وزارة التعليم العالي والبحث العلمي


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# THÈSE

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# Qualitative Study of Certain Second Order Quadratic Difference Equations

**Filière**
Mathématiques
**Spécialité**
Mathématiques et Applications
**Par**


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# Abstract

**This thesis is dedicated to the study of the qualitative behavior of some classes of non-linear difference equations and system of difference equations. We focused on the study of the boundedness character, the asymptotic stability of equilibrium points, oscillation, existence of periodic solutions and the global attractivity about equilibrium points of second order quadratic autonomous rational difference equation as well as the non-autonomous higher order rational difference equation. In addition to that we are interested to solve an open problem concerning the system of higher order non-autonomous difference equations. As well as another third order non-autonomous system. As an application of difference equations and their system, we have giving two biological models. We conclude our work by giving some numerical examples which permit to confirm and illustrate our contributions.**

**Keywords: Difference equations, Periodic solutions, Global asymptotic stability, Boundedness.**

# Résumé

**Cette thèse est consacrée à l'étude du comportement qualitative de certaines classes d'équations aux différences non linéaires et de système d'équations aux différences non linéaires. Nous nous sommes concentrés sur l'étude du caractère de la bornitude, de la stabilité asymptotique des points d'équilibres, l'oscillation, de l'existence des solutions périodiques et de l'attractivité globale autour des points d'équilibres de l'équations aux différences rationnelles autonomes quadratiques du second ordre. Ainsi que l'équation aux différences rationnelles non autonome d'ordre supérieur. Comme application des équations aux différences et leur système. Nous concluons notre travail en donnant quelques exemples numérique qui permettent de confirmer et d'illustrer nos contributions.**

**Mots clés: Equations aux différences, Solutions périodiques, Stabilité asymptotique globale, bornitude.**

# الملخص

هذه الأطروحة كرست لدراسة السلوك النوعي لبعض أنواع معادلات و جملة معادلات الفروق غير الخطية، ركزنا على دراسة خاصية المحدودية، الاستقرار المقارب لنقاط التوازن، التذبذب ووجود الحلول الدورية، بالإضافة الى الجاذبية الكلية لنقاط التوازن لمعادلة الفروق الكسرية التربيعية الذاتية من الرتبة الثانية و معادلة الفروق غير الذاتية من الرتب العليا، وكذلك اهتممنا بحل مسألة مفتوحة تخص جملة معادلتي فروق غير الذاتية ذات الرتب العليا. كتطبيق لمعادلات الفروق وجمل معادلاتها قدمنا نموذجين بيولوجيين. نختم عملنا بإعطاء بعض الأمثلة العددية التي تسمح بتأكيد وتوضيح مساهمتنا .

الكلمات المفتاحية: معادلات الفروق، الحلول الدورية، الاستقرار المقارب الكلي، المحدودية .

# Contents

# Introduction

Difference equations, in the form of recursions and finite differences have recently been a subject of big attraction for mathematicians thanks to their ancient appearance, richness and appreciable flexibility for use. This use dated back to the beginning of the year 2000s before Jesus Christ (B.C) by ancient civilizations like the Babylonians while studying numbers.

Around 250.B.C., Archimedes employed the nonlinear difference equations to calculate the circumference of a circle. In 1202, Leonardo de Pisa known as "Fibonacci" formulated his problem of rabbits which led to the Fibonacci sequence 1,1,2,3,5,8,13... It is also known that Isaac Newton made use of their calculus in the late 1600s. Then, around 1634, Gerard Albert gave the general expression of the sequence of Fibonacci $F_{n+1} = F_n + F_{n-1}$.

In 1769, Leonhard Euler had used for the first time linear difference equation to approximate the solutions of differential equations. After that, George Boole wrote a definitive treatise on the calculus of finite differences in 1872.

In addition to inertia gained by centuries-long research in differential equations, the continued widespread modeling of scientific phenomena in terms of differential equations and systems today leads to the growth of this field at a higher rate than difference equations.

Difference equations have become a valuable tool of a big importance in many fields and scientific disciplines and this is due to their various applications in many domains such as economics, ecology, biology, theory of probability,...See [13, 14, 22, 24, 25, 23].

In fact, difference equations are used for the stimulation of ordinary differential equations or the partial derivation of differential equations in numerical analysis to solve the equations by using sequences with the research of approximate value of the solution. In addition, they are also used in modeling real life phenomena.

In 2005, Saber Elaydi in his book [22] Introduction to Difference equations, set the fundamental principles and notions for the theory of difference equations.

Non-autonomous difference equations constitute a special class of difference equations so as the coefficients are variable. By comparison with those with constant coefficients (also called autonomous), their studies looks more difficult and complicated and there are no many works in this way. Sometimes, in nature, the vital in a changing phenomenon vary, and this explain well the consideration of discrete models represented by non-autonomous difference equations, in particular the case of periodicity.

The main objective of this research work [Qualitative Study of Certain Second Order Quadratic Difference Equations, 2022] is the qualitative study of behaviors of the solution of certain equations and system of autonomous and non-autonomous difference equations. The present thesis comprises four chapters.

In the first chapter, we start by giving some notions about difference equations and their systems as well as the tools needed in this research. Next, we give the application of two biological models. The first is the Beverton Model with periodic environment which arises in the study of response of population to periodically fluctuating forced environment such as seasonal fluctuations in carrying-capacity or demographic parameters such as birth or death rates. The second is concerned with the modeling interaction between species, describing the intra-species dynamics when the biological life cycle of the species creates distinct forms of the organism. Such is the case for many insects and one of the preeminent examples in the literature is the larva-pupa-adult (LPA) or Flour Beetle Model.

In the second chapter, we study the quadratic fractional difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C},$$

we investigate the boundedness of solutions, the global stability of the positive equilibrium point and the occurrence of periodic solutions with non-negative parameters and initial values.

In the third chapter, we are interested in the generalization of the works of kerker *et al.*[37] where we investigated the global behavior of higher-order non-autonomous rational difference equation

$$x_{n+1} = \frac{\alpha_n + x_{n-r}}{\alpha_n + x_{n-k}}, \quad n = 0, 1, ...,$$

where $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence of positive numbers and $r < k$ are positive integers. We study the oscillation about the equilibrium point $\bar{y} = 1$, then the boundedness of the positive solutions. After, the analysis of the global attractor and finally the global asymptotic stability (see [48]).

In the fourth chapter , we study the dynamics of higher-order difference equations with period two coefficient

$$x_{n+1} = \alpha_n + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = \alpha_n + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \ldots,$$

where $\{\alpha_n\}$ is a periodic sequence of non-negative real numbers and the initial conditions $x_i$, $y_i$ are arbitrary positive numbers for $i = -k, -k+1, -k+2, \ldots, 0$ and $k \in \mathbb{Z}^+$. The purpose of the study here is to transform this non-autonomous system to an equivalent fourth-order autonomous system and to discuss its behavior starting by the boundedness character, the local stability, global stability as well as the rate of convergence of the solutions.(see [49]).

# Chapter 1 Preliminaries and Applications to Biology

## 1.1 Preliminaries

In this preliminary section, we recall some general notions about difference equations and the stability with the linearization method. As well as some theorems that proved to be useful to our thesis. For more details, we refer readers to [11, 22, 43, 54].

### 1.1.1 Definition of Stability of Difference Equations

**Definition 1.1.** *(**Difference Equations**): A difference equation of order $(k+1)$ is an equation of the form*

$$x_{n+1} = f(n, x_n, x_{n-1}, ..., x_{n-k}), \qquad n \geq 0, \tag{1.1}$$

*where $f : \mathbb{N} \times I^{k+1} \longrightarrow I$ be a continuously differentiable function. The set $I$ is usually an interval of real numbers, or union of intervals. The solution of equation (1.1) obtained from initial point $( x_0, x_{-1}, ..., x_{-k})$ is a sequence $\{x_n\} \in I$ such that $x_n$ satisfies (1.1) for all $n > 0$. An initial point $( x_0, x_{-1}, ..., x_{-k})$ generates a (forward) solution $\{x_n\}$ by iteration of the function*

$$(n, x_n, x_{n-1}, ..., x_{n-k}) \longrightarrow f(n, x_n, x_{n-1}, ..., x_{n-k}) : \mathbb{N} \times I^{k+1} \longrightarrow I,$$

*so long as each iterate $x_n$ stays in $I$. When the function $f$ does not depend on index $n$, the difference equation in (1.1) is autonomous, i.e;*

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n \geq 0, \tag{1.2}$$

*otherwise, it is non-autonomous. Solutions of (1.1) or (1.2) are also called orbits or trajectories.*

**Definition 1.2.** *(**Equilibrium Point**): A point $\bar{x} \in I$ such that $\bar{x} = f(n, \bar{x}, \bar{x}, ..., \bar{x})$ for all $n \geq 0$, is called an **equilibrium point** of equation (1.1). In particular, $\bar{x} \in I$ is an equilibrium point of equation* (1.2) *if it satisfies the equation*

$$\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x}).$$

**Definition 1.3.** *(**Stability**): An equilibrium point $\bar{x}$ of (1.1) is said to be*

1. ***Locally stable** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0, x_{-1}, ..., x_{-k} \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta$ then $|x_n - \bar{x}| < \varepsilon$, for all $n \geq -k$. Otherwise, the equilibrium $\bar{x}$ is called **unstable**.*

2. ***Attractive** if there exists $\mu > 0$ such that for all $x_0, x_{-1}, ..., x_{-k} \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \mu$, then*

$$\lim_{n \to \infty} x_n = \bar{x}.$$

   *If $\mu = \infty$, $\bar{x}$ is called **globally attractive**.*

3. ***Locally asymptotically stable** if it is locally stable and attractive.*

4. ***Globally asymptotically stable** if it is stable and globally attractive.*

**Definition 1.4.** *(**Periodicity**): A solution $\{x_n\}_{n \geq -k}$ of equation* (1.1) *is called periodic with period $p$ if there exists an integer $p \geq 1$ such that*

$$x_{n+p} = x_n, \quad \text{for all} \quad n \geq -k. \tag{1.3}$$

*A solution is called periodic with prime period $p$ if $p$ is the smallest positive integer for which equation* (1.3) *holds.*

**Definition 1.5.** *(**Semi-cycle**):*

- *A string of sequential terms $\{x_l, \ldots, x_m\}$, $l \geq -k, m \leq \infty$ is said to be a positive semi-cycle if $x_i \geq \bar{x}$, $i \in \{l, \ldots, m\}$, $x_{l-1} < \bar{x}$ and $x_{m+1} < \bar{x}$.*

- *A string of sequential terms $\{x_l, \ldots, x_m\}$, $l \geq -k, m \leq \infty$ is said to be a negative semi-cycle if $x_i < \bar{x}$, $i \in \{l, \ldots, m\}$, $x_{l-1} \geq \bar{x}$ and $x_{m+1} \geq \bar{x}$.*

**Definition 1.6.** *(**Oscillation**): A solution $\{x_n\}_{n\geq -k}$ of Eq. (1.1) is called non-oscillatory if there exists $p \geq -k$ such that either*

$$x_n > \bar{x}, \quad \forall n \geq p \qquad or \quad x_n < \bar{x}, \quad \forall n \geq p,$$

*and it is called **oscillatory** if it is not non-oscillatory.*

Now, we give a reminder about the comparison principle for non-autonomous difference equations (see [43]).

**Theorem 1.1.** *Let $z \geq 0$ be a real number, $g(n, z)$ be a non-decreasing function with respect to $z$ for any fixed natural number $n \geq n_0$, $n_0 \in \mathbb{N}$. Suppose that for $n \geq n_0$, we have*

$$x_{n+1} \leq g(n, x_n),$$
$$y_{n+1} \geq g(n, y_n).$$

*Then,*

$$x_{n_0} \leq y_{n_0}$$

*implies that*

$$x_n \leq y_n, \qquad \forall n \geq n_0.$$

### 1.1.2 Linearized Stability Analysis

The linearized equation of equation (1.2) about the equilibrium point $\bar{x}$ is

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k}, \quad n \in \mathbb{N} \tag{1.4}$$

where

$$p_0 = \frac{\partial f}{\partial x_n}(\bar{x}, \bar{x}, \ldots, \bar{x}), p_1 = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \bar{x}, \ldots, \bar{x}), \ldots, p_k = \frac{\partial f}{\partial x_{n-k}}(\bar{x}, \bar{x}, \ldots, \bar{x}).$$

The characteristic equation of equation (1.4) is

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1}\lambda - p_k = 0 \tag{1.5}$$

Next, we set the theorem about Linearized Stability

**Theorem 1.2.** *Let $\bar{x}$ be an equilibrium point of equation (1.2). Then, the following statements are true*

*(i) If all roots of equation (1.5) lie inside the open unit disk $|\lambda| < 1$. then $\bar{x}$ is locally asymptotically stable.*

*(ii) If at least one root of equation (1.5) has absolute value greater than one, then $\bar{x}$ is unstable.*

**Definition 1.7.** *The equilibrium point $\bar{x}$ of equation (1.2) is called*

1. ***Hyperbolic*** *if, no root of equation (1.5) has absolute value equal to one. If there exists a root of equation (1.5) with absolute value equal to one, then the equilibrium point $\bar{x}$ is called non-hyperbolic.*

2. ***A saddle point*** *if, it is hyperbolic and if there exists a root of equation (1.5) with absolute value less than one and another root of equation (1.5) with absolute value greater than one.*

3. ***Repeller*** *if, all roots of equation (1.5) have absolute value greater than one.*

### 1.1.3 Linearized Stability of the Higher Order Systems

Let $f$ and $g$ be two continuously differentiable functions:

$$f : I^{k+1} \times J^{k+1} \longrightarrow I, \quad g : I^{k+1} \times J^{k+1} \longrightarrow J,$$

where $I$, $J$ are some interval of real numbers. For $n \in \mathbb{N}$, consider the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}) \\ y_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}) \end{cases} \tag{1.6}$$

where $n$, $k \in \mathbb{N}$, $(x_{-k}, x_{-k+1}, \dots, x_0) \in I^{k+1}$ and $(y_{-k}, y_{-k+1}, \dots, y_0) \in J^{k+1}$.
Define the map $F : I^{k+1} \times J^{k+1} \longrightarrow I^{k+1} \times J^{k+1}$ by

$$F(X) = (f_0(X), f_1(X), \dots, f_k(X), g_0(X), g_1(X), \dots, g_k(X))$$

where

$$\begin{gathered} X = (u_0, u_1, \dots, u_k, v_0, v_1, \dots, v_k)^T, \\ f_0(X) = f(X), f_1(X) = u_0, \dots, f_k(X) = u_{k-1}, \\ g_0(X) = g(X), g_1(X) = v_0, \dots, g_k(X) = v_{k-1}. \end{gathered}$$

Let

$$X_n = (x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k})^T.$$

Then, we can easily see that system (1.6) is equivalent to the system written in vector form

$$X_{n+1} = F(X_n), \quad n \in \mathbb{N}, \tag{1.7}$$

that is

$$\begin{cases} x_{n+1} &= f(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}), \\ x_n &= x_n, \\ &\vdots \\ x_{n-k+1} &= x_{n-k+1}, \\ y_{n+1} &= g(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}), \\ y_n &= y_n, \\ &\vdots \\ y_{n-k+1} &= y_{n-k+1}. \end{cases}$$

**Definition 1.8.** *(**Equilibrium Point**): An equilibrium point* $(\bar{x}, \bar{y}) \in I \times J$ *of system* (1.7) *is a solution of the system*

$$\begin{cases} x = f(x, x, \dots, x, y, y, \dots, y) \\ y = g(x, x, \dots, x, y, y, \dots, y). \end{cases}$$

*Furthermore, an equilibrium point* $\bar{X} \in I^{k+1} \times J^{k+1}$ *of system* (1.7) *is a solution of the system*

$$X = F(X).$$

**Definition 1.9.** *(**Stability**): Let* $\bar{X}$ *be an equilibrium point of system* (1.7) *and* $||.||$ *be any norm (e.g. the Euclidean norm).*

1. *The equilibrium point* $\bar{X}$ *is called stable (or locally stable) if for every* $\varepsilon > 0$ *there exist* $\delta$ *such that* $||X_0 - \bar{X}|| < \delta$ *implies* $||X_n - \bar{X}|| < \varepsilon$ *for all* $n \geq 0$.

2. *The equilibrium point* $\bar{X}$ *is called asymptotically stable (or locally asymptotically stable) if it is stable and there exist* $\delta > 0$ *such that* $||X_0 - \bar{X}|| < \delta$ *implies*

$$\lim_{n\to\infty} X_n = \bar{X}.$$

3. *The equilibrium point* $\bar{X}$ *is said to be global attractor (respectively global attractor with basin of attraction a set* $G \subset I^{k+1} \times J^{k+1}$*), if for every* $X_0$ *(respectively for every* $X_0 \in G$*)*

$$\lim_{n\to\infty} X_n = \bar{X}.$$

4. *The equilibrium point $\bar{X}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to $G$) if it is asymptotically stable, and if for every $X_0$ (respectively for every $X_0 \in G$),*

$$\lim_{n\to\infty} X_n = \bar{X}.$$

5. *The equilibrium point $\bar{X}$ is called unstable if it is not stable.*

**Remark 1.1.** *Clearly, $(\bar{x}, \bar{y}) \in I \times J$ is an equilibrium point of system (1.6) if and only if $\bar{X} = (\bar{x}, \bar{x}, \ldots, \bar{x}, \bar{y}, \bar{y}, \ldots, \bar{y}) \in I^{k+1} \times J^{k+1}$ is an equilibrium point of system (1.7).*

**Remark 1.2.** *From here on, by the stability of the equilibrium points of system (1.6), we mean the stability of the corresponding equilibrium points of the equivalent system (1.7).*

The linearized system associated to system (1.7) at the equilibrium point $\bar{X} = (\bar{x}, \bar{x}, \ldots, \bar{x}, \bar{y}, \bar{y}, \ldots, \bar{y})$, is given by

$$X_{n+1} = AX_n, \quad n \in \mathbb{N},$$

where $A$ is the Jacobian matrix of the map $F$ at the equilibrium point $\bar{X}$ given by

$$A = \begin{pmatrix}
\frac{\partial f_0}{\partial u_0}(\bar{X}) & \frac{\partial f_0}{\partial u_1}(\bar{X}) & \ldots & \frac{\partial f_0}{\partial u_k}(\bar{X}) & \frac{\partial f_0}{\partial v_0}(\bar{X}) & \frac{\partial f_0}{\partial v_1}(\bar{X}) & \ldots & \frac{\partial f_0}{\partial v_k}(\bar{X}) \\
\frac{\partial f_1}{\partial u_0}(\bar{X}) & \frac{\partial f_1}{\partial u_1}(\bar{X}) & \ldots & \frac{\partial f_1}{\partial u_k}(\bar{X}) & \frac{\partial f_1}{\partial v_0}(\bar{X}) & \frac{\partial f_1}{\partial v_1}(\bar{X}) & \ldots & \frac{\partial f_1}{\partial v_k}(\bar{X}) \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
\frac{\partial f_k}{\partial u_0}(\bar{X}) & \frac{\partial f_k}{\partial u_1}(\bar{X}) & \ldots & \frac{\partial f_k}{\partial u_k}(\bar{X}) & \frac{\partial f_k}{\partial v_0}(\bar{X}) & \frac{\partial f_k}{\partial v_1}(\bar{X}) & \ldots & \frac{\partial f_k}{\partial v_k}(\bar{X}) \\
\frac{\partial g_0}{\partial u_0}(\bar{X}) & \frac{\partial g_0}{\partial u_1}(\bar{X}) & \ldots & \frac{\partial g_0}{\partial u_k}(\bar{X}) & \frac{\partial g_0}{\partial v_0}(\bar{X}) & \frac{\partial g_0}{\partial v_1}(\bar{X}) & \ldots & \frac{\partial g_0}{\partial v_k}(\bar{X}) \\
\frac{\partial g_1}{\partial u_0}(\bar{X}) & \frac{\partial g_1}{\partial u_1}(\bar{X}) & \ldots & \frac{\partial g_1}{\partial u_k}(\bar{X}) & \frac{\partial g_1}{\partial v_0}(\bar{X}) & \frac{\partial g_1}{\partial v_1}(\bar{X}) & \ldots & \frac{\partial g_1}{\partial v_k}(\bar{X}) \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
\frac{\partial g_k}{\partial u_0}(\bar{X}) & \frac{\partial g_k}{\partial u_1}(\bar{X}) & \ldots & \frac{\partial g_k}{\partial u_k}(\bar{X}) & \frac{\partial g_k}{\partial v_0}(\bar{X}) & \frac{\partial g_k}{\partial v_1}(\bar{X}) & \ldots & \frac{\partial g_k}{\partial v_k}(\bar{X})
\end{pmatrix}$$

**Theorem 1.3.** 1. *If all the eigenvalues of the Jacobian matrix $A$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\bar{X}$ of system (1.7) is asymptotically stable.*

2. *If at least one eigenvalue of the Jacobian matrix $A$ have absolute value greater than one, then the equilibrium point $\bar{X}$ of system (1.7) is unstable.*

We give, in the following, two theorems (see [22, 54]) concerning the rate of convergence of the solutions of the system (1.6).

$$X_{n+1} = (A + B_n)X_n, \quad n \in \mathbb{N} \tag{1.8}$$

where $X_n$ is an k-dimensional vector, $A \in C^{k\times k}$ is a constant matrix and $B : \mathbb{Z}^+ \longrightarrow C^{k\times k}$ is a matrix function satisfying

$$||B_n|| \to 0, \ \textit{when } n \longrightarrow \infty, \tag{1.9}$$

where $||.||$ denotes any matrix norm which is associated with the vector norm.

**Theorem 1.4** (Perron's First Theorem)**.** *Consider system* (1.8) *and suppose condition* (1.9) *holds. If $X_n$ is a solution of* (1.8)*, then either $X_n = 0$ for all large $n$ or*

$$\theta = \lim_{n\to\infty} \sqrt[n]{||X_n||}$$

*exists and $\theta$ is equal to the modulus of one of the eigenvalues of the matrix $A$.*

**Theorem 1.5** (Perron's Second Theorem)**.** *Consider system* (1.8) *and suppose condition* (1.9) *holds. If $X_n$ is a solution of* (1.8)*, then either $X_n = 0$ for all large $n$ or*

$$\theta = \lim_{n\to\infty} \frac{||X_{n+1}||}{||X_n||}$$

*exists and $\theta$ is equal to the modulus of one of the eigenvalues of the matrix $A$.*

## 1.2 Applications to Biology

The study of natural phenomena and social sciences that develop in space and/or time by using the dynamical system is done by looking at the dynamic behavior or the geometrical and topological properties of the solution, whether a particular system results from Economics, Biology, Physics, Chemistry, or even social science such as population models, disease and infection model, etc. In this section, we will include two examples as specific cases derived from population modeling in biology. The first application is the Beverton-Holt Model with periodic environment and the other one is the Flour Beetle Model.

### 1.2.1 The Beverton-Holt Model With Periodic Environment

The Beverton-Holt model is a classical population model which has been considered in the literature for the discrete-time case. Its continuous-time analogue is the well-known logistic model.

The Beverton-Holt difference equation has wide applications in population growth [6], and it has been studied extensively in [13, 14, 23]. Firstly, in [13] Cushing and Henson studied the following difference equation

$$x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \quad x_0 \geq 0, \quad n = 0, 1, \ldots, \tag{1.10}$$

It is known that for $\mu > 1$, $K > 0$, all non-zero solutions converge to the positive equilibrium point $\bar{x} = K$, and for $\mu < 1$, $K > 0$, all solutions converge to the equilibrium point $\bar{x} = 0$. Later, a modification of this equation that emerged in the study of populations living in a periodically (seasonally) fluctuating environment replaces the constant carrying capacity $K$ with a periodic sequence $\{K_n\}$. Thus, the Beverton-Holt model with periodic environment is given by the following difference equation :

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, \quad x_0 \geq 0, \quad n = 0, 1, \ldots, \tag{1.11}$$

where $\mu > 1$ is a rate of change (growth or decay), $K_n > 0$ is a periodic sequence of period $p$ modeling periodicity of environment (periodic supply of food, energy, etc.), and $x_n$ is the size of population at $nth$ generation.

Assuming $x_n > 0$ and rewriting (1.11) as

$$\frac{1}{x_{n+1}} = \frac{K_n + (\mu - 1)x_n}{\mu K_n x_n} \tag{1.12}$$

the substitution $y_n = \dfrac{1}{x_n}$ reduces (1.11) to the linear non-autonomous equation

$$y_{n+1} = \frac{1}{\mu} y_n + p_n, \quad x_0 \geq 0, \quad n = 0, 1, \ldots, \tag{1.13}$$

where $p_n = \frac{(\mu-1)}{\mu K_n}$. The solution of (1.13) is given as

$$y_n = \frac{1}{\mu^n} y_0 + \sum_{k=0}^{n-1} \frac{1}{\mu^{n-k-1}} p_k \tag{1.14}$$

and it is well studied and understood and shows the following properties.

**Theorem 1.6.** *[8] Equation (1.13) has the following properties:*

1. *Equation (1.13) has the unique nonnegative periodic solution $\bar{y}_n$, with period equal to $p$.*

2. *The periodic solution $\{\bar{y}_n\}$ is the global attractor of all solutions of (1.13).*

3. *The periodic environment is deleterious in the sense that the size of population in periodic environment is smaller than the average of sizes in p constant environments. We say that in this case the periodic solution is an attenuant cycle. Mathematically, this means that*

$$\frac{1}{p}(\bar{y}_1 + \bar{y}_2 + \cdots + \bar{y}_p) < \frac{1}{p}((K_1 - 1) + \cdots + (K_p - 1)) \tag{1.15}$$

### 1.2.2 The Flour Beetle Model

In mathematical biology, the model of flour beetle (Tribolium) has attracted many researchers during the last few decades. In [19], B.Dennis *et al.* have proposed and studied the flour beetle population growth. They conducted both theoretical studies as well as experimental studies in laboratory. The life cycle of the flour beetles consists of larval and pupal stages each lasting approximately two weeks, followed by an adult stage, cannibalism occurs among the various groups. Adults feed on eggs, larvae, pupae, callows (young adults) while larvae eat eggs, pupae, and callows. Neither larvae nor adults eat mature adults and larvae do not feed on larvae. Cannibalism of larvae by adults and of pupae and callows by larvae typically occurs at much reduced rates and is assumed negligible in the model.

Let $L_n$, $P_n$ and $A_n$ are the number of feeding larvae, pupae, and non-feeding larvae, and adults, respectively, at time $n$, the unit of time is taken to be the feeding larval maturation period so that after one unit of time, a larva either dies or survives and pupates. This unit of time is also the time spent as a non-feeding larva, pupa and callow. Then the larval-pupal-adult (LPA) model, is a system of three difference equations:

$$\begin{cases} L_{n+1} = bA_n \exp(-c_{ea}A_n - c_{el}L_n), \\ P_{n+1} = L_n(1-\mu_l), \\ A_{n+1} = P_n \exp(-c_{pa}A_n) + A_n(1-\mu_a) \end{cases} \tag{1.16}$$

Where $b$ is a positive constant describing the number of eggs laid per adult per unit of time in the absence of cannibalism. The constants $\mu_l$ and $\mu_a$ are the larval and adult probability of dying form causes other than cannibalism, respectively. Thus $0 \le \mu_l \le 1$ and $0 \le \mu_a \le 1$. The term $\exp(-c_{ea}A_n)$ represents the probability that an egg is not eaten in the presence of $A_n$ adults, $\exp(-c_{el}L_n)$ represents the probability that an egg is not eaten in the presence of $L_n$ larvae and $\exp(-c_{pa}A_n)$ is the survival probability of a pupa in the presence of $A_n$ adults. The constants $c_{ea} \ge 0$, $c_{el} \ge 0$, $c_{pa} \ge 0$ are called the cannibalism "coefficients". It is assumed here that the only significant source of pupal mortality is adult cannibalism. Note that, the number $N = \frac{b(1-\mu_l)}{\mu_a}$ is called the inherent net reproductive number.

The LPA model of flour beetle with no larval cannibalism on eggs, that is the case when $c_{el} = 0$, the system (1.16) is equivalent to the following difference equation

$$A_{n+1} = (1-\mu_a)A_n + b(1-\mu_l)A_{n-2}\exp(-c_{ea}A_{n-2} - c_{pa}A_n). \quad n \ge 2, \tag{1.17}$$

when $L_0$, $P_0$ and $A_0$ are given and non-negative, we have

$$\begin{aligned} A_1 &= P_0 \exp(-c_{pa}A_0) + (1-\mu_a)A_0 \\ A_2 &= (1-\mu_l)L_0 \exp(-c_{pa}A_1) + (1-\mu_a)A_1 \end{aligned}$$

Set $\alpha = 1-\mu_a$, $\beta = b(1-\mu_l)$, $c_1 = c_{ea}$, $c_2 = c_{pa}$ $x_n = A_{n+2}$ for $n \geq -2$. Then equation (1.17) becomes

$$x_{n+1} = \alpha x_n + \beta x_{n-2} \exp(-c_1 x_{n-2} - c_2 x_n). \quad n \geq 0, \tag{1.18}$$

where

$$\begin{aligned} x_{-2} &= A_0 \\ x_{-1} &= A_1 = P_0 \exp(-c_{pa} A_0) + (1-\mu_a) A_0 \\ x_0 &= A_2 = (1-\mu_l) L_0 \exp(-c_{pa} A_1) + (1-\mu_a) A_1. \end{aligned}$$

In [41], Kuang and Cushing studied the global asymptotic stability of equilibrium points of the equation (1.18) as demonstrated in following theorem.

**Theorem 1.7.** *Equation (1.18) has the following properties:*

1. *If $\alpha + \beta \leq 1$, then every solution of the equation (1.18) converges to the zero equilibrium point.*

2. *If $\alpha + \beta > 1$, $\beta < \min\{e(1-\alpha), e\alpha \frac{c_1}{c_2}\}$ and $\max\{x_{-2}, x_{-1}, x_0\} > 0$, then*

$$\lim_{n\to\infty} \sup x_n \leq \frac{\beta}{c_1 e(1-\alpha)},$$

*and every solution of the equation (1.18) converges to the positive equilibrium point $\bar{x} = \frac{1}{c_1+c_2} \ln(\frac{\beta}{1-\alpha})$.*

1. *Equation (1.18) has a periodic solution of prime two $\{\ldots, \tau\theta, \theta, \tau\theta, \theta, \ldots\}$, $\tau \neq 1$, if and only if*

$$\beta^{\tau-1} = \frac{\tau(\tau-\alpha)^{\tau}}{(1-\alpha\tau)}. \tag{1.19}$$

# Chapter 2 Global Stability of Second Order Quadratic Rational Difference Equation

## 2.1 Introduction

Second order rational difference equation with quadratic terms show a wide variety of dynamic behaviors. It is shown that relying on the parameters and initial values there can be globally attracting equilibrium points.

In [44] Lazaryan *et al.* investigated the dynamic of the second order equation

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{2.1}$$

where

$$0 \leq a < 1, \quad \alpha, \beta, \gamma, A, B \geq 0, \quad \alpha + \beta + \gamma, A + B > 0, \quad C > 0, \tag{2.2}$$

with non-negative initial values. In their work, they demonstrated that when (2.2) holds then equation (2.1) typically does not have periodic solutions of period greater than two. At the end, they concluded with the following conjecture:
Study the following equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{2.3}$$

**Conjecture :** Let (2.2) hold and further assume that $b \geq 0$ and $a + b < 1$. Then, the equation (2.3) does not have any prime periodic solutions of period greater than two, investigated the equation (2.3) Dehghan *et al.* in [16]. They studied the global attractivity of the positive equilibrium point, the occurrence of periodic solution and they gave conditions for the occurrence of chaotic behavior.

In this chapter, we investigate the boundedness and local stability of solutions, the global attractivity of the positive equilibrium point and the existence of periodic solutions for the quadratic rational difference equation (2.3) where

$$0 \leq a < 1, \quad a+b < 1, \quad b \geq 0, \quad \alpha, \beta, \gamma, A, B \geq 0, \quad \alpha+\beta+\gamma, A+B > 0, \quad C > 0 \tag{2.4}$$

with non-negative parameters and initial values. We obtain sufficient conditions that imply the global asymptotic stability of the equilibrium point. We also obtain necessary and sufficient conditions for the occurrence of solutions of prime period two solution when $\gamma > 0$, and $aA + B > bB$.

## 2.2 Existence and Boundedness of Solutions

When (2.4) holds, we can assume that $C = 1$ in (2.3) without losing the generality by dividing the numerator and denominator of the fractional part by $C$ and relabeling the parameters. Thus, we consider

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1} \tag{2.5}$$

Note that the underlying function

$$f(u, v) = au + bv + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

is continuous on $[0, \infty) \times [0, \infty)$. The following result gives sufficient conditions for the positive solutions of (2.5) to be uniformly bounded from above and below by positive bounds.

**Lemma 2.1.** *Let* (2.4) *hold and assume further that*

$$\alpha = 0 \quad \text{if} \quad A = 0 \quad \text{and} \quad \beta = 0 \quad \text{if} \quad B = 0. \tag{2.6}$$

*Then every solution* $\{x_n\}$ *of* (2.5) *with non-negative initial values is bounded.*

*Proof.* Let

$$\delta_1 = \begin{cases} \frac{\alpha}{A} & \text{if} \quad A > 0 \\ 0 & \text{if} \quad A = 0 \end{cases} \tag{2.7}$$

$$\delta_2 = \begin{cases} \frac{\beta}{B} & \text{if} \quad B > 0 \\ 0 & \text{if} \quad B = 0 \end{cases} \tag{2.8}$$

By (2.4), $\rho = \delta_1 + \delta_2 + \gamma > 0$ *and for all* $n \geq 0$, we have

$$\begin{aligned}
x_{n+1} &= ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1} \\
&= ax_n + bx_{n-1} + \frac{\alpha x_n}{Ax_n + Bx_{n-1} + 1} + \frac{\beta x_{n-1}}{Ax_n + Bx_{n-1} + 1} + \frac{\gamma}{Ax_n + Bx_{n-1} + 1} \\
&\leq ax_n + bx_{n-1} + \frac{\alpha x_n}{Ax_n} + \frac{\beta x_{n-1}}{Bx_{n-1}} + \gamma \\
&\leq ax_n + bx_{n-1} + \delta_1 + \delta_2 + \gamma \\
&\leq ax_n + bx_{n-1} + \rho.
\end{aligned}$$

By using comparison, we can write the right hand side as follows

$$y_{n+1} = ay_n + by_{n-1} + \rho$$

and this equation is locally asymptotically stable if $a + b < 1$, and converges to the equilibrium point $\bar{y} = \frac{\rho}{1-(a+b)}$. Therefore,

$$\lim_{n\to\infty} \sup x_n \leq \frac{\rho}{1-(a+b)} = M.$$

Now, suppose that $\gamma > 0$. Then for all $n \geq N$

$$x_n \geq \frac{\gamma}{(A+B)M+1} = L.$$

□

**Theorem 2.1.** *Every solution of equation* (2.5) *is unbounded if* $a > 1$ *(or* $b > 1$*).*

*Proof.* Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (2.5), we see that

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1} > ax_n$$

also we notice that the right hand side can be written as follows

$$y_{n+1} = ay_n \quad \textit{implies that} \quad y_n = a^n y_0$$

and this equation is unstable because $a > 1$, and $\lim_{n\to\infty} y_n = \infty$. Then, by using ratio test,we find that $\{x_n\}_{n=-1}^{\infty}$ is unbounded from above (when $b > 1$ is similar). □

## 2.3 Existence and Local Stability of Unique Positive Equilibrium Points

**Lemma 2.2.** *If the condition (2.4) holds and $\gamma > 0$ then, equation (2.5) has a positive equilibrium point $\bar{x}$ that is uniquely given by*

$$\bar{x} = \frac{\alpha+\beta-(1-(a+b))+\sqrt{[\alpha+\beta-(1-(a+b))]^2+4(1-(a+b))(A+B)\gamma}}{2(1-(a+b))(A+B)}.$$

*Proof.* The equilibrium point of equation (2.5) must satisfy the following equation :

$$\bar{x} = a\bar{x} + b\bar{x} + \frac{\alpha\bar{x}+\beta\bar{x}+\gamma}{A\bar{x}+B\bar{x}+1}. \tag{2.9}$$

Then,

$$(1-(a+b))\bar{x} - \frac{\alpha\bar{x}+\beta\bar{x}+\gamma}{A\bar{x}+B\bar{x}+1} = 0$$

$$(1-(a+b))[(A+B)\bar{x}^2+\bar{x}] - (\alpha+\beta)\bar{x} - \gamma = 0$$

$$(1-(a+b))(A+B)\bar{x}^2 - [(\alpha+\beta)-(1-(a+b)]\bar{x} - \gamma = 0.$$

Let

$$d_1\bar{x}^2 - d_2\bar{x} - d_3 = 0, \qquad a+b<1, \qquad b\geq 0,$$

where

$$d_1 = (1-(a+b))(A+B), \quad d_2 = \alpha+\beta-1+(a+b), \quad d_3 = \gamma.$$

That is to say, the equilibrium points must be the roots of the quadratic equation:

$$s(t) = d_1t^2 - d_2t - d_3 \tag{2.10}$$

If the condition (2.4) holds, then $d_1 > 0$ and $d_3 \geq 0$. There are two more cases to consider.

Case 1: If $d_2 = 0$, then equation (2.10) has two roots given by:

$$t_{\pm} = \pm\sqrt{\frac{d_3}{d_1}}.$$

Thus, if $\gamma > 0$, then the unique positive fixed point of equation (2.5) is

$$\bar{x} = \sqrt{\frac{\gamma}{(1-(a+b))(A+B)}}.$$

Case 2: When $d_2 \neq 0$, then the roots of (2.10) are given by

$$t_{\pm} = \frac{d_2 \pm \sqrt{d_2^2 + 4d_1 d_3}}{2d_1}.$$

In particular, if $\gamma > 0$, then the unique positive fixed point of (2.5) is

$$\bar{x} = \frac{\alpha + \beta - (1-(a+b)) + \sqrt{[\alpha+\beta-(1-(a+b))]^2 + 4(1-(a+b))(A+B)\gamma}}{2(1-(a+b))(A+B)}. \tag{2.11}$$

□

Next, we consider the local stability of $\bar{x}$ under the hypotheses of the above Lemma. The characteristic equation associated with the linearization of equation (2.5) at the point $\bar{x}$ is given by

$$\lambda^2 - f_u(\bar{x},\bar{x})\lambda - f_v(\bar{x},\bar{x}) = 0 \tag{2.12}$$

where

$$f(u,v) = au + bv + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

with

$$\begin{aligned} f_u(u,v) &= a + \frac{(\alpha B - A\beta)v + \alpha - A\gamma}{(Au+Bv+1)^2} \\ &= a + \frac{\alpha}{Au+Bv+1} - \frac{A(\alpha u + \beta v + \gamma)}{Au+Bv+1} \times \frac{1}{Au+Bv+1} \end{aligned}$$

and

$$\begin{aligned} f_v(u,v) &= b + \frac{(A\beta - \alpha B)u + \beta - B\gamma}{(Au+Bv+1)^2} \\ &= b + \frac{\beta}{Au+Bv+1} - \frac{B(\alpha u + \beta v + \gamma)}{Au+Bv+1} \times \frac{1}{Au+Bv+1} \end{aligned}$$

Now, from the equation (2.9), we have

$$(1-(a+b))\bar{x} = \frac{(\alpha+\beta)\bar{x} + \gamma}{(A+B)\bar{x}+1}.$$

Yields

$$\begin{aligned} f_u(\bar{x},\bar{x}) &= a + \frac{\alpha}{(A+B)\bar{x}+1} - \frac{A((\alpha+\beta)\bar{x}+\gamma)}{(A+B)\bar{x}+1} \times \frac{1}{(A+B)\bar{x}+1} \\ &= a + \frac{\alpha}{(A+B)\bar{x}+1} - \frac{A((1-(a+b))\bar{x}}{(A+B)\bar{x}+1} \end{aligned}$$

and

$$\begin{aligned} f_v(\bar{x},\bar{x}) &= b + \frac{\beta}{(A+B)\bar{x}+1} - \frac{B((\alpha+\beta)\bar{x}+\gamma)}{(A+B)\bar{x}+1} \times \frac{1}{(A+B)\bar{x}+1} \\ &= b + \frac{\beta}{(A+B)\bar{x}+1} - \frac{B((1-(a+b))\bar{x}}{(A+B)\bar{x}+1} \end{aligned}$$

We define

$$f_u(\bar{x},\bar{x}) = a + \frac{\alpha - (1-(a+b))A\bar{x}}{(A+B)\bar{x}+1} = p$$

and

$$f_v(\bar{x},\bar{x}) = b + \frac{\beta - (1-(a+b))B\bar{x}}{(A+B)\bar{x}+1} = q.$$

Then, the equation (2.12) is equivalent to the equation

$$\lambda^2 - p\lambda - q = 0. \tag{2.13}$$

To solve the equation (2.13), we calculate $p^2+4q$ and we have two cases:

Case 1: when $p^2+4q<0$, the two roots of the equation (2.13) are complex if $p^2+4q<0$ or $q < -(\frac{p}{2})^2$, namely,

$$\lambda_1 = \frac{p - i\sqrt{-(p^2+4q)}}{2}$$

and

$$\lambda_2 = \frac{p + i\sqrt{-(p^2+4q)}}{2}.$$

Note that the fixed point $\bar{x}$ is locally asymptotically stable if both roots of equation (2.13) are inside the unit disk of the complex plain. In this case

$$|\lambda_1| = \sqrt{(\frac{p}{2})^2 + (\frac{\sqrt{-(p^2+4q)}}{2})^2} = \sqrt{-q} = |\lambda_2|.$$

So both roots have modulus less then 1 if and only if $q > -1$ or equivalently, $q+1>0$, i,e.,

$$b + \frac{\beta - (1-(a+b))B\bar{x}}{(A+B)\bar{x}+1} + 1 > 0$$

$$((a+b)B + b(A+B) + A)\bar{x} + b + \beta + 1 > 0.$$

This is clearly true if the condition (2.4) holds. So if the condition(2.4) holds, and $\gamma > 0$ and if $-1 < q < -\frac{p^2}{4}$ then $\bar{x}$ is locally asymptotically stable with complex roots.

Case 2: when $p^2+4q \geq 0$, the two roots of the equation (2.13) are real if $p^2+4q \geq 0$ or $q \geq -(\frac{p}{2})^2$, namely,

$$\lambda_1 = \frac{p - \sqrt{p^2+4q}}{2}$$

and

$$\lambda_2 = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

Now, $\bar{x}$ is locally asymptotically stable if and only if $|\lambda_1| < 1$, and $|\lambda_2| < 1$. First, observe that $\lambda_2 < 1$ if and only if $p + q < 1$, or equivalently

$$a + \frac{\alpha - (1 - (a+b))A\bar{x}}{(A+B)\bar{x} + 1} + b + \frac{\beta - (1 - (a+b))B\bar{x}}{(A+B)\bar{x} + 1} < 1$$

and

$$a + b + \frac{\alpha - (1 - (a+b))A\bar{x} + \beta - (1 - (a+b))B\bar{x}}{(A+B)\bar{x} + 1} < 1$$

$$(a+b)(A+B)\bar{x} + (a+b) + \alpha - (1 - (a+b))A\bar{x} + \beta - (1 - (a+b))B\bar{x} - (A+B)\bar{x} - 1 < 0$$

$$2(A+B)[(a+b) - 1]\bar{x} + (\alpha + \beta) + (a+b) - 1 < 0$$

$$2(A+B)[1 - (a+b)]\bar{x} > (\alpha + \beta) - [1 - (a+b)] \tag{2.14}$$

which is true if (2.4) holds and $\gamma > 0$, see (2.11). Next, note that $p < 2$. To see this, $p - 2 < 0$ if and only if

$$a + \frac{\alpha - (1 - (a+b))A\bar{x}}{(A+B)\bar{x} + 1} - 2 < 0$$

$$(a - 2)[(A+B)\bar{x} + 1] + \alpha - (1 - (a+b))A\bar{x} < 0$$

$$\alpha - (1 - (a+b))A\bar{x} - (2 - a)[(A+B)\bar{x} + 1] < 0. \tag{2.15}$$

From (2.14), we have

$$(2 - a)(A+B)\bar{x} = (2 - 2a + a + 2b - 2b)[(A+B)\bar{x}]$$

$$= 2(1 - (a+b))[(A+B)\bar{x}] + (a + 2b)[(A+B)\bar{x}]$$

$$(2 - a)(A+B)\bar{x} > (\alpha + \beta) - [1 - (a+b)] + (a + 2b)(A+B)\bar{x}.$$

It follows that

$$\begin{aligned}
&\alpha - (1 - (a+b))A\bar{x} - (2 - a)[(A+B)\bar{x} + 1] \\
&= -(1 - (a+b))A\bar{x} - (2 - a)(A+B)\bar{x} - (2 - a) + \alpha \\
&< -\alpha - \beta + [1 - (a+b)] - (a + 2b)(A+B)\bar{x} - (1 - (a+b))A\bar{x} - (2 - a) + \alpha \\
&< -(1 - (a+b))A\bar{x} - 1 - b - \beta - (a + 2b)(A+B)\bar{x} \\
&< 0.
\end{aligned}$$

This proves that (2.15) is true. Finally, $p > -2$. Since this is equivalent to

$$a + \frac{\alpha - (1-(a+b))A\bar{x}}{(A+B)\bar{x}+1} > -2$$

$$\alpha - (1-(a+b))A\bar{x} > -(2+a)[(A+B)\bar{x}+1]$$

$$-A\bar{x} + aA\bar{x} + bA\bar{x} + (2+a)(A+B)\bar{x} > -\alpha - (2+a)$$

$$(-1+a+2+a+b)A\bar{x} + (2+a)B\bar{x} > -\alpha - (2+a)$$

or

$$(1+2a+b)A\bar{x} + (2+a)B\bar{x} > -\alpha - (2+a)$$

which is true if (2.4) holds and $\gamma > 0$. Now, a routine calculation shows that $\lambda_2 < 1$ if and only if

$$\frac{p+\sqrt{p^2+4q}}{2} < 1$$

$$\sqrt{p^2+4q} < 2-p$$

$$p^2+4q < 4+p^2-4p$$

$$p+q < 1$$

$$q < 1-p$$

which is indeed the case shown by the above calculations.
Next, $\lambda_2 > -1$ if and only if

$$p + \sqrt{p^2+4q} > -2 \tag{2.16}$$

If $p > -2$, then (2.16) holds trivially. On the other hand, if $p \leq -2$ or $p+2 \leq 0$, then

$$(a+2)[(A+B)\bar{x}+1] + \alpha - (1-(a+b))A\bar{x} \leq 0$$

$$(1+2a+b)A\bar{x} + (2+a)[B\bar{x}+1] + \alpha \leq 0$$

which is impossible if (2.4) holds. It follows that $|\lambda_2| < 1$ if (2.4) holds and $\gamma > 0$.
Next, we consider $\lambda_1$ and note that $\lambda_1 < 1$ if and only if $p - \sqrt{p^2+4q} < 2$. This is clearly true if $p < 2$ which is in fact the case and we conclude that $\lambda_1 < 1$ if (2.4) holds and $\gamma > 0$.

Next, $\lambda_1 > -1$ if and only if

$$p - \sqrt{p^2+4q} > -2.$$

This requires that $p > -2$, which is true if (2.4) holds and $\gamma > 0$. Now the above inequality reduces to $p + 1 > q$ or

$$1 + a + \frac{\alpha - (1 - (a+b))A\bar{x}}{(A+B)\bar{x} + 1} > b + \frac{\beta - (1 - (a+b))B\bar{x}}{(A+B)\bar{x} + 1}$$

$$\beta - (1 - (a+b))B\bar{x} - \alpha + (1 - (a+b))A\bar{x} < (a + 1 - b)[(A+B)\bar{x} + 1]$$

$$\beta - \alpha - (1 + a - b) < 2(B + aA - bB)\bar{x}. \tag{2.17}$$

We also note that if the reverse of the above inequality holds, i.e.,

$$2(Aa + B - bB)\bar{x} < \beta - \alpha - (1 + a - b). \tag{2.18}$$

Then the above calculation shows that $\lambda_1 < -1$ while $|\lambda_2| < 1$. Therefore, in this case $\bar{x}$ is a saddle point. If $\beta - \alpha - (1 + a - b) \leq 0$ then (2.18) does not hold and $\bar{x}$ is locally asymptotically stable.

The proceeding calculations in particular prove the following

**Lemma 2.3.** *let the condition* (2.4) *holds and* $\gamma > 0$. *Then the positive equilibrium point* $\bar{x}$ *of* (2.5) *is locally asymptotically stable if and only if* (2.17) *holds and a saddle point if and only if the reverse inequality, i.e.,* (2.18) *holds.*

Since $\bar{x}$ is non-hyperbolic if neither (2.17) nor (2.18) holds, Lemma(2.3) gives a complete picture of the local stability of $\bar{x}$ under its stated hypotheses.

## 2.4 Global Attractivity of the Positive Equilibrium Point

In this section, we give sufficient conditions for the global attractivity of the positive fixed point. The following general result from [30]

**Lemma 2.4.** *let $I$ be an open interval of real numbers and suppose that $f \in (I^m, \mathbb{R})$ is non-decreasing in each coordinate. Let $\bar{x} \in I$ be fixed point of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, ....., x_{n-m+1}) \tag{2.19}$$

*and assume that the function $h(t) = f(t, ....., t)$ satisfies the conditions*

$$h(t) > t \quad \text{if} \quad t < \bar{x} \quad \text{and} \quad h(t) < t \quad \text{if} \quad t > \bar{x} \quad t \in I. \tag{2.20}$$

*Then $I$ is an invariant interval of (2.19) and $\bar{x} > 0$ attracts all solutions with initial values in $I$.*

We now use the preceding result to obtain the sufficient conditions for the global attractivity of the positive equilibrium point.

**Theorem 2.2.** *(a) Assume that (2.4) holds with $\gamma > 0$ and suppose that $f(u, v)$ is non-decreasing in both arguments. Then (2.5) has unique fixed point $\bar{x} > 0$ that is asymptotically stable and attracts all positive solutions of (2.5).*

*(b) Assume that (2.4) holds with $\gamma > 0$ and*

$$B\alpha - 2Ab \leq A\beta \leq 2aB + \alpha B, \quad B\gamma \leq b + \beta, \quad A\gamma \leq a + \gamma. \tag{2.21}$$

*Then, equation (2.5) has a unique equilibrium point $\bar{x} > 0$ that is asymptotically stable and attracts all positive solutions of (2.5).*

*Proof.* (a) The existence and uniqueness of $\bar{x} > 0$ follows from Lemma (2.2).

Next, the function $h$ in (2.20) takes the form

$$h(t) = (a + b)t + \frac{(\alpha + \beta)t + \gamma}{(A + B)t + 1}.$$

Note that the equilibrium point $\bar{x}$ of (2.5) is a solution of the equation $h(t) = t$. So, we verify that conditions (2.20) hold for $t > 0$. The function $h$ may be written as

$$h(t) = t\phi(t), \quad \textit{where} \quad \phi(t) = a + b + \frac{\alpha + \beta + \frac{\gamma}{t}}{(A+B)t + 1}.$$

Note that $\phi(\bar{x}) = \frac{h(\bar{x})}{\bar{x}} = 1$. Further,

$$\phi'(t) = \frac{-[(A+B)t+1]\frac{\gamma}{t^2} - (A+B)[\alpha + \beta + \frac{\gamma}{t}]}{[(A+B)t+1]^2}$$

so $\phi$ is decreasing (strictly) for all $t > 0$. Therefore,

$$t < \bar{x} \quad \textit{implies} \quad \phi(t) > \phi(\bar{x})$$

$$t\phi(t) > t\phi(\bar{x})$$

$$h(t) > t$$

and

$$t > \bar{x} \quad \textit{implies} \quad \phi(t) < \phi(\bar{x})$$

$$t\phi(t) < t\phi(\bar{x})$$

$$h(t) < t.$$

Now, by Lemma (2.4) $\bar{x}$ attracts all positive solutions of (2.5). In particular, $\bar{x}$ is not a saddle point so by lemma (2.3), it is asymptotically stable.

(b) We show that if the inequalities (2.21) hold, then the function

$$f(u,v) = au + bv + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

is non-decreasing in each of its two coordinates $u$, $v$. This is demonstrated by computing the partial derivatives $f_u$ and $f_v$ to show that, $f_u \geq 0$ and $f_v \geq 0$. By direct calculation $f_u \geq 0$ iff

$$a + \frac{\alpha A u + \alpha B v + \alpha - A\alpha u - A\beta v - A\gamma}{(Au + Bv + 1)^2} \geq 0$$

$$a(Au + Bv)^2 + (2aB + \alpha B - A\beta)v + 2aAu + a + \alpha - A\gamma \geq 0.$$

The above inequality holds for all $u$, $v > 0$ if

$$A\gamma \leq a + \alpha, \quad A\beta \leq 2aB + \alpha B. \tag{2.22}$$

Similarly, $f_v \geq 0$ iff

$$f_v = b + \frac{\beta(Au + Bv + 1) - B(\alpha u + \beta v + \gamma)}{(Au + Bv + 1)^2}$$

$$b[(Au + Bv)^2 + 1 + 2(Au + Bv)] + \beta Au + \beta Bv + \beta - B\alpha u - B\beta v - B\gamma \geq 0$$

$$b(Au + Bv)^2 + 2bBv + (2Ab + \beta A - B\alpha)u + b + \beta - B\gamma \geq 0$$

the above inequality holds for all $u, v > 0$ if

$$B\gamma \leq b + \beta, \quad A\beta \geq -2Ab + B\alpha. \tag{2.23}$$

By the inequality (2.22) and (2.23), conditions (2.21) are sufficient for the function $f$ to be nondecreasing in each of its coordinates.

□

## 2.5 Periodic Solutions

The following theorem gives necessary and sufficient conditions for the existence of positive period two solutions of equation (2.5) when $aA + B > bB$.

**Theorem 2.3.** *Assume that (2.4) holds with $\gamma > 0$, and $aA + B > bB$. Then (2.5) has a positive prime period two solution if and only if the following conditions are satisfied.*

1. $\beta - \alpha - (1 + a - b) > 0$;
2. $(A - B) > 0$;
3. $\dfrac{4\gamma}{(A - B)(1 + a - b)}$
$< \dfrac{\beta - \alpha - (1 + a - b)}{(aA + B - bB)} \left[ \dfrac{\beta - \alpha - (1 + a - b)}{(aA + B - bB)} - 4\dfrac{aA(\beta + b - 1) + B(1 - b)(a + \alpha)}{(A - B)(1 + a - b)(aA + B - bB)} \right].$

*Proof.* Suppose that there exists prime period two solution

$$\dots, \phi, \psi, \phi, \psi, \dots$$

of equation (2.5), with $\phi, \psi > 0$ and $\phi \neq \psi$ we see from equation (2.5) that

$$\phi = a\psi + b\phi + \frac{\alpha\psi + \beta\phi + \gamma}{A\psi + B\phi + 1}$$

$$\psi = a\phi + b\psi + \frac{\alpha\phi + \beta\psi + \gamma}{A\phi + B\psi + 1}.$$

Then

$$(1-b)\phi - a\psi = \frac{\alpha\psi + \beta\phi + \gamma}{A\psi + B\phi + 1} \quad \textit{and} \quad (1-b)\psi - a\phi = \frac{\alpha\phi + \beta\psi + \gamma}{A\phi + B\psi + 1}.$$

Yields

$$(1-b)A\phi\psi + (1-b)B\phi^2 + (1-b)\phi - aA\psi^2 - aB\phi\psi - a\psi = \alpha\psi + \beta\phi + \gamma \tag{2.24}$$

$$(1-b)A\phi\psi + (1-b)B\psi^2 + (1-b)\psi - aA\phi^2 - aB\phi\psi - a\phi = \alpha\phi + \beta\psi + \gamma \tag{2.25}$$

Subtracting (2.24) from (2.25) gives

$$(1-b)(\phi^2 - \psi^2) + (1-b)(\phi - \psi) + aA(\phi^2 - \psi^2) + a(\phi - \psi) = \alpha(\psi - \phi) + \beta(\phi - \psi).$$

Yields

$$(\phi - \psi)[((1-b)B + aA)(\phi + \psi) + 1 - b + a + \alpha - \beta] = 0.$$

Since $\phi \neq \psi$, it follows that

$$\phi + \psi = \frac{\beta - \alpha - (1 + a - b)}{aA + B - bB} \tag{2.26}$$

Since $aA + B > bB$, we infer from (2.26) that $\beta - \alpha - (1 + a - b) > 0$ is a necessary condition for the existence of positive period two solutions.
Again, adding (2.24) and (2.25) yields
$2(1-b)A\phi\psi + (1-b)B(\phi^2 + \psi^2) + (1-b)(\phi + \psi) - aA(\phi^2 + \psi^2) - 2aB\phi\psi - a(\phi + \psi)$
$= \alpha(\phi + \psi) + \beta(\phi + \psi) + 2\gamma,$
then,

$$2((1-b)A - aB)\phi\psi + ((1-b)B - aA)(\phi^2 + \psi^2) + (1 - b - \alpha - a - \beta)(\phi + \psi) = 2\gamma, \tag{2.27}$$

it follows by (2.26), (2.27) and the relation

$$(\phi^2 + \psi^2) = (\phi + \psi)^2 - 2\phi\psi \quad \textit{for all} \quad \phi, \psi \in \mathbb{R},$$

that

$$2((1-b)A - aB)\phi\psi + ((1-b)B - aA)(\phi + \psi)^2 - 2((1-b)B - aA)\phi\psi + (1 - b - \alpha - a - \beta)(\phi + \psi) = 2\gamma$$

and

$$2((1-b)A-aB-(1-b)B+aA)\phi\psi = (\phi+\psi)[(aA-(1-b)B)(\phi+\psi)+(b+\beta-1+(a+\alpha))]+2\gamma,$$

then,

$$\begin{aligned}
&2(A-B)(1+a-b)\phi\psi \\
&= \frac{\phi+\psi}{aA+B-bB} \\
&[(aA+bB-B)(\beta-\alpha-(1+a-b))+(\alpha+\beta-(1-a-b))(aA+B-bB)]+2\gamma \\
&= \frac{\phi+\psi}{aA+B-bB} \\
&[(aA+bB-B)(\beta+b-1-(a+\alpha))+(aA-(bB-B))(\beta+b-1+(a+\alpha))]+2\gamma \\
&= \frac{\phi+\psi}{aA+B-bB}[2aA(\beta+b-1)+2B(a+\alpha)(1-b)]+2\gamma.
\end{aligned}$$

Thus,

$$(A-B)(1+a-b)\phi\psi = \left[\frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)^2}\right][aA(\beta+b-1)+B(1-b)(a+\alpha)]+\gamma. \tag{2.28}$$

Since from (2.26) we have $\beta-\alpha-(1+a-b)>0$, then $\beta+b-1>0$. Thus, the right hand side of (2.28) is positive and therefore, $A-B>0$ is another necessary condition for the existence of positive period two solutions and

$$\phi\psi = \frac{1}{(A-B)(1+a-b)}\left[\left(\frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)^2}\right)(aA(\beta+b-1)+B(1-b)(a+\alpha))+\gamma\right]. \tag{2.29}$$

Let

$$S = \frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)}$$

and

$$P = \frac{1}{(A-B)(1+a-b)}\left[\left(\frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)^2}\right)(aA(\beta+b-1)+(B-bB)(a+\alpha))+\gamma\right]$$

with $S,P>0$. Now, it is clear from (2.26) and (2.29) that $\phi$ and $\psi$ are the two distinct positive real roots of the quadratic equation

$$t^2-St+P=0$$

with

$$t = \frac{S \pm \sqrt{S^2 - 4P}}{2}$$

which will be the case if and only if

$$S^2 - 4P > 0$$

implies that

$$\frac{4\gamma}{(A-B)(1+a-b)} < \frac{\beta - \alpha - (1+a-b)}{(aA+B-bB)} \left[ \frac{\beta - \alpha - (1+a-b)}{(aA+B-bB)} - 4\frac{aA(\beta+b-1)+B(1-b)(a+\alpha)}{(A-B)(1+a-b)(aA+B-bB)} \right]. \quad \square$$

We also study the nonexistence of period two solutions.

**Theorem 2.4.** *Let $D$ be a subset of real numbers and assume that $f : D \times D \to D$ is non-decreasing in $x \in D$ for each $y \in D$ and non-increasing in $y \in D$ for each $x \in D$. Then, the difference equation $x_{n+1} = f(x_n, x_{n-1})$ has no prime period two solution.*

*Proof.* Assume that the above difference equation has prime period two solution. Then there exist real numbers $\phi$ and $\psi$, such that

$$f(\phi, \psi) = \psi \quad \textit{and} \quad f(\psi, \phi) = \phi.$$

When $\phi = \psi$, we are done. So assume that $\phi \neq \psi$. If $\phi < \psi$, then by the hypothesis

$$f(\phi, \psi) \leq f(\psi, \psi) \leq f(\psi, \phi)$$

which implies that $\psi \leq \phi$, which is contradiction. Similarly, if $\phi > \psi$, then by hypothesis

$$f(\psi, \phi) \leq f(\psi, \psi) \leq f(\phi, \psi)$$

which implies that $\phi \leq \psi$, which is also contradiction. $\square$

Now, we establish the connection between existence of prime period two solutions and stability of equilibrium point.

**Theorem 2.5.** *Let (2.4) hold with $\gamma$, $aA + B - bB > 0$. Then (2.5) has a positive prime period two solution if and only if $\bar{x}$ is saddle.*

*Proof.* First, when $\alpha + \beta - (1 - (a+b)) = 0$, then the equilibrium point

$$\bar{x} = \sqrt{\frac{\gamma}{(1-(a+b))(A+B)}}.$$

This implies that

$$\beta - \alpha - (1 + a - b) = -2a - 2\alpha < 0$$

and $\bar{x}$ must be stable so (2.5) has no prime period two solution.

Now assume that $\alpha + \beta - (1 - (a + b)) \neq 0$. Then the equilibrium point is given by

$$\bar{x} = \frac{\alpha + \beta - (1 - (a + b)) + \sqrt{[\alpha + \beta - (1 - (a + b))]^2 + 4(1 - (a + b))(A + B)\gamma}}{2(1 - (a + b))(A + B)}.$$

By Lemma (2.3), $\bar{x}$ is saddle if and only if

$$\bar{x} < \frac{\beta - \alpha - (1 + a - b)}{2(aA + B - bB)}$$

which implies that $\beta - \alpha - (1 + a - b) > 0$.

Now,

$$\bar{x} < \frac{\beta - \alpha - (1 + a - b)}{2(aA + B - bB)},$$

implies that

$$\frac{\alpha + \beta - (1 - (a + b)) + \sqrt{[\alpha + \beta - (1 - (a + b))]^2 + 4(1 - (a + b))(A + B)\gamma}}{2(1 - (a + b))(A + B)}$$
$$< \frac{\beta - \alpha - (1 + a - b)}{2(aA + B - bB)}$$

iff

$$\frac{\sqrt{(\alpha + \beta - (1 - (a + b)))^2 + 4(1 - (a + b))(A + B)\gamma}}{(1 - (a + b))(A + B)}$$
$$< \frac{\beta - \alpha - (1 + a - b)}{(aA + B - bB)} - \frac{\alpha + \beta - (1 - (a + b))}{(1 - (a + b))(A + B)}$$

iff

$$\sqrt{(\alpha + \beta - (1 - (a + b)))^2 + 4(1 - (a + b))(A + B)\gamma}$$
$$< \frac{(\beta - \alpha - (1 + a - b))(1 - (a + b))(A + B)}{(aA + B - bB)} - [\alpha + \beta - (1 - (a + b))]$$

iff

$$
\begin{aligned}
&(\alpha+\beta-(1-(a+b)))^2+4(1-(a+b))(A+B)\gamma \\
&\quad<\frac{(\beta-\alpha-(1+a-b))^2(1-(a+b))^2(A+B)^2}{(aA+B-bB)^2} \\
&\quad-2\frac{(\beta-\alpha-(1+a-b))(1-(a+b))(A+B)(\alpha+\beta-(1-(a+b)))]}{(aA+B-bB)} \\
&\quad+(\alpha+\beta-(1-(a+b)))^2
\end{aligned}
$$

$$
\begin{aligned}
&4(1-(a+b))(A+B)\gamma \\
&\quad<\frac{(\beta-\alpha-(1+a-b))^2(1-(a+b))^2(A+B)^2}{(aA+B-bB)^2} \\
&\quad-2\frac{(\beta-\alpha-(1+a-b))(1-(a+b))(A+B)(\alpha+\beta-(1-(a+b)))}{(aA+B-bB)}
\end{aligned}
$$

iff

$$
\begin{aligned}
4\gamma&<\frac{(\beta-\alpha-(1+a-b))^2(1-(a+b))(A+B)}{(aA+B-bB)^2} \\
&-2\frac{(\beta-\alpha-(1+a-b))(\alpha+\beta-(1-(a+b)))}{(aA+B-bB)} \\
&=\frac{(\beta-\alpha-(1+a-b))}{(aA+B-bB)} \\
&\left[\frac{(\beta-\alpha-(1+a-b))(1-(a+b))(A+B)}{(aA+B-bB)}-2(\alpha+\beta-(1-(a+b)))\right] \\
&=\frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)} \\
&\times\left[\frac{(\beta-\alpha-(1+a-b))(1-(a+b))(A+B)-2(aA+B-bB)(\alpha+\beta-(1-(a+b)))}{(aA+B-bB)}\right].
\end{aligned}
$$

Adding and subtracting $(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]$ to the numerator of the second fraction in previous equation yields

$$
\begin{aligned}
&(\beta-\alpha-(1+a-b))(1-(a+b))(A+B)-2(aA+B-bB)(\alpha+\beta-(1-(a+b)))\\
&=(\beta-\alpha-(1+a-b))(1-(a+b))(A+B)-2(aA+B-bB)(\alpha+\beta-(1-(a+b)))\\
&+(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]-(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]\\
&=(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]+[\beta-\alpha-(1+a-b)][(1-(a+b))(A+B)\\
&-(1+(a-b))(A-B)]-2(aA+B-bB)(\alpha+\beta-(1-(a+b)))\\
&=(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]+(\beta+b-1+(a+\alpha))(-2aA-2B+2bB)\\
&+(\beta+b-1-(a+\alpha))(-2aA+2B-2bB)\\
&=(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]-4aA(\beta+b-1)-4B(a+\alpha)+4bB(a+\alpha)\\
&=(1+(a-b))(A-B)[\beta-\alpha-(1+a-b)]-4[aA(\beta+b-1)+B(1-b)(a+\alpha)].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
4\gamma &< \frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)}\\
&\times\frac{(1+(a-b))(A-B)\,[\beta-\alpha-(1+a-b)]-4[aA(\beta+b-1)+B(1-b)(a+\alpha)]}{(aA+B-bB)}.
\end{aligned}
$$

Since $\gamma>0$, it must be the case that the right hand side of last expression is positive, which implies $A-B>0$. Dividing both sides of the above expression by $(A-B)(1+a-b)$ then yields:

$$
\begin{aligned}
&\frac{4\gamma}{(A-B)(1+a-b)}\\
&<\frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)}\left[\frac{\beta-\alpha-(1+a-b)}{(aA+B-bB)}-4\frac{aA(\beta+b-1)+B(1-b)(a+\alpha)}{(A-B)(1+a-b)(aA+B-bB)}\right]
\end{aligned}
$$

and the proof is complete, since the conditions of Theorem (2.3) are satisfied. □

We end our discussion in this section with the following immediate consequence of the results already established.

**Lemma 2.5.** *Let (2.4) hold with $\gamma$, $aA+B-bB>0$, and suppose that $f(u,v)$ is non-decreasing in $u$ and either non-decreasing or non-increasing in $v$.*

1. *Equation (2.5) has no periodic solution of period greater than two.*
2. *If equation (2.5) has no prime period two solution then all solutions of (2.5) converge to the positive fixed point $\bar{x}$.*

## 2.6 Numerical Examples

For confirming the theoretical results, we consider some numerical examples which represent different types of solutions to equation (2.5) as follows:

**Example 2.1.** *Consider the equation*

$$x_{n+1} = 0.2x_n + 0.3x_{n-1} + \frac{0.5x_n + 4.8x_{n-1} + 3.3}{5.9x_n + 7.5x_{n-1} + 1} \tag{2.30}$$

*with the initial conditions* $x_{-1} = 0.3$, $x_0 = 3.5$. *In this case, the condition* (2.4) *holds and the inequality* (2.17) *holds. Then, from Lemma* (2.3) *the equilibrium point* $\bar{x} = 1.15$ *which was given by the formula calculated in Lemma* (2.2) *of the equation* (2.30) *is locally asymptotically stable. See Figure 2.1.*

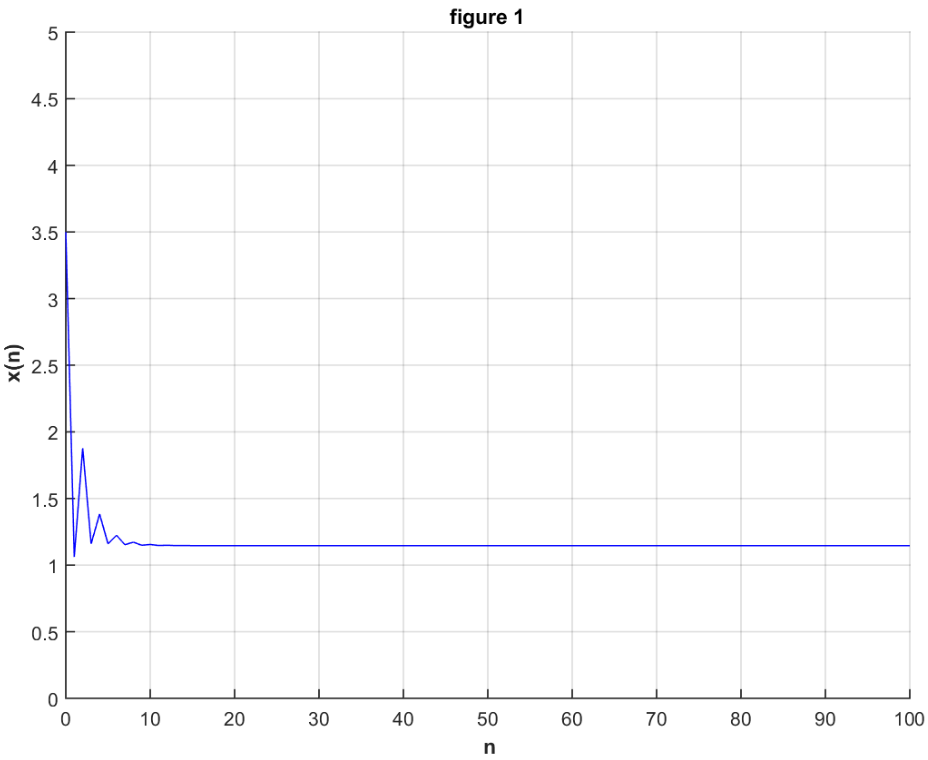


Figure 2.1: Plot of the solution $\{x_n\}_{n\geq 0}$ of the equation (2.30) with the initial values $x_{-1} = 0.3$, $x_0 = 3.5$.

**Example 2.2.** *Consider the equation*

$$x_{n+1} = 0.5x_n + 0.3x_{n-1} + \frac{2.5x_n + 1.8x_{n-1} + 0.3}{2.4x_n + 1.5x_{n-1} + 1} \tag{2.31}$$

*with the initial conditions* $x_{-1} = 0.03$, $x_0 = 0.5$. *In this case, the conditions* (2.21) *are satisfied. Then, the equilibrium point* $\bar{x}$ *of the equation* (2.31) *is asymptotically stable and attracts all positive solutions of equation* (2.31). *See Figure 2.2.*

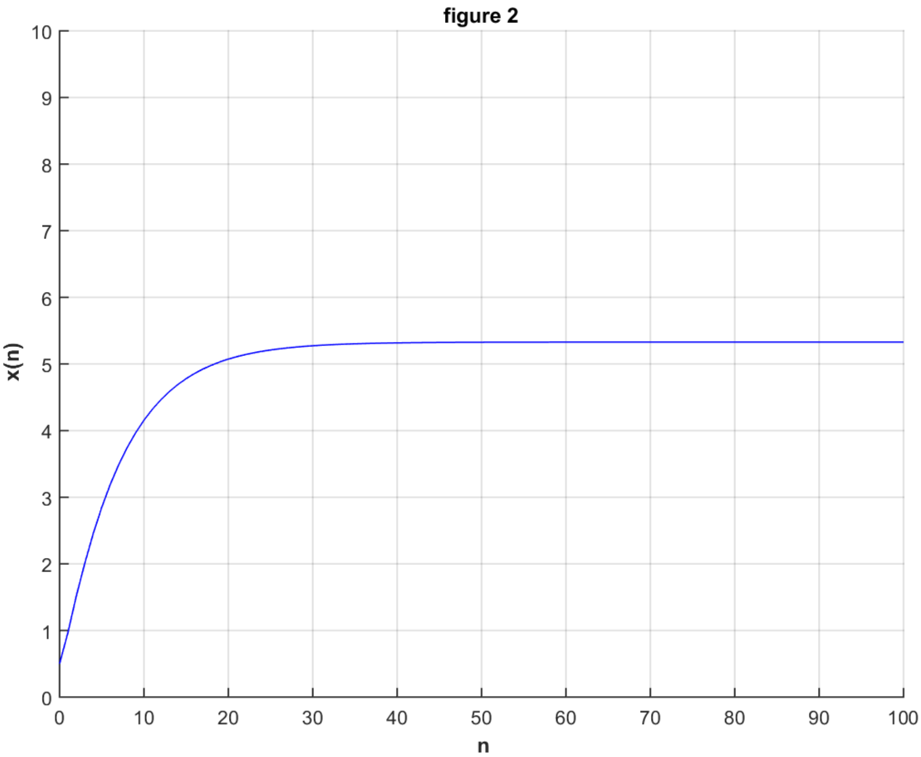


Figure 2.2: Plot of the solution $\{x_n\}_{n\geq 0}$ of the equation (2.31) with the initial values $x_{-1} = 0.03$, $x_0 = 0.5$.

**Example 2.3.** *Consider the equation*

$$x_{n+1} = 0.5x_n + 0.6x_{n-1} + \frac{0.02x_n + 0.03x_{n-1} + 0.06}{3.04x_n + 0.4x_{n-1} + 1} \tag{2.32}$$

*with the initial conditions* $x_{-1} = 1.03$, $x_0 = 0.05$. *In this case,* $1-(a+b) < 0$ *the condition* (2.4) *is not satisfied. Then, the equilibrium point* $\bar{x}$ *of the equation* (2.32) *is unstable. See Figure 2.3.*

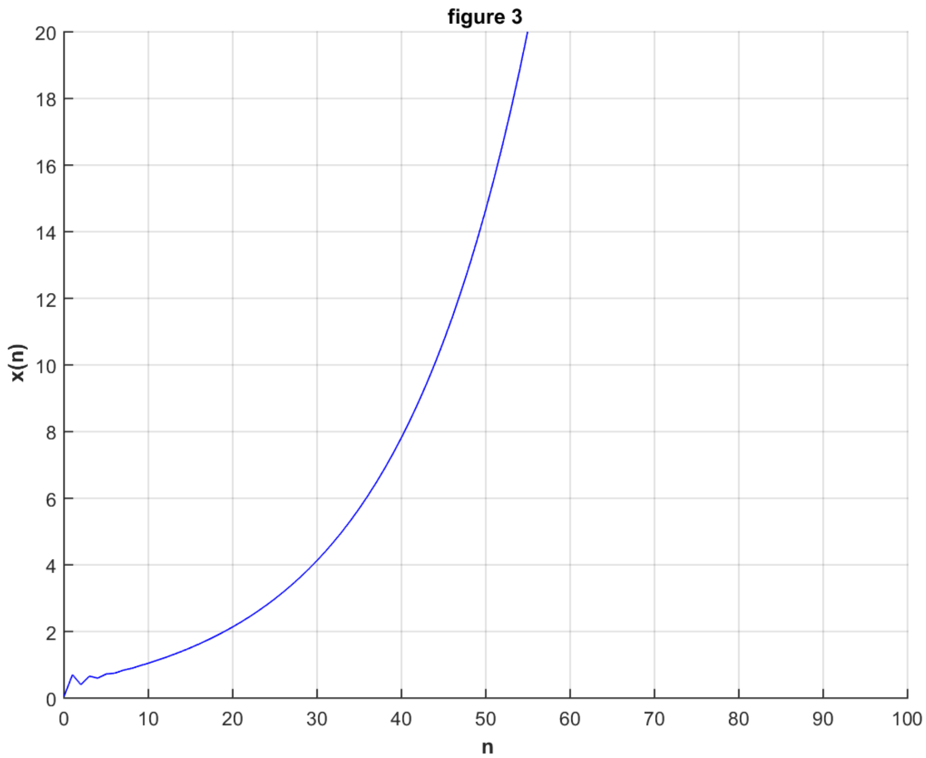


Figure 2.3: Plot of the solution $\{x_n\}_{n\geq 0}$ of the equation (2.32) with the initial values $x_{-1} = 1.03$, $x_0 = 0.05$.

**Example 2.4.** *Consider the equation*

$$x_{n+1} = 0.01x_n + 0.8x_{n-1} + \frac{0.5x_n + 1.8x_{n-1} + 0.02}{6x_n + 3x_{n-1} + 1} \tag{2.33}$$

*with the initial conditions $x_{-1} = 4.4$, $x_0 = 0.07$. In this case, the conditions of Theorem (2.3) are satisfied. Then, the solutions of the equation (2.33) have a positive prime period two solution. See Figure 2.4.*

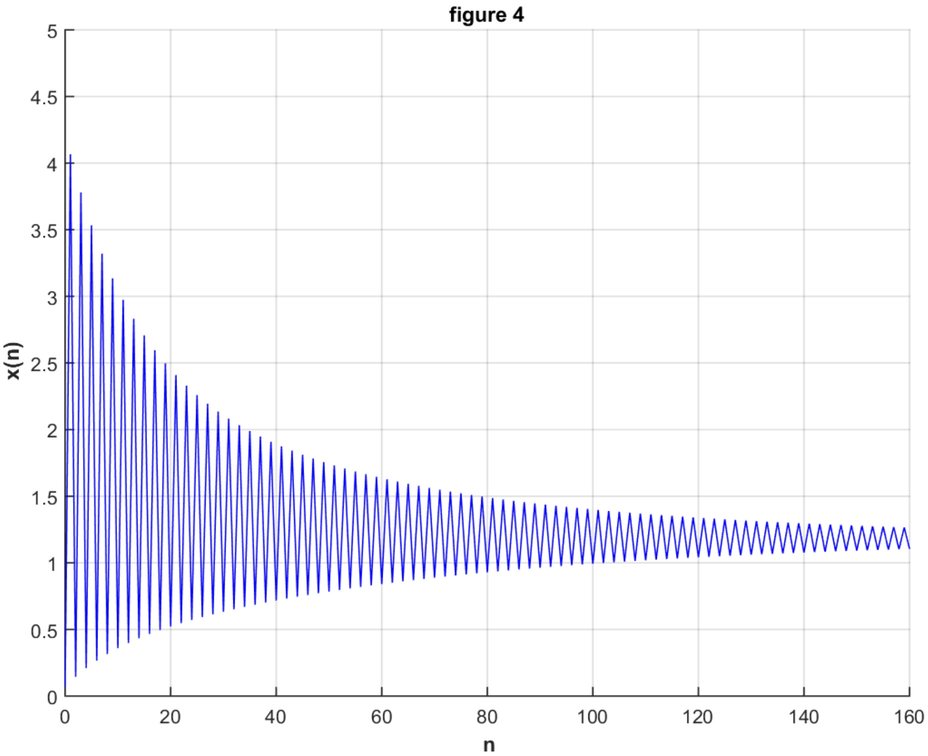


Figure 2.4: Plot of the solution $\{x_n\}_{n\geq 0}$ of the equation (2.33) with the initial values $x_{-1} = 4.4$, $x_0 = 0.07$.

# Chapter 3 On the Global Behavior of Higher-Order Non-autonomous Rational Difference Equation

## 3.1 Introduction

In this chapter, we study the global behavior of the more general rational difference equation

$$x_{n+1} = \frac{\alpha_n + x_{n-r}}{\alpha_n + x_{n-k}}, \quad n = 0, 1, ..., \tag{3.1}$$

where $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence of positive numbers and $r < k$ are positive integers. by taking the sequence $\{\alpha_n\}_{n\geq 0}$ to be bounded. The study of this equation is extension of the works of the mathematicians Dekkar *et al.* [18] and Kerker *et al.* [37] which was the study first proposed by E.Camouzis and G.Ladas in their monograph [11] in which they treated the global asymptotic behavior of higher order rational difference equations. Besides, the importance of this equation itself, the study of these rational difference equations has offered prototypes that played an essential role in the development of the theory of nonlinear difference equations.

Recently, a huge interest was accorded to the study of difference equations

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + C x_{n-k}}, \quad n \in \mathbb{N}, \tag{3.2}$$

with their different particular cases, where the parameters $\alpha$, $\beta$, $A$, $C$ are non-negative real numbers, and $k$ is positive integer (see for example [11, 39, 42]).

In [39], Kocic *et al.* studied the following higher order difference equation

$$y_{n+1} = \frac{a + b y_n}{A + y_{n-k}}, \quad n \in \mathbb{N}, \tag{3.3}$$

with $a, b, A$ are nonnegative real numbers and $k$ is a positive integer. They showed, among others, that the positive equilibrium point of the Eq. (3.3) is globally asymptotically stable. These results were extended in [18] and [37] to the non-autonomous rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \quad n = 0, 1, ... \tag{3.4}$$

Precisely, Dekkar *et al.* [18] considered Eq. (3.4) in the case where $\{\alpha_n\}_{n\geq 0}$ is a periodic sequence of positive numbers with period $T$, while Kerker *et al.* [37] studied Eq. (3.4) when $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence. Recently, Kerker and Bouaziz [36] studied the oscillation and the global attractivity for the more general difference equation (3.1) with $\{\alpha_n\}_{n\geq 0}$ is a convergent sequence of positive numbers and $r < k$ are positive integers. For more related works see [2, 5, 11, 28, 29, 45, 47, 56, 57].

In our work, we deal with a generalization of Eq. (3.1) by taking the sequence $\{\alpha_n\}_{n\geq 0}$ to be bounded. Our discussion starts with the oscillation about the equilibrium point $\bar{x} = 1$ (Theorem 3.1), then the boundedness of the positive solutions (Theorem 3.2), after, analysis the global attractor (Theorem 3.5) and finally the global asymptotic stability ( Theorem 3.6).

## 3.2 Oscillation of Positive Solutions

In this section, we study the oscillatory behavior of positive solutions of Eq. (3.1).

**Theorem 3.1.** *Every positive solution of (3.1) oscillates about $\bar{x} = 1$.*

*Proof.* Assume that Eq. (3.1) has a nonoscillatory solution. Then, there exists $n_0 \geq -k$ such that

$$x_n > 1, \quad \text{for all } n \geq n_0$$

or

$$x_n < 1, \quad \text{for all } n \geq n_0.$$

Suppose that $x_n > 1$, $\forall\, n \geq n_0$. So, for $n \geq n_0 + k$, we have

$$x_{n+1} = x_{n-r}\frac{(\alpha_n/x_{n-r} + 1)}{\alpha_n + x_{n-k}} < x_{n-r}\frac{\alpha_n + 1}{\alpha_n + x_{n-k}} < x_{n-r}. \tag{3.5}$$

Let $p$ be the smallest integer in $\{n_0 + k, ..., n_0 + k + r\}$ such that

$$x_p = \max\{x_i,\ i = n_0 + k, ..., n_0 + k + r\}.$$

Therefore, there exists a non-negative integer $m$ and $j \in \{0, ..., r\}$, such that

$$k - r + p = m(r + 1) + j.$$

Hence, we get

$$x_{n_0+2k+p+1} = \frac{\alpha_{n_0+2k+p} + x_{n_0+2k-r+p}}{\alpha_{n_0+2k+p} + x_{n_0+k+p}}$$

$$= \frac{\alpha_{n_0+2k+p} + x_{n_0+k+m(r+1)+j}}{\alpha_{n_0+2k+p} + x_{n_0+k+p}}.$$

Consequently, by using (3.5) we have two cases:

Case 1: If $m = 0$, we obtain

$$x_{n_0+2k+p+1} = \frac{\alpha_{n_0+2k+p} + x_{n_0+k+j}}{\alpha_{n_0+2k+p} + x_{n_0+k+p}}$$

$$\leq \frac{\alpha_{n_0+2k+p} + x_{n_0+k+p}}{\alpha_{n_0+2k+p} + x_{n_0+k+p}} = 1.$$

Case 2: If $m \geq 1$, we obtain

$$x_{n_0+2k+p+1} < \frac{\alpha_{n_0+2k+p} + x_{n_0+k+j}}{\alpha_{n_0+2k+p} + x_{n_0+k+p}}$$

$$\leq \frac{\alpha_{n_0+2k+p} + x_{n_0+k+p}}{\alpha_{n_0+2k+p} + x_{n_0+k+p}} = 1.$$

Then, in both cases we have a contradiction, and the proof is complete.

□

To confirm our result on the oscillatory behavior of the positive solutions of Eq. (3.1), we consider the two following numerical examples.

**Example 3.1.** *We consider the following fifth order difference equation*

$$y_{n+1} = \frac{[\frac{(-1)^n+4}{4}]^2 + y_{n-2}}{[\frac{(-1)^n+4}{4}]^2 + y_{n-4}} \tag{3.6}$$

*with the initial values $y_{-4} = 1.2$, $y_{-3} = 0.9$, $y_{-2} = 0.7$, $y_{-1} = 1.9$, $y_0 = 1.3$. The solution of Eq. (3.6) is oscillatory about $\bar{y} = 1$, see Fig. 3.1.*

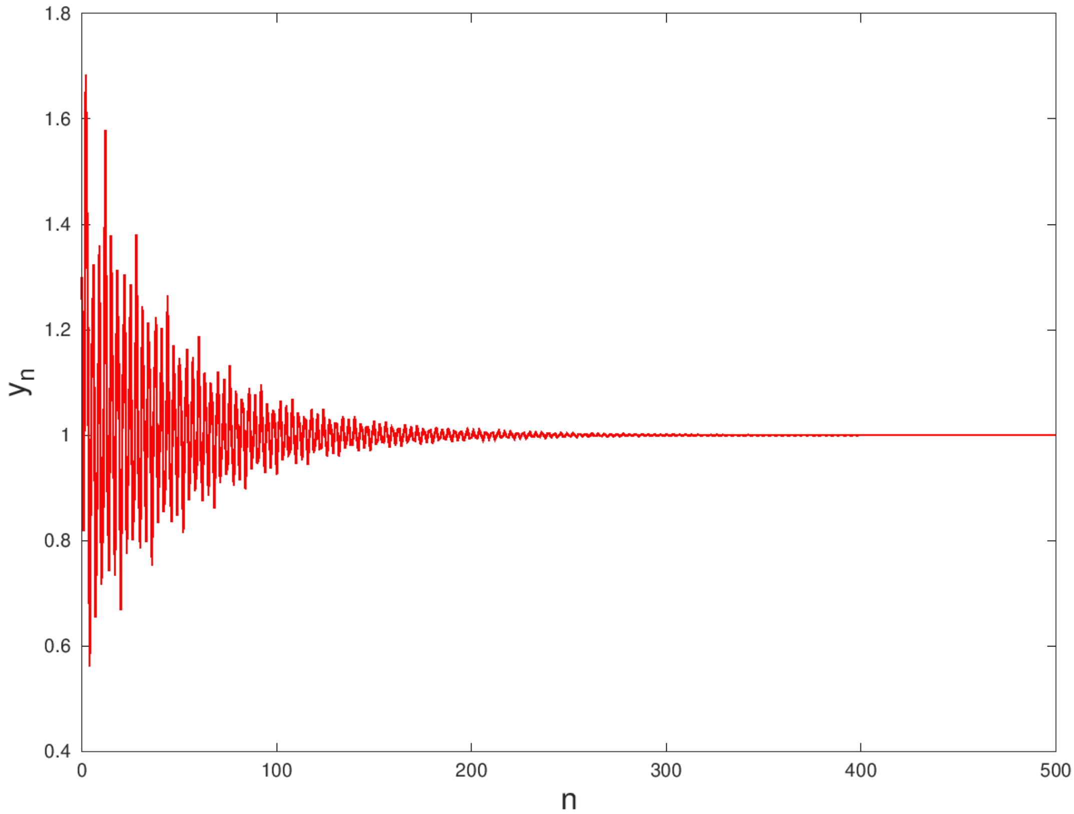


Figure 3.1: Plot of the solution of equation Eq. (3.6) with the initial values $y_{-4} = 1.2$, $y_{-3} = 0.9$, $y_{-2} = 0.7$, $y_{-1} = 1.9$, $y_0 = 1.3$.

**Example 3.2.** *We consider the following rational difference equation*

$$y_{n+1} = \frac{\frac{n+12}{7n+8} + y_{n-2}}{\frac{n+12}{7n+8} + y_{n-11}} \tag{3.7}$$

*with the initial values* $y_{-11} = 0.02$, $y_{-10} = 3.5$, $y_{-9} = 9.1$, $y_{-8} = 1.2$, $y_{-7} = 3.3$, $y_{-6} = 0.5$, $y_{-5} = 2.8$, $y_{-4} = 1.5$, $y_{-3} = 1.9$, $y_{-2} = 2.7$, $y_{-1} = 11.5$, $y_0 = 0.3$. *The solution of Eq. (3.7) is oscillatory about* $\bar{y} = 1$, *see Fig. 3.2.*

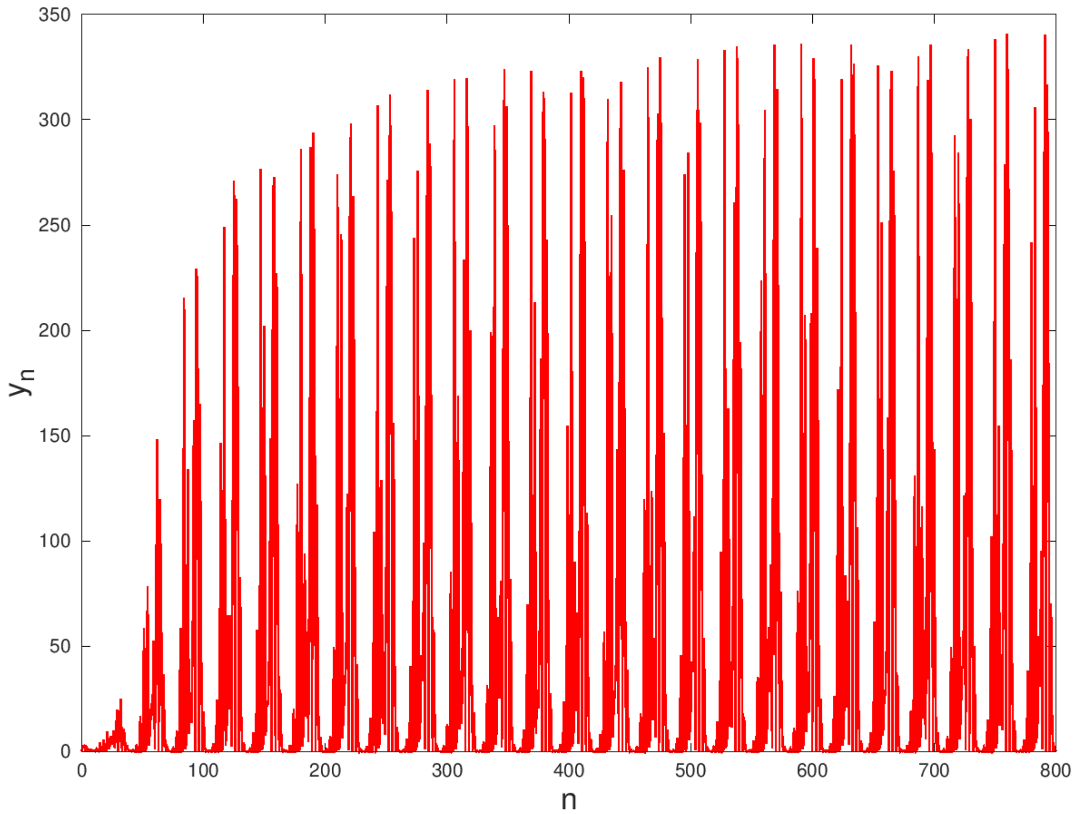


Figure 3.2: Plot of the solution of Eq. (3.7) with the initial values $y_{-11} = 0.02$, $y_{-10} = 3.5$, $y_{-9} = 9.1$, $y_{-8} = 1.2$, $y_{-7} = 3.3$, $y_{-6} = 0.5$, $y_{-5} = 2.8$, $y_{-4} = 1.5$, $y_{-3} = 1.9$, $y_{-2} = 2.7$, $y_{-1} = 11.5$, $y_0 = 0.3$

## 3.3 Boundedness of Positive Solutions

Hereafter, we shall use the following notations

$$\alpha = \lim \alpha_n, \qquad a = \inf\{\alpha_n\} \qquad \text{and} \qquad A = \sup\{\alpha_n\}.$$

We have the following result.

**Theorem 3.2.** *Assume that*

$$a > 1. \tag{3.8}$$

*Then, every positive solution of (3.1) is bounded.*

*Proof.* We have

$$x_{n+1} = \frac{\alpha_n + x_{n-r}}{\alpha_n + x_{n-k}} \leq \frac{A}{a} + \frac{1}{a} x_{n-r}, \tag{3.9}$$

which gives (see [22, p. 77])

$$x_n \leq \frac{A}{a-1} + a^{-\frac{n}{r+1}} \sum_{i=0}^{r} c_i n^i. \tag{3.10}$$

Since $a > 1$, the right hand side of inequality (3.10) tends to $\frac{A}{a-1}$ as $n \to \infty$. Then, by Theorem 1.1, there exists $M > 0$, such that

$$x_n \leq M, \qquad \forall n \geq k.$$

Hence, Eq. (3.1) yields

$$x_{n+1} \geq \frac{a}{A+M} = m > 0.$$

□

**Theorem 3.3.** *Assume that $a \leq 1$ and that there exists a positive integer $m$ such that*

$$k = r + m(r+1). \tag{3.11}$$

*Then, every positive solution of (3.1) is bounded.*

*Proof.* Let $\{x_n\}_{n \geq -k}$ be a positive solution of (3.1). Assume, for the sake of contradiction, that the solutions are unbounded. Then, there exists a sub-sequence $\{x_{n_i+1}\}$ such that

$$\lim_{i\to\infty} x_{n_i+1} = +\infty, \qquad x_{n_i+1} = \max\{x_n : n \leq n_i + 1\}.$$

From (3.9) we have

$$\lim_{i\to\infty} x_{n_i-r} = +\infty. \tag{3.12}$$

Furthermore,

$$0 \leq x_{n_i+1} - x_{n_i-r} = \frac{\alpha_{n_i}(1 - x_{n_i-r}) + (1 - x_{n_i-k})x_{n_i-r}}{\alpha_{n_i} + x_{n_i-k}}$$

which implies that

$$x_{n_i-k} \leq \frac{\alpha_{n_i}}{x_{n_i-r}} + 1 - \alpha_{n_i}. \tag{3.13}$$

In view of (3.12) and (3.13), we see that $\{x_{n_i-k}\}$ is bounded. Hence, by applying (3.9) repeatedly we obtain

$$x_{n_i-r} \leq \begin{cases} m + x_{n_i-k}, & \text{if } a = 1, \\ \frac{A}{1-a}(a^{-m} - 1) + a^{-m}x_{n_i-k}, & \text{if } a < 1, \end{cases}$$

which implies that $\{x_{n_i-r}\}$ is also bounded. This is a contradiction. □

## 3.4 Global Asymptotic Stability

In this section, we investigate the global asymptotic stability of the equilibrium point. First, we have the following local stability result.

**Theorem 3.4.** *Assume that (3.8) holds. Then, $\bar{x} = 1$ is stable.*

*Proof.* Choose $M > A/(a-1)$ such that

$$x_{-k}, ..., x_0 \in \left(\frac{1}{A+M}, M\right).$$

Since $a > 1$ and from Theorem (3.2), we have

$$\frac{a}{A+M} \leq x_n \leq M$$

then

$$\frac{1}{A+M} < x_n \leq M$$

Therefore,

$$x_n \in \left(\frac{1}{A+M}, M\right), \qquad \forall n \geq -k. \tag{3.14}$$

Next, setting

$$M(\varepsilon) = \min\left\{1 + \varepsilon, \frac{1}{1-\varepsilon} - A\right\}$$

and

$$\delta(\varepsilon) = \min\left\{M(\varepsilon) - 1, 1 - \frac{1}{A + M(\varepsilon)}\right\},$$

for $\varepsilon \in (0,1)$, we obtain

$$(1-\delta, 1+\delta) \subseteq \left(\frac{1}{A+M}, M\right) \subseteq (1-\varepsilon, 1+\varepsilon). \tag{3.15}$$

Now, if we take $x_{-k}, ..., x_0 \in \mathbb{R}_+$ with $|x_{-k} - 1| + |x_{-k+1} - 1| + ... + |x_0 - 1| < \delta$, then (3.14), combined with (3.15), yields

$$|x_n - 1| < \varepsilon, \qquad \forall n \geq -k,$$

and so $\bar{x}$ is stable. □

In the next theorem, we establish the global attractivity of the equilibrium point.

**Theorem 3.5.** *Assume that (3.8) holds. Then, $\bar{x} = 1$ is the global attractor of all positive solutions of Eq. (3.1).*

*Proof.* Let $\{x_n\}_{n \geq -k}$ be an arbitrary positive solution of (3.1). Set

$$I = \liminf_{n\to\infty} x_n \qquad \text{and} \qquad S = \limsup_{n\to\infty} x_n$$

which by Theorem 3.2 exist. Let $\{n_p\}$ and $\{n_q\}$ be an infinite increasing sequences of positive integers such that

$$\lim_{q\to\infty} x_{n_q+1} = I \quad \text{and} \quad \lim_{p\to\infty} x_{n_p+1} = S.$$

By taking sub-sequences, if necessary, we assume that $\{\alpha_{n_p}\}_p$, $\{\alpha_{n_q}\}_q$, $\{x_{n_p-r}\}_p$, $\{x_{n_q-r}\}_q$, $\{x_{n_p-k}\}_p$ and $\{x_{n_q-k}\}_q$ converge to $A_0, a_0, L_r, l_r, L_k$ and $l_k$ respectively. Clearly

$$l_r, L_r, l_k, L_k \in [I, S] \quad \text{and} \quad a_0, A_0 \in [a, A].$$

Then, the Eq. (3.1) yields

$$I = \frac{a_0 + l_r}{a_0 + l_k} \geq \frac{a_0 + I}{a_0 + S}$$

and

$$S = \frac{A_0 + L_r}{A_0 + L_k} \leq \frac{A_0 + S}{A_0 + I}.$$

Since the function $(x+I)/(x+S)$ is non-decreasing, we have

$$I \geq \frac{a+I}{a+S}. \tag{3.16}$$

Similarly, since $(x+S)/(x+I)$ is non-increasing, we obtain

$$S \leq \frac{a+S}{a+I}. \tag{3.17}$$

Combining (3.16) with (3.17) gives

$$a+(1-a)I \leq IS \leq a+(1-a)S.$$

Consequently, since $a>1$ we obtain $I \geq S$, and so the sequence $\{x_n\}$ is convergent to the unique limit $l=1$. □

From Theorems 3.4 and 3.5 we obtain the following result.

**Theorem 3.6.** *Assume that (3.8) holds. Then, the equilibrium point $\bar{x}=1$ of Eq. (3.1) is globally asymptotically stable.*

Next, when condition (3.8) does not hold, we have

**Theorem 3.7.** *Assume that $a>0$ and $k=2r+1$. Then $\bar{x}$ is the global attractor of all positive solutions of Eq. (3.1).*

*Proof.* Let $\{x_n\}_{n\geq -k}$ be an arbitrary positive solution of (3.1). In view of Theorem 3.1, it suffices to show that all positive solutions of Eq. (3.1) which are oscillatory about $\bar{x}$ are attracted to it. Setting

$$z_n^{(i)} = x_{n(r+1)+i}, \qquad \text{for all } n \geq n_0+k \text{ and } i=0,...,r,$$

it follows from Eq. (3.1) that

$$z_{n+1}^{(i)} = x_{(n+1)(r+1)+i} = x_{n(r+1)+i+r+1}$$

$$= \frac{\alpha_{n(r+1)+i+r} + x_{n(r+1)+i+r-r}}{\alpha_{n(r+1)+i+r} + x_{n(r+1)+i+r-k}},$$

we have $k = 2r + 1$ yields:

$$z_{n+1}^{(i)} = \frac{\alpha_{n(r+1)+i+r} + x_{n(r+1)+i}}{\alpha_{n(r+1)+i+r} + x_{n(r+1)+i+r-2r-1}}$$

$$= \frac{\alpha_{n(r+1)+i+r} + x_{n(r+1)+i}}{\alpha_{n(r+1)+i+r} + x_{(n-1)(r+1)+i}}$$

$$= \frac{\beta_n^{(i)} + z_n^{(i)}}{\beta_n^{(i)} + z_{n-1}^{(i)}}, \quad \text{for all } n \geq 0 \text{ and } i = 0, ..., r,$$

where $\beta_n^{(i)} = \alpha_{n(r+1)+i+r}$. Hence, all sub-sequences $\left\{z_n^{(i)}\right\}_n$, $i = 0, ..., r$, satisfy the same second order difference equation

$$z_{n+1} = \frac{\beta_n + z_n}{\beta_n + z_{n-1}}. \tag{3.18}$$

In the sequel, we will show that $\bar{x} = 1$ is the global attractor of all positive solutions of Eq. (3.18). Let $\{z_n\}_{n\geq -k}$ be an oscillatory positive solution of (3.18), and let $\{p_i\}$ and $\{q_i\}$ be sequences of integers such that $x_{p_0} < 1$ and for $i = 0, 1, 2, ...$

$$\{z_{p_i+1}, ..., z_{q_i}\} \quad \text{is a positive semicycle,} \tag{3.19}$$

i.e.,

$$z_n \geq 1, \quad n \in (p_i + 1, \ldots, q_i), \quad z_{p_i} < 1, \quad \text{and} \quad z_{q_i+1} < 1,$$

and

$$\{z_{q_i+1}, ..., z_{p_{i+1}}\} \quad \text{is a negative semicycle,} \tag{3.20}$$

i.e.,

$$z_n < 1, \quad n \in (q_i + 1, \ldots, p_{i+1}), \quad z_{q_i} \geq 1, \quad \text{and} \quad z_{p_{i+1}+1} \geq 1.$$

For each $i = 0, 1, 2, ...$, let $P_i$ and $Q_i$ be the smallest integers in $\{p_i + 1, ..., q_i\}$ and $\{q_i + 1, ..., p_{i+1}\}$, respectively, such that

$$z_{P_i} = \max\{z_{p_i+1}, ..., z_{q_i}\} \quad \text{and} \quad z_{Q_i} = \min\{z_{q_i+1}, ..., z_{p_{i+1}}\}.$$

From (3.18) it follows that the extreme point in any semicycle occurs in one of the first two terms of the semicycle. Consequently, we have $\forall i \geq 0$

$$p_i + 1 \leq P_i \leq p_i + 2 \quad \text{and} \quad q_i + 1 \leq Q_i \leq q_i + 2. \tag{3.21}$$

Let

$$I = \liminf_{n\to\infty} z_n = \liminf_{i\to\infty} z_{Q_i} \quad \text{and} \quad S = \limsup_{n\to\infty} z_n = \limsup_{i\to\infty} z_{P_i}. \tag{3.22}$$

Next, by Theorems 3.2 and 3.3, $I$ and $S$ exist, and they satisfy

$$0 < I \leq 1 \leq S < \infty.$$

By definition of $I$ and $S$, $\forall\varepsilon,\ 0 < \varepsilon < I$ and $\delta > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$I - \varepsilon \leq z_n \leq S + \delta, \quad \forall n \geq n_0.$$

Now, we consider the positive semicycle (3.19) with $p_i \geq n_0 + 1$. Then, we have

$$\begin{aligned} z_{n-1} &\geq I - \varepsilon, \quad \text{for } n = p_i, ..., P_i - 1, \\ z_n &\geq 1, \quad \text{for } n = p_i + 1, P_i - 1. \end{aligned}$$

From (3.21) we distinguish two cases:

**Case 1**: $P_i = p_i + 1$. In this case we have

$$\begin{aligned} z_{P_i} = z_{p_i+1} &= \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}} \\ &\leq \frac{\beta_{p_i} + 1}{\beta_{p_i} + I - \varepsilon}. \end{aligned}$$

Since the function $(x+1)/(x+I-\varepsilon)$ is non-increasing, we obtain

$$z_{P_i} \leq \frac{a+1}{a+I-\varepsilon}, \quad \forall\varepsilon > 0,$$

and so

$$S \leq \frac{a+1}{a+I}. \tag{3.23}$$

**Case 2**: $P_i = p_i + 2$. In this case, we have

$$\begin{aligned} z_{P_i} = z_{p_i+2} &= \frac{z_{p_i+2}}{z_{p_i+1}} \times z_{p_i+1} \\ &= \frac{\beta_{p_i+1}/z_{p_i+1} + 1}{\beta_{p_i+1} + z_{p_i}} \times \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}} \\ &\leq \frac{\beta_{p_i+1} + 1}{\beta_{p_i+1} + z_{p_i}} \times \frac{\beta_{p_i} + z_{p_i}}{\beta_{p_i} + z_{p_i-1}} \end{aligned}$$

and since the function $(x+1)/(x+z_{p_i})$ is non-increasing, we have

$$z_{P_i} \leq \frac{a+1}{a+z_{p_i}} \times \frac{\beta_{p_i}+z_{p_i}}{\beta_{p_i}+z_{p_i-1}}.$$

To estimate the last term in the right hand side of this inequality, we have:

*Case 2-a*: If $z_{p_i-1} \leq z_{p_i}$, then the function $(x+z_{p_i})/(x+z_{p_i-1})$ is non-increasing, and then

$$\begin{aligned} z_{P_i} &\leq \frac{a+1}{a+z_{p_i}} \times \frac{a+z_{p_i}}{a+z_{p_i-1}} \\ &\leq \frac{a+1}{a+I-\varepsilon}, \quad \forall \varepsilon > 0. \end{aligned}$$

*Case 2-b*: If $z_{p_i-1} \geq z_{p_i}$, then

$$\begin{aligned} z_{P_i} &\leq \frac{a+1}{a+z_{p_i}} \times \frac{\beta_{p_i}+z_{p_i}}{\beta_{p_i}+z_{p_i}} \\ &\leq \frac{a+1}{a+I-\varepsilon}, \quad \forall \varepsilon > 0. \end{aligned}$$

Therefore, in the two sub-cases we obtain the inequality (3.23).

Similarly, we consider the negative semicycle, (3.20) with $q_i \geq n_0+1$. Then, we have

$$\begin{aligned} z_{n-1} &\leq S+\delta, \quad \text{for } n = q_i, ..., Q_i - 1, \\ z_n &< 1, \quad \text{for } n = q_i+1, Q_i - 1. \end{aligned}$$

From (3.21) we distinguish two cases:

**Case 1**: $Q_i = q_i + 1$. In this case we have

$$\begin{aligned} z_{Q_i} = z_{q_i+1} &= \frac{\beta_{q_i}+z_{q_i}}{\beta_{q_i}+z_{q_i-1}} \\ &\geq \frac{\beta_{q_i}+1}{\beta_{q_i}+S+\delta}. \end{aligned}$$

Since the function $(x+1)/(x+S+\delta)$ is non-decreasing, we obtain

$$z_{Q_i} \geq \frac{a+1}{a+S+\delta}, \quad \forall \varepsilon > 0,$$

and so

$$I \geq \frac{a+1}{a+S}. \tag{3.24}$$

**Case** 2: $Q_i = q_i + 2$. In this case, we have

$$\begin{aligned} z_{Q_i} = z_{q_i+2} &= \frac{z_{q_i+2}}{z_{q_i+1}} \times z_{q_i+1} \\ &= \frac{\beta_{q_i+1}/z_{q_i+1} + 1}{\beta_{q_i+1} + z_{q_i}} \times \frac{\beta_{q_i} + z_{q_i}}{\beta_{q_i} + z_{q_i-1}} \\ &> \frac{\beta_{q_i+1} + 1}{\beta_{q_i+1} + z_{q_i}} \times \frac{\beta_{q_i} + z_{q_i}}{\beta_{q_i} + z_{q_i-1}} \end{aligned}$$

and since the function $(x+1)/(x+z_{q_i})$ is non-decreasing, we have

$$z_{Q_i} \geq \frac{a+1}{a+z_{q_i}} \times \frac{\beta_{q_i} + z_{q_i}}{\beta_{q_i} + z_{q_i-1}}.$$

To estimate the last term in the right hand side of this inequality, we have:

*Case 2-a*: If $z_{q_i-1} \geq z_{q_i}$, then the function $(x+z_{q_i})/(x+z_{q_i-1})$ is non-decreasing, and then

$$\begin{aligned} z_{Q_i} &\geq \frac{a+1}{a+z_{q_i}} \times \frac{a+z_{q_i}}{a+z_{q_i-1}} \\ &\geq \frac{a+1}{a+S+\delta}, \quad \forall \delta > 0. \end{aligned}$$

*Case 2-b*: If $z_{q_i-1} \leq z_{p_i}$, then

$$\begin{aligned} z_{P_i} &\geq \frac{a+1}{a+z_{q_i}} \times \frac{\beta_{q_i} + z_{q_i}}{\beta_{q_i} + z_{q_i}} \\ &\geq \frac{a+1}{a+S+\delta}, \quad \forall \delta > 0. \end{aligned}$$

Therefore, in the two sub-cases we obtain the inequality (3.24).

Combining (3.23) with (3.24) gives

$$a + 1 - aI \leq IS \leq a + 1 - aS.$$

Consequently, since $a > 0$ we obtain $I \geq S$, and so the sequence $\{z_n\}$ is convergent to the unique limit $l = 1$. $\square$

Now, we give four illustrative examples:

**Example 3.3.** *We consider the following fifth order difference equation*

$$y_{n+1} = \frac{5 + \cos(n\pi) + \frac{2}{n+1} + y_{n-2}}{5 + \cos(n\pi) + \frac{2}{n+1} + y_{n-4}}, \tag{3.25}$$

*with the initial values* $y_{-4} = 0.3$*,* $y_{-3} = 5.5$*,* $y_{-2} = 0.9$*,* $y_{-1} = 4.2$ *and* $y_0 = 1.2$*. From Theorem 3.5, the equilibrium point* $\bar{y} = 1$ *is the global attractor of all positive solution of Eq. (3.25), see Fig. 3.3.*

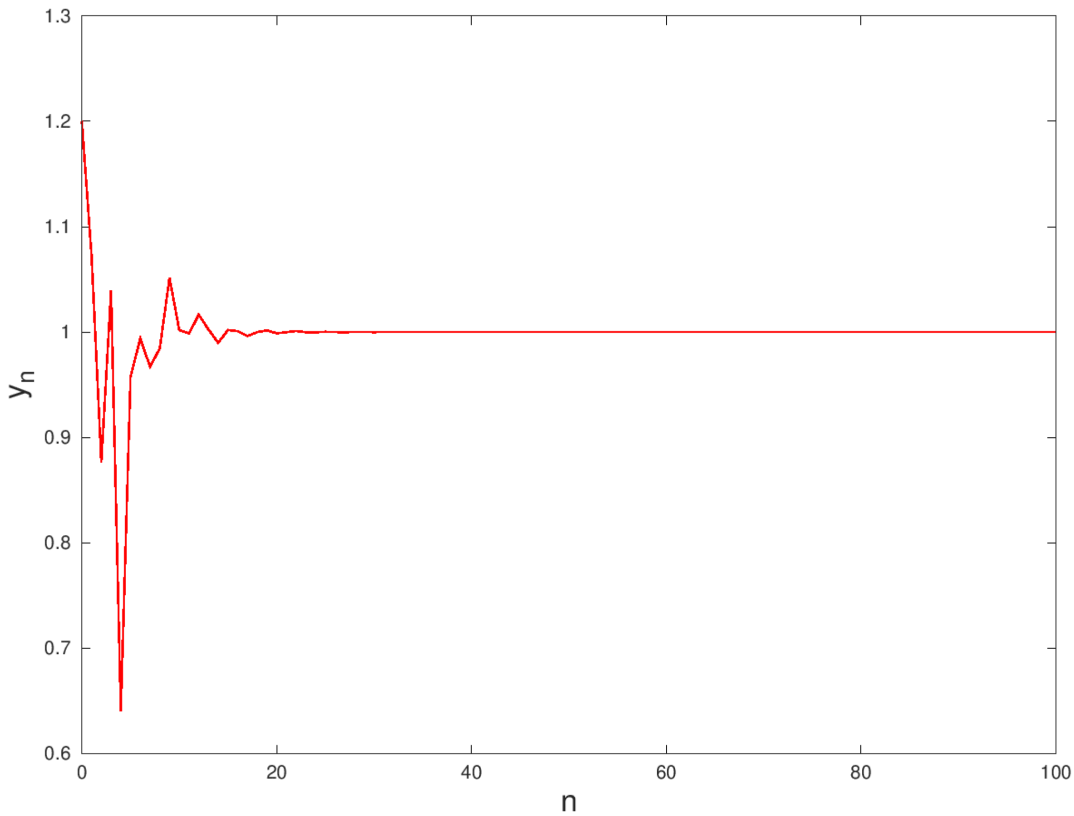


Figure 3.3: Plot of the solution of Eq. (3.25) with the initial values $y_{-4} = 0.3$, $y_{-3} = 5.5$, $y_{-2} = 0.9$, $y_{-1} = 4.2$ and $y_0 = 1.2$.

**Example 3.4.** *We consider the following eighth order rational difference equation*

$$y_{n+1} = \frac{\alpha_n + y_{n-3}}{\alpha_n + y_{n-7}}, \tag{3.26}$$

*where $\alpha_n = 1 - 2^{-(n+1)}$. Here, we take the initial values as follows: $y_{-7} = 0.3$, $y_{-6} = 0.1$, $y_{-5} = 3.4$, $y_{-4} = 0.5$, $y_{-3} = 1.9$, $y_{-2} = 1.7$, $y_{-1} = 1.5$ and $y_0 = 2.3$. From Theorem 3.7, the point $\bar{y} = 1$ is the global attractor of all positive solution of Eq. (3.26), see Fig. 3.4.*

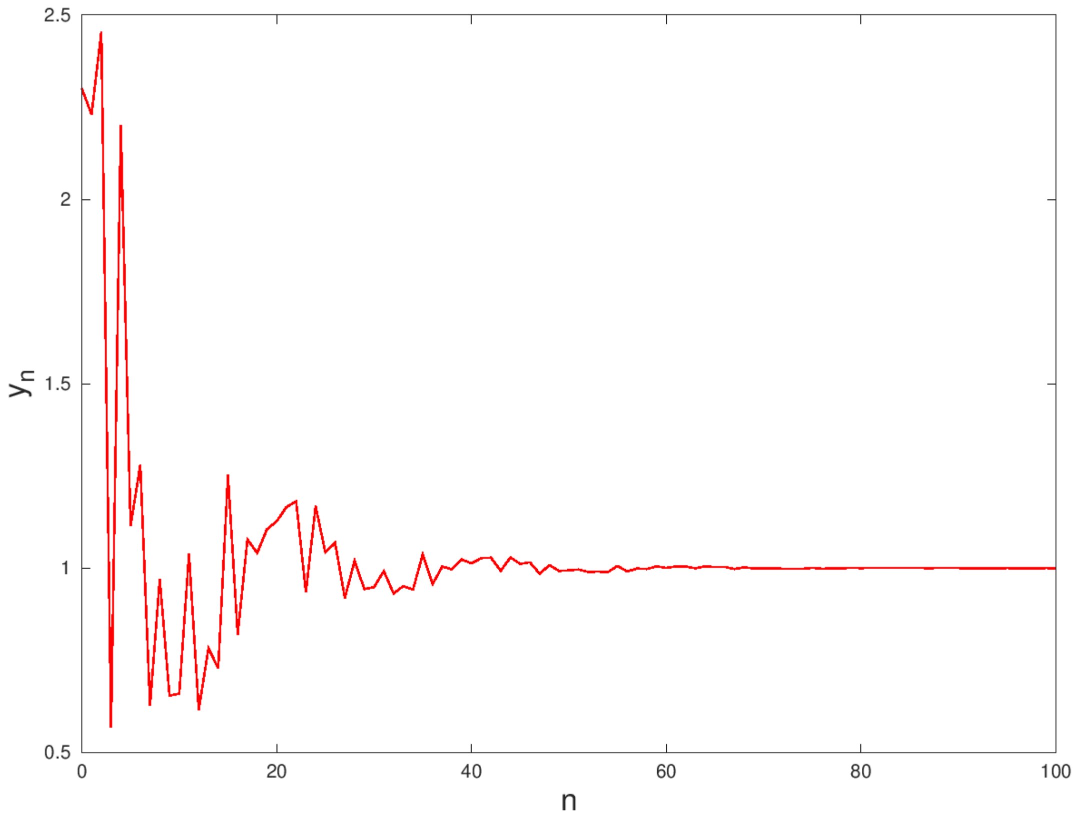


Figure 3.4: Plot of the solution of Eq. (3.26) with the initial values $y_{-7} = 0.3$, $y_{-6} = 0.1$, $y_{-5} = 3.4$, $y_{-4} = 0.5$, $y_{-3} = 1.9$, $y_{-2} = 1.7$, $y_{-1} = 1.5$ and $y_0 = 2.3$.

**Example 3.5.** *We consider the following sixth order rational difference equation*

$$y_{n+1} = \frac{\alpha_n + y_{n-2}}{\alpha_n + y_{n-5}}, \tag{3.27}$$

*where*

$$\alpha_n = \begin{cases} 1/5, & \text{if } n = 3j, \\ 3/10, & \text{if } n = 3j+1, \\ 2/5, & \text{if } n = 3j+2. \end{cases}$$

*Here, we take the initial values as follows:* $y_{-5} = 0.8$, $y_{-4} = 0.3$, $y_{-3} = 2.9$, $y_{-2} = 1.7$, $y_{-1} = 1.5$ *and* $y_0 = 1.3$. *From Theorem 3.7, the point* $\bar{y} = 1$ *is the global attractor of all positive solution of Eq. (3.27), see Fig. 3.5.*

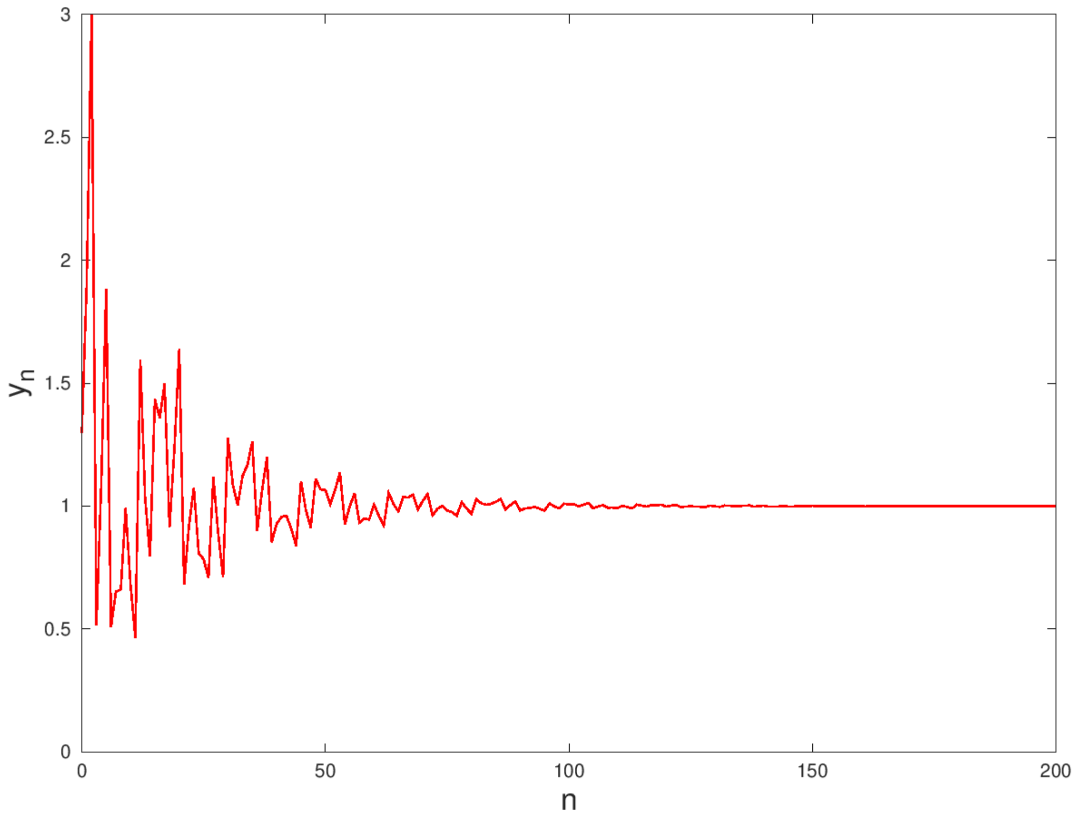


Figure 3.5: Plot of the solution of Eq. (3.27) with the initial values $y_{-5} = 0.8$, $y_{-4} = 0.3$, $y_{-3} = 2.9$, $y_{-2} = 1.7$, $y_{-1} = 1.5$ and $y_0 = 1.3$.

**Example 3.6.** *We consider the following rational difference equation*

$$y_{n+1} = \frac{2 + (n+1)^{-2} + y_{n-1}}{2 + (n+1)^{-2} + y_{n-10}}, \tag{3.28}$$

*with the initial values* $y_{-10} = 2.7$, $y_{-9} = 1.1$, $y_{-8} = 0.3$, $y_{-7} = 0.3$, $y_{-6} = 0.3$, $y_{-5} = 3.4$, $y_{-4} = 2.5$, $y_{-3} = 1.1$, $y_{-2} = 0.5$, $y_{-1} = 0.5$, $y_0 = 1.8$. *From Theorem 3.7, the point* $\bar{y} = 1$ *is the global attractor of all positive solution of Eq. (3.28), see Fig. 3.6.*

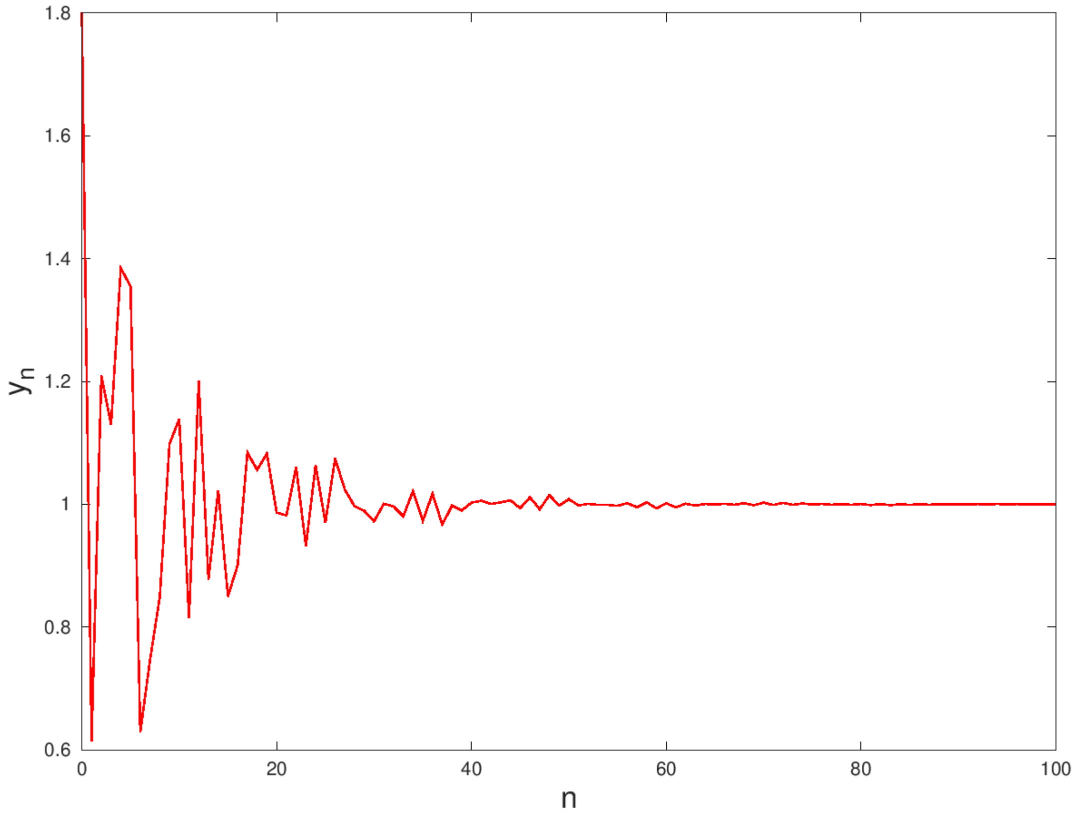


Figure 3.6: Plot of the solution of Eq. (3.28) with the initial values $y_{-10} = 2.7$, $y_{-9} = 1.1$, $y_{-8} = 0.3$, $y_{-7} = 0.3$, $y_{-6} = 0.3$, $y_{-5} = 3.4$, $y_{-4} = 2.5$, $y_{-3} = 1.1$, $y_{-2} = 0.5$, $y_{-1} = 0.5$, $y_0 = 1.8$.

# Chapter 4

# Dynamics of a System of Higher Order Difference Equations with a Period-Two Coefficient

## 4.1 Introduction

In this chapter, we tackle one open problem of the three problems proposed by Gümüs in his article [27], 2018, in which he treated the semi-cycles of the positive solutions for the system

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \ldots \tag{4.1}$$

with parameter $A > 0$, and the initial conditions $x_i$, $y_i$ are arbitrary positive real numbers for $i = -k, -k+1, \ldots, 0$ and $k \in \mathbb{Z}^+$. He also proved that if $A > 1$ then the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ is globally asymptotically stable. The study of Gümüs is an extension of the work of Zhang *et al.* in [58], who studied the asymptotic behavior of the positive solutions of the symmetrical system of the rational difference equation (4.1) in the cases $0 < A < 1$, $A = 1$ and $A > 1$. When $0 < A < 1$, they found out that when $0 < A < 1$ there exist unbounded solutions of the system (4.1) , and when $A = 1$ they proved that the system (4.1) has two periodic solutions. Also, they found that any positive solution is bounded and persists. In the same study, they have shown that the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ attracts all the positive solutions when $A > 1$. After the study of Gümüs, in [4], Abualrub and Aloqeili examined the first open problem. They studied the oscillatory behavior, the boundedness, the persistence of the positive solutions and the global asymptotic stability of the unique positive equilibrium point of the system of two rational difference equations:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \ldots \tag{4.2}$$

with the parameters $A > 0, B > 0$ and the initial conditions $x_i$, $y_i$ are arbitrary positive real numbers for $i = -k, -k+1, \ldots, 0$ and $k \in \mathbb{Z}^+$.

In this work, we investigate the dynamical behavior of the system of difference equations

$$x_{n+1} = \alpha_n + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = \alpha_n + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \ldots \tag{4.3}$$

where $\{\alpha_n\}$ is a periodic sequence of non-negative real numbers and the initial conditions $x_i$, $y_i$ are arbitrary positive numbers for $i = -k, -k+1, -k+2, \ldots, 0$ and $k \in \mathbb{Z}^+$.

We consider the system (4.3) when the period of $\{\alpha_n\}$ is two; namely, $\alpha_{2n} = \alpha$ and $\alpha_{2n+1} = \beta$. Then, we obtain

$$x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}}, \tag{4.4}$$

$$x_{2n+2} = \beta + \frac{y_{2n+1-k}}{y_{2n+1}}, \tag{4.5}$$

$$y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}}, \tag{4.6}$$

$$y_{2n+2} = \beta + \frac{x_{2n+1-k}}{x_{2n+1}}. \tag{4.7}$$

If $\alpha_n = \alpha = \beta = A$, then the system (4.3) turns into the symmetrical system (4.1)

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \dots$$

with the parameter $A > 0$, and the initial conditions $x_i$, $y_i$ are arbitrary positive real numbers for $i = -k, -k+1, ...., 0$ and $k \in \mathbb{Z}^+$, which was studied in ([27],[58]).

Throughout this chapter, we assume that $\alpha \neq \beta$. We study the boundedness character of the system (4.3) in the cases: $0 < \alpha,\ \beta < 1$ and $\alpha,\ \beta > 1$. We use the linearization method to give a necessary and sufficient conditions for the local stability. In addition, we investigate the global behavior of the system (4.3). Furthermore, we determine the rate of the convergence of the solutions and we give some numerical examples that support our theoretical results.

## 4.2 Boundedness Character

In this section, we investigate the boundedness character of (4.3). We show that if $k \in \mathbb{Z}^+$, $\alpha,\ \beta > 1$, then every positive solution of the system (4.3) is bounded. When $0 < \alpha,\ \beta < 1$ and $k$ is odd, then there exist unbounded solutions of the system (4.3).

**Theorem 4.1.** *Suppose that*

$$\alpha > 1 \quad \text{and} \quad \beta > 1. \tag{4.8}$$

*Then every positive solution of the system (4.3) is bounded.*

*Proof.* It is clear from equations (4.4), (4.5), (4.6) and (4.7) that

$$x_{2n} > \beta, \quad y_{2n} > \beta, \quad x_{2n-1} > \alpha, \quad y_{2n-1} > \alpha, \quad \text{for every} \quad n \geq k. \tag{4.9}$$

We assume that $k$ is odd. Then, from the equations (4.4), (4.5), (4.6), (4.7) and (4.8), we obtain

$$x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}} < \alpha + \frac{y_{2n-k}}{\beta}, \tag{4.10}$$

$$x_{2n} = \beta + \frac{y_{2n-k-1}}{y_{2n-1}} < \beta + \frac{y_{2n-k-1}}{\alpha}, \tag{4.11}$$

$$y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}} < \alpha + \frac{x_{2n-k}}{\beta}, \tag{4.12}$$

$$y_{2n} = \beta + \frac{x_{2n-k-1}}{x_{2n-1}} < \beta + \frac{x_{2n-k-1}}{\alpha}. \tag{4.13}$$

From (4.10), (4.12) and using induction we get

$$\begin{aligned} x_{2n+1} &< \alpha \left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \dots\right) + \mu_1 \\ &= \frac{\alpha\beta}{\beta - 1} + \mu_1, \end{aligned}$$

$$\begin{aligned} y_{2n+1} &< \alpha \left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \dots\right) + \mu_1 \\ &= \frac{\alpha\beta}{\beta - 1} + \mu_1, \end{aligned}$$

where $\mu_1 = \max\{x_{-k}, y_{-k}, x_{-k+2}, y_{-k+2}, x_{-k+4}, y_{-k+4}, \dots, x_k, y_k\}$.
Similarly, we get

$$\begin{aligned} x_{2n+2} &< \beta \left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \dots\right) + \mu_2 \\ &= \frac{\alpha\beta}{\alpha - 1} + \mu_2, \end{aligned}$$

$$\begin{aligned} y_{2n+2} &< \beta \left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \dots\right) + \mu_2 \\ &= \frac{\alpha\beta}{\alpha - 1} + \mu_2, \end{aligned}$$

where $\mu_2 = \max\{x_{-k+1}, y_{-k+1}, x_{-k+3}, y_{-k+3}, x_{-k+5}, y_{-k+5}, \dots, x_{k+1}, y_{k+1}\}$.

Now, we suppose that $k$ is even and $\alpha, \beta > 1$. Then, from the equations (4.10), (4.11), (4.12), (4.13) and using induction, we obtain

$$\begin{aligned}
x_{2n+1} &< \alpha + 1 + \frac{1}{\alpha\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] \\
&+ \frac{1}{\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] + \mu_3 \\
&= \alpha + 1 + \frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \mu_3 \\
&= \frac{\alpha\beta(\alpha+1)}{\alpha\beta - 1} + \mu_3,
\end{aligned}$$

$$\begin{aligned}
y_{2n+1} &< \alpha + 1 + \frac{1}{\alpha\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] \\
&+ \frac{1}{\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] + \mu_4 \\
&= \alpha + 1 + \frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \mu_4 \\
&= \frac{\alpha\beta(\alpha+1)}{\alpha\beta - 1} + \mu_4,
\end{aligned}$$

$$\begin{aligned}
x_{2n+2} &< \beta + 1 + \frac{1}{\alpha\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] \\
&+ \frac{1}{\alpha}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] + \mu_4 \\
&= \beta + 1 + \frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\alpha}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \mu_4 \\
&= \frac{\alpha\beta(\beta+1)}{\alpha\beta - 1} + \mu_4,
\end{aligned}$$

$$\begin{aligned}
y_{2n+2} &< \beta + 1 + \frac{1}{\alpha\beta}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] \\
&+ \frac{1}{\alpha}\left[1 + \frac{1}{\alpha\beta} + \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\beta)^3} + \dots\right] + \mu_3 \\
&= \beta + 1 + \frac{1}{\alpha\beta}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \frac{1}{\alpha}\left(\frac{\alpha\beta}{\alpha\beta - 1}\right) + \mu_3 \\
&= \frac{\alpha\beta(\beta+1)}{\alpha\beta - 1} + \mu_3,
\end{aligned}$$

where

$$\mu_3 = \max\{x_{-k+1}, y_{-k}, x_{-k+3}, y_{-k+2}, x_{-k+5}, y_{-k+4}, \dots, x_{k+1}, y_k\}$$

and

$$\mu_4 = \max\{x_{-k}, y_{-k+1}, x_{-k+4}, y_{-k+3}, x_{-k+6}, y_{-k+5}, \dots, x_k, y_{k+1}\}.$$

The proof now is completed. $\square$

Next, we study the existence of unbounded positive solutions of system (4.3) when $0 < \alpha < 1$ and $0 < \beta < 1$.

**Theorem 4.2.** *Suppose that* $0 < \alpha < 1$ *and* $0 < \beta < 1$. *Let* $\gamma = \max\{\alpha, \beta\}$ *and* $\{x_n, y_n\}_{n=-k}^{\infty}$ *be a positive solution of* (4.3). *Then the following statements are true:*

*(a) If* $k$ *is odd,* $0 < x_{-k}, x_{-k+2}, \dots, x_{-1}, y_{-k}, y_{-k+2}, \dots, y_{-1} < 1$ *and* $x_{-k+1}, x_{-k+3}, \dots, x_0, y_{-k+1}, y_{-k+3}, \dots, y_0 > \frac{1}{1-\gamma}$, *then*

$$\lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} x_{2n+1} = \alpha, \quad \lim_{n\to\infty} y_{2n+1} = \alpha.$$

*(b) If* $k$ *is odd,* $0 < x_{-k+1}, x_{-k+3}, \dots, x_0, y_{-k+1}, y_{-k+3}, \dots, y_0 < 1$ *and* $x_{-k}, x_{-k+2}, \dots, x_{-1}, y_{-k}, y_{-k+2}, \dots, y_{-1} > \frac{1}{1-\gamma}$, *then*

$$\lim_{n\to\infty} x_{2n+1} = \infty, \quad \lim_{n\to\infty} y_{2n+1} = \infty, \quad \lim_{n\to\infty} x_{2n} = \beta, \quad \lim_{n\to\infty} y_{2n} = \beta.$$

*Proof.* (a) Since $\gamma \geq \alpha$ and $\gamma \geq \beta$ then

$$0 < x_1 = \alpha + \frac{y_{-k}}{y_0} < \alpha + \frac{1}{y_0} < \alpha + 1 - \gamma < 1,$$

$$0 < y_1 = \alpha + \frac{x_{-k}}{x_0} < \alpha + \frac{1}{x_0} < \alpha + 1 - \gamma < 1,$$

$$x_2 = \beta + \frac{y_{-k+1}}{y_1} > \beta + y_{-k+1} > y_{-k+1} > \frac{1}{1-\gamma},$$

$$y_2 = \beta + \frac{x_{-k+1}}{x_1} > \beta + x_{-k+1} > x_{-k+1} > \frac{1}{1-\gamma}.$$

By induction, we get

$$0 < x_{2n-1}, y_{2n-1} < 1 \text{ and } x_{2n}, y_{2n} > \frac{1}{1-\gamma} \quad \text{for} \quad n = 1, 2, \dots$$

So, from $x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}}$ implies that $x_{2n} = \alpha + \frac{y_{2n-(k+1)}}{y_{2n-1}}$ and $y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}}$, for $l > \frac{k+3}{2}$ and $k$ is odd we have

$$\begin{aligned} x_{2l} &= \alpha + \frac{y_{2l-(k+1)}}{y_{2l-1}} > \alpha + y_{2l-(k+1)} = \alpha + \alpha + \frac{x_{2l-(2k+2)}}{x_{2l-(k+2)}} \\ &> 2\alpha + x_{2l-(2k+2)}, \end{aligned}$$

$$\begin{aligned} x_{4l} &= \alpha + \frac{y_{4l-(k+1)}}{y_{4l-1}} > \alpha + y_{4l-(k+1)} = \alpha + \alpha + \frac{x_{4l-(2k+2)}}{x_{4l-(k+2)}} \\ &> 2\alpha + x_{4l-(2k+2)} = 2\alpha + \alpha + \frac{y_{4l-(3k+3)}}{y_{4l-2k-3}} \\ &> 3\alpha + y_{4l-(3k+3)} = 3\alpha + \alpha + \frac{x_{4l-(4k+4)}}{x_{4l-(3k+4)}} \\ &> 4\alpha + x_{4l-(4k+4)}. \end{aligned}$$

Similarly, we obtain $x_{6l} > 6\alpha + x_{6l-(6k+6)}$. So for all $r = 1, 2, \ldots$

$$x_{2rl} > 2r\alpha + x_{2rl-2r(k+1)}$$

Hence, if $n = rl$, then since $r \to \infty$ and so $\lim_{n\to\infty} x_{2n} = \infty$. In the same, we get $\lim_{n\to\infty} y_{2n} = \infty$.
We consider the system (4.3) and we take the limits on both sides of each equation in the system

$$x_{2n+1} = \alpha + \frac{y_{2n-k}}{y_{2n}}, \quad y_{2n+1} = \alpha + \frac{x_{2n-k}}{x_{2n}}$$

we obtain $\lim_{n\to\infty} x_{2n+1} = \alpha$ and $\lim_{n\to\infty} y_{2n+1} = \alpha$. This completes the proof of statement (a).
Now, we prove the statement (b). Since $\gamma \geq \alpha$ and $\gamma \geq \beta$ then, we have

$$0 < x_2 = \beta + \frac{y_{-k+1}}{y_1} < \beta + \frac{1}{y_1} < \beta + 1 - \gamma < 1,$$

$$0 < y_2 = \beta + \frac{x_{-k+1}}{x_1} < \beta + \frac{1}{x_1} < \beta + 1 - \gamma < 1,$$

$$x_1 = \alpha + \frac{y_{-k}}{y_0} > \alpha + y_{-k} > y_{-k} > \frac{1}{1-\gamma},$$

$$y_1 = \alpha + \frac{x_{-k}}{x_0} > \alpha + x_{-k} > x_{-k} > \frac{1}{1-\gamma}.$$

By induction, we get

$$0 < x_{2n}, y_{2n} < 1 \text{ and } x_{2n-1}, y_{2n-1} > \frac{1}{1-\gamma} \quad \text{for} \quad n = 1, 2, \ldots$$

From $x_{2n} = \beta + \frac{y_{2n-k-1}}{y_{2n-1}}$ implies that $x_{2n+1} = \beta + \frac{y_{2n-k}}{y_{2n}}$ and $y_{2n} = \beta + \frac{x_{2n-k-1}}{x_{2n-1}}$, so for $l > \frac{k+3}{2}$ and $k$ is odd yields

$$\begin{aligned} x_{2l+1} &= \beta + \frac{y_{2l-k}}{y_{2l}} > \beta + y_{2l-k} = \beta + \beta + \frac{x_{2l-(2k+1)}}{x_{2l-(k+1)}} \\ &> 2\beta + x_{2l-(2k+1)}, \end{aligned}$$

$$\begin{aligned} x_{4l+1} &= \beta + \frac{y_{4l-k}}{y_{4l}} > \beta + y_{4l-k} = \beta + \beta + \frac{x_{4l-(2k+1)}}{x_{4l-(k+1)}} \\ &> 2\beta + x_{4l-(2k+1)} = 2\beta + \beta + \frac{y_{4l-(3k+2)}}{y_{4l-2k-2}} \\ &> 3\beta + y_{4l-(3k+2)} = 3\beta + \beta + \frac{x_{4l-(4k+3)}}{x_{4l-(3k+3)}} \\ &> 4\beta + x_{4l-(4k+3)}. \end{aligned}$$

Similarly, we get $x_{6l+1} > 6\beta + x_{6l-(6k+5)}$. So for all $r = 1, 2, \ldots$

$$x_{2rl+1} > 2r\beta + x_{2rl-(2r(k+1)-1)}.$$

Consequently, if $n = rl$, then since $r \to \infty$, $\lim_{n\to\infty} x_{2n+1} = \infty$. Similarly, we get $\lim_{n\to\infty} y_{2n+1} = \infty$. Now, we consider the system (4.3) and we take the limits on both sides of each equation in the system,

$$x_{2n+2} = \beta + \frac{y_{2n+1-k}}{y_{2n+1}}, \quad y_{2n+2} = \beta + \frac{x_{2n+1-k}}{x_{2n+1}},$$

we obtain

$$\lim_{n\to\infty} x_{2n+2} = \beta \quad \text{and} \quad \lim_{n\to\infty} y_{2n+2} = \beta.$$

This completes the proof of statement (b). □

## 4.3 Local Asymptotic Stability

The system (4.3) can be converted into a four-dimensional discrete system with constant coefficients. To this end, let

$$u_n = x_{2n-1}, \quad v_n = x_{2n}, \quad t_n = y_{2n-1}, \quad w_n = y_{2n}, \quad n = 0, 1, 2, \ldots$$

We consider the case $k = 2m$. Hence, for $n \geq 0$ we have

$$\begin{aligned} u_{n+1} &= x_{2n+1} = \alpha + \frac{y_{2n-2m}}{y_{2n}} = \alpha + \frac{w_{n-m}}{w_n}, \\ t_{n+1} &= y_{2n+1} = \alpha + \frac{x_{2n-2m}}{x_{2n}} = \alpha + \frac{v_{n-m}}{v_n}, \\ v_{n+1} &= x_{2n+2} = \beta + \frac{y_{2n+1-2m}}{y_{2n+1}} = \beta + \frac{t_{n-m+1}}{t_{n+1}}, \\ w_{n+1} &= y_{2n+2} = \beta + \frac{x_{2n+1-2m}}{x_{2n+1}} = \beta + \frac{u_{n-m+1}}{u_{n+1}}. \end{aligned}$$

So, for $n = 0, 1, \ldots$, the system (4.3) is equivalent to the system

$$\begin{cases} u_{n+1} = \alpha + \frac{w_{n-m}}{w_n} \\ v_{n+1} = \beta + \frac{t_{n-m+1}v_n}{\alpha v_n + v_{n-m}} \\ t_{n+1} = \alpha + \frac{v_{n-m}}{v_n} \\ w_{n+1} = \beta + \frac{u_{n-m+1}w_n}{\alpha w_n + w_{n-m}} \end{cases} \tag{4.14}$$

where the initial conditions are $w_0 = y_0$, $w_{-1} = y_{-2}, \ldots$, $w_{-m} = y_{-2m}$, $v_0 = x_0$, $v_{-1} = x_{-2}, \ldots$, $v_{-m} = x_{-2m}$, $t_0 = y_{-1}$, $t_{-1} = y_{-3}, \ldots$, $t_{-m+1} = y_{-2m+1}$, $u_0 = x_{-1}$, $u_{-1} = x_{-3}, \ldots$, $u_{-m+1} = x_{-2m+1}$. One can easily see that the system (4.14) has a unique equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$. In this section, we use the linearization method to give necessary and sufficient conditions for the local asymptotic stability.

**Theorem 4.3.** *If $\alpha > 1$, $\beta > 1$ and $k$ is even, then the unique positive equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$ of the system (4.14) is locally asymptotically stable.*

*Proof.* The system (4.14) can be formulated as a system of first order recurrence equations as follows:

$$\begin{aligned} u_n^{(1)} &= u_n, u_n^{(2)} = u_{n-1}, \ldots, u_n^{(m)} = u_{n-m+1}, \\ v_n^{(1)} &= v_n, v_n^{(2)} = v_{n-1}, \ldots, v_n^{(m+1)} = v_{n-m}, \\ t_n^{(1)} &= t_n, t_n^{(2)} = t_{n-1}, \ldots, t_n^{(m)} = t_{n-m+1}, \\ w_n^{(1)} &= w_n, w_n^{(2)} = w_{n-1}, \ldots, w_n^{(m+1)} = w_{n-m}. \end{aligned}$$

The linearization of the system (4.14) about the equilibrium point $E$ is given by

$$Z_{n+1} = AZ_n, \tag{4.15}$$

where

$$Z_n = (u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(m)}, v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m+1)}, t_n^{(1)}, t_n^{(2)}, \ldots, t_n^{(m)}, w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(m+1)})^T$$

and

$$\begin{aligned}
Z_{n+1} = &\\
&(u_{n+1}^{(1)}, u_{n+1}^{(2)}, \ldots, u_{n+1}^{(m)}, v_{n+1}^{(1)}, v_{n+1}^{(2)}, \ldots, v_{n+1}^{(m+1)}, t_{n+1}^{(1)}, t_{n+1}^{(2)}, \ldots, t_{n+1}^{(m)}, w_{n+1}^{(1)}, w_{n+1}^{(2)}, \ldots, w_{n+1}^{(m+1)})^T\\
&= (\alpha + \frac{w_n^{(m+1)}}{w_n^{(1)}}, u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(m-1)}, \beta + \frac{t_n^{(m)} v_n^{(1)}}{\alpha v_n^{(1)} + v_n^{(m+1)}}, v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, \alpha + \frac{v_n^{(m+1)}}{v_n^{(1)}},\\
&t_n^{(1)}, t_n^{(2)}, \ldots, t_n^{(m-1)}, \beta + \frac{u_n^{(m)} w_n^{(1)}}{\alpha w_n^{(1)} + w_n^{(m+1)}}, w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(m)})^T.
\end{aligned}$$

With

$$\begin{aligned}
&\frac{\partial u_{n+1}^{(1)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial u_{n+1}^{(1)}}{\partial u_n^{(m)}} = 0, \frac{\partial u_{n+1}^{(1)}}{\partial v_n^{(1)}} = 0, \ldots, \frac{\partial u_{n+1}^{(1)}}{\partial v_n^{(m+1)}} = 0, \frac{\partial u_{n+1}^{(1)}}{\partial t_n^{(1)}} = 0, \ldots, \frac{\partial u_{n+1}^{(1)}}{\partial t_n^{(m)}} = 0,\\
&\frac{\partial u_{n+1}^{(1)}}{\partial w_n^{(1)}} = \frac{-w_n^{m+1}}{(w_n^{(1)})^2}, \frac{\partial u_{n+1}^{(1)}}{\partial w_n^{(2)}} = 0, \ldots, \frac{\partial u_{n+1}^{(1)}}{\partial w_n^{(m)}} = 0, \frac{\partial u_{n+1}^{(1)}}{\partial w_n^{(m+1)}} = \frac{1}{w_n^{(1)}},\\
&\frac{\partial u_{n+1}^{(2)}}{\partial u_n^{(1)}} = 1, \frac{\partial u_{n+1}^{(2)}}{\partial u_n^{(2)}} = 0, \ldots, \frac{\partial u_{n+1}^{(2)}}{\partial w_n^{(m+1)}} = 0,\\
&\qquad\vdots\\
&\frac{\partial u_{n+1}^{(m)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial u_{n+1}^{(m)}}{\partial u_n^{(m-2)}} = 0, \frac{\partial u_{n+1}^{(m)}}{\partial u_n^{(m-1)}} = 1, \frac{\partial u_{n+1}^{(m)}}{\partial u_n^{(m)}} = 0, \ldots, \frac{\partial u_{n+1}^{(m)}}{\partial w_n^{(m+1)}} = 0,
\end{aligned}$$

$$\frac{\partial v_{n+1}^{(1)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial v_{n+1}^{(1)}}{\partial u_n^{(m)}} = 0, \frac{\partial v_{n+1}^{(1)}}{\partial v_n^{(1)}} = \frac{t_n^{(m)} v_n^{(m+1)}}{(\alpha v_n^{(1)} + v_n^{(m+1)})^2}, \frac{\partial v_{n+1}^{(1)}}{\partial v_n^{(2)}} = 0, \ldots, \frac{\partial v_{n+1}^{(1)}}{\partial v_n^{(m)}} = 0,$$
$$\frac{\partial v_{n+1}^{(1)}}{\partial v_n^{(m+1)}} = \frac{-t_n^{(m)} v_n^{(1)}}{(\alpha v_n^{(1)} + v_n^{(m+1)})^2}, \frac{\partial v_{n+1}^{(1)}}{\partial t_n^{(1)}} = 0, \ldots, \frac{\partial v_{n+1}^{(1)}}{\partial t_n^{(m-1)}} = 0, \frac{\partial v_{n+1}^{(1)}}{\partial t_n^{(m)}} = \frac{v_n^{(1)}}{\alpha v_n^{(1)} + v_n^{(m+1)}},$$
$$\frac{\partial v_{n+1}^{(1)}}{\partial w_n^{(1)}} = 0, \ldots, \frac{\partial v_{n+1}^{(1)}}{\partial w_n^{(m+1)}} = 0,$$
$$\frac{\partial v_{n+1}^{(2)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial v_{n+1}^{(2)}}{\partial v_n^{(1)}} = 1, \frac{\partial v_{n+1}^{(2)}}{\partial v_n^{(2)}} = 0, \ldots, \frac{\partial v_{n+1}^{(2)}}{\partial w_n^{(m+1)}} = 0,$$
$$\vdots$$
$$\frac{\partial v_{n+1}^{(m+1)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial v_{n+1}^{(m+1)}}{\partial v_n^{(m-1)}} = 0, \frac{\partial v_{n+1}^{(m+1)}}{\partial v_n^{(m)}} = 1, \frac{\partial v_{n+1}^{(m+1)}}{\partial v_n^{(m+1)}} = 0, \ldots, \frac{\partial v_{n+1}^{(m+1)}}{\partial w_n^{(m+1)}} = 0,$$
$$\frac{\partial t_{n+1}^{(1)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial t_{n+1}^{(1)}}{\partial u_n^{(m)}} = 0, \frac{\partial t_{n+1}^{(1)}}{\partial v_n^{(1)}} = \frac{-v_n^{m+1}}{(v_n^{(1)})^2}, \frac{\partial t_{n+1}^{(1)}}{\partial v_n^{(2)}} = 0, \ldots, \frac{\partial t_{n+1}^{(1)}}{\partial v_n^{(m)}} = 0,$$
$$\frac{\partial t_{n+1}^{(1)}}{\partial v_n^{(m+1)}} = \frac{1}{v_n^{(1)}}, \frac{\partial t_{n+1}^{(1)}}{\partial t_n^{(1)}} = 0, \ldots, \frac{\partial t_{n+1}^{(1)}}{\partial w_n^{(m+1)}} = 0,$$
$$\frac{\partial t_{n+1}^{(2)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial t_{n+1}^{(1)}}{\partial v_n^{(m+1)}} = 0, \frac{\partial t_{n+1}^{(2)}}{\partial t_n^{(1)}} = 1, \frac{\partial t_{n+1}^{(2)}}{\partial t_n^{(2)}} = 0 \ldots, \frac{\partial t_{n+1}^{(2)}}{\partial w_n^{(m+1)}} = 0,$$
$$\vdots$$
$$\frac{\partial t_{n+1}^{(m)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial t_{n+1}^{(m)}}{\partial t_n^{(m-2)}} = 0, \frac{\partial t_{n+1}^{(m)}}{\partial t_n^{(m-1)}} = 1, \frac{\partial t_{n+1}^{(m)}}{\partial t_n^{(m)}} = 0 \ldots, \frac{\partial t_{n+1}^{(m)}}{\partial w_n^{(m+1)}} = 0,$$

$$\frac{\partial w_{n+1}^{(1)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial w_{n+1}^{(1)}}{\partial u_n^{(m-1)}} = 0, \frac{\partial w_{n+1}^{(1)}}{\partial u_n^{(m)}} = \frac{w_n^{(1)}}{\alpha w_n^{(1)} + w_n^{(m+1)}}, \frac{\partial w_{n+1}^{(1)}}{\partial v_n^{(1)}} = 0, \ldots, \frac{\partial w_{n+1}^{(1)}}{\partial t_n^{(m)}} = 0,$$
$$\frac{\partial w_{n+1}^{(1)}}{\partial w_n^{(1)}} = \frac{u_n^{(m)} w_n^{(m+1)}}{(\alpha w_n^{(1)} + w_n^{(m+1)})^2}, \frac{\partial w_{n+1}^{(1)}}{\partial w_n^{(2)}} = 0, \ldots, \frac{\partial w_{n+1}^{(1)}}{\partial w_n^{(m)}} = 0, \frac{\partial w_{n+1}^{(1)}}{\partial w_n^{(m+1)}} = \frac{-u_n^{(m)} w_n^{(1)}}{(\alpha w_n^{(1)} + w_n^{(m+1)})^2},$$
$$\frac{\partial w_{n+1}^{(2)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial w_{n+1}^{(2)}}{\partial t_n^{(m)}} = 0, \frac{\partial w_{n+1}^{(2)}}{\partial w_n^{(1)}} = 1, \frac{\partial w_{n+1}^{(2)}}{\partial w_n^{(2)}} = 0, \ldots, \frac{\partial w_{n+1}^{(2)}}{\partial w_n^{(m+1)}} = 0$$
$$\vdots$$
$$\frac{\partial w_{n+1}^{(m+1)}}{\partial u_n^{(1)}} = 0, \ldots, \frac{\partial w_{n+1}^{(m+1)}}{\partial w_n^{(m-1)}} = 0, \frac{\partial w_{n+1}^{(m+1)}}{\partial w_n^{(m)}} = 1, \frac{\partial w_{n+1}^{(m+1)}}{\partial w_n^{(m+1)}} = 0,$$

and $A$ is the jacobian matrix at the equilibrium point $E$ under the above linearized system (4.15), it is as follows:

$$A =$$

$$\begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}_{(4m+2)\times(4m+2)}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{4m+2}$ be the eigenvalues of the matrix $A$. Define

$$D = diag(d_1, d_2, \dots, d_{4m+2})$$

be a diagonal matrix such that $d_1 = d_{m+1} = d_{2m+2} = d_{3m+2} = 1$ and

$$d_i = d_{2m+1+i} = 1 - i\varepsilon, \qquad \text{for each } i \in \{2, 3, \dots, m, m+2, \dots, 2m+1\}.$$

Since $\alpha, \beta > 1$, we can take a positive number $\varepsilon$ such that

$$0 < \varepsilon < \min\left\{\frac{\beta - 1}{(\beta+1)(2m+1)}, \quad \frac{(\alpha+1)(\beta+1) - 3}{(\alpha+1)(\beta+1)(2m+1)}\right\}. \tag{4.16}$$

Hence, for all $i$, $1 - i\varepsilon > 0$, and its clear that $D$ is an invertible matrix. Computing matrix $DAD^{-1}$ where

$$D =$$

$$\begin{pmatrix}
d_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & d_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & d_3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & d_{m+1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & d_{m+2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & d_{2m+1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{2m+2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & d_{2m+3} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & d_{3m+1} & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{3m+2} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & d_{3m+3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & d_{4m+1} & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{4m+2}
\end{pmatrix}$$

$$D^{-1} =$$

$$\begin{pmatrix}
\frac{1}{d_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \frac{1}{d_2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \frac{1}{d_3} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \frac{1}{d_{m+1}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \frac{1}{d_{m+2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \frac{1}{d_{2m+1}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{d_{2m+2}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{d_{2m+3}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{1}{d_{3m+1}} & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{d_{3m+2}} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{d_{3m+3}} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{1}{d_{4m+1}} & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{d_{4m+2}}
\end{pmatrix}$$

$$DAD^{-1} =$$

$$\begin{pmatrix}
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_1^{(3m+2)} & \dots & 0 & \delta_1^{(4m+2)} \\
\delta_2^{(1)} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \delta_m^{(m-1)} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \delta_{m+1}^{(m+1)} & \dots & 0 & \delta_{m+1}^{(2m+1)} & 0 & \dots & 0 & \delta_{m+1}^{(3m+1)} & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \delta_{m+2}^{(m+1)} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & \delta_{2m+1}^{(2m)} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \delta_{2m+2}^{(m+1)} & \dots & 0 & \delta_{2m+2}^{(2m+1)} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_{2m+3}^{(2m+2)} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta_{3m+1}^{(3m)} & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & \delta_{3m+2}^{(m)} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_{3m+2}^{(3m+2)} & \dots & 0 & \delta_{3m+2}^{(4m+2)} \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \delta_{3m+3}^{(3m+2)} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta_{4m+2}^{(4m+1)} & 0
\end{pmatrix}$$

where

$$\delta_1^{(3m+2)} = \frac{-d_1}{(\beta+1)d_{3m+2}}, \qquad \delta_1^{(4m+2)} = \frac{d_1}{(\beta+1)d_{4m+2}}, \qquad \delta_2^{(1)} = \frac{d_2}{d_1},$$

$$\delta_m^{(m-1)} = \frac{d_m}{d_{m-1}}, \quad \delta_{m+1}^{(m+1)} = \frac{d_{m+1}}{(\beta+1)(\alpha+1)d_{m+1}}, \quad \delta_{m+1}^{(2m+1)} = \frac{-d_{m+1}}{(\beta+1)(\alpha+1)d_{2m+1}},$$

$$\delta_{m+1}^{(3m+1)} = \frac{d_{m+1}}{(\alpha+1)d_{3m+1}}, \qquad \delta_{m+2}^{(m+1)} = \frac{d_{m+2}}{d_{m+1}}, \qquad \delta_{2m+1}^{(2m)} = \frac{d_{2m+1}}{d_{2m}},$$

$$\delta_{2m+2}^{(m+1)} = \frac{-d_{2m+2}}{(\beta+1)d_{m+1}}, \qquad \delta_{2m+2}^{(2m+1)} = \frac{d_{2m+2}}{(\beta+1)d_{2m+1}}, \qquad \delta_{2m+3}^{(2m+2)} = \frac{d_{2m+3}}{d_{2m+2}},$$

$$\delta_{3m+1}^{(3m)} = \frac{d_{3m+1}}{d_{3m}}, \qquad \delta_{3m+2}^{(m)} = \frac{d_{3m+2}}{(\alpha+1)d_m}, \qquad \delta_{3m+2}^{(3m+2)} = \frac{d_{3m+2}}{(\beta+1)(\alpha+1)d_{3m+2}}$$

$$\delta_{3m+2}^{(4m+2)} = \frac{-d_{3m+2}}{(\beta+1)(\alpha+1)d_{4m+2}}, \qquad \delta_{3m+3}^{(3m+2)} = \frac{d_{3m+3}}{d_{3m+2}}, \qquad \delta_{4m+2}^{(4m+1)} = \frac{d_{4m+2}}{d_{4m+1}}.$$

From the following four inequalities

$$1 = d_1 > d_2 > \cdots > d_m > 0,$$

$$1 = d_{m+1} > d_{m+2} > \cdots > d_{2m} > d_{2m+1} > 0,$$

$$1 = d_{2m+2} > d_{2m+3} > \cdots > d_{3m} > d_{3m+1} > 0,$$

$$1 = d_{3m+2} > d_{3m+3} > \cdots > d_{4m+1} > d_{4m+2} > 0,$$

we get

$$\frac{d_2}{d_1} < 1, \quad \frac{d_3}{d_2} < 1, \ldots, \frac{d_m}{d_{m-1}} < 1, \quad \frac{d_{m+2}}{d_{m+1}} < 1, \ldots, \frac{d_{2m+1}}{d_{2m}} < 1,$$
$$\frac{d_{2m+3}}{d_{2m+2}} < 1, \ldots, \frac{d_{3m+1}}{d_{3m}} < 1, \quad \frac{d_{3m+3}}{d_{3m+2}} < 1, \ldots, \frac{d_{4m+2}}{d_{4m+1}} < 1.$$

Furthermore, since $\alpha, \beta > 1$ and by using (4.16) we have

$$\begin{aligned}\frac{1}{\beta+1} + \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)} &< \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)} + \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)}\\ &< \frac{2}{(1-(2m+1)\varepsilon)(\beta+1)}\\ &< 1\end{aligned}$$

and

$$\begin{aligned}\frac{1}{(\alpha+1)(\beta+1)} &+ \frac{1}{(\beta+1)(\alpha+1)(1-(2m+1)\varepsilon)} + \frac{1}{(\alpha+1)(1-m\varepsilon)}\\ &< \frac{3}{(\beta+1)(\alpha+1)(1-(2m+1)\varepsilon)}\\ &< 1.\end{aligned}$$

Since $A$ and $DAD^{-1}$ have the same eigenvalues, we have

$$\begin{aligned}\max\{|\lambda_j|\} &\leq \|DAD^{-1}\|_\infty\\ &= \max\left\{\frac{1}{(\beta+1)} + \frac{1}{(\beta+1)(1-(2m+1)\varepsilon)}, \quad \frac{d_2}{d_1}, \frac{d_3}{d_2}, \ldots, \frac{d_m}{d_{m-1}},\right.\\ &\frac{d_{m+2}}{d_{m+1}}, \ldots, \frac{d_{2m+1}}{d_{2m}}, \quad \frac{d_{2m+3}}{d_{2m+2}}, \ldots, \frac{d_{3m+1}}{d_{3m}}, \quad \frac{d_{3m+3}}{d_{3m+2}}, \ldots, \frac{d_{4m+2}}{d_{4m+1}},\\ &\left.\frac{1}{(\beta+1)(\alpha+1)} + \frac{1}{(\beta+1)(\alpha+1)(1-(2m+1)\varepsilon)} + \frac{1}{(\alpha+1)(1-m\varepsilon)}\right\}\\ &< 1.\end{aligned}$$

So, the modulus of every eigenvalue of $A$ is less than one. Hence, the unique equilibrium point $E = (\alpha+1, \beta+1, \alpha+1, \beta+1)$ of the system (4.14) is locally asymptotically stable. Thus, the proof is completed. □

## 4.4 Global Asymptotic Stability

In this section, we show that all positive solutions of (4.3) are attracted by a period-two solution.

**Theorem 4.4.** *If $\alpha > 1$, $\beta > 1$, then every positive solution of the system (4.3) converges to the period-two solution $(\alpha+1, \alpha+1)$, $(\beta+1, \beta+1)$,... as $n \to \infty$.*

*Proof.* Let $\{x_n, y_n\}$ be an arbitrary positive solution of the system (4.3) and let

$$u_1 = \limsup_{n\to\infty} x_{2n+1}, \quad l_1 = \liminf_{n\to\infty} x_{2n+1} \quad u_2 = \limsup_{n\to\infty} x_{2n}, \quad l_2 = \liminf_{n\to\infty} x_{2n}$$

$$u_3 = \limsup_{n\to\infty} y_{2n+1}, \quad l_3 = \liminf_{n\to\infty} y_{2n+1}, \quad u_4 = \limsup_{n\to\infty} y_{2n}, \quad l_4 = \liminf_{n\to\infty} y_{2n}$$

Using Theorem (4.1), we get

$$l_1 \le u_1 < +\infty, \quad l_2 \le u_2 < +\infty, \quad l_3 \le u_3 < +\infty, \quad l_4 \le u_4 < +\infty.$$

Now, we assume that $k$ is even. Then the system (4.3) implies that

$$\begin{aligned} u_1 &\le \alpha + \frac{u_4}{l_4}, \quad & u_2 &\le \beta + \frac{u_3}{l_3}, \quad & u_3 &\le \alpha + \frac{u_2}{l_2}, \quad & u_4 &\le \beta + \frac{u_1}{l_1}, \\ l_1 &\ge \alpha + \frac{l_4}{u_4}, \quad & l_2 &\ge \beta + \frac{l_3}{u_3}, \quad & l_3 &\ge \alpha + \frac{l_2}{u_2}, \quad & l_4 &\ge \beta + \frac{l_1}{u_1}. \end{aligned}$$

which implies that

$$\beta u_1 + l_1 \le l_4 u_1 \le \alpha l_4 + u_4, \quad \alpha u_4 + l_4 \le l_1 u_4 \le \beta l_1 + u_1$$

and

$$\alpha u_2 + l_2 \le l_3 u_2 \le \beta l_3 + u_3, \quad \beta u_3 + l_3 \le l_2 u_3 \le \alpha l_2 + u_2.$$

Therefore, we obtain

$$(\beta - 1)(u_1 - l_1) + (\alpha - 1)(u_4 - l_4) \le 0$$

and

$$(\beta - 1)(u_3 - l_3) + (\alpha - 1)(u_2 - l_2) \le 0.$$

Since $\alpha > 1$, $\beta > 1$ and $u_1 - l_1$, $u_2 - l_2$, $u_3 - l_3$, $u_4 - l_4 \geq 0$, we get

$$u_1 - l_1 = 0, \quad u_2 - l_2 = 0, \quad u_3 - l_3 = 0 \text{ and } u_4 - l_4 = 0.$$

Now, we assume that $k$ is odd. Then the system (4.3) implies that

$$u_1 \leq \alpha + \frac{u_3}{l_4}, \quad u_2 \leq \beta + \frac{u_4}{l_3}, \quad u_3 \leq \alpha + \frac{u_1}{l_2}, \quad u_4 \leq \beta + \frac{u_2}{l_1},$$
$$l_1 \geq \alpha + \frac{l_3}{u_4}, \quad l_2 \geq \beta + \frac{l_4}{u_3}, \quad l_3 \geq \alpha + \frac{l_1}{u_2}, \quad l_4 \geq \beta + \frac{l_2}{u_1}$$

which implies that

$$\beta u_1 + l_2 \leq l_4 u_1 \leq \alpha l_4 + u_3, \quad \alpha u_4 + l_3 \leq l_1 u_4 \leq \beta l_1 + u_2$$

and

$$\alpha u_2 + l_1 \leq l_3 u_2 \leq \beta l_3 + u_4, \quad \beta u_3 + l_4 \leq l_2 u_3 \leq \alpha l_2 + u_1.$$

Consequently, we obtain

$$(\beta - 1)u_1 + (1 - \alpha)l_2 \leq (\alpha - 1)l_4 + (1 - \beta)u_3$$

and

$$(1 - \beta)l_1 + (\alpha - 1)u_2 \leq (\beta - 1)l_3 + (1 - \alpha)u_4.$$

By addition, we get

$$(\beta - 1)(u_1 - l_1) + (\alpha - 1)(u_2 - l_2) + (\alpha - 1)(u_4 - l_4) + (\beta - 1)(u_3 - l_3) \leq 0.$$

But $\alpha - 1, \beta - 1 > 0$ and $u_1 - l_1, u_2 - l_2, u_3 - l_3, u_4 - l_4 \geq 0$. Thus

$$u_1 - l_1 = 0, \quad u_2 - l_2 = 0, \quad u_3 - l_3 = 0 \text{ and } u_4 - l_4 = 0.$$

So, we use (4.14) to get

$$l_1 = u_1 = \alpha + 1, \quad l_2 = u_2 = \beta + 1, \quad l_3 = u_3 = \alpha + 1, \quad l_4 = u_4 = \beta + 1$$

Moreover, it is obvious that since $\alpha \neq \beta$, then from equations (4.4), (4.5), (4.6) and (4.7)

$$\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}, \quad \lim_{n\to\infty} y_{2n+1} \neq \lim_{n\to\infty} y_{2n}.$$

Finally, since $l_1 = u_1$, $l_2 = u_2$, $l_3 = u_3$, $l_4 = u_4$, it is clear that $\{x_n, y_n\}$ converges to the period-two solution $(\alpha+1, \alpha+1)$, $(\beta+1, \beta+1)$,... as $n \to \infty$. □

From Theorems (4.3) and (4.4) we obtain the following result.

**Theorem 4.5.** *If $\alpha, \beta > 1$ and $k$ is even then the period-two solution $\{(\alpha+1, \alpha+1), (\beta+1, \beta+1), ...\}$ of the system (4.3) is globally asymptotically stable.*

## 4.5 Rate of Convergence

Let $\{(u_n, v_n, t_n, w_n)\}$ a solution of the system (4.14) converging to the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$. We seek here to determinate the speed of convergence of $\{(u_n, v_n, t_n, w_n)\}$, called "Rate of convergence", and to do this we will build a system of errors, we have

$$\begin{aligned}
u_{n+1} - \bar{u} &= \sum_{i=0}^{k} A_i(u_{n-i} - \bar{u}) + \sum_{i=0}^{k} B_i(w_{n-i} - \bar{w}),\\
v_{n+1} - \bar{v} &= \sum_{i=0}^{k} E_i(v_{n-i} - \bar{v}) + \sum_{i=0}^{k} F_i(t_{n-i} - \bar{t}),\\
t_{n+1} - \bar{t} &= \sum_{i=0}^{k} C_i(t_{n-i} - \bar{t}) + \sum_{i=0}^{k} D_i(v_{n-i} - \bar{v}),\\
w_{n+1} - \bar{w} &= \sum_{i=0}^{k} G_i(w_{n-i} - \bar{w}) + \sum_{i=0}^{k} H_i(u_{n-i} - \bar{u}).
\end{aligned}$$

Set

$$e_n^{(1)} = u_n - \bar{u},\ e_n^{(2)} = v_n - \bar{v},\ e_n^{(3)} = t_n - \bar{t},\ e_n^{(4)} = w_n - \bar{w}.$$

Hence we obtain

$$\begin{aligned}
e_{n+1}^{(1)} &= \sum_{i=0}^{k} A_i e_{n-i}^{(1)} + \sum_{i=0}^{k} B_i e_{n-i}^{(4)}, \quad & e_{n+1}^{(2)} &= \sum_{i=0}^{k} E_i e_{n-i}^{(2)} + \sum_{i=0}^{k} F_i e_{n-i}^{(3)},\\
e_{n+1}^{(3)} &= \sum_{i=0}^{k} C_i e_{n-i}^{(3)} + \sum_{i=0}^{k} D_i e_{n-i}^{(2)}, \quad & e_{n+1}^{(4)} &= \sum_{i=0}^{k} G_i e_{n-i}^{(4)} + \sum_{i=0}^{k} H_i e_{n-i}^{(1)},
\end{aligned}$$

where

$$A_i = 0, i \in \{0, 1, \ldots, m-1\}, \quad B_0 = \frac{-w_{n-m}}{w_n^2}, \quad B_i = 0, i \in \{1, 2, \ldots, m-1\},$$
$$B_m = \frac{1}{w_n}, \quad E_0 = \frac{t_{n-m+1} v_{n-m}}{(\alpha v_n + v_{n-m})^2}, \quad E_i = 0, i \in \{1, 2, \ldots, m-1\}, \quad E_m = \frac{-t_{n-m+1} v_n}{(\alpha v_n + v_{n-m})^2},$$
$$F_i = 0, i \in \{0, 1, \ldots, m-2, m\}, \quad F_{m-1} = \frac{v_n}{\alpha v_n + v_{n-m}}, \quad C_i = 0, i \in \{0, 1, \ldots, m-1\},$$
$$D_0 = \frac{-v_{n-m}}{v_n^2}, \quad D_i = 0, i \in \{1, 2, \ldots, m-1\}, \quad D_m = \frac{1}{v_n}, \quad G_0 = \frac{u_{n-m+1} w_{n-m}}{(\alpha w_n + w_{n-m})^2},$$
$$G_i = 0, i \in \{1, 2, \ldots, m-1\}, \quad G_m = \frac{-u_{n-m+1} w_n}{(\alpha w_n + w_{n-m})^2}, \quad H_i = 0, i \in \{0, 1, \ldots, m-2, m\}$$
$$H_{m-1} = \frac{w_n}{\alpha w_n + w_{n-m}}.$$

Taking the limits, we have

$$\lim_{n\to\infty} A_i = 0 \text{ for } i \in \{0, 1, \ldots, m-1\}, \quad \lim_{n\to\infty} B_0 = \frac{-1}{\bar{w}},$$
$$\lim_{n\to\infty} B_i = 0 \text{ for } i \in \{1, \ldots, m-1\}, \quad \lim_{n\to\infty} B_m = \frac{1}{\bar{w}}, \quad \lim_{n\to\infty} E_0 = \frac{\bar{t}}{(\alpha+1)^2 \bar{v}},$$
$$\lim_{n\to\infty} E_i = 0 \text{ for } i \in \{1, \ldots, m-1\}, \quad \lim_{n\to\infty} E_m = \frac{-\bar{t}}{(\alpha+1)^2 \bar{v}},$$
$$\lim_{n\to\infty} F_i = 0 \text{ for } i \in \{0, 1, \ldots, m-2, m\}, \quad \lim_{n\to\infty} F_{m-1} = \frac{\bar{v}}{(\alpha+1)\bar{v}},$$
$$\lim_{n\to\infty} C_i = 0 \text{ for } i \in \{0, 1, \ldots, m-1\}, \quad \lim_{n\to\infty} D_0 = \frac{-1}{\bar{v}}, \quad \lim_{n\to\infty} D_i = 0 \text{ for } i \in \{1, \ldots, m-1\},$$
$$\lim_{n\to\infty} D_m = \frac{1}{\bar{v}}, \quad \lim_{n\to\infty} G_0 = \frac{\bar{u}}{(\alpha+1)^2 \bar{w}}, \quad \lim_{n\to\infty} G_i = 0 \text{ for } i \in \{1, \ldots, m-1\},$$
$$\lim_{n\to\infty} G_m = \frac{-\bar{u}}{(\alpha+1)^2 \bar{w}}, \quad \lim_{n\to\infty} H_i = 0 \text{ for } i \in \{0, 1, \ldots, m-2, m\}, \quad \lim_{n\to\infty} H_{m-1} = \frac{\bar{w}}{(\alpha+1)\bar{w}}.$$

Hence

$$B_0 = \frac{-1}{\bar{w}} + a_n, \quad B_m = \frac{1}{\bar{w}} + b_n, \quad E_0 = \frac{\bar{t}}{(\alpha+1)^2 \bar{v}} + c_n, \quad E_m = \frac{-\bar{t}}{(\alpha+1)^2 \bar{v}} + d_n,$$
$$D_0 = \frac{-1}{\bar{v}} + f_n, \quad D_m = \frac{1}{\bar{v}} + g_n, \quad G_0 = \frac{\bar{u}}{(\alpha+1)^2 \bar{w}} + h_n, \quad G_m = \frac{-\bar{u}}{(\alpha+1)^2 \bar{w}} + k_n,$$
$$F_{m-1} = \frac{\bar{v}}{(\alpha+1)\bar{v}} + p_n, \quad H_{m-1} = \frac{\bar{w}}{(\alpha+1)\bar{w}} + q_n,$$

where $a_n, b_n, c_n, d_n, f_n, g_n, h_n, k_n, p_n, q_n \longrightarrow 0$ for $n \longrightarrow \infty$. Consequently, we obtain a system of the form

$$e_{n+1} = (A + B(n)) e_n,$$

where

$$
\begin{aligned}
e_n &= \left(e_n^{(1)}, ..., e_{n-m+1}^{(1)}, e_n^{(2)}, ..., e_{n-m}^{(2)}, e_n^{(3)}, ..., e_{n-m+1}^{(3)}, e_n^{(4)}, ..., e_{n-m}^{(4)}\right)^T \\
&= (u_n - \bar{u}, ..., u_{n-m+1} - \bar{u}, v_n - \bar{v}, ..., v_{n-m} - \bar{v}, t_n - \bar{t}, ..., t_{n-m+1} - \bar{t}, w_n - \bar{w}, ..., w_{n-m} - \bar{w})^T
\end{aligned}
$$

and

$$
B(n) = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & a_n & 0 & \dots & 0 & b_n \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & c_n & 0 & \dots & 0 & d_n & 0 & \dots & 0 & p_n & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & f_n & 0 & \dots & 0 & g_n & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & q_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & h_n & 0 & \dots & 0 & k_n \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}
$$

$$
A = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix}
$$

with $||B(n)|| \longrightarrow 0$. Therefore, we can write the limiting system of error terms about the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$ as follows:

$$e_{n+1} =$$

$$\left(\begin{array}{ccccccccccccccccccc}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & \frac{-1}{\beta+1} & 0 & \dots & 0 & \frac{1}{\beta+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \frac{1}{\alpha+1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{(\beta+1)(\alpha+1)} & 0 & \dots & 0 & \frac{-1}{(\beta+1)(\alpha+1)} \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{array}\right)
\begin{pmatrix} e_n^{(1)} \\ e_{n-1}^{(1)} \\ \vdots \\ e_{n-m+1}^{(1)} \\ e_n^{(2)} \\ e_{n-1}^{(2)} \\ \vdots \\ e^2_{(n-m)} \\ e_n^{(3)} \\ e_{n-1}^{(3)} \\ \vdots \\ e_{n-m+1}^{(3)} \\ e_n^{(4)} \\ e_{n-1}^{(4)} \\ \vdots \\ e_{n-m}^{(4)} \end{pmatrix}$$

which is the same as the linearized system of system (4.14) about equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$. Finally, we apply Perron's theorems to obtain the following result.

**Theorem 4.6.** *Assume that a solution $\{(u_n, v_n, t_n, w_n)\}$ of the system (4.14) converges to the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$ which is globally asymptotically stable when $\alpha, \beta > 1$ and $k$ is even . Then, the error vector*

$$e_n = \left(e_n^{(1)}, ..., e_{n-m+1}^{(1)}, e_n^{(2)}, ..., e_{n-m}^{(2)}, e_n^{(3)}, ..., e_{n-m+1}^{(3)}, e_n^{(4)}, ..., e_{n-m}^{(4)}\right)^T$$
$$= (u_n - \bar{u}, ..., u_{n-m+1} - \bar{u}, v_n - \bar{v}, ..., v_{n-m} - \bar{v}, t_n - \bar{t}, ..., t_{n-m+1} - \bar{t}, w_n - \bar{w}, ..., w_{n-m} - \bar{w})^T$$

*of every solution of the system (4.14) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{||e_n||} = |\lambda_i J_F(\bar{u}, \bar{v}, \bar{t}, \bar{w})|, \ \textit{for some } i = 1, 2, \dots, k$$

*or*

$$\lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = |\lambda_i J_F(\bar{u}, \bar{v}, \bar{t}, \bar{w})|, \ \textit{for some } i = 1, 2, \dots, k$$

*where $|\lambda_i J_F(\bar{u}, \bar{v}, \bar{t}, \bar{w})|$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point $(\bar{u}, \bar{v}, \bar{t}, \bar{w})$.*

## 4.6 Numerical Examples

In this section, in order to confirm our theoretical results, we consider some numerical examples.

**Example 4.1.** *Consider the system* (4.3) *with* $k = 9$ *and the initial conditions* $x_{-9} = 3$, $x_{-8} = 4$, $x_{-7} = 0.6$, $x_{-6} = 1.3$, $x_{-5} = 0.1$, $x_{-4} = 2.7$ $x_{-3} = 9$ , $x_{-2} = 5$, $x_{-1} = 2.8$, $x_0 = 5.7$, $y_{-9} = 2$, $y_{-8} = 0.4$, $y_{-7} = 3$, $y_{-6} = 1.01$, $y_{-5} = 7$, $y_{-4} = 4.2$, $y_{-3} = 1.9$ , $y_{-2} = 7$, $y_{-1} = 6.7$, $y_0 = 3$. *Moreover, we take the parameters* $\alpha = \frac{5}{6}$, $\beta = \frac{1}{6}$, *i.e.,*

$$\alpha_n = \begin{cases} \frac{5}{6} & \text{if } n \text{ even },\\ \frac{1}{6} & \text{if } n \text{ odd }. \end{cases}$$

*In this case* $0 < \alpha < 1$, $0 < \beta < 1$ *and* $k$ *is odd. Then, by virtue of Theorem (4.2), the solution of the system* (4.3) *is unbounded (see Figure 4.1).*

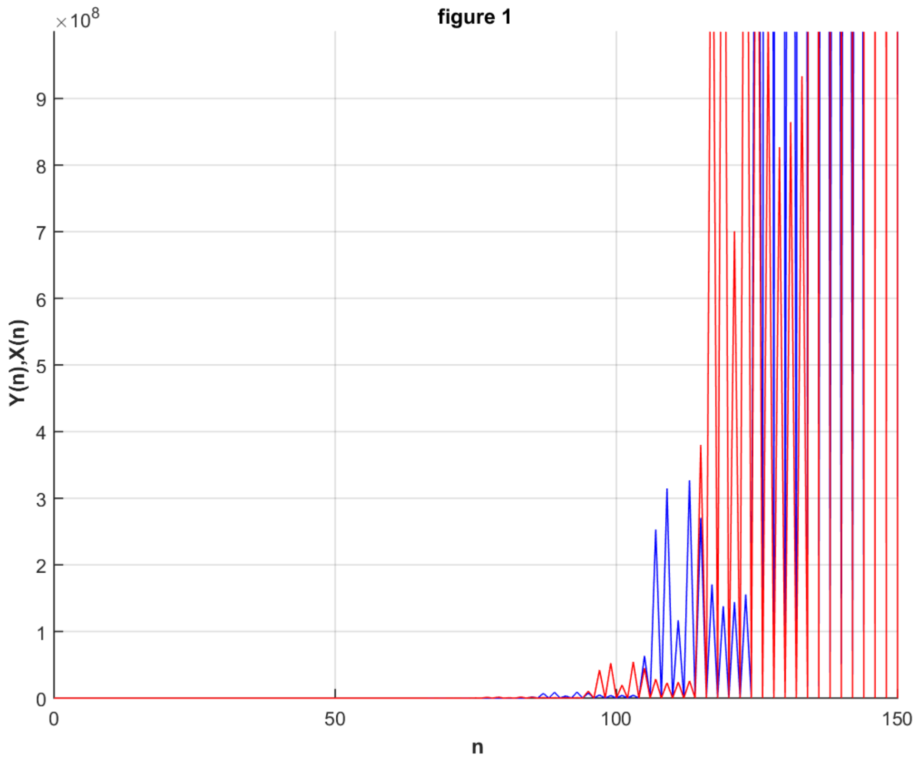


Figure 4.1: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (4.3) with $k = 9$ and the initial values $x_{-9} = 3$, $x_{-8} = 4$, $x_{-7} = 0.6$, $x_{-6} = 1.3$, $x_{-5} = 0.1$, $x_{-4} = 2.7$ $x_{-3} = 9$ , $x_{-2} = 5$, $x_{-1} = 2.8$, $x_0 = 5.7$, $y_{-9} = 2$, $y_{-8} = 0.4$, $y_{-7} = 3$, $y_{-6} = 1.01$, $y_{-5} = 7$, $y_{-4} = 4.2$, $y_{-3} = 1.9$ , $y_{-2} = 7$, $y_{-1} = 6.7$, $y_0 = 3$.

**Example 4.2.** *Consider the system* (4.3) *with* $k = 4$ *and the initial conditions* $x_{-4} = 4$, $x_{-3} = 3$ , $x_{-2} = 1.06$, $x_{-1} = 2$, $x_0 = 0.8$, $y_{-4} = 2$, $y_{-3} = 1.4$ , $y_{-2} = 4$, $y_{-1} = 1$, $y_0 = 4$. *Moreover, we take the parameters* $\alpha = \frac{7}{3}$, $\beta = \frac{5}{3}$. *In this case* $\alpha$, $\beta > 1$ *and* $k$ *is even. Then, by virtue of Theorem (4.4) the solution of the system* (4.3) *converges to the period two solution* $\{(\frac{10}{3}, \frac{10}{3}), (\frac{8}{3}, \frac{8}{3}), ...\}$ *(see Figure 4.2).*

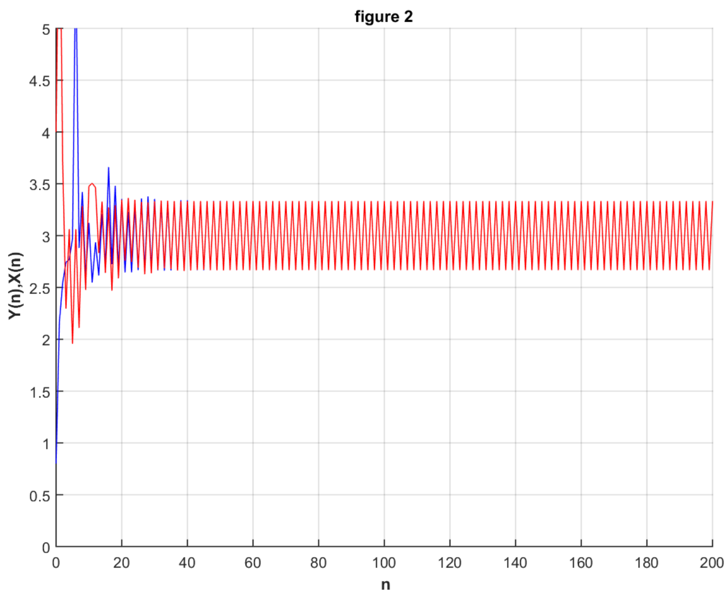


Figure 4.2: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (4.3) with $k = 4$ and the initial values $x_{-4} = 4$, $x_{-3} = 3$, $x_{-2} = 1.06$, $x_{-1} = 2$, $x_0 = 0.8$, $y_{-4} = 2$, $y_{-3} = 1.4$, $y_{-2} = 4$, $y_{-1} = 1$, $y_0 = 4$.

**Example 4.3.** *Consider the system* (4.3) *with* $k = 3$ *and the initial conditions* $x_{-3} = 1.4$ , $x_{-2} = 2.6$, $x_{-1} = 1.4$, $x_0 = 1.1$, $y_{-3} = 2.1$ , $y_{-2} = 1.4$, $y_{-1} = 3.1$, $y_0 = 0.8$. *In addition, we take the parameters* $\alpha = 5$, $\beta = 3$. *In this case* $\alpha$, $\beta > 1$ *and* $k$ *is odd. Then, by virtue of Theorem (4.4) the solution of the system* (4.3) *converges to the period two solution* $\{(6, 6), (4, 4), ...\}$ *(see Figure 4.3).*

**Example 4.4.** *Consider the system* (4.3) *with* $k = 5$ *and the initial conditions* $x_{-5} = 1.4$, $x_{-4} = 2.6$, $x_{-3} = 1.4$, $x_{-2} = 1.1$, $x_{-1} = 2.4$, $x_0 = 1.8$, $y_{-5} = 2.01$, $y_{-4} = 1.4$, $y_{-3} = 3.1$, $y_{-2} = 0.8$, $y_{-1} = 2.3$, $y_0 = 5.9$. *Moreover, we take the sequence* $\{\alpha_n\}$ *as follows*

$$\alpha_n = \begin{cases} 10 & \textit{if } n \textit{ even} \\ 8 & \textit{if } n \textit{ odd} \end{cases}$$

*In this case* $\alpha, \beta > 1$ *and* $k$ *is odd. Then, by virtue of Theorem (4.4) the solution of the system* (4.3) *converges to the period two solution* $\{(11, 11), (9, 9), ...\}$ *(see Figure 4.4).*

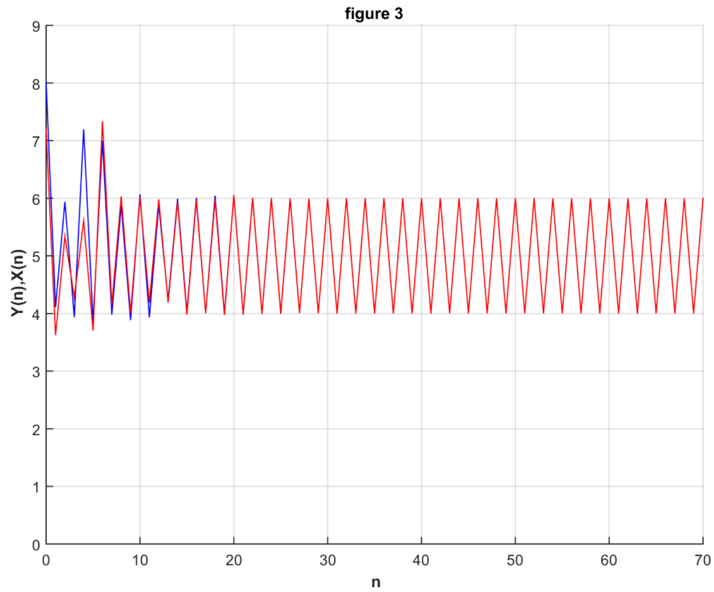


Figure 4.3: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (4.3) with $k = 3$ and the initial values $x_{-3} = 1.4$, $x_{-2} = 2.6$, $x_{-1} = 1.4$, $x_0 = 1.1$, $y_{-3} = 2.1$, $y_{-2} = 1.4$, $y_{-1} = 3.1$, $y_0 = 0.8$.

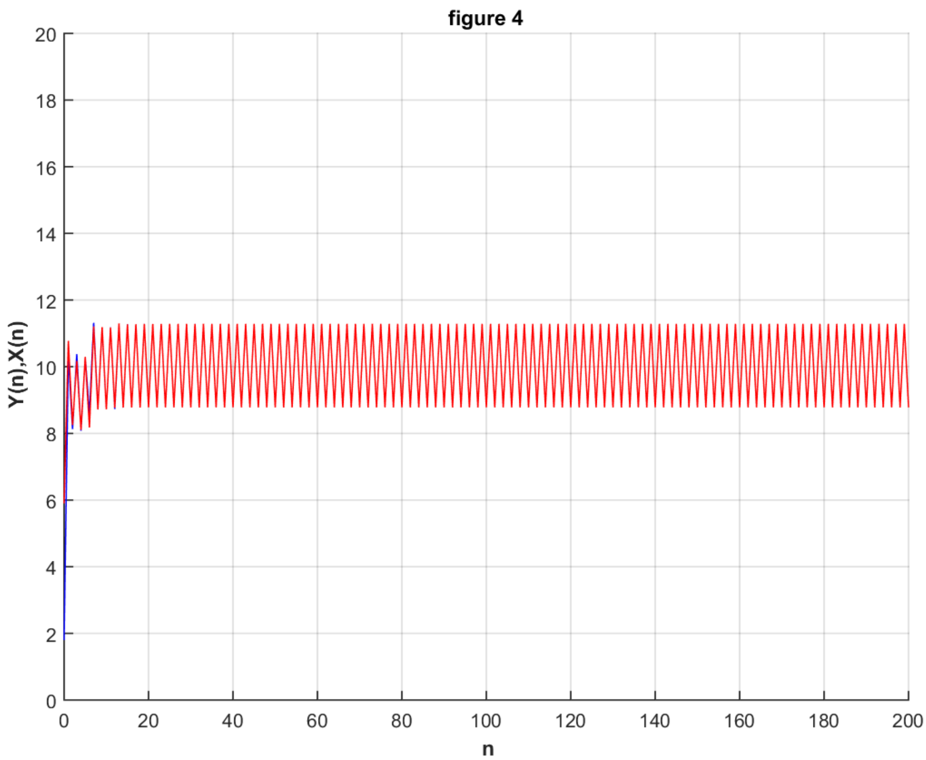


Figure 4.4: Plot of the solution $\{(x_n, y_n)\}_{n\geq 0}$ of the system (4.3) with $k = 5$ and the initial values $x_{-5} = 1.4$,$x_{-4} = 2.6$ , $x_{-3} = 1.4$ $x_{-2} = 1.1$, $x_{-1} = 2.4$, $x_0 = 1.8$, $y_{-5} = 2.01$, $y_{-4} = 1.4$,$y_{-3} = 3.1$, $y_{-2} = 0.8$, $y_{-1} = 2.3$, $y_0 = 5.9$.

# Conclusion and Perspective

This thesis combines some basic notions with some studies and their results that we have achieved about the behavior of some types of difference equations and their systems. More precisely non-linear, autonomous and non-autonomous difference equations of second and higher order besides a system of difference equations of higher order.

In the first chapter, we have given two examples by describing two biological phenomena as an application of difference equations and their systems which were studied by mathematicians. Next, we moved to the study of difference quadratic equation where we discussed according to some parameters the local and global stability as well as the existence of periodic solutions. After that, our study was oriented to generalize the results of non-autonomous difference equation more specifically the equation with bounded coefficient. The latter is the main goal of our study in the third chapter. The results obtained in this chapter were published in an international journal see [48].

In the fourth chapter, we are interested in the study of an open problem proposed in [27]. This open problem is a system of difference equations with periodic coefficient

$$x_{n+1} = \alpha_n + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = \alpha_n + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \ldots$$

where $\{\alpha_n\}$ is a periodic sequence of non-negative real numbers and the initial conditions $x_i$, $y_i$ are arbitrary positive numbers for $i = -k, -k+1, -k+2, \ldots, 0$ and $k \in \mathbb{Z}^+$. This system was difficult to study that is why we transformed it to a system of difference equations with constant coefficient. And we obtained the sufficient results that prove the behavior stability of solutions. It was published in an international journal see [49].

As a future perspective, we are interested in the study of the following system of difference equations

$$x_{n+1} = \alpha_n + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = \beta_n + \frac{x_{n-k}}{x_n}, \qquad n = 0, 1, \ldots$$

where $\{\alpha_n\}$, $\{\beta_n\}$, are periodic sequences or convergent sequences or bounded sequences of non-negative real numbers and the initial conditions $x_i$, $y_i$ are arbitrary positive numbers for $i = -k, -k+1, -k+2, \ldots, 0$ and $k \in \mathbb{Z}^+$.

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