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PERIODIC SOLUTIONS OF SOME CLASSES OF ORDINARY DIFFERENTIAL EQUATIONS

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إهداء

بسم الله الرحمن الرحيم

الحمد لله رب العالمين، وصلى الله على صاحب الشفاعة سيدنا محمد النبي الكريم، وعلى آله وصحبه الميامين، ومن تبعهم بإحسان إلى يوم الدين وبعد:

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Abstract

The objective of this thesis is to provide sufficient conditions for the existence of periodic solutions for various differential systems perturbed by a small parameter using the averaging theory.

We consider the problem of finding the limit cycles for some classes of Duffing differential equation and for two differential systems in $\mathbb{R}^5$ and $\mathbb{R}^6$, using averaging theory of the first order. Further, we study the limit cycles, which bifurcate from the origin of the cubic isochronous Liénard center $\dot{x} = -y$, $\dot{y} = x + x^3 - 3xy$, when we perturb it inside a class of all the cubic polynomial differential systems in $\mathbb{R}^2$, using averaging theory up to the sixth order.

Moreover, we illustrate the results obtained by some examples. We mention that all the computations of this thesis has been done with the help of the algebraic manipulators "Maple" and "Mathematica".

Keywords: averaging theory, differential system, Duffing equation, Liénard equation, limit cycle, periodic solution.

Résumé

L'objectif de cette thèse est de fournir des conditions suffisantes pou l'existence des solutions périodiques pour quelques systèmes différentiels perturbés par un petit paramètre en utilisant la théorie de la moyennisation.
Nous considérons le problème de la recherche des cycles limites pour certaines classes d'équation différentielle de Duffing et pour deux systèmes différentiels dans $\mathbb{R}^5$ et $\mathbb{R}^6$, en utilisant la théorie de moyennisation du premier ordre. En outre, nous étudions les cycles limites qui bifurquent à partir de l'origine du centre isochrone pour le système cubique de Liénard $\dot{x} = -y$, $\dot{y} = x + x^3 - 3xy$, lorsqu'on introduit une perturbation à l'intérieur d'une classe de tous les systèmes différentiels polynomiaux cubiques de $\mathbb{R}^2$, en appliquant la théorie de moyennisation jusqu'au sixième ordre. Par ailleurs, nous illustrons les résultats obtenus par des exemples. Notons que tous les calculs de cette thèse ont été effectués à l'aide des logiciels de calcul "Maple" et "Mathematica".

Mots clés: cycle limite, équation de Duffing, équation de Liénard, solution périodique, système différentiel, théorie de moyennisation.

الملخص

الهدف من هذه الاطروحة هو توفير الشروط الكافية لوجود الحلول الدورية لانواع مختلفة من المعادلات التفاضلية مضطربة بواسطة معلمة صغيرة باستعمال نظرية المتوسط.

نأخذ في الاعتبار مشكلة إيجاد دورات الحد لبعض فئات المعادلة التفاضلية لدوفنغ ولنظامين تفاضليين في $\mathbb{R}^5$ و $\mathbb{R}^6$، باستعمال نظرية المتوسط من الرتبة الأولى. ندرس أيضا الحلول لدورات الحد، والتي تتشعب من أصل مركز المتساوي الزمن "isochronous center" لنظام لينارد المكعب $\dot{x} = -y, \dot{y} = x + x^3 - 3xy$، عندما ندخل اضطراب داخل فئة لجميع الأنظمة التفاضلية بكثيرات حدود من الدرجة الثالثة في $\mathbb{R}^5$، وذلك بتطبيق نظرية المتوسط حتى الرتبة السادسة.

علاوة على ذلك، قمنا بتوضيح النتائج التي تحصلنا عليها بأمثلة. نذكر أن جميع حسابات هذه الأطروحة قد تم الحصول عليها باستخدام برماجي "Maple" و **"Mathematica"**.

كلمات مفتاحية: حل دوري، نظرية المتوسط، دورة الحد، نظام تفاضلي، معادلة دوفنغ، معادلة لينارد.

Introduction

Dynamical systems is an exciting and very active field in pure and applied mathematics that involves tools and techniques from many areas, such as analysis, geometry, etc. Generally, a system is said to be dynamic when it evolves over time. Thus, the study of dynamical systems has applications to a wide variety of fields, including physical, chemical, biological, or economic systems. This evolution is represented by differential equations or applications.

The term “dynamic system” appeared at the beginning of the 20th century between the works of **Poincaré** [49], titled “The New Methods of Celestial Mechanics” in **1892**, and that, in **1927**, of the **Birkhoff** [7], precisely entitled “Dynamical Systems”. One of the main objectives of researchers in the study of dynamical systems is the qualitative study of ordinary differential equations.

Differential equations first came into existence with the invention of calculus in the 17th century by **Newton** and **Leibniz**. These two brilliant minds built the foundations for the theory of ordinary differential equations. An ordinary differential equations is defined by

$$F(t, x, x', x'', \cdots, x^n) = 0, \tag{1}$$

where x^n denote the $n-$th derivative of x with respect to t. When F does not depends in t, we say that differential equation (1) is autonomous. If x is a vector instead of a real function, equation (1) is called a differential system. These equations have a great importance in the development of many areas of science, such as engineering, biology, electronics, economy, etc.

Almost two centuries later, more precisely around **1881**, the work of **Poinacaré** implied a new point of view in the study of ordinary differential equations, which led to the beginning of what is today known as *the qualitative theory of differential equations*, in his series of works "Mémoire sur les courbes définies par une équation différentielle" (see [48]). Using geometric and topological techniques, this brilliant mathematician was able to investigate the qualitative properties of differential equation solutions without explicitly determining such solutions. Thus, instead of looking for a solution, **Poinacaré** turned to a qualitative approach.

One of the important problems in the qualitative theory of real planar differential systems is the determination of limit cycles. The notion of limit cycle was also introduced by **Poincaré**, it is a closed orbit isolated from the other periodic orbits. Years later, in **1901**, contemporaneous with **Poincaré** [48] and using his contributions, **Bendixon** presented the well-known *Poincaré-Bendixon Theorem* which states that under certain conditions, every solution tends to a equilibrium solution which can be either an equilibrium point, or periodic orbit (for more details see [23]). Simulated by this result, in **1907**, **Lyapunov** [44], studied the behavior of the solutions in the neighborhood of an equilibrium position. Due to his work, **Lyapunov** is will know as the founder of the modern theory of *stability of motion.*

Mention that, limit cycle describes the periodic phenomena in nature, and there are a lot of applications such as in physics [15, 16, 28], population dynamics [17, 18, 29, 61] , mechanics [14], astronomy [24], economics [50] and so on.

Limit cycles are also an important topic inside the dynamical systems. Thus the second part of the famous Hilbert 16th Problem is about the number and the configurations of the limit cycles of the planar polynomial systems, for more information see [63] and about the possible configurations of the limit cycles of the polynomial differential systems see [41].

The problems related to the periodic behavior of solutions of higher order differential systems or equations have been discussed by many authors. The papers [6, 22, 31, 45, 58, 62, 64] can be given as good examples in this subject. In the same context, many results have been published on the periodic solutions of different classes of Duffing differential equations under variant conditions.

In **1992**, **Ortega** [47], study the existence of periodic solutions of twist type of a time-dependent differential equation of the second order of the form

$$x'' + f(t, x) = 0,$$

using the relation between topological degree and stability.

In **2015**, **Wang** and **Zhu** [60], study he existence, uniqueness and stability of periodic solutions for the Duffing-type equation

$$x'' + cx' + g(t, x) = h(t),$$

where $c > 0$ is fixed, h is a $T-$periodic function and $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a $T-$periodic function in t; using the Leray-Schauder method.

In **2019**, **Benterki** and **Llibre** in [4], study the existence of periodic solutions for the well known class of **Duffing** differential equation of the form

$$x'' + cx'' + a(t)x + b(t)x^3 = h(t),$$

where c is a real parameter, $a(t), b(t)$ and $h(t)$ are continuous $T-$periodic functions. The results are proved using three different results on the averaging theory of first order.

They also study the existence of new periodic solutions of the two **Duffing** differential equations of the form:

$$y'' + a \sin y = b \sin t, \quad \textit{and} \quad y'' + ay - cy^3 = b \sin t,$$

where a, b and c are real parameters; using averaging theory of first order (see [3]).

In **2020**, **Feddaoui**, **Llibre** and **Makhlouf** [26], study the existence of periodic solutions of the class of Duffing differential equations

$$x'' + c(t)x' + a(t)x + b(t)x^3 = h(t, x),$$

where the functions $a(t), b(t), c(t)$ and $h(t, x)$ are C^2 and $T-$periodic in the variable t.

In the same year, **Cheng** and **Yuan** [20], study the following damped Duffing equation with a equilibriumity

$$x'' + Cx' + g(x) = p(t),$$

where the damped coefficient $C(\geqslant 0)$ is a constant, elastic restoring force $g : (0, +\infty) \longrightarrow \mathbb{R}$ is locally Lipschitz continuous and has a strong equilibriumity of repulsive type at the origin, external force $p : \mathbb{R} \to \mathbb{R}$ is continuous periodic function with a minimal period T, using the twist theorem of nonarea-preserving map.

In **2021**, **Šremr** [57], study a bifurcation of positive solutions of the parameter-dependent periodic Duffing problem

$$u'' = p(t)u - h(t)|u|^{\lambda} sgn(u) + \mu f(t); \;\; u(0) = u(\omega), u'(0) = u'(\omega),$$

where $\lambda > 1$, $p, h, f \in L([0, \omega])$, and $\mu \in \mathbb{R}$ is a parameter.

These last years many papers tried to give partial answers to the 16th Hilbert problem for different classes of polynomial differential systems, see for instance [27, 36–38, 42] and the hundred of references quoted therein.

Smale in [56] proposed the class of classical Liénard differential systems of the form

$$\dot{x} = y, \quad \dot{y} = x - f(x)y, \tag{2}$$

where $f(x)$ is polynomial, or equivalent to the form

$$\dot{x} = y - F(x), \quad \dot{y} = x, \text{ where } F(x) = \int f(x)\,dx.$$

In **1977**, **Lins**, **De Melo** and **Pugh** [35], stated the conjecture that if $f(x)$ has degree $n \geqslant 1$, then the system (2) has at most $\left[\frac{n}{2}\right]$ limit cycles. They prove this conjecture for $n = 1, 2$. Moreover, research continued in the same context, see for instance [32].

Other authors studied the limit cycles of generalized Liénard polynomial differential equations which was introduced in [34] of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3}$$

where $f(x)$ and $g(x)$ are polynomials in the variables x, see for instance [1, 39].

Many results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate equilibrium point (i.e. from a Hopf bifurcation), that are called small amplitude limit cycles, see for instance **Lloyd** [43]. There are partial results concerning the number of small amplitude limit cycles for different classes of Liénard polynomial differential equations or systems see [10, 11].

To obtain analytically periodic solutions is in general a difficult work, many times an impossible work. The averaging theory reduces this difficult problem for some ordinary differential equation to find the zeros of nonlinear functions.

Averaging theory was introduced by **Bogoliubov** and **Krylov** in **1934** [9], and **Bogoliubov** and **Mitropolsky (1961)** [8]. It was then developed by

Verhulst [59], **Sanders** and **Verhulst** [54], **Malkin (1956)** [46], **Roseau (1985)** [52], **Buică** and **Llibre (2004)** [13], etc.

This thesis is presented in the following chapters:

✦ First chapter: Preliminary Notions.

This chapter gives a reminder of the classic preliminary notions and the mathematical tools that are necessary for the study of this thesis.

✦ Second chapter: Averaging theory.

We present the different theorems of the averaging theory for finding the periodic solutions of the differential equations.

✦ Third chapter: Periodic solutions for two classes of Duffing differential equations.

We provide sufficient conditions for the existence of periodic solutions of two classes of Duffing differential equation. The first class is

$$\ddot{x} + \varepsilon p(t)\dot{x} + (1 + \varepsilon q(t))x = \varepsilon f(t, x) + \varepsilon c(t),$$

where $p(t), q(t), f(t, x)$ and $c(t)$ are $2\pi-$periodic functions in the variable t, ε is a small parameter and $x \in \mathbb{R}$. The second class is

$$\ddot{x} + (1 + \varepsilon\mu(t))\dot{x} + \varepsilon \sum_{i=0}^{n} \rho_{2i+1}(t)x^{2i+1} = \varepsilon f(t, x)$$

where $\mu(t), \rho_{2i+1}(t)$ with $i = 0, ..., n$ and $f(t, x)$ are C^2 functions $T-$periodic in the variable t, ε is a small parameter and $x \in \mathbb{R}$, using the averaging theory of the first order. Mention that this study is submitted for publication.

✦ Fourth chapter: Periodic solutions for a generalized Duffing differential equations.

We study the existence of periodic solutions for a class of the well known Duffing differential equations of the form

$$\ddot{x} + c(t)\dot{x} + g(t, x) = p(t),$$

where $c(t), g(t, x)$ and $p(t)$ are $\mathcal{C}^2$ and T–periodic in the variable t, using the averaging theory of the first order.
This chapter is submitted for publication in the international journal "**Differential Equations and Dynamical Systems**".

✦ Fifth chapter: Limit cycles of cubic polynomial differential systems in $\mathbb{R}^2$ via averaging theory of order 6.

Here, we study the limit cycles which bifurcate from the origin of the cubic isochronous Liénard center $\dot{x} = -y$, $\dot{y} = x + x^3 - 3xy$, when we perturb it inside the class of all the cubic polynomial differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = -y + \sum_{i=1}^{6} \varepsilon^i P_i(x, y), \quad \dot{y} = x + x^3 - 3xy + \sum_{i=1}^{6} \varepsilon^i Q_i(x, y),$$

where P_i and Q_i with $i = 1, \ldots, 6$ are polynomials of degree 3 and ε is a small parameter. The tool for doing this study is the averaging theory up to order six. Moreover, we illustrate with some examples the results obtained. Mention that this study is also submitted for publication.

✦ Sixth chapter: Periodic solutions for differential systems in $\mathbb{R}^5$ and $\mathbb{R}^6$.

Based on two different results of the averaging theory of the first order, we provide sufficient conditions for the existence of periodic solutions for two differential systems. The first one in $\mathbb{R}^5$ is of the form

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u, \quad \dot{u} = v,$$
$$\dot{v} = -\alpha\beta\mu x - \beta\mu y - \alpha(\beta+\mu)z - (\beta+\mu)u - \alpha v + \varepsilon f(t,x,y,z,u,v),$$

where α, β and μ are rational numbers different from 0 such that $\alpha \neq \pm\beta$, $\alpha \neq \pm\mu$, and $\beta \neq \pm\mu$ with $|\varepsilon|$ sufficiently small, and f is non-autonomous periodic function. The second differential system in $\mathbb{R}^6$ is given by

$$\begin{aligned}
\dot{x} &= y, \quad \dot{y} = -x - \varepsilon F(t,x,y,z,u,v,w),\\
\dot{z} &= u, \quad \dot{u} = -z - \varepsilon G(t,x,y,z,u,v,w),\\
\dot{v} &= w, \quad \dot{w} = -v - \varepsilon H(t,x,y,z,u,v,w),
\end{aligned}$$

where F, G and H are 2π–periodic functions in the variable t, and ε is a small parameter.

This study was published in the international journal "**Journal of Dynamical and Control Systems**" titled "**Periodic solutions for differential systems in $\mathbb{R}^5$ and $\mathbb{R}^6$**", for more details see [51].

Mention that in what follows, we denote "Chp." which means chapter, and "Sec." means section.

Chapter 1 Preliminary Notions

Contents

In this chapter, we introduce some general and main notions for the qualitative study of dynamical systems and polynomial differential systems.

1.1 Dynamical systems

Definition 1.1.1. *A dynamical system on* $\mathbb{R}^n$ *is a map* $U : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *such that*

(a) $U(.,x) : \mathbb{R} \to \mathbb{R}^n$ *is continuous.*

(b) $U(t,.) : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous.*

(c) $U(0,x) = x$.

(d) $U(t+s,x) = U(t,U(s,x))$, $\forall t,s \in \mathbb{R}$, $\forall x \in \mathbb{R}^n$.

Example 1.1. *Consider the linear system*

$$\begin{cases} \dot{x} = Ax, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

The solution of system (1.1) is of the form $x(t) = e^{tA}x_0$ *where* $t \geqslant 0$, $x \in \mathbb{R}^n$ *and* A *a constant matrix.*

The system (1.1) engender a dynamical system $U(t,x)$ *such that*

$$U : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$$

$$U(t,x) = e^{tA}x.$$

Definition 1.1.2. *A dynamical system* U *on* $\mathbb{R}^n$ *is said to be linear if*

$$U(t, \alpha x + \beta y) = \alpha U(t,x) + \beta U(t,y), \tag{1.2}$$

$\forall \alpha, \beta \in \mathbb{R}, \forall x, y \in \mathbb{R}^n$ *and* $\forall t \geqslant 0$.

1.2 Polynomial differential systems

Definition 1.2.1. *A polynomial differential system in* $\mathbb{R}^n$ *is a system of the form*

$$\begin{cases} \dfrac{dx_1}{dt}(t) &= P_1(x_1(t), x_2(t), ..., x_n(t)), \\ \dfrac{dx_2}{dt}(t) &= P_2(x_1(t), x_2(t), ..., x_n(t)), \\ . & \\ . & \\ . & \\ \dfrac{dx_n}{dt}(t) &= P_n(x_1(t), x_2(t), ..., x_n(t)), \end{cases} \tag{1.3}$$

where $P_1, P_2, \ldots$ *and* P_n *are polynomials with real coefficients.*

- *If* $d = max(degP_1, degP_2, ..., degP_n)$, *then system* (1.3) *is called of degree* d.
- *If* $P_1, P_2, ..., P_n$ *do not depend on* t *explicitly, then system* (1.3) *is said to be autonomous.*

1.3 Solution of a differential system

We call solution of the system (1.3) every derivable application

$$X : I \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

$$t \mapsto X(t) = (X_1(t), X_2(t), ..., X_n(t)),$$

where I is a non-empty interval such that, for any $t \in I$, $(X_1(t), X_2(t), ..., X_n(t))$ satisfies the system.

1.4 Equilibrium point and linearization

The equilibrium points have an important role in studying nonlinear differential systems. **Poincaré** showed that it is enough to know the behavior of the solution through the study of the equilibrium points instead of solving these differential systems.

The most of systems that model natural phenomena are nonlinear. To study the behavior of the trajectories of these systems, in the neighbourhood of an equilibrium point x_0, we study the associated linearized systems.

1.4.1 Equilibrium point

Definition 1.4.1. *Consider the differential system*

$$\dot{x} = f(x). \tag{1.4}$$

A point x_0 is said to be equilibrium or equilibrium of the system (1.4) if

$$f(x_0) = 0.$$

1.4.2 Linearization

Definition 1.4.2. *Consider the nonlinear differential system (1.4).*
Let x_0 be an equilibrium point of the system (1.4).
The system

$$\dot{x} = Ax, \tag{1.5}$$

where

$$A = Df(x_0) = \frac{\partial f_i}{\partial x_j}(x_0), \qquad 1 \leqslant i, j \leqslant n, \tag{1.6}$$

is said linearized system of the system (1.4) at the point x_0.

A *is called jacobian matrix associated to the system (1.4) evaluated at* x_0.

Example 1.2. *Consider the system*

$$\begin{cases} \dot{x} = x^4 - 2y, \\ \dot{y} = 2x + 3y^4. \end{cases} \tag{1.7}$$

$X_0 = (0,0)$ *is an equilibrium point of system (1.7).*

The jacobian matrix associated to the system (1.7) calculated at $(0,0)$ *is given by*

$$Df(0,0) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Thus, the linearized system of the system (1.7) is

$$\begin{cases} \dot{x} = -2y, \\ \dot{y} = 2x. \end{cases} \tag{1.8}$$

Definition 1.4.3. *The equilibrium point* x_0 *of the system (1.4) is said to be hyperbolic if none of the eigenvalues of the jacobian matrix* $A = Df(x_0)$ *has zero real part.*

1.4.3 Classification of equilibrium points

Definition 1.4.4. *Consider the differential system (1.4) with* $x \in \mathbb{R}^2$. *Let* A *the jacobian matrix calculated at the point* $X_0 = (0,0)$, *and let* λ_1 *and* λ_2 *the eigenvalues of this matrix. We distinguish the different cases according to these eigenvalues:*

1. *If λ_1 and λ_2 are real, nonzero and of different sign, then the equilibrium point X_0 is a saddle point. It is always unstable (see Fig. 1.1).*

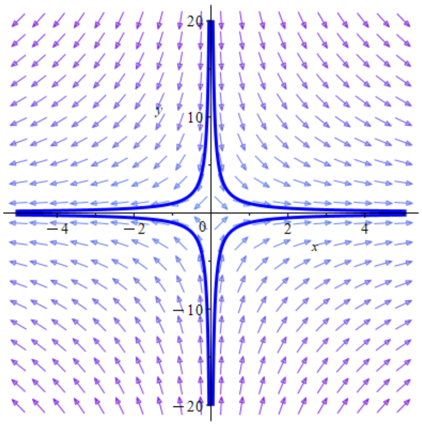

Figure 1.1 – $(0, 0)$ is a saddle point.

2. *If λ_1 and λ_2 are real of the same sign, we have three cases:*

   ✧ *If $\lambda_1 < \lambda_2 < 0$, then X_0 is a stable node (see Fig. 1.2).*

   ✧ *If $0 < \lambda_1 < \lambda_2$, then X_0 is an unstable node (see Fig. 1.3).*

   ✧ *If $\lambda_1 = \lambda_2 = \lambda$, we have two cases:*

      ☆ *If A is diagonalizable, then X_0 is a proper node(PN). It is stable if $\lambda < 0$ and unstable if $\lambda > 0$ (see Fig. 1.4* and *Fig. 1.5 respectively).*

      ☆ *If A If A is not diagonalizable, then X_0 is an exceptional kind of node. It is exceptional stable node (ESN) if $\lambda < 0$ and exceptional unstable node (EUN) if $\lambda > 0$ (see Fig. 1.6 and Fig. 1.7 respectively).*

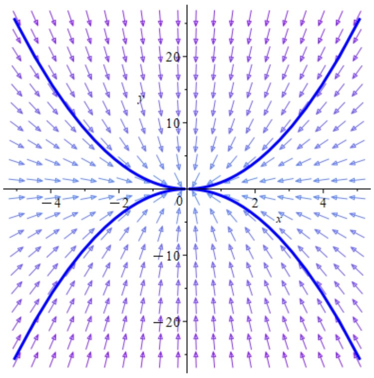

Figure 1.2 – $(0,0)$ is a stable node.

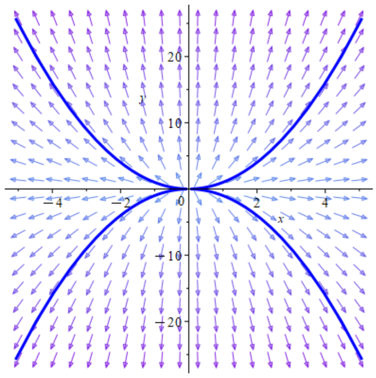

Figure 1.3 – $(0,0)$ is an unstable node.

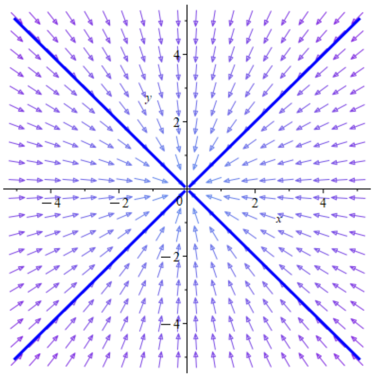

Figure 1.4 – $(0,0)$ is a stable PN.

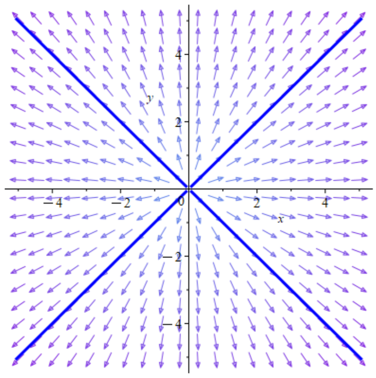

Figure 1.5 – $(0,0)$ is an unstable PN.

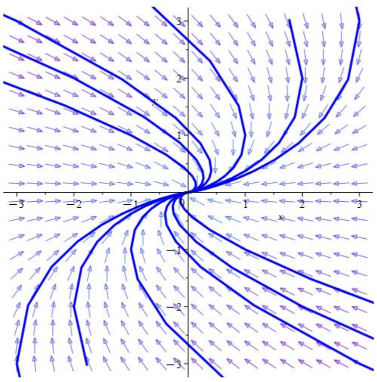

Figure 1.6 – $(0,0)$ is ESN.

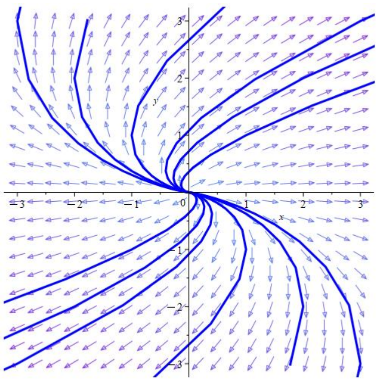

Figure 1.7 – $(0,0)$ is EUN.

3. *If λ_1 and λ_2 are complex conjugated with a nonzero imaginary part, then X_0 is a focus. It is stable if $Re(\lambda_{1,2}) < 0$ and unstable if $Re(\lambda_{1,2}) > 0$ (see Fig. 1.8 and Fig. 1.9 respectively).*

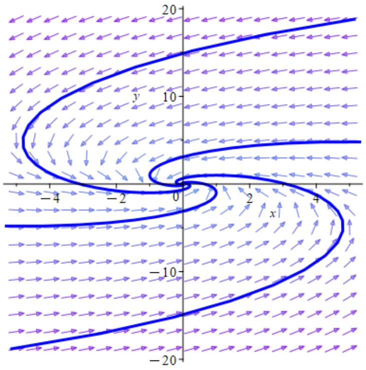

Figure 1.8 – $(0, 0)$ is a stable focus.

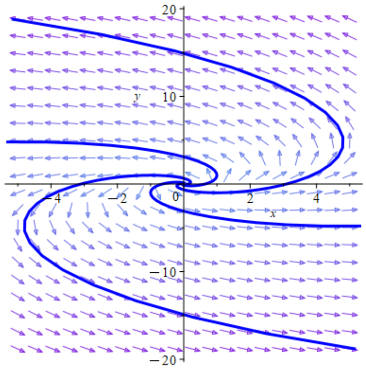

Figure 1.9 – $(0, 0)$ is an unstable focus.

4. *If λ_1 et λ_2 are pure imaginary, then X_0 is a center. It is stable but it is not asymptotically stable (see Fig. 1.10).*

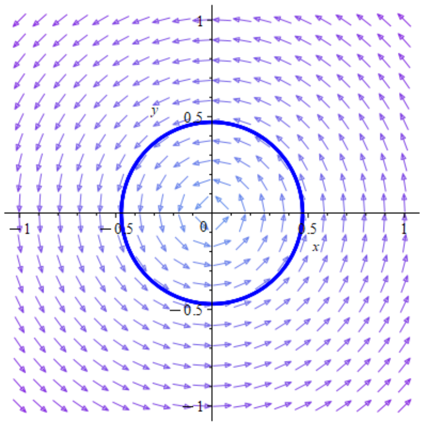

Figure 1.10 – $(0, 0)$ is a center.

1.5 Stability of equilibrium points

A nonlinear system can have many equilibrium points. These points can be stable or unstable.

Consider the system

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.9}$$

Let p be an equilibrium point of the system (1.9) and $\phi(t)$ the solution of this system.

Definition 1.5.1. *We say that*

i) *p is stable if and only if*

$$\forall \varepsilon > 0, \exists \delta > 0, \|\phi(t_0) - p\| < \delta \Rightarrow \|\phi(t) - p\| < \varepsilon, \forall t \geqslant t_0.$$

ii) *p is asymptotically stable if and only if p is stable and if there exists a neighborhood of p such that for all x in this neighborhood*

$$\lim_{t \to \infty} \phi(t) = p.$$

We can study the stability of the system (1.9) according to the eigenvalues of the jacobian matrix $Df(p)$, using the following theorem.

Theorem 1.5.1. *Let p be the equilibrium point of the system (1.9).*

a) *If all the eigenvalues of the jacobian matrix $Df(p)$ have negative real parts, then the equilibrium point p is said to be asymptotically stable.*

b) *If there exists at least one eigenvalue of $Df(p)$ with a positive real part, then the equilibrium point p is said to be unstable.*

c) *If $Df(p)$ has eigenvalues with negative real parts and others with zero real parts, then nothing can be said about the stability of the equilibrium point p.*

1.6 Phase portrait

Definition 1.6.1. *A trajectory is a curve traced by the solution of a differential equation.*

Definition 1.6.2. *Consider the planar system*

$$\begin{cases} \dfrac{dx}{dt} = P_1(x(t), y(t)), \\ \dfrac{dy}{dt} = P_2(x(t), y(t)). \end{cases} \tag{1.10}$$

A phase portrait is the set of trajectories in phase space. In particular, for autonomous systems of ordinary differential equations of two variables, the solutions $(x(t), y(t))$ of the system (1.10) represent in the plane (x, y) curves called orbits.

The equilibrium points of this system are constant solutions and the complete figure of the orbits of this system together with these equilibrium points represent the phase portrait and the $(x \circ y)$ plane is called the phase plane.

1.7 Periodic orbits and limit cycles

1.7.1 Periodic orbits

Definition 1.7.1. *A trajectory $\phi(t, x)$ of the system (1.3) is called periodic orbit if there exists a number $T > 0$ such that*

$$\phi(t + T, x) = \phi(t, x), \forall x \in \mathbb{R}^n. \tag{1.11}$$

The smallest real T satisfying (1.11) is called the period.

1.7.2 limit cycles

Definition 1.7.2. *A limit cycle is a closed periodic orbit isolated in a set of periodic orbits.*

1.8 Stability of limit cycles

Theorem 1.8.1. *Let C being the trajectory corresponding to the limit cycle, and let all the interior and exterior trajectories close to C wind up in spirals around C for $t \to +\infty$ or $t \to -\infty$.*

1. *The limit cycle is said to be stable, if all neighboring trajectories are attracted towards C.*

2. *The limit cycle is said to be unstable, if all neighboring trajectories are pushed away from C.*

Example 1.3. *Consider the system*

$$\begin{cases} \dot{x} &= \frac{1}{2}x - y - x(2x^2 + 2y^2), \\ \dot{y} &= x + \frac{1}{2}y - y(2x^2 + 2y^2). \end{cases} \tag{1.12}$$

In polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ with $r > 0$, system (1.12) becomes

$$\begin{cases} \dot{r} &= \frac{1}{2}r(1 - 4r^2), \\ \dot{\theta} &= 1. \end{cases} \tag{1.13}$$

We obtain

$$f(r) = \frac{dr}{d\theta} = \frac{1}{2}r(1 - 4r^2).$$

So

$$\dot{r} = 0 \Rightarrow r = 0 \quad \textit{or} \quad r = \pm\frac{1}{2}.$$

Since $r > 0$, we only accept the positive root $r = \frac{1}{2}$. Then the periodic solution is written in the following form

$$(x(t), y(t)) = (\frac{1}{2}\cos(t+\theta_0), \frac{1}{2}\sin(t+\theta_0)),$$

with $\theta(0) = \theta_0$.

In the phase plane, there is only one equation limit cycle $x^2 + y^2 = \frac{1}{2}$ whose amplitude $r = \frac{1}{2}$ (see Fig. 1.11).

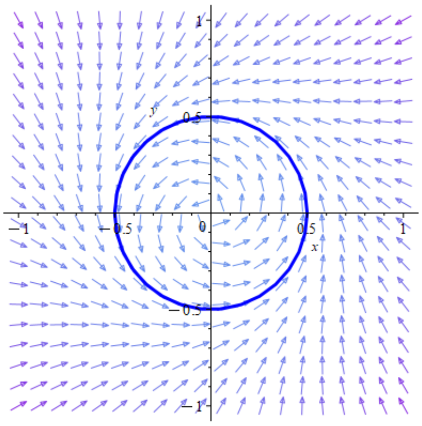

Figure 1.11 – Limit cycle of system (1.12).

1.9 Existence and non-existence of limit cycles

The study of the existence of limit cycles plays an important role in the study of the behavior of trajectories of nonlinear differential systems.

Theorem 1.9.1. ***(Poincaré-Bendixon)***

Consider the following planar system

$$\begin{cases} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y). \end{cases} \tag{1.14}$$

Suppose that f and g are two functions of class C^1 on an open subset of $\mathbb{R}^2$ denoted by E, the system (1.14) has an orbit γ such that the positive orbit $\gamma_+(p) = \Phi(p,t), t \geqslant 0$ passing through the point p is contained in a compact subset F of E. Then we are in one of the following three cases:

- $\gamma_+(p)$ *tends to an equilibrium point.*
- $\gamma_+(p)$ *tends to a periodic orbit.*
- $\gamma_+(p)$ *is a periodic orbit.*

If F does not contain equilibrium points then there is a periodic orbit of the system (1.14).

Theorem 1.9.2. ***(Bendixon criterion)***

Consider the system

$$\begin{cases} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \end{cases}$$

and let $F = (f,g)^T \in C^1(E)$ where E is a simply connected region in $\mathbb{R}^2$. If the divergence of the vector field F (denoted ∇F) is not identically zero and does not change sign in E, then this system does not have a closed orbit entirely contained in E.

Example 1.4. *Consider the following planar differential system*

$$\begin{cases} \dot{x} = 2xy - 2y^4 - x, \\ \dot{y} = x^2 - y^2 - x^2y^3. \end{cases}$$

Let $F = (2xy - 2y^4 - x, x^2 - y^2 - x^2y^3)^T$.

We calculate the divergence of the vector field F, *we obtain*

$$\begin{aligned} divF &= \nabla F = \frac{\partial}{\partial x}(2xy - 2y^4 - x) + \frac{\partial}{\partial y}(x^2 - y^2 - x^2y^3) \\ &= 2y - 1 - 2y - 3x^2y^2 = -1 - 3x^2y^2 < 0. \end{aligned}$$

Hence, according to the ***Bendixon criterion*** *this system has no limit cycle in* $\mathbb{R}^2$.

1.10 Isochronous set

The isochronous set is a set formed only by periodic solutions, which have the same period.

1.11 Descartes Theorem

Consider the real polynomial

$$p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r},$$

with $0 \leqslant i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \cdots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign.If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r - 1$ positive real roots.

For more information see [5].

1.12 Bifurcation

Definition 1.12.1. *we say that a differential equation system*

$$\dot{x} = f(x(t), \mu), \tag{1.15}$$

has a bifurcation at the value $\mu = \mu_0$*, if there is a change in trajectory structure as the parameter* μ *crosses the value* μ_0*. That is, there is a change in the number and/or stability of equilibria of the system at the bifurcation value (see [59], p. 173).*

1.13 Hopf bifurcation

Theorem 1.13.1. *Consider the planar differential system*

$$\begin{cases} \dot{x} &= f_\mu(x, y), \\ \dot{y} &= g_\mu(x, y), \end{cases} \tag{1.16}$$

where μ *is a parameter. Suppose that* $(x, y) = (x_0, y_0)$ *is an equilibrium point of the system (1.16) which depends on* μ*.*

Let $\lambda(\mu) = \alpha(\mu) + i\beta(\mu)$ *and* $\overline{\lambda(\mu)} = \alpha(\mu) - i\beta(\mu)$ *be the eigenvalues of the linearized system in the neighborhood of* (x_0, y_0)*.*

Suppose further that for a certain value of $\mu = \mu_0$*, the following conditions are satisfied:*

1. $\alpha(\mu_0) = 0$, $\beta(\mu_0) = w \neq 0$ *where* $sgn(w) = sgn(\partial g_\mu / \partial x \mid_{\mu=\mu_0} ((x_0, y_0)))$,

2. $\dfrac{d\alpha(\mu)}{d\mu} \mid_{\mu=\mu_0} = d \neq 0$,

3. $a \neq 0$ *where*

$$\begin{aligned} a = & \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16w}(f_{xy}(f_{xx} + f_{yy}) \\ & - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}), \end{aligned}$$

with $f_{xy} = \partial^2 f/\partial x \partial y \mid_{\mu=\mu_0} (x_0, y_0)$*, etc.*

Then, a periodic orbit bifurcate from the equilibrium point for $\mu > \mu_0$ *if* $ad < 0$ *or for* $\mu < \mu_0$ *if* $ad > 0$.

The equilibrium point (x_0, y_0) *is stable for* $\mu > \mu_0$ *(resp. for* $\mu < \mu_0$*) and an unstable equilibrium point for* $\mu < 0$ *(resp.* $\mu > 0$*) if* $d < 0$ *(resp.* $d > 0$*).*

The periodic orbit is stable (resp. unstable) if the equilibrium point is unstable (resp. stable).

The amplitude of the periodic orbits are equal to $\sqrt{\mid \mu - \mu_0 \mid}$ *whilst their periods is* $T = 2\pi/\mid w \mid$ *when* $\mu \longrightarrow \mu_0$.

The bifurcation is said to be supercritical if the periodic orbit is stable and subcritical if the periodic orbit is unstable.

Example 1.5. *Consider the oscillator* $\ddot{x} - (\mu - x^2)\dot{x} + x = 0$ *(an example of a so-called Liénard differential system), which, with* $\dot{x} = y$*, we can write as the first-order system*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + (\mu - x^2)y. \end{cases} \tag{1.17}$$

$(0, 0)$ *is the only equilibrium point of system (1.17).*

The eigenvalues of the jacobian matrix of the linearized system calculated in the neighborhood of $(0, 0)$ *are* $\lambda(\mu) = \frac{1}{2}\left(\mu + i\sqrt{4 - \mu^2}\right)$*,* $\overline{\lambda(\mu)} = \frac{1}{2}\left(\mu - i\sqrt{4 - \mu^2}\right)$.

The system has a Hopf bifurcation at $\mu_0 = 0$*. We have* $w = 1$*,* $d = \frac{1}{2}$ *and* $a = -\frac{1}{8}$.

The equilibrium point $(0,0)$ *is unstable for* $\mu > 0$*, so the bifurcation is supercritical and there is a stable isolated periodic orbit (limit cycle) if* $\mu > 0$ *for each sufficiently small* μ *(see Fig. 1.12 for* $\mu = 0.3$*).*

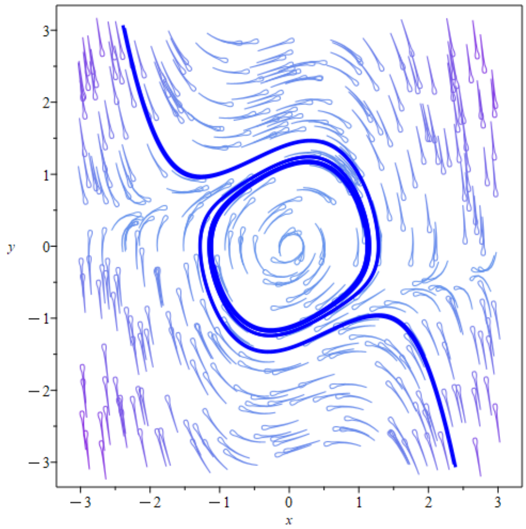

Figure 1.12 – Phase portraits of system (1.17).

1.14 Zero-Hopf bifurcation

Definition 1.14.1. *The equilibrium point of a given differential system, with dimension greater than two, is referred to as Zero-Hopf equilibrium point its associated Jacobian matrix has a zero eigenvalue and a pair of purely imaginary eigenvalues. This kind of bifurcation is thoroughly analyzed by many author see for example* ***Guckenheimer*** *and* ***Holmes*** *in [28] and a references quoted therein.*

1.15 Liénard and Duffing equation

1.15.1 Liénard equation

Definition 1.15.1. *Let f and g be two continuously differentiable functions on $\mathbb{R}$. Then the second order ordinary differential equation of the form*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.18}$$

is called the ***Liénard*** *equation.*

1.15.2 Duffing equation

Definition 1.15.2. *The* ***Duffing*** *equation is a nonlinear second-order differential equation of the form*

$$\ddot{x} + c\dot{x} + g(x) = p(t), \tag{1.19}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and locally Lipschitz function, c is a constant and $c \geqslant 0$, $p : \mathbb{R} \to \mathbb{R}$ is continuous, and T-periodic function.

1.16 Auxiliary results

For $m, n \in \mathbb{N}$, we define

$$I_{m,n} = \int_{0}^{2\pi} \cos^m \theta \sin^n \theta d\theta, \qquad (1.20)$$

then

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n},$$

and

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}.$$

✧ If m and n are evens, then $I_{m,n} = \mathit{coeff}(m,n) \times I_{0,0} = \mathit{coeff}(m,n) \times 2\pi$ (reducing m and n, of 2 in 2 until reaching 0 and 0).

✧ Otherwise $I_{m,n} = coeff(m,n) \times I_{1,0}$, or $I_{1,1}$, or else $I_{0,1} = 0$ everytime. Thus, $I_{m,n} \neq 0$ if and only if m and n are both even.

Chapter

2 Averaging theory

Contents

The method of averaging is a classical and mature tool that allows us to study the dynamics of nonlinear differential systems under periodic forcing. The method of averaging has a long history that starts with the classical works of **Lagrange** and **Laplace**, who provided an intuitive justification of the method. The first formalization of this theory was done in **1928** by **Fatou** [25]. Important practical and theoretical contributions to the averaging theory were made in **1930** by **Bogoliubov** and **Krylov** [9], etc. In **2004**, **Llibre**, **Novaes** and **Teixeira** [40] extended the averaging theory for computing periodic solutions to an arbitrary order in ε for continuous differential equations with n variables. We refer to the book of **Sanders** and **Verhulst** [54] for a general introduction to this subject.

In this chapter, we introduce the theory of averaging, and we give its essential theorems used to achieve the work of this thesis.

2.1 A first order averaging theory

We consider the differential system

$$\dot{x}(t) = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{2.1}$$

with $x \in D \subset \mathbb{R}^n$, D bounded domain, and $t \geqslant 0$. Moreover, we assume that $F(t,x)$ and $R(t,x,\varepsilon)$ are $T-$periodic in t.

The averaged system associated to system (2.1) is defined by

$$\dot{y}(t) = \varepsilon f^0(x), \quad y_0 = x_0, \tag{2.2}$$

where

$$f^0(y) = \frac{1}{T}\int_0^T F(s,y)ds. \tag{2.3}$$

The next theorem says under which conditions the equilibrium points of the averaged system (2.2) provide $T-$periodic orbits of system (2.1).

Theorem 2.1.1. *We consider system (2.1) and assume that*

(i) *F, R, D_xF, D_x^2F and D_xR are continuous and bounded by a constant M (independent of ε) in $[0,+\infty)\times D$, with $-\varepsilon_0<\varepsilon<\varepsilon_0$.*

(ii) *F and R are $T-$periodic in t, with T independent of ε.*

Then, we have:

(a) *If $p\in D$ is a equilibrium point of the averaged system (2.2) such that*

$$\det\Big(D_x f^0(p)\Big)\neq 0, \tag{2.4}$$

Then for $\varepsilon>0$ sufficiently small, there exists a $T-$periodic solution $x_\varepsilon(t)$ of the system (2.1) such that $x_\varepsilon(t)\to p$ as $\varepsilon\to 0$.

(b) *If the equilibrium point $y=p$ of the averaged system (2.2) is hyperbolic, then, for $|\varepsilon|>0$ sufficiently small, the corresponding periodic solution of the system (2.1) is unique and of the same stability as p.*

Proof of Theorem 2.1.1. See [54]. □

Example 2.1. *Consider the* ***Van Der Pol*** *differential equation*

$$\ddot{x}+x=\varepsilon(1-x^2)\dot{x},$$

which can be written as the differential system

$$\begin{cases}\dot{x}=y\\ \dot{y}=-x+\varepsilon(1-x^2)y.\end{cases}$$

In polar coordinates (r,θ) where $x=r\cos\theta$, $y=r\sin\theta$ with $r>0$, this system becomes

$$\begin{cases}\dot{r}=\varepsilon r(1-r^2\cos^2\theta)\sin^2\theta\\ \dot{\theta}=-1+\varepsilon(1-r^2\cos^2\theta)\sin\theta\cos\theta.\end{cases} \tag{2.5}$$

or equivalently

$$\frac{dr}{d\theta} = -\varepsilon r(1 - r^2\cos^2\theta)\sin^2\theta + O(\varepsilon^2). \tag{2.6}$$

Note that the previous differential system is in the normal form (2.1) for applying the averaging theory described in Theorem if we take $x = r$, $t = \theta$, $T = 2\pi$ and $F(t, x) = F(\theta, r) = -r(1 - r^2\cos^2\theta)\sin^2\theta$.
From (2.3) we get that

$$f^0(r) = \frac{1}{2\pi}\int_0^{2\pi} F(\theta, r)d\theta = \frac{1}{8}r(r^2 - 4).$$

The unique positive root of $f^0(r)$ is $r = 2$. Since $\left(\frac{df^0}{dr}\right)(2) = 1 \neq 0$, by statement (a) of Theorem 2.1.1, it follows that system (2.5) has for $|\varepsilon| \neq 0$ sufficiently small a limit cycle bifurcating from the periodic orbit of radius 2 of the unperturbed system (2.5) with $\varepsilon = 0$. Moreover, since $\left(\frac{df^0}{dr}\right)(2) = 1 > 0$, by statement (b) of Theorem 2.1.1, this limit cycle is unstable, (see Fig. 2.1 bellow for $\epsilon = 10^{-2}$).

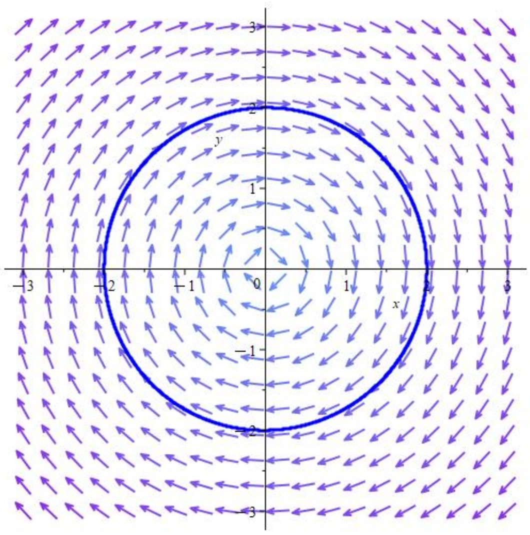

Figure 2.1 – The unstable limit cycle of equation (2.1).

2.2 Another first order averaging theory

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\mathbf{x}' = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{2.7}$$

with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, for ε_0 sufficiently small. Here the functions $F_0, F_1 : \mathbb{R}\times\Omega \mapsto \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \mapsto \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\mathbf{x}' = F_0(t,\mathbf{x}), \tag{2.8}$$

has a submanifold of periodic solutions.

Let $\mathbf{x}(t,\mathbf{z})$ be the solution of system (2.8) such that $\mathbf{x}(0,\mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t,\mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{2.9}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (2.9), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \mapsto \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_k)$.

Theorem 2.2.1. *Let $V \in \mathbb{R}^k$ be open bounded with its closure contained in Ω i.e. $\mathrm{Cl}(V) \subset \Omega$, and let $\beta_0 : \mathrm{Cl}(V) \mapsto \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume*

(i) *$Z = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)),\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in Z$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha)$ of (2.8) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in Z$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (2.9) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the right up corner the $k \times (n-k)$*

zero matrix, and in the right down corner a $(n-k)\times(n-k)$ *matrix* Δ_α *with* $\det(\Delta_\alpha)\neq 0$.

We consider the function $\mathcal{F}:\mathrm{Cl}(V)\mapsto\mathbb{R}^k$ defined by

$$\mathcal{F}(\alpha)=\xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha))dt\right). \tag{2.10}$$

If there exists $a\in V$ with $\mathcal{F}(a)=0$ and $\det((\frac{d\mathcal{F}}{d\alpha})(a))\neq 0$, then there is a T–periodic solution $\mathbf{x}(t,\varepsilon)$ of system (2.7) such that $\mathbf{x}(0,\varepsilon)\to\mathbf{z}_\alpha$ as $\varepsilon\to 0$.

Proof of Theorem 2.2.1. The proof goes back to **Malkin** [46] and **Roseau** [52], and for shorter proof see [12]. □

We assume that there exists an open set V with $\mathrm{Cl}(V)\subset\Omega$ such that for each $\mathbf{z}\in\mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, where $\mathbf{x}(t,\mathbf{z},0)$ denotes the solution of the unperturbed system (2.8) with $\mathbf{x}(t,\mathbf{z},0)=\mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (2.7); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 2.2.2. *[Perturbations of an isochronous set] We assume that there exists an open and bounded set* V *with* $\mathrm{Cl}(V)\subset\Omega$ *such that for each* $\mathbf{z}\in\mathrm{Cl}(V)$, *the solution* $\mathbf{x}(t,\mathbf{z})$ *is* T*–periodic, then we consider the function* $\mathcal{F}:\mathrm{Cl}(V)\to\mathbb{R}^n$ *as*

$$\mathcal{F}(\mathbf{z})=\frac{1}{T}\int_0^T M_{\mathbf{z}}^{-1}(t,\mathbf{z})F_1(t,\mathbf{x}(t,\mathbf{z}))dt. \tag{2.11}$$

If there exists $a\in V$ *with* $\mathcal{F}(a)=0$ *and*

$$\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(a)\right)\neq 0, \tag{2.12}$$

then there exists a T*–periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system* (2.7) *such that* $\mathbf{x}(0,\varepsilon)\to a$ *as* $\varepsilon\to 0$.

Proof of Theorem 2.2.2. It follows immediately from Theorem 2.2.1, taking k = n. □

Theorem 2.2.3. *Under the assumptions of Theorem 2.2.2, for small ε the condition (2.12) ensures the existence and uniqueness of a T–periodic solution $x(t,\varepsilon)$ of system (2.7) such that $\mathbf{x}(0,\varepsilon) \to \alpha$ as $\varepsilon \to 0$, and if all eigenvalues of the matrix $(d\mathcal{F}/d\mathbf{z})\,(\mathbf{a})$ have negative real parts, then the periodic solution $(x(t,\varepsilon)$ is stable. If some of the eigenvalue have a positive real part, then the periodic solution $x(t,\varepsilon)$ is unstable.*

Example 2.2. *Consider the **Michelson** system of the form*

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = c^2 - y - \frac{x^2}{2}, \tag{2.13}$$

*with $(x,y,z) \in \mathbb{R}^3$ and the parameter $c \geqslant 0$. For any $\varepsilon \neq 0$, we take the change of variables $x = \varepsilon\bar{x}$, $y = \varepsilon\bar{y}$, $z = \varepsilon\bar{x}$ and $c = \varepsilon d$, then the **Michelson** system (2.13) becomes*

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -y + \varepsilon\left(d^2 - \frac{1}{2}x^2\right), \tag{2.14}$$

where we still use x, y, z instead of $\bar{x}$, $\bar{y}$, $\bar{z}$. Now doing the change of variables $x = x$, $y = r\sin\theta$ and $z = r\cos\theta$, system (2.14) goes over to

$$\dot{x} = r\sin\theta, \quad \dot{r} = \frac{\varepsilon}{2}\left(2d^2 - x^2\right)\cos\theta, \quad \dot{\theta} = 1 - \frac{\varepsilon}{2r}\left(2d^2 - x^2\right)\sin\theta. \tag{2.15}$$

This system can be written as

$$\begin{aligned} \frac{dx}{d\theta} &= r\sin\theta + \frac{\varepsilon}{2}(2d^2 - x^2)\sin^2\theta + \varepsilon^2 f_1(\theta, e, \varepsilon), \\ \frac{dr}{d\theta} &= \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta + \varepsilon^2 f_2(\theta, e, \varepsilon), \end{aligned} \tag{2.16}$$

where f_1 and f_2 are analytic functions in their variables.

For arbitrary $(x_0, r_0) \neq (0,0)$, system $(2.16)_{\varepsilon=0}$ has the 2π–periodic solution

$$x(\theta) = r_0 + x_0 - r_0\cos\theta, \quad r(\theta) = r_0, \tag{2.17}$$

such that $x(0) = x_0$ *and* $r(0) = r_0$. *It is easy to see that the first variational equation of* $(2.16)_{\varepsilon=0}$ *along the solution (2.17) is*

$$\begin{pmatrix} \dfrac{dy_1}{d\theta} \\ \dfrac{dy_2}{d\theta} \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

It has the fundamental solution matrix

$$M = \begin{pmatrix} 1 & 1-\cos\theta \\ 0 & 1 \end{pmatrix}, \tag{2.18}$$

which is independent of the initial condition (x_0, r_0). *Applying Theorem 2.2.2 to the differential system (2.16) we have that*

$$\mathcal{F}(x_0, r_0) = \frac{1}{2}\int_0^{2\pi} M^{-1} \left(\begin{array}{c} (2d^2 - x^2)\sin^2\theta \\ (2d^2 - x^2)\cos\theta \end{array} \right)\Bigg|_{(2.17)} d\theta.$$

Then $\mathcal{F}(x_0, r_0) = (g_1(x_0, r_0), g_2(x_0, r_0))$ *with*

$$g_1(x_0, r_0) = \frac{1}{4}\left(4d^2 - 5r_0^2 - 6r_0x_0 - 2x_0^2\right), \quad g_2(x_0, r_0) = \frac{1}{2}r_0\left(x_0 + r_0\right).$$

We can check that $\mathcal{F} = 0$ *has a unique solution* $x_0 = -2d$ *and* $r_0 = 2d$, *and that* $detD\mathcal{F}(x_0, r_0)|_{x_0=-2d, r_0=2d} = d^2$. *Hence by Theorem 2.2.2 it follows that for any given* $d > 0$ *and for* $|\,\varepsilon\,| > 0$ *sufficiently small system (2.16) has a periodic orbit* $(x(\theta, \varepsilon), r(\theta, \varepsilon))$ *of period* 2π, *such that* $(x(0, \varepsilon), r(0, \varepsilon)) \to (-2d, 2d)$ *as* $\varepsilon \to 0$. *We note that the eigenvalues of* $detD\mathcal{F}(x_0, r_0)|_{x_0=-2d, r_0=2d}$ *are* $\pm di$. *This shows that the periodic orbit is stable.*

Going back to system (2.13) we get that for $c > 0$ *sufficiently small the* ***Michelson*** *system has a periodic orbit of period close to* 2π *given by* $x(t) = -2c\cos t$, $y(t) = 2c\sin t$ *and* $z(t) = 2c\cos t$, *(see Fig. 2.2 bellow for* $\epsilon = 10^{-2}$*).*

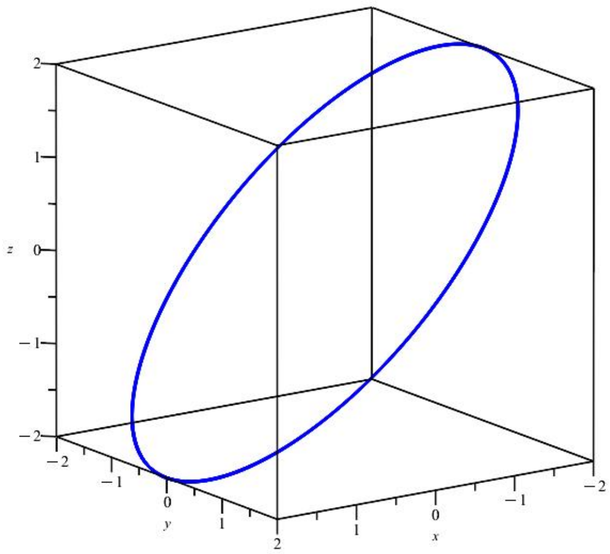

Figure 2.2 – The stable limit cycle of system (2.13).

2.3 A second order averaging theory

consider the differential system

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon) + \varepsilon^3 R(t,x,\varepsilon), \tag{2.19}$$

where the functions $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$ and $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t,.) \in \mathcal{C}^1(D)$ for all $t \in \mathbb{R}, F_1,\ \ F_2,\ \ R, D_x F_1$ are locally Lipshitz with respect to x and R is differential with respect to ε. We define

$F_{k0} : D \longrightarrow \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(z) &= \frac{1}{T} \int_0^T F_1(s,z)\, ds, \\ F_{20}(z) &= \frac{1}{T} \int_0^T [D_z F_1(s,z) \cdot y_1(s,z) + F_2(s,z)]\, ds, \end{aligned} \tag{2.20}$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\, dt.$$

(ii) for $V \in D$ an open bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$, there exists $a \in V$ such that $F_{10}(a)+\varepsilon F_{20}(a) = 0$ and $d_B(F_{10}+\varepsilon F_{20}, V, a) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(.,\varepsilon)$ of the system (2.20) such that $\varphi(.,\varepsilon) \mapsto a$ when $\varepsilon \mapsto 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a) \neq 0$ means that the Brower degree of the function $F_{10} + \varepsilon F_{20} : V \longrightarrow \mathbb{R}^n$ at the fixed point a is not zero. A sufficient condition for inequality to be true is that the jacobian of the function $F_{10}+\varepsilon F_{20}$ at a be non-zero.

If F_{10} is not identically zero, then the zero of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the averaging theory of this first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the averaging theory of second order.

Example 2.3. *Consider the following system*

$$\begin{cases} \dot{x} &= y + \varepsilon x^2 + \varepsilon^2 x^2, \\ \dot{y} &= -x + \varepsilon^2(x^4 + x^3 + y^3 - \frac{1}{3}y). \end{cases} \tag{2.21}$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with $r > 0$, system (2.21) becomes

$$
\begin{cases}
\dot{r} & = & r^2\cos(\theta)^3\,\epsilon + \Big(r^2\cos(\theta)^3 + \sin(\theta)\,r^4\cos(\theta)^4 + \sin(\theta)\,r^3\cos(\theta)^3 \\
 & & +r^3\sin(\theta)^4 - \dfrac{r\sin(\theta)^2}{3}\Big)\,\epsilon^2, \\
\dot{\theta} & = & -1 - r\sin(\theta)\cos(\theta)^2\,\epsilon + \Big(-r\sin(\theta)\cos(\theta)^2 + r^3\cos(\theta)^5 \\
 & & +r^2\cos(\theta)^4 + r^2\cos(\theta)\sin(\theta)^3 - \dfrac{\cos(\theta)\sin(\theta)}{3}\Big)\,\epsilon^2.
\end{cases}
\tag{2.22}
$$

Now consider θ as an independent variable, we get the following system

$$
\begin{aligned}
\frac{dr}{d\theta} = & -r^2\cos(\theta)^3\,\epsilon + \Big(-r^2\cos(\theta)^3 - \sin(\theta)\,r^4\cos(\theta)^4 - \sin(\theta)\,r^3\cos(\theta)^3 \\
& -r^3\sin(\theta)^4 + \frac{r\sin(\theta)^2}{3} + r^3\cos(\theta)^5\sin(\theta)\Big)\,\epsilon^2.
\end{aligned}
$$

It is equivalent to

$$
\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + O(\varepsilon^2),
$$

with $F_1(\theta, r) = -r^2\cos(\theta)^3$, *and*

$$
F_2(\theta, r) = -r^2\cos(\theta)^3 - \sin(\theta)\,r^4\cos(\theta)^4 - \sin(\theta)\,r^3\cos(\theta)^3 - r^3\sin(\theta)^4
$$

$$
+\frac{1}{3}r\sin(\theta)^2 + r^3\cos(\theta)^5\sin(\theta)\,.
$$

Now we calculate the first averaged function, we obtain

$$
F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, r)d\theta = \frac{1}{2\pi}\int_0^{2\pi} r^2\cos^3\theta d\theta = 0.
$$

Since $F_{10}(r) = 0$, we can move on to the second–order averaging theory. we get

$$
F_{20}(r) = \frac{-1}{24}r(9r^2 - 4).
$$

The equation $F_{20}(r) = 0$ has only one positive root $r = \frac{2}{3}$. Since $F_{20}'(\frac{2}{3}) = -\frac{1}{3}$, then system (2.21) has a single stable limit cycle of amplitude $\frac{2}{3}$ for $\mid \varepsilon \mid \neq 0$ sufficiently small, (see Fig. 2.3 bellow for $\epsilon = 10^{-2}$).

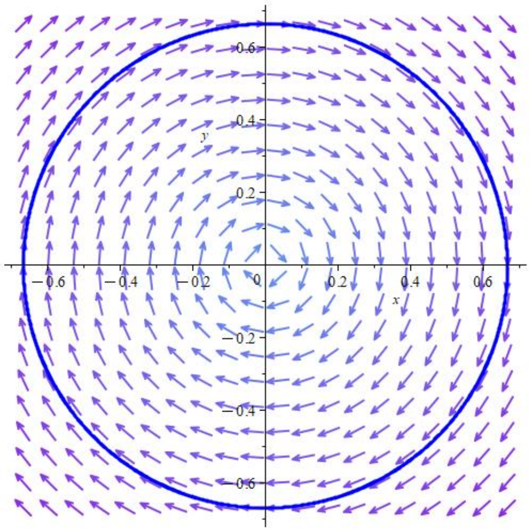

Figure 2.3 – The stable limit cycle of system (2.21).

2.4 A sixth order averaging theory

In this section we present the basic results from the averaging theory up to order 6 that we need for proving our results. It can summarized as follows.

We consider the differential systems given by

$$\dot{x} = \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{2.23}$$

where the functions $F_i : \mathbb{R}\times D \to \mathbb{R}$ for $i = 1, ..., k$, and $R : \mathbb{R}\times D\times(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are continuous, and $T-$periodic in the variable t, D is an open interval of $\mathbb{R}$, and ε a small parameter. We define the functions $y_j(t,z)$ for $j = 1, ..., 6$ associated to system (2.23) by sing the results of [40] as

$$\begin{aligned}
y_1(t,z) = &\int_0^t F_1(s,z)ds, \\
y_2(t,z) = &\int_0^t (2\,\partial F_1(s,z)y_1(s,z) + 2\,F_2(s,z))\,ds, \\
y_3(t,z) = &\int_0^t (6\,\partial F_2(s,z)y_1(s,z) + 6\,F_3(s,z)
\end{aligned}$$

$$
\begin{aligned}
&+3\,\partial F_1(s,z)y_2(s,z) + 3\,\partial^2 F_1(s,z)y_1(s,z)^2)\,ds,\\
y_4(t,z) = &\int_0^t (24\,\partial F_3(s,z)y_1(s,z) + 24\,F_4(s,z)\\
&+12\,\partial F_2(s,z)y_2(s,z) + +12\,\partial^2 F_2(s,z)y_1(s,z)^2\\
&+12\,\partial^2 F_1(s,z)y_1(s,z)y_2(s,z)\\
&+4\,\partial F_1(s,z)y_3(s,z) + 4\,\partial^3 F_1(s,z)y_1(s,z)^3)\,ds,\\
y_5(t,z) = &\int_0^t (120\,\partial F_4(s,z)y_1(s,z) + 120\,F_5(s,z)\\
&+60\,\partial F_3(s,z)y_2(s,z) + 60\,\partial^2 F_3(s,z)y_1(s,z)^2\\
&+20\,\partial^3 F_2(s,z)y_1(s,z)^3 + 60\,\partial^2 F_2(s,z)y_1(s,z)y_2(s,z)\\
&+20\,\partial^2 F_1(s,z)y_1(s,z)y_3(s,z) + 20\,\partial F_2(s,z)y_3(s,z)\\
&+30\,\partial^3 F_1(s,z)y_1(s,z)^2 y_2(s,z) + 15\,\partial^2 F_1(s,z)y_2(s,z)^2\\
&+5\,\partial F_1(s,z)y_4(s,z) + 5\,\partial^4 F_1(s,z)y_1(s,z)^4)\,ds,\\
y_6(t,z) = &\int_0^t (360\,\partial F_4(s,z)y_2(s,z) + 720\,\partial F_5(s,z)y_1(s,z) + 720\,F_6(s,z)\\
&+360\,\partial^2 F_3(s,z)y_1(s,z)y_2(s,z) + 120\,\partial F_3(s,z)y_3(s,z) + 360\,\partial^2 F_4(s,z)y_1(s,z)^2\\
&+120\,\partial^2 F_2(s,z)y_1(s,z)y_3(s,z) + 30\,\partial F_2(s,z)y_4(s,z) + 120\,\partial^3 F_3(s,z)y_1(s,z)^3\\
&+180\,\partial^3 F_2(s,z)y_1(s,z)^2 y_2(s,z) + 90\,\partial^2 F_2(s,z)y_2(s,z)^2 + 30\,\partial^4 F_2(s,z)y_1(s,z)^4\\
&+60\,\partial^2 F_1(s,z)y_2(s,z)y_3(s,z) + 30\,\partial^2 F_1(s,z)y_1(s,z)y_4(s,z) + 6\,\partial F_1(s,z)y_5(s,z)\\
&+60\,\partial^4 F_1(s,z)y_1(s,z)^3 y_2(s,z) + 60\,\partial^3 F_1(s,z)y_1(s,z)^2 y_3(s,z)\\
&+6\,\partial^5 F_1 s,z)y_1(s,z)^5 + 90\,\partial^3 F_1(s,z)y_1(s,z)y_2(s,z)^2)\,ds.
\end{aligned}
$$

Note that $\partial^k F_l(s,z)$ means the k–th partial derivative of the function $F_l(s,z)$ with respect to the variable z. From [40] the first six averaged functions are

$$
f_K(z) = \frac{1}{k!} y_k(T,z), \quad \text{for } k = 1,\ldots,6.
$$

The averaging theory for the differential system (2.23) works as follows. Assume that the averaged function $f_j(z) = 0$ for $j = 1, ..., k-1$ and $f_k(z) \neq 0$ for some $k \geqslant 1$. If $\bar{z}$ is a simple zero of $f_k(z)$, then there is a limit cycle $r(\theta, \varepsilon)$ of system (2.23) such that $r(0, \varepsilon) \longrightarrow \bar{z}$ when $\varepsilon \longrightarrow 0$. Moreover if the derivative $f_k'(z) > 0$ (respectively $f_k'(z) < 0$) the limit cycle $r(\theta, \varepsilon)$ is unstable (respectively stable). For more details on the stability of these limit cycles see Theorem 11.6 of [59].

Chapter

3 Periodic solutions for two classes of Duffing differential equations

Contents

These last years many results have been published on the periodic solutions of different classes of **Duffing** differential equations. These results are on the existence of periodic solutions, in their multiplicity, in their kind of stability, in their bifurcations,... see for instance [3, 4, 19, 21, 26, 30, 33, 55].

In this chapter, using averaging theory, we provide sufficient conditions for the existence of periodic solutions in two classes of **Duffing** differential equations.

3.1 Periodic solutions for a class of Duffing differential equations

In this part, we provide sufficient conditions for the existence of periodic solutions for the class of **Duffing** differential equations in $\mathbb{R}$ of the form

$$\ddot{x} + \varepsilon p(t)\dot{x} + (1 + \varepsilon q(t))x = \varepsilon f(t, x) + \varepsilon c(t), \tag{3.1}$$

where $p(t)$, $q(t)$, $f(t, x)$ and $c(t)$ are 2π–periodic functions in the variable t, ε is a small parameter, and $x \in \mathbb{R}$. Some extensions of these results can be found in [19, 21, 33, 55].

Our main result on the periodic solutions of the first class of **Duffing** differential equations (3.1) is the following.

Theorem 3.1.1. *We define the functions*

$$\mathcal{F}_1(x_0, y_0) = -\frac{1}{2\pi}\int_0^{2\pi} \Big(c(t) + f(t, x_0\cos t + y_0\sin t) - p(t)(y_0\cos t - x_0\sin t) - q(t)(x_0\cos t + y_0\sin t)\Big)\sin t \; dt,$$

$$\mathcal{F}_2(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} \Big(c(t) + f(t, x_0\cos t + y_0\sin t) - p(t)(y_0\cos t - x_0\sin t) - q(t)(x_0\cos t + y_0\sin t)\Big)\cos t \; dt.$$

Then for $\varepsilon \neq 0$ sufficiently small and for every (x_0^*, y_0^*) solution of the system

$$\mathcal{F}_1(x_0, y_0) = 0, \qquad \mathcal{F}_2(x_0, y_0) = 0, \tag{3.2}$$

satisfying

$$\det \left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)} \bigg|_{(x_0,y_0)=(x_0^*,y_0^*)} \right) \neq 0, \tag{3.3}$$

the Duffing differential equation (3.1) has a 2π–periodic solution $x(t, \varepsilon)$ which tends to the 2π–periodic solution $x(t) = x_0^* \cos t + y_0^* \sin t$ of the differential equation $\ddot{x} + x = 0$, when $\varepsilon \to 0$.

Proof of Theorem 3.1.1. If $\dot{x} = y$, then the first class of **Duffing** differential equation (3.1) can be written as the following first–order differential system in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(- p(t)y - q(t)x + f(t, x) + c(t)\big). \end{cases} \tag{3.4}$$

The solution $(x(t), y(t))$ of the unperturbed system (3.4) with $\varepsilon = 0$ such that $(x(0), y(0)) = (x_0, y_0)$ is

$$(x(t), y(t)) = (x_0 \cos t + y_0 \sin t, y_0 \cos t - x_0 \sin t). \tag{3.5}$$

Of course all these periodic orbits have period 2π.

Using the notation of Section 2.1, we have $\mathbf{x} = (x, y)$, $\mathbf{z} = (x_0, y_0)$, $F_0(\mathbf{x}, t) = (y, -x)$, $F_1(\mathbf{x}, t) = (0, -p(t)y - q(t)x + f(t, x) + c(t))$ and $F_2(\mathbf{x}, t, \varepsilon) = (0, 0)$. Since fundamental matrix $M_{\mathbf{z}}(t)$ is independent of $\mathbf{z}$, we denote it simply by $M(t)$. An easy computation provides

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{3.6}$$

From Theorem 2.2.2 we must study the zeros $\mathbf{z} = (x_0, y_0)$ of the function $\mathcal{F}(\mathbf{z})$ defined in (3.4), i.e. of the function $\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(\mathbf{z}), \mathcal{F}_2(\mathbf{z}))$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given in the statement of Theorem 3.1.1. The rest of the proof of Theorem 3.1.1 follows directly from Theorem 2.2.2. □

Two applications of Theorem 3.1.1 are given in the next examples.

Example 3.1. *The particular Duffing differential equation (3.1) with* $p(t) = -\sin^2 t$, $q(t) = 1$, $f(t,x) = (1+x)\sin t$ *and* $c(t) = \cos t$ *becomes*

$$\ddot{x} - \epsilon\,\text{sin}t^2\dot{x} + (1+\epsilon)\,x = \epsilon\,(\text{sin}t\,(x+1) + \text{cos}t)\,. \tag{3.7}$$

After some computations the functions $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *of Theorem 3.1.1 are*

$$\mathcal{F}_1(x_0, y_0) = \frac{1}{8}(3x_0 + 4y_0 - 4), \quad \textit{and} \quad \mathcal{F}_2(x_0, y_0) = \frac{1}{8}(-4x_0 + y_0 + 4).$$

The function $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ *has a unique zero* $(x_0^*, y_0^*) = \left(\frac{20}{19}, \frac{4}{19}\right)$. *Since the jacobian (3.3) is* $\frac{19}{64} \neq 0$, *then for* $\varepsilon \neq 0$ *sufficiently small the differential equation (3.7) has the periodic solution* $x(t, \varepsilon)$, *tending to the periodic solution* $\frac{20}{19}\cos t + \frac{4}{19}\sin t$ *of the differential equation* $\ddot{x} + x = 0$, *when* $\varepsilon \to 0$.

Example 3.2. *The Duffing differential equation (3.1) with* $p(t) = -\sin^2 t$, $q(t) = -\sin t$, $f(t,x) = (5 - x^2)\cos t$ *and* $c(t) = -\cos t$ *becomes*

$$\ddot{x} - \epsilon\,\text{sin}t^2\dot{x} + (1 - \epsilon\,\text{sin}t)\,x = \epsilon\,\text{cos}t\left(4 - x^2\right). \tag{3.8}$$

After some computations we get that

$$\mathcal{F}_1(x_0, y_0) = \frac{1}{8}(2x_0y_0 + 3x_0), \quad \textit{and} \quad \mathcal{F}_2(x_0, y_0) = \frac{1}{8}(-3x_0^2 - y_0^2 + y_0 + 16).$$

The function $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ *has the following four zeros* (x_0^*, y_0^*):

$$\left(0, \frac{1}{2} + \frac{\sqrt{65}}{2}\right), \quad \left(0, \frac{1}{2} - \frac{\sqrt{65}}{2}\right), \quad \left(\frac{7\sqrt{3}}{6}, -\frac{3}{2}\right), \quad \left(-\frac{7\sqrt{3}}{6}, -\frac{3}{2}\right).$$

Since the jacobians (3.3) at these four zeros are respectively $-\frac{\sqrt{65}}{16}-\frac{65}{64}$, $\frac{\sqrt{65}}{16}-\frac{65}{64}$, $\frac{49}{64}$ *and* $\frac{49}{64}$. *Then for* $\varepsilon \neq 0$ *sufficiently small, then the differential equation (3.8) has four periodic solutions which tend to the periodic solutions* $\frac{1}{2}-\frac{\sqrt{65}}{2}\sin t$, $\frac{1}{2}+\frac{\sqrt{65}}{2}\sin t$, $\frac{7\sqrt{3}}{6}\cos t-\frac{3}{2}\sin t$ *and* $-\frac{7\sqrt{3}}{6}\cos t-\frac{3}{2}\sin t$ *of the differential equation* $\ddot{x}+x=0$, *when* $\varepsilon \to 0$.

3.2 Periodic solutions for another class of Duffing differential equations

I. Khatami, E. Zahedi, and M. Zahedi in [30] studied the approximate solutions of the **Duffing** differential equations

$$\ddot{x}+\mu\dot{x}+\sum_{i=0}^{n}\rho_{2i+1}x^{2i+1}=f\cos(\Omega t), \tag{3.9}$$

where μ is the damping parameter, ρ_1 is the linear stiffness coefficient, $\rho_3, \rho_5, ..., \rho_{2n+1}$ are nonlinear arbitrary constants in the restoring force, f is the amplitude, and Ω is the angular frequency of the periodic driving force. The authors obtained numerically information about the solutions of the differential equations (3.9). The periodic solutions of particular differential equations of type (3.9) with $i=0,1$ have been studied in [3, 4, 26].

Here we shall study analytically the periodic solutions of the following class of Duffing differential equations

$$\ddot{x}+(1+\varepsilon\mu(t))\dot{x}+\varepsilon\sum_{i=0}^{n}\rho_{2i+1}(t)x^{2i+1}=\varepsilon f(t,x), \tag{3.10}$$

where the functions $\mu(t)$, $\rho_{2i+1}(t)$ with $i=0,...,n$ and $f(t,x)$ are C^2, $T-$periodic in the variable t, and ε is a small parameter.

Now our result on the periodic solutions of the second class of **Duffing** differential equation (3.10) is summarized in the next theorem.

Theorem 3.2.1. *We define the functions*

$$\mathcal{F}_1(x_0, y_0) = -\frac{1}{2\pi}\int_0^{2\pi} (\mu(t)(x_0 \sin t - y_0 \cos t) - \sum_{i=0}^{n} \rho_{2i+1}(t)$$
$$(y_0 \sin t + x_0 \cos t)^{2i+1} + f(t, y_0 \sin t + x_0 \cos t)) \sin t \; dt,$$
$$\mathcal{F}_2(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} (\mu(t)(x_0 \sin t - y_0 \cos t) - \sum_{i=0}^{n} \rho_{2i+1}(t)$$
$$(y_0 \sin t + x_0 \cos t)^{2i+1} + f(t, y_0 \sin t + x_0 \cos t)) \cos t \; dt.$$

Then for $\varepsilon \neq 0$ sufficiently small and for every (x_0^*, y_0^*) solution of the system

$$\mathcal{F}_1(x_0, y_0) = 0, \qquad \mathcal{F}_2(x_0, y_0) = 0, \tag{3.11}$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)} \right) \neq 0, \tag{3.12}$$

the Duffing differential equation (3.10) has a 2π–periodic solution $x(t, \varepsilon)$ which tends to the 2π–periodic solution $x(t) = x_0^* \cos t + y_0^* \sin t$ of the differential equation $\ddot{x} + x = 0$, when $\varepsilon \to 0$.

Proof of Theorem 3.2.1. If $\dot{x} = y$, then the second class of **Duffing** differential equation (3.10) can be written as the following first–order differential system in $\mathbb{R}^2$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon\Big(- \mu(t)y - \sum_{i=0}^{n} \rho_{2i+1}(t)x^{2i+1} + f(t, x)\Big). \end{aligned} \tag{3.13}$$

From the proof of Theorem 3.1.1 the solution $(x(t), y(t))$ of the unperturbed system (3.13) with $\varepsilon = 0$ such that $(x(0), y(0)) = (x_0, y_0)$ is given in (3.5).

Again using the notation of Section 2.1, we have $\mathbf{x} = (x, y)$, $\mathbf{z} = (x_0, y_0)$, $F_0(\mathbf{x}, t) = (y, -x)$, $F_1(\mathbf{x}, t) = (0, -\mu(t)y - \sum_{i=0}^{n} \rho_{2i+1}(t)x^{2i+1} + f(t, x))$ and

$F_2(\mathbf{x}, t, \varepsilon) = (0, 0)$. From the proof of Theorem 3.1.1 the fundamental matrix $M(t)$ is given in (3.6).

From Theorem 2.2.2 we must study the zeros $\mathbf{z} = (x_0, y_0)$ of the function $\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(\mathbf{z}), \mathcal{F}_2(\mathbf{z}))$ defined in (2.11). For system (3.13) a computation shows that the functions $\mathcal{F}_1(\mathbf{z})$ and $\mathcal{F}_2(\mathbf{z})$ are the ones given in the statement of Theorem 3.2.1. Again the rest of the proof of Theorem 3.2.1 follows directly from the statement of Theorem 2.2.2. □

Applications of Theorem 3.2.1 are the following.

Example 3.3. *The Duffing differential equation (3.10) with $n = 2$, $\rho_1(t) = 1 + \cos t$, $\rho_3(t) = -1 - \sin t$, $\rho_5(t) = \sin t$, $\mu(t) = \sin^2 t \cos t$ and $f(t, x) = -x \cos t \sin t$ becomes*

$$\ddot{x} + \epsilon\, \mathrm{sin} t^2\, \mathrm{cos} t \dot{x} + x + \epsilon\left((1 + \mathrm{cos} t)\, x + (-1 - \mathrm{sin} t)\, x^3 + \mathrm{sin} t x^5\right) = -\epsilon x \mathrm{cos} t\, \mathrm{sin} t. \tag{3.14}$$

After some computations we obtain that

$$\mathcal{F}_1(x_0, y_0) = -\frac{3}{8}x_0^2 y_0 + \frac{1}{8}x_0 + \frac{1}{2}y_0 - \frac{3}{8}y_0^3,$$
$$\mathcal{F}_2(x_0, y_0) = -\frac{1}{2}x_0 + \frac{3}{8}x_0^3 + \frac{3}{8}x_0 y_0^2 - \frac{1}{8}y_0.$$

The function $(\mathcal{F}_1, \mathcal{F}_2)$ has four zeros (x_0^*, y_0^*) given by

$$\left(\frac{\sqrt{30}}{6}, \frac{\sqrt{30}}{6}\right), \quad \left(-\frac{\sqrt{30}}{6}, -\frac{\sqrt{30}}{6}\right), \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

Since the jacobians (3.12) at these four zeros are respectively $\frac{\sqrt{5}}{16}$, $\frac{\sqrt{5}}{16}$, $-\frac{\sqrt{3}}{16}$ and $-\frac{\sqrt{3}}{16}$, then for $\varepsilon \neq 0$ sufficiently small the differential equation (3.14) has four periodic solutions which tend to the periodic solutions $\frac{\sqrt{30}}{6}\cos t + \frac{\sqrt{30}}{6}\sin t$, $-\frac{\sqrt{30}}{6}\cos t - \frac{\sqrt{30}}{6}\sin t$, $-\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{2}\sin t$ and $\frac{\sqrt{2}}{2}\cos t - \frac{\sqrt{2}}{2}\sin t$ of the differential equation $\ddot{x} + x = 0$, when $\varepsilon \to 0$.

Example 3.4. *The differential equation* (3.10) *with* $n=4$, $\rho_1(t) = -\frac{1}{2}+\frac{3}{2}\sin t$, $\rho_3(t) = \frac{3}{2}-\frac{1}{2}\cos t$, $\rho_5(t) = \frac{\cos^3 t}{2}$, $\rho_7(t) = -\frac{3}{2}\sin^3 t$, $\rho_9(t) = -\frac{\sin t}{2}+\cos t$, $\mu(t) = \cos^2 t\sin t$ *and* $f(t,x) = -x(\frac{1}{2}+\cos t)\sin t$ *becomes*

$$\begin{aligned}\ddot{x}+\epsilon\,\text{cos}t^2\,\text{sin}t\dot{x}+x+&\frac{\epsilon}{2}\left((-1+3\,\text{sin}t)\,x+(3-\text{cos}t)\,x^3\right.\\ &\left.+\text{cos}t^3x^5-3\,\text{sin}t^3x^7+(-\,\text{sin}t+2\,\text{cos}t)\,x^9\right)=-\epsilon x\left(\frac{1}{2}+\text{cos}t\right)\text{sin}t.\end{aligned} \tag{3.15}$$

Doing some computations from Theorem 3.2.1 *we obtain*

$$\begin{aligned}\mathcal{F}_1(x_0,y_0)&=\frac{9}{16}x_0^2y_0+\frac{1}{8}x_0-\frac{1}{4}y_0+\frac{9}{16}y_0^3,\\ \mathcal{F}_2(x_0,y_0)&=\frac{1}{4}x_0-\frac{9}{16}x_0^3-\frac{9}{16}x_0y_0^2-\frac{1}{8}y_0.\end{aligned}$$

The function $(\mathcal{F}_1,\mathcal{F}_2)$ *has four zeros* (x_0^*,y_0^*) *given by*

$$\left(\frac{1}{3},\frac{1}{3}\right),\quad \left(-\frac{1}{3},-\frac{1}{3}\right),\quad \left(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right),\quad \left(\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right).$$

Since the jacobians (3.12) *at these four zeros are respectively* $-\frac{1}{16}$, $-\frac{1}{16}$, $\frac{3}{16}$ *and* $\frac{3}{16}$, *then for* $\varepsilon\neq 0$ *sufficiently small the differential equation* (3.15) *has four periodic solutions which tend to the periodic solutions* $\frac{\cos t}{3}+\frac{\sin t}{3}$, $-\frac{\cos t}{3}+\frac{\sin t}{3}$, $-\frac{\sqrt{3}}{3}\cos t+\frac{\sqrt{3}}{3}\sin t$ *and* $\frac{\sqrt{3}}{3}\cos t-\frac{\sqrt{3}}{3}\sin t$ *of the differential equation* $\ddot{x}+x=0$, *when* $\varepsilon\to 0$.

Chapter 4

Periodic solutions for a generalized Duffing differential equations

Contents

Based on the averaging theory, we provide sufficient conditions for the existence of periodic solutions for a class of the well–known **Duffing** differential equations of the form

$$\ddot{x} + c(t)\dot{x} + g(t,x) = p(t), \tag{4.1}$$

where $c(t), g(t,x)$ and $p(t)$ are $\mathcal{C}^2$ and T–periodic in the variable t. This kind of equation have been studied by many authors under variant conditions, see for instance [4, 20, 26, 47, 57, 60].

If $\dot{x} = y$, then the T–periodic **Duffing** differential equation (4.1) can be written as the T–periodic differential system of the first order of the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -c(t)y - g(t,x) + p(t). \end{cases} \tag{4.2}$$

4.1 Statement of the main results

To state our main results, we need some preliminaries. We define the functions

$$\begin{gathered} r(t) = \textstyle\int_0^t c(s)\,ds, \qquad a(t,x) = \int_0^t g(s,x)e^{r(s)}\,ds, \qquad b(t) = \int_0^t p(s)e^{r(s)}\,ds, \\ m(t,x) = \textstyle\int_0^t g(s,x)\,ds, \qquad\qquad n(t) = \int_0^t p(s)\,ds. \end{gathered} \tag{4.3}$$

Our main results on the periodic solutions of the class of **Duffing** differential equation (4.3) are the following.

Theorem 4.1.1. *We consider the differential system (4.2) where the functions $c(t), g(t,x)$ and $p(t)$ are $\mathcal{C}^2$ and T–periodic in the variable t. Assume that the functions of (4.3) are T–periodic in the variable t. Then for every simple zero (x_0^*, y_0^*) of each one of the following 7 systems in the variables x_0 and y_0 of the form $f_1(x_0,y_0) = f_2(x_0,y_0) = 0$, given as follows*

$$\begin{cases} -\int_0^T a(t,x_0)e^{-r(t)}\,dt + y_0\int_0^T e^{-r(t)}\,dt + \int_0^T b(t)e^{-r(t)}\,dt &= 0, \\ -\int_0^T a_x(t,x_0)a(t,x_0)e^{-r(t)}\,dt + y_0\int_0^T a_x(t,x_0)e^{-r(t)}\,dt & \\ \qquad + \int_0^T a_x(t,x_0)b(t)e^{-r(t)}\,dt &= 0. \end{cases} \tag{4.4}$$

$$\begin{cases} y_0\int_0^T e^{-r(t)}\,dt + \int_0^T b(t)e^{-r(t)}\,dt &= 0, \\ -\int_0^T g(t,x_0)e^{r(t)}\,dt &= 0, \end{cases} \tag{4.5}$$

$$\begin{cases} -\int_0^T a(t,x_0)e^{-r(t)}\,dt + y_0\int_0^T e^{-r(t)}\,dt &= 0, \\ -\int_0^T a_x(t,x_0)a(t,x_0)e^{-r(t)}\,dt + y_0\int_0^T a_x(t,x_0)e^{-r(t)}\,dt & \\ \qquad + \int_0^T p(t)e^{r(t)}\,dt &= 0, \end{cases} \tag{4.6}$$

$$\begin{cases} y_0\int_0^T e^{-r(t)}\,dt &= 0, \\ -\int_0^T g(t,x_0)e^{r(t)}\,dt + \int_0^T p(t)e^{r(t)}\,dt &= 0, \end{cases} \tag{4.7}$$

$$\begin{cases} -\int_0^T m(t,x_0)\,dt + Ty_0 + \int_0^T n(t)\,dt = 0, \\ \int_0^T (m_x(t,x_0) - c(t))(n(t) - m(t,x_0) + y_0)\,dt = 0, \end{cases} \tag{4.8}$$

$$\begin{cases} Ty_0 + \int_0^T n(t)\,dt &= 0, \\ -\int_0^T g(t,x_0)\,dt - y_0\int_0^T c(t)\,dt - \int_0^T c(t)n(t)\,dt &= 0, \end{cases} \tag{4.9}$$

$$\begin{cases} -\int_0^T m(t,x_0)\,dt + Ty_0 &= 0, \\ \int_0^T (m_x(t,x_0) - c(t))(-m_x(t,x_0) + y_0)\,dt + \int_0^T p(t)\,dt &= 0, \end{cases} \tag{4.10}$$

the differential system (4.2) has a T–periodic solution $(x(t), y(t))$ such that $(x(0), y(0))$ is close to the (x_0^*, y_0^*).

4.2 Proof of the main results

Proof of Theorem 4.1.1. We do the following rescaling of the functions and the variables which appear in the differential system (4.2)

$$\begin{aligned} x &= X, \\ y &= \varepsilon^{m_1} Y, \\ c(t) &= \varepsilon^{n_1} C(t), \\ g(t,x) &= \varepsilon^{n_2} G(t,X), \\ p(t) &= \varepsilon^{n_3} P(t), \end{aligned} \tag{4.11}$$

where $\varepsilon > 0$ is a small parameter and m_1, n_1, n_2 and n_3 are non–negative integers, then the differential system (4.2) becomes

$$\begin{cases} \dot{X} = \varepsilon^{m_1} Y, \\ \dot{Y} = -\varepsilon^{n_1} C(t) Y - \varepsilon^{n_2 - m_1} G(t,X) + \varepsilon^{n_3 - m_1} P(t), \end{cases} \tag{4.12}$$

where the functions C, G and P are C^2 and T–periodic in the variable t. We distinguish the following two cases with their corresponding subcases.

$$\begin{aligned} &Case \quad I : m_1 = 1 \ and \ n_1 = 0. \\ &Case \ II : m_1 = 1 \ and \ n_1 = 1. \end{aligned}$$

Then we have the following subcases.

$$\begin{aligned} &(\alpha.1) \quad n_2 - m_1 = 0, \quad n_3 - m_1 = 0, \\ &(\alpha.2) \quad n_2 - m_1 = 1, \quad n_3 - m_1 = 0, \\ &(\alpha.3) \quad n_2 - m_1 = 0, \quad n_3 - m_1 = 1, \\ &(\alpha.4) \quad n_2 - m_1 = 1, \quad n_3 - m_1 = 1, \end{aligned}$$

where $\alpha \in \{I, II\}$.

The system (4.12) is in normal form (2.7) for applying the averaging theory. Mention that we do not consider the case $(II.4)$, because it has only equilibrium points instead of periodic orbits, and consequently the averaging theory described in Theorem 2.2.2 cannot be applied.

We shall prove Theorem 4.1.1 statement by statement.

We assume that the functions given by

$$R(t) = \int_0^t C(s)\,ds, \quad A(t,X_0) = \int_0^t G(s,X_0)e^{R(s)}\,ds, \quad B(t) = \int_0^t P(s)e^{R(s)}\,ds,$$
$$M(t,X_0) = \int_0^t G(s,X_0)\,ds, \qquad N(t) = \int_0^t P(s)\,ds,$$

are T–periodic in the variable t.

- *Case* $(I.1)$, i.e; for $n_1 = 0$ and $m_1 = n_2 = n_3 = 1$, the system (4.12) reads

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -C(t)Y - G(t,X) + P(t). \end{cases} \tag{4.13}$$

The system (4.13) for $\varepsilon = 0$, has the periodic solutions

$$(X(t),Y(t)) = \left(X_0, (Y_0 - A(t,X_0) + B(t))e^{-R(t)}\right),$$

for all $(X_0,Y_0) \in \mathbb{R}^2$. Now taking $z = (X_0,Y_0)$, and solving the variational differential equation (2.9), we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ -A_X(t,X_0)e^{-R(t)} & e^{-R(t)} \end{pmatrix},$$

where $A_X(t,X_0) = \partial A/\partial X(t,X_0)$. Now compute the averaged function $F(\mathbf{z}) = (\mathcal{F}_1(\mathbf{X_0},\mathbf{Y_0}), \mathcal{F}_2(\mathbf{X_0},\mathbf{Y_0}))$ given in (2.11), and we get

$$\begin{aligned} \mathcal{F}_1 &= -\int_0^T A(t,X_0)e^{-R(t)}\,dt + Y_0\int_0^T e^{-R(t)}\,dt + \int_0^T B(t)e^{-R(t)}\,dt, \\ \mathcal{F}_2 &= -\int_0^T A_X(t,X_0)A(t,X_0)e^{-R(t)}\,dt + Y_0\int_0^T A_X(t,X_0)e^{-R(t)}\,dt \\ &\quad + \int_0^T A_X(t,X_0)B(t)e^{-R(t)}\,dt = \int_0^T A_X(t,X_0)Y(t)\,dt. \end{aligned}$$

The zeros (X_0^*,Y_0^*) of the system $\mathcal{F}_1 = \mathcal{F}_2 = 0$, whose Jacobian is different from zero,provide periodic orbits of system (4.13) with $\varepsilon \neq 0$ sufficiently small.

Going back to the differential system (4.2) through the rescaling (4.11) the polynomial system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ in the variables X_0 and Y_0 becomes the system (4.4) in the variable x_0 and y_0. Consequently the theorem is proved for system (4.4).

- *Case* $(I.2)$, i.e. for $n_1 = 0,\ m_1 = n_3 = 1$ and $n_2 = 2$, the system (4.12) becomes

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -C(t)Y - \varepsilon G(t,X) + P(t). \end{cases} \tag{4.14}$$

Solving the differential system (4.14) for $\varepsilon = 0$, we obtain the T–periodic solutions

$$(X(t), Y(t)) = \left(X_0, (Y_0 + B(t))e^{-R(t)}\right),$$

for all $(X_0, Y_0) \in \mathbb{R}^2$. Solving the variational differential equation (2.9) we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-R(t)} \end{pmatrix}.$$

We compute the averaged function, we get

$$\mathcal{F}_1 = Y_0 \int_0^T e^{-R(t)}\,dt + \int_0^T B(t)e^{-R(t)}\,dt,$$
$$\mathcal{F}_2 = \int_0^T -G(t, X_0)e^{R(t)}\,dt.$$

By Theorem 2.2.2, the differential system (4.14) has a periodic solution $(X(t,\varepsilon), Y(t,\varepsilon))$ such that $(X(0,\varepsilon), Y(0,\varepsilon)) \to (X_0^*, Y_0^*)$ when $\varepsilon \to 0$, for each simple zero (X_0^*, Y_0^*) of the system $\mathcal{F}_1 = \mathcal{F}_2 = 0$, whose Jacobian is different from zero.

Going back to the differential system (4.2) through the rescaling (4.11) the theorem follows for system (4.4).

• *Case* $(I.3)$ i.e. for $n_1 = 0,\ m_1 = n_2 = 1$ and $n_3 = 2$, the system (4.12) becomes

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -C(t)Y - G(t,X) + \varepsilon P(t). \end{cases} \tag{4.15}$$

Solving the differential system (4.15) for $\varepsilon = 0$, we obtain the T–periodic solutions

$$(X(t), Y(t)) = \left(X_0, (Y_0 - A(t,X_0))e^{-R(t)}\right),$$

for all $(X_0, Y_0) \in \mathbb{R}^2$. Solving the variational differential equation (2.9) we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ -A_X(t,X_0)e^{-R(t)} & e^{-R(t)} \end{pmatrix}.$$

We compute the averaged function given in (2.11), and we get

$$\begin{aligned} \mathcal{F}_1 &= Y_0 \textstyle\int_0^T e^{-R(t)}\,dt - \int_0^T A(t,X_0)e^{-R(t)}\,dt, \\ \mathcal{F}_2 &= -\textstyle\int_0^T A_X(t,X_0)A(t,X_0)e^{-R(t)}\,dt + Y_0 \int_0^T A_X(t,X_0)e^{-R(t)}\,dt \\ &\quad + \textstyle\int_0^T P(t)e^{R(t)}\,dt. \end{aligned}$$

As in the proofs of the theorem for the previous systems it follows the proof for the system (4.6).

• *Case* $(I.4)$ i.e. for $n_1 = 0,\ m_1 = 1$ and $n_2 = n_3 = 2$, the system (4.12) becomes

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -C(t)Y - \varepsilon G(t,X) + \varepsilon P(t), \end{cases} \tag{4.16}$$

Solving the differential system (4.16) for $\varepsilon = 0$, we obtain the T–periodic solutions

$$(X(t), Y(t)) = \left(X_0, Y_0 e^{-R(t)}\right),$$

for all $(X_0, Y_0) \in \mathbb{R}^2 \backslash \{(0,0)\}$. Solving the variational differential equation (2.9) we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-R(t)} \end{pmatrix}.$$

We compute the averaged function given in (2.11), and we get

$$\begin{aligned} \mathcal{F}_1 &= \textstyle\int_0^T Y_0 e^{-R(t)}\, dt, \\ \mathcal{F}_2 &= -\textstyle\int_0^T G(t, X_0) e^{R(t)}\, dt + \int_0^T P(t) e^{R(t)}\, dt. \end{aligned}$$

As in the proofs of the theorem for the previous systems, it follows the proof for the system (4.7).

- *Case* $(II.1)$, i.e. for $m_1 = n_1 = n_2 = n_3 = 1$, the system (4.12) becomes

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -\varepsilon C(t) Y - G(t, X) + P(t). \end{cases} \tag{4.17}$$

Solving the differential system (4.17) for $\varepsilon = 0$, we obtain the T–periodic solution

$$(X(t), Y(t)) = (X_0, Y_0 - M(t, X_0) + N(t)),$$

for all $(X_0, Y_0) \in \mathbb{R}^2$. Solving the variational differential equation (2.9) we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ -M_X(t, X_0) & 1 \end{pmatrix}.$$

We compute the averaged function given in (2.11), and we get

$$\begin{aligned} \mathcal{F}_1 &= -\textstyle\int_0^T M(t, X_0)\, dt + T Y_0 + \int_0^T N(t)\, dt, \\ \mathcal{F}_2 &= \textstyle\int_0^T \left(M_X(t, X_0) - C(t)\right)\left(N(t) - M(t, X_0) + Y_0\right)\, dt. \end{aligned}$$

As in the proofs of the theorem for the previous systems, it follows the proof for the system (4.8).

- *Case* $(II.2)$ i.e. for $m_1 = n_1 = n_3 = 1$, and $n_2 = 2$, the system (4.12) becomes

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -\varepsilon C(t)Y - \varepsilon G(t,X) + P(t), \end{cases} \tag{4.18}$$

Solving the differential system (4.18) for $\varepsilon = 0$, we obtain the T–periodic solutions

$$(X(t), Y(t)) = (X_0, Y_0 + N(t)),$$

for all $(X_0, Y_0) \in \mathbb{R}^2$. Solving the variational differential equation (2.9) we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We compute the averaged function given in (2.11), and we get

$$\begin{aligned} \mathcal{F}_1 &= TY_0 + \textstyle\int_0^T N(t)\,dt, \\ \mathcal{F}_2 &= -\textstyle\int_0^T G(t,X_0)\,dt - Y_0 \int_0^T C(t)\,dt - \int_0^T C(t)N(t)\,dt. \end{aligned}$$

As in the proofs of the theorem for the previous systems, it follows the proof for the system (4.9).

- *Case* $(II.3)$ i.e. for $m_1 = n_1 = n_2 = 1$ and $n_3 = 2$, the system (4.12) becomes

$$\begin{cases} \dot{X} = \varepsilon Y, \\ \dot{Y} = -\varepsilon C(t)Y - G(t,X) + \varepsilon P(t), \end{cases} \tag{4.19}$$

Solving the differential system (4.19) for $\varepsilon = 0$, we obtain the T–periodic solutions

$$(X(t), Y(t)) = (X_0, Y_0 - M(t, X_0)),$$

for all $(X_0, Y_0) \in \mathbb{R}^2$. Solving the variational differential equation (2.9) we obtain the fundamental matrix

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ -M_X(t, X_0) & 1 \end{pmatrix}.$$

We compute the averaged function given in (2.11), and we get

$$\mathcal{F}_1 = -\int_0^T M(t, X_0)\, dt + TY_0,$$
$$\mathcal{F}_2 = \int_0^T (M_X(t, X_0) - C(t))(-M(t, X_0) + Y_0)\, dt + \int_0^T P(t)\, dt.$$

As in the proofs of the theorem for the previous systems, it follows the proof for the system (4.10).

This completes the proof of Theorem 4.1.1.

□

4.3 Examples

In this section we provide examples of each one of the statements of Theorem 4.1.1.

Example 4.1. *Consider the differential system (4.2) with*

$$\begin{aligned} c(t) &= \frac{-\cos t}{2 + \sin t}, \\ g(t, x) &= 3(x + 3)(1 + \frac{\sin t}{2}) \sin t, \\ p(t) &= 3\cos t + \frac{3}{2} \sin t \cos t. \end{aligned}$$

All these functions are 2π–periodic in the variable t. Then we get the functions

$$\begin{array}{rclcl} r(t) & = & \int_0^t c(s)\,ds & = & -\ln(\dfrac{2+\sin t}{2}),\\ a(t,x) & = & \int_0^t e^{r(s)}g(s,x)\,ds & = & 3(x+3)(1-\cos t),\\ b(t) & = & \int_0^t e^{r(s)}p(s)\,ds & = & 3\sin t, \end{array}$$

which are also 2π–periodic in the variable t. Then applying the system (4.4) *of Theorem 4.1.1, we have that the system*

$$\mathcal{F}_1(x_0,y_0) = -3x_0 - \frac{33}{4} + y_0 = 0,$$
$$\mathcal{F}_2(x_0,y_0) = -\frac{27}{2}x_0 - \frac{153}{4} + 3y_0 = 0,$$

has a unique solution $(x_0^*,y_0^*) = (-3,-\frac{3}{4})$. Since the Jacobian (2.12) *for this solution is $\frac{9}{2} > 0$, the differential system* (4.2) *for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(x(t,\varepsilon),y(t,\varepsilon))$ such that $(x(0,\varepsilon),y(0,\varepsilon))$ tends to (x_0^*,y_0^*) when $\varepsilon \to 0$.*

The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1,\mathcal{F}_2)$ at the zero (x_0^*,y_0^*) are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\sqrt{2}\,\mathrm{i} \\ -\frac{3}{2}\sqrt{2}\,\mathrm{i} \end{bmatrix}.$$

Since λ_1 and λ_2 are complex such that the real part of them are zero, then we can say nothing about the stability of the solution.

Example 4.2. *Consider the differential system* (4.2) *with*

$$\begin{array}{rcl} c(t) & = & -\dfrac{\cos t}{2+\sin t},\\ g(t,x) & = & \dfrac{2}{3}x(2+\cos t)\sin t,\\ p(t) & = & \cos t. \end{array}$$

These functions are 2π*-periodic in the variable* t*. We have that the functions*

$$\begin{aligned} r(t) &= \int_0^t c(s)\,ds &&= -\ln(\frac{2+\sin t}{2}), \\ b(t) &= \int_0^t e^{r(s)}p(s)\,ds &&= 2\ln(\frac{2+\sin t}{2}), \end{aligned}$$

are also 2π*–periodic in the variable* t*. Now applying the system* (4.5) *of Theorem* 4.1.1*, we have*

$$\mathcal{F}_1(x_0, y_0) = 0.812267 + 6.283185 y_0 = 0,$$

$$\mathcal{F}_2(x_0, y_0) = 2.592032 x_0 = 0.$$

This system has a unique solution $(x_0^*, y_0^*) = (0, -0.129276)$*, and the Jacobian* (2.12) *for this solution is* $-16.286220 < 0$*. The differential system* (4.2) *for* $\varepsilon \neq 0$ *sufficiently small has one periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *such that* $(x(0,\varepsilon), y(0,\varepsilon))$ *tends to* (x_0^*, y_0^*) *when* $\varepsilon \to 0$*.*
The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1, \mathcal{F}_2)$ *at the zero* (x_0^*, y_0^*) *are*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 4.035619 \\ -4.035619 \end{bmatrix}.$$

Since λ_1 *and* λ_2 *are real such that* λ_1 *is positive, by Theorem* 2.2.3 *it follows that the periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *is unstable.*

Example 4.3. *Consider the differential system* (4.2) *with*

$$\begin{aligned} c(t) &= -\frac{\sin t}{2+\cos t}, \\ g(t,x) &= \frac{x}{3}(2+\cos t)\sin t, \\ p(t) &= (1+\sin t)\sin t. \end{aligned}$$

All these functions are 2π*-periodic in the variable* t*. The functions*

$$\begin{array}{rclcl} r(t) &=& \int_0^t c(s)\,ds &=& \ln(\dfrac{2+\cos t}{3}), \\ a(t,x) &=& \int_0^t e^{r(s)} g(s,x)\,ds &=& \dfrac{x}{27}(19-\cos t(cos^2 t + 6\cos t + 12)), \end{array}$$

which are 2π*-periodic in the variable* t*. For this differential system, the system* (4.6) *of Theorem 4.1.1, becomes*

$$\mathcal{F}_1(x_0, y_0) = -7.741204x_0 + 10.882796y_0 = 0,$$

$$\mathcal{F}_2(x_0, y_0) = -6.558258x_0 + 7.741204y_0 + 2.094395 = 0.$$

It has a unique solution $(x_0^*, y_0^*) = (1.991348, 1.416495)$*, with the Jacobian* (2.12) *for this solution is* $11.445955 > 0$*. The differential system* (4.2) *for* $\varepsilon \neq 0$ *sufficiently small has one periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *such that* $(x(0,\varepsilon), y(0,\varepsilon))$ *tends to* (x_0^*, y_0^*) *when* $\varepsilon \to 0$*.*
The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1, \mathcal{F}_2)$ *at the zero* (x_0^*, y_0^*) *are*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 4.035619 \\ -4.035619 \end{bmatrix}.$$

Since λ_1 *and* λ_2 *are real such that* λ_1 *is positive, then by Theorem 2.2.3, the periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *is unstable.*

Example 4.4. *Consider the differential system* (4.2) *with*

$$\begin{array}{rcl} c(t) &=& -\dfrac{\cos t}{2+\sin t} \\ g(t,x) &=& x(1-\dfrac{\sin t}{2})\sin t, \\ p(t) &=& -(\dfrac{1}{2}+\cos t)\sin t. \end{array}$$

These functions are 2π*–periodic in the variable* t*. The functions*

$$r(t) = \int_0^t c(s)\,ds = -\ln(\frac{2+\sin t}{2}),$$

which are 2π*–periodic in the variable* t*. After computation the system* (4.7) *of Theorem 4.1.1, we obtain that the system*

$$\mathcal{F}_1(x_0, y_0) = 6.283185 y_0 = 0,$$
$$\mathcal{F}_2(x_0, y_0) = 3.888049 x_0 + 0.972012 = 0,$$

has a unique solution $(x_0^*, y_0^*) = (0.25, 0)$*, with the Jacobian* (2.12) *for this solution is* $-24.429330 < 0$*. The differential system* (4.2) *for* $\varepsilon \neq 0$ *sufficiently small has one periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *such that* $(x(0,\varepsilon), y(0,\varepsilon))$ *tends to* (x_0^*, y_0^*) *when* $\varepsilon \to 0$*.*

The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1, \mathcal{F}_2)$ *at the zero* (x_0^*, y_0^*) *are*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 4.942604 \\ -4.942604 \end{bmatrix}.$$

Since λ_1 *and* λ_2 *are real such that* λ_1 *is positive, by Theorem 2.2.3 it follows that the periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *is unstable.*

Example 4.5. *Consider the differential system* (4.2) *with*

$$\begin{aligned} c(t) &= 2\sin^2 t + 3\cos^4 t, \\ g(t,x) &= 2x(\cos t + \sin t), \\ p(t) &= \sin t, \end{aligned}$$

which are 2π*–periodic in the variable* t*. The functions*

$$\begin{aligned} m(t,x) &= \int_0^t g(s,x)\,ds = 2x(\sin t - \cos t + 1), \\ n(t) &= \int_0^t p(s)\,ds = 1 - \cos t, \end{aligned}$$

which are also 2π*-periodic in the variable* t*. Here, the system* (4.8) *of Theorem 4.1.1 becomes*

$$\mathcal{F}_1(x_0, y_0) = -2x_0 + y_0 + 1 = 0,$$
$$\mathcal{F}_2(x_0, y_0) = -\frac{15}{4}x_0 + \frac{7}{8} - \frac{1}{8}y_0 = 0.$$

This system has a unique solution $(x_0^*, y_0^*) = (\frac{1}{4}, -\frac{1}{2})$*, with the Jacobian* (2.12) *for this solution is* $4 > 0$*. The differential system* (4.2) *for* $\varepsilon \neq 0$ *sufficiently small has one periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *such that* $(x(0,\varepsilon), y(0,\varepsilon))$ *tends to* (x_0^*, y_0^*) *when* $\varepsilon \to 0$*.*

The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1, \mathcal{F}_2)$ *at the zero* (x_0^*, y_0^*) *are*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -\frac{17}{16} + \frac{7\sqrt{15}}{16}\mathrm{i} \\ -\frac{17}{16} - \frac{7\sqrt{15}}{16}\mathrm{i} \end{bmatrix}.$$

Since λ_1 *and* λ_2 *are complex such that the real part of them are negative, again by Theorem* 2.2.3 *it follows that the periodic solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *is stable.*

Example 4.6. *Consider the differential system* (4.2) *with*

$$\begin{aligned} c(t) &= (1+\sin t)\cos^2 t, \\ g(t,x) &= -x(x-1)\sin^2 t, \\ p(t) &= -\sin t \cos t, \end{aligned}$$

which are 2π*–periodic in the variable* t*. The function*

$$n(t) = \textstyle\int_0^t p(s)\,ds = -\frac{1}{2}\sin^2 t,$$

is also 2π*-periodic in the variable* t*. For this system, the system* (4.9) *of Theorem* 4.1.1*, becomes*

$$\begin{aligned} \mathcal{F}_1(x_0, y_0) &= -\frac{1}{4} + y_0 = 0, \\ \mathcal{F}_2(x_0, y_0) &= \frac{1}{16} - \frac{1}{2}y_0 + \frac{1}{2}x_0^2 - \frac{1}{2}x_0 = 0. \end{aligned}$$

It has two non-zero solutions (x_0^*, y_0^*) *given by* $(\frac{1}{2} - \frac{\sqrt{6}}{4}, \frac{1}{4})$ *and* $(\frac{1}{2} + \frac{\sqrt{6}}{4}, \frac{1}{4})$*, with the Jacobian* (2.12) *for these solutions being* $\frac{\sqrt{6}}{4} > 0$ *and* $\frac{-\sqrt{6}}{4} < 0$

respectively. The differential system (4.2) for $\varepsilon \neq 0$ *sufficiently small has two periodic solutions* $(x(t,\varepsilon), y(t,\varepsilon))$*, such that* $(x(0,\varepsilon), y(0,\varepsilon))$ *tends to* (x_0^*, y_0^*) *when* $\varepsilon \longrightarrow 0$.

The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1, \mathcal{F}_2)$ *at the zeros* (x_0^*, y_0^*) *are*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{-1+4\sqrt{6}}}{4}\mathrm{i} \\ -\frac{1}{4} - \frac{\sqrt{-1+4\sqrt{6}}}{4}\mathrm{i} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{1+4\sqrt{6}}}{4} \\ -\frac{1}{4} - \frac{\sqrt{1+4\sqrt{6}}}{4} \end{bmatrix}.$$

Since λ_1 *and* λ_2 *are complex such that the real part of them are negative, and* λ_3 *and* λ_4 *are real such that* λ_3 *is positive, again by Theorem 2.2.3 it follows that the first periodic solutions* $(x(t,\varepsilon), y(t,\varepsilon))$ *is stable and the other one is unstable.*

Example 4.7. *Consider the differential system (4.2) with*

$$\begin{aligned} c(t) &= \frac{4}{3}\cos^2 t + 3\sin t, \\ g(t,x) &= \frac{1}{2}x^2(x-3)\sin t, \\ p(t) &= \cos t, \end{aligned}$$

are 2π*-periodic in the variable* t*. The function*

$$m(t,x) = \textstyle\int_0^t g(s,x)\,ds = \frac{x^2}{2}(x-3)(1-\cos t),$$

is 2π*–periodic in the variable* t*. Here the system (4.10) of Theorem 4.1.1, becomes*

$$\begin{aligned} \mathcal{F}_1(x_0,y_0) &= -\frac{1}{2}x_0^3 + \frac{3}{2}x_0^2 + y_0 = 0, \\ \mathcal{F}_2(x_0,y_0) &= -\frac{9}{8}x_0^5 + \frac{45}{8}x_0^4 - \frac{77}{12}x_0^3 - x_0^2 + \frac{3}{2}x_0^2 y_0 - 3x_0 y_0 - \frac{2}{3}y_0 = 0. \end{aligned}$$

This system has two non-zero solutions (x_0^*, y_0^*) *given by* $(2,-2), (3,0)$*, with the Jacobian (2.12) for each solution is* $-3 < 0$ *and* $\frac{81}{8} > 0$ *respectively.*

The differential system (4.2) for $\varepsilon \neq 0$ sufficiently small has two periodic solutions $(x(t,\varepsilon), y(t,\varepsilon))$: the first is stable while the second is unstable, such that $(x(0,\varepsilon), y(0,\varepsilon))$ tends to (x_0^*, y_0^*) when $\varepsilon \to 0$.
The eigenvalues of the corresponding Jacobian matrix of the averaged functions $(\mathcal{F}_1, \mathcal{F}_2)$ at the zero (x_0^*, y_0^*) are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} + \frac{2\sqrt{7}}{3} \\ -\frac{1}{3} - \frac{2\sqrt{7}}{3} \end{bmatrix}.$$

Since λ_1 and λ_2 are real such that λ_1 is positive, then by Theorem 2.2.3, the periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ is unstable.

Chapter 5

Limit cycles of cubic polynomial differential systems in $\mathbb{R}^2$ via averaging theory of order 6

Contents

The following polynomial differential system of degree 3

$$\dot{x} = -y, \quad \dot{y} = x + x^3 - 3xy, \tag{5.1}$$

is a generalized Liénard system having an isochronous center at the origin coordinates, see Theorem 1 of [2] with $B(x) = 3x$.

In this chapter, we study the limit cycles which bifurcate from the center $(0, 0)$ of the nonlinear system (5.1) when we perturb it inside the class of all the planar cubic polynomial differential system of the form

$$\dot{x} = -y + \sum_{i=1}^{6} \varepsilon^i P_i(x, y), \quad \dot{y} = x + x^3 - 3xy + \sum_{i=1}^{6} \varepsilon^i Q_i(x, y), \tag{5.2}$$

where

$$P_j = a_{j1}x + a_{j2}y + a_{j3}x^2 + a_{j4}xy + a_{j5}y^2 + a_{j6}x^3 + a_{j7}x^2y + a_{j8}xy^2 + a_{j9}y^3,$$
$$Q_j = b_{j1}x + b_{j2}y + b_{j3}x^2 + b_{j4}xy + b_{j5}y^2 + b_{j6}x^3 + b_{j7}x^2y + b_{j8}xy^2 + b_{j9}y^3,$$

for $j = 1, \ldots, 6$.

We denote by f_k the k-th averaged function of the averaging theory of order k for $k = 1, \ldots, 6$, for a precise definition see Chp. 2, Sec. 2.4.

5.1 Statement of the main results

Our main result on the limit cycles of the differential system (5.2) is the following.

Theorem 5.1.1. *Using the averaging theory up to sixth order, the maximum number of small amplitude limit cycles for the cubic polynomial differential system (5.2) bifurcating from the origin of the center (5.1) for $\varepsilon > 0$ sufficiently small is detected by the averaging function as follows:*

(a) f_1 *is* 0*;*

(b) f_2 *is* 0 *when* $f_1 = 0$*;*

(c) f_3 *is* 1 *when* $f_1 = f_2 = 0$*, see Figure* 1*;*

(d) f_4 *is* 1 *when* $f_1 = f_2 = f_3 = 0$*, see Figure* 2*;*

(e) f_5 *is* 2 *when* $f_1 = f_2 = f_3 = f_4 = 0$*, see Figure* 3*;*

(f) f_6 *is* 4 *when* $f_1 = f_2 = f_3 = f_4 = f_5 = 0$*, see Figure* 4*.*

All the computations of this paper has been done with the help of the algebraic manipulators maple and mathematica.

5.2 Proof of the main results

Proof of Theorem 5.1.1. In what follows we shall study the limit cycles which bifurcate from the origin of the differential system (5.2) using the averaging theory up to order 6 described in the Section 2.4.

First, doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$, we obtain the differential system $(\dot{X}, \dot{Y})$. After that, we consider the change to $X = r\cos\theta$, $Y = r\sin\theta$ and we get the differential system $(\dot{X}, \dot{Y})$ in polar coordinates (r, θ). In order to study the limit cycles which can bifurcate from the origin using the averaging theory, we take θ as the new independent variable, then the last differential system becomes the differential equation $\dfrac{dr}{d\theta}$. Finally, we do a Taylor expansion in the variable ε truncating at 6–th order in ε and we get the differential equation

$$r' = \frac{dr}{d\theta} = \sum_{i=1}^{6} \varepsilon^i F_i(\theta, r) + O(\varepsilon^7). \tag{5.3}$$

The functions $F_i(\theta, r)$ for $i = 1, \ldots, 6$ of the differential system (5.3) are analytic, and since the independent variable θ appears through the sinus and cosinus of θ, they are $2\pi-$periodic in the variable θ. Hence the assumptions for applying the averaging theory described in the Chp. 2, Sec. 2.4 are satisfied.

We do not provide the functions $F_i(r, \theta)$ for $i = 3, \ldots, 6$ because their expressions are long, and they are easy to compute with the help of an algebraic manipulator such as Mathematica or Maple. Therefore we only give the expressions of the functions $F_1(r, \theta)$ and $F_2(r, \theta)$, i.e.

$$
\begin{aligned}
F_1(r,\theta) = \;& a_{11}\cos^2\theta + a_{12}\cos\theta\sin\theta + b_{11}\cos\theta\sin\theta + b_{12}\sin^2\theta + 3r\cos\theta\sin^2\theta, \\
F_2(r,\theta) = \;& -r\left(-b_{11}\cos^2\theta + (a_{11}-b_{12})cos\theta sin\theta + 3\cos^2\theta\sin\theta + a_{12}\sin^2\theta\right) \\
& \left(-a_{11}\cos^2\theta - (a_{12}+b_{11})\cos\theta\sin\theta - b_{12}\sin^2\theta + 3r\cos\theta\sin^2\theta\right) \\
& +r\left(a_{21}\cos^2\theta + (a_{22}+b_{21})\cos\theta\sin\theta + b_{22}\sin^2\theta + a_{13}r\cos^3\theta\right. \\
& +(a_{14}+b_{13})r\cos^2\theta\sin\theta + (a_{15}+b_{14})r\cos\theta\sin^2\theta + b_{15}r\sin^3\theta \\
& \left.+r^2\cos^3\theta\sin\theta\right).
\end{aligned}
$$

Computing the averaged function of first order $f_1(r)$ from Chp. 2, Sec. 2.4 we get

$$f_1(r) = (a_{11} + b_{12})\pi r.$$

Since the unique zero of $f_1(r) = 0$ is $r = 0$, the averaging theory of first order does not provide any information about the limit cycles which bifurcate from the origin of the differential equation (5.3). So statement (a) of Theorem 5.1.1 is proved.

Now we force that the averaged function of first order be identically zero taking $b_{12} = -a_{11}$, and we compute the averaged function of second order and we obtain

$$f_2(r) = (a_{21} + b_{22})\pi r.$$

Again the unique zero of $f_2(r) = 0$ is $r = 0$, and no information about the limit cycles of the differential equation (5.3). Hence statement (b) of Theorem 5.1.1 is proved. Consequently we take $b_{22} = -a_{21}$, and $f_2(r) \equiv 0$.

Computing the averaged function of third order we get

$$f_3(r) = (a_{31} + b_{32})\pi r - \frac{1}{4}(9a_{11} - 3a_{16} - a_{18} - 3b_{13} - 3b_{15} - b_{17} - 3b_{19})\pi r^3.$$

The unique positive real zero of $f_3(r) = 0$ is

$$r_1 = \frac{2\sqrt{a_{31} + b_{32}}}{\sqrt{9a_{11} - 3a_{16} - a_{18} - 3b_{13} - 3b_{15} - b_{17} - 3b_{19}}}.$$

So using the averaging theory of third order described in Chp. 2, Sec. 2.4, we obtain for $\varepsilon > 0$ sufficiently small at most one limit cycle $r_1(\theta, \varepsilon)$ if $(a_{31} + b_{32})(9_{11} - 316 - a_{18} - 3b_{13} - 3b_{15} - b_{17} - 3b_{19}) > 0$ of the differential equation (5.3) such that $r_1(0, \varepsilon) \to r_1$ when $\varepsilon \to 0$.

Going back to the differential system $(\dot{r}, \dot{\theta})$ the limit cycle $r_1(\theta, \varepsilon)$ of the differential equation (5.3) becomes the limit cycle

$$(r(t, \varepsilon), \theta(t, \varepsilon)) = (r_1 + O(\varepsilon), t + O(\varepsilon)), \tag{5.4}$$

of the differential system $(\dot{r}, \dot{\theta})$, because $\dot{\theta} = 1 + O(\varepsilon)$. Now going back to the differential system $(\dot{X}, \dot{Y})$ the limit cycle (5.4) becomes the limit cycle

$$(X(t, \varepsilon), Y(t, \varepsilon)) = (r_1 \cos t + O(\varepsilon), r_1 \sin t + O(\varepsilon)), \tag{5.5}$$

of the differential system $(\dot{X}, \dot{Y})$. Finally going back to the differential system $(\dot{x}, \dot{y})$ the limit cycle (5.5) becomes the limit cycle

$$(x(t, \varepsilon), y(t, \varepsilon)) = (\varepsilon r_1 \cos t + O(\varepsilon^2), \varepsilon r_1 \sin t + O(\varepsilon^2)), \tag{5.6}$$

of the differential system (5.2). So the limit cycle (5.6) tends to the origin of coordinates when $\varepsilon \to 0$. In summary, for the differential system (5.2) we have proved that when $\varepsilon = 0$ at most one small limit cycle bifurcates from the origin of coordinates using the averaging theory of order three. Hence statement (c) of Theorem 5.1.1 is proved.

Now taking $b_{32} = -a_{31}$ and $b_{17} = 9a_{11} - 3a_{16} - a_{18} - 3b_{13} - 3b_{15} - 3b_{19}$, we obtain $f_3(r) \equiv 0$. So we can apply the averaging theory of order four and we compute the averaged function

$$f_4(r) = C_1\pi r - C_3\pi r^3/4,$$

where

$$\begin{aligned} C_1 = \;& a_{41} + b_{42}, \\ C_3 = \;& 9a_{11}a_{12} - 3a_{11}a_{13} - a_{13}a_{14} + 3a_{11}a_{15} - a_{14}a_{15} - 2a_{11}a_{17} - a_{12}a_{18} \\ & +9a_{21} - 3a_{26} - a_{28} - 9a_{11}b_{11} - a_{18}b_{11} + 2a_{13}b_{13} + 3b_{11}b_{13} - 6a_{11}b_{14} \\ & +b_{13}b_{14} - 3a_{12}b_{15} - 2a_{15}b_{15} + b_{14}b_{15} - 2a_{11}b_{18} - 3a_{12}b_{19} - 3b_{11}b_{19} \\ & -3b_{23} - 3b_{25} - b_{27} - 3b_{29}. \end{aligned}$$

Therefore $f_4(r) = 0$ has at most one positive real root, which is $r_1 = 2\sqrt{C_1/C_3}$ if $C_1C_3 > 0$. So using the averaging theory of order four we get at most one limit cycle $r_1(\theta, \varepsilon)$ of the differential equation (5.3) such that $r_1(0, \varepsilon) \to r_1$ when $\varepsilon \to 0$. Using the previous arguments of the limit cycle found from the averaged function of order three, it follows that for the differential system (5.2) we have proved that when $\varepsilon = 0$ at most one small limit cycle bifurcates from the origin of coordinates using the averaging theory of order four. So statement (d) of Theorem 5.1.1 is proved.

For applying the averaging theory of order five we must have $f_4(r) \equiv 0$, so we isolate from $C_1 = 0$ and $C_3 = 0$ the coefficients b_{42} and b_{27} respectively, and we substitute them in the rest of the computations. Then the averaged function of order five is

$$f_5(r) = \left(D_1 r - \frac{D_3}{4}r^3 + \frac{D_5}{8}r^5\right)\pi,$$

where the big expressions of D_i for $i = 1, 3, 5$ are given in Appendix. Therefore the polynomial $f_5(r)$ can have at most the two positive real roots

$$r_1 = \frac{1}{\sqrt{2}}\sqrt{-\frac{D_3 + \sqrt{D_3^2 - 4D_1D_5}}{D_5}} \quad \text{and} \quad r_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{-D_3 + \sqrt{D_3^2 - 4D_1D_5}}{D_5}}.$$

So using the averaging theory of order five we get at most two limit cycles $r_k(\theta, \varepsilon)$ for $k = 1, 2$ of the differential equation (5.3) such that $r_k(0, \varepsilon) \to r_k$ when $\varepsilon \to 0$. Going back through the changes of variables until to reach the differential system (5.2), the limit cycles $r_k(\theta, \varepsilon)$ provides when $\varepsilon = 0$ at most two small limit cycles bifurcating from the origin of coordinates for the differential system (5.2) using the averaging theory of order five. Hence statement (e) of Theorem 5.1.1 is proved.

In order to apply the averaging theory of order six we need to have $f_5(r) \equiv 0$, so we isolated b_{52} from $D_1 = 0$, b_{37} from $D_3 = 0$, and b_{19} from $D_5 = 0$, and we substitute them in the rest of the computations. Therefore computing the sixth averaged function we obtain

$$f_6(r) = \left(S_1 r - \frac{S_2}{8192} r^2 - \frac{S_3}{12288} r^3 - \frac{S_4}{98304} r^4 - \frac{S_5}{61440} r^5 \right) \pi,$$

where S_i for $i = 1, 2, 3, 4, 5$ are given in Appendix. Since the rank of the Jacobian matrix of the function $S = (S_1, S_2, S_3, S_4, S_5)$ with respect to the coefficients a_{ij} and b_{ij} which appear in their expressions is 5, then the coefficients S_i for $i = 1, 2, 3, 4, 5$ which appear in the polynomial $f_6(r)$ are linearly independent. Therefore, by Descartes Theorem described in Chp. 1, Sec. 1.11, and using the averaging theory of order six, the polynomial $f_6(r)$ can have at most four positive real roots, and consequently the differential equation (5.3) can have at most four limit cycles using the averaging theory of order six. Using the previous arguments these three limit cycles of the differential equation (5.3) provide at most four small limit cycles bifurcating from the origin of coordinates for the differential system (5.2) when $\varepsilon = 0$ using the averaging theory of order six. So statement (f) of Theorem 5.1.1 is proved.

This completes the proof of Theorem 5.1.1. □

5.3 Examples

In this section we provide examples for illustrating our results.

Example 5.1. *We use the averaging theory of the third order. Consider the cubic polynomial differential system*

$$\begin{aligned} \dot{x} &= -y + \varepsilon(2\,x + \frac{1}{2}x^3 + \frac{1}{3}x\,y^2) - 2\varepsilon^3 x, \\ \dot{y} &= x + x^3 - 3xy + \varepsilon\left(-2y + 3x^2 - \frac{3}{2}y^2 - \frac{7}{3}x^2 y - 2y^3\right) + 6\varepsilon^3 y. \end{aligned} \tag{5.7}$$

After some computations we obtain $f_1(r) = f_2(r) \equiv 0$, and $f_3(r) = (4r - 5r^3)\pi$. The polynomial $f_3(r)$ has the unique simple positive root $r_1 = 2\sqrt{5}/5$ because $f_3'(r_1) = -8\pi$. Therefore, by the averaging theory described in Chp. 2, Sec. 2.4 the differential system (5.7) has one stable limit cycle bifurcating from the origin of coordinates when $\varepsilon = 0$ using the averaging theory of order three. This confirm the statement (c) of Theorem 5.1.1. See Figure 5.1 for $\epsilon = 10^{-5}$.

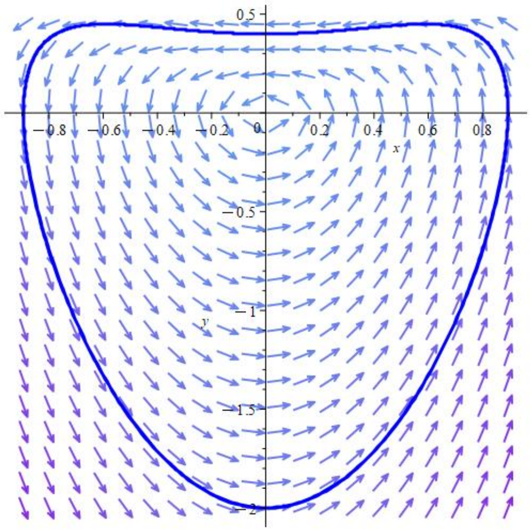

Figure 5.1 – The limit cycle of system (5.7).

Example 5.2. *We use the averaging theory of the fourth order. Consider the cubic polynomial differential system*

$$\begin{aligned}\dot{x} = \;& -y+\varepsilon\left(x-y+2x^2+\frac{1}{2}xy-3y^2+7x^3+2xy^2\right)\\ & +\varepsilon^2\left(4x-y-4x^2+2y^2-x^3+\frac{5}{2}x^2y-\frac{1}{3}xy^2+y^3\right)\\ & +\varepsilon^3\left(2x+12y-x^2+4xy+2x^2y-2xy^2+12y^3\right)\\ & +\varepsilon^4\left(-7x+22y+16xy-10x^3+2xy^2\right),\\ \dot{y} = \;& x+x^3-3xy+\varepsilon(-2x-y-\frac{1}{2}x^2+\frac{4}{3}xy-3y^2-\frac{1}{2}x^2y-y^3)\\ & +\varepsilon^2(-4y+\frac{1}{2}x^2+xy+9y^2+\frac{3}{2}x^3-\frac{3}{2}x^2y)+\\ & \varepsilon^3(-2y+2x^2-xy+\frac{3}{2}x^3+5x^2y+4xy^2-2y^3)\\ & +\varepsilon^4(-2y+x^2+\frac{1}{2}x^3-5x^2y-xy^2+2y^3).\end{aligned} \tag{5.8}$$

After some computations we obtain $f_1(r) = f_2(r) = f_3(r) \equiv 0$, and $f_4(r) = (-9r + 209r^3/24)\pi$. The polynomial $f_4(r)$ has a unique simple positive root $r_1 = 6\sqrt{6/209}$ because $f_4'(r_1) = 18\pi$. Hence, by the averaging theory described in Chp. 2, Sec. 2.4 the differential system (5.8) has one unstable *limit cycle bifurcating from the origin of coordinates when $\varepsilon = 0$ using the averaging theory of order four. This confirm the statement (d) of Theorem 5.1.1. See Figure 5.2 for $\epsilon = 10^{-5}$.*

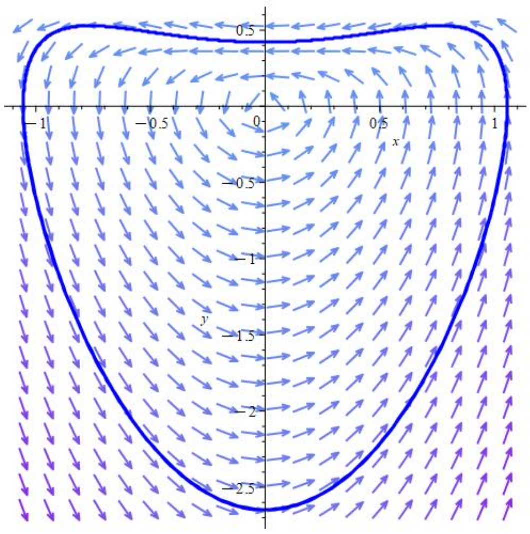

Figure 5.2 – The limit cycle of system (5.8).

Example 5.3. *We use the averaging theory of the fifth order. Consider the cubic differential system*

$$\begin{aligned}\dot{x} &= -y + \varepsilon(2x + 2y - 6xy + y^2 - 3x^3 + 4x^2y + 3xy^2 + y^3)\\ &\quad +\varepsilon^2(4x + 7x^2 + xy - 5x^2y + 12y^3) + \varepsilon^3(3x + y^2 + 5xy^2 + y^3)\\ &\quad +\varepsilon^4(-y + 2x^3 - x^2y + 3y^3) + \varepsilon^5(7x + xy + 7y^2 + 3x^2y + y^3),\\ \dot{y} &= x + x^3 - 3xy + \varepsilon(x - 2y - 3xy + 3y^2 + 2x^3 + 15x^2y + 3xy^2)\\ &\quad +\varepsilon^2(-4y + 2x^2 + 26x^2y - xy^2) + \varepsilon^3(2x - 3y + 2x^2 + 4xy - 2y^2 + x^2y + xy^2)\\ &\quad +\varepsilon^4(-4x + 5x^2 + 2y^2 - 4x^3 + y^3) + \varepsilon^5(2x - 2y - x^2 + 3y^2 + 3x^3 - 2x^2y).\end{aligned} \tag{5.9}$$

Using averaging theory of Chp. 2, Sec. 2.4 we obtain $f_1(r) = f_2(r) = f_3(r) = f_4(r) \equiv 0$, and $f_5(r) = (5r - 41r^3/2 + 51r^5/8)\pi$. The polynomial $f_5(r)$

has the two simple positive roots

$$r_1 = \sqrt{\frac{2}{51}\left(41 - \sqrt{1171}\right)}, \qquad r_2 = \sqrt{\frac{2}{51}\left(41 + \sqrt{1171}\right)}.$$

Since $f_5'(r_1) = -28.58417722$ *and* $f_5'(r_2) = 317.1179024$*, then, the differential system (5.9) has one stable limit cycles for* r_1 *and one unstable limit cycle for* r_2 *bifurcating from the origin.statement (e) of Theorem 5.1.1. See Figure 5.3 for* $\epsilon = 10^{-5}$.

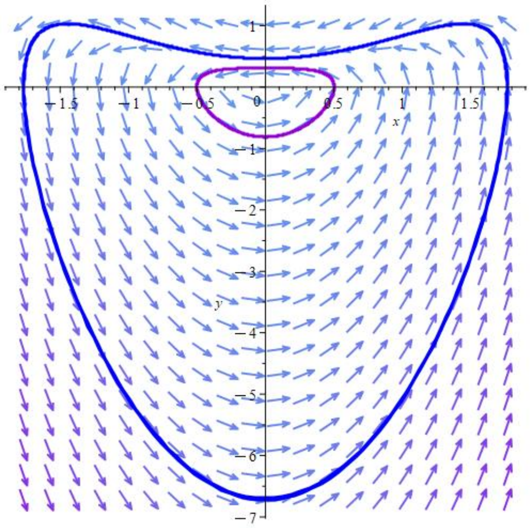

Figure 5.3 – The two limit cycles of system (5.9).

Example 5.4. *We use averaging theory of the sixth order. Consider the differential system*

$$\begin{aligned}
\dot{x} = & \; -y + \varepsilon x - \frac{50013}{1085}\varepsilon^2 xy + \frac{992512}{9765}\varepsilon^3 y + 12\varepsilon^6 x, \\
\dot{y} = & \; x + x^3 - 3xy + \varepsilon\left(-y - 6x^2 y + 5y^3\right) \\
& +\varepsilon^2\left(x^2 + \frac{501407}{930}xy - \frac{401}{155}x^2 y - \frac{64}{465}y^3\right) - \frac{500012}{155}\varepsilon^3 x^2 y + 12\varepsilon^6 y.
\end{aligned} \tag{5.10}$$

Using averaging theory defined in the Section 2.4, we get $f_1(r) = f_2(r) = f_3(r) = f_4(r) = f_5(r) \equiv 0$, *and*

$$f_6(r) = \left(24r - 50r^2 + 35r^3 - 10r^4 + r^5\right)\pi.$$

The equation $f_6(r) = 0$ *has four solutions* $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, *and* $r_4 = 4$. *Since,* $f_6'(r_1) = -6\pi$, $f_6'(r_2) = 4\pi$, $f_6'(r_3) = -6\pi$ *and* $f_6'(r_4) = 24\pi$, *then, by Theorem 5.1.1, system (5.10) has four stable limit cycles for* r_1, r_3 *and the other are unstable for* r_2 *and* r_4. *This confirm the statement (f) of Theorem 5.1.1. See Figure 5.4 for* $\epsilon = 10^{-5}$.

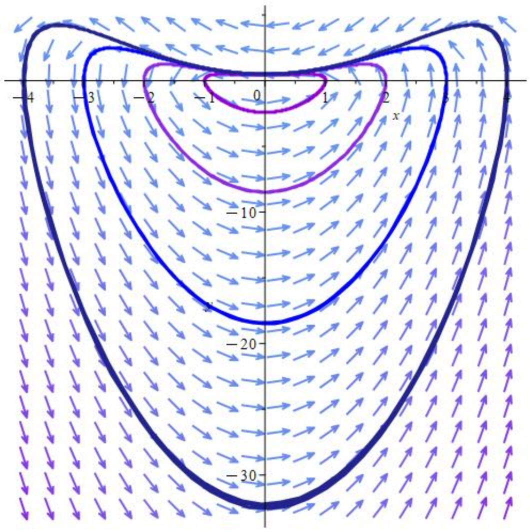

Figure 5.4 – The four limit cycles of system (5.10).

Chapter

6 Periodic solutions for differential systems in $\mathbb{R}^5$ and $\mathbb{R}^6$

Contents

In this chapter, using two different results of the averaging theory of the first order, we study the periodic orbits of two kind of differential systems in $\mathbb{R}^5$ and $\mathbb{R}^6$ that appear frequently in many problems coming from physics, chemistry, economics, engineering, etc.

This chapter was published in the international journal "**Journal of Dynamical and Control Systems**" titled "**Periodic solutions for differential systems in $\mathbb{R}^5$ and $\mathbb{R}^6$**", for more details see [51].

6.1 Periodic solutions for differential systems in $\mathbb{R}^5$

In this section, we shall provide sufficient conditions for the existence of periodic orbits in the differential systems in $\mathbb{R}^5$ of the form

$$\begin{cases} \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u,, \quad \dot{u} = v, \\ \dot{v} = -\alpha\beta\mu x - \beta\mu y - \alpha(\beta+\mu)z - (\beta+\mu)u - \alpha v + \varepsilon f(t,x,y,z,u,v), \end{cases} \tag{6.1}$$

where α, β and μ are rational numbers different from 0 such that $\alpha \neq \pm\beta$, $\alpha \neq \pm\mu$, and $\beta \neq \pm\mu$ with $|\varepsilon|$ sufficiently small, and f is non-autonomous periodic function. These differential systems usually come when we write as a first-order differential system in $\mathbb{R}^5$, the fifth-order differential equation

$$x^{(5)} + \alpha\ddddot{x} + (\beta+\mu)\dddot{x} + \alpha(\beta+\mu)\ddot{x} + \beta\mu\dot{x} + \alpha\beta\mu x = \varepsilon f(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}), \tag{6.2}$$

obtained from (6.1) on setting $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$, $v = \ddddot{x}$.

Fifth-order differential systems do arise in a number of applications, for example, in some three loop electric circuit problems and in control theory (see **Rosenvasser** [53]). Furthermore, there are various papers of such systems and equations, see for instance [22, 62, 64].

Our main result on the periodic solutions of differential system (6.1) or equivalent differential equation (6.2) is the following.

Theorem 6.1.1. *Assume that α, β and μ are rational numbers different from zero such that $\alpha \neq \pm\beta$, $\alpha \neq \pm\mu$, and $\beta \neq \pm\mu$ in differential system (6.1). We define*

$$\begin{aligned}
\mathcal{F}_1(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi k} \int_0^{2\pi k} \cos(\frac{m}{n}t) F_0(t, A(t), B(t), C(t), D(t), E(t))\, dt,\\
\mathcal{F}_2(X_0, Y_0, Z_0, U_0) &= -\frac{1}{2\pi k} \int_0^{2\pi k} \sin(\frac{m}{n}t) F_0(t, A(t), B(t), C(t), D(t), E(t))\, dt,\\
\mathcal{F}_3(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi k} \int_0^{2\pi k} \cos(\frac{p}{q}t) F_0(t, A(t), B(t), C(t), D(t), E(t))\, dt,\\
\mathcal{F}_4(X_0, Y_0, Z_0, U_0) &= -\frac{1}{2\pi k} \int_0^{2\pi k} \sin(\frac{p}{q}t) F_0(t, A(t), B(t), C(t), D(t), E(t))\, dt,
\end{aligned} \tag{6.3}$$

with $\beta = (\frac{m}{n})^2$, $\mu = (\frac{p}{q})^2$, where m, n, p and q are integers, $\beta \neq \mu$,

$(m,n)=(p,q)=1$, let k be the least common multiple of n and q, and

$$
\begin{aligned}
A(t) &= \frac{(-\sqrt{\beta}X_0+\alpha Y_0)\cos(\sqrt{\beta}t)+(\alpha X_0+\beta Y_0)\sin(\sqrt{\beta}t)}{\sqrt{\beta}(\alpha^2+\beta)(\mu-\beta)}\\
&\quad+\frac{(\sqrt{\mu}Z_0-\alpha U_0)\cos(\sqrt{\mu}t)-(\alpha Z_0+\sqrt{\mu}U_0)\sin(\sqrt{\mu}t)}{\sqrt{\mu}(\alpha^2+\mu)(\mu-\beta)},\\
B(t) &= \frac{(\alpha X_0+\sqrt{\beta}Y_0)\cos(\sqrt{\beta}t)+(\sqrt{\beta}X_0-\alpha Y_0)\sin(\sqrt{\beta}t)}{(\alpha^2+\beta)(\mu-\beta)}\\
&\quad-\frac{(\alpha Z_0+\sqrt{\mu}U_0)\cos(\sqrt{\mu}t)+(\sqrt{\mu}Z_0-\alpha U_0)\sin(\sqrt{\mu}t)}{(\alpha^2+\mu)(\mu-\beta)},\\
C(t) &= \frac{(-\beta X_0+\alpha\sqrt{\beta}Y_0)\cos(\sqrt{\beta}t)+(\alpha\sqrt{\beta}X_0+\beta Y_0)\sin(\sqrt{\beta}t)}{(\alpha^2+\beta)(\mu-\beta)}\\
&\quad+\frac{(-\mu Z_0+\alpha\sqrt{\mu}U_0)\cos(\sqrt{\mu}t)+(\alpha\sqrt{\mu}Z_0+\mu U_0)\sin(\sqrt{\mu}t)}{(\alpha^2+\mu)(\mu-\beta)},\\
D(t) &= -\frac{(\alpha\beta X_0+\beta\sqrt{\beta}Y_0)\cos(\sqrt{\beta}t)+(\beta\sqrt{\beta}X_0-\alpha\beta Y_0)\sin(\sqrt{\beta}t)}{(\alpha^2+\beta)(\mu-\beta)}\\
&\quad+\frac{(\alpha\mu Z_0+\mu\sqrt{\mu}U_0)\cos(\sqrt{\mu}t)+(\mu\sqrt{\mu}Z_0-\alpha\mu U_0)\sin(\sqrt{\mu}t)}{(\alpha^2+\mu)(\mu-\beta)},\\
E(t) &= \frac{(-\beta^2X_0+\alpha\beta\sqrt{\beta}Y_0)\cos(\sqrt{\beta}t)+(\alpha\beta\sqrt{\beta}X_0+\beta^2Y_0)\sin(\sqrt{\beta}t)}{(\alpha^2+\beta)(\mu-\beta)}\\
&\quad+\frac{(\mu^2Z_0-\alpha\mu\sqrt{\mu}U_0)\cos(\sqrt{\mu}t)+(\alpha\mu\sqrt{\mu}Z_0+\mu^2U_0)\sin(\sqrt{\mu}t)}{(\alpha^2+\mu)(\mu-\beta)}.
\end{aligned}
\tag{6.4}
$$

If the function f is $2\pi k$–periodic in the variable t, then for every $(X_0^*, Y_0^*, Z_0^*, U_0^*)$ solution of the system

$$\mathcal{F}_i(X_0,Y_0,Z_0,U_0)=0,\quad i=1,\ldots,4, \tag{6.5}$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4)}{\partial(X_0,Y_0,Z_0,U_0)}\right|_{(X_0,Y_0,Z_0,U_0)=(X_0^*,Y_0^*,Z_0^*,U_0^*)}\right)\neq 0, \tag{6.6}$$

the differential system (6.1) has a periodic solution $x(t,\varepsilon)$ that tends to the solution $x_0(t)$ given by

$$x_0(t) = \frac{(-\sqrt{\beta}X_0 + \alpha Y_0)\cos(\sqrt{\beta}t) + (\alpha X_0 + \beta Y_0)\sin(\sqrt{\beta}t)}{\sqrt{\beta}(\alpha^2+\beta)(\mu-\beta)} + \frac{(\sqrt{\mu}Z_0 - \alpha U_0)\cos(\sqrt{\mu}t) - (\alpha Z_0 + \sqrt{\mu}U_0)\sin(\sqrt{\mu}t)}{\sqrt{\mu}(\alpha^2+\mu)(\mu-\beta)},$$

of $x^{(5)} + \alpha\ddddot{x} + (\beta+\mu)\dddot{x} + \alpha(\beta+\mu)\ddot{x} + \beta\mu\dot{x} + \alpha\beta\mu x = 0$ *when* $\varepsilon \to 0$.

Note that this solution is periodic of period $2\pi k$.

Remark 6.1.1. *In the case when one of the statements of Theorem 6.1.1 does not satisfy, then we cannot apply the averaging theory for studying periodic orbits.*

Proof of Theorem 6.1.1. For $\varepsilon = 0$, the unperturbed system of (6.1) has a unique equilibrium point, the origin. The eigenvalues of the linearized system at this equilibrium point are two pairs of imaginary eigenvalues and one real eigenvalue, more precisely the eigenvalues $\pm\sqrt{\beta}i$, $\pm\sqrt{\mu}i$ and $-\alpha$. We shall write system (6.1) in such way that the linear part at the origin will be in its real Jordan normal form. Therefore, by the linear invertible transformation

$$(X,Y,Z,U,V)^T = B\,(x,y,z,u,v)^T,$$

where

$$R = \begin{pmatrix} 0 & \alpha\mu & \mu & \alpha & 1 \\ \alpha\mu\sqrt{\beta} & \sqrt{\beta}\mu & \alpha\sqrt{\beta} & \sqrt{\beta} & 0 \\ 0 & \alpha\beta & \beta & \alpha & 1 \\ \alpha\beta\sqrt{\mu} & \beta\sqrt{\mu} & \alpha\sqrt{\mu} & \sqrt{\mu} & 0 \\ \beta\mu & 0 & \beta+\alpha & 0 & 1 \end{pmatrix},$$

the differential system (6.1) becomes

$$\begin{aligned}
\dot{X} &= -\sqrt{\beta}Y + \varepsilon G_1(t,X,Y,Z,U,V),\\
\dot{Y} &= \sqrt{\beta}X,\\
\dot{Z} &= -\sqrt{\mu}U + \varepsilon G_1(t,X,Y,Z,U,V),\\
\dot{U} &= \sqrt{\mu}Z,\\
\dot{V} &= -\alpha V + \varepsilon G_1(t,X,Y,Z,U,V),
\end{aligned} \tag{6.7}$$

where $G_1(t,X,Y,Z,U,V) = F_0(t,A(t),B(t),C(t),D(t),E(t))$, with $A(t)$, $B(t)$, $C(t)$, $D(t)$ and $E(t)$ given in (6.4).

Note that the linear part of the differential system (6.7) at the origin is in its real normal form of Jordan. We shall apply Theorem 2.2.1 to the differential system (6.7). Therefore, system (6.7) can be written as system (2.7) taking

$$\mathbf{x} = \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix}, \quad F_0(t,\mathbf{x}) = \begin{pmatrix} -\sqrt{\beta}Y \\ \sqrt{\beta}X \\ -\sqrt{\mu}U \\ \sqrt{\mu}Z \\ -\alpha V \end{pmatrix},$$

$$F_1(t,\mathbf{x}) = \begin{pmatrix} G_1(t,X,Y,Z,U,V) \\ 0 \\ G_1(t,X,Y,Z,U,V) \\ 0 \\ G_1(t,X,Y,Z,U,V) \end{pmatrix} \text{ and } F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We shall study the periodic solutions of system (2.8) in our case, i.e. the periodic solutions of system (6.7) with $\varepsilon = 0$. These periodic solutions are

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \\ U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} X_0 \cos(\sqrt{\beta}t) - Y_0 \sin(\sqrt{\beta}t) \\ Y_0 \cos(\sqrt{\beta}t) + X_0 \sin(\sqrt{\beta}t) \\ Z_0 \cos(\sqrt{\mu}t) - U_0 \sin(\sqrt{\mu}t) \\ U_0 \cos(\sqrt{\mu}t) + Z_0 \sin(\sqrt{\mu}t) \\ 0 \end{pmatrix}.$$

This set of periodic solutions has dimension four, all having the some period $2\pi k$, where k be the least common multiple of n and q.

To look for the periodic solutions of our system (6.1) we must calculate the zeros $\mathbf{z} = (X_0, Y_0, Z_0, U_0)$ of the system $\mathcal{F}(\mathbf{z}) = 0$ where $\mathcal{F}(\mathbf{z})$ is given by (6.3). The fundamental matrix $M(t)$ of the differential system (6.7) with $\varepsilon = 0$ along any periodic solution is

$$M(t) = M_{\mathbf{z}}(t) = \begin{pmatrix} \cos(\sqrt{\beta}t) & -\sin(\sqrt{\beta}t) & 0 & 0 & 0 \\ \sin(\sqrt{\beta}t) & \cos(\sqrt{\beta}t) & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{\mu}t) & -\sin(\sqrt{\mu}t) & 0 \\ 0 & 0 & \sin(\sqrt{\mu}t) & \cos(\sqrt{\mu}t) & 0 \\ 0 & 0 & 0 & 0 & e^{-\alpha t} \end{pmatrix}.$$

The inverse matrix of $M(t)$ is

$$M^{-1}(t) = \begin{pmatrix} \cos(\sqrt{\beta}t) & \sin(\sqrt{\beta}t) & 0 & 0 & 0 \\ -\sin(\sqrt{\beta}t) & \cos(\sqrt{\beta}t) & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{\mu}t) & \sin(\sqrt{\mu}t) & 0 \\ 0 & 0 & -\sin(\sqrt{\mu}t) & \cos(\sqrt{\mu}t) & 0 \\ 0 & 0 & 0 & 0 & e^{\alpha t} \end{pmatrix}.$$

It verifies

$$M^{-1}(0) - M^{-1}(2\pi k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - e^{2\alpha\pi k} \end{pmatrix}.$$

Consequently all the assumptions of Theorem 2.2.1 are satisfied.

Now computing the function $\mathcal{F}(\mathbf{z})$ given in (2.10), we got the system $\mathcal{F}(\mathbf{z}) = 0$, which can be written as (6.5), with the function $\mathcal{F}_k(X_0, Y_0, Z_0, U_0)$ given in (6.3).

The zeros $(X_0^*, Y_0^*, Z_0^*, U_0^*)$ of system (6.5) with respect to the variables X_0, Y_0, Z_0 and U_0 provide periodic orbits of system (6.7) with $\varepsilon \neq 0$ sufficiently small if they are simple, i.e. condition (6.9) is satisfied. Going back though the change of variable, for every simple zero $(X_0^*, Y_0^*, Z_0^*, U_0^*)$ of system (6.5) we obtain a $2\pi k$-periodic solution $x(t)$ of the differential system (6.1) for $\varepsilon \neq 0$ sufficiently small such that $x(t)$ tends to the periodic solution, where $x(t)$ is defined in the statement of Theorem 6.1.1, of

$$x^{(5)} + \alpha \ddddot{x} + (\beta + \mu)\dddot{x} + \alpha(\beta + \mu)\ddot{x} + \beta\mu\dot{x} + \alpha\beta\mu x = 0$$

when $\varepsilon \to 0$. Note that this solution is periodic of period $2\pi k$.

This completes the proof of Theorem 6.1.1. □

An example of Theorem 6.1.1 is the following.

Example 6.1. *If* $f(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = (10 - x^2 + \ddot{x}^2)\cos t + 7\dddot{x}\sin t + 3,$ *then the differential equation* (6.2) *with* $\alpha = 2,\ \beta = 1,\ \mu = 4,$ *becomes*

$$\{\overset{.....}{x}(t) + 2\ddddot{x}(t) + 5\dddot{x}(t) + 10\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = \epsilon\,(5 + \cos(t))\,(4\dot{x}(t) + 2)\,.$$

After some computations, the functions $\mathcal{F}_i(X_0, Y_0, Z_0, U_0)$ *for* $i = 1, \ldots, 4$, *of Theorem 6.1.1 are*

$$\begin{aligned}
\mathcal{F}_1(X_0, Y_0, Z_0, U_0) &= (\frac{5}{384}Z_0 + \frac{7}{12})Z_0 + (\frac{5}{384}U_0 - \frac{7}{12})U_0 + 5,\\
\mathcal{F}_2(X_0, Y_0, Z_0, U_0) &= \frac{7}{12}(Z_0 + U_0),\\
\mathcal{F}_3(X_0, Y_0, Z_0, U_0) &= \frac{1}{240}(X_0 - 2Y_0)(-Z_0 + U_0 + 28),\\
\mathcal{F}_4(X_0, Y_0, Z_0, U_0) &= \frac{X_0}{240}(-Z_0 - U_0 + 56) + \frac{Y_0}{120}(Z_0 + U_0 + 14).
\end{aligned}$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$ *has two solutions* $(X_0^*, Y_0^*, Z_0^*, U_0^*)$ *given by* $(0, 0, -40, 40), (0, 0, -\frac{24}{5}, \frac{24}{5})$.

Since the jacobian (6.6) for these two solutions $(X_0^*, Y_0^*, Z_0^*, U_0^*)$ *are* $-\frac{539}{3840}$, $\frac{25333}{518400}$ *respectively, by Theorem 6.1.1, system (6.1) has the two periodic solutions has two periodic solutions* $x_i = (t, \varepsilon)$ *for* $i = 1, 2$ *tending to the periodic solutions* $x_i(t)$ *where*

$$x_1(t) = -\frac{10}{3}\cos(2t), \quad x_2(t) = -\frac{2}{5}\cos(2t),$$

of $x^{(5)} + 2\ddddot{x} + 5\dddot{x} + 10\ddot{x} + 4\dot{x} + 8x = 0$ *when* $\varepsilon \to 0$.

6.2 Periodic solutions for differential systems in $\mathbb{R}^6$

Now our second result on the periodic solutions of the differential system in $\mathbb{R}^6$ of the form

$$\begin{cases} \dot{x} = y, & \dot{y} = -x - \varepsilon F(t, x, y, z, u, v, w), \\ \dot{z} = u, & \dot{u} = -z - \varepsilon G(t, x, y, z, u, v, w), \\ \dot{v} = w, & \dot{w} = -v - \varepsilon H(t, x, y, z, u, v, w), \end{cases} \tag{6.8}$$

where F, G and H are 2π–periodic functions in the variable t, and $|\varepsilon|$ is a small parameter. These systems are a perturbation of the harmonic oscillator in $\mathbb{R}^6$, and these kind of systems have been studied by many authors, see for instance [6, 31, 45, 58].

We summarize our main result on the periodic orbits of the differential system (6.8) as follows.

Theorem 6.2.1. *We define*

$$\begin{aligned}
\mathcal{F}_1(x_0, y_0, z_0, u_0, v_0, w_0) &= \frac{1}{2\pi} \int_0^{2\pi} \sin(t) F_0(t, a(t), b(t), c(t), d(t), e(t), l(t))\, dt, \\
\mathcal{F}_2(x_0, y_0, z_0, u_0, v_0, w_0) &= -\frac{1}{2\pi} \int_0^{2\pi} \cos(t) F_0(t, a(t), b(t), c(t), d(t), e(t), l(t))\, dt, \\
\mathcal{F}_3(x_0, y_0, z_0, u_0, v_0, w_0) &= \frac{1}{2\pi} \int_0^{2\pi} \sin(t) G_0(t, a(t), b(t), c(t), d(t), e(t), l(t))\, dt, \\
\mathcal{F}_4(x_0, y_0, z_0, u_0, v_0, w_0) &= -\frac{1}{2\pi} \int_0^{2\pi} \cos(t) G_0(t, a(t), b(t), c(t), d(t), e(t), l(t))\, dt, \\
\mathcal{F}_5(x_0, y_0, z_0, u_0, v_0, w_0) &= \frac{1}{2\pi} \int_0^{2\pi} \sin(t) H_0(t, a(t), b(t), c(t), d(t), e(t), l(t))\, dt, \\
\mathcal{F}_6(x_0, y_0, z_0, u_0, v_0, w_0) &= -\frac{1}{2\pi} \int_0^{2\pi} \cos(t) H_0(t, a(t), b(t), c(t), d(t), e(t), l(t))\, dt,
\end{aligned} \tag{6.9}$$

where

$$a(t) = x_0 \cos t + y_0 \sin t,\ b(t) = y_0 \cos t - x_0 \sin t,\ c(t) = z_0 \cos t + u_0 \sin t,$$

$$d(t) = u_0 \cos t - z_0 \sin t,\ e(t) = v_0 \cos t + w_0 \sin t,\ l(t) = w_0 \cos t - v_0 \sin t.$$

Then for every $(x_0^*, y_0^*, z_0^*, u_0^*, v_0^*, w_0^*)$ *solution of the system*

$$\mathcal{F}_k(x_0, y_0, z_0, u_0, v_0, w_0) = 0, \ \textit{for } k = 1, \ldots, 6, \tag{6.10}$$

satisfying

$$\det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)}{\partial(x_0, y_0, z_0, u_0, v_0, w_0)}\right|_{(x_0,y_0,z_0,u_0,v_0,w_0)=(x_0^*,y_0^*,z_0^*,u_0^*,v_0^*,w_0^*)} \right) \neq 0, \tag{6.11}$$

the differential system (6.8) *has* 2π*–periodic solution* $(x(t,\varepsilon),\ y(t,\varepsilon),\ z(t,\varepsilon),\ u(t,\varepsilon),\ v(t,\varepsilon),\ w(t,\varepsilon))$ *which when* $\varepsilon \to 0$ *tends to the* 2π*–periodic solution* $(x_0(t), y_0(t), z_0(t), u_0(t), v_0(t), w_0(t))$ *given by*

$$\begin{aligned} x_0(t) &= x_0^* \cos t + y_0^* \sin t, \ \ y_0(t) = y_0^* \cos t - x_0^* \sin t, \\ z_0(t) &= z_0^* \cos t + u_0^* \sin t, \ \ u_0(t) = u_0^* \cos t - z_0^* \sin t, \\ v_0(t) &= v_0^* \cos t + w_0^* \sin t, \ \ w_0(t) = w_0^* \cos t - v_0^* \sin t, \end{aligned}$$

of system (6.8) *with* $\varepsilon = 0$.

Proof of Theorem 6.2.1. Consider the differential system (6.8) in $\mathbb{R}^6$. Its unperturbed system is the system (6.8) with $\varepsilon = 0$, which has the equilibrium point $(0,0,0,0,0,0) = (x,y,z,u,v,w)$. The eigenvalues of the linearized system at this point are $\pm i$, of multiplicity three. The periodic solutions $(x(t), y(t), z(t), u(t), v(t), w(t))$ of the unperturbed system such that $(x(0), y(0), z(0), u(0), v(0), w(0)) = (x_0, y_0, z_0, u_0, v_0, w_0)$ are

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos t + y_0 \sin t \\ y_0 \cos t - x_0 \sin t \\ z_0 \cos t + u_0 \sin t \\ u_0 \cos t - z_0 \sin t \\ v_0 \cos t + w_0 \sin t \\ w_0 \cos t - v_0 \sin t \end{pmatrix}.$$

Of course, all these periodic orbits have period 2π.

Using the notation of Chap. 2, Sec. 2.2, we have

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ u_0 \\ v_0 \\ w_0 \end{pmatrix}, \quad F_0(t,\mathbf{x}) = \begin{pmatrix} y \\ -x \\ u \\ -z \\ w \\ -v \end{pmatrix},$$

$$F_1(t,\mathbf{x}) = \begin{pmatrix} 0 \\ F(t,x,y,z,u,v,w) \\ 0 \\ G(t,x,y,z,u,v,w) \\ 0 \\ H(t,x,y,z,u,v,w) \end{pmatrix} \quad \textit{and} \quad F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since the fundamental matrix $M_{\mathbf{z}}(t)$ is independent of $\mathbf{z}$, we denote it simply by $M(t)$. An easy computation provides

$$M(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 & 0 & 0 \\ -\sin t & \cos t & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos t & \sin t & 0 & 0 \\ 0 & 0 & -\sin t & \cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{pmatrix}.$$

From Theorem 2.2.2, we must study the zeros $\mathbf{z}$ of the function $\mathcal{F}(\mathbf{z})$ defined in (2.11), i.e. of the function $\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(\mathbf{z}), \mathcal{F}_2(\mathbf{z}), \mathcal{F}_3(\mathbf{z}), \mathcal{F}_4(\mathbf{z}), \mathcal{F}_5(\mathbf{z}), \mathcal{F}_6(\mathbf{z}))$ where $\mathcal{F}_k$ for $k = 1, \ldots, 6$ are given in (6.9) of Theorem 6.2.1.

The rest of the proof of Theorem 6.2.1 follows directly from Theorem 2.2.2. □

An example of Theorem 6.2.1 is the following.

Example 6.2. *Consider the differential system (6.8) with F, G and H satisfies*

$$\begin{cases} F(t,x,y,z,u,v,w) = (-1 - y^2 + u^2)\sin t, \\ G(t,x,y,z,u,v,w) = (1 - y^2 + w^2)\cos t, \\ H(t,x,y,z,u,v,w) = (1 - y^2 + w^2)\sin t. \end{cases}$$

After some computations, we get that

$$\begin{aligned}
\mathcal{F}_1(x_0,y_0,z_0,u_0,v_0,w_0) &= -\frac{1}{8}(3x_0^2 + y_0^2 - 3z_0^2 - u_0^2 + 4), \\
\mathcal{F}_2(x_0,y_0,z_0,u_0,v_0,w_0) &= \frac{1}{4}(-x_0y_0 + z_0u_0), \\
\mathcal{F}_3(x_0,y_0,z_0,u_0,v_0,w_0) &= \frac{1}{4}(x_0y_0 - v_0w_0), \\
\mathcal{F}_4(x_0,y_0,z_0,u_0,v_0,w_0) &= \frac{1}{8}(x_0^2 + 3y_0^2 - v_0^2 - 3w_0^2 + 4), \\
\mathcal{F}_5(x_0,y_0,z_0,u_0,v_0,w_0) &= -\frac{1}{8}(3x_0^2 + y_0^2 - 4), \\
\mathcal{F}_6(x_0,y_0,z_0,u_0,v_0,w_0) &= -\frac{1}{4}x_0y_0.
\end{aligned}$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6 = 0$ *has sixty-four solutions* $(x_0^*, y_0^*, z_0^*, u_0^*, v_0^*, w_0^*)$ *given by*

$$
\begin{array}{ll}
(0, \pm 2, 0, \pm 2\sqrt{2}, \pm 4, 0), & (\pm\frac{2\sqrt{3}}{3}, 0, \pm\frac{2\sqrt{6}}{3}, 0, \pm\frac{4\sqrt{3}}{3}, 0), \\
(\pm\frac{2\sqrt{3}}{3}, 0, 0, \pm 2\sqrt{2}, \pm\frac{4\sqrt{3}}{3}, 0), & (0, \pm 2, \pm\frac{2\sqrt{6}}{3}, 0, \pm 4, 0), \\
(0, \pm 2, \pm\frac{2\sqrt{6}}{3}, 0, 0, \pm\frac{4\sqrt{3}}{3}), & (\pm\frac{2\sqrt{3}}{3}, 0, \pm\frac{2\sqrt{6}}{3}, 0, 0, \pm\frac{4}{3}), \\
(\pm\frac{2\sqrt{3}}{3}, 0, 0, \pm 2\sqrt{2}, 0, \pm\frac{4}{3}), & (0, \pm 2, 0, \pm 2\sqrt{2}, 0, \pm\frac{4\sqrt{3}}{3}).
\end{array}
$$

Since the jacobian (6.11) for these solutions $(x_0^*, y_0^*, z_0^*, u_0^*, v_0^*, w_0^*)$ *are* $-\frac{1}{8}, -\frac{1}{24}, \frac{1}{24}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{24}, -\frac{1}{24}, \frac{1}{8}$ *respectively, we obtain using Theorem 2.2.2 sixty-four solutions, but only thirty-two of them are different because all periodic solutions appear repeated when we change* $t \to t + \pi$. *Hence we obtain the thirty-two solutions*

$(x_k(t,\varepsilon), y_k(t,\varepsilon), z_k(t,\varepsilon), u_k(t,\varepsilon), v_k(t,\varepsilon), w_k(t,\varepsilon))$ *for* $k = 1, \ldots, 32$ *tending when* $\varepsilon \to 0$ *the periodic solutions* $(x_k(t), y_k(t), z_k(t), u_k(t), v_k(t), w_k(t))$ *where*

$$(x_{1,2,3,4}(t), y_{1,2,3,4}(t), z_{1,2,3,4}(t), u_{1,2,3,4}(t), v_{1,2,3,4}(t), w_{1,2,3,4}(t)) = \\ (\pm 2\sin t, \pm 2\cos t, \pm\frac{2\sqrt{6}}{3}\cos t, \mp\frac{2\sqrt{6}}{3}\sin t, -4\cos t, 4\sin t),$$

$$(x_{5,6,7,8}(t), y_{5,6,7,8}(t), z_{5,6,7,8}(t), u_{5,6,7,8}(t), v_{5,6,7,8}(t), w_{5,6,7,8}(t)) = \\ (\pm\frac{2\sqrt{3}}{3}\cos t, \mp\frac{2\sqrt{3}}{3}\sin t, \pm\frac{2\sqrt{6}}{3}cost, \mp\frac{2\sqrt{6}}{3}\sin t, -\frac{4\sqrt{3}}{3}\cos t, \frac{4\sqrt{3}}{3}\sin t),$$

$$(x_{9,10,11,12}(t), y_{9,10,11,12}(t), z_{9,10,11,12}(t), u_{9,10,11,12}(t), v_{9,10,11,12}(t), w_{9,10,11,12}(t)) = \\ (\pm 2\sin t, \pm 2\cos t, \pm 2\sqrt{2}\sin t, \pm 2\sqrt{2}\cos t, -\frac{4\sqrt{3}}{3}\sin t, -\frac{4\sqrt{3}}{3}\cos t),$$

$$(x_{13,14,15,16}(t), y_{13,14,15,16}(t), z_{13,14,15,16}(t), u_{13,14,15,16}(t), v_{13,14,15,16}(t), w_{13,14,15,16}(t)) = \\ (\pm\frac{2\sqrt{3}}{3}\cos t, \mp\frac{2\sqrt{3}}{3}\sin t, \pm 2\sqrt{2}\sin t, \pm 2\sqrt{2}\cos t, -\frac{4\sqrt{3}\cos t}{3}, \frac{4\sqrt{3}}{3}\sin t,$$

$$(x_{17,18,19,20}(t), y_{17,18,19,20}(t), z_{17,18,19,20}(t), u_{17,18,19,20}(t), v_{17,18,19,20}(t), w_{17,18,19,20}(t)) = \\ (\mp 2\sin t, \mp 2\cos t, \mp\frac{2\sqrt{6}}{3}\cos t, \pm\frac{2\sqrt{6}}{3}\sin t, -\frac{4\sqrt{3}}{3}\sin t, -\frac{4\sqrt{3}}{3}\cos t),$$

$$(x_{21,22,23,24}(t), y_{21,22,23,24}(t), z_{21,22,23,24}(t), u_{21,22,23,24}(t), v_{21,22,23,24}(t), w_{21,22,23,24}(t)) = (\pm\frac{2\sqrt{3}}{3}\cos t, \mp\frac{2\sqrt{3}}{3}\sin t, \pm\frac{2\sqrt{6}}{3}\cos t, \mp\frac{2\sqrt{6}}{3}\sin t, -\frac{4}{3}\sin t, -\frac{4}{3}\cos t),$$

$$(x_{25,26,27,28}(t), y_{25,26,27,28}(t), z_{25,26,27,28}(t), u_{25,26,27,28}(t), v_{25,26,27,28}(t), w_{25,26,27,28}(t)) = (\pm 2\sin t, \pm 2\cos t, \pm 2\sqrt{2}\sin t, \pm 2\sqrt{2}\cos t, -4\cos t, 4\sin t),$$

$$(x_{29,30,31,32}(t), y_{29,30,31,32}(t), z_{29,30,31,32}(t), u_{29,30,31,32}(t), v_{29,30,31,32}(t), w_{29,30,31,32}(t)) = (\pm\frac{2\sqrt{3}}{3}\cos t, \mp\frac{2\sqrt{3}}{3}\sin t, \pm 2\sqrt{2}\sin t, \pm 2\sqrt{2}\cos t, -\frac{4}{3}\sin t, -\frac{4}{3}\cos t),$$

of $\dot{x} = y,\ \dot{y} = -x,\ \dot{z} = u,\ \dot{u} = -z,\ \dot{v} = w,\ \dot{w} = -v,$ *when* $\varepsilon \to 0.$

Conclusion and Perspectives

Based on different results of the averaging theory, we have provided the sufficient conditions for the existence for periodic solutions of some classes of differential equations and systems such as Duffing differential equations and differential systems of dimension 5 and 6, using averaging theory of the first order. We have also studied the maximum number of limit cycles of planar cubic polynomial differential systems, using averaging theory up to order six. Moreover, we have illustrated our study with examples.

We will continue our research about the existence of periodic solutions for other types of differential systems that model phenomena in biology, physics, mechanics, etc.

We will also continue our research about the limit cycles for differential systems that relate to the second part of sixteen Hilbert problems, using higher-order averaging theory.

Appendix

In this Appendix, we give the constants that we need in the proof of the main results of Chapter 5 (Sect. 5.2).

$D_1 = \; a_{51} + \; b_{52},$

$D_3 = \; 9\,a_{11}^3 - 3\,a_{11}^2\,b_{13} - 9\,a_{11}^2\,b_{15} + 9\,a_{11}\,a_{12}^2 - 3\,a_{11}\,a_{12}\,a_{13} + 6\,a_{11}\,a_{12}\,a_{15}$
$-2\,a_{11}\,a_{12}\,a_{17} - 9\,a_{11}\,a_{12}\,b_{11} - 6\,a_{11}\,a_{12}\,b_{14} - 2\,a_{11}\,a_{12}\,b_{18} - 2\,a_{11}\,a_{13}^2$
$-2\,a_{11}\,a_{13}\,a_{15} + 3\,a_{11}\,a_{13}\,b_{11} + \; a_{11}\,a_{13}\,b_{14} - \; a_{11}\,a_{14}^2 + \; a_{11}\,a_{14}\,b_{13}$
$- \; a_{11}\,a_{14}\,b_{15} - \; a_{11}\,a_{15}\,b_{14} + 9\,a_{11}\,a_{22} - 3\,a_{11}\,a_{23} + 3\,a_{11}\,a_{25} - 2\,a_{11}\,a_{27}$
$+9\,a_{11}\,b_{11}^2 + 6\,a_{11}\,b_{11}\,b_{14} + 2\,a_{11}\,b_{13}\,b_{15} + \; a_{11}\,b_{14}^2 + 2\,a_{11}\,b_{15}^2 - 9\,a_{11}\,b_{21}$
$-6\,a_{11}\,b_{24} - 2\,a_{11}\,b_{28} - \; a_{12}^2\,a_{18} - 3\,a_{12}^2\,b_{15} - 3\,a_{12}^2\,b_{19} - \; a_{12}\,a_{13}\,a_{14}$
$-2\,a_{12}\,a_{14}\,a_{15} - 4\,a_{12}\,a_{15}\,b_{15} - \; a_{12}\,a_{18}\,b_{11} + 9\,a_{12}\,a_{21} - \; a_{12}\,a_{28}$
$-3\,a_{12}\,b_{11}\,b_{19} + \; a_{12}\,b_{14}\,b_{15} - 3\,a_{12}\,b_{25} - 3\,a_{12}\,b_{29} - 3\,a_{13}\,a_{21} - \; a_{13}\,a_{24}$
$-2\,a_{13}\,b_{11}\,b_{13} + 2\,a_{13}\,b_{23} - \; a_{14}\,a_{15}\,b_{11} - \; a_{14}\,a_{23} - \; a_{14}\,a_{25} + 3\,a_{15}\,a_{21}$
$- \; a_{15}\,a_{24} - 2\,a_{15}\,b_{11}\,b_{15} - 2\,a_{15}\,b_{25} - 2\,a_{17}\,a_{21} - \; a_{18}\,a_{22} - \; a_{18}\,b_{21}$
$-9\,a_{21}\,b_{11} - 6\,a_{21}\,b_{14} - 2\,a_{21}\,b_{18} - 3\,a_{22}\,b_{15} - 3\,a_{22}\,b_{19} + 2\,a_{23}\,b_{13}$
$-2\,a_{25}\,b_{15} - \; a_{28}\,b_{11} + 9\,a_{31} - 3\,a_{36} - \; a_{38} - 3\,b_{11}^2\,b_{13} - \; b_{11}\,b_{13}\,b_{14}$
$+3\,b_{11}\,b_{23} - 3\,b_{11}\,b_{29} + 3\,b_{13}\,b_{21} + \; b_{13}\,b_{24} + \; b_{14}\,b_{23} + \; b_{14}\,b_{25} + \; b_{15}\,b_{24}$
$-3\,b_{19}\,b_{21} - 3\,b_{33} - 3\,b_{35} - \; b_{37} - 3\,b_{39},$

$D_5 = \; 15\,a_{11} - 2\,a_{14} - 9\,a_{16} - 2\,a_{18} - 5\,b_{13} - 4\,b_{15} - 3\,b_{19},$

$$
\begin{aligned}
S_1 = {} & a_{61} + b_{62}, \\
S_2 = {} & 23652\, a_{12}\, a_{11}^3 + 1152\, a_{13}\, a_{11}^3 - 1152\, a_{15}\, a_{11}^3 - 1536\, a_{19}\, a_{11}^3 \\
& -3468\, b_{11}\, a_{11}^3 - 1152\, b_{14}\, a_{11}^3 - 1536\, b_{16}\, a_{11}^3 - 832\, a_{12}\, a_{14}\, a_{11}^2 \\
& -6912\, a_{12}\, a_{16}\, a_{11}^2 - 1792\, a_{12}\, a_{18}\, a_{11}^2 + 17520\, a_{21}\, a_{11}^2 + 1536\, a_{26}\, a_{11}^2 \\
& +2240\, a_{14}\, b_{11}\, a_{11}^2 + 2304\, a_{16}\, b_{11}\, a_{11}^2 + 1280\, a_{18}\, b_{11}\, a_{11}^2 - 8128\, a_{12}\, b_{13}\, a_{11}^2 \\
& -1536\, a_{13}\, b_{13}\, a_{11}^2 - 448\, b_{11}\, b_{13}\, a_{11}^2 - 10304\, a_{12}\, b_{15}\, a_{11}^2 + 1536\, a_{15}\, b_{15}\, a_{11}^2 \\
& -4160\, b_{11}\, b_{15}\, a_{11}^2 + 1536\, b_{29}\, a_{11}^2 - 5607\, a_{12}^3\, a_{11} + 34173\, b_{11}^3\, a_{11} \\
& -576\, a_{12}\, a_{13}^2\, a_{11} - 576\, a_{12}\, a_{14}^2\, a_{11} - 1344\, a_{12}\, a_{15}^2\, a_{11} + 17811\, a_{12}\, b_{11}^2\, a_{11} \\
& +1440\, a_{13}\, b_{11}^2\, a_{11} - 2592\, a_{15}\, b_{11}^2\, a_{11} + 2304\, a_{17}\, b_{11}^2\, a_{11} + 3072\, a_{19}\, b_{11}^2\, a_{11} \\
& +576\, a_{12}\, b_{13}^2\, a_{11} + 576\, b_{11}\, b_{13}^2\, a_{11} + 576\, a_{12}\, b_{14}^2\, a_{11} + 576\, b_{11}\, b_{14}^2\, a_{11} \\
& +1344\, a_{12}\, b_{15}^2\, a_{11} + 1344\, b_{11}\, b_{15}^2\, a_{11} + 1440\, a_{12}^2\, a_{13}\, a_{11} + 6624\, a_{12}^2\, a_{15}\, a_{11} \\
& -1152\, a_{12}\, a_{13}\, a_{15}\, a_{11} - 1152\, a_{12}^2\, a_{17}\, a_{11} - 2688\, a_{12}^2\, a_{19}\, a_{11} \\
& -2048\, a_{14}\, a_{21}\, a_{11} - 6144\, a_{16}\, a_{21}\, a_{11} - 2048\, a_{18}\, a_{21}\, a_{11} \\
& -5592\, a_{12}\, a_{22}\, a_{11} + 2304\, a_{15}\, a_{22}\, a_{11} - 1152\, a_{17}\, a_{22}\, a_{11} \\
& -1920\, a_{19}\, a_{22}\, a_{11} + 2304\, a_{12}\, a_{25}\, a_{11} - 1152\, a_{12}\, a_{27}\, a_{11} \\
& -1920\, a_{12}\, a_{29}\, a_{11} + 4032\, a_{32}\, a_{11} - 21969\, a_{12}^2\, b_{11}\, a_{11} \\
& -576\, a_{13}^2\, b_{11}\, a_{11} - 576\, a_{14}^2\, b_{11}\, a_{11} - 1344\, a_{15}^2\, b_{11}\, a_{11} \\
& +2880\, a_{12}\, a_{13}\, b_{11}\, a_{11} + 4032\, a_{12}\, a_{15}\, b_{11}\, a_{11} - 1152\, a_{13}\, a_{15}\, b_{11}\, a_{11} \\
& +1152\, a_{12}\, a_{17}\, b_{11}\, a_{11} + 384\, a_{12}\, a_{19}\, b_{11}\, a_{11} - 22872\, a_{22}\, b_{11}\, a_{11} \\
& +2304\, a_{25}\, b_{11}\, a_{11} - 1152\, a_{27}\, b_{11}\, a_{11} - 1920\, a_{29}\, b_{11}\, a_{11} \\
& -5120\, a_{21}\, b_{13}\, a_{11} - 4896\, a_{12}^2\, b_{14}\, a_{11} + 8928\, b_{11}^2\, b_{14}\, a_{11} \\
& -768\, a_{12}\, a_{15}\, b_{14}\, a_{11} - 3456\, a_{22}\, b_{14}\, a_{11} + 4032\, a_{12}\, b_{11}\, b_{14}\, a_{11} \\
& -768\, a_{15}\, b_{11}\, b_{14}\, a_{11} - 768\, a_{12}\, a_{14}\, b_{15}\, a_{11} - 4096\, a_{21}\, b_{15}\, a_{11} \\
& -768\, a_{14}\, b_{11}\, b_{15}\, a_{11} + 1152\, a_{12}\, b_{13}\, b_{15}\, a_{11} + 1152\, b_{11}\, b_{13}\, b_{15}\, a_{11} \\
& -1152\, a_{12}^2\, b_{16}\, a_{11} + 2304\, b_{11}^2\, b_{16}\, a_{11} - 1152\, a_{22}\, b_{16}\, a_{11} \\
& +1152\, a_{12}\, b_{11}\, b_{16}\, a_{11} - 2688\, a_{12}^2\, b_{18}\, a_{11} + 3072\, b_{11}^2\, b_{18}\, a_{11} \\
& -1920\, a_{22}\, b_{18}\, a_{11} + 384\, a_{12}\, b_{11}\, b_{18}\, a_{11} - 22872\, a_{12}\, b_{21}\, a_{11}
\end{aligned}
$$

$+2304\,a_{15}\,b_{21}\,a_{11} - 1152\,a_{17}\,b_{21}\,a_{11} - 1920\,a_{19}\,b_{21}\,a_{11}$

$-40152\,b_{11}\,b_{21}\,a_{11} - 3456\,b_{14}\,b_{21}\,a_{11} - 1152\,b_{16}\,b_{21}\,a_{11}$

$-1920\,b_{18}\,b_{21}\,a_{11} - 3456\,a_{12}\,b_{24}\,a_{11} - 3456\,b_{11}\,b_{24}\,a_{11}$

$-1152\,a_{12}\,b_{26}\,a_{11} - 1152\,b_{11}\,b_{26}\,a_{11} - 1920\,a_{12}\,b_{28}\,a_{11}$

$-1920\,b_{11}\,b_{28}\,a_{11} + 4032\,b_{31}\,a_{11} - 3344\,a_{14}\,b_{11}^3 - 11520\,a_{16}\,b_{11}^3$

$-2048\,a_{18}\,b_{11}^3 - 3888\,a_{12}\,a_{14}\,b_{11}^2 + 192\,a_{13}\,a_{14}\,b_{11}^2 - 192\,a_{14}\,a_{15}\,b_{11}^2\,b_{11}^2$

$-13824\,a_{12}\,a_{16}\,b_{11}^2 - 2304\,a_{12}\,a_{18}\,b_{11}^2 - 10476\,a_{21} + 576\,a_{24}\,b_{11}^2$

$+1152\,a_{26}\,b_{11}^2 + 2800\,a_{12}^3\,a_{14} + 192\,a_{12}^2\,a_{13}\,a_{14} - 192\,a_{12}^2\,a_{14}\,a_{15}$

$+9216\,a_{12}^3\,a_{16} + 1792\,a_{12}^3\,a_{18} + 6804\,a_{12}^2\,a_{21} + 2304\,a_{12}\,a_{15}\,a_{21}$

$-1152\,a_{12}\,a_{17}\,a_{21} - 1920\,a_{12}\,a_{19}\,a_{21} + 3712\,a_{12}\,a_{14}\,a_{22}$

$+13824\,a_{12}\,a_{16}\,a_{22} + 2560\,a_{12}\,a_{18}\,a_{22} + 4032\,a_{21}\,a_{22} + 576\,a_{12}^2\,a_{24}$

$+1152\,a_{12}^2\,a_{26} + 4032\,a_{12}\,a_{31} + 2256\,a_{12}^2\,a_{14}\,b_{11} + 384\,a_{12}\,a_{13}\,a_{14}\,b_{11}$

$-384\,a_{12}\,a_{14}\,a_{15}\,b_{11} + 6912\,a_{12}^2\,a_{16}\,b_{11} + 1536\,a_{12}^2\,a_{18}\,b_{11}$

$-3672\,a_{12}\,a_{21}\,b_{11} + 2304\,a_{15}\,a_{21}\,b_{11} - 1152\,a_{17}\,a_{21}\,b_{11} - 1920\,a_{19}\,a_{21}\,b_{11}$

$+3712\,a_{14}\,a_{22}\,b_{11} + 13824\,a_{16}\,a_{22}\,b_{11} + 2560\,a_{18}\,a_{22}\,b_{11}$

$+1152\,a_{12}\,a_{24}\,b_{11} + 2304\,a_{12}\,a_{26}\,b_{11} + 4032\,a_{31}\,b_{11} + 4912\,a_{12}^3\,b_{13}$

$-6992\,b_{11}^3\,b_{13} - 9072\,a_{12}\,b_{11}^2\,b_{13} - 960\,a_{13}\,b_{11}^2\,b_{13} - 192\,a_{15}\,b_{11}^2\,b_{13}$

$-960\,a_{12}^2\,a_{13}\,b_{13} - 192\,a_{12}^2\,a_{15}\,b_{13} + 7552\,a_{12}\,a_{22}\,b_{13}$

$+2832\,a_{12}^2\,b_{11}\,b_{13} - 1920\,a_{12}\,a_{13}\,b_{11}\,b_{13} - 384\,a_{12}\,a_{15}\,b_{11}\,b_{13}$

$+7552\,a_{22}\,b_{11}\,b_{13} - 192\,a_{14}\,b_{11}^2\,b_{14} - 192\,a_{12}^2\,a_{14}\,b_{14}$

$-3456\,a_{12}\,a_{21}\,b_{14} - 384\,a_{12}\,a_{14}\,b_{11}\,b_{14} - 3456\,a_{21}\,b_{11}\,b_{14}$

$-192\,a_{12}^2\,b_{13}\,b_{14} - 192\,b_{11}^2\,b_{13}\,b_{14} - 384\,a_{12}\,b_{11}\,b_{13}\,b_{14}$

$+2576\,a_{12}^3\,b_{15} - 2800\,b_{11}^3\,b_{15} - 3024\,a_{12}\,b_{11}^2\,b_{15} - 192\,a_{13}\,b_{11}^2\,b_{15}$

$-1728\,a_{15}\,b_{11}^2\,b_{15} - 192\,a_{12}^2\,a_{13}\,b_{15} - 1728\,a_{12}^2\,a_{15}\,b_{15}$

$+3968\,a_{12}\,a_{22}\,b_{15} + 2352\,a_{12}^2\,b_{11}\,b_{15} - 384\,a_{12}\,a_{13}\,b_{11}\,b_{15}$

$-3456\,a_{12}\,a_{15}\,b_{11}\,b_{15} + 3968\,a_{22}\,b_{11}\,b_{15} + 192\,a_{12}^2\,b_{14}\,b_{15}$

$+192\,b_{11}^2\,b_{14}\,b_{15} + 384\,a_{12}\,b_{11}\,b_{14}\,b_{15} - 1152\,a_{12}\,a_{21}\,b_{16}$

$$
\begin{aligned}
&-1152\, a_{21}\, b_{11}\, b_{16} - 1920\, a_{12}\, a_{21}\, b_{18} - 1920\, a_{21}\, b_{11}\, b_{18} \\
&+3712\, a_{12}\, a_{14}\, b_{21} + 13824\, a_{12}\, a_{16}\, b_{21} + 2560\, a_{12}\, a_{18}\, b_{21} + 4032\, a_{21}\, b_{21} \\
&+3712\, a_{14}\, b_{11}\, b_{21} + 13824\, a_{16}\, b_{11}\, b_{21} + 2560\, a_{18}\, b_{11}\, b_{21} \\
&+7552\, a_{12}\, b_{13}\, b_{21} + 7552\, b_{11}\, b_{13}\, b_{21} + 3968\, a_{12}\, b_{15}\, b_{21} \\
&+3968\, b_{11}\, b_{15}\, b_{21} + 576\, a_{12}^2\, b_{23} + 576\, b_{11}^2\, b_{23} + 1152\, a_{12}\, b_{11}\, b_{23} \\
&-576\, a_{12}^2\, b_{25} - 576\, b_{11}^2\, b_{25} - 1152\, a_{12}\, b_{11}\, b_{25} - 1920\, a_{12}^2\, b_{29} \\
&-1920\, b_{11}^2\, b_{29} - 3840\, a_{12}\, b_{11}\, b_{29},
\end{aligned}
$$

$$
\begin{aligned}
S_3 = \;& 864\, a_{11}^2\, a_{12} + 4932\, a_{11}^3\, a_{12} - 11979\, a_{11}\, a_{12}^3 + 21504\, a_{11}^2\, a_{13} \\
&-22272\, a_{11}^3\, a_{13} - 32976\, a_{11}\, a_{12}^2\, a_{13} - 896\, a_{11}\, a_{12}\, a_{13}^2 \\
&-9888\, a_{11}\, a_{12}\, a_{14} + 1824\, a_{11}^2\, a_{12}\, a_{14} + 1440\, a_{12}^3\, a_{14} \\
&-2304\, a_{11}\, a_{13}\, a_{14} - 7936\, a_{11}^2\, a_{13}\, a_{14} - 1600\, a_{12}^2\, a_{13}\, a_{14} \\
&+3200\, a_{12}\, a_{14}^2 - 6464\, a_{11}\, a_{12}\, a_{14}^2 - 6912\, a_{11}^2\, a_{15} + 5184\, a_{11}^3\, a_{15} \\
&+6240\, a_{11}\, a_{12}^2\, a_{15} - 5760\, a_{11}\, a_{12}\, a_{13}\, a_{15} + 2304\, a_{11}\, a_{14}\, a_{15} \\
&-1536\, a_{11}^2\, a_{14}\, a_{15} - 8576\, a_{12}^2\, a_{14}\, a_{15} + 3584\, a_{11}\, a_{12}\, a_{15}^2 \\
&-36864\, a_{11}\, a_{12}\, a_{16} + 10368\, a_{11}^2\, a_{12}\, a_{16} + 12096\, a_{12}^3\, a_{16} \\
&+12288\, a_{12}\, a_{14}\, a_{16} + 9216\, a_{11}^2\, a_{17} + 288\, a_{11}^3\, a_{17} - 4200\, a_{11}\, a_{12}^2\, a_{17} \\
&-3072\, a_{11}\, a_{14}\, a_{17} - 6144\, a_{11}\, a_{12}\, a_{18} + 2880\, a_{11}^2\, a_{12}\, a_{18} \\
&+48\, a_{12}^3\, a_{18} + 2048\, a_{12}\, a_{14}\, a_{18} + 9216\, a_{11}^2\, a_{19} + 2592\, a_{11}^3\, a_{19} \\
&+4536\, a_{11}\, a_{12}^2\, a_{19} - 3072\, a_{11}\, a_{14}\, a_{19} - 43776\, a_{11}\, a_{21} + 51768\, a_{11}^2\, a_{21} \\
&+21690\, a_{12}^2\, a_{21} - 27072\, a_{12}\, a_{13}\, a_{21} - 3456\, a_{13}^2\, a_{21} + 7296\, a_{14}\, a_{21} \\
&+4224\, a_{11}\, a_{14}\, a_{21} - 3712\, a_{14}^2\, a_{21} + 7104\, a_{12}\, a_{15}\, a_{21} - 3328\, a_{13}\, a_{15}\, a_{21} \\
&+640\, a_{15}^2\, a_{21} + 9216\, a_{11}\, a_{16}\, a_{21} - 4416\, a_{12}\, a_{17}\, a_{21} + 3072\, a_{11}\, a_{18}\, a_{21} \\
&+2880\, a_{12}\, a_{19}\, a_{21} - 20412\, a_{11}\, a_{12}\, a_{22} - 27072\, a_{11}\, a_{13}\, a_{22} \\
&+5856\, a_{12}\, a_{14}\, a_{22} - 2176\, a_{13}\, a_{14}\, a_{22} + 7104\, a_{11}\, a_{15}\, a_{22} \\
&-6016\, a_{14}\, a_{15}\, a_{22} + 34560\, a_{12}\, a_{16}\, a_{22} - 4416\, a_{11}\, a_{17}\, a_{22} \\
&+2304\, a_{12}\, a_{18}\, a_{22} + 2880\, a_{11}\, a_{19}\, a_{22} + 31104\, a_{21}\, a_{22} - 24576\, a_{11}\, a_{12}\, a_{23} \\
&-6912\, a_{11}\, a_{13}\, a_{23} - 2176\, a_{12}\, a_{14}\, a_{23} - 3328\, a_{11}\, a_{15}\, a_{23} - 17664\, a_{21}\, a_{23}
\end{aligned}
$$

$+7296\, a_{11}\, a_{24} + 576\, a_{11}^2\, a_{24} - 1200\, a_{12}^2\, a_{24} - 2176\, a_{12}\, a_{13}\, a_{24}$

$-7424\, a_{11}\, a_{14}\, a_{24} - 6016\, a_{12}\, a_{15}\, a_{24} - 384\, a_{22}\, a_{24} - 3072\, a_{23}\, a_{24}$

$+8448\, a_{11}\, a_{12}\, a_{25} - 3328\, a_{11}\, a_{13}\, a_{25} - 6016\, a_{12}\, a_{14}\, a_{25}$

$+1280\, a_{11}\, a_{15}\, a_{25} + 5376\, a_{21}\, a_{25} - 3072\, a_{24}\, a_{25} - 2304\, a_{11}^2\, a_{26}$

$-1728\, a_{12}^2\, a_{26} - 4416\, a_{11}\, a_{12}\, a_{27} - 6144\, a_{21}\, a_{27} - 3072\, a_{12}^2\, a_{28}$

$-3072\, a_{22}\, a_{28} + 2880\, a_{11}\, a_{12}\, a_{29} + 29088\, a_{12}\, a_{31} - 17664\, a_{13}\, a_{31}$

$+5376\, a_{15}\, a_{31} - 6144\, a_{17}\, a_{31} - 16992\, a_{11}\, a_{32} + 5760\, a_{14}\, a_{32}$

$+27648\, a_{16}\, a_{32} + 3072\, a_{18}\, a_{32} - 17664\, a_{11}\, a_{33} - 3072\, a_{14}\, a_{33}$

$-384\, a_{12}\, a_{34} - 3072\, a_{13}\, a_{34} - 3072\, a_{15}\, a_{34} + 5376\, a_{11}\, a_{35} - 3072\, a_{14}\, a_{35}$

$-6144\, a_{11}\, a_{37} - 3072\, a_{12}\, a_{38} + 33408\, a_{41} - 9216\, a_{46} - 3072\, a_{48}$

$+119520\, a_{11}^2\, b_{11} - 61236\, a_{11}^3\, b_{11} - 71775\, a_{11}\, a_{12}^2\, b_{11}$

$+28704\, a_{11}\, a_{12}\, a_{13}\, b_{11} + 3328\, a_{11}\, a_{13}^2\, b_{11} - 42144\, a_{11}\, a_{14}\, b_{11}$

$-4512\, a_{11}^2\, a_{14}\, b_{11} + 3120\, a_{12}^2\, a_{14}\, b_{11} + 256\, a_{12}\, a_{13}\, a_{14}\, b_{11}$

$+3200\, a_{14}^2\, b_{11} + 1600\, a_{11}\, a_{14}^2\, b_{11} + 1152\, a_{11}\, a_{12}\, a_{15}\, b_{11}$

$-1920\, a_{11}\, a_{13}\, a_{15}\, b_{11} - 5248\, a_{12}\, a_{14}\, a_{15}\, b_{11} + 1664\, a_{11}\, a_{15}^2\, b_{11}$

$-36864\, a_{11}\, a_{16}\, b_{11} - 3456\, a_{11}^2\, a_{16}\, b_{11} + 12096\, a_{12}^2\, a_{16}\, b_{11}$

$+12288\, a_{14}\, a_{16}\, b_{11} - 1296\, a_{11}\, a_{12}\, a_{17}\, b_{11} - 6144\, a_{11}\, a_{18}\, b_{11}$

$-1728\, a_{11}^2\, a_{18}\, b_{11} - 240\, a_{12}^2\, a_{18}\, b_{11} + 2048\, a_{14}\, a_{18}\, b_{11}$

$+432\, a_{11}\, a_{12}\, a_{19}\, b_{11} - 49644\, a_{12}\, a_{21}\, b_{11} + 16704\, a_{13}\, a_{21}\, b_{11}$

$+192\, a_{15}\, a_{21}\, b_{11} + 1728\, a_{17}\, a_{21}\, b_{11} + 2880\, a_{19}\, a_{21}\, b_{11}$

$-67356\, a_{11}\, a_{22}\, b_{11} + 864\, a_{14}\, a_{22}\, b_{11} + 6912\, a_{16}\, a_{22}\, b_{11}$

$-768\, a_{18}\, a_{22}\, b_{11} + 19200\, a_{11}\, a_{23}\, b_{11} + 896\, a_{14}\, a_{23}\, b_{11}$

$-1248\, a_{12}\, a_{24}\, b_{11} + 896\, a_{13}\, a_{24}\, b_{11} - 2944\, a_{15}\, a_{24}\, b_{11}$

$+1536\, a_{11}\, a_{25}\, b_{11} - 2944\, a_{14}\, a_{25}\, b_{11} - 3456\, a_{12}\, a_{26}\, b_{11}$

$+1728\, a_{11}\, a_{27}\, b_{11} - 3072\, a_{12}\, a_{28}\, b_{11} + 2880\, a_{11}\, a_{29}\, b_{11}$

$-43488\, a_{31}\, b_{11} - 384\, a_{34}\, b_{11} - 3072\, a_{38}\, b_{11} + 31779\, a_{11}\, a_{12}\, b_{11}^2$

$-16656\, a_{11}\, a_{13}\, b_{11}^2 + 5760\, a_{12}\, a_{14}\, b_{11}^2 - 1216\, a_{13}\, a_{14}\, b_{11}^2$

$$-480\, a_{11}\, a_{15}\, b_{11}^2 + 256\, a_{14}\, a_{15}\, b_{11}^2 + 15552\, a_{12}\, a_{16}\, b_{11}^2$$
$$-3240\, a_{11}\, a_{17}\, b_{11}^2 + 2448\, a_{12}\, a_{18}\, b_{11}^2 - 4104\, a_{11}\, a_{19}\, b_{11}^2$$
$$+46170\, a_{21}\, b_{11}^2 - 48\, a_{24}\, b_{11}^2 - 1728\, a_{26}\, b_{11}^2 - 76617\, a_{11}\, b_{11}^3$$
$$+4080\, a_{14}\, b_{11}^3 + 15552\, a_{16}\, b_{11}^3 + 2736\, a_{18}\, b_{11}^3 - 26400\, a_{11}\, a_{12}\, b_{13}$$
$$+13632\, a_{11}^2\, a_{12}\, b_{13} + 7320\, a_{12}^3\, b_{13} - 2304\, a_{11}\, a_{13}\, b_{13}$$
$$+7168\, a_{11}^2\, a_{13}\, b_{13} + 2304\, a_{12}^2\, a_{13}\, b_{13} + 9472\, a_{12}\, a_{14}\, b_{13}$$
$$+1216\, a_{11}\, a_{12}\, a_{14}\, b_{13} + 2304\, a_{11}\, a_{15}\, b_{13} + 512\, a_{12}^2\, a_{15}\, b_{13}$$
$$+12288\, a_{12}\, a_{16}\, b_{13} - 3072\, a_{11}\, a_{17}\, b_{13} + 2048\, a_{12}\, a_{18}\, b_{13}$$
$$-3072\, a_{11}\, a_{19}\, b_{13} + 7296\, a_{21}\, b_{13} - 384\, a_{11}\, a_{21}\, b_{13} + 1792\, a_{14}\, a_{21}\, b_{13}$$
$$+19296\, a_{12}\, a_{22}\, b_{13} + 896\, a_{13}\, a_{22}\, b_{13} + 128\, a_{15}\, a_{22}\, b_{13}$$
$$+896\, a_{12}\, a_{23}\, b_{13} + 1792\, a_{11}\, a_{24}\, b_{13} + 128\, a_{12}\, a_{25}\, b_{13}$$
$$+14976\, a_{32}\, b_{13} + 6144\, a_{33}\, b_{13} - 58656\, a_{11}\, b_{11}\, b_{13} + 9600\, a_{11}^2\, b_{11}\, b_{13}$$
$$+9432\, a_{12}^2\, b_{11}\, b_{13} + 1920\, a_{12}\, a_{13}\, b_{11}\, b_{13} + 9472\, a_{14}\, b_{11}\, b_{13}$$
$$-1088\, a_{11}\, a_{14}\, b_{11}\, b_{13} + 640\, a_{12}\, a_{15}\, b_{11}\, b_{13} + 12288\, a_{16}\, b_{11}\, b_{13}$$
$$+2048\, a_{18}\, b_{11}\, b_{13} + 5088\, a_{22}\, b_{11}\, b_{13} - 5248\, a_{23}\, b_{11}\, b_{13}$$
$$+128\, a_{25}\, b_{11}\, b_{13} + 9960\, a_{12}\, b_{11}^2\, b_{13} + 5760\, a_{13}\, b_{11}^2\, b_{13}$$
$$+128\, a_{15}\, b_{11}^2\, b_{13} + 17064\, b_{11}^3\, b_{13} + 6272\, a_{12}\, b_{13}^2 - 1536\, a_{11}\, a_{12}\, b_{13}^2$$
$$-640\, a_{21}\, b_{13}^2 + 6272\, b_{11}\, b_{13}^2 + 384\, a_{11}\, b_{11}\, b_{13}^2 + 28032\, a_{11}^2\, b_{14}$$
$$-16992\, a_{11}^3\, b_{14} - 15288\, a_{11}\, a_{12}^2\, b_{14} + 8384\, a_{11}\, a_{12}\, a_{13}\, b_{14}$$
$$-6912\, a_{11}\, a_{14}\, b_{14} + 512\, a_{12}^2\, a_{14}\, b_{14} - 2368\, a_{11}\, a_{12}\, a_{15}\, b_{14}$$
$$-19584\, a_{12}\, a_{21}\, b_{14} + 5888\, a_{13}\, a_{21}\, b_{14} - 1792\, a_{15}\, a_{21}\, b_{14}$$
$$-19584\, a_{11}\, a_{22}\, b_{14} + 128\, a_{14}\, a_{22}\, b_{14} + 5888\, a_{11}\, a_{23}\, b_{14} + 128\, a_{12}\, a_{24}\, b_{14}$$
$$-1792\, a_{11}\, a_{25}\, b_{14} - 22272\, a_{31}\, b_{14} + 27984\, a_{11}\, a_{12}\, b_{11}\, b_{14}$$
$$-6208\, a_{11}\, a_{13}\, b_{11}\, b_{14} + 640\, a_{12}\, a_{14}\, b_{11}\, b_{14} - 64\, a_{11}\, a_{15}\, b_{11}\, b_{14}$$
$$+28800\, a_{21}\, b_{11}\, b_{14} + 128\, a_{24}\, b_{11}\, b_{14} - 35064\, a_{11}\, b_{11}^2\, b_{14} + 128\, a_{14}\, b_{11}^2\, b_{14}$$
$$-6912\, a_{11}\, b_{13}\, b_{14} + 1536\, a_{11}^2\, b_{13}\, b_{14} + 384\, a_{12}^2\, b_{13}\, b_{14} + 128\, a_{22}\, b_{13}\, b_{14}$$
$$+384\, a_{12}\, b_{11}\, b_{13}\, b_{14} + 3072\, b_{11}^2\, b_{13}\, b_{14} + 3264\, a_{11}\, a_{12}\, b_{14}^2 + 3712\, a_{21}\, b_{14}^2$$

$-4800\,a_{11}\,b_{11}\,b_{14}^2 - 9408\,a_{11}\,a_{12}\,b_{15} - 25344\,a_{11}^2\,a_{12}\,b_{15} - 120\,a_{12}^3\,b_{15}$

$-4608\,a_{11}\,a_{13}\,b_{15} + 7680\,a_{12}^2\,a_{13}\,b_{15} + 9344\,a_{12}\,a_{14}\,b_{15}$

$-9536\,a_{11}\,a_{12}\,a_{14}\,b_{15} + 4608\,a_{11}\,a_{15}\,b_{15} - 7168\,a_{11}^2\,a_{15}\,b_{15}$

$-13568\,a_{12}^2\,a_{15}\,b_{15} + 24576\,a_{12}\,a_{16}\,b_{15} - 6144\,a_{11}\,a_{17}\,b_{15}$

$+4096\,a_{12}\,a_{18}\,b_{15} - 6144\,a_{11}\,a_{19}\,b_{15} + 14592\,a_{21}\,b_{15} - 37248\,a_{11}\,a_{21}\,b_{15}$

$-5888\,a_{14}\,a_{21}\,b_{15} - 3744\,a_{12}\,a_{22}\,b_{15} + 4224\,a_{13}\,a_{22}\,b_{15}$

$-11392\,a_{15}\,a_{22}\,b_{15} + 4224\,a_{12}\,a_{23}\,b_{15} - 5888\,a_{11}\,a_{24}\,b_{15}$

$-11392\,a_{12}\,a_{25}\,b_{15} + 384\,a_{32}\,b_{15} - 6144\,a_{35}\,b_{15} - 73920\,a_{11}\,b_{11}\,b_{15}$

$+27456\,a_{11}^2\,b_{11}\,b_{15} + 19944\,a_{12}^2\,b_{11}\,b_{15} + 2688\,a_{12}\,a_{13}\,b_{11}\,b_{15}$

$+9344\,a_{14}\,b_{11}\,b_{15} + 5056\,a_{11}\,a_{14}\,b_{11}\,b_{15} - 5248\,a_{12}\,a_{15}\,b_{11}\,b_{15}$

$+24576\,a_{16}\,b_{11}\,b_{15} + 4096\,a_{18}\,b_{11}\,b_{15} + 10464\,a_{22}\,b_{11}\,b_{15}$

$+4224\,a_{23}\,b_{11}\,b_{15} - 5248\,a_{25}\,b_{11}\,b_{15} + 8760\,a_{12}\,b_{11}^2\,b_{15}$

$-4992\,a_{13}\,b_{11}^2\,b_{15} + 2176\,a_{15}\,b_{11}^2\,b_{15} - 2088\,b_{11}^3\,b_{15} + 15488\,a_{12}\,b_{13}\,b_{15}$

$-640\,a_{11}\,a_{12}\,b_{13}\,b_{15} + 3328\,a_{21}\,b_{13}\,b_{15} + 15488\,b_{11}\,b_{13}\,b_{15}$

$-4480\,a_{11}\,b_{11}\,b_{13}\,b_{15} - 13824\,a_{11}\,b_{14}\,b_{15} + 7936\,a_{11}^2\,b_{14}\,b_{15}$

$+4160\,a_{12}^2\,b_{14}\,b_{15} + 3968\,a_{22}\,b_{14}\,b_{15} - 512\,a_{12}\,b_{11}\,b_{14}\,b_{15}$

$-1600\,b_{11}^2\,b_{14}\,b_{15} + 5888\,a_{12}\,b_{15}^2 + 5248\,a_{11}\,a_{12}\,b_{15}^2$

$+3456\,a_{21}\,b_{15}^2 + 5888\,b_{11}\,b_{15}^2 + 1024\,a_{11}\,b_{11}\,b_{15}^2 + 9216\,a_{11}^2\,b_{16}$

$+2592\,a_{11}^3\,b_{16} + 1944\,a_{11}\,a_{12}^2\,b_{16} - 3072\,a_{11}\,a_{14}\,b_{16} + 1728\,a_{12}\,a_{21}\,b_{16}$

$+1728\,a_{11}\,a_{22}\,b_{16} - 1296\,a_{11}\,a_{12}\,b_{11}\,b_{16} + 1728\,a_{21}\,b_{11}\,b_{16}$

$-3240\,a_{11}\,b_{11}^2\,b_{16} - 3072\,a_{11}\,b_{13}\,b_{16} - 6144\,a_{11}\,b_{15}\,b_{16}$

$+9216\,a_{11}^2\,b_{18} + 288\,a_{11}^3\,b_{18} - 1608\,a_{11}\,a_{12}^2\,b_{18} - 3072\,a_{11}\,a_{14}\,b_{18}$

$-3264\,a_{12}\,a_{21}\,b_{18} - 3264\,a_{11}\,a_{22}\,b_{18} - -4032\,a_{11}^2\,b_{23} - 22464\,a_{11}^2\,b_{25}$

$-2304\,a_{11}^2\,b_{29} + 432\,a_{11}\,a_{12}\,b_{11}\,b_{18} - 61308\,a_{11}\,a_{12}\,b_{21}$

$-18240\,a_{11}\,a_{12}\,b_{24} + 1728\,a_{11}\,a_{12}\,b_{26} - 3264\,a_{11}\,a_{12}\,b_{28}$

$+16704\,a_{11}\,a_{13}\,b_{21} + 5888\,a_{11}\,a_{13}\,b_{24} + 1792\,a_{11}\,a_{14}\,b_{23}$

$-5888\,a_{11}\,a_{14}\,b_{25} + 192\,a_{11}\,a_{15}\,b_{21} - 1792\,a_{11}\,a_{15}\,b_{24}$

$+1728\, a_{11}\, a_{17}\, b_{21} + 2880\, a_{11}\, a_{19}\, b_{21} - 4104\, a_{11}\, b_{11}^2\, b_{18}$
$+126756\, a_{11}\, b_{11}\, b_{21} + 30144\, a_{11}\, b_{11}\, b_{24} + 1728\, a_{11}\, b_{11}\, b_{26}$
$+2880\, a_{11}\, b_{11}\, b_{28} - 3072\, a_{11}\, b_{13}\, b_{18} - 1280\, a_{11}\, b_{13}\, b_{23}$
$+3328\, a_{11}\, b_{13}\, b_{25} + 28800\, a_{11}\, b_{14}\, b_{21} + 7424\, a_{11}\, b_{14}\, b_{24}$
$-6144\, a_{11}\, b_{15}\, b_{18} + 3328\, a_{11}\, b_{15}\, b_{23} + 6912\, a_{11}\, b_{15}\, b_{25}$
$+1728\, a_{11}\, b_{16}\, b_{21} + 2880\, a_{11}\, b_{18}\, b_{21} + 7296\, a_{11}\, b_{23} + 14592\, a_{11}\, b_{25}$
$-89568\, a_{11}\, b_{31} - 22272\, a_{11}\, b_{34} - 6144\, a_{11}\, b_{38} - 816\, a_{12}^2\, b_{23}$
$-9456\, a_{12}^2\, b_{25} - 6336\, a_{12}^2\, b_{29} + 896\, a_{12}\, a_{13}\, b_{23} + 4224\, a_{12}\, a_{13}\, b_{25}$
$+864\, a_{12}\, a_{14}\, b_{21} + 128\, a_{12}\, a_{14}\, b_{24} + 128\, a_{12}\, a_{15}\, b_{23}$
$-11392\, a_{12}\, a_{15}\, b_{25} + 6912\, a_{12}\, a_{16}\, b_{21} - 768\, a_{12}\, a_{18}\, b_{21}$
$-480\, a_{12}\, b_{11}\, b_{23} + 7584\, a_{12}\, b_{11}\, b_{25} - 3456\, a_{12}\, b_{11}\, b_{29}$
$+5088\, a_{12}\, b_{13}\, b_{21} + 128\, a_{12}\, b_{13}\, b_{24} + 128\, a_{12}\, b_{14}\, b_{23}$
$+3968\, a_{12}\, b_{14}\, b_{25} + 10464\, a_{12}\, b_{15}\, b_{21} + 3968\, a_{12}\, b_{15}\, b_{24}$
$-384\, a_{12}\, b_{33} - 11904\, a_{12}\, b_{35} - 9216\, a_{12}\, b_{39} + 896\, a_{13}\, a_{14}\, b_{21}$
$-5248\, a_{13}\, b_{11}\, b_{23} + 4224\, a_{13}\, b_{11}\, b_{25} - 5248\, a_{13}\, b_{13}\, b_{21}$
$+4224\, a_{13}\, b_{15}\, b_{21} + 6144\, a_{13}\, b_{33} - 2944\, a_{14}\, a_{15}\, b_{21} - 4128\, a_{14}\, b_{11}\, b_{21}$
$+128\, a_{14}\, b_{11}\, b_{24} + 128\, a_{14}\, b_{14}\, b_{21} + 5760\, a_{14}\, b_{31} + 128\, a_{15}\, b_{11}\, b_{23}$
$-5248\, a_{15}\, b_{11}\, b_{25} + 128\, a_{15}\, b_{13}\, b_{21} - 5248\, a_{15}\, b_{15}\, b_{21}$
$-6144\, a_{15}\, b_{35} - 20736\, a_{16}\, b_{11}\, b_{21} + 27648\, a_{16}\, b_{31} - 3840\, a_{18}\, b_{11}\, b_{21}$
$+3072\, a_{18}\, b_{31} + 2880\, a_{21}\, b_{11}\, b_{18} - 41472\, a_{21}\, b_{21} - 22272\, a_{21}\, b_{24}$
$-6144\, a_{21}\, b_{28} - 384\, a_{22}\, b_{23} - 11904\, a_{22}\, b_{25} - 9216\, a_{22}\, b_{29}$
$+6144\, a_{23}\, b_{23} - 384\, a_{24}\, b_{21} - 6144\, a_{25}\, b_{25} - 3072\, a_{28}\, b_{21} + 6144\, a_{31}\, b_{18}$
$-8880\, b_{11}^2\, b_{23} + 7824\, b_{11}^2\, b_{25} + 2880\, b_{11}^2\, b_{29} - 27552\, b_{11}\, b_{13}\, b_{21}$
$-2944\, b_{11}\, b_{13}\, b_{24} - 2944\, b_{11}\, b_{14}\, b_{23} + 896\, b_{11}\, b_{14}\, b_{25}$
$+6240\, b_{11}\, b_{15}\, b_{21} + 896\, b_{11}\, b_{15}\, b_{24} + 8832\, b_{11}\, b_{33} - 2688\, b_{11}\, b_{35}$
$-9216\, b_{11}\, b_{39} - 2944\, b_{13}\, b_{14}\, b_{21} + 24192\, b_{13}\, b_{31} + 3072\, b_{13}\, b_{34}$
$+896\, b_{14}\, b_{15}\, b_{21} + 3072\, b_{14}\, b_{33} + 3072\, b_{14}\, b_{35} + 9600\, b_{15}\, b_{31}$

$$+3072\, b_{15}\, b_{34} + 8832\, b_{21}\, b_{23} - 2688\, b_{21}\, b_{25} - 9216\, b_{21}\, b_{29}$$
$$+3072\, b_{23}\, b_{24} + 3072\, b_{24}\, b_{25} - 9216\, b_{43} - 9216\, b_{45} - 3072\, b_{47} - 9216\, b_{49},$$
$$S_4 = \quad 154368\, a_{11}^2\, a_{12} - 55296\, a_{11}^2\, a_{13} + 86016\, a_{11}^2\, a_{15} - 73728\, a_{11}^2\, a_{17}$$
$$-73728\, a_{11}^2\, a_{19} - 656640\, a_{11}^2\, b_{11} - 193536\, a_{11}^2\, b_{14} - 73728\, a_{11}^2\, b_{16}$$
$$-73728\, a_{11}^2\, b_{18} + 16128\, a_{11}\, a_{12}\, a_{14} + 294912\, a_{11}\, a_{12}\, a_{16}$$
$$+49152\, a_{11}\, a_{12}\, a_{18} + 148224\, a_{11}\, a_{12}\, b_{13} - 96768\, a_{11}\, a_{12}\, b_{15}$$
$$-444015\, a_{11}\, a_{12} - 20480\, a_{11}\, a_{13}\, a_{14} - 20480\, a_{11}\, a_{13}\, b_{13}$$
$$-40960\, a_{11}\, a_{13}\, b_{15} + 1052448\, a_{11}\, a_{13} - 28672\, a_{11}\, a_{14}\, a_{15}$$
$$+24576\, a_{11}\, a_{14}\, a_{17} + 24576\, a_{11}\, a_{14}\, a_{19} + 228096\, a_{11}\, a_{14}\, b_{11}$$
$$+45056\, a_{11}\, a_{14}\, b_{14} + 24576\, a_{11}\, a_{14}\, b_{16} + 24576\, a_{11}\, a_{14}\, b_{18}$$
$$-28672\, a_{11}\, a_{15}\, b_{13} - 57344\, a_{11}\, a_{15}\, b_{15} + 136992\, a_{11}\, a_{15}$$
$$+294912 a_{11}\, a_{16}\, b_{11} + 24576 a_{11}\, a_{17}\, b_{13} + 49152\, a_{11}\, a_{17}\, b_{15} + 82080 a_{11}\, a_{17}$$
$$+49152\, a_{11}\, a_{18}\, b_{11} + 24576\, a_{11}\, a_{19}\, b_{13} + 49152\, a_{11}\, a_{19}\, b_{15}$$
$$-10656\, a_{11}\, a_{19} + 304128\, a_{11}\, a_{21} - 58368\, a_{11}\, a_{24} + 360192\, a_{11}\, b_{11}\, b_{13}$$
$$+327168\, a_{11}\, b_{11}\, b_{15} + 3026961\, a_{11}\, b_{11} + 45056\, a_{11}\, b_{13}\, b_{14}$$
$$+24576\, a_{11}\, b_{13}\, b_{16} + 24576\, a_{11}\, b_{13}\, b_{18} + 90112\, a_{11}\, b_{14}\, b_{15}$$
$$+943680\, a_{11}\, b_{14} + 49152\, a_{11}\, b_{15}\, b_{16} + 49152\, a_{11}\, b_{15}\, b_{18} + 206496\, a_{11}\, b_{16}$$
$$+279648\, a_{11}\, b_{18} - 58368\, a_{11}\, b_{23} - 116736\, a_{11}\, b_{25} - 22528\, a_{12}\, a_{14}^2$$
$$-98304\, a_{12}\, a_{14}\, a_{16} - 16384\, a_{12}\, a_{14}\, a_{18} - 69632\, a_{12}\, a_{14}\, b_{13}$$
$$-47104\, a_{12}\, a_{14}\, b_{15} - 42624\, a_{12}\, a_{14} - 98304\, a_{12}\, a_{16}\, b_{13}$$
$$-196608\, a_{12}\, a_{16}\, b_{15} - 72576\, a_{12}\, a_{16} - 16384\, a_{12}\, a_{18}\, b_{13}$$
$$-32768\, a_{12}\, a_{18}\, b_{15} - 143232\, a_{12}\, a_{18} - 47104\, a_{12}\, b_{13}^2 - 96256\, a_{12}\, b_{13}\, b_{15}$$
$$-31680\, a_{12}\, b_{13} - 4096\, a_{12}\, b_{15}^2 + 115776\, a_{12}\, b_{15} - 135168\, a_{13}\, a_{14}$$
$$-442368\, a_{13}\, a_{16} - 79872\, a_{13}\, a_{18} - 296448\, a_{13}\, b_{13} - 247296\, a_{13}\, b_{15}$$
$$-22528\, a_{14}^2\, b_{11} - 69120\, a_{14}\, a_{15} - 98304\, a_{14}\, a_{16}\, b_{11} - 4608\, a_{14}\, a_{17}$$
$$-16384\, a_{14}\, a_{18}\, b_{11} - 4608\, a_{14}\, a_{19} - 43008\, a_{14}\, a_{21} - 69632\, a_{14}\, b_{11}\, b_{13}$$
$$-47104\, a_{14}\, b_{11}\, b_{15} - 388224\, a_{14}\, b_{11} - 82944\, a_{14}\, b_{14} - 4608\, a_{14}\, b_{16}$$

$$-4608\, a_{14}\, b_{18} - 221184\, a_{15}\, a_{16} - 43008\, a_{15}\, a_{18} - 133632\, a_{15}\, b_{13}$$
$$-133632\, a_{15}\, b_{15} - 98304\, a_{16}\, b_{11}\, b_{13} - 196608\, a_{16}\, b_{11}\, b_{15}$$
$$-1067904\, a_{16}\, b_{11} - 221184\, a_{16}\, b_{14} - 4608\, a_{17}\, b_{13} - 23040\, a_{17}\, b_{15}$$
$$-16384\, a_{18}\, b_{11}\, b_{13} - 32768\, a_{18}\, b_{11}\, b_{15} - 336768\, a_{18}\, b_{11} - 43008\, a_{18}\, b_{14}$$
$$-4608\, a_{19}\, b_{13} - 41472\, a_{19}\, b_{15} - 43008\, a_{21}\, b_{13} - 86016\, a_{21}\, b_{15}$$
$$-362592\, a_{21} + 36864\, a_{24} + 41472\, a_{26} - 47104\, b_{11}\, b_{13}^2 - 96256\, b_{11}\, b_{13}\, b_{15}$$
$$-667584\, b_{11}\, b_{13} - 4096\, b_{11}\, b_{15}^2 - 533952\, b_{11}\, b_{15} - 147456\, b_{13}\, b_{14}$$
$$-4608\, b_{13}\, b_{16} - 4608\, b_{13}\, b_{18} - 161280\, b_{14}\, b_{15} - 23040\, b_{15}\, b_{16}$$
$$-41472\, b_{15}\, b_{18} + 36864\, b_{23} + 59904\, b_{25} - 96768\, b_{29},$$

$$S_5 = \; 409389\, a_{11}\, a_{12} + 151740\, a_{11}\, a_{13} + 233772\, a_{11}\, a_{15} - 18780\, a_{11}\, a_{17}$$
$$-112356\, a_{11}\, a_{19} - 823755\, a_{11}\, b_{11} - 420576\, a_{11}\, b_{14} - 331164\, a_{11}\, b_{16}$$
$$-174372\, a_{11}\, b_{18} - 14472\, a_{12}\, a_{14} - 93744\, a_{12}\, a_{16} + 36816\, a_{12}\, a_{18}$$
$$-92400\, a_{12}\, b_{13} - 70464\, a_{12}\, b_{15} + 28640\, a_{13}\, a_{14} + 46080\, a_{13}\, a_{16}$$
$$+28160\, a_{13}\, a_{18} - 42400\, a_{13}\, b_{13} - 51680\, a_{13}\, b_{15} + 7200\, a_{14}\, a_{15}$$
$$+7040\, a_{14}\, a_{17} + 32640\, a_{14}\, a_{19} + 139848\, a_{14}\, b_{11} + 48800\, a_{14}\, b_{14}$$
$$+32640\, a_{14}\, b_{16} + 7040\, a_{14}\, b_{18} - 46080\, a_{15}\, a_{16} + 12800\, a_{15}\, a_{18}$$
$$-35040\, a_{15}\, b_{13} - 20640\, a_{15}\, b_{15} + 7680\, a_{16}\, a_{17} + 69120\, a_{16}\, a_{19}$$
$$+390096\, a_{16}\, b_{11} + 138240\, a_{16}\, b_{14} + 69120\, a_{16}\, b_{16} + 7680\, a_{16}\, b_{18}$$
$$-2560\, a_{17}\, a_{18} + 14720\, a_{17}\, b_{13} + 12160\, a_{17}\, b_{15} + 7680\, a_{18}\, a_{19}$$
$$+125136\, a_{18}\, b_{11} + 23040\, a_{18}\, b_{14} + 7680\, a_{18}\, b_{16} - 2560\, a_{18}\, b_{18}$$
$$+40320\, a_{19}\, b_{13} + 55680\, a_{19}\, b_{15} - 14220\, a_{21} + 1680\, a_{24} + 51840\, a_{26}$$
$$+15360\, a_{28} + 205920\, b_{11}\, b_{13} + 175536\, b_{11}\, b_{15} + 98720\, b_{13}\, b_{14}$$
$$+78720\, b_{13}\, b_{16} + 22400\, b_{13}\, b_{18} + 85600\, b_{14}\, b_{15} + 71040\, b_{15}\, b_{16}$$
$$+42880\, b_{15}\, b_{18} + 24720\, b_{23} + 24240\, b_{25} + 63360\, b_{29}.$$

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