.

Lemma 18. *For any $v \in C^1(0,T;H_0^1)$, we have*

$$\begin{aligned}
&\int_0^1\int_0^\infty g(s)v_{xx}(t-s)v'(t)\,ds\,dx \\
=\ & \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty g(s)\int_0^1 |v_x(t-s)-v_x(t)|^2\,dx\,ds - \int_0^\infty g(s)\,ds\int_0^1 |v_x(t)|^2\,dx\right) \\
&-\frac{1}{2}\int_0^\infty g'(s)\int_0^1 |v_x(t-s)-v_x(t)|^2\,dx\,ds
\end{aligned}$$

Proof. It's not hard to see

$$\begin{aligned}
\int_0^1\int_0^\infty g(s)v_{xx}(t-s)v'(t)\,ds\,dx &= \int_0^1\int_{-\infty}^t g(t-s)v_{xx}(s)v'(t)\,ds\,dx \\
&= -\int_{-\infty}^t g(t-s)\int_0^1 v_x(s)v_x'(t)\,dx\,ds \\
&= -\int_{-\infty}^t g(t-s)\int_0^1 v_x'(t)\left(v_x(s)-v_x(t)\right)\,dx\,ds \\
&\quad -\int_{-\infty}^t g(t-s)\,ds\int_0^1 v_x'(t)v_x(t)\,dx.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_0^1\int_0^\infty g(s)v_{xx}(t-s)v'(t)\,ds\,dx &= \frac{1}{2}\int_{-\infty}^t g(t-s)\frac{d}{dt}\int_0^1 |v_x(s)-v_x(t)|^2\,dx\,ds \\
&\quad -\frac{1}{2}\int_{-\infty}^t g(t-s)\,ds\left(\frac{d}{dt}\int_0^1 |v_x(t)|^2\,dx\right) \\
&= \frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^t g(t-s)\int_0^1 |v_x(s)-v_x(t)|^2\,dx\,ds\right) \\
&\quad -\frac{1}{2}\frac{d}{dt}\left(\int_0^\infty g(s)\,ds\int_0^1 |v_x(t)|^2\,dx\right) \\
&\quad -\frac{1}{2}\int_{-\infty}^t g'(t-s)\int_0^1 |v_x(s)-v_x(t)|^2\,dx\,ds \\
&= \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty g(s)\int_0^1 |v_x(t-s)-v_x(t)|^2\,dx\,ds\right) \\
&\quad -\frac{1}{2}\frac{d}{dt}\left(\int_0^\infty g(s)\,ds\int_0^1 |v_x(t)|^2\,dx\right) \\
&\quad -\frac{1}{2}\int_0^\infty g'(s)\int_0^1 |v_x(t-s)-v_x(t)|^2\,dx\,ds.
\end{aligned}$$

First main Theorem

Various existence and uniqueness, as well as Faedo-Galerkin method, have been obtained in the last decades for nonlinear IBVPs in Sobolev spaces (see [51], [55], [58], ...). Now, we consider the existence of a local solution for (4.1.1)–(4.1.4), with $\lambda, \lambda_0, \lambda_1 > 0$.

Theorem 10. *Let $u_0(0), u_1 \in H_0^1 \cap H^2$ be given. Assume that* (Hyp1)–(Hyp4) *hold. Then Problem* (4.1.1)–(4.1.4) *has a unique local solution u and*

$$u, u', u'' \in L^\infty(0, T_*; H_0^1 \cap H^2) \tag{4.2.1}$$

for some $T_* > 0$ small enough.

Proof. In the first step of this proof, we use linearization method for a nonlinear term to construct a linear recurrent sequence $\{u_m\}$. Then, the Faedo-Galerkin method combined with the weak compactness method shows that $\{u_m\}$ converges to u which is exactly a unique local solution of (4.1.1)–(4.1.4).

Step 1. Let $T > 0$ and $M > 0$ be fixed. For $k = 1, 2$ and $\Omega = (0, 1)$, We put

$$\|F_k\|_{C^0(\overline{\Omega}\times[0,T]\times[-M,M]^4)} = \sup_{(x,t,y_1,\dots,y_4)\in\overline{\Omega}\times[0,T]\times[-M,M]^4} |F_k(x, t, y_1, \dots, y_4)|$$

and

$$\begin{aligned} \bar{F}_M &= \sum_{k=1}^{2} \|F_k\|_{C^1(\overline{\Omega}\times[0,T]\times[-M,M]^4)} \\ &= \sum_{k=1}^{2} \|F_k\|_{C^0(\overline{\Omega}\times[0,T]\times[-M,M]^4)} + \sum_{k=1}^{2}\sum_{i=1}^{6} \|D_i F_k\|_{C^0(\overline{\Omega}\times[0,T]\times[-M,M]^4)}. \end{aligned}$$

For some $T_* \in (0, T]$ and $M > 0$, we put

$$\begin{aligned} W(M, T_*) = \Big\{ & v, v' \in L^\infty(0, T_*; H_0^1 \cap H^2) : v'' \in L^\infty(0, T_*; H_0^1), \\ & \text{with } \|v\|_{L^\infty(0,T_*;H_0^1\cap H^2)}, \\ & \|v'\|_{L^\infty(0,T_*;H_0^1\cap H^2)},\ \|v''\|_{L^\infty(0,T_*;H_0^1)} \le M \Big\}, \end{aligned}$$

$$W_1(M,T_*) = \{v \in W(M,T_*) : v'' \in L^\infty(0,T_*; H_0^1 \cap H^2)\}.$$

Here we adopt the norm in $H_0^1 = H_0^1(\Omega)$ and $H^2 \cap H_0^1 = H^2(\Omega) \cap H_0^1(\Omega)$ as

$$\|v\|_{H_0^1} = \|\nabla v\|_{L^2} \quad \text{and} \quad \|u\|_{H^2 \cap H_0^1} = \|\Delta u\|_{L^2},$$

respectively, where $\|\cdot\|_{L^2}$ denotes the standard L^2 norm in Ω. To establish the linear recurrent sequence $\{u_m\}$, we choose $u_0(t) \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T_*) \tag{4.2.2}$$

and associate with problem (4.1.1)–(4.1.4) the following problem.
Find $u_m \in W_1(M,T_*)$ $(m \geq 1)$ which satisfies

$$\begin{aligned}
&\int_0^1 u_m'' w\,dx + \int_0^1 \Big(\lambda_0 \nabla u_m + \lambda_1 \nabla u_m' + \nabla u_m''\Big) \nabla w\,dx \\
&- \lambda \int_0^\infty g(s)\,ds \int_0^1 \nabla u_m \nabla w\,dx - \lambda \int_0^\infty g(s) \int_0^1 (\nabla u_m(t-s) - \nabla u_m)\nabla w\,dx\,ds \\
&= \int_0^1 f w dx + \int_0^1 F_{1,m}[u] w\,dx + \int_0^1 F_{2,m}[u] \nabla w\,dx, \quad \forall w \in H_0^1, \\
&u_m(-t) = u_0(t), \quad u_m'(0) = u_1, t \in [0,T],
\end{aligned} \tag{4.2.3}$$

where we denote $\nabla u = u_x$ and

$$F_{k,m}[u] = F_k[u_{m-1}] = F_k\Big(x,t,u_{m-1},\nabla u_{m-1},u_{m-1}',\nabla u_{m-1}'\Big) \tag{4.2.4}$$

for $k = 1,2$.

Proposition 1. *Let $u_0(0), u_1 \in H_0^1 \cap H^2$ be given and the first term of sequence $u_0(t) \equiv 0$. Assume that* (Hyp1)–(Hyp4) *hold. Then there exist positive constants M, $T_* > 0$ such that there exists a recurrent sequence $\{u_m\} \subset W_1(M,T_*)$ defined by* (4.2.2)–(4.2.4).

Proof. We use the standard Faedo-Galerkin method to prove our result. Consider a special orthonormal basis $\{w_j\}_{j=1}^\infty$ on H_0^1, formed by the eigenfunctions of the operator $-\frac{\partial^2}{\partial x^2}$.

Let $V_k = span\{w_1, w_2, \dots, w_k\}$ and the projections of the history and initial data on the finite-dimensional subspace V_k are given by

$$u_{0k}(t) = \sum_{j=1}^{k} \alpha_j^{(k)}(t) w_j,$$
$$u_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j,$$

where

$$\alpha_j^{(k)}(t) = \int_0^1 u_0(t) w_j \, dx,$$
$$\beta_j^{(k)}(t) = \int_0^1 u_1 w_j \, dx.$$

We seek k functions $\varphi_{mj}^{(k)}(t) \in C^2[0,T]$, $1 \le j \le k$, such that the expression in form

$$u_m^{(k)} = \sum_{j=1}^{k} \varphi_{mj}^{(k)} w_j$$

solves the problem

$$\begin{aligned}
&\int_0^1 u_m^{''(k)} w_j dx + \int_0^1 \left(\lambda_0 \nabla u_m^{(k)} + \lambda_1 \nabla u_m^{'(k)} + \nabla u_m^{''(k)}\right) \nabla w_j \, dx \\
&-\lambda \int_0^\infty g(s)\, ds \int_0^1 \nabla u_m^{(k)} \nabla w_j \, dx \\
&-\lambda \int_0^\infty g(s) \int_0^1 \left(\nabla u_m^{(k)}(t-s) - \nabla u_m^{(k)}(t)\right) \nabla w_j \, dx \, ds \\
&= \int_0^1 f w_j dx + \int_0^1 F_{1,m} w_j \, dx + \int_0^1 F_{2,m} \nabla w_j \, dx, \quad 1 \le j \le k, \\
&u_m^{(k)}(-t) = u_{0k}(t), \quad u_m^{'(k)}(0) = u_{1k},
\end{aligned} \tag{4.2.5}$$

in which

$$\begin{aligned}
u_{0k}(t) &\to u_0(t) \quad \text{strongly in } H_0^1 \cap H^2, \\
u_{1k} &\to u_1 \quad \text{strongly in } H_0^1 \cap H^2.
\end{aligned} \tag{4.2.6}$$

This leads to a system of ODEs for unknown functions $\varphi_{mj}^{(k)}$. Based on standard existence theory for ODE, System (4.2.5) admits a unique solution $\varphi_{mj}^{(k)}$, $1 \leq j \leq k$ on interval $[0, T]$ by (4.2.2) and the argument in [52].

A priori estimates. The next estimates prove that there exist positive constants M, $T_* > 0$ such that $u_m^{(k)} \in W(M, T_*)$ for all m and k. We partially estimate the terms of the associated energy. Taking $w = u_m^{'(k)}$ in (4.2.5), we get

$$\begin{aligned}
&\int_0^1 u_m^{''(k)} u_m^{'(k)}\, dx + \int_0^1 \left(\lambda_0 \nabla u_m^{(k)} + \lambda_1 \nabla u_m^{'(k)} + \nabla u_m^{''(k)}\right) \nabla u_m^{'(k)}\, dx \\
&-\lambda \int_0^1 \int_0^\infty g(s) \nabla u_m^{(k)}(t-s)\, ds \nabla u_m^{'(k)}\, dx \\
&= \int_0^1 F_1[u_{m-1}^{(k)}] u_m^{'(k)}\, dx + \int_0^1 F_2[u_{m-1}^{(k)}] \nabla u_m^{'(k)}\, dx + \int_0^1 f u_m^{'(k)}\, dx.
\end{aligned} \tag{4.2.7}$$

Using results in Lemma 18, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\int_0^1 \left(|u_m^{'(k)}|^2 + l |\nabla u_m^{(k)}|^2 + |\nabla u_m^{'(k)}|^2 \right) dx \\
&+\lambda \int_0^\infty g(s) \int_0^1 |\nabla u_m^{(k)}(t-s) - \nabla u_m^{(k)}(t)|^2\, dx\, ds \Big] \\
&+2\lambda_1 \int_0^1 |\nabla u_m^{'(k)}|^2\, dx - \lambda \int_0^\infty g'(s) \int_0^1 |\nabla u_m^{(k)}(t-s) - \nabla u_m^{(k)}(t)|^2\, dx\, ds \\
&= 2\int_0^1 F_1[u_{m-1}^{(k)}] u_m^{'(k)}\, dx + 2\int_0^1 F_2[u_{m-1}^{(k)}] \nabla u_m^{'(k)}\, dx + 2\int_0^1 f u_m^{'(k)}\, dx,
\end{aligned} \tag{4.2.8}$$

where l is defined in (4.1.5). Let us denote the integrand on LHS of (4.2.8) as $e^{(k)}(u_m)$, where

$$\begin{aligned}
e^{(k)}(v) &= \int_0^1 \left(|v^{'(k)}|^2 + l |\nabla v^{(k)}|^2 + |\nabla v^{'(k)}|^2 \right) dx + 2\lambda_1 \int_0^t \int_0^1 |\nabla v^{'(k)}|^2\, dx\, ds \\
&+ \lambda \int_0^\infty g(s) \int_0^1 |\nabla v^{(k)}(t-s) - \nabla v^{(k)}(t)|^2\, dx\, ds \\
&- \lambda \int_0^t \int_0^\infty g'(s) \int_0^1 |\nabla v^{(k)}(\tau - s) - \nabla v^{(k)}(\tau)|^2\, dx\, ds\, d\tau.
\end{aligned}$$

We obtain the similar estimates to (4.2.8), whose integrands are given by $e^{(k)}(\nabla u_m)$ and $e^{(k)}(u_m')$, respectively. Put

$$E_m^{(k)}(t) = e^{(k)}(u_m) + e^{(k)}(\nabla u_m) + e^{(k)}(u_m'). \tag{4.2.9}$$

Then

$$
\begin{aligned}
&E_m^{(k)}(t)\\
&= E_m^{(k)}(0) + 2\int_0^t\int_0^1 f(s)u_m^{'(k)}(s)\,dx\,ds + 2\int_0^t\int_0^1 f_x(s)\nabla u_m^{'(k)}(s)\,dx\,ds\\
&\quad + 2\int_0^t\int_0^1 f'(s)u_m^{''(k)}(s)\,dx\,ds + 2\int_0^t\int_0^1 F_{1,m}(s)u_m^{'(k)}(s)\,dx\,ds\\
&\quad + 2\int_0^t\int_0^1 F_{2,m}(s)\nabla u_m^{'(k)}(s)\,dx\,ds + 2\int_0^t\int_0^1 \nabla F_{1,m}(s)\nabla u_m^{'(k)}(s)\,dx\,ds\\
&\quad + 2\int_0^t\int_0^1 \nabla F_{2,m}(s)\Delta u_m^{'(k)}(s)\,dx\,ds + 2\int_0^t\int_0^1 F'_{1,m}(s)u_m^{''(k)}(s)\,dx\,ds\\
&\quad + 2\int_0^t\int_0^1 F'_{2,m}(s)\nabla u_m^{''(k)}(s)\,dx\,ds.
\end{aligned}
\tag{4.2.10}
$$

We need, now, to estimate

$$A_m^{(k)} = \int_0^1 |u_m^{''(k)}(0)|^2\,dx + \int_0^1 |\nabla u_m^{''(k)}(0)|^2\,dx.$$

Let $w_j = u_m^{''(k)}$ in (4.2.5) and integrate, taking $t \to 0_+$ in the first term, to obtain

$$
\begin{aligned}
&\int_0^1 |u_m^{''(k)}(0)|^2\,dx + \int_0^1 |\nabla u_m^{''(k)}(0)|^2\,dx + \int_0^1 \Big(l\nabla u_{0k} + \lambda_1 \nabla u_{1k}\Big)\nabla u_m^{''(k)}(0)\,dx\\
&+\lambda\int_0^\infty g(s)\int_0^1 \left(\nabla u_{0k}(0) - \nabla u_{0k}(-s)\right) u_m^{''(k)}(0)\,dx\,ds\\
&= \int_0^1 f(0)u_m^{''(k)}(0)\,dx + \int_0^1 F_{1,m}(0)u_m^{''(k)}(0)\,dx + \int_0^1 F_{2,m}(0)\nabla u_m^{''(k)}(0)\,dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
A_m^{(k)} \;\le\; & \int_0^1 \Big(-l\nabla u_{0k} - \lambda_1\nabla u_{1k} + F_{2,m}(0)\Big)\nabla u_m^{''(k)}(0)\,dx\\
- \; & \lambda\int_0^\infty g(s)\int_0^1 \left(\nabla u_{0k}(0) - \nabla u_{0k}(-s)\right) u_m^{''(k)}(0)\,dx\,ds\\
+ \; & \int_0^1 f(0)u_m^{''(k)}(0)\,dx + \int_0^1 F_{1,m}(0)u_m^{''(k)}(0)\,dx
\end{aligned}
$$

$$\begin{aligned} A_m^{(k)} &\leq \Big(l\,\|u_{0k}\|_{H_0^1} + \lambda_1\,\|u_{1k}\|_{H_0^1} + \|F_{1,m}(0)\|_{L^2} + \|F_{2,m}(0)\|_{L^2} + \|f(0)\|_{L^2} \\ &+ \lambda \Big\| \int_0^\infty g(s)(u_{0k}(0) - u_{0k}(-s))\, ds \Big\|_{H_0^1} \Big) \Big(A_m^{(k)} \Big)^{1/2} \\ &\leq \Big(l\,\|u_{0k}\|_{H_0^1} + \lambda_1\,\|u_{1k}\|_{H_0^1} + \|F_{1,m}(0)\|_{L^2} + \|F_{2,m}(0)\|_{L^2} + \|f(0)\|_{L^2} \\ &+ \lambda \Big\| \int_0^\infty g(s)(u_{0k}(0) - u_{0k}(-s))\, ds \Big\|_{H_0^1} \Big)^2 \\ &\leq \xi \quad \text{for all } m,\ k, \end{aligned} \tag{4.2.11}$$

because $\|F_{1,m}(0)\|_{L^2}$ and $\|F_{2,m}(0)\|_{L^2}$ are constant independent of m, where ξ is a constant depending only on f, u_0, u_1, F_1, F_2, λ, λ_0, λ_1 and $\int_0^\infty g(s)\,ds$. Equations (4.2.6), (4.2.9) and (4.2.11) imply that

$$\begin{aligned} E_m^{(k)}(0) &= \int_0^1 \Big(|u_{1k}|^2 + l|\nabla u_{0k}|^2 + |\nabla u_{1k}|^2 \Big)\, dx + \lambda \int_0^\infty g(s) \int_0^1 |\nabla u_{0k}(-s) - \nabla u_{0k}(0)|^2\, dx\, ds \\ &+ \int_0^1 \Big(|\nabla u_{1k}|^2 + l|\Delta u_{0k}|^2 + |\Delta u_{1k}|^2 \Big)\, dx + \lambda \int_0^\infty g(s) \int_0^1 |\Delta u_{0k}(-s) - \Delta u_{0k}(0)|^2\, dx\, ds \\ &+ A_m^{(k)} + \int_0^1 l|\nabla u_{1k}|^2\, dx + \lambda \int_0^\infty g(s) \int_0^1 |\nabla u_{1k}(-s) - \nabla u_{1k}(0)|^2\, dx\, ds \\ &\leq \xi_0 \quad \text{for all } m,\ k \in \mathbb{N}, \end{aligned}$$

where ξ_0 is also a constant depending only on f, u_0, u_1, F_1, F_2, λ, λ_0, λ_1 and $\int_0^\infty g(s)\,ds$. We then now estimate the other terms of (4.2.10). By (4.2.9), (4.2.10) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &E_m^{(k)}(t) \leq \xi_0 + \|f\|^2_{L^2(\Omega\times(0,T))} + \|f_x\|^2_{L^2(\Omega\times(0,T))} + \|f'\|^2_{L^2(\Omega\times(0,T))} \\ &+ 2\int_0^t \int_0^1 |u_m^{\prime(k)}|^2\, ds\, dx + 2\int_0^t \int_0^1 |\nabla u_m^{\prime(k)}|^2\, dx\, ds + 2\int_0^t \int_0^1 |u_m^{\prime\prime(k)}|^2\, dx\, ds \\ &+ \int_0^t \int_0^1 |\nabla u_m^{\prime(k)}|^2\, dx\, ds + \int_0^t \int_0^1 |\Delta u_m^{\prime(k)}|^2\, dx\, ds + \int_0^t \int_0^1 |\nabla u_m^{\prime\prime(k)}|^2\, dx\, ds \\ &+ \int_0^t \int_0^1 |F_{1,m}|^2\, dx\, ds + \int_0^t \int_0^1 |F_{2,m}|^2\, dx\, ds + \int_0^t \int_0^1 |\nabla F_{1,m}|^2\, dx\, ds \\ &+ \int_0^t \int_0^1 |\nabla F_{2,m}|^2\, dx\, ds + \int_0^t \int_0^1 \left|F'_{1,m}\right|^2\, dx\, ds + \int_0^t \int_0^1 \left|F'_{2,m}\right|^2\, dx\, ds \end{aligned}$$

Then,

$$E_m^{(k)}(t) \leq \xi_0 + \|f\|^2_{H^1(\Omega\times(0,T))} + 2\int_0^t E_m^{(k)}(s)\,ds + T_*\bar{F}_M^2 + \int_0^t \int_0^1 |\nabla F_{1,m}|^2 \,dx\,ds$$
$$+ \int_0^t \int_0^1 |\nabla F_{2,m}|^2 \,dx\,ds + \int_0^t \int_0^1 \left|F'_{1,m}\right|^2 \,dx\,ds + \int_0^t \int_0^1 \left|F'_{2,m}\right|^2 \,dx\,ds.$$

Remarking from (4.1.2)

$$\begin{aligned}\nabla F_{k,m}(t) &= D_1F_k[u_{m-1}] + D_3F_k[u_{m-1}]\nabla u_{m-1} + D_4F_k[u_{m-1}]\Delta u_{m-1} \\ &\quad + D_5F_k[u_{m-1}]\nabla u'_{m-1} + D_6F_k[u_{m-1}]\Delta u'_{m-1}\end{aligned}$$

and

$$\begin{aligned}F'_{k,m}(t) &= D_2F_k[u_{m-1}] + D_3F_k[u_{m-1}]u'_{m-1} + D_4F_k[u_{m-1}]\nabla u'_{m-1} \\ &\quad + D_5F_k[u_{m-1}]u''_{m-1} + D_6F_k[u_{m-1}]\nabla u''_{m-1}\end{aligned}$$

for $k = 1, 2$, then we have

$$E_m^{(k)}(t) \leq \xi_0 + \|f\|^2_{H^1(\Omega\times(0,T))} + T_*\left[1 + 4(1+M)^2\right]\bar{F}_M^2 + 2\int_0^t E_m^{(k)}(s)\,ds.$$

We choose $M > 0$ sufficiently large such that

$$\xi_0 + \|f\|^2_{H^1(\Omega\times(0,T))} \leq \frac{1}{2}M^2$$

and then choose $T_* \in (0, T]$ small enough such that

$$\left(\frac{1}{2}M^2 + T_*\left[1 + 4(1+M)^2\right]\bar{F}_M^2\right)\exp[2T_*] \leq \min(1,l)M^2$$

and

$$k_{T_*} = 4\sqrt{2}\bar{F}_M\sqrt{T_*\exp[T_*]\min(1,l)^{-1}} < 1, \tag{4.2.12}$$

Then we have

$$E_m^{(k)}(t) \leq \min(1,l)\exp[-2T_*]M^2 + 2\int_0^t E_m^{(k)}(s)\,ds.$$

Finally by Gronwall's Lemma, we obtain

$$E_m^{(k)}(t) \leq \min(1,l)M^2 \text{ for } t \in (0,T_*) \tag{4.2.13}$$

and hence

$$u_m^{(k)} \in W(M,T_*) \text{ for all } m \text{ and } k.$$

Pass to the limit.

By (4.2.9) and (4.2.13), there exists a subsequence of $\{u_m^{(k)}\}$ such that

$$\begin{aligned} u_m^{(k)} &\to u_m \quad \text{in } L^\infty(0,T_*;H_0^1\cap H^2) \text{ weakly*},\\ u_m^{\prime(k)} &\to u_m' \quad \text{in } L^\infty(0,T_*;H_0^1\cap H^2) \text{ weakly*},\\ u_m^{\prime\prime(k)} &\to u_m'' \quad \text{in } L^\infty(0,T_*;H_0^1) \text{ weakly*},\\ u_m &\in W(M,T_*). \end{aligned} \tag{4.2.14}$$

Passing to limit in (4.2.5) and (4.2.6), it is clear to see that u_m is satisfying (4.2.3) and (4.2.4) in $L^2(0,T_*)$. Furthermore, (4.2.3) and $(4.2.14)_4$ imply that

$$\begin{aligned} \left(u_m''\right)_{xx} &= -\Big(\lambda_0 u_m + \lambda_1 u_m' - \lambda\int_0^\infty g(s)u_m(t-s)\,ds\Big)_{xx}\\ &\quad + u_m'' - F_1[u_m] - \Big(F_2[u_m]\Big)_x - f\\ &\equiv \Psi_m \in L^\infty(0,T_*;L^2). \end{aligned}$$

We deduce that, if $u_m \in L^\infty(0,T_*;H_0^1\cap H^2)$, then $u_m', u_m'' \in L^\infty(0,T_*;H_0^1\cap H^2)$. So we obtain $u_m \in W_1(M,T_*)$. This completes the proof of Proposition 1. □

Step 2. Let the Banach space

$$W_1(T_*) = \{v \in L^\infty(0,T_*;H_0^1) : v' \in L^\infty(0,T_*;H_0^1)\},$$

with respect to the norm

$$\|v\|_{W_1(T_*)} = \|v\|_{L^\infty(0,T_*;H_0^1)} + \|v'\|_{L^\infty(0,T_*;H_0^1)}.$$

We will show the convergence of $\{u_m\}$ to the solution of our problem in the next Lemma.

Lemma 19. *Let* (Hyp1)–(Hyp4) *hold. Then*

(i) Problem (4.1.1)–(4.1.4) *has a unique weak solution* $u \in W_1(M, T_*)$, *where* $M > 0$ *and* $T_* > 0$ *are chosen constants as in Proposition 1.*

(ii) The linear recurrent sequence $\{u_m\}$ *defined by* (4.2.2)–(4.2.4) *converges to the solution* u *of* (4.1.1)–(4.1.4) *strongly in the space* $W_1(T_*)$.

Proof. We use the result obtained in Proposition 1 and the compact embedding Theorems.

Existence. We proved that $\{u_m\}$ is a Cauchy sequence in $W_1(T_*)$. In order to do this, let $w_m = u_{m+1} - u_m$. Then w_m satisfies

$$\begin{aligned}
&\int_0^1 w_m'' w\,dx + \int_0^1 (\lambda_0 \nabla w_m + \lambda_1 \nabla w_m' + \nabla w_m'')\nabla w\,dx \\
&- \lambda \int_0^1 \int_0^\infty g(s) \nabla w_m(t-s)\,ds \nabla w\,dx \\
&= \int_0^1 \Big(F_{1,m+1}[u] - F_{1,m}[u]\Big) w\,dx + \int_0^1 \Big(F_{2,m+1}[u] - F_{2,m}[u]\Big)\nabla w\,dx, \\
&w_m(0) = w_m'(0) = 0.
\end{aligned} \tag{4.2.15}$$

Considering (4.2.15) with $w = w_m'$, and then integrating in t, we obtain by result in Lemma 18

$$\begin{aligned}
&\int_0^1 \Big(|w_m'|^2 + l|\nabla w_m|^2 + |\nabla w_m'|^2\Big)\,dx \\
&+\lambda \int_0^\infty g(s) \int_0^1 |\nabla w_m(t-s) - \nabla w_m(t)|^2\,dx\,ds + 2\lambda_1 \int_0^t \int_0^1 |\nabla w_m'|^2\,dx\,ds \\
&-\lambda \int_0^t \int_0^\infty g'(s) \int_0^1 |\nabla w_m(\tau - s) - \nabla w_m(\tau)|^2\,dx\,ds\,d\tau \\
&= 2\int_0^t \int_0^1 \Big(F_{1,m+1}(s) - F_{1,m}(s)\Big) w_m'(s)\,dx\,ds + 2\int_0^t \int_0^1 \Big(F_{2,m+1}(s) - F_{2,m}(s)\Big)\nabla w_m'(s)\,dx\,ds.
\end{aligned}$$

By the regularity (Hyp4), (4.2.2) and (4.2.14), we have, for $k = 1, 2$,

$$\begin{aligned}
&\int_0^1 |F_{k,m+1}(s) - F_{k,m}(s)|^2\,dx \\
&\le \bar{F}_M^2 \int_0^1 \left(|w_{m-1}| + |\nabla w_{m-1}| + \left|w'_{m-1}\right| + \left|\nabla w'_{m-1}\right|\right)^2\,dx \\
&\le 8\bar{F}_M^2 \left(\int_0^1 |\nabla w_{m-1}|^2\,dx + \int_0^1 \left|\nabla w'_{m-1}\right|^2\,dx\right) \\
&\le 16\bar{F}_M^2 \|w_{m-1}\|^2_{W_1(T_*)}
\end{aligned} \tag{4.2.16}$$

owing to Poincaré inequality

$$\|v\|_{L^2} \le \|v\|_{H_0^1}$$

for $v \in H_0^1(\Omega)$, then

$$\begin{aligned}
E_m(t) &\le 32T_*\bar{F}_M^2\|w_{m-1}\|^2_{W_1(T_*)} + \int_0^t\int_0^1 |w'_m|^2\,dx\,ds + \int_0^t\int_0^1 |\nabla w'_m|^2\,dx\,ds \\
&\le 32T_*\bar{F}_M^2\|w_{m-1}\|^2_{W_1(T_*)} + \int_0^t E_m(s)ds,
\end{aligned} \tag{4.2.17}$$

where

$$\begin{aligned}
&E_m(t) = \int_0^1 \left(|w'_m|^2 + l|\nabla w_m|^2 + |\nabla w'_m|^2\right)dx \\
&+\lambda\int_0^\infty g(s)\int_0^1 |\nabla w_m(t-s) - \nabla w_m(t)|^2\,dx\,ds + 2\lambda_1\int_0^t\int_0^1 |\nabla w'_m|^2\,dx\,ds \\
&-\lambda\int_0^t\int_0^\infty g'(s)\int_0^1 |\nabla w_m(\tau - s) - \nabla w_m(\tau)|^2\,dx\,ds\,d\tau.
\end{aligned}$$

Thanks to Gronwall's Lemma and (4.2.17), we get

$$E_m(t) \le 32\bar{F}_M^2 T_* e^{T_*}\|w_{m-1}\|^2_{W_1(T_*)}$$

so

$$\|w_m\|_{W_1(T_*)} \le k_{T_*}\|w_{m-1}\|_{W_1(T_*)} \quad \forall m \in \mathbb{N}$$

by (4.2.12). Thus we have

$$\|u_m - u_{m+p}\|_{W_1(T_*)} \leq 4M(1-k_{T_*})^{-1}k_{T_*}^m, \quad \forall m, p \in \mathbb{N}.$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T_*)$, so there exists $u \in W_1(T_*)$ such that

$$u_m \to u \text{ strongly in } W_1(T_*). \tag{4.2.18}$$

Note that $u_m \in W_1(M, T_*)$, so there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{gathered} u_{m_j} \to u \quad \text{in } L^\infty(0,T_*; H_0^1 \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' \quad \text{in } L^\infty(0,T_*; H_0^1 \cap H^2) \text{ weakly*}, \\ u''_{m_j} \to u'' \quad \text{in } L^\infty(0,T_*; H_0^1) \text{ weakly*}, \\ u \in W(M, T_*). \end{gathered} \tag{4.2.19}$$

In the same way as (4.2.16), we obtain

$$\|F_{k,m}(t) - F_k[u](t)\|_{L^2}^2 \leq 16\bar{F}_M^2 \|u_{m-1} - u\|_{W_1(T_*)}^2 \tag{4.2.20}$$

for $k = 1, 2$. Then (4.2.18) and (4.2.20) imply

$$F_{k,m} \to F_k[u] \quad \text{strongly in } L^\infty(0,T_*; L^2), \tag{4.2.21}$$

Let us passing to limit in (4.2.3) and (4.2.4) as $m = m_j \to \infty$ by (4.2.18), (4.2.19) and (4.2.21), there exists $u \in W(M, T_*)$ satisfying

$$\begin{aligned} &\int_0^1 u'' w\, dx + \int_0^1 (\lambda_0 u_x + \lambda_1 u'_x + u''_x) w_x\, dx \\ &- \lambda \int_0^1 \int_0^\infty g(s) u_x(t-s)\, ds w_x\, dx \\ &= \int_0^1 f w\, dx + \int_0^1 F_1[u] w\, dx + \int_0^1 F_2[u] w_x\, dx, \end{aligned}$$

for all test functions $w \in H_0^1$, for almost all $t \in (0, T_*)$ and satisfying the initial conditions.

Uniqueness. Let u_1, u_2 be two weak solutions of (4.1.1)–(4.1.4) such that

$$u_1, u_2 \in W_1(M, T_*). \tag{4.2.22}$$

Then $v = u_1 - u_2$ satisfies

$$\begin{aligned}
&\int_0^1 v'' w \, dx + \int_0^1 (\lambda_0 v_x + \lambda_1 v_x' + v_x'') w_x \, dx \\
&- \lambda \int_0^1 \int_0^\infty g(s) v_x(t-s) \, ds w_x \, dx \\
&= \int_0^1 \Big(F_1[u_1] - F_1[u_2] \Big) w \, dx + \int_0^1 \Big(F_2[u_1] - F_2[u_2] \Big) w_x \, dx
\end{aligned} \tag{4.2.23}$$

for all test functions $w \in H_0^1$, for almost all $t \in [0, T_*]$. Taking $w = v'$ in (4.2.23) and integrating with respect to t, for

$$\begin{aligned}
E(t) \equiv &\int_0^1 \Big(|v'|^2 + l|v_x|^2 + |v_x'|^2 \Big) dx + 2\lambda_1 \int_0^t \int_0^1 |v_x'|^2 \, dx \, ds \\
&+ \lambda \int_0^\infty g(s) \int_0^1 |v_x(t-s) - v_x(t)|^2 \, dx \, ds \\
&- \lambda \int_0^t \int_0^\infty g'(s) \int_0^1 |v_x(\tau - s) - v_x(\tau)|^2 \, dx \, ds \, d\tau,
\end{aligned}$$

we obtain

$$\begin{aligned}
E(t) &= 2\int_0^t \int_0^1 \Big(F_1[u_1] - F_1[u_2] \Big) v' \, dx \, ds + 2\int_0^t \int_0^1 \Big(F_2[u_1] - F_2[u_2] \Big) v_x' \, dx \, ds \\
&\le \int_0^t \| F_1[u_1] - F_1[u_2] \|_{L^2}^2 \, ds + \int_0^t \| F_2[u_1] - F_2[u_2] \|_{L^2}^2 \, ds + \int_0^t E(s) \, ds \\
&\le 32 \bar{F}_M^2 \int_0^t \| u_1 - u_2 \|_{W_1(T_*)}^2 \, ds + \int_0^t E(s) \, ds \\
&\le \left(\frac{32 \bar{F}_M^2}{\min(1, l)} + 1 \right) \int_0^t E(s) \, ds.
\end{aligned}$$

Thanks again to Gronwall's Lemma, we have $E \equiv 0$, i.e., $u_1 \equiv u_2$. □

Theorem 10 is completely proved. □

4.3 Blow up

We further prove that if (4.3.8) holds, then the blow up of any weak solution (4.3.1) for a finite time occurs when the initial energy is negative.

Here, we consider (4.1.1)–(4.1.4) with $f = 0$, $F_1 = f_1(u, u_x) \in C^1(\mathbb{R}^2; \mathbb{R})$ and $F_2 = f_2(u, u_x) \in C^1(\mathbb{R}^2; \mathbb{R})$ as follows

$$\begin{cases} u'' - \left(\lambda_0 u_x + \lambda_1 u'_x + u''_x\right)_x + \lambda \int_{-\infty}^{t} g(t-s) u_{xx}(s)\, ds \\ = f_1(u, u_x) - \Big(f_2(u, u_x)\Big)_x, \quad x \in (0,1),\ 0 < t < T_*, \end{cases} \tag{4.3.1}$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 < t < T_*, \tag{4.3.2}$$

and the following initial conditions:

$$u(x, -t) = u_0(x,t), \quad u'(x,0) = u_1(x), \qquad t > 0. \tag{4.3.3}$$

We have proved in the previous section, the existence of local weak solution of (4.1.1)–(4.1.4) in Theorem 10. Furthermore, let us assume that there exist $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants p, $q > 2$, d_1 $d_2 > 0$ such that

$$\begin{aligned} &\frac{\partial \mathcal{F}}{\partial u}(u,v) = f_1(u,v), \qquad \frac{\partial \mathcal{F}}{\partial v}(u,v) = f_2(u,v), \\ &u f_1(u,v) + v f_2(u,v) \geq d_1 \mathcal{F}(u,v), \quad \forall (u,v) \in \mathbb{R}^2, \\ &\mathcal{F}(u,v) \geq d_2(|u|^p + |v|^q), \quad \forall (u,v) \in \mathbb{R}^2. \end{aligned} \tag{4.3.4}$$

We introduce the energy functional $E(t)$ associated with system (4.3.1)-(4.3.3)

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 |u'|^2\, dx + \frac{1}{2} l \int_0^1 |u_x|^2\, dx + \frac{1}{2}\int_0^1 |u'_x|^2\, dx \\ &+ \frac{1}{2}\lambda \int_0^1 \int_0^{\infty} g(s) |u_x(t) - u_x(t-s)|^2\, ds\, dx - \int_0^1 \mathcal{F}(u, u_x)\, dx. \end{aligned} \tag{4.3.5}$$

It is not hard to see this Lemma (Using Lemma 18).

Lemma 20. *Suppose that* (*Hyp*1) *holds. Let* u *be solution of system* (4.3.1)-(4.3.3). *Then the energy functional* (4.3.5) *is a non-increasing function, i.e., for all* $t \geq 0$,

$$\frac{d}{dt}E(t) = -\lambda_1 \int_0^1 |u_x'|^2\, dx + \frac{1}{2}\lambda \int_0^1 \int_0^\infty g'(s)|u_x(t) - u_x(t-s)|^2\, ds\, dx.$$

For a reader, we state this Lemma with its proof.

Lemma 21. *Let* $\nu, \xi > 0$ *be real positive numbers and let* $L(t)$ *be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t), \tag{4.3.6}$$

defined in $[0, \infty)$. *If* $L(0) > 0$, *then the solution does not exist for* $t \geq L(0)^{-\nu}\xi^{-\nu}\nu^{-1}$.

Proof. The direct integration of (4.3.6) gives

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi\nu t.$$

Thus, we get the following estimate:

$$L^{\nu}(t) \geq \Big[L^{-\nu}(0) - \xi\nu t\Big]^{-1}. \tag{4.3.7}$$

It is clear that the right-hand side of (4.3.7) is unbounded for

$$\xi\nu t = L^{-\nu}(0).$$

Lemma 21 is proved. □

Our goal is to prove that when the initial energy is negative, the solution of system (4.3.1) blows up in finite time under (4.3.4) and (4.3.8).

Second main Theorem

Our result here reads as follows.

Theorem 11. *Assume that* (3.4) *holds. Assume further that* $E(0) < 0$ *for any* $u_0(0), u_1 \in H_0^1 \cap H^2$ *holds. There exists a number* r *satisfying* $2 < r < \min(p,q)$ *and* $r < d_1$ *such that*

$$\int_0^\infty g(s)\,ds < \frac{\lambda_0(\frac{r}{2}-1)}{\lambda\Big(\frac{r}{2}-1+\frac{1}{2r}\Big)}. \tag{4.3.8}$$

Then, the unique weak solution u *of* $(3.1)-(3.3)$ *blows up in finite time.*

Proof. Let

$$H(t) = -E(t), \qquad \forall t \in [0, T_*).$$

By Lemma 20, we obtain

$$\begin{aligned} \frac{d}{dt}H(t) &= \lambda_1 \int_0^1 |u_x'|^2\,dx - \frac{1}{2}\lambda \int_0^1 \int_0^\infty g'(s)|u_x(t)-u_x(t-s)|^2\,ds\,dx \\ &\geq 0, \qquad \forall t \in [0, T_*). \end{aligned} \tag{4.3.9}$$

Consequently, $E(0) < 0$ and (4.3.9) imply that

$$0 < H(0) \leq H(t) \quad \forall t \in [0, T_*). \tag{4.3.10}$$

Using (Hyp1), we get

$$\begin{aligned} H(t) - \int_0^1 \mathcal{F}(u,u_x)dx &= -\frac{1}{2}\int_0^1 |u'|^2\,dx - \frac{1}{2}l\int_0^1 |u_x|^2\,dx - \frac{1}{2}\int_0^1 |u_x'|^2\,dx \\ &- \frac{1}{2}\lambda \int_0^1 \int_0^\infty g(s)|u_x(t)-u_x(t-s)|^2\,ds\,dx \\ &\leq 0, \quad \forall t \in [0, T_*). \end{aligned}$$

One implies

$$0 < H(0) \leq H(t) \leq \int_0^1 \mathcal{F}(u, u_x)\, dx.$$

Then, we define functionals

$$\begin{aligned} M(t) &= \frac{1}{2}\int_0^1 u^2\, dx, \\ N(t) &= \frac{1}{2}\lambda_1 \int_0^1 |u_x|^2\, dx + \int_0^1 u_x u_x'\, dx, \end{aligned}$$

and introduce

$$L(t) = H^{1-\sigma}(t) + \varepsilon M'(t) + \varepsilon N(t), \tag{4.3.11}$$

for ε small enough and

$$0 < \sigma < 1/2,\ 2/(1-2\sigma) \leq \min(p, q). \tag{4.3.12}$$

We now show that $L(t)$ satisfies the differential inequality in Lemma 21. By taking the derivative of (4.3.11) and using (4.3.1), we obtain

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_0^1 |u'|^2\, dx + \varepsilon \int_0^1 |u_x'|^2\, dx \\ &- \varepsilon l \int_0^1 |u_x|^2 dx + \varepsilon\lambda \int_0^\infty g(s) \int_0^1 [u_x(t-s) - u_x(t)]u_x\, dx\, ds \\ &+ \varepsilon \int_0^1 f_1(u, u_x) u dx + \varepsilon \int_0^1 f_2(u, u_x) u_x dx. \end{aligned}$$

By the Cauchy-Schwartz and Young inequalities, we find

$$\begin{aligned} &\left| \int_0^\infty g(s) \int_0^1 [u_x(t-s) - u_x(t)]u_x\, dx\, ds \right| \\ &\leq \int_0^\infty g(s) \left(\int_0^1 |u_x(t-s) - u_x(t)|^2\, dx \int_0^1 |u_x|^2\, dx \right)^{1/2} ds \\ &\leq \gamma \int_0^\infty g(s) \int_0^1 |u_x(t-s) - u_x(t)|^2\, dx\, ds + \frac{\int_0^\infty g(s) ds}{4\gamma} \int_0^1 |u_x|^2\, dx \end{aligned}$$

for any $\gamma > 0$. Therefore,

$$\begin{aligned}
L'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_0^1 |u'|^2 \, dx + \varepsilon \int_0^1 |u'_x|^2 \, dx \\
&- \varepsilon \left(l + \lambda \frac{\int_0^\infty g(s)\, ds}{4\gamma} \right) \int_0^1 |u_x|^2 \, dx \\
&- \varepsilon \lambda \gamma \int_0^\infty g(s) \int_0^1 |u_x(t-s) - u_x(t)|^2 \, dx \, ds \\
&+ \varepsilon \int_0^1 f_1(u, u_x) u \, dx + \varepsilon \int_0^1 f_2(u, u_x) u_x \, dx.
\end{aligned}$$

By (4.3.4), we obtain

$$\int_0^1 f_1(u, u_x) u \, dx + \int_0^1 f_2(u, u_x) u_x \, dx \geq d_1 \int_0^1 \mathcal{F}(u, u_x) \, dx.$$

Then, it follows from (4.3.12) and the inequality

$$(1-\sigma)H^{-\sigma}(t)H'(t) > 0, \forall t \in [0, T_*),$$

that

$$\begin{aligned}
L'(t) &\geq \varepsilon \int_0^1 |u'|^2 \, dx + \varepsilon \int_0^1 |u'_x|^2 \, dx - \varepsilon \Big(l + \lambda \frac{\int_0^\infty g(s)\, ds}{4\gamma} \Big) \int_0^1 |u_x|^2 \, dx \\
&- \varepsilon \lambda \gamma \int_0^\infty g(s) \int_0^1 |u_x(t-s) - u_x(t)|^2 \, dx \, ds + \varepsilon d_1 \int_0^1 \mathcal{F}(u, u_x) \, dx.
\end{aligned}$$

Adding $\varepsilon r E(t) + \varepsilon r H(t)$ and using (4.3.4) and the definition of $H(t)$, we get

$$\begin{aligned}
L'(t) &\geq \varepsilon \left(1 + \frac{r}{2}\right) \int_0^1 |u'|^2 \, dx + \varepsilon \left(1 + \frac{r}{2}\right) \int_0^1 |u'_x|^2 \, dx + \varepsilon r H(t) \\
&+ \varepsilon \Big[- \Big(l + \lambda \frac{\int_0^\infty g(s)\, ds}{4\gamma} \Big) + \frac{r}{2} l \Big] \int_0^1 |u_x|^2 \, dx \\
&+ \varepsilon \lambda \left(\frac{r}{2} - \gamma \right) \int_0^\infty g(s) \int_0^1 |u_x(t-s) - u_x(t)|^2 \, dx \, ds \\
&+ \varepsilon \, (d_1 - r) \, d_2 \Big(\int_0^1 |u_x(t)|^q \, dx + \int_0^1 |u(t)|^p \, dx \Big). \qquad (4.3.13)
\end{aligned}$$

Now choosing

$$\gamma = \frac{r}{2},$$

we take

$$a_1 = \lambda_0 \left(-1 + \frac{r}{2}\right) - \lambda \left(\frac{1}{2r} - 1 + \frac{r}{2}\right) \int_0^\infty g(s)\, ds > 0,$$

and

$$a_2 = (d_1 - r)d_2 > 0$$

by (4.1.5) and the assumptions of Theorem 11. Then, estimate (4.3.13) becomes

$$\begin{aligned} L'(t) \;&\geq\; \varepsilon \left(1 + \frac{r}{2}\right) \int_0^1 |u'|^2\, dx + \varepsilon \left(1 + \frac{r}{2}\right) \int_0^1 |u'_x|^2\, dx + \varepsilon r H(t) \\ &+\; \varepsilon a_1 \int_0^1 |u_x|^2\, dx + \varepsilon a_2 \Big(\int_0^1 |u_x|^q\, dx + \int_0^1 |u|^p\, dx \Big). \end{aligned}$$

At this point, we can find a positive constant $\nu = \min(1 + r/2, a_1, a_2)$ such that

$$\begin{aligned} L'(t) \;&\geq\; \varepsilon\nu \Big(H(t) + \int_0^1 |u_x|^q\, dx + \int_0^1 |u|^p\, dx \\ &\quad + \int_0^1 |u_x|^2\, dx + \int_0^1 |u'_x|^2\, dx + \int_0^1 |u'|^2\, dx \Big). \end{aligned} \tag{4.3.14}$$

Thus, we can choose $\varepsilon > 0$ small enough such that

$$L(t) \geq L(0) > 0, \quad \forall t \in [0, T_*).$$

Further, by (4.3.12), Hölder and Young inequalities, we obtain

$$\Big| \int_0^1 uu'\, dx \Big|^{1/(1-\sigma)} \leq C_1 \Big[\Big(\int_0^1 |u|^p\, dx \Big)^{\tau/p(1-\sigma)} + \Big(\int_0^1 |u'|^2\, dx \Big)^{s/2(1-\sigma)} \Big]$$

for $1/\tau + 1/s = 1$, where $C_1 = \max(1/s, 1/\tau)$. We take $s = 2(1 - \sigma)$ to get

$$\tau = \frac{2(1-\sigma)}{1 - 2\sigma} \quad \text{and} \quad C_1 = \frac{1}{s} = \frac{1}{2(1-\sigma)}.$$

By using the algebraic inequality

$$z^\nu \leq (z+1) \leq \Big(1+\frac{1}{a}\Big)(z+a), \qquad \forall z \geq 0, \quad 0<\nu\leq 1, a\geq 0,$$

we find

$$\Big(\int_0^1 |u|^p\,dx\Big)^{2/\{p(1-2\sigma)\}} \leq C_2\Big(\int_0^1 |u|^p\,dx + H(t)\Big), \quad \forall t\in[0,T_*),$$

where $C_2 = 1+1/H(0) > 1$ by (4.3.10). Then we obtain

$$\Big|\int_0^1 uu'\,dx\Big|^{1/(1-\sigma)} \leq C_3\Big(\int_0^1 |u|^p\,dx + \int_0^1 |u'|^2\,dx + H(t)\Big), \tag{4.3.15}$$

where $C_3 = C_1C_2$. Similarly, we obtain

$$\Big|\int_0^1 u_x u'_x\,dx\Big|^{1/(1-\sigma)} \leq C_3\Big(\int_0^1 |u_x|^q\,dx + \int_0^1 |u'_x|^2\,dx + H(t)\Big) \tag{4.3.16}$$

and

$$\Big|\int_0^1 |u_x|^2\,dx\Big|^{1/(1-\sigma)} \leq C_3\Big(\int_0^1 |u_x|^q\,dx + \int_0^1 |u_x|^2\,dx + H(t)\Big), \tag{4.3.17}$$

respectively. Moreover, by (4.3.11), (4.3.15), (4.3.16) and (4.3.17) we note that

$$\begin{aligned} &L^{1/(1-\sigma)}(t) \\ \leq\ & C_4\Big(H(t) + \Big|\int_0^1 uu'\,dx\Big|^{1/(1-\sigma)} + \Big|\int_0^1 |u_x|^2\,dx\Big|^{1/(1-\sigma)} + \Big|\int_0^1 u_x u'_x\,dx\Big|^{1/(1-\sigma)}\Big) \\ \leq\ & C_5\Big(H(t) + \int_0^1 |u|^p\,dx + \int_0^1 |u'|^2\,dx + \int_0^1 |u_x|^2\,dx + \int_0^1 |u_x|^q\,dx + \int_0^1 |u'_x|^2\,dx\Big) \end{aligned}$$

owing to

$$(a+b+c+d)^\alpha \leq 2^{2(\alpha-1)}\,(a^\alpha + b^\alpha + c^\alpha + d^\alpha)$$

for $a,b,c,d\geq 0$ and $\alpha\geq 1$, where

$$C_4 = 2^{2\sigma/(1-\sigma)} \max\left(1, \varepsilon, \frac{1}{2}\varepsilon\lambda_1\right)^{1/(1-\sigma)} \quad \text{and} \quad C_5 = C_4\left(3C_3 + 1\right).$$

This yields

$$L'(t) \geq \xi L^{1/(1-\sigma)}(t), \qquad \forall t \geq 0,$$

along with (4.3.14), where $\xi = \varepsilon\nu/C_5$. Finally, Lemma 21 completes the proof of Theorem 11 for $T_* = C_5(1-\sigma)/(\varepsilon\nu\sigma)L^{-\sigma/(1-\sigma)}(0)$. □

Conclusion

In our work, we have obtained new stability results for solutions to a class of wave equations with weak and strong damping terms, as well as a logarithmic source in $\mathbb{R}^n$. We have demonstrated general stability estimates by introducing a suitable Lyapunov function. These results enhance our understanding of solution stability in this context and offer promising prospects for future applications.

On the other hand, we have studied a dynamical system with variable delay described by a partial differential equation of hyperbolic type. We have demonstrated the $\kappa(t)$-stability of the weak solution under suitable initial conditions in $\mathbb{R}^n$, with $n > 4$, by introducing appropriate Lyapunov functions.

Finally, we studied a boundary value problem for a nonlinear equation called the "Love equation" with infinite memory. By combining the linearization method, the Faedo-Galerkin method, and the weak compactness method, we demonstrated the local existence and uniqueness of the weak solution. We also investigated the possibility of finite-time blow-up of the weak solution.

Bibliography

[1] L. C. Evans, *Partial differential equations*, American Mathematical Society, (1998).

[2] H. Brézis, *Functional analysis sobolev spaces and partial differential equations*, Springer New York, (2010).

[3] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Springer, (1993).

[4] D. Pino, M. D. Pavel and R. Manasevich, *The fredholm alternative at the first eigenvalue for the one-dimensional p-laplacian*, Comptes Rendus De L'academie Des Sciences Series I Mathematics, (1998), 461–465.

[5] F. Le, *Global existence and blow-up of solutions for a higher-order Kirchhoff-type equation with nonlinear dissipation*, Appl. Math. Lett., **17** (2004), 1409-1414.

[6] M. García-H., R. Manásevich and F. Zanolin, *Infinitely many solutions for a Dirichlet problem with a nonhomogeneous p-Laplacian-like operator in a ball*, Advances in Differential Equations, Khayyam Publishing, Inc., **2** (1997), 203-230.

[7] K. Schmitt, *Nonlinear Analysis and Differential Equations An Introduction*, (2004).

[8] G. A. Afrouzi, N. T. Chung, and S. Shakeri, *Existence of positive solutions for Kirchhoff type equations*, Electronic Journal of Differential Equations, (2013).

[9] G. Dai, *Three solutions for a nonlocal Dirichlet boundary value problem involving the p(x)-Laplacian*, Applicable Analysis, **92**, (2013), 191-210.

[10] G. Dai and R. M, *Solutions for a p(x)-Kirchhoff type equation with Neumann boundary data*, Nonlinear Analysis: Real World Applications, **12**, (2011), 2666-2680.

[11] T. Cazenave and A. Haraux, *Equations d'evolution avec non-linearite logarithmique*, Ann. Fac. Sci. Toulouse Math. **2(1)**, (1980), 21-51.

[12] H. Chen, P. Luo and G.W. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl. **422**, (2015), 84-98.

[13] X. Han, *Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics*, Bull. Korean Math. Soc. **50**, (2013), no. 1, 275-283.

[14] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97(4)**, (1975) 1061-1083.

[15] M. Al-Gharabli, *New general decay results for a viscoelastic plate equation with a logarithmic nonlinearity*, Boundary Value Problems, **194** , (2019), 1-21.

[16] W. Lian and R. Xu, *Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term*, Adv. Nonlinear Anal, **9**,(2020), 613-632.

[17] T. Miyasita and K. Zennir, *A sharper decay rate for a viscoelastic wave equation with power nonlinearity*, Math. Meth. Appl. Sci. (2019), 1-7.

[18] M. Al-Gharabli, S. A. Messaoudi, *The existence and the asymptotic behavior of a plate equation with frictional damping and a logarithmic source term*, J. Math. Anal. Appl 454, **5**,(2017), 1-14.

[19] M. Kafini, *Uniform decay of solutions to Cauchy viscoelastic problems with density*, Electron. J. Differential Equations **93**, (2011), 1-9.

[20] N. I. Karachalios and N. M. Stavrakakis, *Global existence and blow-up results for some nonlinear wave equations on* $\mathbb{R}^N$, Adv. Differential Equations, **6**, (2001), 155-174.

[21] I. Lasiecka, S. A. Messaoudi and M. Mustafa, *Note on intrinsic decay rates for abstract wave equations with memory*, J. Math. Phys., **54**, (2013), 031504-18.

[22] Q. Li and L. He, *General decay and blow-up of solutions for a nonlinear viscoelastic wave equation with strong damping*, Bound. Value Probl., **2018:153**, (2018).

[23] P. G. Papadopoulos and N. M. Stavrakakis, *Global existence and blow-up results for an equation of Kirchhoff type on* $\mathbb{R}^N$", Topol. Methods Nonlinear Anal., **17**, (2001), 91-109.

[24] Kh. Zennir, M. Bayoud and S. Georgiev, Decay of solution for degenerate wave equation of Kirchhoff type in viscoelasticity, Int. J. Appl. Comput. Math., **4**:54, (2018).

[25] S. Zitouni and Kh. Zennir, *On the existence and decay of solution for viscoelastic wave equation with nonlinear source in weighted spaces*, Rend. Circ. Mat. Palermo (2), **66**, (2017), 337-353.

[26] F. Alabau-Boussouira, S. Nicaise and C. Pignotti, *Exponential stability of the wave equation with memory and time delay. In: New prospects in direct,* Inverse Cont. Prob. Evol. Equ., **10**, (2014), 1-22.

[27] A. Beniani, A. Benaissa and Kh. Zennir,*Polynomial Decay Of Solutions To The Cauchy Problem For A Petrowsky-Petrowsky System In* $\mathbb{R}^n$*,* J. Acta Appl. Math., **146**, (2016), 67-79.

[28] A. Benaissa and S. A. Messaoudi,*Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks,* J. Math. Phys., **53**, (2012), 123514, 19pp.

[29] A. Benaissa and N. Lohiubi,*Global existence and energy decay of solutions to a nonlinear wave equation with a delay term,* Georgian Math. J., **20**, (2013), 1-24.

[30] A. Benaissa, A. Benguessoum and S. A. Messaoudi, *Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the nonlinear internal feedback,* Int. J. Dyna. Sys. Differ. Equa., **5**(1), (2014), 1-26.

[31] Q. Dai and Z. Yang, *Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay,* Z. Angew. Math. Phys., **65**, (2014), 885-903.

[32] R. Datko, J. Lagnese and M. P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations,* SIAM J. Control Optim., **24**, (1986), 152-156.

[33] B. Feng, *Well-posedness and exponential stability for a plate equation with time-varying delay and past history,* Z. Angew. Math. Phys., **68**, (2017), DOI 10.1007/s00033-016-0753-9

[34] B. Feng, *Global well-posedness and stability for a viscoelastic plate equation with a time delay,* Math. Probl. Eng., **2015**, (2015), 585021, 10pp.

[35] M. Kafini, S. A. Messaoudi and S. Nicaise, *A blow-up result in a nonlinear abstract evolution system with delay,* Nonl. Diff. Equ. Appl., **23**(13), (2016), DOI 10.1007/s00030-016-0371-4

[36] N. I. Karachalios and N. M. Stavrakakis, *Existence of global attractor for semilinear dissipative wave equations on $\mathbb{R}^n$,* J. Diff. Equ., **157**, (1999), 183-205

[37] W. J. Liu, *General decay of the solution for viscoelastic wave equation with a time-varying delay term in the internal feedback,* J. Math. Phys., **54**, (2013), 043504.

[38] W. J. Liu, *General decay rate estimate for the energy of a weak viscoelastic equation with an internal time-varying delay term,* Taiwanese J. Math., **17**, (2013), 2101-2115.

[39] G. Liu and H. Zhang, *Well-posedness for a class of wave equation with past history and a delay,* Z. Angew. Math. Phys., **67**(6), (2016), DOI 10.1007/s00033-015-0593-z

[40] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks,* SIAM J. Control Optim., **45**, (2006), 1561-1585.

[41] S. Nicaise and C. Pignotti, *Intetior feedback stabilization of wave equations with time dependent delay,* Electron. J. Diff. Equ., **2011**(41), (2011), 1-20.

[42] S. Nicaise, J. Valein and E. Fridman, *Stabilization of the heat and the wave equations with boundary time-varying delays,* Discrete Conti. Dyna. Sys.- Series S, **2**, (2009), 559-581.

[43] S. H. Park, *Decay rate estimates for a weak viscoelastic beam equation with time-varying delay,* Appl. Math. Lett., **31**,(2014), 46-51.

[44] R. C. A. M Van Der Vorst, *Best constant for the embedding of the space* $H^2 \cap H_0^1(\Omega)$ *into* $L^{2n/n-4}(\Omega)$, Diff. Int. Equ. **6**, (1993), 259-276.

[45] S. T. Wu, *Asymptotic behavior for a viscoelastic wave equation with a delay term,* Taiwanese J. Math., **17**, (2013), 765-784.

[46] Z. Yang, *Existence and energy decay of solutions for the Euler-Bernoulli viscoelastic equation with a delay,* Z. Angew. Math. Phys., **66**, (2015), 727-745.

[47] Kh. Zennir, *General decay of solutions for damped wave equation of Kirchhoff type with density in* $\mathbb{R}^n$*,* Ann. Univ. Ferrara, **61**, (2015), 381-394.

[48] J. M. Ball, *Remarks on blow-up and nonexistence Theorems for nonlinear evolution equations*, Quart. J. Math. Oxford Ser. (2) **28** (1977) 473-486.

[49] A. Braik, A. Beniani and Kh. Zennir, *Polynomial stability for system of* 3 *wave equations with infinite memories*, Math. Methods Appl. Sci. **41** (2018) 112-126.

[50] A. Braik, Y. Miloudi and Kh. Zennir, *A finite-time blow-up result for a class of solutions with positive initial energy for coupled system of heat equations with memories*, Math. Methods Appl. Sci. **41** (2018) 1674-1682.

[51] M. M. Cavalcanti, V. N. Domingos Cavalcanti and I. Lasiecka, *Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping —source interaction*, J. Differential Equations (**236**) (2007) 407-459.

[52] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955, xii+429 pp.

[53] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970) 297-308.

[54] C. Giorgi, J. Mũenoz Rivera and V. Pata, *Global attractors for a semilinear hyperbolic equation in viscoelasticity*, J. Math. Anal. Appl. **260**(1) (2001) 83-99.

[55] X. Han and M. Wang, *Global existence and blow-up of solutions for a system of nonlinear viscoelastic wave equations with damping and source*, Nonlinear Anal. **71** (2009) 5427-5450.

[56] J. Hao and P. Wang, *General stability result of abstract thermoelastic system with infinite memory*, Bull. Malays. Math. Sci. Soc. (2018). https://doi.org/10.1007/s40840-018-0615-z.

[57] W. Hrusa, *Global existence and asymptotic stability for a semilinear hyperbolic Volterra equation with large initial data*, SIAM J. Math. Anal. **16**(1) (1985) 110-134.

[58] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential Integral Equations **6**(3) (1993) 507-533.

[59] W. Liu and H. Zhuang, *Global existence, asymptotic behavior and blow-up of solutions for a suspension bridge equation with nonlinear damping and source terms*, NoDEA Nonlinear Differential Equations Appl. (2017) **24**(6) Art. 67, 35.

[60] L. T. P. Ngoc, N. T. Duy and N. T. Long, *A linear recursive scheme associated with the Love equation*, Acta Math. Vietnam. **38**(4) (2013) 551-562.

[61] L. T. P. Ngoc, N. T. Duy and N. T. Long, *On a high-order iterative scheme for a nonlinear Love equation*, Appl. Math. **60**(3) (2015) 285-298.

[62] L. T. P. Ngoc and N. T. Long, *Existence, blow-up and exponential decay for a nonlinear Love equation associated with Dirichlet conditions*, Appl. Math. **61**(2) (2016) 165-196.

[63] K. Ono, *On global solutions and blow-up solutions of nonlinear Kirchhoff strings with non-linear dissipation*, J. Math. Anal. Appl. **216**(1) (1997) 321-342.

[64] D. Ouchenane, Kh. Zennir and M. Bayoud, *Global nonexistence of solutions for a system of nonlinear viscoelastic wave equations with degenerate damping and source terms*, Ukrainian Math. J. **65**(5) (2013) 723-739.

[65] S. H. Park, *Stability of a von Karman equation with infinite memory*, Acta Math. Sci. Ser. B (Engl. Ed.) **37**(4) (2017) 965-973.

[66] F. Sun, L. Liu and Y. Wu, *Blow-up of solutions for a nonlinear viscoelastic wave equation with initial data at arbitrary energy level*, Appl. Anal. (2018). https://doi.org/10.1080/00036811.2018.1460812.

[67] N. A. Triet, V. T. T. Mai, L. T. P. Ngoc and N. T. Long, *Existence, blow-up and exponential decay for Kirchhoff-Love equations with Dirichlet conditions*, Electron. J. Differential Equations (2018) Paper No. **167**, 26.

[68] Z. Yang and Z. Gong, *Blow-up of solutions for viscoelastic equations of Kirchhoff type with arbitrary positive initial energy*, Electron. J. Differential Equations (2016) Paper No. **332**, 8.