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THÈSE

Présenté en vue de l'obtention du diplôme de Doctorat en mathématiques

INVERSE PROBLEM OF DETERMINATION OF UNKNOWNS IN A T-FRACTIONAL REACTION-DIFFUSION EQUATION

Filière
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Par

FAIZI Rima

DIRECTEUR DE THÈSE :	Atmania Rahima	Prof	U.B.M. ANNABA

Devant le jury

PRESIDENT :	Chorfi Lahcène	Prof	U.B.M. ANNABA
EXAMINATEUR :	Alem Leila	Prof	U.B.M. ANNABA
EXAMINATEUR :	Berhail Amel	MCA	Univ de GUELMA
EXAMINATEUR :	Nisse Lamine	Prof	U.E.H.L. El-Oued
EXAMINATEUR :	Zouyed Fairouz	Prof	U.B.M. ANNABA

Dedication

To

My mother

My father

My brother

My sisters

Thank you for supporting me and my dreams, and never trying to hold me back from what I want in life.

I dedicate this modest work

Rima Faizi

Acknowledgments

First and foremost, I would like to praise and thank **Allah** who gave me the strength and the will to complete this thesis.

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Abstract

In this thesis, we treat inverse problems of finding some time dependent coefficients in parabolic equations involving a time fractional derivative and over-determination conditions. We are concerned by the well-posedness of these problems, meaning the existence, uniqueness and continuous dependence on the data of the solution.
We start with the determination of the source coefficient in a generalized time fractional diffusion equation, in the case of non-local boundary conditions and an integral constraint by using the Fourier method with a bi-orthogonal system.
Then, we study two inverse coefficient problems for the Caputo time-fractional reaction-diffusion equation, the proof of the obtained results is based on the iteration method and Gronwall's Lemma.
At the end, we consider a direct Caputo time-fractional parabolic initial-boundary value problem. The results of existence, uniqueness and stability are obtained for a weak solution in view of fixed point theory followed by an application to an inverse source coefficient problem with an output data measured at a fixed space point.

Key words : Parabolic equation ; fractional derivative ; inverse problem ; iteration method ; Fourier method ; bi-orthogonal system ; Gronwall's Lemma ; fixed point theory.

AMS Subject Classification : 35K10 ; 26A33 ; 35R30 ; 42A16 ; 47H10.

Résumé

Dans cette thèse, nous traitons quelques problèmes inverses de détermination de coefficients dépendant du temps dans des équations paraboliques impliquant une dérivée partielle d'ordre fractionnaire en temps à partir de donnée supplémentaire. Nous nous intéressons à le caractère bien-posés, c'est-à-dire de l'existence, l'unicité et de la solution dépendance continue aux données.
Nous commençons par déterminer un coefficient du terme source d'une équation de diffusion fractionnaire en temps généralisée, dans le cas de conditions aux limites non locales et d'une contrainte intégrale, en utilisant la méthode de Fourier avec un système bi-orthogonal.
Ensuite, nous étudions deux problèmes inverses d'identification de coefficients pour une équation de réaction-diffusion fractionnaire en temps au sens de Caputo. La preuve des résultats obtenus est basée sur la méthode d'itération et le Lemme de Gronwall.
En dernier lieu, nous considérons un problème parabolique fractionnaire en temps au sens de Caputo direct à valeurs initiale et aux bords. Les résultats sont obtenus pour une solution faible via la théorie du point fixe et sont suivis d'une application à un problème inverse de détermination du terme source à partir d'une donnée de sortie mesurée en un point fixe de l'espace.

Mots clés : Équation parabolique ; dérivée fractionnaire ; problème inverse ; méthode d'itération ; méthode de Fourier ; système bi-orthogonal ; Lemme de Gronwall ; théorie du point fixe.

ملخص

في هذه الرسالة، درسنا بعض المسائل العكسية لتحديد معاملات معتمدة على الوقت في المعادلات المكافئة التي تنطوي على مشتق كسري زمني وشروط تحديد اضافية. نوضح أن هذه المسائل العكسية لها حل كلاسيكي وحيد يعتمد باستمرار على البيانات، وهذا هو معنى المسائل جيدة الطرح.
لقد بدأنا بإيجاد معامل المصدر لمعادلة الانتشار ذات مشتق كسري جزئي للزمن معمم، في حالة شروط الحدودية غير المحلية وشرط مقيد على شكل تكامل. نحن مهتمون بالوجود، التفرد و استقرار الحل وذلك باستخدام طريقة فورييه ونظام الثنائي المتعامد.
بعد ذلك، درسنا الطرح الجيد لحل مشكلتي المعامل معكوستين لمعادلة تفاعل انتشار ذات مشتق كسري جزئي للزمن لكابوتو وتستند الافكار الاساسية للبراهين إلى طريقة التكرار وخاصية جرونوال.
في نهاية المطاف، اعتبرنا معادلة القطع المكافئ ذات مشتق كسري جزئي للزمن لكابوتو تخضع لشروط الحدودية لديركلي والبيانات الأولية. في ضوء نظرية النقطة الصامدة نحصل على الوجود، التفرد ونتائج الاستقرار لحل ضعيف. علاوة على ذلك، درسنا مسئلة معامل عكسي مع بيانات الإخراج المقاسة في نقطة معينة من الفضاء.

الكلمات المفتاحية : معادلة القطع المكافئ؛ مشتق كسري؛ مسئلة عكسية؛ طريقة التكرار؛ طريقة فورييه؛ نظام الثنائي المتعامد؛ خاصية جرونوال؛ نظرية النقطة الصامدة.

Notations

Sets

$\mathbb{N}$	Set of natural numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^n$	Space of n-dimensional real vectors.
$\mathbb{C}$	Set of complex numbers.
Ω	Open bounded subset of $\mathbb{R}^n$.
I	Compact interval of $\mathbb{R}$.

Functions, Spaces of Functions

$\Gamma(.)$	Gamma function.
$B(.)$	Beta function.
$E_{\alpha,\beta}(.)$	Mittag-Leffler function.
$C(I)$	Space of continuous functions on I.
$C^\infty(I)$	Space of infinitely differentiable functions on I.
$C^n(I)$	Space of n time continuous functions on I.
$AC(I)$	Space of absolutely continuous functions on I.
$L^p(\Omega)$	Space of Lebesgue integrable functions on Ω.
$L^\infty(\Omega)$	Space of functions u that are essentially bounded on Ω.
$W^{k,p}(\Omega)$	Sobolev space on Ω.
$H^r(\Omega)$	Sobolev space $W^{r,2}(\Omega)$.
$H^{-r}(\Omega)$	Dual space of $H_0^r(\Omega)$.

Mathematical operators

$\Re(.)$	Real part of the complex number.
$\Im(.)$	Imaginary part of the complex number.
I^α	The Riemann-Liouville fractional integral operator of order α.
∂^m	The partial derivation operator.
D^m	The weak m^{th} derivative.
${}^{RL}D^\alpha$	The Riemann-Liouville fractional derivative of order $\alpha > 0$.
${}^{C}D^\alpha$	The Caputo fractional derivative of order $\alpha > 0$.
$D^{\alpha,\beta}$	The Hilfer fractional derivative of order $\alpha > 0$ and type β.
$n!$	Factorial(n), $n \in \mathbb{N}$: The product of all the integers from 1 to n.
$\langle .,. \rangle_{L^2(\Omega)}$	Usual inner product in $L^2(\Omega)$.

Contents

General Introduction

When we manipulate the notion of derivative, we quickly realize that we can apply the concept of derivative to the derivative function itself, and by the same to introduce the second derivative, then the successive derivatives said of integer order. The integration is an inverse operator to the right of the derivation, can optionally be considered as a derivative of order ”minus one”. So, one can also wonder if this notion can be generalized to fractional or even non-integer orders. The first note about this idea of differentiation for non-integer numbers, dates back to 30 September 1695, with a famous correspondence between Leibniz and L'Hôpital. In a letter, L'Hôpital asked Leibniz about the possibility of the order $n \in \mathbb{N}$, in the notation $\dfrac{d^n f}{dx^n}$ for the n-th derivative of the function f, to be a non-integer, $n = \frac{1}{2}$.

The first serious attempt to give a logical definition for the fractional derivative is due to Liouville, who published nine documents in this subject between 1832 and 1837. Independently, Riemann proposed an approach which proved essentially that of Liouville, and it is since she wears the no. ”Riemann-Liouville approach”. Later, other approaches appeared such as that of Grunwald-Leitnikov, Weyl, Caputo, Hilfer, Hadamard and Caputo-Hadamard. At that time there were almost no practical applications of this theory, and it is for this reason that it was considered an abstract containing only useless mathematical manipulations. The transition from pure mathematical formulations to applications began to emerge from the 1990s, when fractional differential equations appeared in several fields such as physics, engineering, biology, mechanics, electrochemistry, biomedicine,...,etc.

A simple example of fractional derivatives of the function $f(t) = t^2$, is plotted for different values of the fractional order in Figure 1.

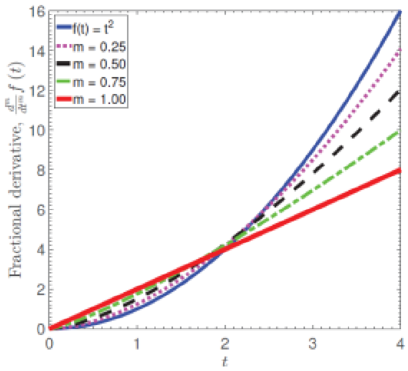


Figure 1: *Fractional derivatives* $\frac{d^m}{dt^m} f(t)$*, of a quadratic function,* $f(t) = t^2$ *(blue, solid line) with the order m which has values* 0.25 *(magenta, dotted line),* 0.50 *(black, dashed line),* 0.75 *(green, dash-dot line), and* 1 *(red, thicker solid line)*

The different values of the fractional order of the function f are obtained by using the following expression

$$\frac{d^m}{dt^m}t^2 = \frac{\gamma(3)}{\gamma(3-m)}t^{2-m},$$

where m is a real number, for more see Example 1.2.

Actually, when modeling some real physical phenomena, fractional derivatives can provide more accurate results than integer order derivatives [60]. The advantages of fractional derivatives are that they have a greater degree of flexibility in the model and provide an excellent instrument for the description of the reality. This is because of the fact that the realistic modeling of a physical phenomenon with memory and hereditary properties does not depend only on the instant time, but also on the history of the previous time, i.e., calculating time fractional derivative at some time requires all the previous processes [49]. Fractional derivatives are involved in the viscoelastic modeling of the behavior of gums and rubbers, more generally in the shaping of memory materials, thanks to the associated integral in its definition. For example, in [5, 7], the authors introduced the fractional derivation to model damping in equations of motion where materials used as insulators or dampers are viscoelastic. The benefit of this introduction is the reduction of the number of model parameters. Effectively, for modeling the stress-strain behavior of a solid, Bagley and Torvik [6] introduced a four-parameter model instead of ten

$$(1+bD_t^{\alpha})\sigma(t) = (G_0+G_1D_t^{\alpha})\varepsilon(t),$$

where σ and ε denote stress and strain respectively and b, G_0, G_1, α are the model parameters.

In view of mathematical interests and applications, the partial differential equations in particular of the time fractional parabolic type, have attracted many researchers in exclusive discipline of applied science and engineering, as shown in [22, 45, 62] and references therein.

Now let's talk about the inverse problems that in fact constantly impact our daily lives. According to Keller [37], two problems are said to be the inverse of one another if the formulation of one of them involves the other. This definition contains a part of arbitrariness and confers a symmetric role to both problems under consideration. A more operational definition is that an inverse problem consists determining the causes of a phenomenon according to the observation of its effects. Thus, this problem is the opposite of the so-called direct problem, consisting of the deduction of the effects, the causes being known.
In mathematics, a linear inverse problem can take the form of an equation $Ku = f$, Where u represents the values of the parameters of the phenomenon, f represents the measurements carried out and K is a linear operator called an input-output mapping. K can be a nonlinear operator, hence the class of nonlinear inverse problems which are generally hard to study.

The mathematical theory of inverse problems sometimes referred as identiability problems was essentially ignored until the middle of the twentieth century. The scientists have focused on direct problems, that is, the construction of the model itself rather than the inverse process that often leads to existence, uniqueness and stability difficulties. At the beginning of the twentieth century, the idea of direct problems dominated mathematical physics. Indeed, the French mathematician Hadamard believed that an important physical problem must be well posed, that is to say that the problem must always have a unique solution which depends continuously on the data. This idea persisted into the middle of the twentieth century. However, the advent of quantum mechanics and many problems in areas of classical physics such as heat conduction and geophysics slowly convinced mathematicians and scientists that direct problems were not the only scientific problems and that the mathematical theory of inverse problems began to be developed by mathematicians in Soviet Union led by Tikhonov.

We can see that inverse problems may cause difficulties. In fact, it is reasonable: *"the same causes produce the same effects"*. However, it is easy to imagine that *" the same effects may come from different causes"*. This idea contains the main difficulty of the study of inverse problems, they can have several solutions, and it is necessary to have additional information to discriminate between them.

The resolution of the inverse problem generally passes through an initial step called the direct problem. Then, the resolution can be done by numerical simulation or analytically.

Among the areas in which the inverse problems arise an important role we can mention:

- The medical imaging (ultrasound, scanners, X-rays, ...);
- The hydrogeology (identification of hydraulic permeabilities);
- Chemistry (determination of reaction constants);
- The radar (determining the shape of an obstacle);
- The quantum mechanics (determination of potential);
- The image processing (restoration of blurred images).

The main objective of this research work is studied some inverse problems of determination a different type of unknown coefficients in fractional parabolic equations, by using different methods and tools. The present thesis is arranged as follows:

In **Chapter** 1, we collect necessary tools of functional analysis and Fourier analysis, then we introduce elementary definitions and basic properties relating to fractional analysis and inverse problems that are used throughout this manuscript.

In **Chapter** 2, an inverse source coefficient problem for a Hilfer time fractional diffusion equation is considered. The existence, uniqueness and continuous dependence on the data of the solution are obtained by using Fourier's method with a bi-orthogonal system of functions.

In **Chapter** 3, we obtain same results for two inverse problems concerning the semi-linear time-fractional parabolic equation by using the Fourier method, iteration method and Gronwall's Lemma.

In **Chapter** 4, we study the existence, uniqueness and stability of weak solution for the time-fractional parabolic direct problem in view of fixed point theory. In addition, an inverse problem with a output measured data in a fixed point space is treated as an application.

CHAPTER

1 Preliminaries

In this chapter, we introduce the mathematical tools necessary for a good understanding of the thesis. We first give reminders of functional analysis and Fourier analysis, then we introduce elementary definitions and basic properties relating to fractional analysis and inverse problems.

1.1 Functional Analysis

First of all, we recall the concepts and fundamental results of the theory of functional analysis that are of a particular interest in what follows.

1.1.1 Functional Spaces

Let $I := [a,b]$ be a compact interval of $\mathbb{R}$, we consider Ω an open bounded subset of $\mathbb{R}^n$, where n is a positive integer. We denote by $\partial\Omega$ the boundary of Ω and $\bar{\Omega} = \Omega \cup \partial\Omega$ the closure of Ω. A typical point in $\mathbb{R}^n$ is denoted by $x = (x_1, x_2, ..., x_n)$. The n-tuple of non-negative integers (including zero) $m = (m_1, m_2, .., m_n)$, is called a multi-index. The number $|m| = m_1 + m_2 + ... + m_n$ is called the length of the multi-index m.
For $1 \leq k \leq n$, we denote the classical derivative operator with respect to the k-th variable by $\partial_k = \partial / \partial x_k$ and

$$\partial^m = \partial_1^{m_1} \partial_2^{m_1} ... \partial_n^{m_n} = \frac{\partial^{|m|}}{\partial_{x_1}^{m_1} \partial_{x_2}^{m_1} ... \partial_{x_n}^{m_n}},$$

is the m^{th} partial derivation operator.

In the following, we present some functional spaces, for more details see [38, 40, 58].

Spaces of Continuous Functions

Definition 1.1. *Let $C(I)$ be the set of all real-valued continuous functions in I, equipped with the norm*

$$\|f\|_{C(I)} = \max_{t \in I} |f(t)|.$$

Analogously, $C^n(I)$ is the Banach space of functions $f : I \to \mathbb{R}$, where f is n time continuously differentiable on I such that

$$\|f\|_{C^n(I)} = \sum_{k=0}^{n} \|f^{(k)}\|_{C(I)} = \sum_{k=0}^{n} \max_{t \in I} |f^{(k)}(t)|, \quad n \in \mathbb{N}.$$

Also, we set

$$\begin{aligned} C^{\infty}(I) &= \bigcap_{n=0}^{\infty} C^n(I) \\ &= \{f : I \to \mathbb{R}; f \in C^n(I), \quad \forall n \in \mathbb{N}\}, \end{aligned}$$

the space of infinitely differentiable functions defined on I with values in $\mathbb{R}$.

Spaces of Absolutely Continuous Functions

Definition 1.2. *A function $f : I = [a,b] \to \mathbb{R}$ is said absolutely continuous on I, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \geq 1$ and finite partitions $[a_i, b_i]_{i=1}^n \subset [a,b]$, then $\sum_{i=1}^{n} |b_i - a_i| \leq \delta$ implies that $\sum_{i=1}^{n} |f(b_i) - f(a_i)| \leq \varepsilon$.*

We denote $AC(I)$ the spaces of absolutely continuous functions on I, equipped with the norm

$$\|f\|_{AC(I)} = |f(a) + \|f'\|_{L^1(I)}.$$

It is known that $AC(I)$ coincides with the space of primitives of Lebesgue summable functions, (see [40], p. 338)

$$f(t) \in AC(I) \Leftrightarrow f(t) = c + \int_a^t \varphi(s)ds, \quad \varphi \in L^1(I). \tag{1.1}$$

Therefore, an absolutely continuous function $f(t)$ has a summable derivative $f'(t) = \varphi(t)$ almost everywhere on I. Thus (1.1) yeilds

$$\varphi(s) = f'(s) \text{ and } c = f(a).$$

Definition 1.3. *For $n \in \mathbb{N}^*$, we denote by $AC^n(I)$ the space of real-valued functions $f(t)$ which have continuous derivatives up to order $n-1$ on I such that $f^{(n-1)} \in AC(I)$, i.e.*

$$AC^n(I) = \{f : I \to \mathbb{R} : f^{(n-1)} \in AC(I)\}.$$

This space is characterized by the following assertion:

Lemma 1.1. *A function $f \in AC^n(I)$, $n \in \mathbb{N}^*$, if and only if it is represented in the form*

$$f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f^{(n)}(s)ds + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

Spaces of Weighted Continuous Functions

Definition 1.4. *Let $\gamma \in \mathbb{C}$ with $0 \leq \Re(\gamma) \leq 1$, we introduce the weighted space $C_\gamma(I)$ of functions f given on $(a,b]$, such that the function $(t-a)^\gamma f \in C(I)$, endowed with the norm*

$$\|f\|_{C_\gamma(I)} = \|(t-a)^\gamma f\|_{C(I)}.$$

For $n \in \mathbb{N}$, we denote by $C_\gamma^n(I)$ the Banach space of functions $f(t)$ which are continuously differentiable on I up to order $n-1$ and have the derivative $f^{(n)}(t)$ of order n on $(a,b]$ such that $f^{(n)} \in C_\gamma(I)$, equipped with the norm

$$\|f\|_{C_\gamma^n(I)} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{C(I)} + \|f^{(n)}\|_{C_\gamma(I)}.$$

In particular case, for $n = 0$, $C_\gamma^0(I) = C_\gamma(I)$.

Spaces of Lebesgue's Integrable Functions

Definition 1.5. *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we denote $L^p(\Omega)$ the space of Lebesgue's integrable functions such that:*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ measurable and } \|u\|_{L^p(\Omega)} < \infty\}$$

with

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

For $p = \infty$, we have the following definition:

Definition 1.6. *We denote $L^\infty(\Omega)$, the space of all essentially bounded functions on Ω:*

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ measurable } ; \|u\|_{L^\infty(\Omega)} < \infty \text{ on } \Omega\}$$

with essential supremum

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} &= ess \sup_{x\in\Omega} |u(x)| \\ &= \inf\{M > 0 : |u(x)| \leq M \text{ a.e. on } \Omega\}.\end{aligned}$$

Note that, if $u \in L^\infty(\Omega)$ then we have $|u(x)| \leq \|u\|_{L^\infty(\Omega)}$ almost everywhere on Ω.

The following theorem summarizes some properties of the L^p spaces:

Theorem 1.1. *Let $1 \leq p \leq \infty$. Then*

1. *The space $L^p(\Omega)$ endowed with the norm $\|.\|_{L^p(\Omega)}$ is a Banach space.*
2. *Hölder's inequality: Let $1 \leq q \leq \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$. Then, $u.v \in L^1(\Omega)$ and*

$$\int_\Omega |u(x)v(x)|dx \leq \left(\int_\Omega |u(x)|^p dx\right)^{1/p} \left(\int_\Omega |v(x)|^q dx\right)^{1/q}.$$

3. *For $1 < p, q < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, the dual $(L^p(\Omega))'$ of $L^p(\Omega)$ is the space $L^q(\Omega)$.*

In particular, if $p = q = 2$, we have a simple consequence of the last theorem:

Corollary 1.1. *The space $L^2(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx, \quad \forall u, v \in L^2(\Omega).$$

Moreover, the following Cauchy-Schwarz inequality holds:

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \quad \forall u, v \in L^2(\Omega).$$

where,

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}.$$

Now, we introduce the space of locally integrable functions that will be used in order to introduce the weak derivatives of real valued functions defined on Ω.

Definition 1.7. *A function $u : \Omega \to \mathbb{R}$ is said to be locally integrable, noted $u \in L^1_{loc}(\Omega)$, if for every compact subset B of Ω, then $u \in L^1(B)$, i.e.*

$$\int_B |u(x)|dx < \infty.$$

Next, we recall the following Gronwall's Lemma:

Lemma 1.2. *Let $u(t)$ and $v(t)$ be non-negative continuous functions on an interval $[0,T]$. Also, let the function $\omega(t)$ be a positive, continuous, and monotonically non-decreasing on $[0,T]$ and satisfies the inequality*

$$u(t) \leq \omega(t) + \int_0^t v(s)\, u(s)\, ds,$$

then

$$u(t) \leq \omega(t) \exp\left(\int_0^t v(s)\, ds\right), \quad \text{for } t \in [0,T].$$

The following result is singular kernels of Gronwall's inequality:

Lemma 1.3. *[64] Let $u : [0,T] \to \mathbb{R}^+$ be a real function, v be a non-negative locally integrable function on $[0,T]$. If there are constants $\omega > 0$ and $\beta \in (0,1)$ such that:*

$$u(t) \leq v(t) + \omega \int_0^t \frac{u(s)}{(t-s)^\beta} ds, \quad t \in [0,T],$$

then

$$u(t) \leq v(t) + K_\beta \omega \int_0^t \frac{v(s)}{(t-s)^\beta} ds, \quad t \in [0,T],$$

for some positive constant K_β.

Sequence Spaces

The most important sequence spaces in functional analysis are ℓ^p spaces, consisting of p-power summable sequences, with the p-norm. These are special cases of L^p spaces for the counting measure on the set of natural numbers.

Definition 1.8. *Let $p \in [1,\infty]$, we define the p-norm of a sequence $\{\rho_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}$ as*

$$\|\rho\|_{\ell^p(\mathbb{R})} = \begin{cases} \left(\sum_{i\in\mathbb{N}} |\rho_i|^p\right)^{1/p} & , \text{if } p < \infty \\ \sup_{i\in\mathbb{N}} |\rho_i| & , \text{if } p = \infty \end{cases}$$

and we set

$$\ell^p(\mathbb{R}) = \{\rho = (\rho_0, \rho_1, \rho_2, \ldots) : \|\rho\|_{\ell^p(\mathbb{R})} < \infty, \rho_i \in \mathbb{R}, i \in \mathbb{N}\}.$$

Some properties of ℓ^p spaces are summarized below:

Proposition 1.1. *Let $1 \leq p \leq \infty$. Then*

1. *The space $\ell^p(\mathbb{R})$ equipped with their respective norms is a Banach space.*

2. *Hölder's inequality: Let $1 \leq q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\rho \in \ell^p(\mathbb{R})$ and $\upsilon \in \ell^q(\mathbb{R})$. Then, $\rho.\upsilon \in \ell^1(\mathbb{R})$ and*

$$\sum_{i\in\mathbb{N}} |\rho_i \upsilon_i| \leq \left(\sum_{i\in\mathbb{N}} |\rho_i|^p \right)^{1/p} \left(\sum_{i\in\mathbb{N}} |\upsilon_i|^q \right)^{1/q}.$$

3. *For $1 \leq p < q \leq \infty$, we have the stict inclusion*

$$\ell^p(\mathbb{R}) \subset \ell^q(\mathbb{R}).$$

Moreover, the canonical injection associated with this inclusion is continuous for the respective norms $\ell^p(\mathbb{R})$ and $\ell^q(\mathbb{R})$, and more precisely, we have

$$\|\rho\|_{\ell^q(\mathbb{R})} \leq \|\rho\|_{\ell^p(\mathbb{R})}, \quad \forall \rho \in \ell^q(\mathbb{R}).$$

Sobolev Spaces

The Sobolev space is a vector space of functions that have weak derivatives. The weak derivative is an extension of the classical derivative, i.e., if the classical derivative exists, then the two derivatives coincide. Let us start with the definition of the support of a function:

Definition 1.9. *Given a function $u : \Omega \to \mathbb{R}$, its support is defined to be*

$$supp\ u = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

In other words, the support of u is the smallest closed set of $\mathbb{R}$ such that u is null outside. We have the following useful result ([10], p.153):

Lemma 1.4. *Let $u \in L^1_{loc}(\Omega)$. If*

$$\int_\Omega u(x)\varphi(x)dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega)$$

then $u = 0$ a.e. on Ω.

Later on, we will need the following space of infinitely differentiable functions with compact support:

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : \text{ supp } u \subset_{\text{compact}} \Omega\}.$$

Definition 1.10. *Let $u \in L^1_{loc}(\Omega)$. If there exists a function $v \in L^1_{loc}(\Omega)$ such that:*

$$\int_\Omega u(x)\partial^m \varphi(x)dx = (-1)^{|m|} \int_\Omega v(x)\varphi(x)dx, \quad \forall \varphi \in C_0^\infty(\Omega) \tag{1.2}$$

then, v is called the weak m^{th} derivative of u.

From the definition of the weak derivative and by integration by parts, we see that if u is k-times continuously differentiable on Ω, then, for each m with $|m| \leq k$, the classical partial derivative $\partial^m u$ is also the m^{th} weak derivative of u.

Lemma 1.5. *A weak m^{th}-derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{loc}(\Omega)$ two weak derivatives of u satisfy

$$\int_\Omega u(x)\partial^m \varphi(x)dx = (-1)^{|m|}\int_\Omega v(x)\varphi(x)dx = (-1)^{|m|}\int_\Omega \tilde{v}(x)\varphi(x)dx,$$

for all $\varphi \in C_0^\infty(\Omega)$. This implies that

$$\int_\Omega (v(x) - \tilde{v}(x))\varphi(x)dx = 0,$$

for all $\varphi \in C_0^\infty(\Omega)$. From Lemma 1.4, we deduce that $v(x) - \tilde{v}(x) = 0$ a.e., which is the uniqueness of the weak derivative. □

Notation. For simplicity, if u and v satisfy (1.2) for some positive integer $|m|$, we write $v := D^m u$, the weak m^{th} derivative of u.

Definition 1.11. *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index m with $|m| \leq k$, the weak derivatives $D^m u$ exist and are in $L^p(\Omega)$. Thus*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^m u \in L^p(\Omega), \quad \forall |m| \leq k\}.$$

If $u \in W^{k,p}(\Omega)$, we define its norm as:

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|m|\leq k} \|D^m u\|^p_{L^p(\Omega)}\right)^{1/p} & \text{if} \quad 1 \leq p < \infty; \\ \max_{|m|\leq k} \|D^m u\|_{L^\infty(\Omega)} & \text{if} \quad p = \infty. \end{cases}$$

A semi-norm over the space $W^{k,p}(\Omega)$ is

$$|u|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|m|= k} \|D^m u\|^p_{L^p(\Omega)}\right)^{1/p} & \text{if} \quad 1 \leq p < \infty; \\ \max_{|m|= k} \|D^m u\|_{L^\infty(\Omega)} & \text{if} \quad p = \infty. \end{cases}$$

Notation. When $p = 2$, we usually write

$$W^{k,2}(\Omega) = H^k(\Omega), \quad k = 0, 1, \ldots$$

Theorem 1.2. *Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev spaces $\left(W^{k,p}(\Omega), \|.\|_{W^{k,p}(\Omega)}\right)$ are Banach spaces.*

Next, we have the following result:

Theorem 1.3. *Let* $p \in [0, \infty]$. *Then*

$$W^{1,p}(a,b) \hookrightarrow C[a,b].$$

The previous theorem shows that every element $u \in W^{1,p}(a,b)$ can be identified with an element, still denoted by u, in the space $C[a,b]$. Moreover, there is a positive constant c such that

$$\|u\|_{C[a,b]} \leq c\|u\|_{W^{1,p}(a,b)}.$$

Definition 1.12. *We denote by* $W_0^{k,p}(\Omega)$, *the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$, *i.e.*

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{k,p}(\Omega)}.$$

Thus, $u \in W^{k,p}(\Omega)$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}} \in C_0^\infty(\Omega)$ such that:

$$\|u_n - u\|_{W^{k,p}(\Omega)} \to 0.$$

We interpret $W_0^{k,p}(\Omega)$ as comprising the function $u \in W^{k,p}(\Omega)$ such that:

$$D^m u = 0 \quad \text{on } \partial\Omega \text{ for all } |m| \leq k-1.$$

It can be shown that the semi-norm $|.|_{W^{k,p}(\Omega)}$ is a norm on $W_0^{k,p}(\Omega)$ and there exists a constant $c > 0$ such that:

$$|u|_{W^{k,p}(\Omega)} \leq \|u\|_{W^{k,p}(\Omega)} \leq c|u|_{W^{k,p}(\Omega)}, \quad \forall u \in W_0^{k,p}(\Omega).$$

Notation. It is customary to write

$$W_0^{k,2}(\Omega) = H_0^k(\Omega).$$

1.1.2 Fixed Point Theorems

Fixed point theorems concern the maps F of a set X into itself admiting at least one fixed point; a point x for which $F(x) = x$, under certain conditions on F. The fixed point theory is one of the most powerful and fruitful tools in the study of ordinary or partial differential equations. First, we define what we mean by a Lipschitz mapping.

Definition 1.13. *Let* $(X, \|.\|)$ *be a complete normed space. A map* $F : X \to X$ *is called a Lipschitz mapping on* X*, if there exists a positive constant L such that:*

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in X.$$

The smallest L for which the above inequality holds is the Lipschitz constant of F. *If* $L \geq 1$, F *is said to be non-expansive and if* $L < 1$, F *is called a contraction mapping.*

Next, let us recall the well-known Banach and Schauder fixed point theorems, for more see [57].

Theorem 1.4 (Banach's theorem)**.** *Let F be a contraction mapping on a complete normed space X. Then, F has a unique fixed point $\bar{x} \in X$.*

Theorem 1.5 (Schauder's theorem)**.** *Let B be a nonempty, convex and compact subset of a normed space B. Then, every completely continuous mapping $F : B \to B$ has at least one fixed point in B.*

Finally, we recall the Arzela-Ascoli theorems to characterize the relatively compact parts of the space of continuous functions from a compact space to any space.

Theorem 1.6. *Let X,Y be a Banach spaces such that X is a compact space. A subset M of $C(X,Y)$ is relatively compact if and only if:*

1. *M is equicontinuous.*

2. *For all $x \in X$, the subset $M = \{F(x), F \in M\}$ is relatively compact in Y.*

Theorem 1.7. *Let Ω be a bounded subset of $\mathbb{R}^n$. A subset M in $C(\bar{\Omega})$ is relatively compact if and only if M is bounded and equicontinuous.*

1.1.3 Laplace Transform

The Laplace transform is a mathematical tool that is used in solving differential equations by converting them from one form to another form that is easy to solve. Regularly, it is effective in solving ordinary or partial linear differential equations.

Definition 1.14. *The Laplace transform of a function f of the real variable $s \in \mathbb{R}^+ = (0,\infty)$ is defined by*

$$F(t) = (\mathscr{L} f(s))(t) = \int_0^\infty e^{-st} f(s)ds, \quad t \in \mathbb{C}, \tag{1.3}$$

$f(s)$ is called the original of $F(t)$.

The Laplace transform of a function f exists if the integral (1.3) is convergent, for that the original must be of exponential order σ, which means that there are two positive constants M and L such that

$$|f(s)| \leq Me^{\sigma s}, \quad \text{for } s > L.$$

In this case, the Laplace transform (1.3) exists for $\Re(t) > \sigma$.

We can reconstruct the original f from its Laplace transform F by using the inverse Laplace transform

$$f(s) = (\mathscr{L}^{-1} F(t))(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(t)dt, \quad c = \Re(t) > \sigma.$$

Property 1.1. *Some properties of the Laplace transform are listed below:*

1. *The Laplace transform is a linear operator, that is for functions $f(s), g(s)$ and any constant k, we have*
$$(\mathscr{L}\{kf(s)+g(s)\})(t) = k(\mathscr{L}f(s))(t) + (\mathscr{L}g(s))(t).$$
2. *If $F(t)$ and $G(t)$ are the Laplace transforms of $f(s)$ and $g(s)$ respectively, then*
$$(\mathscr{L}\{(f*g)(s)\})(t) = F(t)G(t).$$
3. *The Laplace transform of the n-order derivative of the function f is*
$$(\mathscr{L}f^{(n)}(s))(t) = t^n(\mathscr{L}f(s))(t) - \sum_{k=0}^{n-1} t^k f^{(n-k-1)}(0).$$

1.2 Fourier Analysis

One of the basic concepts of Fourier analysis is the study of the possibility of representing or approximating periodic functions by trigonometric series. This is a very helpful strategy, specially in solving PDEs because it is always easier to deal with trigonometric functions than general ones.

1.2.1 Trigonometric Fourier Series

The starting point here is the following definition of the Fourier series:

Definition 1.15. *We call the Fourier series associated with an L-periodic function f, the corresponding trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + b_n \sin(n\omega x), \quad \omega = \frac{2\pi}{L}, \quad \forall n \in \mathbb{N}, x \in \mathbb{R} \tag{1.4}$$

with $a_n = \frac{2}{L}\int_0^L f(x)\cos(n\omega x)dx$ *and* $b_n = \frac{2}{L}\int_0^L f(x)\sin(n\omega x)dx.$

Theorem 1.8 (Weierstrass M-test)**.** *Suppose that $\{f_n\}$ is a sequences of real-or complex-valued functions defined on a set S. The series $\sum_{n=0}^{\infty} f_n(x)$ is absolutely and uniformly convergent on a set S, if there exists a numerical sequence $M_n > 0, \forall n \in \mathbb{N}$ such that:*

$$|f_n(x)| \leq M_n, \forall x \in S \text{ and } \sum_{n=0}^{\infty} M_n < \infty.$$

Due to the above theorem, we have the following proposition:

Proposition 1.2. *If the numerical series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then the Fourier series (1.4) is normally convergent on $\mathbb{R}$, therefore absolutely and uniformly convergent on $\mathbb{R}$.*

Remark 1.1. *According to Euler's relations*

$$\cos(n\omega x) = \frac{e^{in\omega x} + e^{-in\omega x}}{2}, \quad \sin(n\omega x) = \frac{e^{in\omega x} - e^{-in\omega x}}{2i}.$$

The series (1.4) *becomes*

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{in\omega x} = c_0 + \sum_{n\geq 1}\left(c_n e^{in\omega x} + c_{-n}e^{-in\omega x}\right), \quad \forall n \in \mathbb{N},$$

with

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}.$$

This expression is called the complex form of a Fourier series, where $c_n(f)$ is the Fourier coefficients of f as defined below:

Definition 1.16. *Let $f : \mathbb{R} \to \mathbb{C}$ be a $L-$periodic and piecewise continuous function. We call Fourier coefficients of f the complex numbers*

$$c_n(f) = \frac{1}{L}\int_0^L f(x)e^{-in\omega x}dx, \quad \omega = \frac{2\pi}{L}, n \in \mathbb{Z}. \tag{1.5}$$

Lemma 1.6 (Riemann-Lebesgue)**.** *Let $c_n(f)$, $n \in \mathbb{Z}$, the Fourier coefficients relating to f. Then, if $f \in L^1(0,L)$ we have*

$$|c_n(f)| \leq \|f\|_{L^1(0,L)}, n \in \mathbb{Z}, \text{ and } \lim_{n\to\pm\infty} c_n(f) = 0.$$

For all function $f \in L^1(0,L)$, we can associate a Fourier series of f, but we can't ensure its convergence. For this, we have the following results about its convergence:

Proposition 1.3. *If $f \in L^2[0,L]$, then $f \in L^1[0,L]$ and $\|f\|_{L^1[0,L]} \leq \|f\|_{L^2[0,L]}$. In addition,*

$$\sum_{n\in\mathbb{Z}} |c_n(f)|^2 \leq \|f\|^2_{L^2[0,L]}.$$

The second assertion is known in the literature as Bessel's inequality, which means that for any function $f \in L^2[0,L]$, its associated Fourier series converges to f in $L^2[0,L]$, it's called quadrature convergence in the sense that:

$$\lim_{n\to\pm\infty} \|f - \sum_{n\in\mathbb{Z}} c_n(f)e^{in\omega x}\|_{L^2[0,L]} = 0.$$

Theorem 1.9. *Let $f : \mathbb{R} \to \mathbb{C}$ be L-periodic such that it is continuous and f' is a piecewise continuous on $\mathbb{R}$. Then*

1. *The Fourier series of f converges absolutely and uniformly to f.*
2. *The Fourier series of f' is obtained by deriving term by term from the Fourier series of f and the Fourier coefficients satisfy*

$$\sum_{n\in\mathbb{Z}} |c_n(f)| < \infty.$$

In fact, the smoother the function involved, the faster the corresponding coefficients tend to 0.

Proposition 1.4. *If $f \in C^k[0,L]$ such that $f^{(k)} \in L^2[0,L]$, then the Fourier coefficients of $f^{(k)}$, for $k \geq 1$ satisfy*

$$c_n(f^{(k)}) = \left(\frac{2i\pi n}{L}\right)^k c_n(f).$$

In addition, the serie $\sum_{n\in\mathbb{Z}} n^{2k}|c_n(f)|^2$ converges and $\lim_{n\to\pm\infty} |n^k c_n(f)| = 0$.

Remark 1.2. *It is clear that: $|c_n(f)e^{in\varpi x}| \leq |c_n(f)|$ and by using the Weierstrass Theorem 1.8, we deduce that the Fourier series is absolutely and uniformly convergent if the series of coefficients is convergent.*

1.2.2 Spectral Problem

A spectral problem linked to an operator A is the search for real or complex values λ and corresponding functions X satisfying $LX = -\lambda X; \quad X \neq 0$.

Sturm-Liouville Problem

Complete orthogonal sets of functions in L^2 space arise naturally as solutions of certain second-order linear differential equations under appropriate boundary conditions, commonly referred as Sturm–Liouville boundary-value problems.

Definition 1.17. *[3] The differential equation on a finite interval $[a,b]$ with homogeneous mixed boundary conditions, that is,*

$$\mathbf{L}X = -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]X(x) + q(x)X(x) = \lambda\vartheta(x)X(x), \quad x \in [a,b] \tag{1.6}$$

$$\alpha_1 X(a) + \alpha_2 X'(a) = 0, \quad |\alpha_1| + |\alpha_1| > 0 \tag{1.7}$$

$$\beta_1 X(b) + \beta_2 X'(b) = 0, \quad |\beta_1| + |\beta_1| > 0 \tag{1.8}$$

where α_i and β_i are real constants for $i = 1,2$, $p(x)$ and $\vartheta(x)$ are positive continuous functions on $[a,b]$, is called as regular Sturm-Liouville problem.

The aim is to find all values λ for which problem (1.6)-(1.8) has nontrivial solution $X(x)$. When values of λ exist, they are called eigenvalues of the problem and the solutions associated with each λ are called eigenfunctions. It is implicitly assumed that $X(x)$ and its derivative are continuous on $[a,b]$, which also means that these are bounded functions.

Remark 1.3. *It can be shown that a regular Sturm-Liouville operator*

$$\mathbf{L} : L^2([a,b]) \cap C^2([a,b]) \to L^2([a,b])$$

defined in (1.6) is self-adjoint [41], i.e., for all X, Y satisfying the boundary conditions (1.7)-(1.8), we have

$$\langle X, \mathbf{L}Y\rangle_{L^2([a,b])} = \langle \mathbf{L}X, Y\rangle_{L^2([a,b])}.$$

It is interesting to note some basic properties of the eigenvalues of Sturm-Liouville problems, see [3, 41]:

Property 1.2. *For a regular Sturm-Liouville problem* (1.6)-(1.8)*, we have:*

1. *The set of eigenvalues is real, countably infinite, and is a monotonically increasing sequence*
$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$
with $\lim_{n\to\infty} \lambda_n = +\infty$.

2. *Each eigenvalue* λ_n *corresponds a unique eigenfunction* $X_n(x)$ *(up to a constant multiple) with exactly* $(n-1)$ *zeros in* (a,b)*, called the nth fundamental solution.*

3. *Let* X_1 *and* X_2 *be eigenfunctions of a regular Sturm-Liouville operator corresponding to distinct eigenvalues* λ_1, λ_2 *respectively. Then* X_1 *and* X_2 *are orthogonal with respect to the weight function* $\vartheta(x)$*, that is*
$$\langle X_1, X_2 \rangle_\vartheta = \int_a^b \overline{X_1(x)} X_2(x) \vartheta(x) dx, \quad x \in [a,b].$$

Let $E_\vartheta(a,b)$ denote the space of piecewise continuous functions on $[a,b]$ with inner product $\langle .,. \rangle_\vartheta$, where $\vartheta(x)$ is the weight function from (1.6).

Proposition 1.5. *A regular Sturm-Liouville problem admits an orthonormal sequence of real-valued eigenfunctions in* $E_\vartheta(a,b)$.

The eigenfunctions of a Sturm-Liouville problem can be used to describe piecewise continuous functions, which is very useful for solving time-dependent PDE for which separation of variables yields a Sturm-Liouville problem.

Proposition 1.6. *The orthonormal set* $\{X_n\}_{n=0}^{\infty}$ *of eigenfunctions of a regular Sturm-Liouville problem is a basis for* $E_\vartheta(a,b)$*, that is* $E_\vartheta(a,b)$ *is complete.*

The expansion of a function $f \in E_\vartheta(a,b)$ in the orthonormal basis of eigenfunctions, given by
$$f = \sum_{n=0}^{\infty} a_n X_n, \quad a_n = \langle X_n, f \rangle_\vartheta = \int_a^b \overline{X_n(x)} f(x) \vartheta(x) dx,$$
is called an eigenfunction expansion of f.

The eigenfunction expansion has these essential properties:

Proposition 1.7. *Let* $\{X_n\}_{n=0}^{\infty}$ *be an orthonormal set of eigenfunctions of a regular Sturm-Liouville problem.*

1. *If* f *is continuous and piecewise differentiable on* $[a,b]$ *and satisfies the boundary conditions of the Sturm-Liouville problem, then the eigenfunction expansion of* f *converges uniformly to* f *on* $[a,b]$.

2. *If* f *is piecewise differentiable on* $[a,b]$*, then for* $x \in (a,b)$ *the eigenfunction expansion of* f *converges to* $[f(x^+) + f(x^-)]/2$*, where* $f(x^-)$ *and* $f(x^+)$ *are the left- and right-hand limits of* f *at* x.

A Special Case of Spectral Problem

The spectral problem involving equation (1.6) with $p(x) = \vartheta(x) = 1$ and $q(x) = 0$ is given as:

$$X''(x) = -\lambda X(x), \quad x \in (0,1) \tag{1.9}$$

equipped with the boundary conditions

$$X'(0) = X'(1); \quad X(1) = 0. \tag{1.10}$$

The first boundary condition here is non-local, since it describes the desired solution values $X(x)$ at several points in the interval. This a non-locality in the boundary condition leads to the violation of the self-adjointness of problem (1.9)-(1.10).
The eigenvalues of problem (1.9)-(1.10) are equal to $\lambda_k = (2\pi k)^2, k = 0,1,2,...$, and the eigenfunctions corresponding to λ_k are given by:

$$X_0(x) = 1 - x \text{ for } k = 0, \quad X_k = \sin(2\pi kx) \text{ for } k \geq 1.$$

The set of functions $\{X_0, X_k; k \geq 1\}$ does not form a complete system, because the eigenfunctions X_k, $k \geq 1$ are not orthogonal to X_0, as we can see

$$\int_0^1 (1-x)\sin(2\pi kx)dx = -\frac{1}{2\pi k} \neq 0.$$

Therefore, it can not be a basis of $L^2(0,1)$ space. We complete the set of eigenfunctions by the so-called associated functions, which are the solutions of the following problem

$$\begin{cases} \tilde{X}''(x) = -\lambda_k \tilde{X}(x) - X_k(x), & x \in (0,1) \\ \tilde{X}'(0) = \tilde{X}'(1); \tilde{X}(1) = 0. \end{cases} \tag{1.11}$$

If $k = 0$, problem (1.11) has no solutions. For $k \geq 1$, the problem (1.11) has the eigenfunctions

$$\tilde{X}_k(x) = \frac{x-1}{4\pi k}\cos(2\pi kx).$$

The system $S_1' = \{X_0, X_k, \tilde{X}_k, k \geq 1\}$ is complete in $L^2(0,1)$, and any function from this space can be approximated (with an arbitrary accuracy in the metric of $L^2(0,1)$) by a linear combination of functions from this system. However, looking that

$$\langle \tilde{X}_k, X_k \rangle_{L^2(0,1)} = \frac{1}{4\pi k}\int_0^1 (x-1)\cos(2\pi kx)\sin(2\pi kx)dx = -\frac{1}{32\pi^2 k^2},$$

mains that the constructed system is not orthogonal to each other, the completeness of such a system in the space $L^2(0,1)$ does not ensure its basis property in this space, i.e., the possibility of uniquely expanding any function from $L^2(0,1)$ into a series in functions of this system that converges in the metric of $L^2(0,1)$.

To remedy this, we use a corrective method to construct another complete set which forms with the set S_1 a bi-orthogonal system for the space $L^2(0,1)$.

For this, we have to consider the following conjugate or adjoint problem of spectral problem (1.9)-(1.10):

$$\begin{cases} Y''(x) = -\lambda Y(x), & x \in (0,1) \\ Y(0) = Y(1); & Y'(0) = 0. \end{cases} \tag{1.12}$$

Indeed, an elementary integration by parts shows that

$$\int_0^1 Y(x)X''(x)dx = X(0)Y'(0) + X'(0)\left[Y(1) - Y(0)\right] + \int_0^1 X(x)Y''(x)dx,$$

the right side of this relation vanishes if $Y'(0) = 0$ and $Y(1) = Y(0)$.
By solving the adjoint problem (1.12), we obtain the eigenvalues $\lambda = \lambda_k^*$ which coincide with λ_k and the corresponding eigenfunctions for this problem are

$$Y_0 = 1 \text{ for } k = 0, \quad Y_k = \cos(2\pi kx), \text{ for } k \geq 1.$$

We encounter the same problem of incompleteness of the set of eigenfunctions. The completion of the set passes through the following boundary-value problem:

$$\begin{cases} \tilde{Y}''(x) = -\lambda_k \tilde{Y}(x) - Y_k(x), & x \in (0,1) \\ \tilde{Y}(0) = \tilde{Y}(1); & \tilde{Y}'(0) = 0, \end{cases} \tag{1.13}$$

whose solutions are given by

$$\tilde{Y}(x) = -\frac{x}{4\pi k}\sin(2\pi kx), \quad k \geq 1$$

and for $k = 0$, problem (1.13) has no solutions. The set $S_2' = \{Y_0, Y_k, \tilde{Y}_k, k \geq 1\}$ is a complete, non-orthogonal system of the space $L^2(0,1)$. The two sets S_1' and S_2' construct bi-orthogonal pair of functions for the space $L^2(0,1)$. Taking into consideration that

$$\langle X_0, Y_0 \rangle_{L^2(0,1)} = 1/2 \text{ and } \langle X_k, \tilde{Y}_k \rangle_{L^2(0,1)} = -\langle \tilde{X}_k, Y_k \rangle_{L^2(0,1)} = -\frac{1}{16\pi k},$$

we renumber and normalize the eigenfunctions of S_1' and S_2' to get

$$S_1 = \{X_0(x) = 2(1-x), \quad X_{2k-1}(x) = 4(1-x)\cos(\sqrt{\lambda_k}x), \quad X_{2k}(x) = 4\sin(\sqrt{\lambda_k}x), \quad k \geq 1\};$$

$$S_2 = \{Y_0(x) = 1, \quad Y_{2k-1}(x) = \cos(\sqrt{\lambda_k}x), \quad Y_{2k}(x) = x\sin(\sqrt{\lambda_k}x), \quad k \geq 1\};$$

respectively, where $\lambda_k = (2\pi k)^2, \quad k \geq 1$.

Lemma 1.7. *[47] The systems S_1 and S_2 form a bi-orthogonal system of functions in $L^2(0,1)$, i.e., for all non-negative integers i and j, we have*

$$\langle X_i, Y_j \rangle = \int_0^1 X_i(x)Y_j(x)dx = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Proof. The proof is carried out directly by computing the corresponding value of integrals, we find that

$$\langle X_i, Y_j \rangle = \int_0^1 X_i(x)Y_j(x)dx = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases}.$$

□

Lemma 1.8. *[47] The systems S_1 and S_2 are complete in $L^2(0,1)$.*

Proof. Let $f \in L^2(0,1)$ orthogonal with the system of functions S_1, so $f(x)$ can be represented as

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos(\sqrt{\lambda_k}x),$$

which converges in $L^2(0,1)$. Further, $f(x)$ is orthogonal with S_1, mains that:

$$\begin{aligned}
0 &= \int_0^1 f(x)4(1-x)\cos(\sqrt{\lambda_k}x)dx \\
&= A_0 \int_0^1 4(1-x)\cos(\sqrt{\lambda_k}x)dx + \sum_{k=1}^{\infty} A_k \int_0^1 4(1-x)\cos(\sqrt{\lambda_k}x)\cos(\sqrt{\lambda_k}x)dx \\
&= A_k, \quad k = 1,2,...
\end{aligned}$$

it follows that $A_k = 0, k = 1,2,...$, then $f(x) = 0$ on $[0,1]$. Thus, the system S_1 is complete in $L^2(0,1)$. In the same manner, we prove the completeness of the system S_2. □

Lemma 1.9. *[47] The systems S_1 and S_2 form a Riez basis in $L^2(0,1)$.*

Proof. Recall that, A sequence of vectors $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space $\mathbb{H}$ is called a Riesz sequence, if there exist constants $0 < A < B < \infty$ such that

$$A\sum_{n=0}^{\infty} |a_n|^2 \leq \| \sum_{n=0}^{\infty} a_n x_n \|^2 \leq B \sum_{n=0}^{\infty} |a_n|^2,$$

for all sequence of scalars $(a_n)_{n\in\mathbb{N}}$ in $l^2(\mathbb{R})$. A Riesz sequence is called Riesz basis if additionally $\overline{span\{x_n : n \in \mathbb{N}\}} = \mathbb{H}$, i.e., that is linearly independent in the space $\mathbb{H}$.
According to the results of the book [23], from Lemma 1.8, it is sufficient to establish the completeness of the systems S_1 and S_2 and by using the Bessel inequality in Proposition 1.3, we get that the following series on S_1 and S_2:

$$\left(2\int_0^1 (1-x)f(x)dx\right)^2 + \sum_{k=1}^{\infty}\left[\left(4\int_0^1 (1-x)f(x)\cos(\sqrt{\lambda_k}x)dx\right)^2 + \left(4\int_0^1 f(x)\sin(\sqrt{\lambda_k}x)\right)^2\right];$$

$$\left(\int_0^1 f(x)dx\right)^2 + \sum_{k=1}^{\infty}\left[\left(\int_0^1 f(x)\cos(\sqrt{\lambda_k}x)dx\right)^2 + \left(\int_0^1 f(x)x\sin(\sqrt{\lambda_k}x)\right)^2\right],$$

respectively, are convergent for any $f \in L^2(0,1)$. □

1.3 Fractional Analysis

Before introducing the basic facts on fractional operators, we define some special functions which play a crucial role in the theory of fractional calculus, one can find more details on the books [38, 49].

1.3.1 Basic Functions

Gamma Function

The Gamma function is a natural extension of the factorial from integers to real (or complex) numbers.

Definition 1.18. *For $z \in \mathbb{C}$, the Gamma function is defined as:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(z) > 0.$$

This integral is convergent for all complex number $z \in \mathbb{C}$ with $\Re(z) > 0$.

Property 1.3. *We notice several properties of Gamma function:*

1. *The Gamma function satisfies the following recurrence formula:*

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C}. \tag{1.14}$$

2. *The Gamma function generalizes the factorial*

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}^*.$$

3. *The Gamma function can be represented by the limit*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)},$$

where we initially suppose $\Re(z) > 0$.

Remark 1.4. *The assertion (1.14), can be easily proved by integration by parts*

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = \left[-e^{-t} t^z\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z).$$

Beta Function

The Beta function is a special type of function which is also known as Euler's integral of the first kind. It is an important function in calculus and analysis due to its close connection to the Gamma function.

Definition 1.19. *For $z, w \in \mathbb{C}$, the Beta function is defined as:*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \quad \Re(z) > 0, \Re(w) > 0.$$

The Beta function is connected with the Gamma function by the following relation:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \Re(z) > 0, \Re(w) > 0. \tag{1.15}$$

Mittag-Leffler Function

The Mittag-Leffler function is a generalization of the exponential function. It plays a key role in the treatment of problems related to integral and differential equations of fractional order. We begin with the more general class of Mittag-Leffler function definition, for more see [24, 49, 51].

Definition 1.20. *The three-parametric Mittag-Leffler (or Prabhakar) function is defined by the series expansion*

$$E^{\rho}_{\alpha,\beta}(z) := \sum_{n\geq 0} \frac{(\rho)_n z^n}{\Gamma(\alpha n+\beta)n!}; \quad z,\alpha,\beta,\rho \in \mathbb{C}, \text{ with } \Re(\alpha) > 0, \Re(\beta) > 0, \rho > 0, \tag{1.16}$$

where $(\rho)_n$ is the Pochhammer symbol defined by:

$$(\rho)_0 = 1, \quad (\rho)_n = \rho(\rho+1)\ldots(\rho+n-1) = \frac{\Gamma(\rho+n)}{\Gamma(\rho)}, \quad n \in \mathbb{N}, \rho \neq 0.$$

For $\rho = 1$, we recover the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n+\beta)}; \quad \Re(\alpha) > 0, \Re(\beta) > 0, \text{ with } z \in \mathbb{C}. \tag{1.17}$$

For $\rho = \beta = 1$, we recover the classical Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n+1)}; \quad \Re(\alpha) > 0, \text{ with } z \in \mathbb{C}. \tag{1.18}$$

Corollary 1.2. *The following properties hold on $(0,T]$ for $0 < \alpha \leq \beta \leq 1, \lambda \in \mathbb{R}^+$:*

1. *$E^{\rho}_{\alpha,\beta}(-\lambda t^{\alpha})$ is an entire function and thus bounded on any finite interval*
2. *$E_{\alpha,\beta}(-\lambda t^{\alpha})$ is a bounded positive monotonic decreasing function, and satisfies the following differentiation formulas for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$:*

$$\left(\frac{d}{dz}\right)^n E_n(\lambda z^n) = \lambda E_n(\lambda z^n);$$
$$\left(\frac{d}{dz}\right)^n \left[z^{\beta-1} E_{n,\beta}(\lambda z^n)\right] = z^{\beta-n-1} E_{n,\beta-n}(\lambda z^n).$$

The following lemmas are well known properties of Mittag-Leffler function and can be found in Podlubny's book [49].

Lemma 1.10. *Let $\beta \in \mathbb{R}$ be arbitrary and μ satisfy that $\pi\alpha/2 < \mu < \min(\pi, \alpha\pi)$. Then there exists a constant C depending on α, β and μ such that $\left|E_{\alpha,\beta}(z)\right| \leq \frac{C}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi$.*

Lemma 1.11. *For $\alpha, \beta, \rho > 0$, $\lambda \in \mathbb{C}$, the Laplace transform of Mittag-Leffler function is*

$$\mathscr{L}\left[z^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda z^{\alpha})\right](t) = \frac{t^{\alpha\rho-\beta}}{(t^{\alpha}-\lambda)^{\rho}}, \quad \Re(t) > 0, \quad |\lambda t^{-\alpha}| < 1.$$

Lemma 1.12. *[24] Let $\nu, \mu, \gamma, \sigma, \rho > 0$, we have*

$$\int_0^t (t-s)^{\nu-1}E^{\gamma}_{\mu,\nu}(\lambda(t-s)^{\mu})s^{\sigma-1}E^{\rho}_{\mu,\sigma}(\lambda s^{\mu})ds = t^{\nu+\sigma-1}E^{\gamma+\rho}_{\mu,\nu+\sigma}(\lambda t^{\mu}). \tag{1.19}$$

1.3.2 Riemann-Liouville Fractional Integral

The Riemann-Liouville fractional integral is a simple generalization of Cauchy's well known representation of an n-fold integral as a convolution integral of a function $f : [a,b] \to \mathbf{R}$

$$\begin{aligned} I^n_a f(t) &= \int_a^t dt_1 \int_a^{t_1} dt_2 \ldots \int_a^{t_{n-1}} f(t_n)dt_n \\ &= \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}f(s)ds, \quad t > a,\ n \in \mathbb{N}, \end{aligned}$$

where $I^n_a f(t)$ is the n-fold integral operator (Cauchy's formula).
By substituting the integer n with a real positive order α and by replacing the factorial function by the Gamma function, we will have the following definition:

Definition 1.21. *The left sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b])$, is defined by:*

$$I^{\alpha}_{a^+} f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad \forall t \in (a,b]$$

where $\Gamma(.)$ denotes the Gamma function.

Example 1.1. *We consider the function f defined by*

$$f(t) = (t-a)^{\beta}, \quad t > a, \quad \beta \in \mathbb{R}.$$

We have

$$I^{\alpha}_{a^+}(t-a)^{\beta} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^{\beta}ds,$$

we use the change of variable $s = a + (t-a)\nu$, we obtain

$$I^{\alpha}_{a^+}(t-a)^{\beta} = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-\nu)^{\alpha-1}\nu^{\beta}d\nu,$$

by using the Definition 1.19 and the relation (1.15), we get

$$\begin{aligned} I^{\alpha}_{a^+}(t-a)^{\beta} &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}B(\alpha,\beta+1) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \end{aligned}$$

The term by term fractional integration of the series (1.17) gives the following lemma:

Lemma 1.13. *For $\alpha, \beta > 0$, we have*

$$I_{a^+}^{\alpha}\left[(t-a)^{\beta-1}E_{\nu,\beta}(\lambda(t-a)^{\nu})\right](z) = (z-a)^{\alpha+\beta-1}E_{\nu,\beta+\alpha}(\lambda(z-a)^{\nu}).$$

The following lemmas provide some basic properties of the fractional integral operator $I_{a^+}^{\alpha}$. The proofs can be found in [38].

Lemma 1.14. .

1. *The semi-group property of the fractional integration operator of function $f \in L^1(a,b)$ is given by*
$$I_{a^+}^{\alpha}I_{a^+}^{\beta}f(x) = I_{a^+}^{\alpha+\beta}f(x), \quad \Re(\alpha) > 0, \Re(\beta) > 0.$$

2. *For $\Re(\alpha) > 0$, $I_{a^+}^{\alpha}$ maps $C(a,b)$ into $C([a,b])$.*

3. *Let $\Re(\alpha) > 0$ and $0 \leq \Re(\gamma) < 1$. Then $I_{a^+}^{\alpha}$ is bounded from $C_{\gamma}(a,b)$ into $C_{\gamma}(a,b)$.*

Lemma 1.15. *let $\alpha > 0, n = [\alpha]+1$ and $f \in L^1(0,b)$, then the Laplace transform of the Riemann-Liouville fractional integral is formulated as:*

$$\mathscr{L}(I_{0^+}^{\alpha}f)(s) = s^{-\alpha}\mathscr{L}[f](s).$$

1.3.3 Fractional Derivatives

Many mathematicians have contributed to the development of the theory of non-integer order derivative and different definitions of this operator have emerged. In this section, we will present the most familiar definitions of fractional derivative, limiting our scope to the Riemann-Liouville, Caputo and the Hilfer versions.

Riemann-Liouville Fractional Derivative

Definition 1.22. *The Riemann-Liouville fractional derivative of order α of a function $f \in L^1([a,b])$ is defined by*

$$\begin{aligned} {}^{RL}D_{a+}^{\alpha}f(t) &:= \left(\frac{d}{dt}\right)^n (I_{a+}^{n-\alpha})f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}f(s)ds, \quad \forall t \in (a,b], \end{aligned}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Example 1.2. *Let $\alpha > 0, \beta > -1$, we have*

$${}^{RL}D_{a^+}^{\alpha}(t-a)^{\beta} = \left(\frac{d}{dt}\right)^n \left(I_{a^+}^{n-\alpha}(t-a)^{\beta}\right),$$

from the result of Example 1.1, we obtain

$$
\begin{aligned}
{}^{RL}D^{\alpha}_{a^+}(t-a)^{\beta} &= \frac{d^n}{dt^n}\left[\frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}(t-a)^{\beta+n-\alpha}\right] \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}\frac{d^n}{dt^n}\left[(t-a)^{\beta+n-\alpha}\right] \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}\frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha},
\end{aligned}
$$

where we used the following formula

$$
\begin{aligned}
\left(\frac{d}{dt}\right)^n (t-a)^m &= m(m-1)...(m-n+1)(t-a)^{m-n} \\
&= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}(t-a)^{m-n}. \qquad (1.20)
\end{aligned}
$$

Remark 1.5. *If we let $\beta = 0$ in the previous example, we see that the Riemann-Liouville fractional derivative of a constant is not 0. In fact*

$$
{}^{RL}D^{\alpha}_{a^+}1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}.
$$

Lemma 1.16. *Let $z > a$ ($a \in \mathbb{R}^+$), α, β and $\lambda \in \mathbb{C}$, we have*

$$
{}^{RL}D^{\alpha}_{a^+}\left[(t-a)^{\beta-1}E_{\nu,\beta}(\lambda(t-a)^{\nu})\right](z) = (z-a)^{\beta-\alpha-1}E_{\nu,\beta-\alpha}(\lambda(z-a)^{\nu}).
$$

The following Lemmas can be found in [38]:

Lemma 1.17. *Let $\Re(\alpha) > 0, n = [\Re(\alpha)]+1$ and $I^{n-\alpha}_{a^+}f$ the Riemann-Liouville fractional integral of order $n-\alpha$ of a function f.*

1. *If $f \in L^p(a,b)$ with $1 \le p \le \infty$, we have*

$$
\left({}^{RL}D^{\alpha}_{a^+}I^{\alpha}_{a^+}f\right)(t) = f(t).
$$

2. *If $f \in L^1(a,b)$ and $I^{n-\alpha}_{a^+}f \in AC^n[a,b]$, then*

$$
\left(I^{\alpha}_{a^+}\,{}^{RL}D^{\alpha}_{a^+}f\right)(t) = f(t) - \sum_{j=1}^{n}\frac{\left[I^{n-\alpha}_{a^+}f\right]^{(n-j)}(a^+)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j},
$$

hold almost everywhere on $[a,b]$.

Remark 1.6. *In particular case, when $0 < \alpha < 1$ we have*

$$
\left(I^{\alpha}_{a^+}\,{}^{RL}D^{\alpha}_{a^+}f\right)(t) = f(t) - \frac{I^{n-\alpha}_{a^+}f(a^+)}{\Gamma(\alpha)}(t-a)^{\alpha-1}.
$$

Lemma 1.18. *If $f \in L^1(0,b)$, then the Laplace transform of the fractional derivative of Riemann-Liouville is given by:*

$$\mathscr{L}[{}^{RL}D_{0^+}^{\alpha} f](s) = s^{\alpha}\mathscr{L}[f](s) - \sum_{j=0}^{n-1} s^{n-1-j}\left[\left(\frac{d}{dt}\right)^{j}\left(I_{0^+}^{n-\alpha} f(t)\right)\right]_{t=0}, \tag{1.21}$$

where $n-1<\alpha<n$.

At the end, we give a result concerning the commutativity between the infinite sum and the fractional derivation which can be found in Samko et all's book [54].

Theorem 1.10. *Let $(f_i(t))_{i\geq 0}$ be a sequence of functions defined on $(0,T]$. Suppose the following conditions are fulfilled:*

(i) For a given $\alpha>0$, the α-derivatives ${}^{RL}D_{0^+}^{\alpha} f_i(t), i\geq 0, \quad t\in(0,T]$ exist.

(ii) $\sum_{i=1}^{\infty} f_i(t)$ and $\sum_{i=1}^{\infty} {}^{RL}D_{0^+}^{\alpha} f_i(t)$ are uniformly convergent series on the interval $[\varepsilon,T]$ for any $\varepsilon>0$.

Then, the function defined by the series $\sum_{i=1}^{\infty} f_i(t)$ is α-differentiable and satisfies

$${}^{RL}D_{0^+}^{\alpha}\sum_{i=1}^{\infty} f_i(t) = \sum_{i=1}^{\infty} {}^{RL}D_{0^+}^{\alpha} f_i(t).$$

Caputo Fractional Derivative

Definition 1.23. *For a function $f\in AC^n([a,b])$, the Caputo fractional derivative of order α of f is defined by*

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha} f(t) &= I_{a^+}^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad \forall t\in(a,b]. \end{aligned}$$

Example 1.3. *Let $\alpha>0$ such that $n-1<\alpha<n$ with $\beta>n-1$. We have from (1.20)*

$$\frac{d^n}{dt^n}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}(t-a)^{\beta-n}.$$

Then

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha}(t-a)^{\beta} &= I_{a^+}^{(n-\alpha)}\left[\frac{d^n}{dt^n}(t-a)^{\beta}\right] \\ &= I_{a^+}^{(n-\alpha)}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}(t-a)^{\beta-n}\right] \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} I_{a^+}^{(n-\alpha)}[(t-a)^{\beta-n}], \end{aligned}$$

from the result of Example 1.1, we get

$${}^{C}D_{a^+}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha}.$$

Remark 1.7. *In particular, if $\beta = 0$, then*

$$^{C}D^{\alpha}_{a^+} 1 = 0.$$

Lemma 1.19. *Let $\Re(\alpha) > 0, n = [\Re(\alpha)] + 1$ and $f \in L^{\infty}(a,b)$ or $f \in C(a,b)$*

1. *If $\Re(\alpha) \in \mathbb{N}$ or $\alpha \in \mathbb{R}$ then*

$$\left(^{C}D^{\alpha}_{a^+} I^{\alpha}_{a^+} f \right)(t) = f(t).$$

2. *If $\Re(\alpha) \in \mathbb{N}$ and $\Im(\alpha) \neq 0$, then*

$$\left(I^{\alpha}_{a^+} \, ^{C}D^{\alpha}_{a^+} f \right)(t) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a^+)}{j!} (t-a)^j.$$

Remark 1.8. *Clearly, we see that if $0 < \alpha \leq 1$ and $f \in AC(a,b)$ or $f \in C(a,b)$, then*

$$\left(I^{\alpha}_{a^+} \, ^{C}D^{\alpha}_{a^+} f \right)(t) = f(t) - f(a).$$

Some of interesting Lemmas are given below:

Lemma 1.20. *The Mittag-Leffer function $E_{\alpha}(\lambda t^{\alpha})$ is invariant with respect to the left Caputo fractional derivative, that is*

$$^{C}D^{\alpha}_{0^+} E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha}), \quad \text{for } \alpha > 0, \lambda \in \mathbb{R}.$$

Lemma 1.21. *If $f \in C[0,b]$, then the Laplace transform of the Caputo fractional derivative is given by*

$$\mathscr{L}[^{C}D^{\alpha}_{0^+} f](s) = s^{\alpha} \mathscr{L}[f](s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} f^{(j)}(0^+), \tag{1.22}$$

where $n-1 < \alpha < n$.

The relation between the fractional derivative of Caputo and that of Riemann-Liouville on the interval $[a,b]$ is described by the following theorem:

Theorem 1.11. *Let $\Re(\alpha) > 0, n = [\Re(\alpha)] + 1$. If f has $n-1$ derivatives in point a and $^{RL}D^{\alpha}_{a^+} f$ exists, then*

$$^{C}D^{\alpha}_{a^+} f(t) = ^{RL} D^{\alpha}_{a^+} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a), \tag{1.23}$$

almost everywhere on $[a,b]$.

Remark 1.9. *If $f^{(k)}(a) = 0$ for $k = 0,1,2,...,n-1$, then the fractional derivative of Riemann-Liouville and Caputo coincide, i.e. :*

$$^{C}D^{\alpha}_{a^+} f(x) = ^{RL}D^{\alpha}_{a^+} f(x).$$

Theorem 1.12. *Let $\alpha \in (0,1)$, $f \in L^1[0,T]$, $c \in \mathbb{R}$. Then the solution of the problem*

$$\begin{cases} ^{C}D^{\alpha}_{0^+} u(t) + \lambda u(t) & = f(t), \quad t > 0 \\ u(0) & = c \end{cases}$$

is given in $AC[0,T]$ and satisfies

$$u(t) = cE_{\alpha}(-\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^{\alpha}) f(s) ds.$$

Hilfer Fractional Derivative

More recently, in 2000, Hilfer introduced a new form of fractional derivative operator, known as the generalized Riemann–Liouville fractional derivative or the Hilfer fractional derivative of order α and a type β, which interpolates the both Riemann–Liouville as well as Caputo sense. For more details about this operator, see [25, 27].

Definition 1.24. *The Hilfer fractional derivative of order α and parameter β for an integrable function $f : [a,b] \to \mathbb{R}$ is defined by*

$$D_{a^+}^{\alpha,\beta} f(t) := I_{a^+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} I_{a^+}^{(1-\beta)(n-\alpha)} f(t),$$

where $n-1 < \alpha < n, \quad 0 \leq \beta \leq 1$.

Remark 1.10. *When $\beta = 0$, the Hilfer fractional derivative corresponds to the Riemann–Liouville fractional derivative*

$$D_{a^+}^{\alpha,0} f(t) = \frac{d^n}{dt^n} I_{a^+}^{(n-\alpha)} f(t)$$

while, for $\beta = 1$, the Hilfer fractional derivative corresponds to the Caputo fractional derivative

$$D_{a^+}^{\alpha,1} f(t) = I_{a^+}^{(n-\alpha)} \frac{d^n}{dt^n} f(t).$$

Example 1.4. *Let $\alpha, \beta, \gamma > 0$. In light of Definition 1.24, we have*

$$D_{a^+}^{\alpha,\beta} (t-a)^{\gamma} = I_{a^+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} \left[I_{a^+}^{(1-\beta)(n-\alpha)} (t-a)^{\gamma} \right].$$

We observe from the result of Example 1.1 that

$$I_{a^+}^{(1-\beta)(n-\alpha)} (t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma((1-\beta)(n-\alpha)+\gamma+1)} (t-a)^{(1-\beta)(n-\alpha)+\gamma}$$

and from (1.20) we get

$$\begin{aligned} \frac{d^n}{dt^n} \left[I_{a^+}^{(1-\beta)(n-\alpha)} (t-a)^{\gamma} \right] &= \frac{\Gamma(\gamma+1)}{\Gamma((1-\beta)(n-\alpha)+\gamma+1)} \frac{d^n}{dt^n} \left[(t-a)^{(1-\beta)(n-\alpha)+\gamma} \right] \\ &= \frac{\Gamma(\gamma+1)}{\Gamma((1-\beta)(n-\alpha)+\gamma-n+1)} (t-a)^{(1-\beta)(n-\alpha)+\gamma-n}. \end{aligned}$$

Then, the Example 1.1 yields that

$$\begin{aligned} D_{a^+}^{\alpha,\beta} (t-a)^{\gamma} &= \frac{\Gamma(\gamma+1)}{\Gamma((1-\beta)(n-\alpha)+\gamma-n+1)} I_{a^+}^{\beta(n-\alpha)} \left[(t-a)^{(1-\beta)(n-\alpha)+\gamma-n} \right] \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}. \end{aligned}$$

Remark 1.11. *If we let $\gamma = 0$ in the previous example, we see that the Hilfer fractional derivative of a constant is not 0. In fact*

$$D_{a^+}^{\alpha,\beta} 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Lemma 1.22. *Let* $0<\alpha<1$, $0\le\beta\le 1$, $a,\nu,\mu\in\mathbb{R}^+$ *and* $\lambda\in\mathbb{C}$, *we have*

$$D_{a^+}^{\alpha,\beta}\left[(t-a)^{\mu-1}E_{\nu,\mu}(\lambda(t-a)^{\nu})\right]=(t-a)^{\mu-\alpha-1}E_{\nu,\mu-\alpha}(\lambda(t-a)^{\nu}),\quad t>a.$$

We have the following relationship between the Hilfer and Riemann-Lioville fractional derivatives (see [33]):

Theorem 1.13. *Let* $\alpha\in(0,1),\beta\in[0,1],\gamma=\alpha+\beta(1-\alpha)$ *and* $f\in L^1((a,b),\mathbb{R}^n)$. *Then the left-sided Hilfer derivative of order* α *and a type* β *of* f *can be expressed equivalently as:*

$$\left(D_{a^+}^{\alpha,\beta}f\right)(t)=D_{a^+}^{\alpha}\left(f(.)-\frac{(I_{a^+}^{1-\gamma}f)(a)}{\Gamma(\gamma)}(t-a)^{\gamma-1}\right)(t),\quad t\in(a,b].\tag{1.24}$$

The difference between fractional derivatives of different types becomes apparent from their Laplace transformations. It is found in [25, 59] that:

$$\mathscr{L}\left[\left(D_{a^+}^{\alpha,\beta}f\right)(x)\right](s)=s^{\alpha}\mathscr{L}\left[f(x)\right](s)-s^{\beta(1-\alpha)}I_{a^+}^{(1-\beta)(1-\alpha)}f(0^+),\quad \text{for}<\alpha<1.\tag{1.25}$$

The solution of the linear Cauchy-type problem with constant coefficient involving the Hilfer fractional derivative on a bounded interval can be obtained by using the method of successive approximations or the operational method, see [14, 27].

Lemma 1.23. *The solution of the following the Hilfer fractional differential equation of order* $0<\alpha<1$ *and type* $0\le\beta\le 1$ *with initial condition*

$$\begin{cases}D_{0^+}^{\alpha,\beta}y(t) &=\lambda y(t)+f(t),\quad t\in(0,T],\lambda\in\mathbb{R}\\ I^{1-\gamma}y(0^+) &=b,\quad \gamma=\alpha+\beta(1-\alpha),b\in\mathbb{R}\end{cases}$$

for $t^{1-\gamma}f\in C[0,T]$ *is given by*

$$y(t)=t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^{\alpha})b+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds,$$

with $t^{1-\gamma}y\in C[0,T]$.

Due to relation (1.24) and the linearity of fractional differential operators, we can extend Theorem 1.10 to the following lemma:

Lemma 1.24. *Let* $(f_i(t))_{i\ge 0}$ *be a sequence of functions defined on* $(0,T]$ *for each* $i\in\mathbb{N}$. *Suppose the following conditions are fulfilled:*

(i) the fractional derivative $D_{0+}^{\alpha,\beta}f_i(t), i\ge 0$ *exists for all* $i\in\mathbb{N}, t\in(0,T]$;

(ii) both series $\sum_{i=1}^{\infty}f_i(t)$ *and* $\sum_{i=1}^{\infty}D_{0+}^{\alpha,\beta}f_i(t)$ *are uniformly convergent series on the interval* $[\varepsilon,T]$ *for any* $\varepsilon>0$.

Then,

$$D_{0+}^{\alpha,\beta}\sum_{i=1}^{\infty}f_i(t)=\sum_{i=1}^{\infty}D_{0+}^{\alpha,\beta}f_i(t),\quad 0<\alpha\le\beta<1,\quad 0<t<T.$$

1.4 Inverse Problems

1.4.1 Problem Classification

Assume that we have a mathematical model of a physical, biological or other process. We presume that this model describes the system behind the process, its operating conditions and explains the principal quantities of the model as follows:

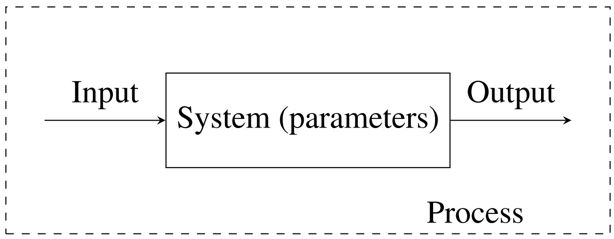


Figure 1.1: An input-output process

Any mathematical model of a given process can be divided into three different types of parameters. We can then have three types of problems:

(A) The direct problem: Given the input and system parameters, determine the output of the model.

(B) The reconstruction problem: Given the system parameters and output, find out which input led to this output.

(C) The identification problem: Given the input and output, determine the system parameters which are in agreement with the input-output relationship.

Now, let's use mathematical terminology to describe the input, output and parameters:

$$\begin{aligned} \mathbb{X}: &\ \text{space of input quantities;} \\ \mathbb{Y}: &\ \text{space of output quantities;} \\ \mathbb{P}: &\ \text{space of system parameters;} \\ A(p): &\ \text{an operator from } \mathbb{X} \text{ into } \mathbb{Y} \text{ associated to } p \in \mathbb{P}. \end{aligned}$$

In these terms, we formulate the above problems (A), (B) and (C) in the following way:

(A) Given $x \in \mathbb{X}$ and $p \in \mathbb{P}$, find $y := A(p)x$.

(B) Given $y \in \mathbb{Y}$ and $p \in \mathbb{P}$, find $x \in \mathbb{X}$ such that $A(p)x = y$.

(C) Given $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, find $p \in \mathbb{P}$ such that $A(p)x = y$.

We call a problem of type (A) a direct (or forward) problem since it is oriented towards a cause-effect sequence. However, problems of type (B) and (C) are referred to be inverse problems, because they involve determining the unidentified causes of well-known consequences.

1.4.2 Well-posed and Ill-posed Problems

Jacques Hadamard has proposed a definition of well-posed problems that assembles the conditions for their robust resolution.

Definition 1.25. *Let $(X,\|.\|_X)$, $(Y,\|.\|_Y)$ are two normed spaces, $f \in Y$ and $u \in X$ is a solution of the abstract problem*

$$\tilde{K}(u) = f, \tag{1.26}$$

where, $\tilde{K}$ is an operator from X to Y. We say that problem (1.26) is well posed if and only if:

(i) for all $f \in Y$, there exists a solution u of (1.26),

(ii) this solution is unique in X,

(iii) the solution u depends continuously on the data f for the norms of Y and X, i.e.

$$\forall \varepsilon > 0, \exists \delta > 0, \|f_1 - f_2\|_Y < \delta \Rightarrow \|u_1 - u_2\|_X < \varepsilon,$$

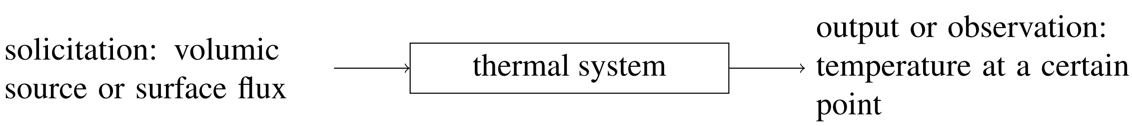
where $\tilde{K}(u_1) = f_1, \tilde{K}(u_2) = f_2$.

Remark 1.12. *We observe that:*

1. *The well-posed character qualifies in fact the triplet $(\tilde{K};X;Y)$.*
2. *Obviously:*
   - *The condition (i) requires $\tilde{K}$ to be surjective: $\tilde{K}(X) = Y$.*
   - *The condition (ii) means that $\tilde{K}$ is injective, so that the set of the two conditions (i) and (ii) induce that $\tilde{K}$ is invertible.*
   - *The last condition is the continuity of the inverse operator of $\tilde{K}$ for the norms of X and Y.*
3. *A problem which does not satisfy the three Hadamard conditions is said to be an ill-posed problem.*

1.4.3 Mathematical Model of Inverse Problems in Heat Transfer

We consider a rectangular iron bar that we heat at one of its ends.



The diffusion of heat inside in a homogeneous bar is modeled by a boundary problem for a heat equation. To calculate the temperature distribution in an inhomogeneous material occupying an open connected domain $\Omega \subset \mathbb{R}^3$, firstly the conservation of energy takes the form

$$\rho c \frac{\partial \tilde{T}}{\partial t} + div(\overrightarrow{q}) = f(x,y,z) \quad \text{in } \Omega, \tag{1.27}$$

where $\tilde{T}$ is the temperature, ρ is the density of the fluid, c is the specific heat, $\overrightarrow{q}$ represents a heat flux and f is a volume source. Then, Fourier's law connects density to the temperature gradient

$$\overrightarrow{q} = -K_t \text{ grad } \tilde{T},$$

where K_t is the thermal conductivity (which may be a tensor, and depends on the position). Substituting $\overrightarrow{q}$ into (1.27) , we obtain the heat equation, in a heterogeneous medium

$$\rho c \frac{\partial \tilde{T}}{\partial t} - div(K_t \text{ grad } \tilde{T}) = f(x,y,z) \quad \text{ in } \Omega. \tag{1.28}$$

This equation must be complemented by boundary conditions on the boundary of the domain Ω and an initial condition.

The direct problem involves to determination the temperature distribution T from the physical coefficients and the source term f.

Several inverse problems can be established:

- given a measurement of the temperature at an instant $t_f > 0$, determine the initial temperature.
- given a (partial) temperature measurement, determine certain coefficients in the equation.
- given a (partial) temperature measurement, determine the source term f.

Then, we will give two examples, the first deals with an ill posed inverse problem and the second with a well posed inverse problem.

Example 1.5. *The inverse problem consists in determining the unknown initial condition $u(x,0) = \varphi(x)$ such that the temperature field $u(x,t)$ satisfies*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial^2 x} & = 0, \quad 0 < x < \pi, 0 < t < T \\ u(x,T) & = \psi(x), \quad 0 \le t \le T \\ u(0,t) = u(\pi,t) & = 0 \end{cases} \tag{1.29}$$

where $\psi \in L^2(0,\pi)$ is a given function.

We will see that the inverse problem (1.29) is ill posed.

Formal solution. First, we seek a solution with separated variable $u(x,t) = T(t)X(x)$. This enable us to split into two differential equations, one is a Sturm-Liouville problem

$$\begin{cases} X''(x) & = -\lambda X(x), \quad x \in]0,\pi[\\ X(0) & = X(\pi) = 0, \end{cases} \tag{1.30}$$

this yields the following eignvalues and eignfunctions:

$$\lambda_n = n^2 \text{ and } X_n(x) = \sin(nx); \quad n \ge 1. \tag{1.31}$$

The second step is to solve the equation $T_n'(t) = -n^2 T_n(t); n \ge 1$, we obtain that

$$T_n(t) = C_n e^{-n^2 t}, \quad n \ge 1. \tag{1.32}$$

Thus, the general form of the solution is given by:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} X_n(x).$$

Moreover, to recover the initial condition we have from the additional condition

$$u(x,T) = \sum_{n=1}^{\infty} T_n(T) X_n(x) = \psi(x),$$

we deduce that $T_n(T)$ is the Fourier coefficient of $\psi(x)$

$$T_n(T) = \psi_n = \langle \psi, X_n \rangle_{L^2(0,\pi)} = \frac{2}{\pi} \int_0^{\pi} \psi(x) \sin(nx) dx, \quad n \geq 1$$

in view of (1.32) we get $C_n = e^{Tn^2} \psi_n$. Then, we can explain the solution $u(x,t)$ of problem (1.29) in the form

$$u(x,t) = \sum_{n=1}^{\infty} e^{(T-t)n^2} \psi_n X_n(x).$$

Let $u(x,0) = \varphi(x)$ be the initial temperature, then we deduce

$$\varphi(x) = \sum_{n=1}^{\infty} e^{Tn^2} \psi_n X_n(x).$$

Existence and uniqueness of the solution. According to the theory of semi-groups, in particular the Hille-Yosida theorem, the problem (1.29) being linear, admits a unique solution in $L^2(0,T;H_0^1(0,\pi)) \cap C([0,T];L^2(0,\pi))$ if φ is in $L^2(0,\pi)$ and it has the series form $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$.

Instability of the solution. We consider the problem (1.29) with the noisy data $\psi_k(x) = \psi(x) + \frac{1}{k} X_k(x)$

$$\begin{aligned}
\varphi_k(x) &= \sum_{n=1}^{\infty} e^{Tn^2} \langle \psi + \frac{1}{k} X_k, X_n \rangle_{L^2(0,\pi)} X_n(x) \\
&= \sum_{n=1}^{\infty} e^{Tn^2} \left[\langle \psi, X_n \rangle_{L^2(0,\pi)} + \langle \frac{1}{k} X_k, X_n \rangle_{L^2(0,\pi)} \right] X_n(x) \\
&= \sum_{n=1}^{\infty} e^{Tn^2} \langle \psi, X_n \rangle_{L^2(0,\pi)} X_n(x) + \frac{1}{k} e^{Tk^2} X_k(x),
\end{aligned}$$

clear that $\|\psi - \psi_k\|_{L^\infty(0,\pi)} = \frac{1}{k} \to 0$, when $k \to +\infty$ and $\|\varphi - \varphi_k\|_{L^\infty(0,\pi)} = \frac{1}{k} e^{Tk^2} \to +\infty$, when $k \to +\infty$. Then, we deduce that the solution of problem (1.29) is not stable. Hence the problem (1.29) is ill-posed.

Remark 1.13. *When the derivative with respect to the time variable of fractional type, the inverse problem will be also ill-posed.*

Example 1.6. *We are concerned with the fractional inverse problem of determining the source term $a(t)$ such that the distribution $u(x,t)$ satisfies the following problem*

$$\begin{cases} \frac{{}^C\partial^\alpha}{\partial t}u(x,t) - \frac{\partial^2}{\partial^2 x}u(x,t) & = a(t)c(x) := f(x,t), \quad 0 < x < \pi, 0 < t < T \\ u(x,0) & = \varphi(x), \quad 0 \le t \le T \\ u(0,t) & = u(\pi,t) = 0 \\ \int_0^\pi u(x,t)dx & = E(t), \quad 0 \le t \le T \end{cases} \tag{1.33}$$

where $\frac{{}^C\partial^\alpha}{\partial t}$ *is stands for the Caputo fractional derivative of order* $0 < \alpha < 1$ *in the time variable. The functions* φ, c *and* E *satisfying the following assumptions:*

(A_1) $c \in C^2[0,\pi]$*;* $c(0) = c(\pi) = 0$ *and for positive constant* M_0 *such that* $\int_0^\pi c(x)dx > \frac{1}{M_0}$.

(A_2) $\varphi \in C^2[0,\pi]; \varphi(0) = \varphi(\pi) = 0.$

(A_3) $E \in AC[0,T]$ *and* $\int_0^\pi \varphi(x)dx = E(0)$.

(A_4) $M_0 K_2 \frac{T^\alpha}{\alpha} < 1$, *where* K_2 *is definded in* (1.38).

Theorem 1.14. *Let* (A_1)*-* (A_4) *be satisfied. Then the inverse problem* (1.33) *of determining* $a(t)$ *is well posed.*

Proof. **Existence of the solution.** It is possible to use the variable separation method, the first step is done for $f(x,t) = 0$, we obtain the same Sturm-Liouville problem (1.30), thus the same eignvalues and eignfunctions (1.31). For the second step, we decompose the initial condition and source term according to the basis $(X_n(x))_{n\ge1}$ of $L^2(0,\pi)$ in the Fourier series

$$\varphi_n = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin(nx)dx, \quad c_n = \frac{2}{\pi}\int_0^\pi c(x)\sin(nx)dx, \qquad n \ge 1$$

respectively. Then, substituting the general form of the solution $u(x,t) = \sum_{n=1}^\infty T_n(t)X_n(x)$ in the FPDE and initial condition of problem (1.33) and we integrate over $(0,\pi)$, we get the following fractional Cauchy problem, for $n \ge 1$

$$\begin{cases} \frac{{}^C\partial^\alpha}{\partial t}T_n(t) + \lambda_n T_n(t) & = a(t)c_n, \quad 0 < \alpha < 1, 0 < t < T \\ T_n(0) & = \varphi_n. \end{cases} \tag{1.34}$$

From Theorem 1.12, we find

$$T_n(t) = \varphi_n E_{\alpha,1}(-\lambda_n t^\alpha) + c_n \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)a(s)ds. \tag{1.35}$$

Consequently

$$u(x,t) = \sum_{n=1}^\infty T_n(t)\sin(nx).$$

For determining the source term $a(t)$, applying ${}^C D_{0^+}^\alpha$ to the last condition of problem (1.33) we obtain

$$\begin{aligned} {}^C D_{0^+}^\alpha E(t) &= \int_0^\pi \frac{{}^C\partial^\alpha}{\partial t}u(x,t)dx \\ &= \int_0^\pi \left[\frac{\partial^2 u(x,t)}{\partial^2 x} + a(t)c(x)\right]dx \\ &= u_x(\pi,t) - u_x(0,t) + a(t)\int_0^\pi c(x)dx. \end{aligned}$$

Then, we deduce that

$$a(t) = \frac{{}^{C}D_{0^+}^{\alpha}E(t) + u_x(0,t) - u_x(\pi,t)}{\int_0^{\pi} c(x)dx}, \tag{1.36}$$

where

$$u_x(0,t) = \sum_{n=1}^{\infty} T_n(t)X_n'(0) = \sum_{n=1}^{\infty} nT_n(t),$$
$$u_x(\pi,t) = \sum_{n=1}^{\infty} T_n(t)X_n'(\pi) = \sum_{n=1}^{\infty} nT_n(t)\cos(n\pi).$$

In view of (1.35), the equation (1.36) can be written as:

$$\begin{aligned} a(t) &= H_0(t) + H_1(t) \\ &+ \sum_{n=1}^{\infty} \frac{n - n\cos(n\pi)}{\int_0^{\pi} c(x)dx} c_n \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}) a(s) ds, \end{aligned} \tag{1.37}$$

where

$$H_0(t) = \frac{{}^{C}D_{0^+}^{\alpha}E(t)}{\int_0^{\pi} c(x)dx},$$
$$H_1(t) = \sum_{n=1}^{\infty} \frac{n - n\cos(n\pi)}{\int_0^{\pi} c(x)dx} \varphi_n E_{\alpha,1}(-\lambda_n t^{\alpha}).$$

From the assumption $(A_1), (A_3)$ and the fact $|E_{\alpha,1}(-\lambda_n t^{\alpha})| \leq 1$, we have

$$\|H_0\|_{C[0,T]} \leq M_0 \|E\|_{AC[0,T]}.$$

Using (A_2) with integration by parts, we get

$$|\varphi_n| = \frac{2}{\pi}\left|\int_0^{\pi} \varphi(x)\sin(nx)dx\right| = \frac{2}{\pi}\left|\int_0^{\pi} \varphi''(x)\frac{\sin(nx)}{n^2}dx\right| = \frac{|\varphi_n''|}{n^2}.$$

The Chauchy-Schwartz and Bessel's inequalities give that

$$K_1 := \sum_{n=1}^{\infty} n|\varphi_n| = \sum_{n=1}^{\infty} \frac{|\varphi_n''|}{n} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} |\varphi_n''|^2\right)^{1/2} < \infty,$$

which indicates that

$$\|H_1\|_{C[0,\pi]} \leq M_0 \sum_{n=1}^{\infty} n|\varphi_n| = M_0 K_1.$$

Then, from (A_1)-(A_4) and the fact $|E_{\alpha,\alpha}(-\lambda_n t^{\alpha})| \leq 1$ we get

$$\|a\|_{C[0,T]} \leq \frac{M_0}{1 - M_0 K_2 \frac{T^{\alpha}}{\alpha}} \left(\|E\|_{AC[0,T]} + K_1\right), \quad \text{with } M_0 K_2 \frac{T^{\alpha}}{\alpha} < 1$$

where

$$K_2 := \sum_{n=1}^{\infty} n|c_n| = \sum_{n=1}^{\infty} \frac{|c_n''|}{n} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} |c_n''|^2\right)^{1/2} < \infty. \tag{1.38}$$

This means that $a(t)$ is bounded in $C[0,T]$.

Now, we define an operator $\tilde{B} : C([0,T]) \to C([0,T])$ as

$$\tilde{B}(a(t)) := \frac{{}^{C}D^{\alpha}_{0^+}E(t) + u_x(0,t) - u_x(\pi,t)}{\int_0^{\pi} c(x)dx},$$

and we show that the mapping $\tilde{B}$ is a contraction. Indeed, combining (A_1)-(A_3) and $|E_{\alpha,\alpha}(-\lambda_n t^{\alpha})| \leq 1$, we infer that

$$\|\tilde{B}(a) - \tilde{B}(b)\|_{C[0,T]} \leq M_0 K_2 \frac{T^{\alpha}}{\alpha} \|a-b\|_{C[0,T]}.$$

Thus, the mapping $\tilde{B}$ is a contraction for $M_0 K_2 \frac{T^{\alpha}}{\alpha} < 1$. This ensures the uniqueness of $a \in C([0,T])$, by using the Banach fixed point theorem.

Stability of the solution. We consider the problem (1.33) with the noisy data $E_k(t) = E(t) + \dfrac{t}{k}$, we have

$$a_k(t) = \frac{{}^{C}D^{\alpha}_{0^+}E(t) + u_x(0,t) - u_x(\pi,t)}{\int_0^{\pi} c(x)dx} + \frac{{}^{C}D_{0^+}t}{k\int_0^{\pi} c(x)dx}.$$

From Example 1.3, we get

$$a_k(t) = \frac{{}^{C}D^{\alpha}_{0^+}E(t) + u_x(0,t) - u_x(\pi,t)}{\int_0^{\pi} c(x)dx} + \frac{\Gamma(2)t^{1-\alpha}}{k\Gamma(2-\alpha)\int_0^{\pi} c(x)dx}.$$

Then

$$|a(t) - a_k(t)| = \frac{\Gamma(2)t^{1-\alpha}}{k\Gamma(2-\alpha)\int_0^{\pi} c(x)dx}.$$

Clear that $|E - E_k| = \frac{t}{k} \to 0$, when $k \to +\infty$ and $\|a - a_k\|_{C[0,T]} \leq \dfrac{M_0\Gamma(2)}{k\Gamma(2-\alpha)} T^{1-\alpha} \to 0$, when $k \to +\infty$. Then, we deduce that the problem (1.33) is well-posed. □

CHAPTER

2 An Inverse Problem for the Hilfer Time Fractional Diffusion Equation

2.1 Introduction

The fractional diffusion equations, which are obtained from the classical diffusion equations in mathematical physics by replacing the first order of time derivative by fractional derivative of order α satisfying $0 < \alpha < 1$, have been developed to describe anomalous diffusion process. The diffusion equation with Riemann-Liouville fractional derivative coincides with classical diffusion equation from both sides when $\alpha \to 1^-$ and when $\alpha \to 1^+$, while in the case of the derivative of Caputo, there is no coincidence with the classical case except for the left-side limit when $\alpha \to 1^-$, see [44]. The Hilfer fractional derivative has the important physical properties of both Riemann–Liouville and Caputo fractional derivatives which could be used as a better model to explain anomalies by tuning the order and type of fractional derivatives. In [26], the Hilfer fractional derivative is used to explain the relaxation spectra that occur in the formation of glass, other applications can be found in [25, 27].

In the recent years, inverse problems for time fractional diffusion equations have been considered by many authors, see [4, 30, 53, 56, 61, 63, 65]. Let us mention that in [39], the authors considered the inverse problem of determination the temperature distribution and a source term, which is independent of the time variable for the fractional heat equation

$$\begin{cases} \partial^{\alpha}_{0^+,t}(u(x,t)-u(x,0)) - u_{xx} = f(x), & (x,t) \in Q_T := I \times (0,T) \\ u(x,0) = \varphi(x), \quad u(x,T) = \psi(x), & x \in I \\ u_x(0,t) = u_x(1,t), \quad u(1,t) = 0, & t \in [0,T] \end{cases}$$

where $I = [0,1], T > 0$, $\partial^{\alpha}_{0^+,t}$ stands for the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $\varphi(x)$ and $\psi(x)$ are the initial and final temperatures respectively. The choice of the term $\partial^{\alpha}_{0^+,t}(u(x,t)-u(x,0))$ rather then the usual term $\partial^{\alpha}_{0^+,t}u(x,t)$ is not to avoid only the singularity at zero but also to impose meaningful initial condition. The authors propose a method for determining the solution and source term of the fractional heat equation, this method is based on selecting a bi-orthogonal basis of $L^2(0,1)$ space corresponding to a nonself-adjoint boundary value problem.

In [21], Furati and al. generalize the above inverse problem in the following form:

$$\begin{cases} \partial^{\alpha,\gamma}_{0^+,t}u(x,t) - u_{xx} = f(x), & (x,t) \in Q_T, \quad 0 < \alpha < \gamma \leq 1 \\ I^{1-\gamma}_{0^+,t}u(x,0) = g(x), \quad u(x,T) = h(x), & x \in [0,1] \\ u_x(0,t) = u_x(1,t), \quad u(1,t) = 0, & t \in [0,T] \end{cases}$$

where, $Q_T := [0,1] \times (0,T)$, $\partial^{\alpha,\gamma}_{0^+,t}$ represents the Hilfer fractional derivative of order α and type β, $I^{1-\gamma}_{0^+,t}$ stands the Riemann-Liouville integral of order $1-\gamma$, g and h are the initial and final

conditions, respectively and assumed to be in $L^2(0,1)$.
As in [39], the authors used two sets of Riesz basis, which form a bi-orthogonal system for the space $L^2(0,1)$ in order to prove the existence and uniqueness for the solution of the inverse problem. They utilize the asymptotic behavior of the generalized Mittag–Leffler function to obtain an existence result under some smoothness requirements of the initial and final conditions.
In [2], the authors considered the inverse problem of heat equation involving a fractional derivative in time, with initial and non-local boundary conditions:

$$\begin{cases} \partial^{\alpha}_{0^+,t}(u(x,t)-u(x,0))-\rho u_{xx}=F(x,t)=a(t)f(x,t), & (x,t)\in(0,1)\times(0,T], \\ u(x,0)=\varphi(x), \quad x\in[0,1] \\ u_x(0,t)=u_x(1,t), \quad u(1,t)=0, \quad t\in[0,T] \end{cases}$$

where, ρ is a positive constant, $\partial^{\alpha}_{0^+,t}$ stands for the Riemann-Liouville fractional derivative of order $0<\alpha<1$ in the time variable. The inverse problem consists of determining a source term $a(t)$ and the temperature distribution $u(x,t)$ such that the following integral condition:

$$\int_0^1 u(x,t)dx = E(t), \quad t\in[0,T]$$

holds, which is necessary in order to ensure the unique solvability of the inverse problem. The authors proved the existence and uniqueness of the solution by using the Fourier method and bi-orthogonal system, and its continuous dependence on the data is also presented.

Motivated by the above works, we study in this chapter an inverse problem of finding the time-dependent coefficient of a generalized time fractional diffusion equation, in the case of non-local boundary and integral over-determination conditions, that is a generalization of the inverse problem considered in [2] and presented in [18].

2.2 Statement of the Problem

We address an inverse source coefficient problem for the following generalized time fractional heat equation

$$\partial^{\alpha,\beta}_{0^+,t}u(x,t)=u_{xx}(x,t)+a(t)F(x,t), \quad (x,t)\in D_T \tag{2.1}$$

with the initial integral condition

$$I^{1-\gamma}_{0^+,t}u(x,0)=\varphi(x), \quad \gamma=\alpha+\beta(1-\alpha), \quad 0<x<1 \tag{2.2}$$

and the non-local boundary conditions

$$u_x(0,t)=u_x(1,t); \quad u(1,t)=0, \quad 0<t\le T \tag{2.3}$$

where, $D_T=\{0<x<1, 0<t<T\}$, $\partial^{\alpha,\beta}_{0^+,t}$ stands the Hilfer time fractional derivative of order $0<\alpha<1$ and type $0\le\beta\le1$, $I^{1-\gamma}_{0^+,t}$ is the left sided Riemann-Liouville fractional integral of order $1-\gamma>0$, $\varphi(x)$ and $F(x,t)$ are given functions on $[0,1]$ and $\bar{D}_T$ respectively. The non-local boundary conditions (2.3) are commonly referred to as the boundary conditions describing the relationship between the desired solution values on multiple points, which can be more useful than the standard classical boundary conditions for describing some applications of heat conduction or thermo-elasticity, see [11, 12].

The direct problem is the determination of a function $u(x,t)$ which satisfies the initial and boundary conditions, whenever $a(t)$ is given function. However, the inverse problem consists of determining a source term $a(t)$ and the temperature distribution $u(x,t)$, from the initial temperature $\varphi(x)$ and boundary conditions (2.3). This problem is not uniquely solvable, to ensure unique solvability we impose an additional constraint for the integral of the solution with respect to the space variable, namely

$$\int_0^1 u(x,t)dx = E(t), \quad t \in [0,T] \tag{2.4}$$

where $E(t)$ is a given function which is the total amount of heat in space.

We intend to solve the direct problem (2.1)-(2.3) by using the Fourier method, frequently known as separation of variables method. The spectral problem for the corresponding homogeneous equation of the problem (2.1)-(2.3) is the boundary-value problem (1.9)-(1.10).
We recall that this boundary-value problem is non-self-adjoint and the set of eigenfunctions of the spectral problem (1.9)-(1.10) is not complete in the space $L^2(0,1)$. we supplement the set of eigenfunctions with the associated eigenfunctions making the set complete on $L^2(0,1)$. Another complete set of eigenfunctions and associated eigenfunctions of the adjoint problem (1.12) is obtained in previous chapter 1.2.2 to construct bi-orthogonal system of functions.

A regular solution of the inverse problem is a pair of functions $\{u(x,t);a(t)\}$ such that

$$t^{1-\gamma}u \in C(\bar{D_T}), \quad t^{1-\gamma}\partial_{0^+,t}^{\alpha,\beta}u \in C(\bar{D_T}) \text{ and } a \in C[0,T],$$

which satisfy equations (2.1)-(2.4). Our approach for the solvability of the inverse problem is based on the expansion of the solution $u(x,t)$ by using the following bi-orthogonal system of functions:

$$S_1 = \{X_0(x) = 2(1-x), \quad X_{2k-1}(x) = 4(1-x)\cos(\sqrt{\lambda_k}x), \quad X_{2k}(x) = 4\sin(\sqrt{\lambda_k}x); \quad k \geq 1\}$$

$$S_2 = \{Y_0(x) = 1, \quad Y_{2k-1}(x) = \cos(\sqrt{\lambda_k}x), \quad Y_{2k}(x) = x\sin(\sqrt{\lambda_k}x); \quad k \geq 1\},$$

where $\lambda_k = (2\pi k)^2$.

2.3 The Main Result

Now, we turn to the main result of this chapter, namely the well-posedness in the sense of Hadamard of the inverse source coefficient problem (2.1)-(2.4), we show the existence, uniqueness and continuous dependence upon the data of the solution of this problem.

First, let us assume the following assumptions on the data of problem (2.1)-(2.4) hold:

(A_1) $\varphi \in C^4[0,1]$ such that $\varphi(1) = 0, \varphi'(0) = \varphi'(1), \varphi''(1) = 0, \varphi^{(3)}(0) = \varphi^{(3)}(1)$.

(A_2) The function $t^{1-\gamma}F(x,.) \in C[0,T]$ for $x \in [0,1]$ and for $t \in [0,T]$, $F(.,t) \in C^4[0,1]$ satisfies the following conditions:

(1) $F(1,t) = 0, \quad F_x(0,t) = F_x(1,t), \quad F_{xx}(1) = 0, \quad F_{xxx}(0,t) = F_{xxx}(1,t);$

(2) there exists a positive constant M_F such that $t^{1-\gamma}|\int_0^1 F(x,t)dx| > M_F$ for each $t \in [0,T]$.

(A_3) $t^{1-\gamma}E \in C[0,T]$, $t^{1-\gamma}D_{0^+}^{\alpha,\beta}E \in C[0,T]$ with $\int_0^1 \varphi(x)dx = I_{0^+}^{1-\gamma}E(0)$.

From the properties of Mittag-Leffler function, there exists a positive constant M such that

$$M = \max(M_{\alpha,\gamma}; M_{\alpha,\alpha+\gamma}; M_{\alpha,2\alpha}; M_{\alpha,\alpha}) \tag{2.5}$$

where,

$$M_{\alpha,\gamma} := \max_{0\le t\le T} |E_{\alpha,\gamma}(-\lambda_k t^\alpha)|, \quad M_{\alpha,\alpha+\gamma} := \max_{0\le t\le T} |E^2_{\alpha,\alpha+\gamma}(-\lambda_k t^\alpha)|,$$
$$M_{\alpha,2\alpha} := \max_{0\le s<t\le T} |E^2_{\alpha,2\alpha}(-\lambda_k (t-s)^\alpha)|, \quad M_{\alpha,\alpha} := \max_{0\le s<t\le T} |E_{\alpha,\alpha}(-\lambda_k (t-s)^\alpha)|.$$

Lemma 2.1. *If (A_1) and (A_2) are satisfied, then for some positive constants C, G, we have*

$$\sum_{k\ge1} \sqrt{\lambda_k}|\varphi_{2k-1}| \le C\|\varphi\|_{C^4[0,1]}; \tag{2.6}$$
$$\sum_{k\ge1} \sqrt{\lambda_k}|F_{2k-1}(t)| \le G\|F(.,t)\|_{C^4[0,1]}; \tag{2.7}$$

where, $\varphi_{2k-1} = \langle \varphi, Y_{2k-1}\rangle_{L^2(0,1)}$ and $F_{2k-1}(t) = \langle F(t), Y_{2k-1}\rangle_{L^2(0,1)}$.

Proof. Since $\varphi \in C^4[0,1]$, by integration by parts we have

$$\begin{aligned}\varphi_{2k-1} = \langle \varphi, Y_{2k-1}\rangle_{L^2(0,1)} &= \int_0^1 \varphi(x)\cos(\sqrt{\lambda_k}x)dx \\ &= \frac{1}{\lambda_k^2}\int_0^1 \varphi^{(4)}(x)\cos(\sqrt{\lambda_k}x)dx \\ &= \frac{1}{\lambda_k^2}\langle \varphi^{(4)}, Y_{2k-1}\rangle_{L^2(0,1)} = \frac{\varphi^{(4)}_{2k-1}}{\lambda_k^2}\end{aligned}$$

which yields, by the Hölder inequality

$$\sum_{k\ge1} \sqrt{\lambda_k}|\varphi_{2k-1}| = \left(\sum_{k\ge1}\frac{1}{(\lambda_k)^3}\right)^{1/2}\left(\sum_{k\ge1}|\varphi^{(4)}_{2k-1}|^2\right)^{1/2}$$

and from the Bessel inequality, we get

$$\sum_{k\ge1} \sqrt{\lambda_k}|\varphi_{2k-1}| \le \tilde{C}\|\varphi^{(4)}\|_{L^2(0,1)} \le C\|\varphi\|_{C^4[0,1]}.$$

Similarly, for $F(.,t) \in C^4[0,1]$ we find (2.7). □

Remark 2.1. *In the same way, we can deduce the uniform convergence of the series: $\sum_{k\ge1}\lambda_k|\varphi_{2k}|$, $\sum_{k\ge1}\lambda_k|F_{2k}(t)|$ and $\sum_{k\ge1}(\sqrt{\lambda_k})^n|\varphi_{2k-1}|$, $\sum_{k\ge1}(\sqrt{\lambda_k})^n|F_{2k-1}(t)|$, for $n = 2,3$ where, $\varphi_{2k} = \langle \varphi, Y_{2k}\rangle_{L^2(0,1)}$ and $F_{2k}(t) = \langle F(t), Y_{2k}\rangle_{L^2(0,1)}$.*

Next, our main result is stated in the following theorem:

Theorem 2.1. *Let* (A_1)*-*(A_3) *be satisfied. Then, the inverse problem (2.1)-(2.4) is well-posed.*

2.4 Proof of the Result

For the proof of the main result, i.e., Theorem 2.1 we will use properties of the bi-orthogonal system of functions S_1 and S_2. Our proof consists of 4 sections.

2.4.1 Formal Solution and Source Coefficient Term

By applying the standard procedure of the Fourier method, we obtain the following representation of the solution of direct problem (2.1)-(2.3) for arbitrary $a \in C[0,T]$:

$$u(x,t) = u_0(t)X_0(x) + \sum_{k\geq 1} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k\geq 1} u_{2k}(t)X_{2k}(x), \tag{2.8}$$

where the first coefficient

$$u_0(t) = \int_0^1 u(x,t)Y_0(x)dx = E(t) \tag{2.9}$$

is known and for $k \geq 1$

$$u_{2k-1}(t) = \int_0^1 u(x,t)Y_{2k-1}(x)dx, \quad u_{2k}(t) = \int_0^1 u(x,t)Y_{2k}(x)dx$$

have to be determined. Let $\{\varphi_0, \varphi_{2k-1}, \varphi_{2k}\}$ be the coefficients of the series expansion of $\varphi(x)$ in the basis S_1 which satisfy:

$$\varphi(x) = \varphi_0 X_0(x) + \sum_{k\geq 1} \varphi_{2k-1}X_{2k-1}(x) + \sum_{k\geq 1} \varphi_{2k}X_{2k}(x), \tag{2.10}$$

where

$$\varphi_0 = \int_0^1 \varphi(x)Y_0(x)dx, \quad \varphi_{2k-1} = \int_0^1 \varphi(x)Y_{2k-1}(x)dx, \quad \varphi_{2k} = \int_0^1 \varphi(x)Y_{2k}(x)dx.$$

Then, replacing $u(x,t)$ in the equation (2.1) by the representation (2.8), we find

$$\begin{aligned} &D_{0+}^{\alpha,\beta} u_0(t)X_0(x) + \sum_{k\geq 1} D_{0+}^{\alpha,\beta} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k\geq 1} D_{0+}^{\alpha,\beta} u_{2k}(t)X_{2k}(x) \\ &= u_0(t)X_0''(x) + \sum_{k\geq 1} u_{2k-1}(t)X_{2k-1}''(x) + \sum_{k\geq 1} u_{2k}(t)X_{2k}''(x) + a(t)F(x,t), \end{aligned} \tag{2.11}$$

where

$$X_0''(x) = 0, \quad X_{2k-1}''(x) = 2\sqrt{\lambda_k}X_{2k}(x) - \lambda_k X_{2k-1}(x), \quad X_{2k}''(x) = -\lambda_k X_{2k}(x).$$

Also, substituting (2.8) and (2.10) in the initial condition (2.2), we obtain

$$\begin{aligned} &I_{0+}^{1-\gamma} u_0(0)X_0(x) + \sum_{k\geq 1} I_{0+}^{1-\gamma} u_{2k-1}(0)X_{2k-1}(x) + \sum_{k\geq 1} I_{0+}^{1-\gamma} u_{2k}(0)X_{2k}(x) \\ &= \varphi_0 X_0(x) + \sum_{k\geq 1} \varphi_{2k-1}X_{2k-1}(x) + \sum_{k\geq 1} \varphi_{2k}X_{2k}(x). \end{aligned} \tag{2.12}$$

Multiplying the equations (2.11) and (2.12) by $Y_0(x)$, integrating the both on $(0,1)$ and from the Lemma 1.7, we get the following system:

$$\begin{cases} D_{0+}^{\alpha,\beta} u_0(t) = a(t)F_0(t), \\ I_{0+}^{1-\gamma} u_0(0) = \varphi_0; \end{cases} \tag{2.13}$$

where

$$F_0(t) = \int_0^1 F(x,t)Y_0(x)dx. \tag{2.14}$$

Alike, we obtain the following system of coupled fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha,\beta} u_{2k-1}(t) + \lambda_k u_{2k-1}(t) = a(t)F_{2k-1}(t), \\ I_{0+}^{1-\gamma} u_{2k-1}(0) = \varphi_{2k-1}; \end{cases} \tag{2.15}$$

$$\begin{cases} D_{0+}^{\alpha,\beta} u_{2k}(t) + \lambda_k u_{2k}(t) - 2\sqrt{\lambda_k} u_{2k-1}(t) = a(t)F_{2k}(t), \\ I_{0+}^{1-\gamma} u_{2k}(0) = \varphi_{2k}; \end{cases} \tag{2.16}$$

where

$$F_{2k-1}(t) = \int_0^1 F(x,t)Y_{2k-1}(x)dx, \tag{2.17}$$

$$F_{2k}(t) = \int_0^1 F(x,t)Y_{2k}(x)dx. \tag{2.18}$$

In view of Lemma 1.23, the solutions of problems (2.15) and (2.16) satisfy the following integral equations:

$$\begin{aligned} u_{2k-1}(t) &= t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^\alpha)\varphi_{2k-1} \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)F_{2k-1}(s)ds \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} u_{2k}(t) &= t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^\alpha)\varphi_{2k} \\ &\quad + 2\sqrt{\lambda_k}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2k-1}(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)F_{2k}(s)ds, \end{aligned} \tag{2.20}$$

respectively. By using (2.19), we note the first integral in (2.20) as follows:

$$\begin{aligned} J &= \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2k-1}(s)ds \\ &= \varphi_{2k-1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)s^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k s^\alpha)ds \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)\left[\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(s-\tau)^\alpha)a(\tau)F_{2k-1}(\tau)d\tau\right]ds \\ &= \varphi_{2k-1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)s^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k s^\alpha)ds \\ &+ \int_0^t a(\tau)F_{2k-1}(\tau)\left[\int_\tau^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)(s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(s-\tau)^\alpha)ds\right]d\tau. \end{aligned}$$

Then, taking into account Lemma 1.12, we transform J as follows:

$$\begin{aligned}
J &= \varphi_{2k-1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)s^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k s^\alpha)ds \\
&+\int_0^t a(\tau)F_{2k-1}(\tau)\left[\int_\tau^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)(s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(s-\tau)^\alpha)ds\right]d\tau \\
&= \varphi_{2k-1}t^{\alpha+\gamma-1}E^2_{\alpha,\alpha+\gamma}(-\lambda_k t^\alpha)+\int_0^t (t-s)^{2\alpha-1}E^2_{\alpha,2\alpha}(-\lambda_k(t-s)^\alpha)a(s)F_{2k-1}(s)ds. \quad (2.21)
\end{aligned}$$

Substituting (2.21) into (2.20), we get the following form:

$$\begin{aligned}
u_{2k}(t) &= t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^\alpha)\varphi_{2k}+2\sqrt{\lambda_k}t^{\alpha+\gamma-1}E^2_{\alpha,\alpha+\gamma}(-\lambda_k t^\alpha)\varphi_{2k-1} \quad (2.22)\\
&+2\sqrt{\lambda_k}\int_0^t (t-s)^{2\alpha-1}E^2_{\alpha,2\alpha}(-\lambda_k(t-s)^\alpha)a(s)F_{2k-1}(s)ds \\
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)F_{2k}(s)ds.
\end{aligned}$$

In view of the goal to determine the function $a(t)$, using (2.13) and (2.9) to find that

$$a(t)=\frac{D^{\alpha,\beta}_{0^+}u_0(t)}{\int_0^1 F(x,t)dx}=\frac{D^{\alpha,\beta}_{0^+}E(t)}{F_0(t)}. \quad (2.23)$$

2.4.2 Existence of a Regular Solution

First, from (2.23), (A_2) and (A_3) we have

$$a(t)=\frac{t^{1-\gamma}D^{\alpha,\beta}_{0^+}E(t)}{t^{1-\gamma}F_0(t)},$$

which is a quotient of continuous functions. Hence

$$\|a\|_{C[0,T]}\le\frac{1}{M_F}\|t^{1-\gamma}D^{\alpha,\beta}_{0^+}E\|_{C[0,T]}<\infty, \quad (2.24)$$

and we deduce that $a\in C[0,T]$.
Then, we will show that the series corresponding to $t^{1-\gamma}u(x,t), t^{1-\gamma}u_{xx}(x,t)$ and $t^{1-\gamma}D^{\alpha,\beta}_{0^+}u(x,t)$ are uniformly convergent in $[0,1]\times[\varepsilon,T]$, for any $\varepsilon>0$. Before we proceed further, (2.9) implies that

$$t^{1-\gamma}|u_0(t)|\le\|t^{1-\gamma}E\|_{C[0,T]},\quad \forall t\in[0,T]. \quad (2.25)$$

Next, from (2.19) and the result of Example 1.1, we obtain

$$\begin{aligned}
t^{1-\gamma}|u_{2k-1}(t)| &\le M_{\alpha,\gamma}|\varphi_{2k-1}|+t^{1-\gamma}M_{\alpha,\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}|F_{2k-1}(s)|ds \\
&\le M_{\alpha,\gamma}|\varphi_{2k-1}|+t^{1-\gamma}M\|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k-1}\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}ds \\
&\le M|\varphi_{2k-1}|+\frac{\Gamma(\alpha)\Gamma(\gamma)T^\alpha}{\Gamma(\alpha+\gamma)}M\|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k-1}\|_{C[0,T]}.
\end{aligned}$$

Taking the sum of the last inequality and by Bessel's inequality, we find

$$t^{1-\gamma}\sum_{k\geq 1}|u_{2k-1}(t)| \leq M\|\varphi\|_{C^4[0,1]} + \frac{\Gamma(\alpha)\Gamma(\gamma)T^{\alpha}}{\Gamma(\alpha+\gamma)}M\|a\|_{C[0,T]}\|t^{1-\gamma}F\|_{C([0,T];C^4[0,1])}. \qquad (2.26)$$

The same for (2.22), we find after summation

$$t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)| \leq M_{\alpha,\gamma}\sum_{k\geq 1}|\varphi_{2k}| + 2t^{\alpha}M_{\alpha,\alpha+\gamma}\sum_{k\geq 1}\sqrt{\lambda_k}|\varphi_{2k-1}|$$
$$+2t^{1-\gamma}M_{\alpha,2\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{2\alpha-1}s^{\gamma-1}s^{1-\gamma}\sum_{k\geq 1}\sqrt{\lambda_k}|F_{2k-1}(s)|\,ds$$
$$+t^{1-\gamma}M_{\alpha,\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}\sum_{k\geq 1}|F_{2k}(s)|\,ds.$$

Then, from Bessel's inequality and Lemma 2.1 we obtain

$$t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)| \leq M_{\alpha,\gamma}\|\varphi\|_{C^4[0,1]} + 2t^{\alpha}M_{\alpha,\alpha+\gamma}C\|\varphi\|_{C^4[0,1]}$$
$$+2t^{1-\gamma}M_{\alpha,2\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{2\alpha-1}s^{\gamma-1}s^{1-\gamma}G\|F(.,s)\|_{C^4[0,1]}ds$$
$$+t^{1-\gamma}M_{\alpha,\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}\|F(.,s)\|_{C^4[0,1]}ds.$$

Therefore, using estimation (2.5) we get

$$t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)| \leq (1+2CT^{\alpha})M\|\varphi\|_{C^4[0,1]}$$
$$+2t^{1-\gamma}M\|a\|_{C[0,T]}G\|t^{1-\gamma}F\|_{C([0,T];C^4[0,1])}\int_0^t (t-s)^{\alpha}(t-s)^{\alpha-1}s^{\gamma-1}ds$$
$$+t^{1-\gamma}M\|a\|_{C[0,T]}\|t^{1-\gamma}F\|_{C([0,T];C^4[0,1])}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}ds.$$

Thus, according to Example 1.1 we find

$$t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)| \leq L_{\varphi}\|\varphi\|_{C^4[0,1]} + L_F\|a\|_{C[0,T]}\|t^{1-\gamma}F\|_{C([0,T];C^4[0,1])}, \qquad (2.27)$$

where

$$L_{\varphi} = M(1+2CT^{\alpha}), \qquad L_F = \frac{T^{\alpha}M\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left[2GT^{\alpha}+1\right]. \qquad (2.28)$$

Since, the series expression (2.8) of the solution $u(x,t)$ gives

$$t^{1-\gamma}|u(x,t)| \leq 2t^{1-\gamma}|u_0(t)| + 4t^{1-\gamma}\sum_{k\geq 1}|u_{2k-1}(t)| + 4t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)|, \qquad (2.29)$$

we conclude by (2.25),(2.26) and (2.27) that the series $t^{1-\gamma}u(x,t)$ is uniformly convergent on $[0,1]\times[\varepsilon,T]$ for any $\varepsilon>0$. Similarly, we can obtain the uniform convergence of $t^{1-\gamma}u_{xx}(x,t)$. From (2.9),(2.15) and (2.16), we get the following inequalities:

$$t^{1-\gamma}|D_{0^+}^{\alpha,\beta}u_0(t)| \leq \|t^{1-\gamma}D_{0^+}^{\alpha,\beta}E\|_{C[0,T]};$$
$$t^{1-\gamma}|D_{0^+}^{\alpha,\beta}u_{2k-1}(t)| \leq t^{1-\gamma}\lambda_k|u_{2k-1}(t)| + \|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k}\|_{C[0,T]};$$
$$t^{1-\gamma}|D_{0^+}^{\alpha,\beta}u_{2k}(t)| \leq t^{1-\gamma}\lambda_k|u_{2k}(t)| + 2t^{1-\gamma}\sqrt{\lambda_k}|u_{2k-1}(t)| + \|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k-1}\|_{C[0,T]}.$$

Due to Lemma 2.1 and Remark 2.1, we deduce the uniform convergence of the series

$$t^{1-\gamma}\sum_{k\geq 1}\sqrt{\lambda_k}|u_{2k-1}(t)|,\quad t^{1-\gamma}\sum_{k\geq 1}\lambda_k|u_{2k-1}(t)| \text{ and } t^{1-\gamma}\sum_{k\geq 1}\lambda_k|u_{2k}(t)|,$$

the Weierstrass M-test Theorem 1.8 will ensure the uniform convergence of the series

$$t^{1-\gamma}\sum_{k\geq 1}D_{0^+}^{\alpha,\beta}u_{2k-1}(t) \text{ and } t^{1-\gamma}\sum_{k\geq 1}D_{0^+}^{\alpha,\beta}u_{2k}(t), \text{ on } [\varepsilon,T], \text{ for each } \varepsilon>0.$$

In view of (2.29) and Lemma 1.24, the series corresponding to $t^{1-\gamma}D_{0^+}^{\alpha,\beta}u(x,t)$ is uniformly convergent.
Now, we show that $u(x,t)$ satisfies the initial condition, recall that we have for $t=0$

$$I_{0^+}^{1-\gamma}u(x,0)=\lim_{t\to 0^+}I_{0^+}^{1-\gamma}u(x,t).$$

Keeping in mind (2.9) and (A_3), it follows that

$$I_{0^+}^{1-\gamma}u_0(0)=I_{0^+}^{1-\gamma}E(0)=\varphi_0.$$

By applying $I_{0^+}^{1-\gamma}$ to the formula (2.19) we get

$$\begin{aligned}
I_{0^+}^{1-\gamma}u_{2k-1}(t) &= I_{0^+}^{1-\gamma}\left[t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^\alpha)\right]\varphi_{2k-1}\\
&\quad + I_{0^+}^{1-\gamma}\left[\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)F_{2k-1}(s)ds\right]\\
&\leq I_{0^+}^{1-\gamma}\left[t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^\alpha)\right]\varphi_{2k-1}\\
&\quad + M_{\alpha,\alpha}\|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k-1}\|_{C[0,T]}\Gamma(\alpha)I_{0^+}^{1-\gamma}I_{0^+}^{\alpha}\left[t^{\gamma-1}\right].
\end{aligned}$$

From the semi-group property in the Lemma 1.14, then using Lemma 1.13 and Example 1.1, we obtain for $\gamma=\alpha+\beta(1-\alpha)$

$$\begin{aligned}
I_{0^+}^{1-\gamma}u_{2k-1}(t) &\leq I_{0^+}^{1-\gamma}\left[t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^\alpha)\right]\varphi_{2k-1}\\
&\quad + M_{\alpha,\alpha}\|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k-1}\|_{C[0,T]}\Gamma(\alpha)I_{0^+}^{1-\beta(1-\alpha)}\left[t^{\gamma-1}\right]\\
&\leq \varphi_{2k-1}+\frac{\Gamma(\gamma)}{\alpha}M_{\alpha,\alpha}\|a\|_{C[0,T]}\|t^{1-\gamma}F_{2k-1}\|_{C[0,T]}t^\alpha,
\end{aligned}$$

passing the limit when $t\to 0^+$, we get

$$\lim_{t\to 0^+}I_{0^+}^{1-\gamma}u_{2k-1}(t)=I_{0^+}^{1-\gamma}u_{2k-1}(0)=\varphi_{2k-1}.$$

The same, from (2.22) we have

$$I_{0^+}^{1-\gamma}u_{2k}(0)=\varphi_{2k}.$$

Therefore, we deduce that

$$\begin{aligned}
I_{0^+,t}^{1-\gamma}u(x,0) &= I_{0^+}^{1-\gamma}u_0(0)X_0(x)+\sum_{k\geq 1}I_{0^+}^{1-\gamma}u_{2k-1}(0)X_{2k-1}(x)+\sum_{k\geq 1}I_{0^+}^{1-\gamma}u_{2k}(0)X_{2k}(x)\\
&= \varphi_0X_0(x)+\sum_{k\geq 1}\varphi_{2k-1}X_{2k-1}(x)+\sum_{k\geq 1}\varphi_{2k}X_{2k}(x)\\
&= \varphi(x).
\end{aligned}$$

2.4.3 Uniqueness of the Solution

We suppose that $\{u_1(x,t), a_1(t)\}$ and $\{u_2(x,t), a_2(t)\}$ are two solution sets of inverse problem (2.1)-(2.4). Let $\bar{u}(x,t) = u_1(x,t) - u_2(x,t)$ and $\bar{a}(t) = a_1(t) - a_2(t)$, then $\bar{u}(x,t)$ satisfies the following problem:

$$\partial_{0^+,t}^{\alpha,\beta}\bar{u}(x,t) = \bar{u}_{xx}(x,t) + \bar{a}(t)F(x,t), \qquad 0<x<1, 0<t<T \tag{2.30}$$

$$I_{0^+,t}^{1-\gamma}\bar{u}(x,0) = 0, \quad \gamma = \alpha+\beta(1-\alpha) \qquad 0<x<1 \tag{2.31}$$

$$\bar{u}_x(0,t) = \bar{u}_x(1,t); \quad \bar{u}(1,t) = 0, \qquad 0<t\leq T \tag{2.32}$$

and the integral over-determination data

$$\int_0^1 \bar{u}(x,t)dx = 0, \qquad t\in[0,T]. \tag{2.33}$$

Using the basis S_1, we can write

$$\bar{u}(x,t) = \bar{u}_0(t)X_0(x) + \sum_{k\geq 1}\bar{u}_{2k-1}(t)X_{2k-1}(x) + \sum_{k\geq 1}\bar{u}_{2k}(t)X_{2k}(x),$$

where the coefficient

$$\bar{u}_0(t) = \int_0^1 \bar{u}(x,t)Y_0(x)dx = 0 \tag{2.34}$$

is obtained from (2.33) and for $k\geq 1$

$$\bar{u}_{2k-1}(t) = \int_0^1 \bar{u}(x,t)Y_{2k-1}(x)dx, \tag{2.35}$$

$$\bar{u}_{2k}(t) = \int_0^1 \bar{u}(x,t)Y_{2k}(x)dx. \tag{2.36}$$

Taking the fractional derivative $D_{0^+}^{\alpha,\beta}$ under the integral sign of equation (2.34) and (2.14) into account, then by using (2.30) and integration by parts, we get

$$D_{0^+}^{\alpha,\beta}\bar{u}_0(t) = \bar{a}(t)F_0(t). \tag{2.37}$$

Hence, from (2.37) and (2.34) we obtain

$$\bar{a}(t) = \frac{D_{0^+}^{\alpha,\beta}\bar{u}_0(t)}{F_0(t)} = 0. \tag{2.38}$$

Taking the fractional derivative $D_{0^+}^{\alpha,\beta}$ under the integral sign of equation (2.35) and (2.17) into account, then from (2.30),(2.31) and integration by parts, we get

$$D_{0^+}^{\alpha,\beta}\bar{u}_{2k-1}(t) + \lambda_k\bar{u}_{2k-1}(t) = \bar{a}(t)F_{2k-1}(t), \tag{2.39}$$

the associated initial condition is

$$I_{0^+}^{1-\gamma}\bar{u}_{2k-1}(0) = 0. \tag{2.40}$$

In view of Lemma 1.23, the solution of the problem (2.39)-(2.40) is

$$\bar{u}_{2k-1}(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-s)^{\alpha}) \bar{a}(s) F_{2k-1}(s) ds,$$

then, from (2.38) we deduce that $\bar{u}_{2k-1}(t) = 0$.
In the same manner from (2.36), we obtain the equation

$$D_{0^+}^{\alpha,\beta} \bar{u}_{2k}(t) + \lambda_k \bar{u}_{2k}(t) - 2\sqrt{\lambda_k} \bar{u}_{2k-1}(t) = \bar{a}(t) F_{2k}(t) \tag{2.41}$$

with the associated initial condition

$$I_{0^+}^{1-\gamma} \bar{u}_{2k}(0) = 0, \tag{2.42}$$

according to Lemma 1.23 with $\bar{u}_{2k-1}(t) = 0$ and (2.38), we get $\bar{u}_{2k}(t) = 0$. Therefore, we conclude the uniqueness of the solution of the inverse problem.

2.4.4 Continuous Dependence of the Solution Upon the Data

Let $\{u(x,t), a(t)\}$ and $\{\bar{u}(x,t), \bar{a}(t)\}$ be two solutions of inverse problem (2.1)-(2.4), corresponding to two sets of the data $\Phi(t) = \{\varphi(x), F(x,t), E(t)\}$ and $\bar{\Phi}(t) = \{\bar{\varphi}(x), \bar{F}(x,t), \bar{E}(t)\}$ respectively. Denote

$$\|\Phi\| = \|\varphi\|_{C^4[0,1]} + \|t^{1-\gamma} E\|_* + \|t^{1-\gamma} F\|_{C\left([0,T];C^4[0,1]\right)},$$

where

$$\|t^{1-\gamma} E\|_* = \max(\|t^{1-\gamma} E\|_{C[0,T]}; \|t^{1-\gamma} D_{0^+}^{\alpha;\gamma} E\|_{C[0,T]}). \tag{2.43}$$

First, in view of (2.9) and (2.43) we have

$$\begin{aligned} t^{1-\gamma} |u_0(t) - \bar{u}_0(t)| &\leq \|t^{1-\gamma}(E - \bar{E})\|_{C[0,T]} \\ &\leq \|t^{1-\gamma}(E - \bar{E})\|_*. \end{aligned} \tag{2.44}$$

Next, we have from (2.19) for $k \geq 1$

$$\begin{aligned} t^{1-\gamma} |u_{2k-1}(t) - \bar{u}_{2k-1}(t)| &\leq M_{\alpha,\gamma} |\varphi_{2k-1} - \bar{\varphi}_{2k-1}| \\ &+ t^{1-\gamma} M_{\alpha,\alpha} \|a - \bar{a}\|_{C[0,T]} \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} s^{1-\gamma} |F_{2k-1}(s)| ds \\ &+ t^{1-\gamma} M_{\alpha,\alpha} \|\bar{a}\|_{C[0,T]} \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} s^{1-\gamma} |F_{2k-1}(s) - \bar{F}_{2k-1}(s)| ds. \end{aligned}$$

Using estimation (2.5) and Example 1.1, we get

$$\begin{aligned} t^{1-\gamma} |u_{2k-1}(t) - \bar{u}_{2k-1}(t)| &\leq M |\varphi_{2k-1} - \bar{\varphi}_{2k-1}| \\ &+ \frac{\Gamma(\alpha)\Gamma(\gamma) T^{\alpha}}{\Gamma(\alpha+\gamma)} M \|a - \bar{a}\|_{C[0,T]} \|t^{1-\gamma} F_{2k-1}\|_{C[0,T]} \\ &+ \frac{\Gamma(\alpha)\Gamma(\gamma) T^{\alpha}}{\Gamma(\alpha+\gamma)} M \|\bar{a}\|_{C[0,T]} \|t^{1-\gamma}(F_{2k-1} - \bar{F}_{2k-1})\|_{C[0,T]}. \end{aligned}$$

By Bessel's inequality, there exist positive constants $\beta_i, i = 1,2$ such that

$$\begin{aligned} t^{1-\gamma} \sum_{k \geq 1} |u_{2k-1}(t) - \bar{u}_{2k-1}(t)| &\leq M \|\varphi - \bar{\varphi}\|_{C^4[0,1]} + \beta_1 \|a - \bar{a}\|_{C[0,T]} \\ &+ \beta_2 \|t^{1-\gamma}(F - \bar{F})\|_{C\left([0,T];C^4[0,1]\right)}, \end{aligned} \tag{2.45}$$

where

$$\beta_1 = \frac{\Gamma(\alpha)\Gamma(\gamma)T^\alpha}{\Gamma(\alpha+\gamma)} M\|t^{1-\gamma}F\|_{C\left([0,T];C^4[0,1]\right)}; \quad \beta_2 = \frac{\Gamma(\alpha)\Gamma(\gamma)T^\alpha}{\Gamma(\alpha+\gamma)} M\|\bar{a}\|_{C[0,T]}. \tag{2.46}$$

Then, by similar previous computations, we find from (2.22) after summation

$$\begin{aligned} t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)-\bar{u}_{2k}(t)| &\leq M_{\alpha,\gamma}\sum_{k\geq 1}|\varphi_{2k}-\bar{\varphi}_{2k}| + 2t^\alpha M_{\alpha,\alpha+\gamma}\sum_{k\geq 1}\sqrt{\lambda_k}|\varphi_{2k-1}-\bar{\varphi}_{2k-1}| \\ &+ 2t^{1-\gamma}M_{\alpha,2\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{2\alpha-1}s^{\gamma-1}s^{1-\gamma}\sum_{k\geq 1}\sqrt{\lambda_k}\,|F_{2k-1}(s)-\bar{F}_{2k-1}(s)|\,ds \\ &+ 2t^{1-\gamma}M_{\alpha,2\alpha}\|a-\bar{a}\|_{C[0,T]}\int_0^t (t-s)^{2\alpha-1}s^{\gamma-1}s^{1-\gamma}\sum_{k\geq 1}\sqrt{\lambda_k}\,|\bar{F}_{2k-1}(s)|\,ds \\ &+ t^{1-\gamma}M_{\alpha,\alpha}\|a\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}\sum_{k\geq 1}|F_{2k}(s)-\bar{F}_{2k}(s)|\,ds \\ &+ t^{1-\gamma}M_{\alpha,\alpha}\|a-\bar{a}\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}\sum_{k\geq 1}|\bar{F}_{2k}(s)|\,ds. \end{aligned}$$

According to (2.5), Lemma 2.1 and Bessel's inequality

$$\begin{aligned} t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)-\bar{u}_{2k}(t)| &\leq (1+2CT^\alpha)M\|\varphi-\bar{\varphi}\|_{C^4[0,1]} \\ &+ 2t^{1-\gamma}M\|a\|_{C[0,T]}\int_0^t (t-s)^{2\alpha-1}s^{\gamma-1}s^{1-\gamma}G\|F(.,s)-\bar{F}(.,s)\|_{C^4[0,1]}ds \\ &+ 2t^{1-\gamma}M\|a-\bar{a}\|_{C[0,T]}\int_0^t (t-s)^{2\alpha-1}s^{\gamma-1}s^{1-\gamma}G\|\bar{F}(.,s)\|_{C^4[0,1]}ds \\ &+ t^{1-\gamma}M\|a\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}\|F(.,s)-\bar{F}(.,s)\|_{C^4[0,1]}ds \\ &+ t^{1-\gamma}M\|a-\bar{a}\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}s^{1-\gamma}\|\bar{F}(.,s)\|_{C^4[0,1]}ds. \end{aligned}$$

From Example 1.1, we find

$$\begin{aligned} t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)-\bar{u}_{2k}(t)| &\leq L_\varphi\|\varphi-\bar{\varphi}\|_{C^4[0,1]} + \beta_3\|a-\bar{a}\|_{C[0,T]} \\ &+ \beta_4\|t^{1-\gamma}(F-\bar{F})\|_{C\left([0,T];C^4[0,1]\right)}, \end{aligned} \tag{2.47}$$

where

$$\beta_3 = L_F\|t^{1-\gamma}\bar{F}\|_{C\left([0,T];C^4[0,1]\right)}; \quad \beta_4 = L_F\|a\|_{C[0,T]} \tag{2.48}$$

and L_φ, L_F are defined in (2.28).
Thus, in view of the series expression (2.8) we get

$$\begin{aligned} t^{1-\gamma}|u(x,t)-\bar{u}(x,t)| &\leq 2t^{1-\gamma}|u_0(t)-\bar{u}_0(t)| + 4t^{1-\gamma}\sum_{k\geq 1}|u_{2k-1}(t)-\bar{u}_{2k-1}(t)| \\ &+ 4t^{1-\gamma}\sum_{k\geq 1}|u_{2k}(t)-\bar{u}_{2k}(t)|. \end{aligned}$$

Then, from (2.44), (2.45) and (2.47) we deduce for some positive constants $\rho_i, i = 1,2,3$

$$\begin{aligned} t^{1-\gamma}|u(x,t) - \bar{u}(x,t)| \leq 2\|t^{1-\gamma}(E-\bar{E})\|_* + 4\rho_1\|\varphi - \bar{\varphi}\|_{C^4[0,1]} + 4\rho_2\|a-\bar{a}\|_{C[0,T]} \\ + 4\rho_3\|t^{1-\gamma}(F-\bar{F})\|_{C\left([0,T];C^4[0,1]\right)}, \end{aligned}$$

where

$$\rho_1 = M + L_\varphi; \quad \rho_2 = \beta_1 + \beta_3; \quad \rho_3 = \beta_2 + \beta_4;$$

with $\beta_i, i = 1:4$ are defined in (2.46) and (2.48).
Consequently, we get

$$\|t^{1-\gamma}(u-\bar{u})\|_{C([0,T]\times[0,1])} \leq \Pi_0\|\Phi - \bar{\Phi}\| + 4\rho_2\|a-\bar{a}\|_{C[0,T]}, \tag{2.49}$$

where $\Pi_0 = \max(2, 4\rho_1, 4\rho_3)$. On the other hand, from (2.23) we have

$$\begin{aligned} |a(t) - \bar{a}(t)| &= \left| \frac{D_{0^+}^{\alpha,\beta}E(t)}{\int_0^1 F(x,t)dx} - \frac{D_{0^+}^{\alpha,\beta}\bar{E}(t)}{\int_0^1 \bar{F}(x,t)dx} \right| \\ &= \left| \frac{D_{0^+}^{\alpha,\beta}(E-\bar{E})(t)\int_0^1 \bar{F}(x,t)dx}{\int_0^1 F(x,t)dx\int_0^1 \bar{F}(x,t)dx} + \frac{D_{0^+}^{\alpha,\beta}\bar{E}(t)\int_0^1 [\bar{F}(x,t) - F(x,t)]\,dx}{\int_0^1 F(x,t)dx\int_0^1 \bar{F}(x,t)dx} \right|, \end{aligned}$$

using assumption $(A_2) - 2$, we get

$$\begin{aligned} |a(t) - \bar{a}(t)| \leq \frac{1}{M_F^2}\Big[&\|t^{1-\gamma}D_{0^+}^{\alpha,\beta}(E-\bar{E})\|_{C[0,T]}\|t^{1-\gamma}\bar{F}\|_{C\left([0,T];C^4[0,1]\right)} \\ &+ \|t^{1-\gamma}D_{0^+}^{\alpha,\beta}\bar{E}\|_{C[0,T]}\|t^{1-\gamma}(F-\bar{F})\|_{C\left([0,T];C^4[0,1]\right)}\Big]. \end{aligned}$$

Then, from (2.43) we obtain for some positive constants $\rho_i, i = 4,5$

$$\|a-\bar{a}\|_{C[0,T]} \leq \rho_4\|t^{1-\gamma}(E-\bar{E})\|_* + \rho_5\|t^{1-\gamma}(F-\bar{F})\|_{C\left([0,T];C^4[0,1]\right)},$$

where

$$\rho_4 = \frac{1}{M_F^2}\|t^{1-\gamma}\bar{F}\|_{C\left([0,T];C^4[0,1]\right)}; \quad \rho_5 = \frac{1}{M_F^2}\|t^{1-\gamma}D_{0^+}^{\alpha,\beta}\bar{E}\|_{C[0,T]}.$$

Therefore, we find

$$\|a-\bar{a}\|_{C[0,T]} \leq \Pi_1\|\Phi - \bar{\Phi}\|, \quad \Pi_1 = \max(\rho_4, \rho_5). \tag{2.50}$$

Substituting (2.50) into (2.49), we get

$$\|t^{1-\gamma}(u-\bar{u})\|_{C([0,T]\times[0,1])} \leq \Pi_2\|\Phi - \bar{\Phi}\|,$$

for positive constant $\Pi_2 = \max(\Pi_0, 4\rho_2\Pi_1)$. Hence, $\|t^{1-\gamma}(u-\bar{u})\|_{C([0,T]\times[0,1])} \to 0$ and $\|a-\bar{a}\|_{C[0,T]} \to 0$, when $\|\Phi - \bar{\Phi}\| \to 0$. This induces the continuous dependence of the solution upon the data.

2.5 Examples

In this section, we present two examples to illustrate the obtained result.

Example 1

Consider problem (2.1)-(2.4) with $0 < \alpha+\gamma \leq \mu < 1$ and

$$\varphi(x) = \sin(2\pi x), \quad F(x,t) = 2t^{\gamma-1}(1-x), \quad E(t) = t^{\mu-1} = u_0(t).$$

Then, we have

$$\varphi_0 = \int_0^1 \varphi(x)Y_0(x)dx = \int_0^1 \sin(2\pi x)dx = \left[-\frac{1}{2\pi}\cos(2\pi x)\right]_{x=0}^{x=1} = 0$$

$$F_0(t) = \int_0^1 F(x,t)Y_0(x)dx = \int_0^1 2t^{\gamma-1}(1-x)dx = 2t^{\gamma-1}\left[x-\frac{x^2}{2}\right]_{x=0}^{x=1} = t^{\gamma-1}$$

In view of Theorem 1.7, we can find easily for $k \geq 1$

$$\varphi_{2k-1} = \int_0^1 \varphi(x)Y_{2k-1}(x)dx = \int_0^1 \sin(2\pi x)\cos(2\pi kx)dx = 0,$$

$$F_{2k-1} = \int_0^1 F(x,t)Y_{2k-1}(x)dx = 2t^{\gamma-1}\int_0^1 (1-x)\cos(2\pi kx)dx = 0,$$

$$F_{2k} = \int_0^1 F(x,t)Y_{2k}(x)dx = 2t^{\gamma-1}\int_0^1 (1-x)x\sin(2\pi kx)dx = 0$$

and

$$\varphi_{2k} = \int_0^1 \varphi(x)Y_{2k}(x)dx = \int_0^1 \sin(2\pi x)x\sin(2\pi kx)dx,$$

for $k = 1$

$$\varphi_{2k} = \int_0^1 \sin(2\pi x)x\sin(2\pi x)dx = \frac{1}{4},$$

for $k \geq 2$

$$\varphi_{2k} = \int_0^1 \varphi(x)Y_{2k}(x)dx = \int_0^1 \sin(2\pi x)x\sin(2\pi kx)dx.$$

Using the following trigonometric formula:

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right],$$

we obtain

$$\varphi_{2k} = \frac{1}{2}\int_0^1 x\cos((1-k)2\pi x)dx - \frac{1}{2}\int_0^1 x\cos((1+k)2\pi x)dx,$$

by integration by parts, we get $\varphi_{2k} = 0$.
From the result of Example 1.4, we have

$$D_{0^+}^{\alpha,\beta}E(t) = D_{0^+}^{\alpha,\beta}t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}t^{\mu-\alpha-1}.$$

Using (2.19) and (2.22), we obtain

$$u_{2k-1}(t) = 0, u_{2k}(t) = t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^{\alpha})\varphi_{2k}, \quad k = 1,2,\ldots.$$

Hence, according to (2.8) and (2.23), we obtain the solution of this inverse problem

$$\left\{u(x,t) = 2t^{\mu-1}(1-x) + t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_k t^{\alpha})\sin(2\pi x), a(t) = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}t^{\mu-\alpha-\gamma}\right\}.$$

In addition, Theorem 2.1 implies that this inverse problem is well-posed.

Example 2

Consider problem (2.1)-(2.4) with $0 < \alpha + \gamma \leq \mu < 1$, and

$$\varphi(x) = 0, \quad F(x,t) = t^{\mu-\alpha-1}(1-x), \quad E(t) = \frac{t^{\mu-1}}{2} E_{\alpha,\mu}(t^{\alpha}) = u_0(t).$$

It is easy to show that

$$\varphi_0 = \varphi_{2k-1} = \varphi_{2k} = 0, \quad k = 1, 2, \ldots.$$

and from Lemma 1.7, we get

$$F_0(t) = \frac{1}{2} t^{\mu-\alpha-1}, \quad F_{2k-1}(t) = F_{2k}(t) = 0 \quad , k = 1, 2, \ldots.$$

In the light of Lemma 1.22, we have

$$D_{0^+}^{\alpha,\beta} E(t) = D_{0^+}^{\alpha,\beta} \left[\frac{t^{\mu-1}}{2} E_{\alpha,\mu}(t^{\alpha}) \right] = \frac{t^{\mu-\alpha-1}}{2} E_{\alpha,\mu-\alpha}(t^{\alpha}).$$

By using (2.19) and (2.22), we obtain

$$u_{2k-1}(t) = u_{2k}(t) = 0, \quad k = 1, 2, \ldots.$$

Hence, from (2.8) and (2.23), the solution is this pair of functions

$$\left\{ u(x,t) = (1-x) t^{\mu-1} E_{\alpha,\mu}(t^{\alpha}), a(t) = E_{\alpha,\mu-\alpha}(t^{\alpha}) \right\}.$$

Moreover, in view of Theorem 2.1 we deduce that this inverse problem is well-posed.

CHAPTER

3 An Iteration Method for Some Inverse Problems for a Time-fractional Reaction-diffusion Equation

3.1 Introduction

A modified version of the classical reaction-diffusion equation, called fractional reaction-diffusion equation, is used to describe anomalous diffusion process called sub-diffusion by introducing a new parameter, the fractional index $\alpha \in (0,1)$ into the equation.

In many practical situations, various parameters are unknown such as diffusion coefficient, source or reaction term in reaction-diffusion or fractional reaction-diffusion problems according to the data, see [34, 35, 36, 43]. We were inspired by the paper [35], where Kanca and Baglan considered the inverse problem of finding a pair of functions (u,r) satisfying the following reaction-diffusion problem:

$$\begin{aligned}
&u_t(x,t) = u_{xx} - p(t)u(x,t) + r(t)f(x,t,u(x,t)), \quad (x,t) \in D\\
&u(x,0) = \varphi(x), \quad x \in [0,1]\\
&u(0,t) = u(1,t), \quad u_x(1,t) = 0, \quad t \in [0,T]\\
&\int_0^1 u(x,t)dx = E(t), \quad 0 \le t \le T
\end{aligned}$$

where

$$D := \{0 < x < 1, 0 < t < T\}.$$

The existence and uniqueness of the solution of the above inverse problem is proved by using the iteration method and Fourier analysis. Also, the continuous dependence upon the data of the inverse problem is obtained.

This chapter is devoted to two inverse problems concerning the time-fractional reaction-diffusion equations. We will study the existence, uniqueness and continuous dependence upon the data for these problems with some restrictions on the order of derivation α or the domain of existence, by using the Fourier analysis, the iteration method and Gronwall's Lemma. Some of the results we are going to see have been the subject of our paper [19].

3.2 Statement of the Problem

We are concerned with the semi-linear time-fractional reaction-diffusion equation

$$^C\partial^{\alpha}_{0^+,t}u(x,t) = u_{xx}(x,t) - a(t)u(x,t) + c(t)F(x,t,u(x,t)), \quad (x,t) \in D_T \tag{3.1}$$

with the initial condition

$$u(x,0) = \varphi(x), \qquad x \in [0,1] \tag{3.2}$$

where $D_T = (0,1)\times(0,T]$, ${}^C\partial^{\alpha}_{0^+,t}$ is stands for the Caputo fractional partial derivative of order $0<\alpha<1$ in the time variable, $\varphi(x)$ and $F(x,t,u(x,t))$ are given functions.

Mathematically, the semi-linear reaction-diffusion equation (3.1) is a parabolic partial differential equation where $u = u(x,t)$ represents the concentration of one substance, $u_{xx}(x,t)$ is a diffusion term and $a(t)u(x,t)+c(t)F(x,t,u(x,t))$ represents the reaction term where $F(x,t,u(x,t))$ is the nonlinear source term and $a(t)$ can be regarded as a control parameter.

We focus here our attention on two inverse problems which correspond to recovery of source or control coefficient for the semi-linear reaction-diffusion equation (3.1) with various measurements, as follows:

Problem P1. Given $a(t)$ and find a couple of functions $\{u(x,t),c(t)\}$ satisfying equation (3.1) and initial condition (3.2), under the non-local boundary conditions

$$u(0,t) = u(1,t);\ u_x(1,t) = 0, \qquad t\in[0,T] \tag{3.3}$$

and the integral over-determination data

$$\int_0^1 xu(x,t)dx = E(t), \qquad t\in[0,T]. \tag{3.4}$$

Problem P2. Take $c(t) = 1$, then the equation (3.1) takes the form

$${}^C\partial^{\alpha}_{0^+,t}u(x,t) = u_{xx}(x,t) - a(t)u(x,t) + F(x,t,u(x,t)), \quad (x,t)\in D_T \tag{3.5}$$

and we have to find a pair of functions $\{u(x,t),a(t)\}$ satisfying (3.5)-(3.2), under the non-local boundary conditions

$$u_x(0,t) = u_x(1,t);\ u(1,t) = 0, \qquad t\in[0,T] \tag{3.6}$$

and the integral over-determination data

$$\int_0^1 u(x,t)dx = H(t), \qquad t\in[0,T]. \tag{3.7}$$

Note that, the additional over-determination conditions (3.4) or (3.7) are needed for the unique solvability of our inverse problems. This kind of integral condition can be weighted as in (3.4) or unweighted as in (3.7) which is the total amount of substance in space.

First, let us define the following spaces:

$$C^{1,0}\left(\overline{D}_T\right) = \left\{u(.,t)\in C^1[0,1], \quad t\in[0,T] \text{ and } u(x,.)\in C[0,T], x\in[0,1]\right\};$$
$$C^{2,\alpha}(D_T) = \left\{u(.,t)\in C^2(0,1), \quad t\in(0,T] \text{ and } {}^C D^{\alpha}_{0^+,t}u(x,.)\in C(0,T], x\in(0,1)\right\}.$$

Note that, from the properties of Mittag-leffler type functions, we have for each $t\in[0,T]$, $k\geq 1$ with $\lambda_k = (2\pi k)^2$:

$$E_{\alpha,1}(-\lambda_k t^{\alpha}) \leq N_{\alpha 1}, \quad E_{\alpha,\alpha}(-\lambda_k t^{\alpha}) \leq N_{\alpha}, \quad \lambda_k E_{\alpha,\alpha}(-\lambda_k t^{\alpha}) \leq N_{\lambda}. \tag{3.8}$$

For a better optimization, from the decrease of the Mittag-leffler type functions (see [46, 50, 55]), we deduce that

$$\max\{N_{\alpha 1}, N_{\alpha}\} \leq 1. \tag{3.9}$$

From the third property in Proposition 1.1 we have:

$$\left(\sum_{k\geq 1}|y_k|^2\right)^{1/2} \leq \sum_{k\geq 1}|y_k|, \quad \sum_{k\geq 1}\frac{1}{k^2}=\frac{\pi^2}{6}, \quad \sum_{k\geq 1}\frac{1}{k^4}=\frac{\pi^4}{90}. \tag{3.10}$$

The starting point is to explain two techniques noted **[Tec K]** and **[Tec G]**, that we will use further:

[Tec K]: For a given function $W_n(x)$, adding and subtracting $\int_0^t\int_0^1(t-s)^{\alpha-1}F(x,s,0)W_n(x)dxds$ to the first member of the following inequality, we obtain

$$\begin{aligned}&\left|\int_0^t\int_0^1(t-s)^{\alpha-1}F(x,s,u(x,s))W_n(x)dxds\right|\\ &\leq \int_0^t(t-s)^{\alpha-1}\left|\int_0^1[F(x,s,u(x,s))-F(x,s,0)]W_n(x)dx\right|ds\\ &+\int_0^t(t-s)^{\alpha-1}\left|\int_0^1 F(x,s,0)W_n(x)dx\right|ds.\end{aligned}$$

Using the Cauchy-Schwartz inequality, we find

$$\begin{aligned}&\left|\int_0^t\int_0^1(t-s)^{\alpha-1}F(x,s,u(x,s))W_n(x)dxds\right|\\ &\leq\left(\int_0^t(t-s)^{2\alpha-2}ds\right)^{1/2}\left(\int_0^t\left(\int_0^1[F(x,s,u(x,s))-F(x,s,0)]W_n(x)dx\right)^2ds\right)^{1/2}\\ &+\left(\int_0^t(t-s)^{2\alpha-2}ds\right)^{1/2}\left(\int_0^t\left(\int_0^1 F(x,s,0)W_n(x)dx\right)^2ds\right)^{1/2}\\ &\leq L_\alpha\left(\int_0^t\left(\int_0^1[F(x,s,u(x,s))-F(x,s,0)]W_n(x)dx\right)^2ds\right)^{1/2}\\ &+L_\alpha\left(\int_0^t\left(\int_0^1 F(x,s,0)W_n(x)dx\right)^2ds\right)^{1/2},\end{aligned}$$

where

$$L_\alpha=\frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}} \text{ for } \alpha \text{ having to satisfy} \frac{1}{2}<\alpha<1. \tag{3.11}$$

[Tec G]: For the following form of inequality on $[0,T]$ with continuous functions $v(t)$, $P_1(t)$ and positive constant $P2$

$$v(t)\leq P_1(t)+P_2\left(\int_0^t(v(s))^2ds\right)^{1/2},$$

we take the square with the fact that $(a(t)+b(t))^2<2a^2(t)+2b^2(t)$, we have

$$(v(t))^2\leq 2P_1^2(t)+2P_2^2\int_0^t(v(s))^2ds. \tag{3.12}$$

Applying the Gronwall's Lemma 1.2 on (3.12), we get

$$(v(t))^2\leq 2P_1^2(t)\exp\left(2P_2^2t\right),$$

therefore, we obtain

$$v(t)\leq\sqrt{2}P_1(t)\exp\left(P_2^2t\right).$$

3.3 Determination of an Unknown Source Coefficient Problem

In this section, we will give the results about existence, uniqueness of the solution and the continuous dependence upon the data of the inverse problem (P1).

We observe that the operator A in the spectral problem associated to (3.1) and (3.3)

$$\begin{cases} AX = -X'' = \lambda X \\ X(0) = X(1); \quad X'(1) = 0 \end{cases}$$

with the eigenvalues $\lambda_k = (2k\pi)^2$; $k \geq 0$, is non-self-adjoint. Then, analogously to section 1.2.2 in special case of spectral problem, we construct two sets of functions to form a bi-orthogonal normalized system for the space $L^2(0,1)$ in the following form:

$$\Phi = \{\phi_0(x), \phi_{1,k}(x), \phi_{2,k}(x)\}_{k=1}^{\infty}, \quad \Psi = \{\psi_0(x), \psi_{1,k}(x), \psi_{2,k}(x)\}_{k=1}^{\infty}, \tag{3.13}$$

where

$$\phi_0(x) = 2, \quad \phi_{1,k}(x) = 4\cos(\sqrt{\lambda_k}x), \quad \phi_{2,k}(x) = 4(1-x)\sin(\sqrt{\lambda_k}x), \tag{3.14}$$

$$\psi_0 = x, \quad \psi_{1,k}(x) = x\cos(\sqrt{\lambda_k}x), \quad \psi_{2,k} = \sin(\sqrt{\lambda_k}x). \tag{3.15}$$

The system (3.13) forms a Riez basis in $L^2(0,1)$. As a consequence, we can expand the solution in terms of the functions of bi-orthogonal system (3.13). For more details, see [28, 29].

Now, let us define the classical solution of the inverse problem $(P1)$, as follows:

Definition 3.1. *The pair* $\{u(x,t), c(t)\} \in [C^{2,\alpha}(D_T) \cap C^{1,0}(\bar{D}_T)] \times C[0,T]$ *which satisfies (3.1)-(3.4) is called classical solution of the inverse problem* $(P1)$.

Next, let us give the following assumptions on the data of problem $(P1)$:

(A_1) $a \in C[0,T]$ is positive and $M_a := \|a\|_{C[0,T]}$.

(A_2) $\varphi \in C^4[0,1]$ such that $\varphi(0) = \varphi(1), \varphi'(1) = 0, \varphi''(0) = \varphi''(1), \varphi^{(3)}(1) = 0$.

(A_3) Let the function $F(x,t,u)$ be continuous with respect to all its arguments on $\bar{D}_T \times \mathbb{R}$ and satisfying the following conditions:

(1) $F(.,t,u) \in C^4[0,1], t \in [0,T], F(x,t,u)|_{x=0} = F(x,t,u)|_{x=1}$,
$F_x(x,t,u)|_{x=1} = 0, \quad F_{xx}(x,t,u)|_{x=0} = F_{xx}(x,t,u)|_{x=1}, \quad F_{xxx}(x,t,u)|_{x=1} = 0$;

(2) there exists a non-negative function $b(x,t)$ such that for each $u, \tilde{u} \in \mathbb{R}$ and each $(x,t) \in D_T$

$$\left|\frac{\partial^n}{\partial x^n}F(x,t,u) - \frac{\partial^n}{\partial x^n}F(x,t,\tilde{u})\right| \leq b(x,t)|u-\tilde{u}|, \quad n = 0,1,2$$

with $b \in L^2(D_T)$, $\max\limits_{0\leq t\leq T} \|b(.,t)\|_{L^2(0,1)} < \infty$;

(3) $M_F = \max\{\|\frac{\partial^n}{\partial x^n}F(.,.,0)\|_{L^2(D_T)}; n = 0,1,2\}$;

(4) there exists a positive constant F_m such that $|\int_0^1 xF(x,t,u(x,t))dx| > F_m$ for each $t \in [0,T]$ and uniformly to $u \in \mathbb{R}$.

(A_4) $E \in AC([0,T])$ and $E(0) = \int_0^1 x\varphi(x)dx$.

Below, we use will the following notations:

$$B = \|b\|_{L^2(D_T)}, \quad B_1 = \max_{0\leq t\leq T} \|b(.,t)\|_{L^2(0,1)}. \tag{3.16}$$

3.3.1 Existence and Uniqueness of the Solution

First, for arbitrary $c \in C[0,T]$, we intend to apply the Fourier's method to construct the formal solution of the direct problem (3.1)-(3.3). The series expansion of the solution $u(x,t)$ in the basis (3.14) in $L^2(0,1)$ has the following representation:

$$u(x,t) = u_0(t)\phi_0(x) + \sum_{k\geq 1} u_{1,k}(t)\phi_{1,k}(x) + \sum_{k\geq 1} u_{2,k}(t)\phi_{2,k}(x), \tag{3.17}$$

where the coefficients

$$u_0(t) = \int_0^1 u(x,t)\psi_0(x)dx, \quad u_{n,k}(t) = \int_0^1 u(x,t)\psi_{n,k}(x)dx, \quad k \geq 1, n = 1,2$$

are to be determined. The coefficients of the series expansion of $F(x,t,u(x,t))$ and $\varphi(x)$ in the basis (3.15) for $k \geq 1$ are given by

$$F_0(t,u) = \int_0^1 F(x,t,u(x,t))\psi_0(x)dx, \quad F_{n,k}(t,u) = \int_0^1 F(x,t,u(x,t))\psi_{n,k}(x)dx, \quad n = 1,2$$

and

$$\varphi_0(t) = \int_0^1 \varphi(x)\psi_0(x)dx, \quad \varphi_{n,k}(t) = \int_0^1 \varphi(x)\psi_{n,k}(x)dx, \quad n = 1,2$$

respectively. By properties of the bi-orthogonal system (3.13), we obtain from (3.1) the following Cauchy problems, for $k \geq 1$:

$$\begin{cases} {}^C D^\alpha_{0+} u_0(t) = -a(t)u_0(t) + c(t)F_0(t,u), \\ u_0(0) = \varphi_0; \end{cases} \tag{3.18}$$

$$\begin{cases} {}^C D^\alpha_{0+} u_{2,k}(t) = -\lambda_k u_{2,k}(t) - a(t)u_{2,k}(t) + c(t)F_{2,k}(t,u), \\ u_{2,k}(0) = \varphi_{2,k}; \end{cases} \tag{3.19}$$

$$\begin{cases} {}^C D^\alpha_{0+} u_{1,k}(t) = -\lambda_k u_{1,k}(t) - (4\pi k)u_{2,k}(t) - a(t)u_{1,k}(t) + c(t)F_{1,k}(t,u), \\ u_{1,k}(0) = \varphi_{1,k}. \end{cases} \tag{3.20}$$

Thus, by applying the integral operator I^α_{0+} to the fractional differential equation of the Cauchy problem (3.18), the solution is obtained as:

$$u_0(t) = \varphi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [-a(s)u_0(s)ds + c(s)F_0(s,u)]ds.$$

From Theorem 1.12, the solutions of the Cauchy problems (3.19) and (3.20) satisfy:

$$\begin{aligned} u_{2,k}(t) &= \varphi_{2,k}E_{\alpha,1}(-\lambda_k t^{\alpha}) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})[-a(s)u_{2,k}(s)+c(s)F_{2,k}(s,u)]ds; \\ u_{1,k}(t) &= \varphi_{1,k}E_{\alpha,1}(-\lambda_k t^{\alpha}) - (4\pi k)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})u_{2,k}(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})[-a(s)u_{1,k}(s)+c(s)F_{1,k}(s,u)]ds; \end{aligned}$$

for $k \geq 1$, respectively. Note that from the additional condition (3.4), we get

$$u_0(t) = E(t). \tag{3.21}$$

Then, from (3.18) and (3.21) we obtain

$$c(t) = \frac{{}^{C}D_{0+}^{\alpha}E(t) + a(t)E(t)}{\int_0^1 xF(x,t,u(x,t))dx}. \tag{3.22}$$

Definition 3.2. *We denote $\mathscr{B}$ the set of continuous functions on $[0,T]$*

$$\{u(t)\} = \{u_0(t), u_{1,k}(t), u_{2,k}(t), k = 1,2,..\}$$

satisfying

$$2|u_0(t)| + 4\sum_{k\geq 1}\left(|u_{1,k}(t)| + |u_{2,k}(t)|\right) < \infty, \text{ for each } t \in [0,T].$$

It can be shown that $(\mathscr{B}, \|.\|_{\mathscr{B}})$ is a Banach space with the norm

$$\|u\|_{\mathscr{B}} = 2\max_{0\leq t\leq T}|u_0(t)| + 4\sum_{k\geq 1}\left(\max_{0\leq t\leq T}|u_{1,k}(t)| + \max_{0\leq t\leq T}|u_{2,k}(t)|\right). \tag{3.23}$$

Lemma 3.1. *If (A_2) is satisfied, then for some positive constant C^*, we have*

$$|\varphi_0| + \sum_{k\geq 1}\left(|\varphi_{1,k}| + |\varphi_{2,k}|\right) \leq C^*\|\varphi\|_{C^4[0,1]}. \tag{3.24}$$

Proof. First, by the Cauchy-Schwartz inequality we get

$$\begin{aligned} |\varphi_0| = \left|\int_0^1 \varphi(x)xdx\right| &\leq \left(\int_0^1 |x|^2dx\right)^{1/2}\left(\int_0^1 |\varphi(x)|^2dx\right)^{1/2} \\ &\leq \frac{1}{\sqrt{3}}\|\varphi\|_{L^2[0,1]} \leq \frac{1}{\sqrt{3}}\|\varphi\|_{C^2[0,1]}. \end{aligned}$$

The integration by parts of $\varphi_{2,k}$ two times and assumption (A_2) give

$$\varphi_{2,k} = \int_0^1 \varphi(x)\sin(2\pi k)dx = \frac{-1}{(2k\pi)^2}\int_0^1 \varphi''(x)\sin(2\pi k)dx = \frac{-1}{(2k\pi)^2}\varphi''_{2,k}.$$

Using the Cauchy-Schwartz and the Bessel inequalities, we obtain

$$\begin{aligned} \sum_{k\geq 1}|\varphi_{2,k}| &\leq \frac{1}{4\pi^2}\left(\sum_{k\geq 1}\frac{1}{k^4}\right)^{1/2}\left(\sum_{k\geq 1}|\varphi''_{2,k}|^2\right)^{1/2} \\ &\leq C_1\|\varphi''\|_{L^2(0,1)} \leq C_1\|\varphi\|_{C^2[0,1]}. \end{aligned}$$

Similarly, for $\varphi_{1,k}$, we have

$$\varphi_{1,k} = -\frac{1}{(2k\pi)^2}\varphi''_{1,k} + \frac{2}{(2k\pi)^3}\varphi''_{2,k}.$$

Then

$$\begin{aligned}\sum_{k\geq 1}|\varphi_{1,k}| &\leq \frac{1}{4\pi^2}\left(\sum_{k\geq 1}\frac{1}{k^4}\right)^{1/2}\left(\sum_{k\geq 1}|\varphi''_{1,k}|^2\right)^{1/2} + \frac{1}{4\pi^3}\left(\sum_{k\geq 1}\frac{1}{k^8}\right)^{1/2}\left(\sum_{k\geq 1}|\varphi''_{2,k}|^2\right)^{1/2}\\ &\leq (C_1+C_2)\,\|\varphi\|_{C^2[0,1]}.\end{aligned}$$

Hence

$$|\varphi_0| + \sum_{k\geq 1}\left(|\varphi_{2k}| + |\varphi_{2k-1}|\right) \leq \left(\frac{1}{\sqrt{3}} + 2C_1 + C_2\right)\|\varphi\|_{C^2[0,1]} \leq C^*\|\varphi\|_{C^4[0,1]}.$$

□

The first main result can be stated as follows:

Theorem 3.1. *Let* (A_1)-(A_4) *be satisfied. Then, the inverse problem* $(P1)$ *has a unique classical solution* $\{u(x,t), c(t)\}$ *for* $1/2 < \alpha < 1$ *and a small* T *.*

Remark 3.1. *We mean by a small* T*, a time limited by condition* (3.31).

Proof. Our proof will be carried out in the six steps:
Step 1: Let us define an iteration for the Fourier coefficient of (3.17) as follows:

$$u_0^{(N+1)}(t) = E(t); \tag{3.25}$$

$$\begin{aligned}u_{2,k}^{(N+1)}(t) &= u_{2,k}^{(0)}(t) - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)u_{2,k}^{(N+1)}(s)ds \\ &+ \int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)c^{(N)}(s)F(x,s,u^{(N)}(x,s))\sin(2\pi kx)dxds;\end{aligned} \tag{3.26}$$

$$\begin{aligned}u_{1,k}^{(N+1)}(t) &= u_{1,k}^{(0)}(t) - (4\pi k)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2,k}^{(N+1)}(s)ds \\ &- \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)u_{1,k}^{(N+1)}(s)ds \\ &+ \int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)c^{(N)}(s)F(x,s,u^{(N)}(x,s))x\cos(2\pi kx)dxds;\end{aligned} \tag{3.27}$$

for $N \geq 0$ and $k \geq 1$ with

$$u_0^{(0)}(t) = \varphi_0, \quad u_{2,k}^{(0)}(t) = \varphi_{2,k}E_{\alpha,1}(-\lambda_k t^\alpha), \quad u_{1,k}^{(0)}(t) = \varphi_{1,k}E_{\alpha,1}(-\lambda_k t^\alpha). \tag{3.28}$$

Again, we define an iteration for (3.22) as:

$$c^{(N)}(t) = \frac{{}^C D_{0+}^\alpha E(t) + a(t)E(t)}{\int_0^1 xF(x,t,u^{(N)}(x,t))dx}, \text{ for } N \geq 0.$$

Step 2: Next, we will show that the iterations $u^{(N)}(t)$ and $c^{(N)}(t)$ are in $\mathscr{B}$ and $C[0,T]$ respectively. In view of $(A_1),(A_3)-4$ and (A_4), we get easily that

$$\|c^{(N)}\|_{C[0,T]} \leq \frac{\|E\|_{AC[0,T]} + \|a\|_{C[0,T]}\|E\|_{C[0,T]}}{F_m} < \infty,$$

which implies that $c^{(N)} \in C[0,T]$ for all $N \geq 0$ and we denote

$$M_c = \max\{\|c^{(N)}\|_{C[0,T]}; N \geq 0\}.$$

Now, using (3.23) and (3.28), then in view of (3.9) and (3.24) we find

$$\begin{aligned}\|u^{(0)}\|_{\mathscr{B}} &= 2\max_{0\leq t\leq T}|u_0^{(0)}(t)| + 4\sum_{k\geq 1}\left(\max_{0\leq t\leq T}|u_{1,k}^{(0)}(t)| + \max_{0\leq t\leq T}|u_{2,k}^{(0)}(t)|\right)\\ &\leq 2|\varphi_0| + 4N_{\alpha 1}\sum_{k\geq 1}(|\varphi_{1,k}| + |\varphi_{2,k}|) \leq 4C^*\|\varphi\|_{C^4[0,1]} < \infty,\end{aligned}$$

we conclude that $u^{(0)} \in \mathscr{B}$.
It's clear that, from (3.25) we have for $N \geq 0$

$$\max_{0\leq t\leq T}|u_0^{(N+1)}(t)| \leq \|E\|_{C[0,T]}. \tag{3.29}$$

Taking $N = 0$ in (3.26), we have

$$\begin{aligned}u_{2,k}^{(1)}(t) &= u_{2,k}^{(0)}(t) - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)u_{2,k}^{(1)}(s)ds\\ &+ \int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)c^{(0)}(s)F(x,s,u^{(0)}(x,s))\sin(2\pi kx)dxds,\end{aligned}$$

which yields

$$\begin{aligned}|u_{2,k}^{(1)}(t)| &\leq N_{\alpha 1}|\varphi_{2,k}| + \frac{N_\alpha T^\alpha}{\alpha}\|a\|_{C[0,T]}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)|\\ &+ N_\alpha\|c^{(0)}\|_{C[0,T]}\int_0^t\int_0^1 (t-s)^{\alpha-1}F(x,s,u^{(0)}(x,s))\sin(2\pi kx)dxds.\end{aligned}$$

Applying **[Tec K]** (see p. 54) to the last equation and from (3.9), we obtain

$$\begin{aligned}|u_{2,k}^{(1)}(t)| &\leq |\varphi_{2,k}| + \frac{T^\alpha}{\alpha}\|a\|_{C[0,T]}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)|\\ &+ L_\alpha\|c^{(0)}\|_{C[0,T]}\left(\int_0^t\left(\int_0^1[F(x,s,u^{(0)}(x,s)) - F(x,s,0)]\sin(2\pi kx)dx\right)^2 ds\right)^{1/2}\\ &+ L_\alpha\|c^{(0)}\|_{C[0,T]}\left(\int_0^t\left(\int_0^1 F(x,s,0)\sin(2\pi kx)dx\right)^2 ds\right)^{1/2}.\end{aligned}$$

Integrating by parts the two last terms twice with respect to x, then taking the maximum and the sum of both sides of the inequality, we get

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)| \leq \frac{1}{\Psi}\sum_{k\geq 1}|\varphi_{2,k}| + \frac{L_{c,T}}{\Psi 4\pi^2}\sum_{k\geq 1}\frac{1}{k^2} \times \left(\int_0^T \left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\sin(2\pi kx)dx\right)^2 ds\right)^{1/2} + \frac{L_{c,T}}{\Psi 4\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\left(\int_0^T \left(\int_0^1 F_{xx}(x,s,0)\sin(2\pi kx)dx\right)^2 ds\right)^{1/2},$$

where

$$L_{c,T} = M_c L_\alpha, \quad \Psi = \left(1 - \frac{M_a T^\alpha}{\alpha}\right), \tag{3.30}$$

$$\text{with } \frac{M_a T^\alpha}{\alpha} < 1 \text{ and from (3.11) } \frac{1}{2} < \alpha < 1. \tag{3.31}$$

By Hölder's inequality, we obtain

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)| \leq \frac{1}{\Psi}\sum_{k\geq 1}|\varphi_{2,k}| + \frac{L_{c,T}}{\Psi 4\pi^2}\left(\sum_{k\geq 1}\frac{1}{k^4}\right)^{1/2} \times \left(\sum_{k\geq 1}\int_0^T \left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\sin(2\pi kx)dx\right)^2 ds\right)^{1/2} + \frac{L_{c,T}}{\Psi 4\pi^2}\left(\sum_{k\geq 1}\frac{1}{k^4}\right)^{1/2}\left(\sum_{k\geq 1}\int_0^T \left(\int_0^1 F_{xx}(x,s,0)\sin(2\pi kx)dx\right)^2 ds\right)^{1/2}.$$

Keeping in mind, the third fact of (3.10), we deduce

$$\frac{1}{4\pi^2}\left(\sum_{k\geq 1}\frac{1}{k^4}\right)^{1/2} = \frac{1}{12\sqrt{10}} < 1,$$

with the Bessel inequality, we get

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)| \leq \frac{1}{\Psi}\sum_{k\geq 1}|\varphi_{2,k}| + \frac{L_{c,T}}{\Psi} \times \left(\int_0^T\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]^2 dxds\right)^{1/2} + \frac{L_{c,T}}{\Psi}\left(\int_0^T\int_0^1 |F_{xx}(x,s,0)|^2 dxds\right)^{1/2}.$$

Then, the Lipschitz condition and $(A_3)-2$, lead to

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)| \leq \frac{1}{\Psi}\sum_{k\geq 1}|\varphi_{2,k}| + \frac{L_{c,T}}{\Psi}\left[\|b\|_{L^2(D_T)}\|u^{(0)}\|_{\mathscr{B}} + \|F_{xx}(.,.,0)\|_{L^2(D_T)}\right] \leq \frac{1}{\Psi}\sum_{k\geq 1}|\varphi_{2,k}| + \frac{L_{c,T}}{\Psi}\left[B\|u^{(0)}\|_{\mathscr{B}} + M_F\right]. \tag{3.32}$$

Next, we take $N=0$ in (3.27) to get

$$u_{1,k}^{(1)}(t) = u_{1,k}^{(0)}(t) - (4\pi k)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2,k}^{(1)}(s)ds$$
$$-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)u_{1,k}^{(1)}(s)ds$$
$$+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)c^{(0)}(s)F(x,s,u^{(0)}(x,s))x\cos(2\pi kx)dxds.$$

In the same way and by applying **[Tec K]** (see p. 54) to the last equation, we find

$$|u_{1,k}^{(1)}(t)| \le |\varphi_{1,k}|N_{\alpha 1} + \frac{N_\lambda T^\alpha}{\alpha\pi k}\max_{0\le t\le T}|u_{2,k}^{(1)}(t)|$$
$$+\frac{N_\alpha T^\alpha}{\alpha}\|a\|_{C[0,T]}\max_{0\le t\le T}|u_{1,k}^{(1)}(t)| + L_\alpha N_\alpha \|c^{(0)}\|_{C[0,T]}$$
$$\times\left(\int_0^t\left(\int_0^1 [F(x,s,u^{(0)}(x,s)) - F(x,s,0)]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}$$
$$+L_\alpha\|c^{(0)}\|_{C[0,T]}N_\alpha\left(\int_0^t\left(\int_0^1 F(x,s,0)\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}.$$

We integrate by parts the two last terms with respect to x, taking the maximum and the sum of both sides of the last inequality, by Hölder's inequality and (3.9), we get

$$\sum_{k\ge 1}\max_{0\le t\le T}|u_{1,k}^{(1)}(t)| \le \frac{1}{\Psi}\sum_{k\ge 1}|\varphi_{1,k}| + \frac{L_{\lambda,T}}{\pi\Psi}\left(\sum_{k\ge 1}\frac{1}{k^2}\right)^{1/2}$$
$$\times\left(\sum_{k\ge 1}\left(\max_{0\le t\le T}|u_{2,k}^{(1)}(t)|\right)^2\right)^{1/2} + \frac{L_{c,T}}{2\pi\Psi}\left(\sum_{k\ge 1}\frac{1}{k^2}\right)^{1/2}$$
$$\times\left(\int_0^t\sum_{k\ge 1}\left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)]\sin(2\pi kx)dx\right)^2 ds\right)^{1/2}$$
$$+\frac{L_{c,T}}{2\pi\Psi}\left(\sum_{k\ge 1}\frac{1}{k^2}\right)^{1/2}\left(\int_0^t\sum_{k\ge 1}\left(\int_0^1 F_x(x,s,0)\sin(2\pi kx)dx\right)^2 ds\right)^{1/2}$$

where,
$$L_{\lambda,T} = \frac{N_\lambda T^\alpha}{\alpha}. \tag{3.33}$$

By the first and second facts of (3.10), we conclude that

$$\left(\sum_{k\ge 1}\left(\max_{0\le t\le T}|u_{2,k}^{(1)}(t)|\right)^2\right)^{1/2} \le \sum_{k\ge 1}\max_{0\le t\le T}|u_{2,k}^{(1)}(t)|;$$
$$\frac{1}{2\pi}\left(\sum_{k\ge 1}\frac{1}{k^2}\right)^{1/2} = \frac{1}{2\sqrt{6}} < 1,$$

respectively.

Using this with estimation (3.32), Bessel's inequality, Lipschitz condition and $(A_3)-3$, we get

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{1,k}^{(1)}(t)| \leq \frac{1}{\Psi}\sum_{k\geq 1}\left(|\varphi_{1,k}|+\frac{L_{\lambda,T}}{\Psi}|\varphi_{2,k}|\right) + \frac{L_{c,T}}{\Psi}\left(\frac{L_{\lambda,T}}{\Psi}+1\right)\left[B\|u^{(0)}\|_{\mathscr{B}}+M_F\right]. \tag{3.34}$$

Finally, in view of the norm (3.23) and the estimations (3.29), (3.32), (3.34) with Lemma 3.1, we get

$$\begin{aligned}\|u^{(1)}\|_{\mathscr{B}} &\leq 2\|E\|_{C[0,T]}+\frac{4}{\Psi}\sum_{k\geq 1}\left[|\varphi_{1,k}|+\left(1+\frac{L_{\lambda,T}}{\Psi}\right)|\varphi_{2,k}|\right] \\ &+4\frac{L_{c,T}}{\Psi}\left(\frac{L_{\lambda,T}}{\Psi}+2\right)\left[B\|u^{(0)}\|_{\mathscr{B}}+M_F\right] \\ &\leq 2\|E\|_{C[0,T]}+D_0\|\varphi\|_{C^4[0,1]}+D_c\left[B\|u^{(0)}\|_{\mathscr{B}}+M_F\right]<\infty,\end{aligned}$$

where

$$D_0=4\frac{C^*}{\Psi}\left(1+\frac{L_{\lambda,T}}{\Psi}\right),\quad D_c=4\frac{L_{c,T}}{\Psi}\left(\frac{L_{\lambda,T}}{\Psi}+2\right).$$

As a result $u^{(1)}\in\mathscr{B}$. In the same way, by induction and the norm (3.23) we obtain for all $N\geq 1$

$$\|u^{(N)}\|_{\mathscr{B}}\leq 2\|E\|_{C[0,T]}+D_0\|\varphi\|_{C^4[0,1]}+D_c\left[B\|u^{(N-1)}\|_{\mathscr{B}}+M_F\right]. \tag{3.35}$$

Since, by induction $u^{(N-1)}(t)$ is bounded, then $u^{(N)}(t)$ is in $\mathscr{B}$ and then we denote

$$M_u=\max\{\|u^{(N)}\|_{\mathscr{B}},\quad N\geq 0\}.$$

It is easy to prove (3.35) and later all the induction cases by using the recurrence method.
Step 3: Now, we prove that the iterations $u^{(N)}(t)$ and $c^{(N)}(t)$ converge in $\mathscr{B}$ and $C[0,T]$ respectively, as $N\to\infty$. First, we have for each $N\geq 0$

$$\begin{aligned}|c^{(N+1)}(t)-c^{(N)}(t)| &= |{}^{C}D_{0^+}^{\alpha}E(t)+a(t)E(t)| \\ &\times\left|\frac{\int_0^1 x\left[F(x,t,u^{(N)}(x,t))-F(x,t,u^{(N+1)}(x,t))\right]dx}{\int_0^1 xF(x,t,u^{(N+1)}(x,t))dx\int_0^1 xF(x,t,u^{(N)}(x,t))dx}\right|.\end{aligned}$$

Using $(A_3)-4$ and applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned}|c^{(N+1)}(t)-c^{(N)}(t)| &\leq \frac{|{}^{C}D_{0^+}^{\alpha}E(t)+a(t)E(t)|}{F_m^2} \\ &\times\left(\int_0^1 x^2dx\right)^{1/2}\left(\int_0^1\left[F(x,t,u^{(N)}(x,t))-F(x,t,u^{(N+1)}(x,t))\right]^2dx\right)^{1/2}.\end{aligned}$$

Then, from the Lipschitz condition and (A_4), we find

$$|c^{(N+1)}(t)-c^{(N)}(t)| \leq \mathscr{E}\left(\int_0^1 b^2(x,t)|u^{(N)}(x,t)-u^{(N+1)}(x,t)|^2dx\right)^{1/2}, \tag{3.36}$$

where

$$\mathscr{E} = \frac{\|E\|_{AC[0,T]}+M_a\|E\|_{C[0,T]}}{F_m^2\sqrt{3}}. \tag{3.37}$$

Hence

$$\|c^{(N+1)}-c^{(N)}\|_{C[0,T]} \leq \mathscr{E}\max_{0\leq t\leq T}\|b(.,t)\|_{L^2(0,1)}\|u^{(N+1)}-u^{(N)}\|_{\mathscr{B}}. \tag{3.38}$$

We turn now to estimate $\|u^{(N+1)}-u^{(N)}\|_{\mathscr{B}}$ for $N=0$, first we have

$$|u_0^{(1)}(t)-u_0^{(0)}(t)| = |E(t)-\varphi_0|,$$

thus

$$\max_{0\leq t\leq T}|u_0^{(1)}(t)-u_0^{(0)}(t)| \leq \|E\|_{C[0,T]}+C^*\|\varphi\|_{C^4[0,1]}. \tag{3.39}$$

Then, applying the same estimations used in **step 2** on

$$u_{2,k}^{(1)}(t)-u_{2,k}^{(0)}(t) = -\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})a(s)u_{2,k}^{(1)}(s)ds$$
$$+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})c^{(0)}(s)F(x,s,u^{(0)}(x,s))\sin(2\pi kx)dxds,$$

taking the maximum and the sum, we obtain

$$\begin{aligned}\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)-u_{2,k}^{(0)}(t)| &\leq M_a\frac{T^{\alpha}}{\alpha}\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)| \\ &+L_{c,T}\left[\|b\|_{L^2(D_T)}\|u^{(0)}\|_{\mathscr{B}}+\|F_{xx}(.,.,0)\|_{L^2(D_T)}\right] \\ &\leq M_a\frac{T^{\alpha}}{\alpha}M_u+L_{c,T}\left[BM_u+M_F\right].\end{aligned} \tag{3.40}$$

In the same manner, we estimate

$$u_{1,k}^{(1)}(t)-u_{1,k}^{(0)}(t) = -(4\pi k)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})u_{2,k}^{(1)}(s)ds$$
$$-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})a(s)u_{1,k}^{(1)}(s)ds$$
$$+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})c^{(0)}(s)F(x,s,u^{(0)}(x,s))x\cos(2\pi kx)dxds,$$

then, the sum gives

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{1,k}^{(1)}(t)-u_{1,k}^{(0)}(t)| \leq L_{\lambda,T}\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2,k}^{(1)}(t)| \tag{3.41}$$

$$+M_a\frac{T^{\alpha}}{\alpha}\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{1,k}^{(1)}(t)|+L_{c,T}\left[\|b\|_{L^2(D_T)}\|u^{(0)}\|_{\mathscr{B}}+\|F_x(.,.,0)\|_{L^2(D_T)}\right]$$

$$\leq\left(L_{\lambda,T}+\frac{M_aT^{\alpha}}{\alpha}\right)M_u+L_{c,T}\left[BM_u+M_F\right]. \tag{3.42}$$

Thus, we use (3.39), (3.40) and (3.42) to see that

$$\begin{aligned}\|u^{(1)}-u^{(0)}\|_{\mathscr{B}} &\leq 2(\|E\|_{C[0,T]}+C^*\|\varphi\|_{C^4[0,1]})\\ &+4\left(L_{\lambda,T}+2\frac{M_aT^\alpha}{\alpha}\right)M_u+8L_{c,T}\left(BM_u+M_F\right)\\ &:=\mathscr{K}.\end{aligned}$$

Next, to estimate $\|u^{(N+1)}-u^{(N)}\|_{\mathscr{B}}$ for $N=1$, Firstly we have

$$|u_0^{(N+1)}(t)-u_0^{(N)}(t)|=0, \quad N\geq 1. \tag{3.43}$$

Then, we consider

$$\begin{aligned}u_{2,k}^{(2)}(t)-u_{2,k}^{(1)}(t) &= -\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a(s)\left[u_{2,k}^{(2)}(s)-u_{2,k}^{(1)}(s)\right]ds\\ &+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)\\ &\times\left[c^{(1)}(s)F(x,s,u^{(1)}(x,s))-c^{(0)}(s)F(x,s,u^{(0)}(x,s))\sin(2\pi kx)\right]dxds.\end{aligned}$$

Applying the same techniques used previously, adding and subtracting $c^{(1)}(s)F(x,s,u^{(0)}(x,s))$ and $\left[c^{(1)}(s)-c^{(0)}(s)\right]F(x,s,0)$ under the integral with respect with x and integrating F twice with respect to x, we obtain

$$\begin{aligned}&|u_{2,k}^{(2)}(t)-u_{2,k}^{(1)}(t)|\leq N_\alpha\int_0^t (t-s)^{\alpha-1}a(s)\left|u_{2,k}^{(2)}(s)-u_{2,k}^{(1)}(s)\right|ds\\ &+N_\alpha\int_0^t (t-s)^{\alpha-1}\left|c^{(1)}(s)\right|\left|\int_0^1\left[F_{xx}(x,s,u^{(1)}(x,s))-F_{xx}(x,s,u^{(0)}(x,s))\right]\frac{\sin(2\pi kx)}{(2k\pi)^2}dx\right|ds\\ &+N_\alpha\int_0^t (t-s)^{\alpha-1}\left|c^{(1)}(s)-c^{(0)}(s)\right|\left|\int_0^1\left[F_{xx}(x,s,u^{(0)}(x,s))-F_{xx}(x,s,0)\right]\frac{\sin(2\pi kx)}{(2k\pi)^2}\right|dxds\\ &+N_\alpha\int_0^t (t-s)^{\alpha-1}\left|c^{(1)}(s)-c^{(0)}(s)\right|\left|\int_0^1 F_{xx}(x,s,0)\frac{\sin(2\pi kx)}{(2k\pi)^2}dx\right|ds.\end{aligned}$$

From (3.9) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&|u_{2,k}^{(2)}(t)-u_{2,k}^{(1)}(t)|\leq M_a\frac{T^\alpha}{\alpha}\max_{0\leq t\leq T}|u_{2,k}^{(2)}(t)-u_{2,k}^{(1)}(t)|\\ &+L_{c,T}\left(\int_0^t\left(\int_0^1\left[F_{xx}(x,s,u^{(1)}(x,s))-F_{xx}(x,s,u^{(0)}(x,s))\right]\frac{\sin(2\pi kx)}{(2k\pi)^2}dx\right)^2ds\right)^{1/2}\\ &+L_\alpha\left(\int_0^t|c^{(1)}(s)-c^{(0)}(s)|^2\right.\\ &\times\left.\left(\int_0^1\left[F_{xx}(x,s,u^{(0)}(x,s))-F_{xx}(x,s,0)\right]\frac{\sin(2\pi kx)}{(2k\pi)^2}dx\right)^2ds\right)^{1/2}\\ &+L_\alpha\left(\int_0^t|c^{(1)}(s)-c^{(0)}(s)|^2\left(\int_0^1 F_{xx}(x,s,0)\frac{\sin(2\pi kx)}{(2k\pi)^2}dx\right)^2ds\right)^{1/2}.\end{aligned}$$

Taking the sum of both sides, then by using Hölder's and Bessel's inequalities, the Lipschitz condition and the estimation (3.36), we get

$$\sum_{k\geq 1}|u_{2,k}^{(2)}(t)-u_{2,k}^{(1)}(t)| \leq \frac{L_{c,T}}{\Psi}\left(\int_0^t\int_0^1 b^2(x,s)\left|u^{(1)}(x,s))-u^{(0)}(x,s)\right|^2 dxds\right)^{1/2}$$
$$+\frac{L_\alpha}{\Psi}\mathscr{E}\left(\int_0^t\int_0^1 b^2(x,s)\left|u^{(1)}(x,s)-u^{(0)}(x,s)\right|^2 dx\int_0^1 b^2(x,s)\left|u^{(0)}(x,s)\right|^2 dxds\right)^{1/2}$$
$$+\frac{L_\alpha}{\Psi}\mathscr{E}\left(\int_0^t\int_0^1 b^2(x,s)\left|u^{(1)}(x,s)-u^{(0)}(x,s)\right|^2 dx\int_0^1 (F_{xx}(x,s,0))^2 dxds\right)^{1/2}.$$

Thus,

$$\sum_{k\geq 1}|u_{2,k}^{(2)}(t)-u_{2,k}^{(1)}(t)| \leq H_c\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2 dxds\right)^{1/2}, \tag{3.44}$$

where

$$H_c=\frac{1}{\Psi}\left[L_{c,T}+L_\alpha\mathscr{E}\left(M_u B+M_F\right)\right]. \tag{3.45}$$

Again, using arguments similar to those used previously, applying the Cauchy-Schwartz inequality, then taking the sum and using the Lipschitz condition, we obtain

$$\sum_{k\geq 1}|u_{1,k}^{(2)}(t)-u_{1,k}^{(1)}(t)| \leq \frac{N_\lambda}{\Psi\pi}\int_0^t (t-s)^{\alpha-1}\sum_{k\geq 1}\frac{1}{k}|u_{2,k}^{(2)}(s)-u_{2,k}^{(1)}(s)|ds$$
$$+\frac{L_{c,T}}{\Psi}\sum_{k\geq 1}\left(\int_0^t\left[\int_0^1 b(x,s)\,|u^{(1)}(x,s)-u^{(0)}(x,s)|\frac{\sin(2\pi kx)}{2\pi k}dx\right]^2 ds\right)^{1/2}$$
$$+\frac{L_\alpha}{\Psi}\sum_{k\geq 1}\left(\int_0^t |c^{(1)}(s)-c^{(0)}(s)|^2\left[\int_0^1 b(x,s)\,|u^{(0)}(x,s)|\frac{\sin(2\pi kx)}{2\pi k}dx\right]^2 ds\right)^{1/2}$$
$$+\frac{L_\alpha}{\Psi}\sum_{k\geq 1}\left(\int_0^t |c^{(1)}(s)-c^{(0)}(s)|^2\left[\int_0^1 F_x(x,s,0)\frac{\sin(2\pi kx)}{2\pi k}dx\right]^2 ds\right)^{1/2}.$$

Concerning the first term, by means of Hölder's inequality, (3.10) and (3.44), we find

$$\frac{N_\lambda}{\Psi\pi}\int_0^t (t-s)^{\alpha-1}\left(\sum_{k\geq 1}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k\geq 1}|u_{2,k}^{(2)}(s)-u_{2,k}^{(1)}(s)|^2\right)^{1/2} ds$$
$$\leq \frac{N_\lambda}{\Psi}H_c\int_0^t (t-s)^{\alpha-1}\left(\int_0^s\int_0^1 b^2(x,r)\,|u^{(1)}(x,r)-u^{(0)}(x,r)|^2 dxdr\right)^{1/2} ds$$
$$\leq \frac{N_\lambda}{\Psi}H_c\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2 dxds\right)^{1/2}\int_0^t (t-s)^{\alpha-1}ds$$
$$\leq \frac{N_\lambda T^\alpha}{\alpha\Psi}H_c\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2 dxds\right)^{1/2}.$$

Then, from (3.36), the Hölder and Bessel inequalities, we get

$$\begin{aligned}&\sum_{k\geq 1}|u^{(2)}_{1,k}(t)-u^{(1)}_{1,k}(t)|\\ &\leq\left[\frac{N_\lambda T^\alpha}{\alpha\Psi}H_c+\frac{L_{c,T}}{\Psi}+\frac{\mathscr{E}L_\alpha}{\Psi}\left(\|b\|_{L^2_{(D_T)}}\|u^{(1)}\|_{\mathscr{B}}+\|F_x(.,.,0)\|_{L^2_{(D_T)}}\right)\right]\\ &\times\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{k\geq 1}|u^{(2)}_{1,k}(t)-u^{(1)}_{1,k}(t)|\\ &\leq\left[\frac{L_{\lambda,T}}{\Psi}+1\right]H_c\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2},\end{aligned}\tag{3.46}$$

where $L_{\lambda,T}$ and H_c are defined by (3.33) and (3.45) respectively.

Finally, we use (3.43), (3.44) and (3.46) to see that

$$|u^{(2)}(t)-u^{(1)}(t)|\leq K\left(\int_0^t\int_0^1 b^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2},\tag{3.47}$$

where

$$K=4H_c\left[\frac{L_{\lambda,T}}{\Psi}+2\right].\tag{3.48}$$

Consequently, from (3.16) we have

$$\|u^{(2)}-u^{(1)}\|_{\mathscr{B}}\leq KB_1\sqrt{T}\|u^{(1)}-u^{(0)}\|_{\mathscr{B}}.$$

Applying the same estimation techniques used to get (3.47), we obtain for $N=2$

$$|u^{(3)}(t)-u^{(2)}(t)|\leq K\left(\int_0^t\int_0^1 b^2(x,s)|u^{(2)}(x,s)-u^{(1)}(x,s)|^2dxds\right)^{1/2}.$$

Combining (3.17) with Definition 3.2, we infer that

$$\begin{aligned}&|u^{(3)}(t)-u^{(2)}(t)|\\ &\leq K\left(\int_0^t\int_0^1 b^2(x,s)|u^{(2)}(s)-u^{(1)}(s)|^2dxds\right)^{1/2}\\ &\leq K^2\left(\int_0^t\int_0^1 b^2(x,s)dx\int_0^s\int_0^1 b^2(x,r)|u^{(1)}(x,r)-u^{(0)}(x,r)|^2dxdrds\right)^{1/2}\\ &\leq K^2\left(\max_{0\leq t\leq T}\|b(.,t)\|_{L^2(0,1)}\right)^2\left(\int_0^t\int_0^s drds\right)^{1/2}\|u^{(1)}-u^{(0)}\|_{\mathscr{B}}\\ &\leq K^2B_1^2\frac{t}{\sqrt{2}}\mathscr{K}.\end{aligned}$$

Thus, for a general value of $N\geq 1$, we get by induction

$$\|u^{(N+1)}-u^{(N)}\|_{\mathscr{B}}\leq K^NB_1^N\frac{T^{N/2}}{\sqrt{N!}}\mathscr{K}.\tag{3.49}$$

It is clear that $\|u^{(N+1)}-u^{(N)}\|_{\mathscr{B}} \to 0$ when $N \to \infty$ and in view of (3.38) we deduce that, $\|c^{(N+1)}-c^{(N)}\|_{C[0,T]} \to 0$ when $N \to \infty$. From this, we conclude that $\left(u^{(N+1)}(t)\right)_{N\geq 0}$ and $\left(c^{(N+1)}(t)\right)_{N\geq 0}$ are Cauchy's type sequences and converge in $\mathscr{B}$ and $C[0,T]$, respectively. Following this, these sequences are uniformly convergent to elements of $\mathscr{B}$ and $C\,[0,T]$ respectively.

Step 4: Let us show that $u^{(N+1)}(t)$ and $c^{(N+1)}(t)$ converge to $u(t)$ and $c(t)$ respectively. We start with the following term:

$$|c(t)-c^{(N+1)}(t)| = |{}^{C}D^{\alpha}_{0+}E(t)+a(t)E(t)| \times \left| \frac{\int_0^1 x\left[F(x,t,u^{(N+1)}(x,t))-F(x,t,u(x,t))\right]dx}{\int_0^1 xF(x,t,u(x,t))dx\int_0^1 xF(x,t,u^{(N+1)}(x,t))dx} \right|.$$

Using $(A_3)-4$, the Cauchy-Schwartz inequality and the Lipschitz condition, we obtain

$$|c(t)-c^{(N+1)}(t)| \leq \mathscr{E}\left(\int_0^1 b^2(x,t)|u(x,t)-u^{(N+1)}(x,t)|^2dx\right)^{1/2}, \tag{3.50}$$

where $\mathscr{E}$ is defined by (3.37). As already seen, we obtain after some estimations

$$\begin{aligned} |u(t)-u^{(N+1)}(t)| &\leq K\left(\int_0^t\int_0^1 b^2(s,x)|u(s)-u^{(N)}(s)|^2dxds\right)^{1/2} \\ &\leq K\max_{0\leq t\leq T}\|b\,(.,t)\,\|_{L^2(0,1)}\left(\int_0^t |u(s)-u^{(N)}(s)|^2dx\right)^{1/2} \\ &\leq KB_1\left(\int_0^t |u(s)-u^{(N)}(s)|^2ds\right)^{1/2}, \end{aligned} \tag{3.51}$$

where K is defined by (3.48), thus

$$\begin{aligned} |u(t)-u^{(N+1)}(t)|^2 &\leq 2\,(KB_1)^2\int_0^t |u(s)-u^{(N+1)}(s)|^2ds \\ &\quad +2\,(KB_1)^2\int_0^t |u^{(N+1)}(s)-u^{(N)}(s)|^2ds. \end{aligned}$$

By Gronwall's inequality and (3.49), we get

$$\begin{aligned} |u(t)-u^{(N+1)}(t)|^2 &\leq 2\,(KB_1)^2\int_0^t |u^{(N+1)}(s)-u^{(N)}(s)|^2ds\exp\left(2\,(KB_1)^2t\right) \\ &\leq 2\,(KB_1)^2\frac{K^{2N}\mathscr{K}^2B_1^{2N}}{N!}\int_0^t s^Nds\exp\left(2\,(KB_1)^2t\right), \end{aligned}$$

which yields that

$$\|u-u^{(N+1)}\|_{\mathscr{B}} \leq \sqrt{2}K^{N+1}\mathscr{K}B_1^{N+1}\frac{T^{\frac{N+1}{2}}}{\sqrt{(N+1)!}}\exp\left((KB_1)^2T\right),$$

tends to 0 when $N \to \infty$. Consequently, from (3.50) we find

$$\|c-c^{(N+1)}\|_{C[0,T]} \leq \mathscr{E}B_1\|u-u^{(N+1)}\|_{\mathscr{B}}, \tag{3.52}$$

tends to 0 when $N \to \infty$.

Step 5: For the uniqueness result, we assume that the problem $(P1)$ has two pairs of solutions $\{u(x,t),c(t)\},\{\bar{u}(x,t),\bar{c}(t)\}$. From (3.22) and by the same way that led us to get (3.52), we find

$$\|c-\bar{c}\|_{C[0,T]} \leq \mathscr{E} B_1 \|u-\bar{u}\|_{\mathscr{B}}, \tag{3.53}$$

where B_1, $\mathscr{E}$ are defined by (3.16) and (3.37) respectively. On other side, remark that, the difference $(u(x,t)-\bar{u}(x,t))$ is just the difference between the coefficients of their respective series defined by (3.17) noted $(u(t)-\bar{u}(t))$. Therefore, the approximation techniques used previously to get (3.51) lead to

$$|u(t)-\bar{u}(t)| \leq KB_1 \left(\int_0^t |u(s)-\bar{u}(s)|^2 ds\right)^{1/2},$$

where B_1, K are defined by (3.16) and (3.48) respectively. Thus, we apply Gronwall's inequality to

$$|u(t)-\bar{u}(t)|^2 \leq (KB_1)^2 \int_0^t |u(s)-\bar{u}(s)|^2 ds,$$

and we get $|u-\bar{u}|=0$ for each $t\in[0,T]$. This implies the uniqueness of $u(t)$ and from (3.53) we deduce the uniqueness of $c(t)$.

Step 6: Let us show that the obtained solution is classical. The fact that $u(t)$ is in $\mathscr{B}$ means that series (3.17) is convergent in $C([0,T],L^2(0,1))$. Under (A_1)-(A_3), series (3.17) and series of its x-partial derivatives $\sum_{k\geq 1}\partial_x.$ are uniformly convergent in $C(\overline{D}_T)$. Therefore, the series of its xx-partial derivatives $\sum_{k\geq 1}\partial_{xx}.$ is uniformly convergent in $(0,1)\times[\varepsilon,T]$, for any $\varepsilon>0$.
The other hand, it is easy to show that the series of the time-fractional partial derivatives $\sum_{k\geq 1}{}^C\partial^{\alpha}_{0+,t}.$ is uniformly convergent on $[\varepsilon,T]$ for any $\varepsilon>0$ by using Theorem 1.8 and the previous procedure for (3.18), (3.19) and (3.20) to prove the convergence of

$$2{}^CD^{\alpha}_{0+}u_0(t)+4\sum_{k=1}^{\infty}\left({}^CD^{\alpha}_{0+}u_{1,k}(t)+{}^CD^{\alpha}_{0+}u_{2,k}(t)\right).$$

By virtue of relation (1.23) and linearity of fractional differential operators, we can extend Theorem 1.10 to the Caputo fractional derivative case. Then, the α-partial derivative ${}^C\partial^{\alpha}_{0+,t}$ of the series (3.17) is uniformly convergent for $t\in[\varepsilon,T]$, for any $\varepsilon>0$ and $x\in(0,1)$. Thus, $u\in C^{2,\alpha}(D_T)\cap C^{1,0}(\overline{D}_T)$ for arbitrary $c\in C[0,T]$, for a small T which must satisfy (3.31). The proof is complete. □

3.3.2 Continuous Dependence of the Solution Upon the Data

In this section, we establish a stability estimate of the solution $\{u(x,t),c(t)\}$ with respect to the additional data $E(t)$. In reality the inverse problem is well posed if we have stability with respect to all the data of the problem. A stability result can be obtained with respect to the initial data φ, the control parameter $a(t)$ and additional data $E(t)$. But we limit ourselves here to the additional data which directly influences the inverse problem, assuming that the direct problem is stable.

Theorem 3.2. *Under assumptions (A_1)-(A_4), the solution $\{u(x,t),c(t)\}$ of the inverse problem $(P1)$ depends continuously upon the data of $E(t)$ for a small T.*

Proof. Let $\{u(x,t),c(t)\}$ and $\{\bar{u}(x,t),\bar{c}(t)\}$ be two solutions of the inverse problem $(P1)$, corresponding to two sets of the data $E(t)$ and $\bar{E}(t)$ respectively, we denote

$$\|E\|_{AC[0,T]} = M_E.$$

First, we have

$$|c(t)-\bar{c}(t)| = \left|\frac{{}^C D^\alpha E(t)+a(t)E(t)}{\int_0^1 xF(x,t,u(x,t))dx} - \frac{{}^C D^\alpha \bar{E}(t)+a(t)\bar{E}(t)}{\int_0^1 xF(x,t,\bar{u}(x,t))dx}\right| \le J_1+J_2,$$

where

$$J_1 = \left|\frac{{}^C D^\alpha E(t)}{\int_0^1 xF(x,t,u(x,t))dx} - \frac{{}^C D^\alpha \bar{E}(t)}{\int_0^1 xF(x,t,\bar{u}(x,t))dx}\right|,$$
$$J_2 = a(t)\left|\frac{E(t)}{\int_0^1 xF(x,s,u(x,t))dx} - \frac{\bar{E}(t)}{\int_0^1 xF(x,s,\bar{u}(x,t))dx}\right|.$$

From $(A_3)-4$ we get

$$\begin{aligned}
J_1 &\le \frac{1}{F_m^2}\left|{}^C D^\alpha E(t)\int_0^1 xF(x,t,\bar{u}(x,t))dx - {}^C D^\alpha \bar{E}(t)\int_0^1 xF(x,t,u(x,t))dx\right| \\
&\le \frac{1}{F_m^2}|{}^C D^\alpha E(t)|\int_0^1 x|F(x,t,\bar{u}(x,t))-F(x,t,u(x,t))|dx \\
&+\frac{1}{F_m^2}|{}^C D^\alpha (E(t)-\bar{E}(t))|\int_0^1 x|F(x,t,u(x,t))|dx.
\end{aligned}$$

Adding and subtracting $x|F(x,t,0)|$ under the last integral and by using the Cauchy-Schwartz inequality, we find

$$\begin{aligned}
J_1 &\le \frac{M_E}{F_m^2}\left(\int_0^1 x^2dx\right)^{1/2}\left(\int_0^1 |F(x,t,\bar{u}(x,t))-F(x,t,u(x,t))|^2dx\right)^{1/2} \\
&+\frac{1}{F_m^2}|{}^C D^\alpha (E(t)-\bar{E}(t))|\left(\int_0^1 x^2dx\right)^{1/2}\left(\int_0^1 |F(x,t,u(x,t))-F(x,t,0)|^2dx\right)^{1/2} \\
&+\frac{1}{F_m^2}|{}^C D^\alpha (E(t)-\bar{E}(t))|\left(\int_0^1 x^2dx\right)^{1/2}\left(\int_0^1 |F(x,t,0)|^2dx\right)^{1/2},
\end{aligned}$$

and the Lipschitz condition gives

$$\begin{aligned}
J_1 &\le \frac{M_E}{\sqrt{3}F_m^2}\left(\int_0^1 b^2(x,t)|u(x,t)-\bar{u}(x,t)|^2dx\right)^{1/2} \\
&+\frac{1}{\sqrt{3}F_m^2}(M_uB_1+M)\|E-\bar{E}\|_{AC[0,T]},
\end{aligned}$$

where $M=\max\limits_{0\le t\le T}\|F(.,t,0)\|_{L^2(0,1)}$. In the same way, we obtain for J_2

$$\begin{aligned}
J_2 &\le M_a\frac{M_E}{\sqrt{3}F_m^2}\left(\int_0^1 b^2(x,t)|u(x,t)-\bar{u}(x,t)|^2dx\right)^{1/2} \\
&+M_a\frac{1}{\sqrt{3}F_m^2}(M_uB_1+M)\|E-\bar{E}\|_{AC[0,T]}.
\end{aligned}$$

Then, we get

$$|c(t)-\bar{c}(t)| \leq \mathscr{E}\left(\int_0^1 b^2(x,t)|u(x,t)-\bar{u}(x,t)|^2dx\right)^{1/2} + \eta_1\|E-\bar{E}\|_{AC[0,T]},$$

where $\mathscr{E}$ is defined by (3.37) and $\eta_1 = \dfrac{1}{\sqrt{3}F_m^2}(1+M_a)(M_uB_1+M)$.

Therefore, we find

$$\|c-\bar{c}\|_{C[0,T]} \leq \mathscr{E}B_1\|u-\bar{u}\|_{\mathscr{B}} + \eta_1\|E-\bar{E}\|_{AC[0,T]}, \tag{3.54}$$

in addition, we have

$$|c(t)-\bar{c}(t)|^2 \leq 2\mathscr{E}^2\int_0^1 b^2(x,t)|u(x,t)-\bar{u}(x,t)|^2dx + 2\eta_1^2\|E-\bar{E}\|^2_{AC[0,T]}. \tag{3.55}$$

To achieve our goal, we need to estimate each coefficient of the series $u(x,t)-\bar{u}(x,t)$, by the similar techniques used in the previous section, we obtain the following estimations

$$\begin{aligned}\sum_{k\geq 1}|u_{2,k}(t)-\bar{u}_{2,k}(t)| \leq{}& \frac{L_{c,T}}{\Psi}\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s)-\bar{u}(x,s)|^2dxds\right)^{1/2}\\ &+\frac{L_\alpha}{\Psi}\left(\int_0^t |c(t)-\bar{c}(t)|^2\int_0^1 b^2(x,s)\,|\bar{u}(x,s)|^2\,dxds\right)^{1/2}\\ &+\frac{L_\alpha}{\Psi}\left(\int_0^t |c(t)-\bar{c}(t)|^2\int_0^1 (F_{xx}(x,s,0))^2\,dxds\right)^{1/2}\end{aligned}$$

and from (3.55), we obtain

$$\begin{aligned}&\sum_{k\geq 1}|u_{2,k}(t)-\bar{u}_{2,k}(t)| \leq \frac{L_{c,T}}{\Psi}\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s))-\bar{u}(x,s)|^2dxds\right)^{1/2}\\ &+\frac{\sqrt{2}\mathscr{E}L_\alpha}{\Psi}\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s))-\bar{u}(x,s)|^2dxds\right)^{1/2}\left(\int_0^t\int_0^1 b^2(x,s)\,|\bar{u}(x,s)|^2\,dxds\right)^{1/2}\\ &+\frac{\sqrt{2}\eta_1L_\alpha}{\Psi}\|E-\bar{E}\|_{AC[0,T]}\left(\int_0^t\int_0^1 b^2(x,s)\,|\bar{u}(x,s)|^2\,dxds\right)^{1/2}\\ &+\frac{\sqrt{2}\mathscr{E}L_\alpha}{\Psi}\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s))-\bar{u}(x,s)|^2dxds\right)^{1/2}\left(\int_0^t\int_0^1 |F_{xx}(x,s,0)|^2\,dxds\right)^{1/2}\\ &+\frac{\sqrt{2}\eta_1L_\alpha}{\Psi}\|E-\bar{E}\|_{AC[0,T]}\left(\int_0^t\int_0^1 |F_{xx}(x,s,0)|^2\,dxds\right)^{1/2},\end{aligned}$$

which indicates that

$$\begin{aligned}\sum_{k\geq 1}|u_{2,k}(t)-\bar{u}_{2,k}(t)| \leq{}&\xi_1\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s))-\bar{u}(x,s)|^2dxds\right)^{1/2}\\ &+\xi_2\|E-\bar{E}\|_{AC[0,T]},\end{aligned} \tag{3.56}$$

where

$$\begin{aligned}\xi_1 &= \frac{1}{\Psi}\left[L_{c,T}+L_\alpha(M_{\bar{u}}B+M_F)\sqrt{2}\mathscr{E}\right],\\ \xi_2 &= \frac{L_\alpha}{\Psi}\sqrt{2}\eta_1(M_{\bar{u}}B+M_F).\end{aligned}$$

Also, we have

$$\begin{aligned}\sum_{k\geq 1}|u_{1,k}(t)-\bar{u}_{1,k}(t)| \leq \frac{N_\lambda}{\Psi}\int_0^t (t-s)^{\alpha-1}\sum_{k\geq 1}|u_{2,k}(s)-\bar{u}_{2,k}(s)|ds \\ +\frac{L_{c,T}}{\Psi}\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s)-\bar{u}(x,s)|^2dxds\right)^{1/2} \\ +\frac{L_\alpha}{\Psi}\left(\int_0^t |c(t)-\bar{c}(t)|^2\int_0^1 b^2(x,s)\,|\bar{u}(x,s)|^2\,dxds\right)^{1/2} \\ +\frac{L_\alpha}{\Psi}\left(\int_0^t |c(t)-\bar{c}(t)|^2\int_0^1 (F_x(x,s,0))^2\,dxds\right)^{1/2},\end{aligned}$$

using (3.55) and (3.56) to find

$$\begin{aligned}\sum_{k\geq 1}|u_{1,k}(t)-\bar{u}_{1,k}(t)| \leq \left(1+\frac{L_{\lambda,T}}{\Psi}\right)\xi_1\left(\int_0^t\int_0^1 b^2(x,s)|u(x,s))-\bar{u}(x,s)|^2dxds\right)^{1/2} \\ +\left(1+\frac{L_{\lambda,T}}{\Psi}\right)\xi_2\|E-\bar{E}\|_{AC[0,T]},\end{aligned}$$

where $L_{\lambda,T}$ is defined by (3.33). Then, by the fact that

$$|u_0(t)-\bar{u}_0(t)| \leq \|E-\bar{E}\|_{C[0,T]},$$

we deduce

$$|u(t)-\bar{u}(t)| \leq \eta_2 B_1\left(\int_0^t |u(s)-\bar{u}(s)|^2ds\right)^{1/2}+\eta_3\|E-\bar{E}\|_{C[0,T]},$$

where

$$\eta_2 = 4\left(2+\frac{L_{\lambda,T}}{\Psi}\right)\xi_1, \quad \eta_3 = 2\left[2\left(2+\frac{L_{\lambda,T}}{\Psi}\right)\xi_2+1\right].$$

Applying the **[Tec G]** (see p. 54) to the last inequality, we get

$$\|u-\bar{u}\|_{\mathscr{B}} \leq \sqrt{2}\eta_3\exp((\eta_2 B_1)^2 T)\|E-\bar{E}\|_{C[0,T]},$$

and from (3.54), we have

$$\|c-\bar{c}\|_{C[0,T]} \leq \left(\mathscr{E}B_1\sqrt{2}\eta_3\exp((\eta_2 B_1)^2T)+\eta_1\right)\|E-\bar{E}\|_{AC[0,T]}.$$

Then $\|u-\bar{u}\|_{\mathscr{B}} \to 0$ and $\|c-\bar{c}\|_{C[0,T]} \to 0$, when $\|E-\bar{E}\|_{AC[0,T]} \to 0$. This induces the continuous dependence of $u(x,t)$ and $c(t)$ on $E(t)$ for T satisfying (3.31). □

Remark 3.2. *By similar method, we can obtain the continuous dependence of the solution* $\{u(x,t),c(t)\}$ *upon the data* $\{\varphi(x),a(t),E(t)\}$.

3.4 Determination of an Unknown Control Coefficient Problem

The main purpose of this section is to establish existence, uniqueness and stability results of the inverse problem $(P2)$.

First of all, we define the mean of classical solution of inverse problem $(P2)$.

Definition 3.3. *The pair $\{u(x,t),a(t)\}$ is said to be a classical solution of inverse problem $(P2)$ if $\{u(x,t),a(t)\} \in \left[C^{2,\alpha}(D_T) \cap C^{1,0}(\bar{D}_T)\right] \times C[0,T]$ and $a(t)$ is positive on $[0,T]$ such that equation (3.5) and conditions (3.2),(3.6)-(3.7) are satisfied.*

The spectral problem for the corresponding homogeneous equation of the problem (3.5)-(3.6) is boundary-value problem (1.9)-(1.10). As stated earlier in Section 1.2.2, the following system of functions:

$$S_1 = \{X_0(x) = 2(1-x), \quad X_{2k-1}(x) = 4(1-x)\cos(\sqrt{\lambda_k}x), \quad X_{2k}(x) = 4\sin(\sqrt{\lambda_k}x); \quad k \geq 1\};$$

$$S_2 = \{Y_0(x) = 1, \quad Y_{2k-1}(x) = \cos(\sqrt{\lambda_k}x), \quad Y_{2k}(x) = x\sin(\sqrt{\lambda_k}x), \quad k \geq 1\}$$

with $\lambda_k = (2\pi k)^2$, form a bi-orthogonal normalized system for the space $L^2(0,1)$.

Throughout this section, we assume that the following conditions on the data of problem $(P2)$ hold:

(H_1) $\varphi \in C^4[0,1]$ such that $\varphi(1) = 0, \varphi'(0) = \varphi'(1), \varphi''(1) = 0, \varphi^{(3)}(0) = \varphi^{(3)}(1)$.

(H_2) $F(x,t,u)$ is continuous on $\bar{D}_T \times \mathbb{R}$ and satisfies the following conditions:

(1) $F(.,t,u) \in C^4[0,1]$, for $t \in [0,T]$, $F(x,t,u)|_{x=1} = 0$,
$F_x(x,t,u)|_{x=0} = F_x(x,t,u)|_{x=1} = 0$, $F_{xx}(x,t,u)|_{x=1} = 0$,
$F_{xxx}(x,t,u)|_{x=0} = F_{xxx}(x,t,u)|_{x=1}$;

(2) there exists a non-negative function $h \in L^2(D_T)$ such that for each $u, \tilde{u} \in \mathbb{R}$

$$\left|\frac{\partial^n}{\partial x^n}F(x,t,u) - \frac{\partial^n}{\partial x^n}F(x,t,\tilde{u})\right| \leq h(x,t)|u-\tilde{u}|, \quad n = 0,1,2$$

with $B_h = \|h\|_{L^2(D_T)}$ and $M_h = \max_{0\leq t\leq T} \|h(.,t)\|_{L^2(0,1)} < \infty$;

(3) $M_F = \max\{\|\frac{\partial^n}{\partial x^n}F(.,.,0)\|_{L^2(D_T)}; n = 0,1,2\}$, $M = \max_{0\leq t\leq T} \|F(.,t,0)\|_{L^2(0,1)} < \infty$;

(4) $\int_0^1 F(x,t,u(x,t))dx > 0$ for $t \in [0,T]$.

(H_3) $H \in AC[0,T]$, $\min_{0\leq t\leq T} H(t) = H_m > 0$, ${}^C D_{0+}^{\alpha} H(t) \leq 0$ for $t \in [0,T]$ and
$H(0) = \int_0^1 \varphi(x)dx$.

3.4.1 Existence and Uniqueness of the Solution

For arbitrary $a \in C[0,T]$, the Fourier series argument shows that the solution of the direct problem (3.2), (3.5)-(3.6) can be written in the following form:

$$u(x,t) = u_0(t)X_0(x) + \sum_{k\geq 1} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k\geq 1} u_{2k}(t)X_{2k}(x), \tag{3.57}$$

where the first coefficient

$$u_0(t) = \int_0^1 u(x,t)Y_0(x)dx = H(t) \tag{3.58}$$

is obtained from (3.7) and for $k \geq 1$

$$u_{2k-1}(t) = \int_0^1 u(x,t)Y_{2k-1}(x)dx, \quad u_{2k}(t) = \int_0^1 u(x,t)Y_{2k}(x)dx$$

have to be determined. These coefficients satisfy the following Cauchy problems on $[0,T]$ with $k \geq 1$:

$$\begin{cases} {}^C D^{\alpha}_{0+} u_0(t) = -a(t)u_0(t) + F_0(t,u), \\ u_0(0) = \varphi_0; \end{cases} \tag{3.59}$$

$$\begin{cases} {}^C D^{\alpha}_{0+} u_{2k-1}(t) = -\lambda_k u_{2k-1}(t) - a(t)u_{2k-1}(t) + F_{2k-1}(t,u), \\ u_{2k-1}(0) = \varphi_{2k-1}; \end{cases} \tag{3.60}$$

$$\begin{cases} {}^C D^{\alpha}_{0+} u_{2k}(t) = -\lambda_k u_{2k}(t) + 2\sqrt{\lambda_k} u_{2k-1}(t) - a(t)u_{2k}(t) + F_{2k}(t,u), \\ u_{2k}(0) = \varphi_{2k}; \end{cases} \tag{3.61}$$

respectively, where

$$F_k(t,u) = \int_0^1 F(x,t,u(x,t))Y_k dx, \quad \varphi_k = \int_0^1 \varphi(x)Y_k dx, \quad \text{for } k \geq 0$$

are the coefficients of the series expansion of $F(x,t,u(x,t))$ and $\varphi(x)$ in the basis (S_2). Applying the integral operator I^{α}_{0+} to the fractional differential equation of the Cauchy problem (3.59) and from Remark 1.8, we get

$$u_0(t) = \varphi_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[a(s)u_0(s) - F_0(s,u)\right] ds. \tag{3.62}$$

According to Theorem 1.12, the solutions of problems (3.60)-(3.61) satisfy

$$\begin{aligned} u_{2k-1}(t) = {} & \varphi_{2k-1}E_{\alpha,1}(-\lambda_k t^{\alpha}) \\ & - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k (t-s)^{\alpha}) \left[a(s)u_{2k-1}(s) - F_{2k-1}(s,u)\right] ds; \end{aligned} \tag{3.63}$$

$$\begin{aligned} u_{2k}(t) = {} & \varphi_{2k}E_{\alpha,1}(-\lambda_k t^{\alpha}) \\ & + 2\sqrt{\lambda_k} \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k (t-s)^{\alpha}) u_{2k-1}(s)ds \\ & - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k (t-s)^{\alpha}) \left[a(s)u_{2k}(s) - F_{2k}(s,u)\right] ds; \end{aligned} \tag{3.64}$$

for $k \geq 1$, respectively.

Now, applying ${}^{C}D^{\alpha}_{0^+}$ to the additional condition (3.7) we obtain

$$\begin{aligned}
{}^{C}D^{\alpha}_{0^+}H(t) &= \int_0^1 {}^{C}\partial^{\alpha}_{0^+,t}u(x,t)dx \\
&= \int_0^1 \left[u_{xx}(x,t) - a(t)u(x,t) + F(x,t,u(x,t))\right]dx \\
&= u_x(1,t) - u_x(0,t) - a(t)H(t) + \int_0^1 F(x,t,u(x,t))dx,
\end{aligned}$$

which yields

$$a(t) = \frac{\int_0^1 F(x,t,u(x,t))dx - {}^{C}D^{\alpha}_{0^+}H(t)}{H(t)}. \tag{3.65}$$

Definition 3.4. *We denote $\mathscr{N}$ the set of continuous functions on $[0,T]$*

$$\{u(t)\} = \{u_0(t), u_{2k-1}(t), u_{2k}(t); \quad k \geq 1\}$$

satisfying

$$2|u_0(t)| + 4\sum_{k\geq 1}\left(|u_{2k-1}(t)| + |u_{2k}(t)|\right) < \infty \; for \; t \in [0,T].$$

It can be shown that $(\mathscr{N}, \|.\|_{\mathscr{N}})$ is a Banach space with

$$\|u\|_{\mathscr{N}} = 2\max_{0\leq t\leq T}|u_0(t)| + 4\sum_{k\geq 1}\left(\max_{0\leq t\leq T}|u_{2k-1}(t)| + \max_{0\leq t\leq T}|u_{2k}(t)|\right).$$

Lemma 3.2. *If (H_1) is satisfied, then the series $\sum_{k\geq 1}|\varphi_{2k-1}|$, $\sum_{k\geq 1}|\varphi_{2k}|$ are uniformly convergent. Moreover for some positive constant $\tilde{C}$, we have*

$$|\varphi_0| + \sum_{k\geq 1}\left(|\varphi_{2k-1}| + |\varphi_{2k}|\right) \leq \tilde{C}\|\varphi\|_{C^4[0,1]}.$$

Proof. First, we have by Cauchy's inequality

$$|\varphi_0| = \left|\int_0^1 \varphi(x)dx\right| \leq \|\varphi\|_{L^2[0,1]}.$$

By integration by parts four times, we get for $k \geq 1$

$$\begin{aligned}
\varphi_{2k-1} &= \int_0^1 \varphi(x)\cos(2k\pi x)dx = \frac{1}{(2k\pi)^4}\varphi^{(4)}_{2k-1}; \\
\varphi_{2k} &= \int_0^1 \varphi(x)x\sin(2k\pi x)dx = \frac{1}{(2k\pi)^4}\varphi^{(4)}_{2k} + \frac{4}{(2k\pi)^5}\varphi^{(4)}_{2k-1}.
\end{aligned}$$

Taking the sum and using Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{k\geq 1}|\varphi_{2k-1}| &\leq \left(\sum_{k\geq 1}\frac{1}{(2k\pi)^8}\right)^{1/2}\left(\sum_{k\geq 1}|\varphi^{(4)}_{2k-1}|^2\right)^{1/2} \leq \tilde{C}_1\|\varphi^{(4)}\|_{L^2(0,1)}; \\
\sum_{k\geq 1}|\varphi_{2k}| &\leq \sum_{k\geq 1}\frac{1}{(2k\pi)^4}|\varphi^{(4)}_{2k}| + \sum_{k\geq 1}\frac{4}{(2k\pi)^5}|\varphi^{(4)}_{2k-1}| \leq \tilde{C}_2\|\varphi^{(4)}\|_{L^2[0,1]}.
\end{aligned}$$

Thus, we conclude the uniform convergence of the above series. Moreover

$$\begin{aligned}|\varphi_0|+\sum_{k\geq 1}|\varphi_{2k-1}|+\sum_{k\geq 1}|\varphi_{2k}| &\leq \|\varphi\|_{L^2[0,1]}+\left(\tilde{C}_1+\tilde{C}_2\right)\|\varphi^{(4)}\|_{L^2[0,1]}\\ &\leq \tilde{C}\|\varphi\|_{C^4[0,1]}.\end{aligned}$$

□

Remark 3.3. *Similarly, from (H_1)-(H_2) we deduce the uniform convergence of the following series:*

$$\sum_{k\geq 1}\lambda_k|\varphi_{2k}|,\ \sum_{k\geq 1}\lambda_k|F_{2k}(t,u)|$$

and

$$\sum_{k\geq 1}(\sqrt{\lambda_k})^n|\varphi_{2k-1}|,\ \sum_{k\geq 1}(\sqrt{\lambda_k})^n|F_{2k-1}(t,u)|\ \text{for}\ n=1,2,3.$$

Theorem 3.3. *Let (H_1)-(H_3) be satisfied. Then the inverse problem $(P2)$ has a unique classical solution $\{u(x,t),a(t)\}$ on $[0,T]$ for a large T and $1/2<\alpha<1$.*

Proof. To establish the proof of the above theorem, we proceed in six steps:
Step 1: We define an iteration for the Fourier coefficient of (3.57) for $N\geq 0$, as follows:

$$u_0^{(N+1)}(t)=H(t); \tag{3.66}$$

$$\begin{aligned}u_{2k-1}^{(N+1)}(t)=u_{2k-1}^{(0)}(t)-\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a^{(N)}(s)u_{2k-1}^{(N+1)}(s)ds\\ +\int_0^t\int_0^1(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)F(x,s,u^{(N)}(x,s))\cos(\sqrt{\lambda_k}x)dxds;\end{aligned} \tag{3.67}$$

$$\begin{aligned}u_{2k}^{(N+1)}(t)=u_{2k}^{(0)}(t)+2\sqrt{\lambda_k}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2k-1}^{(N+1)}(s)ds\\ -\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a^{(N)}(s)u_{2k}^{(N+1)}(s)ds\\ +\int_0^t\int_0^1(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)F(x,s,u^{(N)}(x,s))x\sin(\sqrt{\lambda_k}x)dxds;\end{aligned} \tag{3.68}$$

with

$$u_0^{(0)}(t)=\varphi_0,\quad u_{2k-1}^{(0)}(t)=\varphi_{2k-1}E_{\alpha,1}(-\lambda_k t^\alpha),\quad u_{2k}^{(0)}(t)=\varphi_{2k}E_{\alpha,1}(-\lambda_k t^\alpha). \tag{3.69}$$

Then, we define an iteration for (3.65) on $[0,T]$ for $N\geq 0$, as follows:

$$a^{(N)}(t)=\frac{1}{H(t)}\left[-{}^CD_{0+}^\alpha H(t)+\int_0^1 F(x,t,u^{(N)}(x,t))dx\right]. \tag{3.70}$$

Step 2: Let us show that the iterations $u^{(N)}(t)$ and $a^{(N)}(t)$ are in $\mathscr{N}$ and $C[0,T]$ respectively. It is clear that, by Definition 3.4, the estimation of Mittag-Leffer type function in (3.8) and Lemma 3.2, we get

$$|u^{(0)}|\leq|\varphi_0|+N_{\alpha 1}\sum_{k\geq 1}(|\varphi_{2k-1}|+|\varphi_{2k}|).$$

Then, from (3.9) we find

$$\|u^{(0)}\|_{\mathscr{N}}\leq|\varphi_0|+\sum_{k\geq 1}(|\varphi_{2k-1}|+|\varphi_{2k}|)\leq\tilde{C}\|\varphi\|_{C^4[0,1]},$$

thus $u^{(0)}\in\mathscr{N}$.

We take $N=0$ in (3.70), to get

$$a^{(0)}(t)=\frac{1}{H(t)}\left[-{}^{C}D_{0+}^{\alpha}H(t)+\int_{0}^{1}F(x,t,u^{(0)}(x,t))dx\right]>0.$$

Let us add and subtract $\frac{1}{H(t)}\int_0^1 F(x,t,0)dx$ to the last equation and we use the Lipschitz condition to obtain

$$\begin{aligned}a^{(0)}(t)&\leq\frac{|{}^{C}D_{0+}^{\alpha}H(t)|}{H(t)}+\frac{1}{H(t)}\left|\int_0^1\left[F(x,t,u^{(0)}(x,t))-F(x,t,0)\right]dx\right|\\&+\frac{1}{H(t)}\int_0^1 F(x,t,0)dx\\&\leq\frac{|{}^{C}D_{0+}^{\alpha}H(t)|}{H(t)}+\frac{1}{H(t)}\int_0^1 h(x,t)\left|u^{(0)}(x,t)\right|dx+\frac{1}{H(t)}\int_0^1 F(x,t,0)dx.\end{aligned}$$

From Cauchy's inequality, (H_2) and (H_3), we find

$$a^{(0)}(t)\leq H_m^{-1}\left[\|{}^{C}D_{0+}^{\alpha}H\|_{C[0,T]}+\|h(.,t)\|_{L^2(0,1)}\|u^{(0)}\|_{\mathscr{N}}+\|F(.,t,0)\|_{L^2(0,1)}\right].$$

then, we obtain

$$\|a^{(0)}\|_{C[0,T]}\leq H_m^{-1}\left(\|H\|_{AC[0,T]}+M_h\|u^{(0)}\|_{\mathscr{N}}+M\right).$$

Since $u^{(0)}\in\mathscr{N}$, we deduce that $a^{(0)}\in C[0,T]$.
Now, from (3.66) we get easily

$$\max_{0\leq t\leq T}|u_0^{(N+1)}(t)|\leq\|H\|_{C[0,T]},\quad \text{for } N\geq 0. \tag{3.71}$$

Next, we consider $u_{2k-1}^{(1)}(t)$ and $u_{2k}^{(1)}(t)$ are given by (3.67) and (3.68) for $N=0$ respectively, using (3.8) and $a^{(0)}\in C[0,T]$, we obtain

$$\begin{aligned}|u_{2k-1}^{(1)}(t)|&\leq N_{\alpha 1}|\varphi_{2k-1}|+N_\alpha\|a^{(0)}\|_{C[0,T]}\int_0^t(t-s)^{\alpha-1}\left|u_{2k-1}^{(1)}(s)\right|ds\\&+N_\alpha\left|\int_0^t\int_0^1(t-s)^{\alpha-1}F(x,s,u^{(0)}(x,s))\cos(2\pi kx)dxds\right|,\end{aligned}$$

from Cauchy-Schwartz's inequality and (3.9), we find

$$\begin{aligned}|u_{2k-1}^{(1)}(t)|&\leq|\varphi_{2k-1}|+\|a^{(0)}\|_{C[0,T]}\left(\int_0^t(t-s)^{2\alpha-2}ds\right)^{1/2}\left(\int_0^t|u_{2k-1}^{(1)}(t)|^2ds\right)^{1/2}\\&+\left|\int_0^t\int_0^1(t-s)^{\alpha-1}F(x,s,u^{(0)}(x,s))\cos(2\pi kx)dxds\right|.\end{aligned}$$

Applying **[Tec K]** (see p. 54) to the last integral, then integrating twice by parts the integrals depending on F with respect to x on $[0,1]$, we get

$$\begin{aligned}|u_{2k-1}^{(1)}(t)|&\leq|\varphi_{2k-1}|+\|a^{(0)}\|_{C[0,T]}L_\alpha\left(\int_0^t|u_{2k-1}^{(1)}(t)|^2ds\right)^{1/2}\\&+\frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1[F_{xx}(x,s,u^{(0)}(x,s))-F_{xx}(x,s,0)]\frac{\cos(2\pi kx)}{k^2}dx\right)^2ds\right)^{1/2}\\&+\frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1F_{xx}(x,s,0)\frac{\cos(2\pi kx)}{k^2}dx\right)^2ds\right)^{1/2},\end{aligned}$$

where L_α is given by (3.11). Applying **[Tec G]** (see p. 54) and after that taking the maximum, we get

$$\max_{0\le t\le T}|u^{(1)}_{2k-1}(t)| \le \sqrt{2}\exp\left(\|a^{(0)}\|^2_{C[0,T]}L^2_\alpha T\right)$$
$$\times\left[|\varphi_{2k-1}| + \frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\frac{\cos(2\pi kx)}{k^2}dx\right)^2 ds\right)^{1/2}\right.$$
$$\left.+\frac{L_\alpha}{4\pi^2}\left(\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\frac{\cos(2\pi kx)}{k^2}dx\right)^2 ds\right)^{1/2}\right].$$

Taking the sum of both sides of the last inequality, we find

$$\sum_{k\ge1}\max_{0\le t\le T}|u^{(1)}_{2k-1}(t)| \le A\left(a^{(0)}\right)\sum_{k\ge1}|\varphi_{2k-1}| + A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}$$
$$\times\sum_{k\ge1}\frac{1}{k^2}\left(\int_0^T\left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}$$
$$+A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\sum_{k\ge1}\frac{1}{k^2}\left(\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\cos(2\pi kx)dx\right)^2 ds\right)^{1/2},$$

where

$$A\left(a^{(0)}\right) = \sqrt{2}\exp\left(\|a^{(0)}\|^2_{C[0,T]}L^2_\alpha T\right). \tag{3.72}$$

Then, Hölder's inequality gives

$$\sum_{k\ge1}\max_{0\le t\le T}|u^{(1)}_{2k-1}(t)| \le A\left(a^{(0)}\right)\sum_{k\ge1}|\varphi_{2k-1}| + A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\left(\sum_{k\ge1}\frac{1}{k^4}\right)^{1/2}$$
$$\times\left(\sum_{k\ge1}\int_0^T\left(\int_0^1 [F_{xx}(x,s,u^{(0)}(x,s)) - F_{xx}(x,s,0)]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}$$
$$+A\left(a^{(0)}\right)\frac{L_\alpha}{4\pi^2}\left(\sum_{k\ge1}\frac{1}{k^4}\right)^{1/2}\left(\sum_{k\ge1}\int_0^T\left(\int_0^1 F_{xx}(x,s,0)\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}.$$

Using the third fact of (3.10), then from Bessel's inequality and the Lipschitz condition, we have

$$\sum_{k\ge1}\max_{0\le t\le T}|u^{(1)}_{2k-1}(t)| \le A\left(a^{(0)}\right)\sum_{k\ge1}|\varphi_{2k-1}|$$
$$+A\left(a^{(0)}\right)\frac{L_\alpha}{12\sqrt{10}}\left(\int_0^T\int_0^1 h^2(x,t)|u^{(0)}|^2 dxds\right)^{1/2}$$
$$+A\left(a^{(0)}\right)\frac{L_\alpha}{12\sqrt{10}}\left(\int_0^T\int_0^1 (F_{xx}(x,s,0)dx)^2 ds\right)^{1/2},$$

and by $(H_2)-2-3$, we get

$$\sum_{k\ge1}\max_{0\le t\le T}|u^{(1)}_{2k-1}(t)| \le A\left(a^{(0)}\right)\left[\sum_{k\ge1}|\varphi_{2k-1}| + \frac{L_\alpha}{12\sqrt{10}}\left[B_h\|u^{(0)}\|_{\mathscr{N}} + M_F\right]\right]. \tag{3.73}$$

In the same way, according to (3.8) and $a^{(0)} \in C[0,T]$, we have

$$\begin{aligned}|u_{2k}^{(1)}(t)| \le N_{\alpha 1}|\varphi_{2k}| + \frac{N_\lambda}{\pi k}\max_{0\le t\le T}|u_{2k-1}^{(1)}(t)|\int_0^t (t-s)^{\alpha-1}ds \\ + N_\alpha \|a^{(0)}\|_{C[0,T]}\int_0^t (t-s)^{\alpha-1}\left|u_{2k}^{(1)}(s)\right|ds \\ + N_\alpha\left|\int_0^t\int_0^1 (t-s)^{\alpha-1}F(x,s,u^{(0)}(x,s))x\sin(2\pi kx)dxds\right|.\end{aligned}$$

Applying Cauchy-Schwartz's inequality and **[Tec K]** (see p. 54), after that integrating by parts the integrals depending on F with respect to x on $[0,1]$, we find

$$\begin{aligned}|u_{2k}^{(1)}(t)| \le N_{\alpha 1}|\varphi_{2k}| + \frac{N_\lambda T^\alpha}{\pi k\alpha}\max_{0\le t\le T}|u_{2k-1}^{(1)}(t)| + N_\alpha L_\alpha\|a^{(0)}\|_{C[0,T]}\left(\int_0^t |u_{2k}^{(1)}(s)|^2 ds\right)^{1/2} \\ + \frac{L_\alpha}{2\pi}\left(\int_0^t\left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)]\frac{\cos(2\pi kx)}{k}dx\right)^2 ds\right)^{1/2} \\ + \frac{L_\alpha}{2\pi}\left(\int_0^t\left(\int_0^1 F_x(x,s,0)\frac{\cos(2\pi kx)}{k}dx\right)^2 ds\right)^{1/2}.\end{aligned}$$

Applying **[Tec G]** (see p. 54) to the last inequality and from (3.9), we get

$$\begin{aligned}|u_{2k}^{(1)}(t)| \le \sqrt{2}\exp\left(\|a^{(0)}\|_{C[0,T]}^2 L_\alpha^2 T\right)\Bigg[|\varphi_{2k}| + \frac{N_\lambda T^\alpha}{\alpha\pi}\frac{1}{k}\max_{0\le t\le T}|u_{2k-1}^{(1)}(t)| \\ + \frac{L_\alpha}{2\pi}\left(\int_0^T\left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)]\frac{\cos(2\pi kx)}{k}dx\right)^2 ds\right)^{1/2} \\ + \frac{L_\alpha}{2\pi}\left(\int_0^T\left(\int_0^1 F_x(x,s,0)\frac{\cos(2\pi kx)}{k}dx\right)^2 ds\right)^{1/2}\Bigg].\end{aligned}$$

Taking the maximum and the sum of both sides of the last inequality, by Hölder's inequality, we get

$$\begin{aligned}\sum_{k\ge1}\max_{0\le t\le T}|u_{2k}^{(1)}(t)| \le A\left(a^{(0)}\right)\sum_{k\ge1}|\varphi_{2k}| \\ + A\left(a^{(0)}\right)\frac{N_\lambda T^\alpha}{\alpha\pi}\left(\sum_{k\ge1}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k\ge1}\left(\max_{0\le t\le T}|u_{2k-1}^{(1)}(t)|\right)^2\right)^{1/2} \\ + A\left(a^{(0)}\right)\frac{L_\alpha}{2\pi}\left(\sum_{k\ge1}\frac{1}{k^2}\right)^{1/2} \\ \times\left(\sum_{k\ge1}\int_0^T\left(\int_0^1 [F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2} \\ + A\left(a^{(0)}\right)\frac{L_\alpha}{2\pi}\left(\sum_{k\ge1}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k\ge1}\int_0^T\left(\int_0^1 F_x(x,s,0)\cos(2\pi kx)dx\right)^2 ds\right)^{1/2},\end{aligned}$$

where $A\left(a^{(0)}\right)$ is defined by (3.72). From (3.10) and Bessel's inequality, we get

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2k}^{(1)}(t)| \leq A\left(a^{(0)}\right)\sum_{k\geq 1}|\varphi_{2k}| + A\left(a^{(0)}\right)\frac{L_\lambda}{\sqrt{6}}\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2k-1}^{(1)}(t)|$$
$$+A\left(a^{(0)}\right)\frac{L_\alpha}{2\sqrt{6}}\left(\int_0^T\int_0^1\left|F_x(x,s,u^{(0)}(x,s)) - F_x(x,s,0)\right|^2 dxds\right)^{1/2}$$
$$+A\left(a^{(0)}\right)\frac{L_\alpha}{2\sqrt{6}}\|F_x(.,.,0)\|_{L^2(D_T)},$$

where $L_\lambda = \dfrac{N_\lambda T^\alpha}{\alpha}$. The Lipschitz condition, $(H_2)-3$ and estimation (3.73) give

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2k}^{(1)}(t)| \leq A\left(a^{(0)}\right)\sum_{k\geq 1}|\varphi_{2k}| + \left(A\left(a^{(0)}\right)\right)^2\frac{L_\lambda}{\sqrt{6}}\sum_{k\geq 1}|\varphi_{2k-1}| \tag{3.74}$$
$$+A\left(a^{(0)}\right)L_\alpha\left[A\left(a^{(0)}\right)\frac{L_\lambda}{24\sqrt{15}} + \frac{1}{2\sqrt{6}}\right]\left[B_h\|u^{(0)}\|_{\mathcal{N}} + M_F\right].$$

According the estimates (3.71),(3.73), (3.74) and the fact that

$$\|u^{(1)}\|_{\mathcal{N}} = 2\max_{0\leq t\leq T}|u_0^{(1)}(t)| + 4\sum_{k\geq 1}\left(\max_{0\leq t\leq T}|u_{2k-1}^{(1)}(t)| + \max_{0\leq t\leq T}|u_{2k}^{(1)}(t)|\right),$$

we get

$$\|u^{(1)}\|_{\mathcal{N}} \leq 2\|H\|_{C[0,T]} + Q_1(a^0)\|\varphi\|_{C^4[0,1]} + Q_2(a^0)\left[B_h\|u^{(0)}\|_{\mathcal{N}} + M_F\right],$$

where

$$Q_1(a^0) = 4\left(A\left(a^{(0)}\right) + \frac{A^2\left(a^{(0)}\right)L_\lambda}{\sqrt{6}}\right),$$
$$Q_2(a^0) = 4A\left(a^{(0)}\right)L_\alpha\left[\frac{1}{12\sqrt{10}} + \frac{A\left(a^{(0)}\right)L_\lambda}{24\sqrt{15}} + \frac{1}{2\sqrt{6}}\right].$$

As a result $u^{(1)} \in \mathcal{N}$. In the same way, by induction we get for general value of $N \geq 1$

$$\|u^{(N)}\|_{\mathcal{N}} \leq 2\|H\|_{C[0,T]} + Q_1(a^{(N-1)})\|\varphi\|_{C^4[0,1]} + Q_2(a^{(N-1)})\left[B_h\|u^{(N-1)}\|_{\mathcal{N}} + M_F\right].$$

Since, by induction $u^{(N-1)} \in \mathcal{N}$ and $a^{(N-1)} \in C[0,T]$, then $u^{(N)} \in \mathcal{N}$. Also, $a^{(N)}(t) > 0$ on $[0,T]$ for general value of N, we have

$$\|a^{(N)}\|_{C[0,T]} \leq H_m^{-1}\left(\|H\|_{AC[0,T]} + M_h\|u^{(N)}\|_{\mathcal{N}} + M\right).$$

Finally, since $u^{(N)} \in \mathcal{N}$, we deduce that $a^{(N)} \in C[0,T]$.

Step 3: We will study the convergence of the bounded iterations $\left(u^{(N+1)}(t)\right)_{N\geq 0}$ and $\left(a^{(N)}(t)\right)_{N\geq 0}$ in $\mathcal{N}$ and $C[0,T]$ respectively. Let us denote

$$L_a = \max\{\|a^{(N)}\|_{C[0,T]}; N\geq 0\};\ L_u = \max\{\|u^{(N)}\|_{\mathcal{N}}; N\geq 0\}. \tag{3.75}$$

First, from (3.70), we have for $N \geq 0$

$$a^{(N+1)}(t) - a^{(N)}(t) = \frac{1}{H(t)}\left[\int_0^1 F(x,t,u^{(N+1)}(x,t))dx - \int_0^1 F(x,t,u^{(N)}(x,t))dx\right],$$

by using the Lipschitz condition and (H_3), we find

$$|a^{(N+1)}(t) - a^{(N)}(t)| \leq H_m^{-1}\int_0^1 h(x,t)|u^{(N+1)}(x,t) - u^{(N)}(x,t)|dx. \tag{3.76}$$

Then, Cauchy's inequality gives

$$\|a^{(N+1)} - a^{(N)}\|_{C[0,T]} \leq H_m^{-1}M_h\|u^{(N+1)} - u^{(N)}\|_{\mathscr{N}}. \tag{3.77}$$

Next, to estimate $\|u^{(N+1)} - u^{(N)}\|_{\mathscr{N}}$ for $N = 0$, we start with the first term

$$u_0^{(1)}(t) - u_0^{(0)}(t) = H(t) - \varphi_0,$$

then

$$\max_{0\leq t\leq T}|u_0^{(1)}(t) - u_0^{(0)}(t)| \leq \|H\|_{C[0,T]} + \|\varphi\|_{C^4[0,1]}. \tag{3.78}$$

Then, applying the same estimation used in **Step 2** on

$$u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t) = -\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a^{(0)}(s)u_{2k-1}^{(1)}(s)ds$$
$$+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)F(x,s,u^{(0)}(x,s))\cos(2\pi kx)dxds,$$

we obtain

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t)| \leq \frac{T^\alpha}{\alpha}L_a\|u^{(1)}\|_{\mathscr{N}} + \frac{L_\alpha}{12\sqrt{10}}\left[B_h\|u^{(0)}\|_{\mathscr{N}} + M_F\right]. \tag{3.79}$$

In the same manner for

$$u_{2k}{}^{(1)}(t) - u_{2k}^{(0)}(t) = 2\sqrt{\lambda_k}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2k-1}^{(1)}(s)ds$$
$$-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a^{(0)}(s)u_{2k}^{(1)}(s)ds$$
$$+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)F(x,s,u^{(0)}(x,s))x\sin(2\pi kx)dxds,$$

we get

$$\sum_{k\geq 1}\max_{0\leq t\leq T}|u_{2k}^{(1)}(t) - u_{2k}^{(0)}(t)| \leq \left[\frac{L_\lambda}{\sqrt{6}} + \frac{T^\alpha L_a}{\alpha}\right]\|u^{(1)}\|_{\mathscr{N}} + \frac{L_\alpha}{2\sqrt{6}}\left[B_h\|u^{(0)}\|_{\mathscr{N}} + M_F\right]. \tag{3.80}$$

Consequently, from (3.78),(3.79) and (3.80) we find

$$\|u^{(1)} - u^{(0)}\|_{\mathscr{N}} \leq 2\|H\|_{C[0,T]} + 2\|\varphi\|_{C^4[0,1]} + 4\left(\frac{2L_aT^\alpha}{\alpha} + \frac{L_\lambda}{\sqrt{6}}\right)L_u$$
$$+4\left(\frac{L_\alpha}{12\sqrt{10}} + \frac{L_\alpha}{2\sqrt{6}}\right)\left[B_h\|u^{(0)}\|_{\mathscr{N}} + M_F\right] := \mathscr{F}_0.$$

For $N=1$, to estimate $\|u^{(2)}-u^{(1)}\|_{\mathcal{N}}$, we start with the fact

$$|u_0^{(N+1)}(t)-u_0^{(N)}(t)|=0, \quad N\geq 1. \tag{3.81}$$

Then, to estimate the following expression

$$u_{2k-1}^{(2)}(t)-u_{2k-1}^{(1)}(t)=-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})a^{(1)}(s)\left[u_{2k-1}^{(2)}(s)-u_{2k-1}^{(1)}(s)\right]ds$$
$$-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})u_{2k-1}^{(1)}(s)\left[a^{(1)}(s)-a^{(0)}(s)\right]ds$$
$$+\int_0^t\int_0^1 (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})\left[F(x,s,u^1(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi kx)dxds,$$

we use (3.8), (3.75), Cauchy's inequality and estimate (3.76), we get

$$|u_{2k-1}^{(2)}(t)-u_{2k-1}^{(1)}(t)|\leq N_\alpha L_a L_\alpha\left(\int_0^t\left|u_{2k-1}^{(2)}(s)-u_{2k-1}^{(1)}(s)\right|^2 ds\right)^{1/2}$$
$$+N_\alpha\max_{0\leq t\leq T}\left|u_{2k-1}^{(1)}(t)\right|H_m^{-1}L_\alpha\left(\int_0^t\left(\int_0^1 h(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2 ds\right)^{1/2}$$
$$+N_\alpha L_\alpha\left(\int_0^t\left(\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}.$$

Applying **[Tec G]** (see p. 54) and from (3.9), we obtain

$$|u_{2k-1}^{(2)}(t)-u_{2k-1}^{(1)}(t)|\leq\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)\max_{0\leq t\leq T}\left|u_{2k-1}^{(1)}(t)\right|H_m^{-1}L_\alpha \tag{3.82}$$
$$\times\left(\int_0^t\left(\int_0^1 h(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2 ds\right)^{1/2}+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)L_\alpha$$
$$\times\left(\int_0^t\left(\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2}.$$

Taking the sum of both sides of the obtained inequality, we integrate twice by parts the terms depending on F with respect to x, by Hölder's inequality, we get

$$\sum_{k\geq 1}|u_{2k-1}^{(2)}(t)-u_{2k-1}^{(1)}(t)|\leq\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)\sum_{k\geq 1}\max_{0\leq t\leq T}\left|u_{2k-1}^{(1)}(t)\right|$$
$$\times H_m^{-1}L_\alpha\left(\int_0^t\left(\int_0^1 h(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2 ds\right)^{1/2}$$
$$+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)\frac{L_\alpha}{4\pi^2}\left(\sum_{k\geq 1}\frac{1}{k^4}\right)^{1/2}$$
$$\times\left(\sum_{k\geq 1}\int_0^t\left(\int_0^1\left[F_{xx}(x,s,u^{(1)}(x,s))-F_{xx}(x,s,u^{(0)}(x,s))\right]\cos(2\pi kx)dx\right)^2 ds\right)^{1/2},$$

from Bessel's inequality and the Lipschitz condition, we find

$$\sum_{k\geq 1}|u_{2k-1}^{(2)}(t)-u_{2k-1}^{(1)}(t)| \leq \mathscr{H}_1\left(\int_0^t\int_0^1 h^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2} \tag{3.83}$$

with

$$\mathscr{H}_1=\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)\left(L_uL_\alpha H_m^{-1}+\frac{L_\alpha}{12\sqrt{10}}\right). \tag{3.84}$$

In the same way, using previous approximation techniques for

$$\begin{aligned}
&u_{2k}^{(2)}(t)-u_{2k}^{(1)}(t)\\
&\quad=2\sqrt{\lambda_k}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)\left[u_{2k-1}^{(2)}(s)-u_{2k-1}^{(1)}(s)\right]ds\\
&\quad-\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)a^{(1)}(s)\left[u_{2k}^{(2)}(s)-u_{2k}^{(1)}(s)\right]ds\\
&\quad-\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)u_{2k}^{(1)}(s)\left[a^{(1)}(s)-a^{(0)}(s)\right]ds\\
&\quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)\\
&\quad\times\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]x\sin(2k\pi x)dxds,
\end{aligned}$$

from (3.8), Cauchy's inequality and estimation (3.76), we get

$$\begin{aligned}
&|u_{2k}^{(2)}(t)-u_{2k}^{(1)}(t)|\\
&\quad\leq N_\lambda\int_0^t(t-s)^{\alpha-1}\frac{1}{k\pi}|u_{2k-1}^{(2)}(s)-u_{2k-1}^{(1)}(s)|ds\\
&\quad+N_\alpha L_\alpha L_a\left(\int_0^t|u_{2k}^{(2)}(s)-u_{2k}^{(1)}(s)|^2ds\right)^{1/2}\\
&\quad+N_\alpha H_m^{-1}L_\alpha\max_{0\leq t\leq T}|u_{2k}^{(1)}(t)|\left(\int_0^t\left(\int_0^1h(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\
&\quad+N_\alpha L_\alpha\left(\int_0^t\left(\int_0^1\left[F_x(x,s,u^{(1)}(x,s))-F_x(x,s,u^{(0)}(x,s))\right]\frac{\cos(2k\pi x)}{2\pi k}dx\right)^2ds\right)^{1/2}.
\end{aligned}$$

Applying **[Tec G]** (see p. 54) and using (3.9), we get

$$\begin{aligned}
&|u_{2k}^{(2)}(t)-u_{2k}^{(1)}(t)|\\
&\quad\leq\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)N_\lambda\int_0^t(t-s)^{\alpha-1}\frac{1}{\pi k}|u_{2k-1}^{(2)}(s)-u_{2k-1}^{(1)}(s)|ds\\
&\quad+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)H_m^{-1}L_\alpha\max_{0\leq t\leq T}|u_{2k}^{(1)}(t)|\\
&\quad\times\left(\int_0^t\left(\int_0^1h(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}\\
&\quad+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)L_\alpha\\
&\quad\times\left(\int_0^t\left(\int_0^1\left[F_x(x,s,u^{(1)}(x,s))-F_x(x,s,u^{(0)}(x,s))\right]\frac{\cos(2k\pi x)}{2\pi k}dx\right)^2ds\right)^{1/2}.
\end{aligned} \tag{3.85}$$

Taking estimation (3.82) into consideration, we get concerning the following term

$$\sum_{k\geq 1}\frac{1}{k\pi}|u^{(2)}_{2k-1}(s)-u^{(1)}_{2k-1}(s)| \leq \sqrt{2}\exp\left((L_\alpha L_a)^2T\right)H_m^{-1}L_\alpha$$
$$\times\sum_{k\geq 1}\frac{1}{k\pi}\max_{0\leq t\leq T}\left|u^{(1)}_{2k-1}(t)\right|\left(\int_0^t\left(\int_0^1 h(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|dx\right)^2ds\right)^{1/2}$$
$$+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)L_\alpha\sum_{k\geq 1}\frac{1}{k\pi}$$
$$\times\left(\int_0^t\left(\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]\cos(2\pi kx)dx\right)^2ds\right)^{1/2},$$

then, the Hölder and Bessel inequalities give that

$$\sum_{k\geq 1}\frac{1}{k\pi}|u^{(2)}_{2k-1}(s)-u^{(1)}_{2k-1}(s)| \leq \sqrt{2}\exp\left((L_\alpha L_a)^2T\right)H_m^{-1}\frac{L_\alpha}{\pi}\left(\sum_{k\geq 1}\frac{1}{k^2}\right)^{1/2}$$
$$\times\left(\sum_{k\geq 1}\left(\max_{0\leq t\leq T}\left|u^{(1)}_{2k-1}(t)\right|\right)^2\right)^{1/2}\left(\int_0^t\int_0^1 h^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2}$$
$$+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)L_\alpha\frac{1}{\pi}\left(\sum_{k\geq 1}\frac{1}{k^2}\right)^{1/2}$$
$$\times\left(\int_0^t\int_0^1\left[F(x,s,u^{(1)}(x,s))-F(x,s,u^{(0)}(x,s))\right]^2dxds\right)^{1/2}.$$

At last, from the Lipschitz condition we deduce

$$\sum_{k\geq 1}\frac{1}{k\pi}|u^{(2)}_{2k-1}(s)-u^{(1)}_{2k-1}(s)| \leq \sqrt{\frac{2}{6}}\exp\left((L_\alpha L_a)^2T\right)L_\alpha\left[H_m^{-1}L_u+1\right]$$
$$\times\left(\int_0^t\int_0^1 h^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2},$$

which yields

$$\int_0^t(t-s)^{\alpha-1}\sum_{k\geq 1}\frac{1}{k\pi}|u^{(2)}_{2k-1}(s)-u^{(1)}_{2k-1}(s)|ds \leq \sqrt{\frac{2}{6}}\exp\left((L_\alpha L_a)^2T\right)L_\alpha\left[H_m^{-1}L_u+1\right]$$
$$\times\left(\int_0^t\int_0^1 h^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2}\int_0^t(t-s)^{\alpha-1}ds. \tag{3.86}$$

Taking the sum of both sides of the inequality (3.85), according to (3.86), Hölder's inequality, Bessel's inequality and the Lipschitz condition, we obtain

$$\sum_{k\geq 1}|u^{(2)}_{2k}(t)-u^{(1)}_{2k}(t)| \leq \mathscr{H}_2\left(\int_0^t\int_0^1 h^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2}, \tag{3.87}$$

with

$$\mathscr{H}_2=\frac{2T^\alpha}{\alpha\sqrt{6}}\left(\exp\left((L_\alpha L_a)^2T\right)\right)^2L_\alpha N_\lambda\left(H_m^{-1}L_u+1\right) \tag{3.88}$$
$$+\sqrt{2}\exp\left((L_\alpha L_a)^2T\right)L_\alpha\left(H_m^{-1}L_u+\frac{1}{2\sqrt{6}}\right).$$

Thus, from formula (3.81), (3.83) and (3.87) we deduce

$$|u^{(2)}(t)-u^{(1)}(t)| \leq \mathscr{H}\left(\int_0^t\int_0^1 h^2(x,s)|u^{(1)}(x,s)-u^{(0)}(x,s)|^2dxds\right)^{1/2}, \tag{3.89}$$

where $\mathscr{H}=4[\mathscr{H}_1+\mathscr{H}_2]$, $\mathscr{H}_1$ and $\mathscr{H}_2$ are defined by (3.84) and (3.88) respectively.
As a result

$$\|u^{(2)}-u^{(1)}\|_{\mathscr{N}} \leq \mathscr{H}B_h\|u^{(1)}-u^{(0)}\|_{\mathscr{N}}.$$

Applying the same estimation, we obtain for $N=2$

$$\begin{aligned}
|u^{(3)}(t)-u^{(2)}(t)| &\leq \mathscr{H}\left(\int_0^t\int_0^1 h^2(x,s)|u^{(2)}(s)-u^{(1)}(s)|^2dxds\right)^{1/2}\\
&\leq \mathscr{H}\max_{0\leq t\leq T}\|h(.,t)\|_{L^2(0,1)}\left(\int_0^t\left|u^{(2)}(s)-u^{(1)}(s)\right|^2ds\right)^{1/2}\\
&\leq \mathscr{H}^2\left(\max_{0\leq t\leq T}\|h(.,t)\|_{L^2(0,1)}\right)^2\left(\int_0^t\int_0^s|u^{(1)}(r)-u^{(0)}(r)|^2drds\right)^{1/2}\\
&\leq \mathscr{H}^2M_h^2\mathscr{F}_0\left(\int_0^t\int_0^s drds\right)^{1/2}=\mathscr{H}^2M_h^2\frac{t}{\sqrt{2}}\mathscr{F}_0.
\end{aligned}$$

Similarly, for a general value of N, we get

$$\|u^{(N+1)}-u^{(N)}\|_{\mathscr{N}} \leq \mathscr{H}^NM_h^N\frac{T^{\frac{N}{2}}}{\sqrt{N!}}\mathscr{F}_0. \tag{3.90}$$

It is easy to see that $u^{(N+1)}\to u^{(N)}$ when $N\to\infty$ and from (3.77) we deduce that $a^{(N+1)}\to a^{(N)}$ when $N\to\infty$. Therefore $(u^{(N)})_{N\geq 0}$ and $(a^{(N)})_{N\geq 0}$ converge in $\mathscr{N}$ and $C[0,T]$, respectively.

Step 4: Let us show that there exist u and a in $\mathscr{N}$ and $C[0,T]$ respectively, such that

$$\lim_{N\to\infty}u^{(N+1)}(t)=u(t), \quad \lim_{N\to\infty}a^{(N+1)}(t)=a(t).$$

Similarly to the previous step, we obtain

$$\begin{aligned}
|u(t)-u^{(N+1)}(t)| &\leq \mathscr{H}\left(\int_0^t\int_0^1 h^2(x,s)|u(x,s)-u^{(N)}(x,s)|^2dxds\right)^{1/2}\\
&\leq \mathscr{H}M_h\left(\int_0^t|u(s)-u^{(N)}(s)|^2ds\right)^{1/2},
\end{aligned}$$

which implies that

$$\begin{aligned}
|u(t)-u^{(N+1)}(t)|^2 &\leq 2(\mathscr{H}M_h)^2\int_0^t|u(s)-u^{(N+1)}(s)|^2ds\\
&\quad+2(\mathscr{H}M_h)^2\int_0^t|u^{(N+1)}(s)-u^{(N)}(s)|^2ds.
\end{aligned}$$

Then, by Gronwall's inequality and (3.90), we find

$$\begin{aligned}
|u(t)-u^{(N+1)}(t)|^2 &\leq 2(\mathscr{H}M_h)^2\exp(2(\mathscr{H}M_h)^2t)\int_0^t|u^{(N+1)}(s)-u^{(N)}(s)|^2ds\\
&\leq 2(\mathscr{H}M_h)^2\exp(2(\mathscr{H}M_h)^2t)\frac{(\mathscr{H}M_h)^{2N}\mathscr{F}_0^2}{N!}\int_0^t s^Nds.
\end{aligned}$$

Finally, we conclude that

$$\|u-u^{(N+1)}\|_{\mathscr{N}} \leq \sqrt{2}\exp((\mathscr{H}M_h)^2T)\mathscr{F}_0\frac{(\mathscr{H}M_h)^{N+1}T^{\frac{N+1}{2}}}{\sqrt{(N+1)!}},$$

when $N\to\infty$, we obtain $u^{N+1}(t)\to u(t)$. Additionally, we can find

$$\|a-a^{(N)}\|_{C[0,T]} \leq H_m^{-1}M_h\|u-u^{(N)}\|_{\mathscr{N}},$$

then, we deduce that $a^{(N)}(t)\to a(t)$ when $N\to\infty$. As a consequence, $u\in\mathscr{N}$ and $a\in C[0,T]$.

Step 5 We are going to show that the obtained solution is classical. The series expression (3.57) of the solution $u(x,t)$ gives

$$|u(x,t)| \leq 2|u_0(t)| + 4\sum_{k\geq 1}|u_{2k-1}(t)| + 4\sum_{k\geq 1}|u_{2k}(t)|.$$

From Lemma 3.2 and Remark 3.3, we conclude by (3.63),(3.64) and (3.58) that the majorizing sums of the series (3.57) and its x-partial derivative $\sum_{k\geq 1}\partial_x$ are absolutely convergent. Then, they are uniformly convergent and their sums are continuous in $\bar{D}_T$. In addition, the series of xx-partial derivative $\sum_{k\geq 1}\partial_{xx}$ of (3.57) is uniformly convergent in $(0,1)\times[\varepsilon,T]$, for any $\varepsilon>0$. In addition, from the expressions of fractional derivative (3.59)-(3.60) and (3.61), we get the following inequalities:

$$|{}^CD^{\alpha}_{0+}u_0(t)| \leq \|a\|_{C[0,T]}|u_0(t)| + \|F_0(.,u)\|_{C[0,T]};$$
$$|{}^CD^{\alpha}_{0+}u_{2k-1}(t)| \leq \lambda_k|u_{2k-1}(t)| + \|a\|_{C[0,T]}\||u_{2k-1}(t)| + \|F_{2k-1}(.,u)\|_{C[0,T]};$$
$$|{}^CD^{\alpha}_{0+}u_{2k}(t)| \leq \lambda_k|u_{2k}(t)| + 2\sqrt{\lambda_k}|u_{2k-1}(t)| + \|a\|_{C[0,T]}|u_{2k}(t)| + \|F_{2k}(.,u)\|_{C[0,T]};$$

due to Lemma 3.2 and Remark 3.3, we deduce the uniform convergence of the series

$$\sum_{k\geq 1}\lambda_k|u_{2k-1}(t)|, \quad \sum_{k\geq 1}\sqrt{\lambda_k}|u_{2k-1}(t)|, \quad \sum_{k\geq 1}\lambda_k|u_{2k}(t)|.$$

Then, from the Weirstrass M-test Theorem 1.8, we show that the series $\sum\limits_{k=1}^{\infty}{}^CD^{\alpha}_{0+}u_{2k-1}(t)$ and $\sum_{k=1}^{\infty}{}^CD^{\alpha}_{0+}u_{2k}(t)$ are convergent, also from (3.58) we observe that, ${}^CD^{\alpha}_{0+}u_0(t)$ is bounded. By virtue of relation (1.23) and linearity of fractional differential operators we can extend Theorem 1.10 to the Caputo fractional derivative case. Then, the α-partial derivative ${}^C\partial^{\alpha}_{0+,t}$ of the series (3.57) and the series $\sum_{k\geq 1}{}^C\partial_{0^+,t}$ are uniformly convergent in $(0,1)\times[\varepsilon,T]$, for any $\varepsilon>0$. Thus, $u\in C^{2,\alpha}(D_T)\cap C^{1,0}(\bar{D}_T)$ for arbitrary positive $a\in C[0,T]$.

Step 6: Finally, we prove the uniqueness of these solutions. We assume that problem $(P2)$ has two solution pairs $\{u(t),a_1(t)\},\{v(t),a_2(t)\}$. As in view, from (3.65) we get

$$\|a_1-a_2\|_{C[0,T]} \leq H_m^{-1}M_h\|u-v\|_{\mathscr{N}}. \tag{3.91}$$

Next, as done previously, we have

$$|u(t)-v(t)| \leq \mathscr{H}M_h\left(\int_0^t|u(s)-v(s)|^2ds\right)^{1/2}.$$

Applying **[Tec G]** (see p. 54) , we get $|u(t)-v(t)|\leq 0; t\in[0,T]$ and consequently $u(t)=v(t)$ on $[0,T]$. Furthermore, from (3.91) we deduce that $a_1(t)=a_2(t), t\in[0,T]$. □

3.4.2 Continuous Dependence of the Solution Upon the Data

The following result for continuous dependence upon the data of the solution of the inverse problem $(P2)$ holds.

Theorem 3.4. *Under assumption* (H_1)-(H_4)*, the solution* $\{u(x,t),a(t)\}$ *of the inverse problem* $(P2)$ *depends continuously upon the data of* $\{\varphi(x),H(t)\}$.

Proof. Let $\{u(x,t),a(t)\}$, $\{\bar{u}(x,t),\bar{a}(t)\}$ be two solutions of the inverse problem $(P2)$, corresponding two sets of the data $\{\varphi(x),H(t)\}$ and $\{\bar{\varphi}(x),\bar{H}(t)\}$ respectively. First, we have

$$|a(s)-\bar{a}(s)| \leq \left|\frac{{}^C D_{0+}^{\alpha}\bar{H}(s)}{\bar{H}(s)} - \frac{{}^C D_{0+}^{\alpha}H(s)}{H(s)}\right| + \int_0^1 \left|\frac{F(x,s,u(x,s))}{H(s)} - \frac{F(x,s,\bar{u}(x,s))}{\bar{H}(s)}\right| dx. \tag{3.92}$$

Adding and subtracting $\dfrac{{}^C D_{0+}^{\alpha}\bar{H}(s)}{H(s)}$ to the first term of the right sided of (3.92), we get

$$\begin{aligned}\left|\frac{{}^C D_{0+}^{\alpha}\bar{H}(s)}{\bar{H}(s)} - \frac{{}^C D_{0+}^{\alpha}H(s)}{H(s)}\right| &\leq \frac{|{}^C D_{0+}^{\alpha}(\bar{H}(s)-H(s))|}{|H(s)|} + |{}^C D_{0+}^{\alpha}\bar{H}(s)|\left|\frac{1}{\bar{H}(s)} - \frac{1}{H(s)}\right| \\ &\leq \frac{|{}^C D_{0+}^{\alpha}(\bar{H}(s)-H(s))|}{|H(s)|} + |{}^C D_{0+}^{\alpha}\bar{H}(s)|\frac{|H(s)-\bar{H}(s)|}{|\bar{H}(s)H(s)|} \\ &\leq \frac{1}{H_m}\|H-\bar{H}\|_{AC[0,T]} + \frac{1}{H_m^2}\|\bar{H}\|_{AC[0,T]}\|H-\bar{H}\|_{C[0,T]},\end{aligned}$$

Next, adding and subtracting $\int_0^1 \dfrac{F(x,s,\bar{u}(x,s))}{H(s)}dx$ and $\left(\dfrac{1}{H(s)} - \dfrac{1}{\bar{H}(s)}\right)\int_0^1 F(x,s,0)dx$ to the second term of the right sided of (3.92), we get

$$\begin{aligned}&\int_0^1 \left|\frac{F(x,s,u(x,s))}{H(s)} - \frac{F(x,s,\bar{u}(x,s))}{\bar{H}(s)}\right| dx \\ &\leq \frac{1}{H(s)}\int_0^1 |F(x,s,u(x,s))-F(x,s,\bar{u}(x,s))|dx \\ &+ \left|\frac{1}{H(s)} - \frac{1}{\bar{H}(s)}\right|\int_0^1 |F(x,s,\bar{u}(x,s))-F(x,s,0)|dx \\ &+ \left|\frac{1}{H(s)} - \frac{1}{\bar{H}(s)}\right|\int_0^1 |F(x,s,0)|dx \\ &\leq \frac{1}{H(s)}\int_0^1 h(x,s)\,|u(x,s)-\bar{u}(x,s)|dx + \left|\frac{1}{H(s)} - \frac{1}{\bar{H}(s)}\right|\int_0^1 h(x,s)\,|\bar{u}(x,s))|dx \\ &+ \left|\frac{1}{H(s)} - \frac{1}{\bar{H}(s)}\right|\int_0^1 |F(x,s,0)|dx.\end{aligned}$$

From Cauchy's inequality, we obtain

$$\begin{aligned}
&\int_0^1 \left| \frac{F(x,s,u(x,s))}{H(s)} - \frac{F(x,s,\bar{u}(x,s))}{\bar{H}(s)} \right| dx \\
&\leq \frac{1}{H(s)} \int_0^1 h(x,s)\,|u(x,s)-\bar{u}(x,s)|dx + \frac{|\bar{H}(s)-H(s)|}{H(s)\bar{H}(s)} \left(\int_0^1 h^2(x,s)\,dx \right)^{1/2} \|\bar{u}\|_{\mathcal{N}} \\
&+ \frac{|\bar{H}(s)-H(s)|}{H(s)\bar{H}(s)} \left(\int_0^1 |F(x,s,0)|^2 dx \right)^{1/2} \\
&\leq H_m^{-1} \int_0^1 h(x,s)\,|u(x,s)-\bar{u}(x,s)|dx + H_m^{-2} \|H-\bar{H}\|_{C[0,T]} \left[M_h L_{\bar{u}} + M \right]
\end{aligned}$$

where, $L_{\bar{u}} = \|\bar{u}\|_{\mathcal{N}}$.
Thus, we deduce that there exist a positive constant μ_1 such that

$$|a(s)-\bar{a}(s)| \leq \mu_1 \|H-\bar{H}\|_{AC[0,T]} + H_m^{-1} \int_0^1 h(x,s)\,|u(x,s)-\bar{u}(x,s)|dx \tag{3.93}$$

then

$$\|a-\bar{a}\|_{C[0,T]} \leq \mu_1 \|H-\bar{H}\|_{AC[0,T]} + H_m^{-1} M_h \|u-\bar{u}\|_{\mathcal{N}}. \tag{3.94}$$

Now, to estimate $u(x,t)-\bar{u}(x,t)$ we start with

$$|u_0(t)-\bar{u}_0(t)| \leq \|E-\bar{E}\|_{C[0,T]}.$$

For later, use (3.8) to arrive

$$\begin{aligned}
|u_{2k-1}(t)-\bar{u}_{2k-1}(t)| &\leq N_{\alpha 1} |\varphi_{2k-1}-\bar{\varphi}_{2k-1}| + N_\alpha \int_0^t (t-s)^{\alpha-1} |a(s)-\bar{a}(s)| |u_{2k-1}(s)| ds \\
&+ N_\alpha \int_0^t (t-s)^{\alpha-1} \bar{a}(s) |u_{2k-1}(s)-\bar{u}_{2k-1}(s)| ds \\
&+ N_\alpha \left| \int_0^t \int_0^1 (t-s)^{\alpha-1} \left[F(x,s,u(x,s)) - F(x,s,\bar{u}(x,s)) \right] \cos(2\pi kx) dx ds \right|.
\end{aligned}$$

Then, by (3.93) and the same estimation techniques used previously in **Step 3**, there exist positive constants $\tau_i, i=1,2,3$ such that

$$\begin{aligned}
\sum_{k\geq 1} |u_{2k-1}(t)-\bar{u}_{2k-1}(t)| &\leq \tau_1 \sum_{k\geq 1} |\varphi_{2k-1}-\bar{\varphi}_{2k-1}| + \tau_2 \|H-\bar{H}\|_{AC[0,T]} \\
&+ \tau_3 \left(\int_0^t \int_0^1 h^2(x,s) |u(x,s)-\bar{u}(x,s)|^2 dx ds \right)^{1/2}.
\end{aligned}$$

Repeating the same arguments to get

$$\begin{aligned}
\sum_{k\geq 1} |u_{2k}(t)-\bar{u}_{2k}(t)| &\leq \tau_4 \sum_{k\geq 1} |\varphi_{2k}-\bar{\varphi}_{2k}| + \tau_5 \|H-\bar{H}\|_{AC[0,T]} \\
&+ \tau_6 \sum_{k\geq 1} |\varphi_{2k-1}-\bar{\varphi}_{2k-1}| + \tau_7 \left(\int_0^t \int_0^1 h^2(x,s) |u(x,s)-\bar{u}(x,s)|^2 dx ds \right)^{1/2}
\end{aligned}$$

where, $\tau_i > 0, i=4,..,7$. Thus, we get for some positive constants $\sigma_i, i=1,2,3$

$$\begin{aligned}
|u(t)-\bar{u}(t)| &\leq \sigma_1 \|\varphi-\bar{\varphi}\|_{C^4[0,1]} + \sigma_2 \|H-\bar{H}\|_{AC[0,T]} \\
&+ \sigma_3 M_h \left(\int_0^t |u(s)-\bar{u}(s)|^2 dx ds \right)^{1/2}.
\end{aligned} \tag{3.95}$$

Applying **[Tec G]** (see p. 54) to (3.95), we get

$$|u(t) - \bar{u}(t)| \leq \sqrt{2}\exp\left(\sigma_3^2 M_h^2 T\right)\left[\sigma_1 \|\varphi - \bar{\varphi}\|_{C^4[0,1]} + \sigma_2 \|H - \bar{H}\|_{AC[0,T]}\right].$$

It follows that for some positive constants $\nu_i, i = 1,\ldots,4$ and a large T, we get

$$\|u - \bar{u}\|_{\mathcal{N}} \leq \nu_1 \|\varphi - \bar{\varphi}\|_{C^4[0,1]} + \nu_2 \|H - \bar{H}\|_{AC[0,T]}$$

and by (3.94), we obtain

$$\|a - \bar{a}\|_{C[0,T]} \leq \nu_3 \|\varphi - \bar{\varphi}\|_{C^4[0,1]} + \nu_4 \|H - \bar{H}\|_{AC[0,T]}.$$

This implies that when $\|\varphi - \bar{\varphi}\|_{C^4[0,1]} \leq \varepsilon_1$ and $\|H - \bar{H}\|_{AC[0,T]} \leq \varepsilon_2$, then

$$\|u - \bar{u}\|_{\mathcal{N}} \leq \nu_1 \varepsilon_1 + \nu_2 \varepsilon_2 \text{ and } \|a - \bar{a}\|_{C[0,T]} \leq \nu_3 \varepsilon_1 + \nu_4 \varepsilon_2.$$

This induces the continuous dependence of $\{u(x,t), a(t)\}$ on the data of $\{\varphi(x), H(t)\}$. □

3.5 Examples

Now, we present two examples to illustrate the obtained results for problem $(P1)$ and $(P2)$, respectively.

Example 1

Consider problem $(P1)$ with

$$\varphi(x) = \cos(2\pi x) + 1, \quad F(x,t,u) = (1-x)[\sin(2\pi x)e^{x+t-u} + 1], \quad a(t) = \exp(2t)$$

and

$$E(t) = E_\alpha(-2\pi t^\alpha).$$

It is easy to see that (A_1)-(A_2) are satisfied with

$$\left|\frac{\partial^n}{\partial x^n} F(x,t,u) - \frac{\partial^n}{\partial x^n} F(x,t,\tilde{u})\right| \leq e^{x+t}|u - \tilde{u}|, \quad n = 0,1,2$$

and

$$\int_0^1 x F(x,t,u) dx > 0, \quad \forall x \in (0,1).$$

Also, we have

$$\int_0^1 x \varphi(x) dx = 1 = E(0).$$

It follows, all conditions of Theorem 3.1 are satisfied. Then, this inverse problem has a unique classical solution $\{u(x,t), c(t)\}$ on $[0,T]$ for $\alpha \in (1/2, 1)$. In addition, from Theorem 3.2 the solution of this inverse $\{u(x,t), c(t)\}$ depends continuously upon the data of $\{E(t)\}$.

Example 2

Consider problem $(P2)$ with

$$\varphi(x) = -2\pi sin(2\pi x) + (1-x), \quad F(x,t,u) = (1-x)[cos(2\pi x)e^{x-t-u} + 4]$$

and

$$H(t) = \frac{1}{2}E_\alpha(-2\pi t^\alpha).$$

Clearly that (H_1)-(H_2) are satisfied with

$$\left|\frac{\partial^n}{\partial x^n}F(x,t,u) - \frac{\partial^n}{\partial x^n}F(x,t,\tilde{u})\right| \leq e^{x-t}|u-\tilde{u}|, \quad n = 0,1,2$$

and

$$F_0(t,u) = \int_0^1 F(x,t,u)dx > 0, \quad \forall x \in (0,1).$$

Moreover, From Lemma 1.20 we have

$${}^C D^\alpha H(t) = -\pi E_\alpha(-2\pi t^\alpha) < 0, \quad \forall t \in [0,T]$$

and

$$\int_0^1 \varphi(x)dx = \frac{1}{2} = H(0).$$

Therefore, all conditions of Theorem 3.3 are satisfied. Thus, this inverse problem has a unique classical solution $\{u(x,t), a(t)\}$ on $[0,T]$ for $\alpha \in (1/2,1)$. In addition, Theorem 3.4 implies that the continuous dependence of $\{u(x,t), a(t)\}$ upon the data of $\{\varphi(x), H(t)\}$.

CHAPTER

4 Weak Solution for a Time-fractional Reaction-diffusion Model with Application to an Inverse Problem

4.1 Introduction

It is well-known that the considerable interest in the study of fractional reaction-diffusion equation, has been motivated by applications in different fields of science and engineer. It is worth mentioning that significant development has been made in the study mathematical properties of the solution to the fractional partial differential equation. In [1], Kirane et al studied the time-fractional reaction-diffusion equation and obtained the globally bounded solutions with suitable assumptions on initial data. In [69], the authors established the existence and uniqueness of a weak solution of time-fractional reaction-diffusion model with density-dependent diffusion and non-local boundary condition by using the Faedo-Galerkin method and compactness arguments. Moreover, in [13] the authors obtained the existence of the solution of anti-periodic nonlinear fractional boundary value problem by using Schauder's and Banach's fixed point theorems. Also, in [52] the authors proved the existence and uniqueness of weak solution of super-diffusion equation in the Banach space $C([0,T];H_0^1(\Omega))$ by using fixed point theorem.

Recently, there has been a growing interest in inverse problems with fractional derivatives. Usually, in these works a fractional time derivative is considered and determination of source term under some additional condition(s) is the inverse problem, see [31, 32, 48, 67].
In [16], A. Demir and al. consider the following initial boundary value problem:

$$\begin{cases} {}^C\partial^{\alpha}_{0^+,t}u(x,t) = (k(x)u_x(x,t))_x + r(t)F(x,t), \quad 0<\alpha\leq 1, (x,t)\in\Omega_T \\ u(x,0) = g(x), \quad 0<x<1 \\ u(0,t) = \psi_0(t), \quad u_x(1,t) = \psi_1(t), \quad 0<t<T \end{cases}$$

where $\Omega_T = \{(x,t)\in\mathbb{R}^2 : 0<x<1, 0<t<T\}$ and ${}^C\partial^{\alpha}_{0^+,t}u(x,t)$ is the Caputo fractional derivative of order $0<\alpha\leq 1$ in the time variable. The main goal of their studies is to investigate the distinguishability of the unknown source function $r(t)$ via input-output mappings in a one dimensional time fractional inhomogeneous parabolic equation from the output data at specified given data. By defining the input-output mappings $\Phi[.]:\mathscr{R}\to C^1[0,T]$ and $\Psi[.]:\mathscr{R}\to C[0,T]$, where $\mathscr{R}$ represents the set of admissible source functions, the inverse problem is reduced to the problem of their invertibility.

Inspired by the works mentioned above, in this chapter, we aim at studying the existence, uniqueness and stability of weak solution for the time-fractional reaction-diffusion problem in view of fixed point theory. In addition, an inverse problem with a measured output data $g(t)$ in a fixed point space is studied. Note that main results of this chapter is presented in the paper [17].

4.2 Statement of the Problem

In this chapter, we mainly consider the time-fractional reaction-diffusion equation as follows:

$$^{C}\partial^{\alpha}_{0+,t}u(x,t) = d(t)\Delta u(x,t) + \beta(t)F(u(x,t)), \quad (x,t) \in \Omega_T \tag{4.1}$$

subject to Dirichlet boundary conditions

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in [0,T] \tag{4.2}$$

and the initial data

$$u(x,0) = \phi(x), \quad x \in \Omega \tag{4.3}$$

where $\Omega_T = \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^n, n \geq 1$ is bounded and $\partial\Omega$ is its smooth boundary, $^{C}\partial^{\alpha}_{0+,t}$ denotes the Caputo time fractional derivative of order $\alpha \in (0,1)$, $d(t)$ is the diffusion coefficient, $\beta(t)F(u)$ is the reaction term and $\phi(x)$ is the initial given data.

We will consider two cases, $F(u) = Mu$ and $F(u) = M$, $M \in (0,\infty)$, we discuss the existence, uniqueness and stability of weak solution of direct problem (4.1)-(4.3) in a Banach space $C([0,T];H^1_0(\Omega))$ by using the fixed point theorems.
Next, we consider the inverse problem of determining the unknown time dependent reaction term $\beta(t)$ from the output data at specified given data, by defining the input-output mapping $\Phi[.] : \mathscr{R} \to L^\infty[0,T]$ and $\mathscr{R}$ the set of admissible source coefficients. These formulations reduce the inverse problem of finding $\beta(t)$ to the problem of invertibility of Φ. This leads us to investigating the distinguishability of the source function via the above input-output mappings.

Definition 4.1. *We say that the mapping $\Phi[.] : \mathscr{R} \to L^\infty[0,T]$ has the distinguishability property if $\Phi[\beta_1] \neq \Phi[\beta_2]$ implies $\beta_1(t) \neq \beta_2(t)$.*

This, in particular, means the injectivity of the inverse mapping Φ^{-1}.

4.3 Weak Solution via Fixed Point Theory

The first goal of this study is to reformulate the equation (4.1) from which the weak solutions will be derived.

According to Remark 1.8, we can reduce the problem (4.1)-(4.3) to an equivalent integral equation by applying the fractional integral operator I^{α}_{0+} as follows:

$$\begin{cases} u(x,t) = \phi(x) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} W(u(x,s))ds & \text{in} \quad \Omega_T, \\ u(x,t) = 0, \quad \text{in} \quad \partial\Omega, \end{cases} \tag{4.4}$$

where

$$W(u) = d(s)\Delta u + \beta(s)F(u). \tag{4.5}$$

The functional integral equations describe many physical phenomena in various areas of natural science, mathematical physics, mechanics, and population dynamics [9, 15]. The theory of integral equations have been a source of very interesting problems within the realm of functional analysis, topology and fixed point theory (see, [8, 42]). It serves as a useful tool in turn for other branches of mathematics, for example, for differential equations, see [66].

Now, we introduce the operator Ψ as follows:

$$\begin{cases} \Psi(u(x,t)) = \phi(x) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} W(u(x,s))ds \text{ in } \Omega_T, \\ \Psi u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in [0,T], \end{cases}$$

where $W(u)$ is defined by (4.5).

Definition 4.2. *When $u \in C([0,T], H_0^1(\Omega))$ satisfies*

$$\langle (u(t) - \Psi u(t)), v \rangle_{L^2(\Omega)} = 0, \quad \forall t \in [0,T], v \in H_0^1(\Omega), \text{ i.e.}$$

$$\int_\Omega u(x,t)v(x)dx = \int_\Omega \left[\phi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} W(u(x,s))ds \right] v(x)dx, \tag{4.6}$$

where $\langle .,. \rangle_{L^2(\Omega)}$ is the scalar product in $L^2(\Omega)$ spaces and $W(u)$ is defined in (4.5), we call u is a weak solution of the time fractional diffusion equation (4.1)-(4.3).

In the case $F(u) = M$, we have the following existence result, which is assured by using Schauder's theorem:

Theorem 4.1. *Let $\alpha \in (0,1)$, we assume that d, β and ϕ satisfy the following:*

(A_1) *$d \in L^\infty([0,T], \mathbb{R}^+), \beta \in L^\infty[0,T]$ and $\phi \in H_0^1(\Omega)$.*

Then, if the positive constant C satisfies

$$C = \frac{\|d\|_{L^\infty[0,T]} T^\alpha}{\Gamma(\alpha+1)} < 1, \tag{4.7}$$

the time-fractional reaction-diffusion problem (4.1)-(4.3) has at least one solution $u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T], L^2(\Omega))$ such that ${}^C\partial_{0+,t}^\alpha u \in L^2(0,T; L^2(\Omega))$.

Proof. First, we define the subspace of the standard Banach space $C([0,T]; H_0^1(\Omega))$

$$B(\overline{\Omega}_T) = L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$$

and

$$B_R = \left\{ u \in B(\overline{\Omega}_T) : \|u\|_{B_R} \le R \right\},$$

a closed convex nonempty subset of $C\left([0,T]; H_0^1(\Omega)\right)$ equipped with the induced norm

$$\|u\|_{B_R} = \|u\|_{C\left([0,T]; H_0^1(\Omega)\right)} = \sup_{t \in [0,T]} \|u(.,t)\|_{H_0^1(\Omega)},$$

where we choose R sufficiently big to have

$$R \ge \left(\|\phi\|_{L^2(\Omega)} + \frac{MC_0 \|\beta\|_{L^\infty[0,T]} T^\alpha}{\Gamma(\alpha+1)} \right) \left(1 - \frac{\|d\|_{L^\infty[0,T]} T^\alpha}{\Gamma(\alpha+1)} \right)^{-1},$$

where $C_0 = (\int_\Omega dx)^{1/2}$.

Next, we prove that the operator Ψ makes B_R into itself and is completely continuous. For any $v \in H_0^1(\Omega)$ satisfying $\|v\|_{H_0^1(\Omega)} \leq 1$, using integration by parts and Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\langle W(u(t)),v\rangle_{L^2(\Omega)}| &= |\langle d(t)\Delta u(t)+\beta(t)F(u(t)),v\rangle_{L^2(\Omega)}|\\ &\leq d(t)|\langle \Delta u(t),v\rangle_{L^2(\Omega)}|+|\beta(t)||\langle M,v\rangle_{L^2(\Omega)}|\\ &\leq d(t)|\langle \nabla u(t),\nabla v\rangle_{L^2(\Omega)}|+|\beta(t)||\langle M,v\rangle_{L^2(\Omega)}|\\ &\leq \|d\|_{L^\infty[0,T]}\|\nabla u(.,t)\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\ &\quad +MC_0\|\beta\|_{L^\infty[0,T]}\|v\|_{L^2(\Omega)}.\end{aligned} \tag{4.8}$$

By using the real order Sobolev imbedding theorem, we get that $\|v\|_{L^2(\Omega)} \leq \|v\|_{H_0^1(\Omega)}$ for each $v \in H_0^1(\Omega)$. Then, we obtain

$$\begin{aligned}|\langle W(u(t)),v\rangle_{L^2(\Omega)}| \leq\ & \|d\|_{L^\infty[0,T]}\|u(t)\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}\\ &+MC_0\|\beta\|_{L^\infty[0,T]}\|v\|_{H_0^1(\Omega)}.\end{aligned}$$

From the fact that $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$, we get $\Psi u \in H^{-1}(\Omega)$. Hence

$$\begin{aligned}&\|\Psi u(t)\|_{H^{-1}(\Omega)} = \sup_{\|v\|_{H_0^1(\Omega)}\leq 1} |\langle \Psi u(t),v\rangle_{L^2(\Omega)}|\\ &\leq \sup_{\|v\|_{H_0^1(\Omega)}\leq 1}\left[\left|\langle \phi,v\rangle_{L^2(\Omega)}\right|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\langle W(u(s)),v\rangle_{L^2(\Omega)}\right|ds\right]\\ &\leq \sup_{\|v\|_{H_0^1(\Omega)}\leq 1}\left[\|\phi\|_{L^2(\Omega)}+\frac{T^\alpha}{\alpha\Gamma(\alpha)}\|d\|_{L^\infty[0,T]}\sup_{t\in[0,T]}\|u(t)\|_{H_0^1(\Omega)}\right.\\ &\left.+\frac{T^\alpha}{\alpha\Gamma(\alpha)}MC_0\|\beta\|_{L^\infty[0,T]}\right]\|v\|_{H_0^1(\Omega)}\\ &\leq \|\phi\|_{L^2(\Omega)}+\left[\|d\|_{L^\infty[0,T]}\|u\|_{B_R}+MC_0\|\beta\|_{L^\infty[0,T]}\right]\frac{T^\alpha}{\Gamma(\alpha+1)}\\ &\leq \|\phi\|_{L^2(\Omega)}+\left[\|d\|_{L^\infty[0,T]}R+MC_0\|\beta\|_{L^\infty[0,T]}\right]\frac{T^\alpha}{\Gamma(\alpha+1)} \leq R,\end{aligned}$$

consequently, $\Psi u \in B_R$. After that, we are going to prove that ΨB_R is equicontinuous set. For all $t_1,t_2 \in [0,T]$ such that $t_2 > t_1$, for $u \in B_R$ and every $v \in H_0^1(\Omega)$ such that $\|v\|_{H_0^1(\Omega)} \leq 1$, we have

$$\begin{aligned}&\|\Psi u(t_2)-\Psi u(t_1)\|_{H^{-1}(\Omega)} = \sup_{\|v\|_{H_0^1(\Omega)}\leq 1}|\langle \Psi u(t_2)-\Psi u(t_1),v\rangle_{L^2(\Omega)}|\\ &\leq \sup_{\|v\|_{H_0^1(\Omega)}\leq 1}\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\langle W(u(s)),v\rangle_{L^2(\Omega)}ds\right.\\ &\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|\langle W(u(s)),v\rangle_{L^2(\Omega)}|ds\right|\\ &\leq \left(\frac{\|d\|_{L^\infty[0,T]}}{\Gamma(\alpha+1)}\|u\|_{B_R}+\frac{MC_0\|\beta\|_{L^\infty[0,T]}}{\Gamma(\alpha+1)}\right)\left[|t_2^\alpha-t_1^\alpha|+2\,(t_2-t_1)^\alpha\right],\end{aligned}$$

which tends to 0, when t_1 tends to t_2, this implies the equicontinuity of Ψ.

The Arzela-Ascoli Theorem 1.7 shows that Ψ is completely continuous from its uniformly boundedness and its equicontinuity. Finally, combining this result with Schauder's theorem, we conclude the existence result of a solution in $B\left(\overline{\Omega}_T\right)$. The proof is complete. □

Remark 4.1. *In this case, we can prove uniqueness of the solution by using the standard procedure for the uniqueness result, i.e., let $u_1(x,t)$ and $u_2(x,t)$ are two weak solutions of problem (4.1)-(4.3) in the sense of Definition 4.2. Then from the representation (4.6) of the weak solution, we have*

$$\int_\Omega (u_1(x,t)-u_2(x,t))v(x)dx = 0, \quad \forall t \in [0,T], v \in H_0^1(\Omega),$$

which implies that $u_1(x,t) = u_2(x,t)$.

Now, let us prove the stability result:

Theorem 4.2. *The solution of the problem (4.1)-(4.3) depends continuously on the initial data ϕ, the diffusion coefficient d and the reaction coefficient β.*

Proof. Let u_1 and u_2 be weak solutions of problem (4.1)-(4.3) corresponding to initial conditions ϕ_1 and ϕ_2, diffusion coefficient d_1 and d_2, reaction coefficients β_1 and β_2 respectively. Then, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \|u_1(t)-u_2(t)\|_{H^{-1}(\Omega)} &= \sup_{t\in[0,T]} \sup_{\|v\|_{H_0^1(\Omega)}\le 1} |\langle u_1(t)-u_2(t), v\rangle_{L^2(\Omega)}| \\
&\le \sup_{\|v\|_{H_0^1(\Omega)}\le 1} |\langle (\phi_1-\phi_2), v\rangle_{L^2(\Omega)}| \\
&+ MC_0\|\beta_1-\beta_2\|_{L^\infty[0,T]}\frac{T^\alpha}{\alpha\Gamma(\alpha)} + \|d_1-d_2\|_{L^\infty[0,T]}\frac{T^\alpha}{\alpha\Gamma(\alpha)}\|u_1\|_{B(\overline{\Omega}_T)} \\
&+ \|d_2\|_{L^\infty[0,T]}\frac{T^\alpha}{\alpha\Gamma(\alpha)}\|u_1-u_2\|_{B(\overline{\Omega}_T)} \\
&\le \|\phi_1-\phi_2\|_{L^2(\Omega)} + \frac{T^\alpha}{\Gamma(\alpha+1)}M_1\|d_1-d_2\|_{L^\infty[0,T]} \\
&+ \frac{T^\alpha MC_0}{\Gamma(\alpha+1)}\|\beta_1-\beta_2\|_{L^\infty[0,T]} + C\|u_1-u_2\|_{B(\overline{\Omega}_T)},
\end{aligned}$$

where $M_1 = \|u_1\|_{B(\overline{\Omega}_T)}$ and C is defined in (4.7). Hence,

$$\begin{aligned}
\|u_1-u_2\|_{B(\overline{\Omega}_T)} \le \frac{1}{1-C}\|\phi_1-\phi_2\|_{L^2(\Omega)} + \frac{T^\alpha M_1}{(1-C)\Gamma(\alpha+1)}\|d_1-d_2\|_{L^\infty[0,T]} \\
+ \frac{T^\alpha MC_0\|\beta_1-\beta_2\|_{L^\infty[0,T]}}{(1-C)\Gamma(\alpha+1)},
\end{aligned}$$

which implies the stability result. □

When $F(u) = Mu$, we have the following theorem, which is based on the Banach contraction principle:

Theorem 4.3. *Assume that β, d and ϕ satisfy (A_1). Then, if the positive constant*

$$\mathscr{M} = \left(\|d\|_{L^\infty[0,T]} + M\|\beta\|_{L^\infty[0,T]}\right)$$

satisfies

$$\mathscr{M}\frac{T^\alpha}{\Gamma(\alpha+1)} < 1, \tag{4.9}$$

the problem (4.1)-(4.3) admits a unique solution $u \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C([0,T],L^2(\Omega))$ such that ${}^C\partial_{0+,t}^\alpha u \in L^2((0,T],L^2(\Omega))$.

Proof. First, we have to check that the operator Ψ makes $B\left(\overline{\Omega}_T\right)$ into itself. In view of (4.8) with $F(u) = Mu$, we have

$$|\langle W(u(t)),v\rangle_{L^2(\Omega)}| \leq \left[\|d\|_{L^\infty[0,T]} + M\|\beta\|_{L^\infty[0,T]}\right] \|u(t)\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}. \tag{4.10}$$

Then, we have

$$\begin{aligned}\|\Psi u(t)\|_{H^{-1}(\Omega)} &= \sup_{\|v\|_{H_0^1(\Omega)}\leq 1} |\langle \Psi u(t), v\rangle_{L^2(\Omega)}| \\ &= \sup_{\|v\|_{H_0^1(\Omega)}\leq 1}\left[|\langle\phi,v\rangle_{L^2(\Omega)}| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\langle W(u(s)),v\rangle_{L^2(\Omega)}|ds\right],\end{aligned}$$

using estimation (4.10) , we get

$$\sup_{t\in[0,T]}\|\Psi u(t)\|_{H^{-1}(\Omega)} \leq \|\phi\|_{L^2(\Omega)} + \frac{\left(\|d\|_{L^\infty[0,T]} + M\|\beta\|_{L^\infty[0,T]}\right)T^\alpha}{\Gamma(\alpha+1)}\|u\|_{B\left(\overline{\Omega}_T\right)}.$$

Thus, Ψ is bounded and maps $B\left(\overline{\Omega}_T\right)$ into itself. Then, we will show that Ψ is a contraction. For any $u_1, u_2 \in B\left(\overline{\Omega}_T\right)$ and $v \in H_0^1(\Omega)$ satisfying $\|v\|_{H_0^1(\Omega)} \leq 1$ we have

$$\begin{aligned}&|\langle W(u_1(t)) - W(u_2(t)), v\rangle_{L^2(\Omega)}| \\ &\qquad = |\langle d(t)[\Delta(u_1(t)-u_2(t))] + \beta(t)[F(u_1(t)) - F(u_2(t))], v\rangle_{L^2(\Omega)}| \\ &\qquad \leq \|d\|_{L^\infty[0,T]}\|\nabla(u_1(t)-u_2(t))\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ &\qquad + M\|\beta\|_{L^\infty[0,T]}\|u_1(t)-u_2(t)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\qquad \leq \left[\|d\|_{L^\infty[0,T]} + M\|\beta\|_{L^\infty[0,T]}\right]\|u_1(t)-u_2(t)\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\sup_{t\in[0,T]}\|\Psi u_1(t) - \Psi u_2(t)\|_{H^{-1}(\Omega)} &= \sup_{t\in[0,T]}\sup_{\|v\|_{H_0^1}\leq 1}|\langle \Psi u_1(t) - \Psi u_2(t), v\rangle_{L^2(\Omega)}| \\ &\leq \sup_{t\in[0,T]}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{\|v\|_{H_0^1}\leq 1}\left|\langle W(u_1(s)) - W(u_2(s)), v\rangle_{L^2(\Omega)}\right|ds \\ &\leq \left(\|d\|_{L^\infty[0,T]} + M\|\beta\|_{L^\infty[0,T]}\right)\frac{T^\alpha}{\alpha\Gamma(\alpha)}\|u_1-u_2\|_{B\left(\overline{\Omega}_T\right)} \\ &\leq \mathscr{M}\frac{T^\alpha}{\Gamma(\alpha+1)}\|u_1-u_2\|_{B\left(\overline{\Omega}_T\right)},\end{aligned}$$

knowing that condition (4.9) is satisfied. From the Banach theorem, we deduce that Ψ is a contraction mapping, thus the fractional reaction-diffusion problem (4.1)-(4.3) has a unique solution $u \in B\left(\overline{\Omega}_T\right)$. This completes the proof. □

Now, we prove the stability of the solution.

Theorem 4.4. *The unique solution of the problem* (4.1)-(4.3) *depends continuously upon the initial condition* ϕ.

Proof. Let u_1, u_2 be two solutions of problem (4.1)-(4.3) corresponding to initial conditions ϕ_1 and ϕ_2 respectively, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \|u_1(t)-u_2(t)\|_{H^{-1}(\Omega)} &= \sup_{t\in[0,T]} \sup_{\|v\|_{H_0^1(\Omega)}\le 1} \left|\langle u_1(t)-u_2(t), v\rangle_{L^2(\Omega)}\right| \\
&\le \sup_{\|v\|_{H_0^1(\Omega)}\le 1} \left|\langle (\phi_1-\phi_2), v\rangle_{L^2(\Omega)}\right| \\
&+ (\|d\|_{L^\infty[0,T]} + M\|\beta\|_{L^\infty[0,T]} \frac{T^\alpha}{\alpha\Gamma(\alpha)} \|u_1-u_2\|_{B(\overline{\Omega}_T)} \\
&\le \sup_{\|v\|_{H_0^1(\Omega)}\le 1} \|\phi_1-\phi_2\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \mathscr{M}\|u_1-u_2\|_{B(\overline{\Omega}_T)}.
\end{aligned}$$

Hence

$$\|u_1-u_2\|_{B(\overline{\Omega}_T)} \le \frac{1}{1-\mathscr{M}} \|\phi_1-\phi_2\|_{L^2(\Omega)},$$

which implies the stability. □

4.4 Application to an Inverse Problem

First, we consider problem (4.1)-(4.3) with $F(u) = M, d(t) = d$ a positive constant and the initial data $\phi(x) = 0$. The inverse problem consist in determining $\beta(t)$, by an additional measured data in a fixed space point $x_0 \in \Omega$

$$u(x_0,t) = g(t), \quad t \in [0,T]. \tag{4.11}$$

We just prove a stability estimation of the unknown reaction coefficient $\beta(t)$.

Theorem 4.5. *Let g and ${}^C D^\alpha_{0+} g$ in $L^\infty[0,T]$. Then, there exist two constants $\xi_1, \xi_2 > 0$ such that:*

$$\xi_1 \|g\|_{L^\infty[0,T]} \le \|\beta\|_{L^\infty[0,T]} \le \xi_2 \|{}^C D^\alpha_{0+} g\|_{L^\infty[0,T]}. \tag{4.12}$$

Proof. In view of (4.4) with $F(u) = M$, we get

$$\begin{aligned}
\|u\|_{B(\overline{\Omega}_T)} &\le \|\phi\|_{L^2(\Omega)} + \|d\|_{L^\infty[0,T]} \|u\|_{B(\overline{\Omega}_T)} \frac{T^\alpha}{\Gamma(\alpha+1)} \\
&+ MC_0 \|\beta\|_{L^\infty[0,T]} \frac{T^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

taking into consideration that $\phi(x) = 0$, and $d(t) = d$, from (4.11) we obtain

$$\|g\|_{L^\infty[0,T]} \le \|u\|_{B(\overline{\Omega}_T)} \le \frac{T^\alpha MC_0}{(1-C_d)\Gamma(\alpha+1)} \|\beta\|_{L^\infty[0,T]}.$$

where $C_d = \dfrac{dT^\alpha}{\Gamma(\alpha+1)}$.

Then,

$$\|\beta\|_{L^\infty[0,T]} \geq \frac{(1-C_d)\Gamma(\alpha+1)}{T^\alpha M C_0}\|g\|_{L^\infty[0,T]}. \tag{4.13}$$

On the other hand, we resolve problem (4.1)-(4.3) with $d(t) = d$ a positive constant, $F(u) = M$ and the initial data $\phi(x) = 0$ by Fourier method. We can obtain the solution as an eigenfunctions expansion form $\sum_{n\geq 1} T_n(t)X_n(x)$, where the eigenfunctions satisfy the following spectral problem:

$$\begin{cases} -\Delta X_n = \lambda_n X_n, & x \in \Omega \\ X_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

For second claim, replacing $\sum_{n\geq 1} T_n(t)X_n(x)$ in equation (4.1), multiplying by an eigenfunction and integrating over Ω, we get the following Cauchy problem:

$$\begin{cases} {}^C D_{0+}^\alpha T_n(t) + d\lambda_n T_n(t) = M\beta(t), & t \in [0,T] \\ T_n(0) = 0, \end{cases} \tag{4.15}$$

From Theorem 1.12, the solution of the above problem is

$$T_n(t) = M\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-d\lambda_n(t-s)^\alpha)\beta(s)ds. \tag{4.16}$$

Finally, the unique solution of problem (4.1)-(4.3) satisfies consequently

$$u(x,t) = \sum_{n=1}^\infty \int_0^t M(t-s)^{\alpha-1}\beta(s)E_{\alpha,\alpha}(-d\lambda_n(t-s)^\alpha)dsX_n(x). \tag{4.17}$$

Using the supplementary data, we get

$$\begin{aligned} g(t) &= u(x_0,t) \\ &= \sum_{n=1}^\infty \int_0^t M(t-s)^{\alpha-1}\beta(s)E_{\alpha,\alpha}(-d\lambda_n(t-s)^\alpha)dsX_n(x_0). \end{aligned}$$

Hence, in view of equation (4.1) we get

$$\begin{aligned} |\beta(t)|M \leq & \left|{}^C D_{0+}^\alpha g(t)\right| \\ & + Md\sum_{n=1}^\infty \lambda_n \int_0^t |\beta(s)|(t-s)^{\alpha-1}E_{\alpha,\alpha}(-d\lambda_n(t-s)^\alpha)ds\,|X_n(x_0)|. \end{aligned} \tag{4.18}$$

We use The Lemma 1.10 to get

$$d\lambda_n E_{\alpha,\alpha}(-d\lambda_n(t-s)^\alpha) \leq \frac{d\lambda_n(t-s)^\alpha C}{1+d\lambda_n(t-s)^\alpha} < E, \tag{4.19}$$

and the fact that $\sum_{n=1}^\infty |X_n(x_0)|$ is convergent to denote

$$G(t-s) = d\sum_{n=1}^\infty \lambda_n(t-s)^{\alpha-1}E_{\alpha,\alpha}(-d\lambda_n(t-s)^\alpha)\,|X_n(x_0)|. \tag{4.20}$$

Then, we obtain

$$\begin{aligned}|\beta(t)| &\leq \frac{\left|{}^C D_{0+}^{\alpha} g(t)\right|}{M} + \int_0^t |\beta(s)| G(t-s)\, ds \\ &\leq \frac{\left|{}^C D_{0+}^{\alpha} g(t)\right|}{M} + E \sum_{n=1}^{\infty} |X_n(x_0)| \int_0^t |\beta(s)| (t-s)^{\alpha-1} ds. \qquad (4.21)\end{aligned}$$

Consequently, applying the Gronwall Lemma 1.3, we obtain

$$\begin{aligned}|\beta(t)| \leq &\frac{\left|{}^C D_{0+}^{\alpha} g(t)\right|}{M} \\ &+ K_{(1-\alpha)} E \sum_{n=1}^{\infty} |X_n(x_0)| \int_0^t \frac{\left|{}^C D_{0+}^{\alpha} g(s)\right|}{M} (t-s)^{\alpha-1} ds\end{aligned}$$

and thus,

$$\|\beta\|_{L^\infty[0,T]} \leq \frac{1}{M} \left[1 + K_{(1-\alpha)} E \sum_{n=1}^{\infty} |X_n(x_0)| \frac{T^\alpha}{\alpha} \right] \|{}^C D_{0+}^{\alpha} g\|_{L^\infty[0,T]}. \qquad (4.22)$$

Between (4.13) and (4.22) we obtain (4.12). This completes the proof. □

Now, we reduce the inverse problem (4.1)-(4.3)-(4.11) of determining the unknown coefficient $\beta(t)$ to the problem of invertibility of the input-output mapping $\Phi[.] : \mathscr{R} \to L^\infty[0,T]$ corresponding to the output data $g(t) = u(x_0,t)$ given by

$$\Phi(\beta(t)) = \sum_{n=1}^{\infty} \int_0^t \beta(s) M (t-s)^{\alpha-1} E_{\alpha,\alpha}(-d\lambda_n (t-s)^\alpha) ds X_n(x_0), \qquad (4.23)$$

where $\mathscr{R}$ represents the set of admissible source functions defined by

$$\mathscr{R} = \left\{ \beta \in L^\infty[0,T] : \xi_1 \|g\|_{L^\infty[0,T]} \leq \|\beta\|_{L^\infty[0,T]} \leq \xi_2 \|{}^C D_{0+}^{\alpha} g\|_{L^\infty[0,T]} \right\}.$$

It is shown that the distinguishability of the input-output mapping holds.

Theorem 4.6. *Let g and ${}^C D_{0+}^{\alpha} g \in L^\infty[0,T]$ hold. Then, the mapping $\Phi[\beta]$ defined by (4.23) is distinguishable on $\mathscr{R}$, i.e., for any $\beta_1, \beta_2 \in \mathscr{R}$,*

$$\Phi[\beta_1] \neq \Phi[\beta_2] \Rightarrow \beta_1(t) \neq \beta_2(t); \quad \textit{for } t \in [0,T].$$

Proof. We use the contra-posed reasoning i.e. :

$$\forall \beta_1, \beta_2 \in \mathscr{R} \quad \beta_1(t) = \beta_2(t) \text{ for } \quad t \in [0,T] \Rightarrow \Phi[\beta_1] = \Phi[\beta_2].$$

From the uniqueness of the solution of the direct problem (4.1)-(4.3), we find $\beta_1(t) = \beta_2(t)$ implies that $u_1(x,t) = u_2(x,t)$ with $\phi(x) = 0$. Then, $\beta_1(s)M = \beta_2(s)M$ and we get easily $g_1(t) = g_2(t)$, which leads us to $\Phi[\beta_1] = \Phi[\beta_2]$. □

Conclusion

In this work, our main scientific contributions focused on the existence, uniqueness and continuous dependence on the data of the solution for various classes of inverse parabolic problems involving two types of fractional derivatives (Hilfer and Caputo). The results are based on Fourier's method, a bi-orthogonal system, the iteration method, Gronwall's Lemma and the technique of fixed point. In most case, it is difficult, or infeasible to find the appropriate spaces of the analytical solutions and suitable conditions to ensure the well posed character of the problems. Note that in Chapter 3, the order of fractional derivative is not what we supposed to be $0 < \alpha < 1$, in fact we got a limited order $\alpha \in (1/2, 1)$, this can be improved in the future. Also the results are obtained for small T for problem $(P1)$ and we get an improvement for a large T for problem $(P2)$.

For the perspective, it would be interesting to extend the obtained results by considering the time-space semi-linear parabolic equation with fractional Laplacian operator. Other methods can be tried such as Galerkin's or numerical methods.

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