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THESE

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Contribution à l'étude qualitative et quantitative de certaine classe d'équations integro –différentielles à retard

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ملخص

تهدف هذه الأطروحة الى تعميم نظرية أسكولي أرزيلا من فضاءات التطبيقات المستمرة إلى فضاءات التطبيقات المستمرة والقابلة للاشتقاق من الرتب العليا من جهة. ومن جهة أخرى تطبيق هذا التعميم في دراسة وجود حلول بعض المعادلات التفاضلية التكاملية، معادلات تفاضلية غير خطية من الرتبة الثانية بوجود تأخر زمني وبعض المعادلات التفاضلية الغير خطية من الرتب العليا. كما قمنا كذلك بدراسة الوجود والوحدانية لحلول بعض المعادلات الغير خطية من الرتبة الثانية باستعمال نظرية النقطة الصامدة لبيروف في الفضاءات المترية المعممة.

ونشير إلى أن درستنا قد تمت باستعمال شروط أبسط مقارنة بالشروط الموجودة في الدراسات السابقة. وفي نهاية كل فصل قدمنا امثلة عددية توضيحية.

الكلمات المفتاحية:

المعادلات التفاضلية غير الخطية، المعادلات التفاضلية التكاملية، نظرية أسكولي أرزيلا في C^n، نظرية النقطة الصامدة.

Abstract

This study aims to generalize the Ascoli- Arzelá theorem from the space of continuous functions to the space of functions with continuous of high order derivative on one hand. On the other hand, we use this generalization in studying the existence of the solutions of some integro-differential equations, non-linear differential equations of the second order with delay argument and some non-linear differential equations of high order. Additionally, in this thesis we study also the existence and uniqueness of the solutions of some second order non-linear differential equations by using Perov's fixed point theorem in the generalized metric spaces. It is worth mentioning that this study uses simpler conditions compared to those used in previous studies. Finally, illustrative numerical examples are provided in the end of each chapter.

Keywords : *Non-linear differential equations, Integro-differential equations, Ascoli-Arzelà theorem in C^n, Fixed point theorem.*

Résumé

L'objectif de cette thèse est généralisation le théorème d'Ascoli Arzelá de l'espace des fonctions continues à l'espace des fonctions à dérivées continues d'ordre supérieure d'une part.

D'autre part, nous utilisons cette généralisation pour étudier l'existence des solutions de certaines équations intégro-différentielles, des équations différentielles non linéaires du second ordre avec un retard et de certaines équations différentielles non linéaires d'ordre supérieure. De plus, dans

cette thèse, nous étudions également l'existence et l'unicité des solutions de certaines équations différentielles non linéaires du second ordre en utilisant le théorème du point fixe de Perov dans les espaces métriques généralisés. Il est bien noter que cette étude utilise des conditions plus simples par

rapport à celles utilisées dans les études précédentes. Enfin, nous donnons des exemples numériques

à la fin de chaque chapitre pour illustrer les résultats théoriques.

Mots-clés : *Équations différentielles non linéaires*, *Théorème d'Ascoli-Arzelá dans* C^n, *Équations intgro-differentiel*, *Théorème de point fixe.*

Doctorat en Sciences: CONTRIBUTION à L'ETUDE QUALITATIVE ET QUANTITATIVE DE CERTAINE CLASSE D'EQUATIONS INTEGRO –DIFFERENTIELLES A RETARD

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DEDICATION

I do not have the words to fully express my gratitude to my father and mother for their unconditional love and support. I am immensely grateful to my wife for being with me through this phase of my life. My particular appreciation goes to my heroic children Mohamed Barea and Anes.

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Contents

Introduction

Ascoli's theorem, or Ascoli- Arzelá theorem of functional analysis, demonstrated by the Italian mathematicians Guilio-Ascoli (1843/1896) and Cesare-Arzelá (1847/1912), gives necessary and sufficient conditions to decide whether a given set of continuous functions is relatively compact for certain topologies. In order to establish its proof, we will introduce some notions and auxiliary definitions of precompact spaces, which will give us a criterion of compactness, complete spaces, and also the notion of equicontinuity which was introduced at the same time by Ascoli (1883/1884) and Arzelá (1882/1883). A weak form of the theorem was proved by Ascoli, he established the sufficient condition of compactness, and by Arzelá (1895), who established the necessary condition and gave the first clear presentation of the result.

Delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs are also called time-delay systems, systems with aftereffect or dead-time, hereditary systems, equations with deviating argument, or differential-difference equations. The nonlinear delay differential equations arise in the modeling of many phenomena in physics, mechanics and Biology (see [5, 10, 20, 41, 42, 58] and the references therein).

The existence of such a periodic solution is of quite fundamental importance biologically since it concerns the long time survival of species.

In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. Boundary value problems arise in several branches of physics as any physical differential equation will have them. Problems involving the wave equation, such as the determination of normal modes, are often stated as boundary value problems. A large class of important boundary value problems are the Sturm-Liouville problems. The analysis of these problems involves the eigenfunctions of a differential operator. To be useful in applications, a boundary value problem should be well posed. This means that given the input to the problem there exists a unique solution, which depends continuously on the input. Much theoretical work in the field of partial differential equations is devoted to proving that boundary value problems arising from scientific and engineering applications are in fact well-posed.

The aim of this thesis is to generalize the Ascoli-Arzelá theorem from the space of continuous functions to the space of functions with continuous of high order derivative on one hand. On the other hand, we use this generalization in studying the existence of the solutions of some integro-differential equations, non-linear differential equations of the second order with delay argument and some non-linear differential equations of high order. Additionally, in this thesis we study also the existence and uniqueness of the solutions of some second order non-linear differential equations by using Perov's fixed point theorem in the generalized metric spaces. It is worth mentioning that this study uses simpler conditions compared to those used in previous studies.

This thesis is composed of four chapters. In the first chapter we generalize the Ascoli-Arzelá theorem from the space of continuous functions to the space of functions with continuous of high order C^n. The aim of the second chapter consists of using the generalization of Ascoli- Arzelá theorem in the space C^1 in order to prove the compactness criteria and to use Schauder fixed point theorem in the space C^1 to prove the existences of a periodic solution for some nonlinear delay differential equations. Moreover, we use the well know Perov's fixed point theorem to prove the existence and uniqueness of a periodic solution. In the third chapter, we use a generalization of Ascoli-Arzelá theorem in the space C^n in order to prove the compactness criteria and to use Schauder fixed point theorem in the space C^n to prove the existences of a solution for a higher-order boundary value problem. In the last chapter, We study the existence of a solution of some nonlinear integral equations of the product type.

CHAPTER 1

Generalization of Ascoli- Arzelá theorem in C^n

1.1 Introduction

The problem of proving the compactness of various subsets of a given metric space is encountered quite frequently in analysis[32]. The Ascoli- Arzelá theorem is a fundamental theorem of analysis that allows us to determine if a subset of a space of continuous functions is compact. Although there are a number of ways to determine if a subset of a metric is compact, the Ascoli- Arzelá theorem is often easier to apply than general conditions of completeness and total boundedness when working in spaces of functions[32]. This theorem simplifies checking of compactness for subsets of spaces of continuous functions in much the same way the Heine-Borel theorem does for subsets of $\mathbb{R}^n$ [53]. It also has several applications in other branches of mathematics especially in ordinary differential equations and other branches of science and engineering. An important question: which subsets F of $C([a,b])$ are compact ?
We know that in $\mathbb{R}^n$ that closed and bounded sets are compact. Unfortunately, this is not true in $C([a,b])$.

Example 1.1.1. $\{f \in C([0,1]) / \|f\|_\infty \leq 1\}$ is not compact in $C([0,1])$. Consider the sequence of functions$f_n(x) = x^n$.

Figure 1.1: Cesare Arzelá and Guilio Ascoli

To characterize compactness in $C(X)$, we need a new definition:

Definition 1.1.1. Let X be a compact metric space, and let F be a subset of $C(X, \mathbb{R})$or $C(X, \mathbb{C})$. Then F is called equicontinuous if for every $\epsilon > 0$ there is $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in X$ satisfying $|x - y| < \delta$.
The family F is called equibounded if there is a constant M such that $\|f(x)\| \leq M$ for all $f \in F$ and for all $x \in X$.

The following result gives the Ascoli- Arzelá theorem in the space of continuous functions:

Theorem 1.1.1. *[51] (Ascoli- Arzelá theorem in C) Let X be a compact metric space, and let F be a subset of* $C(X, \mathbb{R})$*or* $C(X, \mathbb{C})$. *Then F is relatively compact if and only if F is equicontinuous and equibounded.*

From Ascoli- Arzelá theorem ,we deduce the following result

Proposition 1.1.1. *: A subset F of* $C(X, \mathbb{R})$ *or* $C(X, \mathbb{C})$ *is compact if and only if it is bounded, and equicontinuous.*

Application of Ascoli-Arzelá

Here we examine one application each to functional analysis and ordinary differential equation.

Application to Functional Analysis[16]

For $f \in C([a,b],\mathbb{R})$ let $(Tf)(x) = \int_a^x f(x)dt$. Then $Tf \in C([a,b],\mathbb{R})$, so T is a linear map from $C([a,b],\mathbb{R})$ to itself. Let $F = \{Tf : f \in C([a,b],\mathbb{R}), \|f\|_\infty \leq \alpha\}$. We would like to see whether F is equicontinuous.

$|(Tf)(x) - (Tf)(y)| = |\int_x^y f(t)dt| \leq |x-y|$. Hence F is an equicontinuous family. Also $|(Tf)(x) \leq (b-a)\alpha$. This implies that F is bounded. Hence $\overline{F}$ is compact.

Application to Ordinary Differential Equation (Peano's Theorem)[32] Let f be a continuous function from a neighborhood U of $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $(0, x_0) \in U$. Then there exists an $\epsilon > 0$ such that the initial-value problem

$$\frac{dx}{dt} = f(t, x(t)), x(0) = x_0 \tag{1.1.1}$$

has a solution x on $[0, \epsilon]$.

Hint: It is easy to see that the Cauchy problem (1.1.1) is equivalent to following Volterra integral equation:

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \tag{1.1.2}$$

and then, we use Schauder and Ascoli- Arzelá to prove the existence of a solution of Equation (1.1.1) on interval $[0, \epsilon]$ [32].

In the following section, we give a generalization of Ascoli-Arzelá theorem 1.1.1.

1.2 Generalization of Ascoli- Arzelá theorem in C^n

Before stating the main result in this chapter, we provide the following notations and definition.

Let E be a finite dimensional Banach space endowed with the norm $\|.\|_1$, and X be a compact subset of $\mathbb{R}$. We note by $C^n(X, E)$ the space of all functions with n continuous derivatives from X to E, this space is endowed with the norm $\|f\| = \sum_{i=0}^{n} \|f^{(i)}\|_\infty$ such that

$\|f\|_\infty = \sup_{x\in X}\{\|f(x)\|_1\}$.
For our purpose, we need the following definition in $C^n(X,E)$.

Definition 1.2.1. The family $F \subset C^n(X,E)$ is called equicontinuous if for every $\epsilon > 0$ there is $\delta > 0$ such that $\left\|f^{(i)}(x) - f^{(i)}(y)\right\|_1 < \varepsilon$ for all $i = 0,...,n$ and for all $x, y \in X$ satisfying $|x-y| < \delta$.
The family $F \subset C^n(X,E)$ is called equibounded if there is a constant M such that $\left\|f^{(i)}(x)\right\|_1 \leq M$ for all $i = 0,...,n$, for all $f \in F$ and for all $x \in X$.

The following result gives the Ascoli- Arzelá theorem in the space $C^n(X,E)$

Theorem 1.2.1. *Let F be a subset of $C^n(X,E)$. Then F is relatively compact if and only if F is equicontinuous and equibounded.*

Proof. Assume that F is relatively compact. This is means that $\overline{F}$ is compact. We claim that F is equicontinuous and equibounded. Since $\overline{F}$ is compact,then it is equibounded and since $F \subset \overline{F}$, we deduce that F is equibounded. To see that F is equicontinuous, let $\varepsilon > 0$, then there exist $f_1,...,f_m \in C^n(X,E)$ such that

$$F \subset B_{\frac{\varepsilon}{3(n+1)}}(f_1) \cup ... \cup B_{\frac{\varepsilon}{3(n+1)}}(f_m).$$

Since $f_j^{(i)}$ are uniformly continues, then there exists $\delta > 0$ such that for all $x, y \in X$, if $|x-y| < \delta$, then for all $i = 0,...,n$ and for all $j = 1,...,m$

$$\|f_j^{(i)}(x) - f_j^{(i)}(y)\|_1 < \frac{\varepsilon}{3}.$$

Let $f \in F$, then there exists $j \in \{1,...,m\}$ such that $f \in B_{\frac{\varepsilon}{3}}(f_j)$.
Hence, for all $i = 0,...,n$

$$\begin{aligned}\|f^{(i)}(x) - f^{(i)}(y)\|_1 \leq \|f^{(i)}(x) - f_j^{(i)}(x)\|_1 + \|f_j^{(i)}(x) - f_j^{(i)}(y)\|_1 \\ + \|f^{(i)}(y) - f_j^{(i)}(y)\|_1 < \varepsilon.\end{aligned}$$

Which implies that F is equicontinuous.
Conversely, assume that F is equicontinuous and equibounded. To show that F is relatively

compact it suffices to show that F is totally bounded, indeed if F is totally bounded, then $\overline{F}$ is also totally bounded, which implies that $\overline{F}$ is compact.
Since F is equicontinuous, then for all $x \in X$ and $\varepsilon > 0$, there exists $\delta_x > 0$ such that if $y \in X$ and $|x-y| < \delta_x$, we have for all $i = 0, ..., n$

$$\|f^{(i)}(x) - f^{(i)}(y)\|_1 < \frac{\varepsilon}{4(n+1)} \text{ for all } f \in F.$$

The collection $\{B_{\delta_x}(x)\}_{x\in X}$ is an open cover of the compact subset X, hence there exist $x_1, x_2, ..., x_m \in X$ such that $X = \bigcup_{j=1}^{m} B_{\delta_{x_j}}$.
Which implies that, for all $x \in B_{\delta_{x_j}}$ and for all $i = 0, ..., n$

$$\|f^{(i)}(x) - f^{(i)}(x_j)\|_1 < \frac{\varepsilon}{4(n+1)} \text{ for all } f \in F.$$

Since F is equibounded, then the set
$\mathcal{F} = \{(f(x_j), f'(x_j), ..., f^{(n)}(x_j)), j = 1, ..., m; f \in F\}$ is bounded.
Since a bounded set in E^{n+1} is totally bounded (because E is a finite dimensional), then there exists a subset
$\{(y_{1,i}, y_{2,i}, ..., y_{n+1,i}), i = 1, ..., k\} \subset E^{n+1}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^{k} B_{\frac{\varepsilon}{4(n+1)}}(y_{1,i}, y_{2,i}, ..., y_{n+1,i})$$

For any application $\varphi : \{1, ..., m\} \longrightarrow \{1, ..., k\}$, we define the set

$$\mathcal{F}_\varphi = \{f \in F : (f(x_j), f'(x_j), ..., f^{(n)}(x_j)) \in B_{\frac{\varepsilon}{4(n+1)}}(y_{1,\varphi_j}, y_{2,\varphi_j}, ..., y_{n+1,\varphi_j}), j = 1, ..., m\}.$$

It is clear that $F = \bigcup \mathcal{F}_\varphi$. Now, we show that the diameter of $\mathcal{F}_\varphi$ is less than ε.
Let $f, g \in \mathcal{F}_\varphi$ and $x \in X$, then there exists $j \in \{1, ..., m\}$ such that $x \in B_{\delta_{x_j}}$.

Hence, for all $i = 1, ..., n$

$$\begin{aligned}\|f^{(i)}(x) - g^{(i)}(x)\|_1 \leq & \|f^{(i)}(x) - f^{(i)}(x_j)\|_1 + \|f^{(i)}(x_j) - y_{i+1,\varphi_j}\|_1 \\ & + \|g^{(i)}(x_j) - y_{i+1,\varphi_j}\|_1 + \|g^{(i)}(x_j) - g^{(i)}(x)\|_1 \leq \varepsilon.\end{aligned}$$

which implies that the diameter of $\mathcal{F}_\varphi$ less than ε. Therefore, F can be covered by finitely many sets of diameter less than ε.

Thus F is totally bounded and the proof is completed. $\square$

CHAPTER 2

On The periodic solution of second order non linear delay differential equation

2.1 Introduction

Motivated by the papers [35, 36, 56] and the references therein, we consider the nonlinear second delay differential equations

$$x''(t) + p(t)x'(t) + q(t)x(t) = f(t, x(t), x(t - \tau(t)), x'(t - \tau(t))), t \in \mathbb{R}, \qquad (2.1.1)$$

The nonlinear delay differential equations arise in the modeling of many phenomena in physics, mechanics and Biology (see [5, 10, 20, 41, 42, 58] and the references therein).
The existence of such a periodic solution is of quite fundamental importance biologically since it concerns the long time survival of species, we refer to [4, 6, 7, 17, 25, 26, 29, 30, 33, 37, 43, 50, 52, 55, 18, 57] for some recent work on the subject of periodic solution for delay differential equations.
We begin with summarizing a few relative results done in the literature on the existence of periodic solutions for first and second delay differential equations.
In [6] Ardjouni and Djoudi have investigated the existence of positive periodic solutions to

the following first order delay differential equations

$$\frac{d}{dt}(x(t) - g(t, x(t - \tau(t)))) = r(t)x(t) - f(t, x(t - \tau(t))).$$

Raffoul ([50]) has considered the nonlinear neutral differential equation of the form

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau(t)) + q(t, x(t - \tau(t))), \tag{2.1.2}$$

which arises in a food-limited population models. The author has studied the existence of a periodic solution by using Krasnoselskii's fixed point theorem.
The second order nonlinear delay differential equations have been investigated by many authors. For example:
In [34] The authors have used the monotone iterative technique to prove the existence of a periodic solution for the following second order delay differential equation,

$$-x''(t) = f(t, x(t), x(t - \tau)).$$

Lia and Cheng in [35] have studied the existence of periodic solutions of the following second order delay differential equation

$$ax''(t) + bx(t) + g(x(t - \tau)) = p(t).$$

Liu and Ge [36] have studied the following second order nonlinear differential equation with delay and variable coefficients:

$$x''(t) + p(t)x'(t) + q(t)x(t) = \lambda h(t)f(t, x(t - \tau(t))) + r(t).$$

Wang et al. in [56] have established the existence and the uniqueness for the following general form of (2.1.2)

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t)x'(t - \tau(t)) + f(t, x(t), x(t - \tau(t)))$$

by using Krasnoselskii's fixed point theorem and Banach's fixed point theorem.
In most of the above works, the authors have transformed the delay differential equation to an integral equation defined on $C(\mathbb{R},\mathbb{R})$ (the space real continuous functions), where the derivative x' does not appear under the the integral sign, and then use Ascoli- Arzelá theorem and some fixed point theorems.
In general the second order nonlinear delay differential equations of the forme (2.1.1) can not be transformed to an integral equation defined on $C(\mathbb{R},\mathbb{R})$ and x' appears under the the integral sign, therefore, we can't use the well known Ascoli- Arzelá theorem in $C(\mathbb{R},\mathbb{R})$.
For this reason, our main task in this chapter consists of using the generalization of Ascoli-Arzelá theorem in the space $C^1(X,E)$ (the space real from a compact subset of $\mathbb{R}$ into a Banach space E with continuous first derivative) in order to prove the compactness criteria and to use Schauder fixed point theorem in the space C^1 to prove the existences of a periodic solution for (2.1.1) in section 2. In Section 3, we transform our problem (2.1.1) into of integral equations and we use the well know Perov's fixed point theorem to prove the existence and uniqueness of a periodic solution of (2.1.1).

2.2 Application of generalization of Ascoli- Arzelá in C^1 to the solution of second order delay differential equation

Let T be a positive constant, we consider the spaces

$$P_T = \{\psi \in C(\mathbb{R}), \psi(t+T) = \psi(t), \forall t \in \mathbb{R}\}.$$
$$P_T^1 = \{\psi \in C^1(\mathbb{R}), \psi(t+T) = \psi(t), \forall t \in \mathbb{R}\}.$$

It is clear that P_T^1 is a Banach space endowed with the norm
$\|x\| = \sup\limits_{t\in[0,T]} |x| + \sup\limits_{t\in[0,T]} |x'|.$
Equation (2.1.1) will be studied under the following assumptions:

(i) $f \in C(\mathbb{R}^4, \mathbb{R})$ and there exists $T > 0$ such that

$$f(t+T, x, y, z) = f(t, x, y, z), \forall (t, x, y, z) \in \mathbb{R}^4.$$

(ii) There exist $\beta, \gamma, \delta \geq 0$ and $\phi \in C(\mathbb{R}, \mathbb{R}^+)$ bounded such that

$$|f(t,u,v,w))| \leq \phi(t) + \beta|u| + \gamma|v| + \delta|w|, \forall (t,u,v,w) \in \mathbb{R}^4.$$

(iii) $p, q : \mathbb{R} \longrightarrow \mathbb{R}^+, \tau : \mathbb{R} \longrightarrow \mathbb{R}$ are all continuous $T-$periodic functions, $\int_0^T p(s) > 0$, $\int_0^T q(s) > 0$, and $\tau'(t) \neq 1$, for all $t \in [0,T]$.

Before stating the main result in this section, we need the following lemmas,

Lemme 2.2.1. *[36]Suppose that (iii) holds and*

$$R_1 \frac{\left(\exp\left(\int\limits_0^T p(u)du\right) - 1\right)}{Q_1 T} \geq 1,$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int\limits_t^{t+T} \frac{\exp\left(\int\limits_t^s p(u)du\right)}{\exp\left(\int\limits_0^T p(u)du\right) - 1} q(s)ds \right|, Q_1 = \left(1 + \exp\left(\int\limits_0^T p(u)du\right)\right)^2 R_1^2.$$

Then there are continuous $T-$periodic functions a and b such that $b(t) > 0$, $\int\limits_0^T a(u)du > 0$ and

$$a(t) + b(t) = p(t), b'(t) + a(t)b(t) = q(t), t \in \mathbb{R}.$$

Lemme 2.2.2. *[56] Suppose the conditions of Lemma 2.2.1 hold and $\psi \in P_T$.*

Then the equation

$$x'' + p(x)x'(t) + q(x)x(t) = \psi(t)$$

has a $T-$periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\psi(s)ds$$

where

$$G(t,s) = \frac{\int_t^s \exp\left(\int_t^u b(v)dv + \int_u^s a(v)dv\right) du + \int_s^{t+T} \exp\left(\int_t^u b(v)dv + \int_u^{s+T} a(v)dv\right) du}{\left(\exp\left(\int_0^T a(u)du\right) - 1\right)\left(\exp\left(\int_0^T b(u)du\right) - 1\right)}.$$

Corollary 2.2.1. *[56] The Green's function $G(t,s)$ satisfies the following properties:*

$$G(t,t+T) = G(t,t), G(t+T,s+T) = G(t,s),$$
$$\frac{\partial G(t,s)}{\partial t} = -b(t)G(t,s) + F(t,s).$$

where $F(t,s) = \dfrac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right) - 1}.$

Lemme 2.2.3. *[56] Let* $A = \int_0^T p(u)du, B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))du\right)$. *If* $A^2 \geq 4B$, *then we have*

$$\min\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := l$$
$$\max\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right) := L$$

Corollary 2.2.2. *[56] The function $G(t,s)$ satisfies*

$$m =: \frac{T}{(e^L-1)^2} \le G(t,s) \le \frac{T\exp\left(\int\limits_0^T p(u)du\right)}{(e^l-1)^2} := M.$$

It is easy to check,by using Lemma 2.2.2, that x is a solution of (2.1.1) in P_T^1 if and only if x is the solution of the following integral equation in P_T^1

$$x(t) = \int\limits_t^{t+T} G(t,s)[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))]ds. \tag{2.2.1}$$

Before stating our main result, we recall the following Schauder fixed point theorem.

Theorem 2.2.1. *[59] Let C be a nonempty bounded, closed and convex subset of a Banach space E and A is a continuous operator from C into itself. If $A(C)$ is relatively compact, then A has a fixed point.*

Under the hypothesis (i), (ii), (iii) and the previous lemmas and corollaries, we will make use of Schauder fixed point theorem to prove the following main result.

Theorem 2.2.2. *If the hypotheses (i), (ii) and (iii) hold, and if*

$$k = T\left(M\left(\|b\|_\infty + 1\right) + \|F\|_\infty\right) Max\left\{\delta, (\beta+\gamma)\right\} < 1. \tag{2.2.2}$$

Then, the second order delay differential equation (2.3.1) *has a periodic solution in $C^1(\mathbb{R})$.*

Proof. Solving Eq. (2.3.1) is equivalent to finding a fixed point of the operator A defined by the following expression

$$Ax(t) = \int\limits_t^{t+T} G(t,s)[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))]ds.$$

It is clear that the operator A is well defined from P_T^1 into itself, moreover

$$(Ax)'(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))]ds.$$

The proof is split into three steps.

Step I. There exists $\alpha > 0$ such that A transforms $C = \{x \in P_T^1, \|x\| \leq \alpha\}$ into itself.

It is clear that C nonempty bounded, closed and convex. Moreover, for all $x \in C$ and $t \in [0,T]$, we have

$$\begin{aligned} |Ax(t)| &= \left| \int_t^{t+T} G(t,s) \left[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))\right] ds \right| \\ &\leq \int_t^{t+T} |G(t,s)| \left[\phi(s) + \beta|x(s)| + \gamma|x(s-\tau(s))| + \delta|x'(s-\tau(s))|\right] ds \\ &\leq MT \left[\|\phi\|_\infty + (\beta+\gamma)\|x\|_\infty + \delta\|x'\|_\infty\right] \end{aligned} \tag{2.2.3}$$

and

$$\begin{aligned} |(Ax)'(t)| &= \left| \int_t^{t+T} \frac{\partial G(t,s)}{\partial t} \left[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))\right] ds \right| \\ &\leq \int_t^{t+T} \left|\frac{\partial G(t,s)}{\partial t}\right| \left[\phi(s) + \beta|x(s)| + \gamma|x(s-\tau(s))| + \delta|x'(s-\tau(s))|\right] ds \\ &\leq \int_t^{t+T} \left|\frac{\partial G(t,s)}{\partial t}\right| \left[\|\phi\|_\infty + (\beta+\gamma)\|x\|_\infty + \delta\|x'\|_\infty\right] ds \\ &\leq \int_t^{t+T} (|b(t)||G(t,s)| + |F(t,s)|) \left[\|\phi\|_\infty + (\beta+\gamma)\|x\|_\infty + \delta\|x'\|_\infty\right] ds \\ &\leq T\left(\|b\|_\infty M + \|F\|_\infty\right) \left[\|\phi\|_\infty + (\beta+\gamma)\|x\|_\infty + \delta\|x'\|_\infty\right], \end{aligned} \tag{2.2.4}$$

where $\|b\|_\infty = \max_{t\in[0,T]}\{|b(t)|\}$ and $\|F\|_\infty = \max_{0\leq t\leq T, t\leq s\leq t+T}\{|F(t,s)|\}$.

Hence, by (4.2.1) and (4.2.2), we obtain

$$\begin{aligned}
\|Ax\| &\leq T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\left[\|\phi\|_\infty+(\beta+\gamma)\|x\|_\infty+\delta\|x'\|_\infty\right] \\
&\leq T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\|\phi\|_\infty+k\|x\| \\
&\leq T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\|\phi\|_\infty+k\alpha,
\end{aligned}$$

where k is defined by (2.3.1).

We deduce that, A transforms C into itself if

$$T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\|\phi\|_\infty+k\alpha \leq \alpha$$

which implies, under the condition (2.3.1), that

$$\frac{T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\|\phi\|_\infty}{1-k} \leq \alpha.$$

Then, A transforms C into itself for

$$\alpha=\frac{T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\|\phi\|_\infty}{1-k}.$$

Step 2. The operator A is continuous.

Let $(x_n) \in C$ be a convergence sequence to $x \in C$, which implies that $(x_n^{(i)})$ converges to $x^{(i)}$ $(i=1,2)$ in the space $C([0,T],[-\alpha,\alpha])$.

Since f is uniformly continuous on the compact set $[0,T]\times[-\alpha,\alpha]^4$,

then the sequence $(f(s,x_n(s),x_n(s-\tau(s)),x_n'(s-\tau(s))))$ converges to

$f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))$ in $C([0,T],\mathbb{R})$. It follows that,

$$\begin{aligned}
\|Ax_n-Ax\| \leq T\left(M\left(\|b\|_\infty+1\right)+\|F\|_\infty\right)\|f(s,x_n(s),x_n(s-\tau(s)),x_n'(s-\tau(s))) \\
-f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)))\|_\infty.
\end{aligned}$$

Which implies that (Ax_n) converges to Ax and the operator A is continuous.

Step 3. $A(C)$ is relatively compact, it is clear that $A(C)$ is equibounded.

Now, to show that $A(C)$ is equicontinuous, take t_1 and t_2 in I.

Let $H(s) = f(s, x(s), x(s-\tau(s)), x'(s-\tau(s)))$, by the assumption (ii), we have

$$\|H\|_\infty \leq \|\phi\|_\infty + \alpha \max(\beta + \gamma, \delta)$$

It follows that

$$
\begin{aligned}
|Ax(t_1) - Ax(t_2)| &= \left| \int_{t_1}^{t_1+T} G(t_1,s)H(s)ds - \int_{t_2}^{t_2+T} G(t_2,s)H(s)ds \right| \\
&\le \left| \int_{t_1}^{t_1+T} G(t_1,s)H(s)ds - \int_{t_1}^{t_1+T} G(t_2,s)H(s)ds \right| \\
&+ \left| \int_{t_1}^{t_1+T} G(t_2,s)H(s)ds - \int_{t_2}^{t_2+T} G(t_2,s)H(s)ds \right| \\
&\le \int_{t_1}^{t_1+T} |G(t_1,s) - G(t_2,s)||H(s)|ds + |\int_{t_1}^{t_2} G(t_2,s)H(s)ds \\
&+ \int_{t_2}^{t_2+T} G(t_2,s)H(s)ds + \int_{t_2+T}^{t_1+T} G(t_2,s)H(s)ds - \int_{t_2}^{t_2+T} G(t_2,s)H(s)ds| \\
&\le \int_{t_1}^{t_1+T} |G(t_1,s) - G(t_2,s)||H(s)|ds + \int_{t_1}^{t_2} |G(t_2,s)H(s)|ds \\
&+ \int_{t_1+T}^{t_2+T} |G(t_2,s)H(s)|ds \\
&\le \|H\| \int_{t_1}^{t_1+T} |(G(t_1,s) - G(t_2,s)|ds \\
&+ \|H\| \int_{t_1}^{t_2} |G(t_2,s)|ds + \|H\| \int_{t_2+T}^{t_1+T} |G(t_2,s)|ds \\
&\le \|H\| \int_{t_1}^{t_1+T} |G(t_1,s) - G(t_2,s)|ds + 2M\|H\||t_1 - t_2| \\
&\le (\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)) \int_{t_1}^{t_1+T} |G(t_1,s) - G(t_2,s)|ds \\
&+ 2M\left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right)|t_1 - t_2|.
\end{aligned}
\tag{2.2.5}
$$

Using a similar argument as above, we prove that

$$\begin{aligned}
|A'x(t_1) - A'x(t_2)| &\leq \|H\| \int_{t_1}^{t_1+T} \left| \frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t} \right| ds \\
&+ \|H\| \int_{t_1}^{t_2} \left| \frac{\partial G(t_2,s)}{\partial t} \right| ds + \|H\| \int_{t_2+T}^{t_1+T} \left| \frac{\partial G(t_2,s)}{\partial t} \right| ds \\
&\leq \|H\| \int_{t_1}^{t_1+T} \left| \frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t} \right| ds \qquad (2.2.6) \\
&+ 2\left(M\|b\|_\infty + \|F\|_\infty\right) \|H\| |t_1 - t_2| \\
&\leq \left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right) \int_{t_1}^{t_1+T} \left| \frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t} \right| ds \\
&+ 2\left(M\|b\|_\infty + \|F\|_\infty\right) \left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right) |t_1 - t_2|.
\end{aligned}$$

Now, let $\varepsilon > 0$, since the functions $G(t,s)$ and $\dfrac{\partial G(t,s)}{\partial t}$ are uniformly continuous on the compact set $[0,T] \times [0,2T]$, then there exists $\delta_1 > 0$ such that, if $|t_2 - t_1| \leq \delta_1$, we have for all $\forall s \in [0,2T]$

$$|G(t_2,s) - G(t_1,s)| \leq \frac{\varepsilon}{2T\left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right)},$$

and

$$\left| \frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t} \right| \leq \frac{\varepsilon}{2T\left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right)}, \forall s \in [0,2T].$$

Then from (2.2.5) and (4.2.7), if $|t_2 - t_1| \leq \delta = \min(\delta_1, \delta_2, \delta_3)$, where

$$\begin{cases} \delta_2 = \dfrac{\varepsilon}{4M\left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right)}, \\ \delta_3 = \dfrac{\varepsilon}{4\left(M\|b\|_\infty + \|F\|_\infty\right)\left(\|\phi\|_\infty + \alpha \max(\beta+\gamma,\delta)\right)}, \end{cases}$$

We deduce, for $i = 0, 1$, that

$$|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| \leq \varepsilon$$

Consequently, the set $A(C)$ is equicontinuous.

Hence, by Theorem (1.2.1), $A(C)$ is relatively compact.

The proof of Theorem 2.2.2 then follows from Schauder fixed point theorem. □

Example 2.2.1. Consider the following second-order delay differential equation:

$$x''(t) + p(t)x'(t) + q(t)x(t) = \cos(6t) + \frac{r}{2}x(t) + \frac{r}{2}x(t - \tau(t)) + rx'(t - \tau(t)), t \in \mathbb{R}, \tag{2.2.7}$$

where $p(t) = \dfrac{1}{2}, q(t) = \dfrac{1}{16}, \tau(t) = 1 + \sin(6t)$.

Hence, by using the notations of Theorem 2.2.2, we have $T = \dfrac{\pi}{3}, \phi(t) = \cos(6t)$, $\beta = \gamma = \dfrac{r}{2}, \delta = r$, where r is a positive number. We may see that the conditions of Lemma 2.2.1 hold, and $a(t) = b(t) = \dfrac{1}{4}$, $F(t, s) = \dfrac{\exp\left(\frac{1}{4}(s-t)\right)}{\exp\left(\frac{1}{4}T\right) - 1}$,

$$G(t, s) = \frac{(s-t)\exp\left(\frac{1}{4}(s-t)\right) + (t + \frac{\pi}{3} - s)\exp\left(\frac{1}{4}(s + \frac{\pi}{3} - t)\right)}{\left(\exp(\frac{\pi}{12})\right)^2}.$$

By using the notations of Lemma 2.2.3, we have $A = \dfrac{\pi}{6}, B = \dfrac{\pi^2}{144}$.

Which implies that $l = \dfrac{\pi}{12}$, $M = \dfrac{\pi\exp\left(\frac{\pi}{6}\right)}{\left(\exp\left(\frac{\pi}{12}\right) - 1\right)^2}$ and $\|F\|_\infty = \dfrac{\exp\left(\frac{\pi}{12}\right)}{\exp\left(\frac{\pi}{12}\right) - 1}$.

Therefore, the inequality in Theorem 2.2.2 takes the form

$$\frac{\pi}{6}\left(\frac{\pi\exp\left(\frac{\pi}{6}\right)}{\left(\exp\left(\frac{\pi}{12}\right) - 1\right)^2}\left(1 + \frac{1}{4}\right) + \frac{\exp\left(\frac{\pi}{12}\right)}{\exp\left(\frac{\pi}{12}\right) - 1}\right) r < 1.$$

Then by Theorem 2.2.2, we conclude that the second-order delay differential equation (2.2.7) has a periodic solution if $r < \dfrac{24\left(\exp\left(\frac{\pi}{12}\right) - 1\right)^2}{5\pi^2\exp\left(\frac{\pi}{6}\right) + 4\pi\exp\left(\frac{\pi}{12}\right)\left(\exp\left(\frac{\pi}{12}\right) - 1\right)}$.

2.3 Existences and uniqueness of period solution by using Perov's fixed point theorem

In this section, we recall the following notations and results in generalized metric spaces.

Definition 2.3.1. [49] Let X be a nonempty set and $d : X \times X \longrightarrow \mathbb{R}^n$ be a mapping such that for all $x, y, z \in X$, one has :
i) $d(x,y) \geq 0_{\mathbb{R}^n}$ and $d(x,y) = 0_{\mathbb{R}^n} \Longleftrightarrow x = y$,
ii) $d(x,y) = d(y,x)$,
iii) $d(x,y) \leq d(x,z) + d(z,y)$,
where for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$, we have $x \leq y \Longleftrightarrow x_i \leq y_i$, for any $i = \overline{1,n}$.
Then d is called a generalized metric and (X,d) is a generalized metric space.

Definition 2.3.2. [49] If (E,d) is a generalized complete metric space and $T : E \rightarrow E$ which satisfies the inequality

$$d(Tx, Ty) \leq Ad(x,y) \text{ for all } x, y \in E,$$

where A is a matrix convergent to zero (the norms of it's eigenvalues are in the interval $[0,1)$). We say that T is a Picard operator or generalized contraction.

We recall the following Perov's fixed point theorem.

Theorem 2.3.1. *[49] Let (E,d) be a complete generalized metric space. If $T : E \rightarrow E$ is a map for which there exists a matrix $A \in M_n(\mathbb{R})$ such that*

$$d(Tx, Ty) \leq Ad(x,y), \ \forall x, y \in E$$

and the norms of the eigenvalues of A are in the interval $[0,1)$, then T has a unique fixed point $x^* \in E$ and the sequence of successive approximations $x_m = T^m(x_0)$ converges to x^* for any $x_0 \in E$. Moreover, the following estimation holds

$$d(x_m, x^*) \leq A^m (I_n - A)^{-1} d(x_0, x_1), \ \forall m \in \mathbb{N}^*.$$

We consider the following functional spaces

$$
\begin{aligned}
P(T) &= \{x \in C(\mathbb{R}) : x(t+T) = x(t),\ \forall t \in \mathbb{R}\} \\
P^1(T) &= \left\{x \in C^1(\mathbb{R}) : x(t+T) = x(t),\ \forall t \in \mathbb{R}\right\} \\
K^+(T) &= \left\{x \in P^1(T) :\ x(t) \geq 0,\ \forall t \in \mathbb{R}\right\}
\end{aligned}
$$

and denote by E the product space $E = K^+(T) \times P(T)$ which is a generalized metric space with the generalized metric $d_C : E \times E \to \mathbb{R}^2$, defined by

$$
d_C((x_1, y_1), (x_2, y_2)) = \left(\|x_1 - x_2\| + \left\|x_1' - x_2'\right\|, \|y_1 - y_2\|\right)
$$

where $\|u\| = \max\{|u(t)| : t \in [0, T]\}$ for any $u \in P(T)$.

Lemme 2.3.1. *[11] (E, d_C) is a complete generalized metric space.*

It is easy to check, under the above assumptions, that x is a solution of (2.1.1) in K^+ (see, [8]) if and only if x is the solution of the following integral equation

$$
x(t) = \int_t^{t+T} G(t,s)[f(s, x(s), x(s-\tau(s)), x'(s-\tau(s)))]ds. \tag{2.3.1}
$$

Equation (2.1.1) will be studied under the following assumptions:

(i) $f \in C(\mathbb{R} \times (\mathbb{R}^+)^2 \times \mathbb{R}, \mathbb{R})$ and there exists $T > 0$ such that

$$
f(t+T, x, y, z) = f(t, x, y, z), \forall (t, x, y, z) \in \mathbb{R} \times (\mathbb{R}^+)^2 \times \mathbb{R}.
$$

(ii) There exist $\alpha, \beta, \gamma \geq 0$ such that

$$
|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2| + \gamma |w_1 - w_2|,
$$

$\forall t \in \mathbb{R}, \forall (u_1, u_2, v_1, v_2) \in (\mathbb{R}^+)^4, \forall (w_1, w_2) \in \mathbb{R}^2.$

(iii) $p, q : \mathbb{R} \longrightarrow \mathbb{R}^+, \tau : \mathbb{R} \longrightarrow \mathbb{R}$ are all continuous $T-$periodic functions, $\int_0^T p(s) > 0$, $\int_0^T q(s) > 0$, and $\tau'(t) \neq 1$, for all $t \in [0, T]$.

Under the hypothesis (i), (ii), (iii) and the previous lemmas and corollaries, we will make use of Perov's fixed point theorem to prove the following main result.

Theorem 2.3.2. *If the hypotheses (i), (ii) and (iii) hold, and if*

$$T\left(\gamma\theta + (\alpha+\beta)(M+\theta)\right) < 1, \tag{2.3.2}$$

such that $\theta = \|b\|M + \dfrac{\exp(L)}{l-1}$. Then, the second order delay differential equation (2.3.1) *has a unique positive periodic solution in $K^+(T)$.*

Proof. If we differentiate the equation (2.3.1) with respect to t and denoting $x'(t) = y(t)$, we obtain, by using Corollary 2.2.1, for all $t \in \mathbb{R}$,

$$y(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds,$$

which leads to,

$$\begin{cases} x(t) = \displaystyle\int_t^{t+T} G(t,s)[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds, \\ y(t) = \displaystyle\int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds \end{cases}$$

Let $A : E \to C(\mathbb{R}) \times C(\mathbb{R})$ the map defined by the following expression

$$A(x,y)(t) = \begin{pmatrix} A_1(x,y)(t) \\ \\ A_2(x,y)(t) \end{pmatrix},$$

where,

$$A_1(x,y)(t) = \int_t^{t+T} G(t,s)[f(s, x(s), x(s-\tau(s)), y(s-\tau(s)))]ds,$$

and,

$$A_2(x,y)(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s,x(s),x(s-\tau(s)),y(s-\tau(s)))]ds. \tag{2.3.3}$$

The rest of the proof is divided into claims.

Claim 1: The operator A transform E into itself.

It is clair, from Conditions (i) and Corollary 2.2.1, that $A_1(E) \subset C^1(\mathbb{R})$.

Moreover, from Conditions (i), (iii) and Corollary 2.2.1, it follows that $\forall t \in \mathbb{R}, \forall (x,y) \in E$, $A_1(x,y)(t) \geq 0$ and

$$\begin{aligned}
A_1(x,y)(t+T) &= \int_{t+T}^{t+2T} G(t+T,s)[f(s,x(s),x(s-\tau(s)),y(s-\tau(s)))]ds \\
&= \int_t^{t+T} G(t+T,s+T) \\
&\times [f(s+T,x(s+T),x(s+T-\tau(s+T)),y(s+T-\tau(s+T)))]ds \\
&= A_1(x,y)(t).
\end{aligned}$$

Hence, $A_1(E) \subset K^+(T)$. Similarly, we have,

$$\begin{aligned}
A_2(x,y)(t+T) &= \int_{t+T}^{t+2T} \frac{\partial G(t+T,s)}{\partial t}[f(s,x(s),x(s-\tau(s)),y(s-\tau(s)))]ds \\
&= \int_t^{t+T} \frac{\partial G(t+T,s+T)}{\partial t} \\
&\times [f(s+T,x(s+T),x(s+T-\tau(s+T)),y(s+T-\tau(s+T)))]ds \\
&= A_2(x,y)(t), \ \forall t \in \mathbb{R}, \forall (x,y) \in E.
\end{aligned}$$

We deduce that, $A(E) \subset E$.

Claim 2: The operator A is a generalized contraction.

From Condition (ii), we have

$$
\begin{aligned}
&|A_1(x_1,y_1)(t) - A_1(x_2,y_2)(t)| + \left|A_1'(x_1,y_1)(t) - A_1'(x_2,y_2)(t)\right| \\
\leq &\int_t^{t+T} G(t,s)\left[\alpha\left|x_1(s)-x_2(s)\right| + \beta\left|x_1(s)-x_2(s)\right| + \gamma\left|y_1(s)-y_2(s)\right|\right] ds \\
+ &\int_t^{t+T} \frac{\partial G(t,s)}{\partial t}\left[\alpha\left|x_1(s)-x_2(s)\right| + \beta\left|x_1(s)-x_2(s)\right| + \gamma\left|y_1(s)-y_2(s)\right|\right] ds \\
\leq &TM(\alpha+\beta)\|x_1-x_2\| + TM\gamma\|y_1-y_2\| + \left(\|b\|M + \frac{\exp(L)}{l-1}\right) \\
&\times T[(\alpha+\beta)\|x_1-x_2\| + \gamma\left(\|b\|M + \frac{\exp(L)}{l-1}\right)(\|y_1-y_2\|)] \\
&\leq T(\alpha+\beta)\left(M + \|b\|M + \frac{\exp(L)}{l-1}\right)\left(\|x_1-x_2\| + \|x_1'-x_2'\|\right) \\
+ &T\gamma\left(M + \|b\|M + \frac{\exp(L)}{l-1}\right)\|y_1-y_2\|
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|A_2(x_1,y_1)(t) - A_2(x_2,y_2)(t)| \leq &\int_t^{t+T} \frac{\partial G(t,s)}{\partial t} \\
&\times\left[\alpha\left|x_1(s)-x_2(s)\right| + \beta\left|x_1(s)-x_2(s)\right| + \gamma\left|y_1(s)-y_2(s)\right|\right] ds \\
\leq &T\left(\underbrace{\|b\|M + \frac{\exp(L)}{l-1}}_{=\theta}\right)\left[(\alpha+\beta)\|x_1-x_2\| + \gamma\|y_1-y_2\|\right] \\
\leq &T(\alpha+\beta)\theta\left(\|x_1-x_2\| + \|x_1'-x_2'\|\right) \\
&+ T\gamma\theta\|y_1-y_2\|]
\end{aligned}
$$

We deduce that,

$$
d_C\left(A\left(x_1,y_1\right), A\left(x_2,y_2\right)\right) \leq K\begin{pmatrix} \|x_1-x_2\| + \|x_1'-x_2'\| \\ \|y_1-y_2\| \end{pmatrix},
$$

where the matrix K is given by,

$$K = \begin{pmatrix} T(\alpha+\beta)(M+\theta) & T\gamma(M+\theta) \\ \\ T(\alpha+\beta)\theta & T\gamma\theta \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\begin{cases} \lambda_1 = T[\gamma\theta + (\alpha+\beta)(M+\theta)] \\ \lambda_2 = 0 \end{cases}$$

Since the norms of the eigenvalues are in the interval $[0,1)$, then, by Perov's fixed point theorem, the operator T has a unique solution $x^* = (x_*, y_*) \in K^+(\omega) \times P(\omega)$, which implies that $x_* \in C^1(\mathbb{R})$, and for all $t \in \mathbb{R}$

$$(x_*)'(t) = \int_t^{t+T} \frac{\partial G(t,s)}{\partial t} [f(s, x_*(s), x_*(s-\tau(s)), y_*(s-\tau(s)))] ds$$

Hence, by using (2.3.3), for all $t \in \mathbb{R}$

$$((x_*)' - y_*)(t) = 0.$$

We deduce, that $(x_*)' = y_*$ and x_* is the unique solution of (2.3.1). $\square$

The following proposition gives an estimation of the error between the exact solution and the approximate solution of (2.3.1).

Proposition 2.3.1. *Under the assumptions of Theorem 2.3.2, the solution of the equation* (2.3.1)*, which is obtained by the successive approximations method starting from any $x^0 = (x_0, y_0) \in E$, verifies the following estimation:*

$$d_C(x^m, x^*) \leq \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times d_C(x^1, x^0),$$

where

$$\begin{cases} e_1 = & \dfrac{(M+\gamma)\lambda_1^{m-1}T(\alpha+\beta)}{\lambda_1-1} \\ e_2 = & \dfrac{\gamma\lambda_1^{m-1}T(\alpha+\beta)}{1-\lambda_1} \\ e_3 = & \dfrac{(\gamma+M)\lambda_1^{m-1}T\theta[2T(M+\gamma)(\alpha+\beta)-1]}{\lambda_1-1} \\ e_4 = & \dfrac{\lambda_1^{m-1}T\theta\gamma[T(1+M+\gamma)(\alpha+\beta)-1]}{\lambda_1-1} \end{cases} \tag{2.3.4}$$

Proof. From Theorem 2.3.1, by the conditions of Theorem 2.3.2, one has that

$$d_C(x^m, x^*) \leq A^m(I-A)^{-1}d_C(x^1, x^0),\ \forall m \in \mathbb{N}^*.$$

We have,

$$A^m = \begin{pmatrix} (M+\gamma)\lambda_1^{m-1}T(\alpha+\beta) & (M+\gamma)\lambda_1^{m-1}T\theta \\ \gamma\lambda_1^{m-1}T(\alpha+\beta) & \lambda_1^{m-1}T\theta\gamma \end{pmatrix},$$

And we find

$$A^m = \lambda_1^{m-1}T \begin{pmatrix} (M+\gamma)(\alpha+\beta) & (M+\gamma)\theta \\ \gamma(\alpha+\beta) & \theta\gamma \end{pmatrix},$$

And we have

$$(I-A)^{-1} = \frac{1}{\lambda_1-1}\begin{pmatrix} -1+T\theta\gamma & -T\theta(M+\gamma) \\ -T(\alpha+\beta)\gamma & -1+T(\alpha+\beta)(M+\gamma) \end{pmatrix},$$

Which implies that,

$$A^m(I-A)^{-1} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix},$$

where $e_i, i = 1, ..., 4$ are given by (2.3.4). □

To illustrate this result, we have the following example.

Example 2.3.1. Consider the following second-order delay differential equation:

$$x''(t) + p(t)x'(t) + q(t)x(t) = \frac{1}{\lambda}x(t) + \frac{1}{\lambda}x(t-\tau(t)) + \frac{10}{\lambda}x'(t-\tau(t)), t \in \mathbb{R}, \lambda \geq 10^3 \tag{2.3.5}$$

where $p(t) = \frac{1}{2}, q(t) = \frac{1}{16}, \tau(t) = 1 + \sin(6t)$.
Hence, by using the notations of Theorem 2.3.2, we have $T = \frac{\pi}{3}$,
$\alpha = \beta = \frac{1}{\lambda}, \gamma = \frac{10}{\lambda}$, where $\lambda = 10^3$ is a positive number. We may see that the conditions of Lemma 2.2.1 hold, and $a(t) = b(t) = \frac{1}{4}$, $F(t,s) = \dfrac{\exp\left(\frac{1}{4}(s-t)\right)}{\exp\left(\frac{1}{4}T\right)-1}$,

$$G(t,s) = \frac{(s-t)\exp\left(\frac{1}{4}(s-t)\right) + (t+\frac{\pi}{3}-s)\exp\left(\frac{1}{4}(s+\frac{\pi}{3}-t)\right)}{\left(\exp(\frac{\pi}{12})\right)^2}.$$

By using the notations of Lemma 2.2.3, we have $A = \frac{\pi}{6}, B = \frac{\pi^2}{144}$.
Which implies that $l = L = \frac{\pi}{12}$, $M = \dfrac{\pi\exp\left(\frac{\pi}{6}\right)}{\left(\exp\left(\frac{\pi}{12}\right)-1\right)^2}$ and $\|F\|_\infty = \dfrac{\exp\left(\frac{\pi}{12}\right)}{\exp\left(\frac{\pi}{12}\right)-1}$.
We find $\theta = 19.149, M = 59.25$and by using the previous values of $T, M, \alpha, \beta, \gamma$ therefore, the inequality in Theorem 2.3.2 takes the form

$$T\left(\gamma\theta + (\alpha+\beta)(M+\theta)\right) < 1.$$

Then by Theorem 2.3.2, we conclude that the second-order delay differential equation (2.3.5) has a periodic solution.

CHAPTER 3

The existence of solution for some classes of Higher order boundary value problem

3.1 Introduction

In this chapter, we consider the following higher-order boundary value problem:

$$\begin{cases} u^{(n)} + f(t, u, u', ..., u^{(n-2)}) = 0, n \geq 2, t \in I = [0,1], \\ u^{(i)}(0) = 0, 0 \leq i \leq n-3. \end{cases} \tag{3.1.1}$$

with the conditions

$$\begin{cases} \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0. \end{cases} \tag{3.1.2}$$

on the one hand and with the conditions

$$\begin{cases} u^{(n-2)}(0) = 0, \\ \alpha u(\eta) = u(1). \end{cases} \tag{3.1.3}$$

on the other hand, where n is a given positive integer, $\alpha, \gamma > 0, \beta, \delta \geq 0$, $0 < \eta < 1, 0 < \alpha\eta^{n-1} < 1$, f is continuous and satisfies $|f(s, u_0, u_1, ..., u_{n-2})| \leq a(s) + \sum_{k=0}^{n-2} b_k|u_k|$ such that a is continuous on I and $b_k \in \mathbb{R}^+, k = 0, ..., n-2$.
Equation (3.1.1) and its particular forms have been studied by many authors (see for example [3, 46, 47, 2, 24, 23, 19, 22, 40, 39, 44] and the references therein).
Wong and Agarwal in [47] and Patricia et al. in [46] have studied the following boundary value problem:

$$\begin{cases} u^{(n)} + \lambda Q(t, u, u', ..., u^{(n-2)}) = \lambda P(t, u, u', ..., u^{(n-2)}) \\ u^{(i)}(0) = 0, 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases}$$

under the following condition: there exist continuous functions $f : (0, +\infty) \longrightarrow (0, +\infty)$ and $p_1, p, q_1, q : (0, 1) \longrightarrow \mathbb{R}$ such that

$$(i)\, q(t) \leq \frac{Q(t, u_0, u_1, ..., u_{n-2})}{f(u)} \leq q_1(t), p(t) \leq \frac{P(t, u_0, u_1, ..., u_{n-2})}{f(u)} \leq p_1(t).$$

$$(ii)\, q(t) - p_1(t) \geq 0.$$

Agarwal and Wong [2] have studied the existence of a positive solution for the problem (3.1.1) under the following condition: there exists $L \geq 0$ such that

$$f(t, u, u', ..., u^{(n-2)}) + L \geq 0 \text{ on } [0, 1] \times [0, \infty)^{n-1},$$
$$\int_0^1 g(s, s)[f(s, u, u', ..., u^{(n-2)}) + L]ds \leq \lambda,$$

and some other conditions, where the function g is defined in (3.2.2).
Chyan and Henderson [19] have studied the existence of a positive solution of the following

problem,

$$\begin{cases} u^{(n)} + \lambda q(t) f(u) = 0, \\ u^{(i)}(0) = u^{(n-2)}(1) = 0, 0 \leq i \leq n-2. \end{cases}$$

such that f and q are continuous and nonnegative functions.
The following analogical problem have studied by Eloe and Ahmad in[21],

$$\begin{cases} u^{(n)} + f(t,u) = 0, t \in (0,1) \\ u^{(i)}(0) = 0, 0 \leq i \leq n-2, \\ \alpha u(\eta) = u(1), 0 < \eta < 1, \end{cases}$$

The following more general form have studied by J. R. Graef and T. Moussaoui in [27],

$$\begin{cases} u^{(n)} + f(t,u) = 0, t \in (0,1) \\ u^{(i)}(0) = 0, 0 \leq i \leq n-2, \\ \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i) = u(1), 0 < \eta < 1, \end{cases}$$

where the derivatives $x^{(i)}, 0 \leq i \leq n-2$ do not appear in the nonlinear terms.
Our main task in this paper consists of using the generalization of Ascoli-Arzelá theorem in the space $C^n(X, E)$ (the space of functions from a compact subset of $\mathbb{R}$ into a Banach space E with continuous nth derivative) in order to prove the compactness criteria and to use Schauder fixed point theorem in the space C^n to prove the existences of a solution for the higher-order boundary value problem (3.1.1).

3.2 Application of generalization of Ascoli- Arzelá theorem in C^n to the solution of a higher-order boundary value problem

In this section, we study the existence of a solution for the problem (3.1.1) with the conditions (3.1.2).
It is easy to check, (see [2]), that u is a solution of (3.1.1) in $C^n(I,\mathbb{R})$ if and only if u is a solution of the following integro-differential equation:

$$u(t)=\int_0^1 G(t,s)f(s,u,u',...,u^{(n-2)})ds, \tag{3.2.1}$$

in $C^{n-2}(I,\mathbb{R})$, such that $g(t,s)=\dfrac{\partial^{n-2}G(t,s)}{\partial t^{n-2}}$ is the Green's function of the second order boundary value problem

$$\begin{cases} -u^{(2)}=0, t\in[0,1],\\ \alpha u(0)-\beta u'(0)=0,\\ \gamma u(1)+\delta u'(1)=0.\end{cases}$$

Moreover,

$$g(t,s)=\frac{1}{\alpha\gamma+\alpha\delta+\beta\gamma}\begin{cases}(\beta+\alpha s)[\delta+\gamma(1-t)], 0\le s\le t,\\ (\beta+\alpha t)[\delta+\gamma(1-s)], t\le s\le 1.\end{cases} \tag{3.2.2}$$

Equation (3.2.1) will be studied under the following assumptions:

(i) $f\in C(I\times\mathbb{R}^{n-1},\mathbb{R})$.

(ii) There exist a function $a\in C(I,\mathbb{R}^+)$ and constants $b_k\in\mathbb{R}^+ (k=0,...,n-2)$ such that

$$|f(s,u_0,u_1,...,u_{n-2})|\le a(s)+\sum_{k=0}^{n-2}b_k|u_k|$$

Under the assumptions $(i)$and (ii), we will make use of Schauder fixed point theorem to prove the following main result.

Theorem 3.2.1. *If the hypotheses (i), (ii) hold, and if*

$$r\sum_{i=0}^{n-2}\|\int_0^1|\partial_1^{(i)}G(t,s)|ds\|_\infty<1$$

such that $r=Max\{b_0,...,b_{n-2}\}$.
Then, the integro-differential equation (3.2.1) has a solution in $C^{n-2}(I,\mathbb{R})$.

Proof. Solving Equation (3.2.1) is equivalent to finding a fixed point of the operator A defined in the space $E=C^{n-2}(I,\mathbb{R})$ by the following expression

$$Ax(t)=\int_0^1 G(t,s)f(s,x,x',...,x^{(n-2)})ds.$$

It is clear that the operator A is well defined from E into itself.
Moreover for all $x\in E$, $t\in I$ and $i=0,...,n-2$, we have

$$(Ax)^{(i)}(t)=\int_0^1 \partial_1^{(i)}G(t,s)f(s,x,x',...,x^{(n-2)})ds.$$

The proof is split into three steps.

Step I. There exists $\alpha>0$ such that A transforms $C=\{x\in E,\|x\|\leq\alpha\}$ into itself. It is clear that C is nonempty, bounded, closed and convex subset of E.

Moreover, for all $x \in C$, $t \in I$ and $i = 0, ..., n-2$, we have

$$\begin{aligned} |(Ax)^{(i)}(t)| &= \left| \int_0^1 \partial_1^{(i)} G(t,s) f(s, x, x', ..., x^{(n-2)}) ds \right| \\ &\leq \int_0^1 |\partial_1^{(i)} G(t,s)| \left(a(s) + \sum_{k=0}^{n-2} b_k |x^{(k)}(s)| \right) ds \qquad (3.2.3) \\ &\leq \left(\|a\|_\infty + \sum_{k=0}^{n-2} b_k \|x^{(k)}\|_\infty \right) \int_0^1 |\partial_1^{(i)} G(t,s)| ds. \end{aligned}$$

Hence, for $r = Max\{b_0, ..., b_{n-2}\}$, we obtain

$$\begin{aligned} \|Ax\| &= \sum_{i=0}^{n-2} \|A^{(i)} x\|_\infty \\ &\leq (\|a\|_\infty + r\alpha) \sum_{i=0}^{n-2} \| \int_0^1 |\partial_1^{(i)} G(t,s)| ds \|_\infty \end{aligned}$$

We deduce that, A transforms C into itself if

$$(\|a\|_\infty + r\alpha) \sum_{i=0}^{n-2} \| \int_0^1 |\partial_1^{(i)} G(t,s)| ds \|_\infty \leq \alpha.$$

which implies, under the condition of Theorem (3.2.1), that

$$\frac{\|a\|_\infty \sum_{i=0}^{n-2} \| \int_0^1 |\partial_1^{(i)} G(t,s)| ds \|_\infty}{1 - r \sum_{i=0}^{n-2} \| \int_0^1 |\partial_1^{(i)} G(t,s)| ds \|_\infty} \leq \alpha.$$

Then, A transforms C into itself for

$$\alpha = \frac{\|a\|_\infty \sum\limits_{i=0}^{n-2} \| \int\limits_0^1 |\partial_1^{(i)} G(t,s)|ds\|_\infty}{1 - r\sum\limits_{i=0}^{n-2} \| \int\limits_0^1 |\partial_1^{(i)} G(t,s)|ds\|_\infty}.$$

Step 2. The operator A is continuous.
Let $(x_m) \in C$ be a convergence sequence to $x \in C$, which implies that $(x_m^{(i)})$ converges to $x^{(i)}$ in the space $C(I, [-\alpha, \alpha])$ for all $i = 0, ..., n-2$.
Since f is uniformly continuous on the compact set $I \times \underbrace{[-\alpha, \alpha] \times ... \times [-\alpha, \alpha]}_{n-1 \text{ times}}$, then the sequence $(f(s, x_m, x_m^{'}, ..., x_m^{(n-2)}))$ converges to $f(s, x, x^{'}, ..., x^{(n-2)})$ in $C(I, \mathbb{R})$.
It follows that,

$$\|Ax_m - Ax\| \leq \|f(s, x_m, x_m', ..., x_m^{n-2}) - f(s, x, x', ..., x^{n-2})\|_\infty \sum_{i=0}^{n-2} \| \int_0^1 \partial_1^{(i)} G(t,s) ds\|_\infty.$$

Which implies that (Ax_m) converges to Ax and the operator A is continuous.
Step 3. $A(C)$ is relatively compact, it is clear that $A(C)$ is equibounded.
Now, to show that $A(C)$ is equicontinuous, take t_1 and t_2 in I.
Then, for all $i = 0, ..., n-3$, there exists ξ_i between t_1 and t_2 such that

$$\partial_1^{(i)} G(t_2, s) - \partial_1^{(i)} G(t_1, s) = (t_2 - t_1)\partial_1^{(i+1)} G(\xi_i, s).$$

Hence, for all $i = 0, ..., n-3$,

$$\begin{aligned}
|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| &= \left| \int\limits_0^1 f(s, x, x', ..., x^{(n-2)})(\partial_1^{(i)} G(t_2, s) - \partial_1^{(i)} G(t_1, s))ds \right| \\
&\leq \int\limits_0^1 |f(s, x, x', ..., x^{(n-2)})\partial_1^{(i+1)} G(\xi_i, s)(t_2 - t_1)|ds \qquad (3.2.4) \\
&\leq |t_2 - t_1| \left(\|a\|_\infty + r\alpha\right) \left\| \int\limits_0^1 |\partial_1^{(i+1)} G(t,s)|ds \right\|_\infty
\end{aligned}$$

Now, let $\varepsilon > 0$. We note $\lambda = \max\limits_{0\leq i\leq n-3} \left\| \int\limits_0^1 |\partial_1^{(i+1)} G(t,s)| ds \right\|_\infty$.

Then from (3.2.4), if $|t_2 - t_1| \leq \delta_1 = \dfrac{\varepsilon}{1 + (\|a\|_\infty + r\alpha)\,\lambda}$, we have for all $i = 0, ..., n-3$,

$$|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| \leq \varepsilon$$

On the other hand, since the function $g(t,s)$ is uniformly continuous on $I \times I$.
Then there exists $\delta_2 > 0$ such that if $|t_2 - t_1| \leq \delta_2$, then for all $s \in I$

$$|g(t_2,s) - g(t_1,s)| < \frac{\varepsilon}{1 + \|a\|_\infty + r\alpha}.$$

Which implies, for $i = n-2$, that

$$\begin{aligned} |(Ax)^{(n-2)}x(t_2) - (Ax)^{(n-2)}x(t_1)| &= |\int\limits_0^1 f(s,x,x',...,x^{(n-2)})(g(t_2,s) - g(t_1,s))ds| \\ &\leq (\|a\|_\infty + r\alpha)\, \|g(t_2,s) - g(t_1,s)\|_\infty \\ &\leq \varepsilon. \end{aligned}$$

Hence, the third step is completed by setting $\delta = \min(\delta_1, \delta_2)$. Therefore, the set $A(C)$ equicontinuous.
The proof of Theorem 3.2.1 then follows from Schauder fixed point theorem. □

Example 3.2.1. Consider the following third-order boundary value problem:

$$\begin{cases} u^{(3)} + \lambda \ln(2 + u^2 + (u')^2) = 0, t \in I = [0,1], \\ u(0) = 0, \\ u'(0) - u^{(2)}(0) = 0, \\ u'(1) + u^{(2)}(1) = 0. \end{cases} \tag{3.2.5}$$

where λ is a positive number. Hence, by using the notations of Theorem 3.2.1,

$$n = 3, f(t,u,u') = \lambda \ln(2 + u^2 + (u')^2), \alpha = \gamma = \beta = \delta = 1, \frac{\partial G(t,s)}{\partial t} = g(t,s),$$

where,

$$g(t,s) = \frac{1}{3}\begin{cases}(1+s)[1+(1-t)], 0 \leq s \leq t, \\ (1+t)[1+(1-s)], t \leq s \leq 1.\end{cases}$$

Which implies that $\int_0^1 |g(t,s)|ds = \frac{1}{2}(1-t+t^2)$ and $\| \int_0^1 |g(t,s)|ds\| = \frac{5}{8}$.
On the other hand, we have

$$G(t,s) = \int_0^t g(r,s)dr = \frac{1}{3}\begin{cases}(1+s)[2t - \frac{t^2}{2}], 0 \leq s \leq t, \\ (2-s)[t + \frac{t^2}{2}], t \leq s \leq 1.\end{cases}$$

Which implies that $\int_0^1 |G(t,s)|ds = \frac{1}{4}(2t+t^2)$ and $\| \int_0^1 |G(t,s)|ds\| = \frac{3}{4}$.
It is easy to see that $|f(s,u_0,u_1)| \leq \lambda \ln(2) + \lambda|u_0| + \lambda|u_1|$.
Hence, the conditions (i) and (ii) are fulfilled with $a(s) = \lambda \ln(2), b_0 = b_1 = \lambda$.
Therefore, the inequality in Theorem 3.2.1 takes the form

$$\lambda\left(\frac{5}{8} + \frac{1}{3}\right) < 1 \Longleftrightarrow \lambda < \frac{8}{11}.$$

Then by Theorem 3.2.1, we conclude that the third-order boundary value problem (3.2.5) has a solution $u \in C^3(I,\mathbb{R})$ if $\lambda < \frac{8}{11}$.

3.3 Application to the solution of a higher-order boundary value problem

In this section, we study the existence of a solution for the problem (3.1.1) with the conditions (3.1.3).

It is easy to check, (see [21]), that u is a solution of (3.1.1) in $C^n(I,\mathbb{R})$ if and only if u is a solution of the following integro-differential equation:

$$u(t) = -\int_0^1 G(t,s)f(s,u,u',...,u^{(n-2)})ds, \tag{3.3.1}$$

in $C^{n-2}(I,\mathbb{R})$, such that the Green's function $G(t,s)$ is defined by

$$G(t,s) = \begin{cases} \dfrac{a(s)t^{n-1}}{(n-1)!}, & if 0 \leq t \leq s \leq 1, \\ \dfrac{a(s)t^{n-1}+(t-s)^{n-1}}{(n-1)!}, & if 0 \leq s \leq t \leq 1. \end{cases}$$

where

$$a(s) = \begin{cases} -\dfrac{(1-s)^{n-1}}{1-\alpha\eta^{n-1}}, & \eta \leq s, \\ -\dfrac{(1-s)^{n-1}-\alpha(\eta-s)^{n-1}}{1-\alpha\eta^{n-1}}, & s \leq \eta. \end{cases}$$

Moreover,

$$\partial_1^{(i)}G(t,s) = \begin{cases} \dfrac{a(s)t^{n-i-1}}{(n-i-1)!}, & if 0 \leq t \leq s \leq 1, \\ \dfrac{a(s)t^{n-i-1}+(t-s)^{n-i-1}}{(n-i-1)!}, & if 0 \leq s \leq t \leq 1. \end{cases}$$

Equation (3.3.1) will be studied under the following assumptions:

(i) $f \in C(I\times\mathbb{R}^{n-1},\mathbb{R})$.

(ii) There exist a function $\varphi \in C(I,\mathbb{R}^+)$ and constants $b_k \in \mathbb{R}^+ (k=0,...,n-2)$ such that

$$|f(s,u_0,u_1,...,u_{n-2})| \leq \varphi(s) + \sum_{k=0}^{n-2} b_k|u_k|$$

Under the assumptions $(i)$and (ii), we will make use of Schauder fixed point theorem to prove the following main result.

Theorem 3.3.1. *If the hypotheses* (i), (ii) *hold, and if*

$$r\left(\frac{1+\alpha\eta^{n}}{n(1-\alpha\eta^{n-1})}+1\right)\sum_{i=1}^{n-1}\frac{1}{i!}<1$$

such that $r = Max\{b_0, ..., b_{n-2}\}$.
Then, the integro-differential equation (3.2.1) *has a solution in* $C^{n-2}(I,\mathbb{R})$.

Proof. Solving Equation (3.3.1) is equivalent to finding a fixed point of the operator A defined in the space $E = C^{n-2}(I,\mathbb{R})$ by the following expression

$$Ax(t) = -\int_{0}^{1} G(t,s)f(s,x,x',...,x^{(n-2)})ds.$$

It is clear that the operator A is well defined from E into itself.
Moreover for all $x \in E$, $t \in I$ and $i = 0, ..., n-2$, we have

$$(Ax)^{(i)}(t) = -\int_{0}^{1} \partial_1^{(i)}G(t,s)f(s,x,x',...,x^{(n-2)})ds.$$

The proof is split into three steps.

Step I. There exists $\beta > 0$ such that A transforms $C = \{x \in E, \|x\| \leq \beta\}$ into itself. It is clear that C is nonempty, bounded, closed and convex subset of E.
Moreover, for all $x \in C$, $t \in I$ and $i = 0, ..., n-2$, we have

$$\begin{aligned}
|(Ax)^{(i)}(t)| &= \left|\int_{0}^{1} \partial_1^{(i)}G(t,s)f(s,x,x',...,x^{(n-2)})ds\right| \\
&\leq \int_{0}^{1} |\partial_1^{(i)}G(t,s)|\left(\varphi(s)+\sum_{k=0}^{n-2} b_k|x^{(k)}(s)|\right)ds \qquad (3.3.2)\\
&\leq \left(\|\varphi\|_\infty+\sum_{k=0}^{n-2} b_k\|x^{(k)}\|_\infty\right)\int_{0}^{1} |\partial_1^{(i)}G(t,s)|ds.
\end{aligned}$$

Hence, for $r = Max\{b_0, ..., b_{n-2}\}$, we obtain

$$\begin{aligned}
\|Ax\| &= \sum_{i=0}^{n-2} \|A^{(i)}x\|_\infty \\
&\leq (\|\varphi\|_\infty + r\beta) \sum_{i=0}^{n-2} \| \int_0^1 |\partial_1^{(i)} G(t,s)| ds\|_\infty \\
&\leq (\|\varphi\|_\infty + r\beta) \sum_{i=0}^{n-2} \frac{1}{(n-i-1)!} \int_0^1 (|a(s)|+1) ds \\
&\leq (\|\varphi\|_\infty + r\beta) \sum_{i=0}^{n-2} \frac{1}{(n-i-1)!} (\int_0^\eta |a(s)| ds + \int_\eta^1 |a(s)| ds + 1) \\
&\leq (\|\varphi\|_\infty + r\beta) \sum_{i=0}^{n-2} \frac{1}{(n-i-1)!} \left(\frac{-(1-s)^n}{n(1-\alpha\eta^{n-1})}|_0^1 + \frac{-\alpha(\eta-s)^n}{n(1-\alpha\eta^{n-1})}|_0^\eta + 1 \right) \\
&\leq (\|\varphi\|_\infty + r\beta) \left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})} + 1 \right) \sum_{i=0}^{n-2} \frac{1}{(n-i-1)!} \\
&\leq (\|\varphi\|_\infty + r\beta) \left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})} + 1 \right) \sum_{i=1}^{n-1} \frac{1}{i!}
\end{aligned}$$

We deduce that, A transforms C into itself if

$$(\|\varphi\|_\infty + r\beta) \left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})} + 1 \right) \sum_{i=1}^{n-1} \frac{1}{i!} \leq \beta.$$

which implies, under the condition of Theorem (3.3.1), that

$$\frac{\|\varphi\|_\infty \left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})} + 1 \right) \sum\limits_{i=1}^{n-1} \frac{1}{i!}}{1 - r \left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})} + 1 \right) \sum\limits_{i=1}^{n-1} \frac{1}{i!}} \leq \beta.$$

Then, A transforms C into itself for

$$\beta = \frac{\|\varphi\|_\infty \left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})}+1\right)\sum\limits_{i=1}^{n-1}\frac{1}{i!}}{1-r\left(\frac{1+\alpha\eta^n}{n(1-\alpha\eta^{n-1})}+1\right)\sum\limits_{i=1}^{n-1}\frac{1}{i!}}.$$

Step 2. The operator A is continuous.

Let $(x_m) \in C$ be a convergence sequence to $x \in C$, which implies that $(x_m^{(i)})$ converges to $x^{(i)}$ in the space $C(I,[-\beta,\beta])$ for all $i = 0, ..., n-2$.

Since f is uniformly continuous on the compact set $I \times \underbrace{[-\beta,\beta]\times ... \times[-\beta,\beta]}_{n-1\text{ times}}$, then the sequence $(f(s, x_m, x_m^{'}, ..., x_m^{(n-2)}))$ converges to $f(s, x, x^{'}, ..., x^{(n-2)})$ in $C(I,\mathbb{R})$.

It follows that,

$$\|Ax_m - Ax\| \leq \|f(s, x_m, x_m', ..., x_m^{n-2}) - f(s, x, x', ..., x^{n-2})\|_\infty \sum_{i=0}^{n-2} \| \int_0^1 \partial_1^{(i)} G(t,s)ds\|_\infty.$$

Which implies that (Ax_m) converges to Ax and the operator A is continuous.

Step 3. $A(C)$ is relatively compact, it is clear that $A(C)$ is equibounded.

Now, to show that $A(C)$ is equicontinuous, take t_1 and t_2 in I.

Then, for all $i = 0, ..., n-3$, there exists ξ_i between t_1 and t_2 such that

$$\partial_1^{(i)} G(t_2, s) - \partial_1^{(i)} G(t_1, s) = (t_2 - t_1)\partial_1^{(i+1)} G(\xi_i, s).$$

Hence, for all $i = 0, ..., n-2$,

$$\begin{aligned} |Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| &= \left| \int_0^1 f(s, x, x', ..., x^{(n-2)})(\partial_1^{(i)} G(t_2, s) - \partial_1^{(i)} G(t_1, s))ds \right| \\ &\leq \int_0^1 |f(s, x, x', ..., x^{(n-2)})\partial_1^{(i+1)} G(\xi_i, s)(t_2 - t_1)|ds \\ &\leq |t_2 - t_1| \left(\|\varphi\|_\infty + r\beta \right) \left\| \int_0^1 |\partial_1^{(i+1)} G(t, s)|ds \right\|_\infty \end{aligned} \tag{3.3.3}$$

Now, let $\varepsilon > 0$. We note $\lambda = \max\limits_{0 \leq i \leq n-2} \left\| \int_0^1 |\partial_1^{(i+1)} G(t, s)|ds \right\|_\infty$.
Then from (3.2.4), if $|t_2 - t_1| \leq \delta = \dfrac{\varepsilon}{1 + (\|\varphi\|_\infty + r\beta)\lambda}$, we have for all $i = 0, ..., n-2$,

$$|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| \leq \varepsilon$$

Hence, the third step is completed. Therefore, the set $A(C)$ equicontinuous.
The proof of Theorem 3.3.1 then follows from Schauder fixed point theorem. □

Example 3.3.1. Consider the following third-order boundary value problems:

$$\begin{cases} u^{(3)} + \lambda \ln(2 + u^2 + (u')^2) = 0, t \in I = [0, 1], \\ u(0) = 0, \\ u'(0) = 0, \\ u\left(\dfrac{1}{2}\right) = u(1). \end{cases} \tag{3.3.4}$$

where λ is a positive number. Hence, by using the notations of Theorem 3.3.1,

$$n = 3, f(t, u, u') = \lambda \ln(2 + u^2 + (u')^2), \alpha = 1, \eta = \frac{1}{2},$$

where,

$$a(s) = \begin{cases} -\dfrac{(1-s)^2}{1-(\frac{1}{2})^2}, & \dfrac{1}{2} \leq s \leq 1, \\ -\dfrac{(1-s)^2-(\frac{1}{2}-s)^2}{1-(\frac{1}{2})^2}, & s \leq \eta. \end{cases}$$

and

$$G(t,s) = \begin{cases} \dfrac{a(s)t^2}{2}, & if 0 \leq t \leq s \leq 1, \\ \dfrac{a(s)t^2+(t-s)^2}{2}, & if 0 \leq s \leq t \leq 1. \end{cases}$$

It is easy to see that $|f(s,u_0,u_1)| \leq \lambda \ln(2) + \lambda|u_0| + \lambda|u_1|$.
Hence, the conditions (i) and (ii) are fulfilled with $\varphi(s) = \lambda \ln(2), b_0 = b_1 = \lambda$ and $r = \lambda$.
Therefore, the inequality in Theorem 3.3.1 takes the form

$$\lambda \left(\frac{1+(\frac{1}{2})^3}{3(1-(\frac{1}{2})^2)} + 1 \right) \times \left(1 + \frac{1}{2} \right) < 1 \Longleftrightarrow \lambda < \frac{4}{9}.$$

Then by Theorem 3.3.1, we conclude that the third-order boundary value problems (3.3.4) has a solution $u \in C^3(I,\mathbb{R})$ if $\lambda < \dfrac{4}{9}$.

CHAPTER 4

Existence of a solution of integral equations of the product type

4.1 Introduction

In this chapter, we concern with the nonlinear functional integral equations of the product type which was presented by many authors in [1, 9, 12, 13, 14, 28, 38, 48] such as B. Boulfoul et al. [13] have studied the existence of a continuous solution to the following integral equation

$$x(t) = f(t, x(t)) + f_1\Big(t, \int_0^t v_1(t, s, x(s))ds\Big) \times f_2\Big(t, \int_0^t v_2(t, s, x(s))ds\Big). \tag{4.1.1}$$

for $t > 0$. This equation arises in the study of the spread of infectious disease, see ([14, 28, 9] and the references therein).

Ardjouni and Djoudi in [9] studied the approximating solution of the following nonlinear Hybrid Caputo fractional intego-differential equation by using Dhage iteration principle:

$$Ax(t) = \Big(p(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s))ds\Big) \times \Big(\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds\Big). \tag{4.1.2}$$

In this work, we will studied the following more general forme integro-differential equations of product type by using the generalized Ascoli-Arzelá theorem in C^1 and Schauder's fixed point theorem.

$$x(t)=\Big[p(t)+\int_0^t h_1(t,s)g(s,x(s),x'(s))ds\Big]\times\Big[q(t)+\int_0^t h_2(t,s)f(s,x(s),x'(s))ds\Big]. \tag{4.1.3}$$

4.2 Application of generalization of Ascoli- Arzelá theorem in C^1 to the solution of integral equations of the product type

Let $a\in\mathbb{R}$ be a positive constant, we consider the space

$$P_a^1=C^1([0,a],\mathbb{R}).$$

It is clear that P_a^1 is a Banach space endowed with the norm

$$\|x\|=\sup_{t\in[0,a]}|x|+\sup_{t\in[0,a]}|x'|$$

.

Equation (4.1.3) will be studied under the following assumptions:

(H_1) $p,q\in C^1(\mathbb{R},\mathbb{R})$.

(H_2) $g\in C(\mathbb{R}^3,\mathbb{R})$,there exists $k_1,k_2\geq 0$ and $\phi_1\in C(\mathbb{R},\mathbb{R})$ bounded such that

$$|g(t,u,v)|\leq\phi_1(t)+k_1|u|+k_2|v|,\quad\forall(t,u,v)\in\mathbb{R}^3.$$

(H_3) $f\in C(\mathbb{R}^3,\mathbb{R})$, there exists $k_3,k_4\geq 0$ and $\phi_2\in C(\mathbb{R},\mathbb{R})$ bounded such that

$$|f(t,u,v)|\leq\phi_2(t)+k_3|u|+k_4|v|,\quad\forall(t,u,v)\in\mathbb{R}^3.$$

(H_4) $h_1, h_2 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are all continuous bounded functions.

In the proof the main result of this chapter, we will use the following Lebesgue's dominated convergence theorem.

Theorem 4.2.1. *[15] Let Ω be a measurable set of $\mathbb{R}$ and (f_k) be a sequence in $L^1(\Omega, \mathbb{R})$ space such that: $f_k(x) \longrightarrow f(x)$ a.e and there exists a functions g in $L^1(\Omega, \mathbb{R})$ such that $|f_k(s)| \leq g(s)$ Then $f \in L^1(\Omega, \mathbb{R})$ and*

$$\int_{\Omega} |f_k - f| \, ds \longrightarrow 0, when\ , k \longrightarrow \infty$$

.

Under the hypothesis $(H_1 - H_4)$ and the previous theorems, we will make use of Schauder fixed point theorem to prove the following main result.

Theorem 4.2.2. *If the hypotheses $(H_1 - H_4)$ hold, and if suppose that conditions:*

$$Max\{\theta + a\theta^2, k_5 a\theta, 3\theta + 3a\theta^2 + 2\theta^2, 2k_5\theta + 3k_5 a\theta\} < \frac{1}{2}$$

Then, nonlinear perturbed integral equation of product type (4.1.3) *has a solution in P_a^1, where θ is defined in the proof.*

Proof. Solving Eq. (4.1.3) is equivalent to finding a fixed point of the operator A such that A are defined by the following expression

$$Ax(t) = \left[p(t) + \int_0^t h_1(t,s) g(s, x(s), x'(s)) ds\right] \times \left[q(t) + \int_0^t h_2(t,s) f(s, x(s), x'(s)) ds\right]$$

It is clear that the operator A is well defined from P^1 into itself, moreover

$$
\begin{aligned}
(Ax)'(t) = & \Big[\frac{\partial p(t)}{\partial t} + h_1(t,t)g(t,x(t),x'(t)) + \int_0^t \frac{\partial h_1(t,s)}{\partial t} g(s,x(s),x'(s))ds\Big] \\
& \times \Big[q(t) + \int_0^t h_2(t,s)f(s,x(s),x'(s))ds] + [p(t) + \int_0^t h_1(t,s)g(s,x(s),x'(s))ds\Big] \\
& \times \Big[\frac{\partial q(t)}{\partial t} + h_2(t,t)f(t,x(t),x'(t)) + \int_0^t \frac{\partial h_2(t,s)}{\partial t} f(s,x(s),x'(s))ds]
\end{aligned}
$$

The proof is split into four steps.

Step I. There exists $\alpha > 0$ such that A transforms $C = \{x \in P^1, \|x\| \leq \alpha\}$ into itself . It is clear that C is nonempty, bounded, convex and closed.

To simplify notations, we introduce the constants

$$
\theta = \max_{t\in[0,a]} \left\{ |p(t)|, |q(t)|, |\phi_1(t)|, |\phi_2(t)|, |\frac{\partial p(t,s)}{\partial t}|, |\frac{\partial q(t,s)}{\partial t}|, |h_1(t,s)|, |h_2(t,s)|, |h_3(t,s)|, |h_4(t,s)| \right\}
$$

$$
h_3(t,s) = \Big|\frac{\partial h_1(t,s)}{\partial t}\Big|, h_4(t,s) = \Big|\frac{\partial h_2(t,s)}{\partial t}\Big|, k_5 = \max\{k_1,k_2,k_3,k_4\}
$$

Moreover, for all $x \in C$ and $t \in [0,a]$, we have

$$
\begin{aligned}
|Ax(t)| =& \left| p(t) + \int_0^t h_1(t,s) g(s, x(s), x'(s)) ds \right| \times \left| q(t) + \int_0^t h_2(t,s) f(s, x(s), x'(s)) ds \right| \\
&\leq \left[|p(t)| + \int_0^t |h_1(t,s)| [|\phi_1(s)| + (k_1 |x(s)| + k_2 |x'(s)|) ds \right] \\
&\times \left[|q(t)| + \int_0^t |h_2(t,s)| [|\phi_2(s)| (k_3 |x(s)| + k_4 |x'(s)| ds \right] \\
&\leq \left[|p(t)| + \int_0^t |h_1(t,s)| \left[|\phi_1(s)| + k_5 (|x(s)| + |x'(s)|)) ds \right] \\
&\times [|q(t)| + \int_0^t |h_2(t,s)| [|\phi_2(s)| + k_5 (|x(s)| + |x'(s)|) ds \Big] \\
&\leq [\theta + \int_0^t \theta [\theta + k_5 (\|x\|_\infty + \|x'\|_\infty) ds] \\
&\times [\theta + \int_0^t \theta [\theta + k_5 (\|x\|_\infty + \|x'\|_\infty) ds \Big] \\
&\leq [\theta + \int_0^t \theta [\theta + k_5 \|x\| ds] \times [\theta + \int_0^t \theta [\theta + k_5 \|x\| ds \Big] \\
&\leq [\theta + \theta a (\theta + k_5 \alpha) \Big] \times \Big[\theta + a\theta (\theta + k_5 \alpha) \Big] \\
&\leq [\theta + \theta a (\theta + k_5 \alpha)]^2
\end{aligned}
\tag{4.2.1}
$$

and

$$
\begin{aligned}
|(Ax)'(t)| =&[|\frac{\partial p(t)}{\partial t}+h_1(t,t)g(t,x(t),x'(t))+\int\limits_0^t \frac{\partial h_1(t,s)}{\partial t}g(s,x(s),x'(s))ds]\\
&\times [q(t)+\int\limits_0^t h_2(t,s)f(s,x(s),x'(s))ds]+[p(t)+\int\limits_0^t h_1(t,s)g(s,x(s),x'(s))ds]\\
&\times \frac{\partial q(t)}{\partial t}+h_2(t,t)f(t,x(t),x'(t))+\int\limits_0^t \frac{\partial h_2(t,s)}{\partial t}f(s,x(s),x'(s))ds]\\
&\leq [|\frac{|\partial p(t)}{\partial t}|+|h_1(t,t)|(\phi_1(t)+k_1|x(t)|+k_2|x'(t)|)+\\
&\int\limits_0^t |\frac{\partial h_1(t,s)}{\partial t}|+(\phi_1(s)+k_1|x(s)|+k_2|x'(s)|)ds]\\
&\times [|q(t)|+\int\limits_0^t |h_2(t,s)|(\phi_2(s)+k_3|x(s)|+k_4|x'(s)|)ds]+\\
&[|p(t)|+\int\limits_0^t |h_1(t,s)|(\phi_1(s)+k_1|x(s)|+k_2|x'(s)|)ds]\times\\
&[|\frac{|\partial q(t)}{\partial t}|+|h_2(t,t)|(\phi_2(t)+k_3|x(t)|+k_4|x'(t)|)+\\
&\int\limits_0^t |\frac{\partial h_2(t,s)}{\partial t}|+(\phi_2(s)+k_3|x(s)|+k_4|x'(s)|)ds]\\
&\leq [\theta+\theta(\theta+k_5(\|x(t)\|_\infty+|x'(t)\|_\infty)+a\theta(\theta+k_5(\|x(s)\|_\infty+\|x'(s)\|_\infty)]\\
&\times [\theta+a\theta(\theta+k_5(\|x(s)\|_\infty+\|x'(s)\|_\infty)]+[\theta+a\theta(\theta+k_5(\|x(s)\|_\infty+\|x'(s)\|_\infty)]\\
&\times [\theta+\theta(\theta+k_5(\|x(t)\|_\infty+|x'(t)\|_\infty)+a\theta(\theta+k_5(\|x(s)\|_\infty+\|x'(s)\|_\infty)]\\
&\leq 2[\theta+\theta(\theta+k_5\|x\|)+a\theta(\theta+k_5\|x\|]\times[\theta+a\theta(\theta+k_5(\|x\|]\\
&\leq 2[\theta+\theta(\theta+k_5\alpha)+a\theta(\theta+k_5\alpha)]\times[\theta+a\theta(\theta+k_5\alpha)]
\end{aligned}
\tag{4.2.2}
$$

Hence, by (4.2.1) , (4.2.2) we obtain

$$\begin{aligned}\|A(x)\| &\leq [\theta+\theta a(\theta+k_5\alpha)]^2+2[\theta+\theta(\theta+k_5\alpha)+a\theta(\theta+k_5\alpha)]\times[\theta+a\theta(\theta+k_5\alpha)]\\ &\leq [\theta+\theta a(\theta+k_5\alpha)]\times[\theta+\theta a(\theta+k_5\alpha)+2[\theta+\theta(\theta+k_5\alpha)+a\theta(\theta+k_5\alpha)]\\ &\leq [\theta+a\theta^2+a\theta k_5\alpha)]\times[3\theta+3a\theta^2+2\theta^2+2k_5\theta\alpha+3k_5a\theta\alpha]\\ &\leq [\theta+a\theta^2+a\theta k_5\alpha)]\times[3\theta+3a\theta^2+2\theta^2+(2\theta k_5+3k_5\theta a)\alpha]\end{aligned}$$

If we put

$$r = Max\{\theta+a\theta^2, k_5a\theta, 3\theta+3a\theta^2+2\theta^2, 2\theta k_5+3k_5a\theta\}$$

We obtain

$$\|Ax\| \leq (r+r\alpha)\times(r+r\alpha).$$

We deduce that, A transforms C into itself if $\|Ax\| \leq (r+r\alpha)^2 \leq \alpha$, which implies

$$r^2+r^2\alpha^2+2r^2\alpha \leq \alpha \Longrightarrow -r^2\alpha^2+(1-2r^2)\alpha-r \geq 0,$$

we have $\Delta = (1-2r^2)^2-4r^4 = (1-2r)\times(1+2r)$.
Since $r<\frac{1}{2}$, then $\Delta>0$, hence we have two solutions:

$$\begin{cases}\alpha_1 = \dfrac{1-2r^2-\sqrt{\Delta}}{2r^2},\\ \alpha_2 = \dfrac{1+2r^2+\sqrt{\Delta}}{2r^2}.\end{cases}$$

Note that $0<\alpha_1\leq\alpha<\alpha_2$
Then, A transforms C into itself for

$$\alpha \in [\alpha_1, \alpha_2].$$

Step 2. The operator A is continuous. Let $(x_n)_n \in C$ be a convergence sequence to $x \in C$, which implies that $(x_n^{(i)})$ converges to $x^{(i)}$ $(i=1,2)$ in the space $C([0,a],[-\alpha,\alpha])$. then by

Lebesgue's dominated convergence theorem 4.2.1 , we have

$$\begin{aligned}
\lim_{n\to+\infty}(Ax_n)(t) &= \lim_{n\to+\infty}[p(t) + \int_0^t h_1(t,s)g(s,x_n(s),x_n'(s))ds] \\
&\times [q(t) + \int_0^t h_2(t,s)f(s,x_n(s),x_n'(s))ds] \\
&= (p(t) + \int_0^t h_1(t,s)[\lim_{n\to+\infty} g(s,x_n(s),x_n'(s))ds]) \\
&\times (q(t) + \int_0^t h_2(t,s)[\lim_{n\to+\infty} f(s,x_n(s),x_n'(s))ds) \\
&= (p(t) + \int_0^t h_1(t,s)g(s,x(s),x'(s))ds]) \\
&\times (q(t) + \int_0^t h_2(t,s)f(s,x(s),x'(s))ds) = (Ax)(t)
\end{aligned}$$

Which implies for $t \in [0,a]$ that (Ax_n) converges to Ax and the operator A is continuous.

Step3.

$A(C)$ is relatively compact, it is clear that $A(C)$ is equibounded.

Now, to show that $A(C)$ is equicontinuous, take t_1 and t_2 in $I = [0,a]$.

We put $Ax(t) = H_1x(t) \times H_2x(t)$ such that

$$H_1x(t) = p(t) + \int_0^t h_1(t,s)g(s,x(s),x'(s))ds$$

and

$$H_2x(t) = q(t) + \int_0^t h_2(t,s)f(s,x(s),x'(s))ds$$

We conclude that

$$Ax(t) = (p(t) + \int_0^t h_1(t,s)g(s,x(s),x'(s))ds) \times (q(t) + \int_0^t h_2(t,s)f(s,x(s),x'(s))ds)$$

For l'equicontinuity if $t_2 < t_1$, we have

$$\begin{aligned} Ax(t_1) - Ax(t_2) =& H_1x(t_1) \times H_2x(t_1) - H_1x(t_{2)} \times H_2x(t_2) \\ &= H_1x(t_1) \times [H_2x(t_1) - H_2x(t_2)] + H_2x(t_2) \times [H_1x(t_1) - H_1x(t_2)] \end{aligned} \tag{4.2.3}$$

Before study the equicontinuity of A we study the equicontinuity of the operator

$$H_1 = p(t) + \int_0^t h_1(t,s)g(s,x(s),x'(s))ds$$

Let $t_1, t_2 \in [0,a]$ we have for $t_2 < t_1$:

$$
\begin{aligned}
|H_1x(t_1) - H_1x(t_2)| =& |p(t_1) + \int_0^{t_1} h_1(t_1,s)g(s,x(s),x'(s)ds - p(t_2) - \int_0^{t_2} h_1(t_2,s)g(s,x(s),x'(s)ds| \\
& \leq |p(t_1) - p(t_2) + \int_0^{t_2} h_1(t_1,s)g(s,x(s),x'(s)ds + \int_{t_2}^{t_1} h_1(t_1,s)g(s,x(s),x'(s)ds \\
& - \int_0^{t_2} h_1(t_2,s)g(s,x(s),x'(s)ds| \\
& \leq |p(t_1) - p(t_2)| + |\int_0^{t_2} (h_1(t_1,s) - h_1(t_2,s))g(s,x(s),x'(s)ds| + \\
& |\int_{t_2}^{t_1} h_1(t_1,s)g(s,x(s),x'(s)ds| \\
& \leq |p(t_1) - p(t_2)| + |\int_0^{t_2} (h_1(t_1,s) - h_1(t_2,s))(|\phi_1(s)| + k_1|x(s)| + k_2|x'(s)|)ds + \\
& |\int_{t_2}^{t_1} h_1(t_1,s)(|\phi_1(s)| + k_1|x(s)| + k_2|x'(s)|)ds \\
& \leq |p(t_1) - p(t_2)| + \int_0^{t_2} |h_1(t_1,s) - h_1(t_2,s)|(\|\phi_1\| + k_5(\|x\|_\infty + \|x'\|_\infty))ds + \\
& \int_{t_2}^{t_1} |h_1(t_1,s)|(\|\phi_1\| + k_5(\|x\|_\infty + \|x'\|_\infty))ds \\
& \leq |p(t_1) - p(t_2)| + \int_0^{t_2} |h_1(t_1,s) - h_1(t_2,s)|(\theta + k_5\|x\|)ds + \int_{t_2}^{t_1} \theta(\theta + k_5\|x\|)ds \\
& \leq |p(t_1) - p(t_2)| + a(\theta + k_5\alpha)\|h_1(t_1,s) - h_1(t_2,s)\|_\infty + \theta(\theta + k_5\alpha)|t_1 - t_2|.
\end{aligned}
$$

Finally we have

$$\begin{aligned}|H_1x(t_1) - H_2x(t_2)| \leq |p(t_1) - p(t_2)| + a(\theta + k_5\alpha)\|h_1(t_1, s) - h_1(t_2, s)\|_\infty \\ +\theta(\theta + k_5\alpha)|t_1 - t_2|\end{aligned} \tag{4.2.4}$$

similarly, we obtain

$$\begin{aligned}|H_2x(t_1) - H_2x(t_2)| \leq |q(t_1) - q(t_2)| + a(\theta + k_5\alpha)\|h_2(t_1, s) - h_2(t_2, s)\|_\infty \\ +\theta(\theta + k_5\alpha)|t_1 - t_2|.\end{aligned} \tag{4.2.5}$$

Now, let $\varepsilon > 0$, since the functions $h_1(t, s), h_2(t, s)$ and p, q are uniformly continuous on the compact set $[0, a] \times [0, 2a]$, then there exists $\delta > 0$ such that, if $|t_2 - t_1| \leq \delta$, we have for all $s \in [0, 2a]$. Consequently, the set $H_1(C), H_2(C)$ is equicontinuous.

We have

$$\begin{aligned}|H_1x(t)| =& |p(t) + \int_0^t h_1(t, s)g(s, x(s), x'(s))ds| \\ &\leq [\theta + \theta a(\theta + k_5\alpha)]\end{aligned}$$

and

$$\begin{aligned}|H_2x(t)| =& |q(t) + \int_0^t h_2(t, s)f(s, x(s), x'(s))ds| \\ &\leq [\theta + \theta a(\theta + k_5\alpha)]\end{aligned}$$

Hence, by (4.2.3),(4.2.4),(4.2.5), we have

$$\begin{aligned}Ax(t_1) - Ax(t_2) =& H_1x(t_1) \times H_2x(t_1) - H_1x(t_{2)} \times H_2x(t_2) \\ =& H_1x(t_1) \times [H_2x(t_1) - H_2x(t_2)] + H_2x(t_2) \times [H_1x(t_1) - H_1x(t_2)]\end{aligned}$$

which implies that

$$\begin{aligned}|Ax(t_1)-Ax(t_2)| =&[\theta+\theta a(\theta+k_5\alpha)]\times[|H_2x(t_1)-H_2x(t_2)|+|H_1x(t_1)-H_1x(t_2)]\\ &\leq[\theta+\theta a(\theta+k_5\alpha)]\times[|q(t_1)-q(t_2)|+a(\theta+k_5\alpha)\|h_2(t_1,s)-h_2(t_2,s)\|_\infty\\ &+\theta(\theta+k_5\alpha)|t_1-t_2|]+|p(t_1)-p(t_2)|+a(\theta+k_5\alpha)\|h_1(t_1,s)-h_1(t_2,s)\|_\infty\\ &+\theta(\theta+k_5\alpha)|t_1-t_2|].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(Ax)'(t) =&[\frac{\partial p(t)}{\partial t}+h_1(t,t)g(t,x(t),x'(t))+\int_0^t\frac{\partial h_1(t,s)}{\partial t}g(s,x(s),x'(s))ds]\\ &\times[q(t)+\int_0^t h_2(t,s)f(s,x(s),x'(s))ds]+[p(t)+\int_0^t h_1(t,s)g(s,x(s),x'(s))ds]\\ &\times\frac{\partial q(t)}{\partial t}+h_2(t,t)f(t,x(t),x'(t))+\int_0^t\frac{\partial h_2(t,s)}{\partial t}f(s,x(s),x'(s))ds]\end{aligned}$$

if we put
$p_1(t)=\dfrac{\partial p(t)}{\partial t}+h_1(t,t)g(t,x(t),x'(t)),q_1(t)=\dfrac{\partial q(t)}{\partial t}+h_2(t,t)f(t,x(t),x'(t))$
We conclude that

$$\begin{aligned}(Ax)'(t) =&[p_1(t)+\int_0^t h_3(t,s)g(s,x(s),x'(s))ds]\times[q(t)+\int_0^t h_2(t,s)f(s,x(s),x'(s))ds]+\\ &[p(t)+\int_0^t h_1(t,s)g(s,x(s),x'(s))ds]\times[q_1(t))+\int_0^t h_4(t,s)f(s,x(s),x'(s))ds]\end{aligned}$$

Let

$$\begin{cases} H_1(t) = p(t) + \int\limits_0^t h_1(t,s)g(s,x(s),x'(s)ds, \\ H_2(t) = q(t) + \int\limits_0^t h_2(t,s)f(s,x(s),x'(s)ds \\ H_3(t) = p_1(t) + \int\limits_0^t h_3(t,s)g(s,x(s),x'(s)ds \\ H_4(t) = q_1(t) + \int\limits_0^t h_4(t,s)f(s,x(s),x'(s)ds. \end{cases} \tag{4.2.6}$$

So

$$(Ax)'(t) = H_3(t) \times H_2(t) + H_1(t) \times H_4(t).$$

Using a similar argument as above, we prove that

$$\begin{aligned} A'x(t_1) - A'x(t_2) &= H_3x(t_1) \times H_2x(t_1) + H_1x(t_1) \times H_4x(t_1) \\ &- H_3x(t_2) \times H_2x(t_2) - H_1x(t_2) \times H_4x(t_2) \\ &= H_3x(t_1) \times H_2x(t_1) - H_3x(t_2) \times H_2x(t_2) \\ &+ H_1x(t_1) \times H_4x(t_1) - H_1x(t_2) \times H_4x(t_2) \\ &= H_3x(t_1) \times [(H_2x(t_1) - H_2x(t_2)] \\ &+ H_2x(t_2) \times [H_3x(t_1) - H_3x(t_2)] + H_1x(t_1) \\ &\times [(H_4x(t_1) - (H_4x(t_2)] + H_4x(t_2) \times [(H_1x(t_1) - H_1x(t_2)]. \end{aligned} \tag{4.2.7}$$

Now, we have

$$\begin{aligned}
|H_3(t)| &= |p_1(t) + \int\limits_0^t h_3(t,s)g(s,x(s),x'(s)ds| \\
&= |\frac{\partial p(t)}{\partial t} + h_1(t,t)g(t,x(t),x'(t)) + \int\limits_0^t \frac{\partial h_1(t)}{\partial t} g(s,x(s),x'(s)ds| \\
&\leq \theta + \theta(\theta + k_5\|x\|x) + \int\limits_0^t \theta(\theta + k_5\|x\|x)ds \\
&\leq \theta + \theta(\theta + k_5\alpha) + \int\limits_0^t (\theta(\theta + k_5\alpha)ds \\
&\leq \theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha)
\end{aligned} \tag{4.2.8}$$

and

$$\begin{aligned}
|H_4(t)| &= |q_1(t) + \int\limits_0^t h_4(t,s)f(s,x(s),x'(s)ds| \\
&= |\frac{\partial q(t)}{\partial t} + h_1(t,t)f(t,x(t),x'(t)) + \int\limits_0^t \frac{\partial h_2(t)}{\partial t} f(s,x(s),x'(s)ds| \\
&\leq \theta + \theta(\theta + k_5\|x\|x) + \int\limits_0^t \theta(\theta + k_5\|x\|x)ds \\
&\leq \theta + \theta(\theta + k_5\alpha) + \int\limits_0^t (\theta(\theta + k_5\alpha)ds \\
&\leq \theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha)
\end{aligned} \tag{4.2.9}$$

Hence, H_1, H_2, H_3, H_4 are bounded and we have

$$\begin{cases} |H_1(t)| = |p(t) + \int\limits_0^t h_1(t,s)g(s,x(s),x'(s)ds| \leq \theta + a\theta(\theta + k_5\alpha) \\ |H_2(t)| = |q(t) + \int\limits_0^t h_2(t,s)f(s,x(s),x'(s)ds| \leq \theta + a\theta(\theta + k_5\alpha) \\ |H_3(t)| \leq \theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha) \\ |H_4(t)| \leq \theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha). \end{cases} \tag{4.2.10}$$

In the following, we prove that $H_3(C)$ and $H_4(C)$ are equicontinuous.

$$\begin{aligned} |H_3x(t_1) - H_3x(x_2)| &= \left| p_1(t_1) + \int\limits_0^{t_1} h_3(t_1,s)g(s,x(s),x'(s)ds - p_1(t_2) + \int\limits_0^{t_2} h_3(t_2,s)g(s,x(s),x'(s)ds \right| \\ &\leq |\frac{\partial p(t_1)}{\partial t} + h_1(t_1,t_1)g(t_1,x(t_1),x'(t_1)) + \int\limits_0^{t_1} \frac{\partial h_1(t_1)}{\partial t} g(s,x(s),x'(s)ds \\ &- \frac{\partial p(t_2)}{\partial t} - h_1(t_2,t_2)g(t_2,x(t_2),x'(t_2)) - \int\limits_0^{t_2} \frac{\partial h_1(t_2)}{\partial t} g(s,x(s),x'(s)ds| \\ &\leq |\frac{\partial p(t_1)}{\partial t} - \frac{\partial p(t_2)}{\partial t}| + |h_1(t_1,t_1)g(t_1,x(t_1),x'(t_1)) - h_1(t_2,t_2)g(t_2,x(t_2),x'(t_2))| + \\ &|\int\limits_0^{t_1} \frac{\partial h_1(t_1)}{\partial t} g(s,x(s),x'(s)ds - \int\limits_0^{t_2} \frac{\partial h_1(t_2)}{\partial t} g(s,x(s),x'(s)ds| \\ &\leq |\frac{\partial p(t_1)}{\partial t} - \frac{\partial p(t_2)}{\partial t}| + |h_1(t_1,t_1)g(t_1,x(t_1),x'(t_1)) - h_1(t_2,t_2)g(t_2,x(t_2),x'(t_2))| + \\ &|\int\limits_0^{t_2} \frac{\partial h_1(t_1)}{\partial t} g(s,x(s),x'(s)ds + \int\limits_{t_2}^{t_1} \frac{\partial h_1(t_1)}{\partial t} g(s,x(s),x'(s)ds - \int\limits_0^{t_2} \frac{\partial h_1(t_2)}{\partial t} g(s,x(s),x'(s))ds| \\ &\leq |\frac{\partial p(t_1)}{\partial t} - \frac{\partial p(t_2)}{\partial t}| + |h_1(t_1,t_1)g(t_1,x(t_1),x'(t_1)) - h_1(t_2,t_2)g(t_2,x(t_2),x'(t_2))| + \\ &|\int\limits_0^{t_2} (\frac{\partial h_1(t_1)}{\partial t} - \frac{\partial h_1(t_2)}{\partial t}) g(s,x(s),x'(s))ds + \int\limits_{t_2}^{t_1} \frac{\partial h_1(t)}{\partial t} g(s,x(s),x'(s))ds| \\ &\leq |\frac{\partial p(t_1)}{\partial t} - \frac{\partial p(t_2)}{\partial t}| + |h_1(t_1,t_1)g(t_1,x(t_1),x'(t_1)) - h_1(t_2,t_2)g(t_2,x(t_2),x'(t_2))| + \\ &(\theta + k_5\alpha) \int\limits_0^{t_2} |\frac{\partial h_1(t_1)}{\partial t} - \frac{\partial h_1(t_2)}{\partial t}| ds + \theta(\theta + k_5\alpha)|t_1 - t_2| \end{aligned} \tag{4.2.11}$$

and we have

$$\begin{aligned}
|H_4x(t_1)-H_4x(x_2)| &= |q_1(t_1)+\int_0^{t_1} h_4(t_1,s)f(s,x(s),x'(s)ds - q_1(t_2)+\int_0^{t_2} h_4(t_2,s)f(s,x(s),x'(s)ds| \\
&\leq |\frac{\partial q(t_1)}{\partial t}+h_2(t_1,t_1)f(t_1,x(t_1),x'(t_1))+\int_0^{t_1}\frac{\partial h_2(t_1)}{\partial t}f(s,x(s),x'(s)ds \\
&- \frac{\partial q(t_2)}{\partial t}-h_2(t_2,t_2)f(t_2,x(t_2),x'(t_2))-\int_0^{t_2}\frac{\partial h_2(t_2)}{\partial t}f(s,x(s),x'(s)ds| \\
&\leq |\frac{\partial q(t_1)}{\partial t}-\frac{\partial q(t_2)}{\partial t}|+|h_2(t_1,t_1)f(t_1,x(t_1),x'(t_1))-h_2(t_2,t_2)f(t_2,x(t_2),x'(t_2))|+ \\
&|\int_0^{t_1}\frac{\partial h_2(t_1)}{\partial t}f(s,x(s),x'(s)ds-\int_0^{t_2}\frac{\partial h_2(t_2)}{\partial t}f(s,x(s),x'(s)ds| \\
&\leq |\frac{\partial q(t_1)}{\partial t}-\frac{\partial q(t_2)}{\partial t}|+|h_2(t_1,t_1)f(t_1,x(t_1),x'(t_1))-h_2(t_2,t_2)g(t_2,x(t_2),x'(t_2))|+ \\
&|\int_0^{t_2}\frac{\partial h_2(t_1)}{\partial t}f(s,x(s),x'(s)ds+\int_{t_2}^{t_1}\frac{\partial h_2(t_1)}{\partial t}f(s,x(s),x'(s)ds-\int_0^{t_2}\frac{\partial h_1(t_2)}{\partial t}f(s,x(s),x'(s))ds| \\
&\leq |\frac{\partial q(t_1)}{\partial t}-\frac{\partial q(t_2)}{\partial t}|+|h_2(t_1,t_1)f(t_1,x(t_1),x'(t_1))-h_2(t_2,t_2)f(t_2,x(t_2),x'(t_2))|+ \\
&(\theta+k_5\alpha)\int_0^{t_2}|\frac{\partial h_2(t_1)}{\partial t}-\frac{\partial h_2(t_2)}{\partial t}|ds+\theta(\theta+k_5\alpha)|t_1-t_2|
\end{aligned} \tag{4.2.12}$$

Now, let $\varepsilon>0$, since the functions $h_3(t,s),h_4(t,s)$ and p_1,q_1 are uniformly continuous on the compact set $[0,a]\times[0,2a]$, then there exists $\delta_{\prime}>0$ such that, if $|t_2-t_1|\leq\delta$, we have for all $s\in[0,2a]$

Consequently, the set $H_3(C),H_4(C)$ are equicontinuous.

Hence, by(4.2.4) , (4.2.5), (4.2.7),(4.2.10) , (4.2.11),(4.2.12) we obtain

$$
\begin{aligned}
|A'x(t_1) - A'x(t_2)| &= |H_3x(t_1) \times (H_2x(t_1) + H_1x(t_1) \times (H_4x(t_1) - H_3x(t_2) \times (H_2x(t_2) - H_1x(t_2) \times (H_4x(t_2)| \\
&= |H_3x(t_1) \times (H_2x(t_1) - H_3x(t_2) \times (H_2x(t_2) + H_1x(t_1) \times (H_4x(t_1) - H_1x(t_2) \times (H_4x(t_2)| \\
&= |H_3x(t_1) \times [(H_2x(t_1) - H_2x(t_2)] + H_2x(t_2) \times [H_3x(t_1) - H_3x(t_2)] + H_1x(t_1) \\
&\quad \times [(H_4x(t_1) - (H_4x(t_2)] + H_4x(t_2) \times [(H_1x(t_1) - H_1x(t_2)]| \\
&\leq [\theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha)] \times |(H_2x(t_1) - H_2x(t_2)| + [\theta + a\theta(\theta + k_5\alpha)] \times \\
&\quad |H_3x(t_1) - H_3x(t_2)| + [\theta + a\theta(\theta + k_5\alpha)]|(H_4x(t_1) - (H_4x(t_2)| + \\
&\quad [\theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha)]|(H_1x(t_1) - H_1x(t_2)| \\
&\leq [\theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha)] \times (|q(t_1) - q(t_2)| + a(\theta + k_5\alpha) \times \\
&\quad \|h_2(t_1, s) - h_2(t_2, s)\|_\infty + \theta(\theta + k_5\alpha)|t_1 - t_2|) + [\theta + a\theta(\theta + k_5\alpha)] \times \\
&\quad (|\frac{\partial p(t_1)}{\partial t} - \frac{\partial p(t_2)}{\partial t}| + |h_1(t_1, t_1)g(t_1, x(t_1), x'(t_1)) - h_1(t_2, t_2)g(t_2, x(t_2), x'(t_2))| + \\
&\quad (\theta + k_5\alpha) \int_0^{t_2} |\frac{\partial h_1(t_1)}{\partial t} - \frac{\partial h_1(t_2)}{\partial t}| ds + \theta(\theta + k_5\alpha)|t_1 - t_2|) + [\theta + a\theta(\theta + k_5\alpha)] \\
&\quad \times (|\frac{\partial q(t_1)}{\partial t} - \frac{\partial q(t_2)}{\partial t}| + |h_2(t_1, t_1)f(t_1, x(t_1), x'(t_1)) - h_2(t_2, t_2)f(t_2, x(t_2), x'(t_2))| + \\
&\quad (\theta + k_5\alpha) \int_0^{t_2} |\frac{\partial h_2(t_1)}{\partial t} - \frac{\partial h_2(t_2)}{\partial t}| ds + \theta(\theta + k_5\alpha)|t_1 - t_2|) + [\theta + \theta(\theta + k_5\alpha) + a\theta(\theta + k_5\alpha)] \\
&\quad \times (|p(t_1) - p(t_2)| + a(\theta + k_5\alpha)\|h_1(t_1, s) - h_1(t_2, s)\|_\infty + \theta(\theta + k_5\alpha)|t_1 - t_2|)
\end{aligned}
\tag{4.2.13}
$$

by(4.2.3),(4.2.4) , (4.2.5), (4.2.13)

We deduce, for $i = 0, 1$, that

$$|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| \leq \varepsilon$$

Consequently, the set $A(C)$ is equicontinuous.

Hence, by Theorem (1.2.1), $A(C)$ is relatively compact.

The proof of Theorem 4.2.2 then follows from Schauder's fixed point theorem. □

Example 4.2.1. Consider the differential equation (4.1.3) :

$$x(t) = [p(t) + \int_0^t h_1(t,s)g(s,x(s),x'(s))ds] \times [q(t) + \int_0^t h_2(t,s)f(s,x(s),x'(s))ds].$$

Hence, by using the notations of Theorem 4.2.2,and for

$$\begin{cases} g(t,u,v) = 2\lambda t + \frac{1}{5}cos(u), \\ f(t,u,v) = \lambda + \frac{1}{6}sin(u) + \frac{1}{5} \\ h_1(t,s) = 2\lambda t cos(s), h_2(t,s) = 3\lambda sin(s) \\ p(t) = 3\lambda t, q(t) = 2\lambda t \end{cases} \tag{4.2.14}$$

if we put $a = 1$ we have $k_1 = k_4 = \frac{1}{5}, k_3 = \frac{1}{6}, k_3 = 0$
so $k_5 = \max\{k_1, k_2, k_3, k_4\} = \frac{1}{5}$
and we have

$$\theta = \max_{t\in[0,a]}\{|p(t)|, |q(t)|, |\phi_1(t)|, |\phi_2(t)|, |\frac{\partial p(t,s)}{\partial t}|, |\frac{\partial q(t,s)}{\partial t}|, |h_1(t,s)|, |h_2(t,s)|, |h_3(t,s)|, |h_4(t,s)|\}$$

so

$$\theta = \max_{t\in[0,1]}\{|3\lambda t|, |2\lambda t|\} = 3\lambda$$

for the condition the Theorem 4.2.2

$$Max\{\theta + a\theta^2, k_5 a\theta, 3\theta + 3a\theta^2 + 2\theta^2, 2\theta k_5 + 3k_5 a\theta\} < \frac{1}{2}$$

and we find

$$Max\{3\lambda + 9\lambda^2, \frac{3}{5}\lambda, 9\lambda + 45\lambda^2, \frac{6\lambda}{5} + \frac{9}{5}\lambda\} < \frac{1}{2}$$

$$\Longleftrightarrow Max\{9\lambda + 45\lambda^2, 3\lambda\} < \frac{1}{2}$$

$$\Longleftrightarrow \begin{cases} 3\lambda < \dfrac{1}{2}, \\ 9\lambda + 45\lambda^2 < \dfrac{1}{2} \end{cases}$$

if

$$3\lambda < \frac{1}{2} \Longrightarrow \lambda < \frac{1}{6} = 0.166$$

and if

$$9\lambda + 45\lambda^2 < \frac{1}{2} \Longrightarrow 90\lambda^2 + 18\lambda - 1 < 0$$

We calculate $\Delta^{'} = \sqrt{171} > 0$, we obtain: $\begin{cases} \lambda_1 = \dfrac{-9+\sqrt{171}}{90} = 0.045 \\ \lambda_2 = \dfrac{-9-\sqrt{171}}{90} = -0.245 \end{cases}$ Then by Theorem 4.2.2, we conclude that if $\lambda \in]\lambda_2, \lambda_1[$ the equation of product type (4.1.3) has a solution $u \in C^1(I, \mathbb{R})$.

Conclusion and Perspective

In this thesis, we have given a new generalization of Ascoli- Arzelá theorem from the space of continuous functions C to the space of functions with continuous of high order derivative C^n. On the other hand, we have used this new generalization to study the existence and the uniqueness of solutions for some integral, integro-differential equations and high order boundary values problems under simple and easy conditions on the data functions. We have also studied the existence and uniqueness of a solution for a second order boundary value problem by using Perov's fixed point in in the generalized metric spaces. Therefore, we have also discussed of existence of a solution of integral equations of the product type. In the end of each chapter, some numerical examples are given to illustrate the theoretical results. Since the general forms of the nonlinear parts of most high order boundary values problems contain the derivatives of the unknown function, then we cannot transform it to an integral equation defined on the space of continuous functions C, therefore, we can't use the well-known Ascoli- Arzelá theorem in C to prove the existence of the solutions. For this reason, our main task in this thesis consists of giving a generalization of the well-knownAscoli- Arzelá.

Further researches on this kind of problems will be conducted by generalizing the Ascoli- Arzelá theorem from $C(\mathbb{R})$ to $C^n(\mathbb{R}^m)$ and use this new generalization to study the existence of solutions for some partial differential equations, partial integro-differential equations.

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