وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR-ANNABA UNIVERSITY

UNIVERSITE BADJI MOKHTAR ANNABA

جامعة باجي مختار

-عنابة-

Faculty of Sciences

Department of Mathematics

Year : 2021/2022

THESIS

Presented with a view to obtaining the doctorate degree

PRICING AND HEDGING OPTIONS

Option
Actuariat
By
MEZDOUD Zaineb

SUPERVISOR : REMITA Mohamed Riad Prof. U.B.M. ANNABA

In front of the jury

PRESIDENT :	ZEGHDOUDI Halim	Prof.	U.B.M. ANNABA
EXAMINER :	BOUTABIA Hacene	Prof.	U.B.M. ANNABA
EXAMINER :	KANDOUCI Abdeldjebbar	Prof.	U.Saida
EXAMINER :	BOUKHETALA Kamel	Prof.	USTHB
INVITED :	KEBIRI Omar	MCB.	BUT Cottbus (Allemagne)

وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR-ANNABA UNIVERSITY

UNIVERSITE BADJI MOKHTAR ANNABA

جامعة باجي مختار

-عنابة-

Faculté des Sciences

Département de Mathématiques

Année : 2021/2022

THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

ÉVALUATION ET COUVERTURE DES OPTIONS

Option
Actuariat
Par
MEZDOUD Zaineb

DIRECTEUR DE THÈSE : REMITA Mohamed Riad Prof. U.B.M. ANNABA

Devant le jury

PRESIDENT :	ZEGHDOUDI Halim	Prof.	U.B.M. ANNABA
EXAMINATEUR :	BOUTABIA Hacène	Prof.	U.B.M. ANNABA
EXAMINATEUR :	KANDOUCI Abdeldjebbar	Prof.	U.Saida
EXAMINATEUR :	BOUKHETALA Kamel	Prof.	USTHB
INVITE :	KEBIRI Omar	MCB.	BUT Cottbus (Allemagne)

Acknowledgement

First of all, I thank Allah for helping me to present this thesis. I am extremely thankful to my advisor, Prof. REMITA Mohamed Riad , who encouraged me to work in the field of pricing and hedging options . I really have the honor to complete this work under his supervision. I would like to thank Dr. KEBIRI Omar and Prof. HARTMANN Carsten for the patience to instruct me through the work, I am also so grateful for the precious time they offered me.
I am indebted to Prof. ZEGHDOUDI Halim, Prof. BOUTABIA Hacene, Prof. KANDOUCI Abdeldjebbar and Prof. BOUKHETALA Kamel. Last but not least, I am deeply appreciative of my beloved family who have always supported me through the difficult times. Thanks to you all for pushing me forward in every step in the journey of my life.

Dedication

This modest work is dedicated to

My loving husband Taqiyeddine.

My wonderful daughters Hidaya Belkis and Farah Halima.

My mother Zohra and my mother in-law Djimaa.

My father in-law Mossa and in the memory of my grand father Ayachi.

My sister Meriem and my sister in-law Afef .

My brothers Mohamed Said, Abobakar and my brother in-law Fahad.

My brother Abderrahman, his wife Chaima

and their sons Bahaeddine and Nadjmeddine.

My grand father in-law Mohamed Zobir and his wife Sohila.

My friends Dr. Khawla, Dr. Thara, Dr. Afef Amina.

In the memory of my mathematic teacher Boukhdana Mohamed.

Everyone who encouraged me, especially those who wanted to disappoint me.

Contents

Abstract

The thesis under study focus on pricing and hedging European options. We propose an α-hypergeometric model with uncertain volatility (UV) by which we derive a worst-case scenario for option pricing. The approach is based on the connection between a certain class of nonlinear partial differential equations of Hamilton-Jacobi-Bellman type (G-HJB), that govern the nonlinear expectation of the UV model [50] and that provide an alternative to the difficult model calibration problem of UV models, and second-order backward stochastic differential equations (2BSDEs). Moreover, we formulate a concrete model that is solved numerically using the deep learning method by Beck et al. [6] and exploiting the link between fully nonlinear G-HJB equations and 2BSDE. Finally we highlight several option Hedging strategies as Delta hedging, Delta-Sigma hedging and the Hedging by perturbation analysis.

Key words and phrases: Options, pricing models, α-hypergeometric stochastic volatility model, uncertain volatility model, 2BSDE, deep learning based discretisation of 2BSDE, hedging strategies.

ملخص

تركز الأطروحة قيد الدراسة على التسعير والتحوط في الخيارات الأوروبية. نقترح نموذج «α-hypergeometric» مع تقلب غير مؤكد (UV) نشتق من خلاله سيناريو أسوأ حالة لتسعير الخيار. يعتمد النهج على الارتباط بين فئة معينة من المعادلات التفاضلية الجزئية غير الخطية من نوع هاملتون-جاكوبي-بيلمان (G-HJB)، والتي تحكم التوقع غير الخطي لنموذج التقلب الغير مؤكد (UV) [50] والتي توفر بديلاً للنموذج الصعب مشكلة معايرة نماذج التقلب الغير مؤكد (UV)، والمعادلات التفاضلية العشوائية العكسية من الدرجة الثانية (2BSDEs). علاوة على ذلك، نقوم بصياغة نموذج ملموس يتم حله عدديًا باستخدام طريقة «deep learning» بواسطة Beck et al. [6] واستغلال الارتباط بين معادلاتG-HJB اللاخطية و2BSDE. أخيرًا، نسلط الضوء على العديد من استراتيجيات التحوط في الخيارات مثل تحوط دلتا، وتحوط دلتا-سيجما، والتحوط عن طريق تحليل الاضطراب.

الكلمات والعبارات المفتاحية: الخيارات، نماذج التسعير، نموذج التقلب العشوائي «α-hypergeometric»، نموذج التقلب غير المؤكد، 2BSDE، لتقدير القائم على «deep learning» لـ 2BSDE استراتيجيات التحوط.

Résumé

Dans la présente thèse, nous nous focalisons sur l'évaluation et la couverture des options européennes. Nous proposons un modèle α-hypergéométrique à volatilité incertaine (UV) par lequel nous dérivons un pire scénario pour la valorisation des options. L'approche est basée sur la connexion entre une certaine classe d'équations aux dérivées partielles non linéaires du type Hamilton-Jacobi-Bellman (G-HJB), qui gouvernent l'espérance non linéaire du modèle UV [50] et qui fournissent une alternative au problème difficile du calibrage des modèles UV et équations différentielles stochastiques rétrogrades du second ordre (2BSDEs). De plus, nous formulons un modèle concret qui est résolu numériquement en utilisant la méthode « Deep Learning » de Beck et al. [6] et en exploitant le lien entre les équations G-HJB entièrement non linéaires et 2BSDE. Enfin, nous mettons en évidence plusieurs stratégies de couverture de l'option telle que, la couverture Delta, la couverture Sigma et la couverture par l'analyse de perturbation.

Mots et phrases clés : Options, modèles d'évaluation, modèle de α-hypergéométrique à volatilité stochastique, modèle de volatilité incertaine, 2BSDE, discrétisation de 2BSDE basée sur « deep learning », stratégies de couverture.

Introdution

A derivative can be defined as a financial instrument whose value depends on the values of other, more basic, underlying variables. Very often the variables underlying derivatives are the prices of traded assets. A stock option, for example, is a derivative whose value is dependent on the price of a stock. The exercise of the option allows its holder to realize a profit equal to $X_T - K$, by buying the derivative at the strike price K and reselling it on the market at the price X_T. We see that at maturity T, the value of the call is given by the quantity:

$$(X_T - K)^+ = \max(X_T - K, 0)$$

For the seller of the option, it is a question, in the event of exercise, of being able to provide a derivative at the price K, and, consequently of being able to produce at maturity a wealth equal to $(X_T - K)^+$. At the time of the sale of the option, which we will take as the origin of the times, the price X_T is unknown and two questions arise:

1- How much should the buyer of the option pay, in other words how to evaluate at the instant $t = 0$ a wealth $(X_T - K)^+$ available at the date T? This is the pricing problem.

2- How will the seller, who receives the premium at time 0, manage to produce wealth $(X_T - K)^+$ on date T? It's the hedging problem.

The classical option pricing problem based on the seminal work by Black and Scholes [10] assumes that the volatility of the underlying asset is constant over time. While the Black-Scholes model is still considered an important paradigm for option pricing, there is plenty of empirical evidences that the assumption of constant volatility is not adequate. In order to come up with more realistic models, various strategies have been proposed to treat the volatility of asset prices as a stochastic process [35]. One of the most famous representatives of the large class of stochastic volatility models is the Heston model [32] that has become the basis of many other models, such as jump diffusion models [39], and various forms of uncertain volatility models (UVM) such as [3, 24, 28], all of which can be considered as extensions of the Black-Scholes model.

In the Heston model, the price hits zero in finite time unless the Filler condition is imposed. As a consequence, the underlying Optimization problems are typically endowed with constraints, which pose Additional problems in model calibration. In view of this, the α-hypergeometric stochastic volatility model has been introduced by Da Fonseca and Martini [16] to ensure strict positivity of volatility. On the other hand, the uncertain volatility model developed by [3] has received intensive attention in mathematical finance for risk management purposes. Where they are proposed for pricing and hedging derivative securities and option portfolios in an environment where the volatility is not known precisely, but is assumed instead to lie between two extreme values $\sigma_{\min}$, and $\sigma_{\max}$ [14] [57]. In our work we consider an α-hypergeometric stochastic volatility model, also, we focus on the uncertain volatility model.

One of the common features of all stochastic volatility models is that the volatility process can only be indirectly observed through the asset price, which poses specific challenges for the parameter estimation (or: *calibration*) of these models. Standard approaches are based on maximum likelihood estimation using (filtered) time series data [1, 36] or fitting of the implied volatility surface [25, 27]. While Jean-Pierre FOUQUE presented in his book [23] the multiscale perturbation analysis in the case of European options, where he uses a combination of singular and regular perturbation techniques to derive approximations for the option prices. Furthermore, in the Markovian framework , option prices are obtained as solutions of linear (or nonlinear) partial differential equations [11] [19]. The solution of the partial differential equations have interesting connections to the solution of the backward stochastic differential equations (BSDEs)and to the solution of the forward-backward stochastic differential equations (FBSDEs) , many scientific articles have dealt with this link, mention to [2][33][34] , whereas Touzi [13] introduced the second-order backward stochastic differential equations (2BSDEs) and show how they are related to fully nonlinear parabolic PDEs see also [46][53].

The Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in the underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The most common of the Greeks are the first order derivatives: delta, Vega, theta and rho as well as gamma, a second-order derivative of the value function.

The Greeks of option play a crucial role in trading and managing portfolios of option. the practitioners use them to quantify the different aspects of the risk inherent in their option portfolios. They attempt to make the portfolio immune to small changes in the price of the underlying asset (delta/gamma hedging) and its volatility (sigma hedging). This is one of the hedging strategies.

This thesis is organized as follows: In chapter 1 we give a brief introduction about options, we present the multiscale perturbation analysis in the case of European options and we highlights on some financial pricing models such as Black-Scholes model and Heston model. in chapter 2 we focus in our chosen model, the α-hypergeometric stochastic volatility model, we present some basic properties to this model, we study the pricing option by using the Mellin transformation method also the approximation of the solution of the partial differential equation corresponding to the model. In chapter 3 we formulate the worst-case price scenario and the corresponding fully nonlinear partial differential equation of G-Hamilton-Jacobi-Bellman type (G-HJB equation), and we derive some basic properties such as moment bounds and the convergence of the worst-case price scenario as $\delta \to 0$ ($\delta > 0$ the rescale time); this chapter also includes some technical results such as convergence of the second derivatives, we consider the formulation of the fully nonlinear PDE for the nonlinear expectation of the price process and derive a uniform corrector result for the limit $\delta \to 0$. We moreover formulate a concrete model that is solved numerically using the deep learning method by Beck et al. [6] and exploiting the link between fully nonlinear G-HJB equations and 2BSDE. In chapter 4 we present some hedging strategies such as Pure delta hedging, Delta-sigma hedging and hedging by perturbation analysis. In the end of this thesis, we give the conclusion and the perspective also the bibliography.

Chapter 1

Background

In this chapter, we give a brief introduction about options, we present the multiscale perturbation analysis in the case of European options [23],[59] and we highlight on some financial pricing models such as Black-Scholes model [10],[17] and Heston model [32],[35].

1.1 Around Options

Derivatives (contingent claims) are contracts based on the underlying asset price (X_t). They go back a long, long way. One of the earliest mentions of derivatives, by Aristotle (384-322 BCE) in his *Politics*, describes the successful trading of the noted Greek philosopher Thales (mid-620s to mid-540s BCE).

> *"as in the contrivance of Thales the Milesian, when they reviled him for his poverty, as if the study of philosophy was useless; for they say that he, perceiving by his skill in astrology that there would be great plenty of olives that year, while it was yet winter, having got a little money he gave earnest for all the oil works that were in Miletus and Chios, which he hired al a low price, there being no one to bid against him; but when the season came for making oil, many persons wanting them, he all at once let them upon what terms he pleased; and raising a large sum of money by that means, conviced them that it was easy for philosophers to be rich if they chose it."*[5]

As seen from [5], Thales gambled asymmetrically on the values of the olives, taking full advantage if the olives had a higher value than the rental of the contract but only losing the earnest money if not. This asymmetric bet on pricing is the essence of the options.

Options are contracts whose price is derived from the current state of the underlying asset. The price in the contract is known as the exercise price or strike price K; the date in the contract is known as the expiration date or maturity T.

There are two types of option: a call option gives the holder the right to buy the underlying asset by a certain date for a certain price; a put option gives the holder the right to sell the underlying asset by a certain date for a certain price.

American options can be exercised at any time up to the expiration date, however European options can be exercised only on the maturity itself witch we will be interested.

The payoff of European call option is

$$h(X_T) = (X_T - K)^+ = \begin{cases} X_T - K & if \quad X_T > K, \\ 0 & if \quad X_T \leq K, \end{cases}$$

since in the first case the holder will exercise the option and make a profit $X_T - K$, by buying the stock for K and selling it immediately at the market price X_T. In the second case the option is not exercised, where the market price of the asset is less than the strike price.

Similarly, the payoff of European put option is

$$h(X_T) = (K - X_T)^+ = \begin{cases} K - X_T & if \quad X_T < K, \\ 0 & if \quad X_T \geq K, \end{cases}$$

in the first case buying the stock at the market price and exercising the put option yields a profit of $K - X_T$, and in the second case the option is simply not exercised.

1.2 important theorems

Feynman-Kac Theorem

The Feynman-Kac formula states that a probabilistic expectation value with respect to some Ito-diffusion can be obtained as a solution of an associated PDE. It may be formulated as follows

Theorem 1. *Let $X(t)$ be a stochastic process driven by a stochastic differential equation*

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dB(t),$$

with an initial value at initial time t , $X(t) = x$, and let $Y(t,x) \in \mathcal{L}^2$ be a deterministic function which satisfies

$$\int_t^T E\left[\sigma(s,X(s))\frac{\partial Y}{\partial x}(s,X(s))\right]^2 ds < \infty,$$

with boundary condition $Y(T,X(T)) = f(X(T))$.
If the function $Y(t,x)$ is a solution to the boundary value problem

$$\frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 Y}{\partial x^2} + \mu(t,x)\frac{\partial Y}{\partial x} - g(t,x)Y(t,x) = 0,$$

then Y has the following representation

$$Y(t,x) = E\left[\exp\left(-\int_t^T g(s,X(s))ds\right)f(X(T)) \mid X(t) = x\right].$$

Girsanov's Theorem

The Girsanov theorem describes the impact of a probability change on stochastic calculus. Let $(\Omega,(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a filtered probability space.We assume that $(\mathcal{F}_t)_{t\geq 0}$ is the usual completion of the filtration of a Brownian motion $(B_t)_{t\geq 0}$. Let $\mathbb{Q}$ be a probability measure on $\mathcal{F}_\infty$ which is equivalent to $\mathbb{P}$. We denote by D the density of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Theorem 2 (Girsanov theorem)**.** *There exists a progressively measurable process $(\Theta_t)_{t\geq 0}$ such that for every $t \geq 0$,*

$$\mathbb{P}\left(\int_0^t \Theta_s^2 ds < +\infty\right) = 1,$$

and

$$\mathbb{E}(D \mid \mathcal{F}_t) = \exp\left(\int_0^t \Theta_s dB_s - \frac{1}{2}\int_0^t \Theta_s^2 ds\right).$$

Moreover, the process $B_t - \int_0^t \Theta_s ds$ is a Brownian motion on the filtered probability space $(\Omega,(\mathcal{F}_t)_{t\geq 0},\mathbb{Q})$. As a consequence, a continuous and adapted process $(X_t)_{t\geq 0}$ is a $\mathbb{P}-$semimartingale if and only if it is a $\mathbb{Q}-$semimartingale.

1.3 First-Order Perturbation Theory

In this section, we present briefly the multiscale perturbation analysis in the case of European options [23]. These models have two stochastic volatility factors, one fast and one slow, but in our chosen model we focus only on the slow stochastic volatility factor [59].

Our objective is to price European derivatives and therefore we consider these models under a risk-neutral pricing probability measure $\mathbb{Q}$, we then write the pricing partial differential equations; where the solution of this end is considered as option prices, and we derive the first order approximation.

1.3.1 Option Pricing Under Slow Stochastic Volatility

The Model Under Risk-Neutral Measure $\mathbb{Q}$

Under risk-neutral pricing measure $\mathbb{Q}$, we give the system of stochastic equations

$$\begin{aligned} dX_t &= rX_t dt + f(V_t)X_t dW_t^1, \\ dV_t &= (\delta c(V_t) - \sqrt{\delta} g(V_t)\Lambda(V_t))dt + \sqrt{\delta} g(V_t) dW_t^2, \end{aligned} \tag{1.1}$$

which present the evolution of the price X_t of the underlying asset where:

1) $\mathbb{Q}$-standard Brownian motions (W_t^1, W_t^2) are correlated, $d < W_t^1, W_t^2 >= \rho dt \quad |\rho| < 1$.

2) The volatility $f(V_t)$ of the underlying asset X_t is driven by the slow volatility factor V_t, where f is a positive function and smooth in v.

3) $r \in \mathbb{R}$ is the instantaneous interest rate.

4) The value X_t of the underlying asset remains positive, as can be seen by applying Itô's formula to deduce

$$X_t = X_0 \exp\left\{ \int_0^t (r - \frac{1}{2} f^2(V_t))ds + \int_0^t f(V_s) dW_s^1 \right\}.$$

5) $\delta > 0$ corresponds the long time scale $\frac{1}{\delta}$ of the slow volatility factor V_t.

6) The coefficients $c(V)$ and $g(V)$ describe the dynamics of the process V_t.

7) $\Lambda(V)$ is combined market prices of volatility risk which determine the risk-neutral pricing measure $\mathbb{Q}$.

The price of the European option $P^{\delta}(t,x,v)$ is a function of the $t < T$, The value X_t of the under asset and the value V_t of the slow volatility factor.

$$P^{\delta}(t,X_t,V_t) = E\left\{e^{-r(T-t)}h(X_T)|X_t,V_t\right\}, \tag{1.2}$$

where under the risk-neutral probability $\mathbb{Q}$, the process (X_t,V_t) is Markovian , $E\{\}$ the expectation value depends on the parameter r and the functions (f,c,g,Λ), and $h(X_T)$ the payoff function of the European option. In order to use 1.2, the parameter r and the functions (f,c,g,Λ) need to be fully specified and estimated, so we use the perturbation approach which will simplify these complicated issues by approximating the price P^{δ} by a quantity depends only on a few group market parameters. This approximation is of the form

$$P^{\delta} = P_0 + P_1^{\delta}, \tag{1.3}$$

where P_0 is Black-Scholes price and P_1^{δ} is the first-order slow scale correction.

1.3.2 Pricing Partial Differential Equation

The function $P^{\delta}(t,x,v)$ defined in 1.2 is also characterized as the solution of the partial differential equation

$$\frac{\partial P^{\delta}}{\partial t} + \mathcal{L}(X,V)P^{\delta} - rP^{\delta} = 0, \tag{1.4}$$

with the terminal condition $P^{\delta}(T,x,v) = h(X_T)$ and where $\mathcal{L}(X,V)$ denotes the infinitesimal generator of the Markov's process (X_t,V_t) given by 1.1.

We define the operator $\mathcal{L}^{\delta}$ by

$$\mathcal{L}^{\delta} = \frac{\partial}{\partial t} + \mathcal{L}(X,V) - r,$$

so we can be written the equation 1.4 and its terminal condition as

$$\begin{aligned} \mathcal{L}^{\delta}P^{\delta} &= 0, & (1.5)\\ P^{\delta}(T,x,v) &= h(X_T). & (1.6) \end{aligned}$$

When the parameter δ is small, it's appropriate to write the operator $\mathcal{L}^{\delta}$ as a sum of components with a view to derive approximation for P^{δ}. This decomposition is

$$\mathcal{L}^{\delta} = \mathcal{L}_{\mathcal{BS}} + \sqrt{\delta}M_1 + \delta M_2 + \ldots, \tag{1.7}$$

where

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2} f^2(v) x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - .), \tag{1.8}$$

$$M_1 = g(v)(\rho f(v) x \frac{\partial^2}{\partial x \partial v} - \Lambda(v) \frac{\partial}{\partial v}), \tag{1.9}$$

$$M_2 = \frac{1}{2} g^2(v) \frac{\partial^2}{\partial v^2} + c(v) \frac{\partial}{\partial v}, \tag{1.10}$$

note that

- $\mathcal{L}_{BS}$: contains the time derivative and is the Black-Scholes operator at the volatility level $f(V)$.
- M_1: contains the mixed derivative due to covariation between X and V and the first derivative with respect to V due to the market price volatility risk Λ.
- M_2: in the infinitesimal generator of process V under the physical measure $\mathbb{P}$.

Lemma 3 ([23])**.** *The process V_t with infinitesimal generator M_2 given by 1.10 admits moments of any order uniformly in $t \leq T$*

$$\sup_{t \leq T} \mathbb{E}\left\{|V_t|^k\right\} \leq C(T,k).$$

In 1.7 we notice that in the small δ limit, the operator terms associated with this parameter are small, it gives rise to regular perturbation problem about Black-Scholes operator $\mathcal{L}_{BS}$ [22].

Now we expand P^δ in powers of $\sqrt{\delta}$ to give a formal derivation of the price approximation when δ is small.

$$P^\delta = P_0 + \sqrt{\delta} P_1 + \delta P_2 + \dots, \tag{1.11}$$

before using 1.7 to collect the terms in the increasing powers of δ , we insert the expansion 1.11 into the partial differential equation 1.5 and also the terminal condition 1.6, so

$$\mathcal{L}_{BS} P_0 + \sqrt{\delta}\{\mathcal{L}_{BS} P_1 + M_1 P_0\} + \cdots = 0. \tag{1.12}$$

Equating to zero first terms independent of δ and then the terms in $\sqrt{\delta}$ in 1.12, and similarly in the terminal condition leads us to define P_0 and P_1 as follows

Definition 1. *We define P_0 as the unique solution to the problem*

$$\mathcal{L}_{BS} P_0 = 0 \tag{1.13}$$

$$P_0(T,x,v) = h(x). \tag{1.14}$$

Definition 2. *The next term P_1 is defined as the unique solution to the problem*

$$\mathcal{L}_{BS} P_1 = -M_1 P_0 \tag{1.15}$$

$$P_1(T,x,v) = 0. \tag{1.16}$$

Thus P_0 is the solution of the homogeneous linear parabolic PDE 1.13 with $h(x)$ terminal condition, while P_1, the first-order term in $\sqrt{\delta}$, solves a similar problem but with a source term and zero terminal condition [19].

1.4 Financial Pricing Models

In this section, we will highlight on some financial pricing model [38] such as Black-Scholes model [10] and the Heston model [32].
As we know option pricing is the most famous problem in financial market [12, 27, 28] , which is based on Black-Scholes model [10] where the volatility is constant over time. So we will define this model and deduce the Black-Scholes formula from its partial differential equation.
The imposition about the volatility constant in this model is unrealistic, so the stochastic volatility models are created to erase this problem [1, 35, 42]. The most well-known is the Heston model [32], where we will define it and also calibrate its partial differential equation to pricing European option.

1.4.1 Black-Scholes Model

The Black Scholes model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes [10]. Under Black-Scholes model, In the risk-neutral measure $\mathbb{Q}$, it is assumed that the asset price follows the stochastic process:

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

where $r \in \mathbb{R}$ is a risk-free rate, $\sigma > 0$ the volatility and W_t is the Brownian motion.

The Black-Scholes Partial Differential Equation

Black-Scholes partial differential equation satisfied by $C(X,t)$

$$-rC(X,t) + r\frac{\partial C}{\partial x}(X,t)X + \frac{\partial C}{\partial t}(X,t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(X,t)\sigma^2 X^2 = 0,$$

such that

$$C(X,t) = \begin{cases} 0 & , \quad t \in [0,T), \\ \max\{0, X-K\} & , \quad t = T, \end{cases}$$

by applying Feynman-Kac formula [38] we obtain

$$C(X,t) = e^{-r(T-t)} E\left(\max\{0, X(T)-K\} / F_t\right), \tag{1.17}$$

so,we have

$$X(t) = X_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W(t)},$$

and

$$\ln \frac{X(t)}{X_0} = (r - \frac{\sigma^2}{2})t + \sigma W(t),$$

the expectation in 1.17 is possible to compute it, because we know the distribution of $X(T)$. This computation gives the next results

$$C(X,t) = XN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{(T-t)}}\left[\ln(\frac{X}{K}) + (r + \frac{\sigma^2}{2})(T-t)\right], \tag{1.18}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{1.19}$$

The result is known as the Black-Scholes formula.

Remark 1. *For the put option, we obtain the following result*

$$P(X,t) = Ke^{-r(T-t)}N(-d_2) - XN(-d_1),$$

N: standard Gaussian distribution.

1.4.2 Heston Model

As a realistic models for the motion of asset prices, models ambiguity have been proposed (we advise [21, 39, 42]) such the Heston model [32], it belongs to class of the stochastic volatility models. In the Heston model, the stock price and the volatility process which under the Feller condition $2\kappa\theta > \sigma^2$ is strictly positive, given by the following SDEs

$$dX_t = X_t r dt + X_t\sqrt{V_t}dW_t^1, \tag{1.20}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2, \tag{1.21}$$

where W_t^1 and W_t^2 are standard Brownian processes with correlation coefficient $\rho > 0$ given by $dW_t^1 dW_t^2 = \rho dt$ and

- θ: the long-run average variance of the price; as t tends to infinity, the expected value of V_t tends to θ.
- κ: the rate at which V_t reverts to θ.
- σ: the 'vol of vol', which determines the volatility of V_t.

European Option Pricing Under the Heston Model

The Girsanov's theorem allow as incorporate the market price of volatility, λ, to switch from probability measure to the risk neutral measure [38]. By using Fiorentini G, Leon A and Rubio G [20]; the premium of volatility risk $\lambda(t, X_t, V_t)$ will be defined as

$$\lambda(t, X_t, V_t) = \lambda V_t.$$

Black-Scholes, Merton (1973) [10] demonstrated that under a market free arbitrage, the value of any asset $U := U(t, X_t, V_t)$ must satisfy the PDE

$$\frac{1}{2}v^2x^2\frac{\partial^2 U}{\partial x^2} + \rho\sigma v x\frac{\partial^2 U}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + rx\frac{\partial U}{\partial x} + \{\kappa[\theta - v] - \lambda v\}\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0 \quad (1.22)$$

European call option with a strike price K and maturing at time T is succumb to the conditions bellow

$$\begin{aligned}
U(X,V,T) &= \max(0, X-K), \\
U(0,V,t) &= 0, \\
\frac{\partial U}{\partial x}(\infty,V,t) &= 1, \\
rx\frac{\partial U}{\partial x}(X,0,t) + \kappa\theta\frac{\partial U}{\partial v}(X,0,t) - rU(X,0,t) + \frac{\partial U}{\partial t}(X,0,t) &= 0, \\
U(X,\infty,t) &= X.
\end{aligned} \quad (1.23)$$

Due to the similar structure to the Black-Scholes model, Heston (1993) [32] suggest that the solution should be of a similar form as

$$C(X,V,t) = XP_1 - KP(X,T)P_2, \quad (1.24)$$

where the first term is the present value of the underlying asset, and the second term is the present value of the strike price. Substituting the proposed solution 1.24 into the original PDE 1.22, shows that P_1 and P_2 must satisfy the PDEs

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\rho\sigma v\frac{\partial P_j}{\partial x\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2}+(r+u_j v)\frac{\partial P_j}{\partial x}+(a_j-b_j v)\frac{\partial P_j}{\partial v}+\frac{\partial P_j}{\partial t}=0, \tag{1.25}$$

where $u_1=\frac{1}{2}$, $u_2=-\frac{1}{2}$, $a=\kappa\theta$, $b_1=\kappa+\lambda-\rho\sigma$ and $b_2=\kappa+\lambda$ for $j=1,2$.
The European option price satisfies the boundary condition 1.23 and the PDEs 1.25 are constrained to the terminal condition

$$P_j(X,V,T,\ln[K])=\mathbf{1}_{\{X>\ln[K]\}}. \tag{1.26}$$

Then characteristic function solution is

$$P_j(X,V,t,N)=e^{C(T-t,N)+D(T-t,N)v+iNX}, \tag{1.27}$$

where

$$\begin{aligned} C(\tau,N) &= rNi\tau+\frac{a}{\sigma^2}(b_j-\rho\sigma Ni+d)\tau-2\ln\left[\frac{1-ge^{d\tau}}{1-g}\right], \\ D(\tau,N) &= \frac{b_j-\rho\sigma Ni+d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right], \\ g &= \frac{b_j-\rho\sigma Ni+d}{b_j-\rho\sigma Ni-d}, \\ d &= \sqrt{(\rho\sigma Ni-b_j)^2-\sigma^2(2u_jNi-N^2)}. \end{aligned} \tag{1.28}$$

After some conversion of the characteristic function 1.27, we obtain the conditional probability that the option expires in-the-money

$$P_j(X,V,T,\ln[K])=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iN\ln[K]}f_j(X,V,T,N)}{iN}\right]dN. \tag{1.29}$$

The final solution consists of 1.24, 1.27 and 1.29. The conditional probability 1.29 may be interpreted as "adjusted" or "risk-neutralized" probability. The integrand in equation 1.29 is a "smooth function that decays rapidly" and it is integrable as shown by Kendall and Stuart (1977) [37]. Its integrand cannot be evaluated analytically, but it can be approximated numerically [1] [36].

Chapter 2

The α-Hypergeometric Stochastic Volatility Model

Stochastic volatility models [38][42] have been introduced as realistic models for the motion of asset prices in financial markets. The most well-known of such models is the Heston model [32], which however has one major drawback as its stochastic volatility may reach zero in finite time unless one imposes the Feller condition, and this poses potential problems in model calibration.
In view of this, the α-hypergeometric stochastic volatility model has been introduced by Da Fonseca and Martini [16] to ensure strict positivity of volatility.

2.1 Generalities About α-Hypergeometric Model

In the α-hypergeometric model the dynamics of the asset price X_t at time t and the log-volatility V_t are governed by

$$dX_t = rX_t dt + e^{V_t} X_t dW_t^1, \tag{2.1}$$

$$dV_t = (a - be^{\alpha V_t})dt + \sigma dW_t^2, \tag{2.2}$$

where $b, \alpha, \sigma > 0$, $a \in \mathbb{R}$ are constants, and W_t^1 , W_t^2 are Brownian motions with correlation ρ.

2.1.1 Basic Properties

Dependency on α

$$\alpha V_{V_0,\alpha,a,b,\sigma} = V_{\alpha V_0,1,\alpha a,\alpha b,\alpha\sigma},$$

Noise Limit

$(V_t)_{t\geq 0}$ the solution of the stochastic differential equation 2.5,

$$V_t - V_0 + b\int_0^t e^{\alpha V_s} ds = at + \sigma W_t^2,$$

$I(t) = \int_0^t e^{\alpha V_s} ds$, we note that $\frac{dI(t)}{dt} = e^{\alpha V_t}$ so that

$$\ln\frac{dI(t)}{dt} + \alpha b I(t) = \alpha(V_0 + at + \sigma W_t^2),$$

$$\frac{dI(t)}{dt} e^{(\alpha b I(t))} = e^{\alpha(V_0+at+\sigma W_t^2)},$$

which gives in turn by integrating

$$e^{\alpha b I(t)} = 1 + \alpha b\int_0^t e^{\alpha(V_0+as+\sigma W_s^2)} ds.$$

Finally, we get

$$I(t) = \frac{\ln\left(1 + \alpha b\int_0^t e^{\alpha(V_0+as+\sigma W_s^2)}\right) ds}{\alpha b}. \tag{2.3}$$

Noiseless Limit

when $\sigma = 0$, the above accounts are standing, and from it the formula 2.3 simplifies to

$$I(t) = \frac{\ln\left(1 + \frac{b}{a} e^{\alpha V_0}(e^{\alpha a t} - 1)\right)}{\alpha b},$$

in particular $\frac{I(t)}{t} \to \frac{a}{b}$ when $t \to \infty$.

For Negative b

It follows from this scaling property that the SDE has a well-defined solution when b and α are negative. Also, from the expression of I(t) the solution is well defined up to the stopping time, if b < 0 and $\alpha > 0$ [16].

2.1.2 Martingality of X_t

In order to show certain martingale properties of X_t, we need the following Lemma.

Lemma 1 ([7]). *For $a \in \mathbb{R}$, $b > 0$ and $c \geq 0$, a Brownian motion W_t and*

$$L_t := \int_0^t a - be^{cW_s} dW_s,$$

for $t \geq 0$, the stochastic exponencial $\mathcal{E}(L)$ is a martingale.

Theorem 4 ([16]). *X_t in the α-hypergeometric model is a martingale if only if $\alpha \geq 2$ or $\alpha < 2$ and one of the following conditions is fulfilled*

- $\rho \leq 0$.
- $\alpha > 1$.
- $\alpha = 1$.
- $b \geq \rho\sigma$.

Theorem 5 ([7]). *Let X_t be a martingale in the α-hypergeometric model, then $X_t \in L^\theta$*

$$\mathbb{E}(\sup_{0 \leq s \leq t} X_s^\theta) < \infty,$$

holds for all $t > 0$ in the cases

- $\alpha < 1$, $\rho < 0$, *and* $1 < \theta \leq \frac{1}{1-\rho^2}$.
- $\alpha = 1$, $b > \rho\sigma$, *and* $1 < \theta \leq \frac{\sigma - 2b\rho + \sqrt{(\sigma - 2b\rho)^2 + 4b^2(1-\rho^2)}}{2\sigma(1-\rho^2)}$.
- $\alpha > 1$ *and* $\theta > 1$.

Conversely

$$\mathbb{E}(X_t^\theta) = \infty,$$

holds for all $t > 0$ in the cases

- $\alpha < 1$, $\rho = 0$, *and* $\theta > 1$.
- $\alpha < 1$, $\rho < 0$, *and* $\theta > \frac{1}{1-\rho^2}$.
- $\alpha = 1$, $b = \rho\sigma$, *and* $\theta > 1$.

2.2 Pricing Option with Mellin Transformation Method

In this section, we will pricing European option with Mellin transformation method, before that we need to compute certain transforms of V_t also X_t (see [7]).

2.2.1 Moment Transform and Laplace Moment Transform of V_t

Proposition 6. *In the 1-hypergeometric model with $\theta > 0$ and $\lambda > \frac{\theta^2}{2\sigma^2} + a\theta$ the Laplace transform in time of the moment transform of V_t is given by*

$$\int_0^\infty e^{-\lambda t}\mathbb{E}(e^{\theta V_t})dt = \frac{1}{\sigma^2}e^{\left(-\frac{a}{\sigma^2}V_0+\frac{b}{\sigma^2}e^{V_0}\right)}(J_1+J_2),$$

with

$$\begin{aligned}
J_1 &= 2\frac{\Gamma(a_1-1)}{\Gamma(b_1)}e^{-\frac{z_0}{2}}z_0^{\eta}U(a_1-1,b_1;z_0)(2\nu_2)^{-\theta-\frac{a}{\sigma^2}}I_1,\\
J_2 &= 2\frac{\Gamma(a_1-1)}{\Gamma(b_1)}e^{-\frac{z_0}{2}}z_0^{\eta}M(a_1-1,b_1;z_0)(2\nu_2)^{-\theta-\frac{a}{\sigma^2}}I_2,\\
I_1 &= \frac{z_0^{b_1-a_1+\theta}}{b_1-a_1+\theta^2}F_2([b_1-a_1+1,b_1-a_1+\theta][b_1-a_1+\theta+1,b_1],-z_0),\\
I_2 &= \frac{\Gamma(b_1-a_1+\theta)\Gamma(\theta-a_1+1)}{\Gamma(\theta)}\\
&- z_0^{\theta-a_1+1}\frac{\Gamma(b_1-1)_2F_2([2-a_1,1+\theta-a_1][2-b_1,2+\theta-a_1],-z_0)}{\Gamma(a_1-1)(1+\theta-a_1)}\\
&- z_0^{\theta-a_1+b_1}\frac{\Gamma(1-b_1)_2F_2([1-a_1+b_1,\theta-a_1+b_1][b_1,1+\theta-a_1+b_1],-z_0)}{\Gamma(a_1-b_1)(\theta-a_1+b_1)},
\end{aligned}$$

where $a_1-1=\eta-\frac{a}{\sigma^2}$, $b_1=1+2\eta$, $\nu_2=\frac{b}{\sigma^2}$, $z_0=2\nu_2e^{V_0}$, $\eta^2=\frac{a^2}{\sigma^4}+\frac{2\lambda}{\sigma^2}$,
U the confluent hypergeometric function and M Whittaker function.

Theorem 7. *In the α-hypergeometric model the Laplace moment transform of V_1 is given by*

$$\int_0^\infty e^{-\lambda t}\mathbb{E}(e^{\theta V_t})dt = \int_0^\infty e^{-\lambda t}\mathbb{E}(e^{\frac{\theta}{\alpha}\tilde{V}_t})dt,$$

where the process $\tilde{V}_t$ with starting value $\tilde{V}_0=\alpha V_0$ follows the SDE

$$d\tilde{V}_t = (\alpha a-\alpha b e^{\tilde{V}_t})dt+\alpha\sigma dW_t,$$

which can be calculated using Proposition 6.

2.2.2 Transforms of X_t

As we have seen for the process V_t, we are able to compute the Laplace moment transform for $\alpha > 0$, but unfortunately, we cannot use the same strategy for X_t. So, in the next Theorem which compute the Laplace transform in time of the Mellin transform of X_t, we focus on the 1-hypergeometric model, because it lies in the class of solvable stochastic volatility model [7].

Theorem 8. *Assume X_t and V_t be given by the 1-hpergeometric model with $\rho\sigma < b$. Furthermore let $\theta \in (\theta^*, \theta_+)$ where*

$$\begin{aligned} \theta^* &= \frac{9\sigma - 16b\rho + 3\sqrt{32b^2 + 9\sigma^2 - 32b\rho\sigma}}{2\sigma(9 - 8\rho^2)}, \\ \theta_+ &= \frac{\sigma - 2b\rho + \sqrt{(\sigma - 2b\rho)^2 + 4b^2(1-\rho^2)}}{2\sigma(1-\rho^2)}, \end{aligned} \tag{2.4}$$

and $\lambda > 0$ such that

$$\left(\frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}\right)^{\frac{1}{2}} - \frac{(b - \theta\rho\sigma)(\frac{a}{\sigma^2} + \frac{1}{2})}{\sqrt{(b-\theta\rho\sigma)^2 + \sigma^2\theta(1-\theta)}} + \frac{1}{2} > 0.$$

Then the Laplace transform in time of Mellin transform of X_t is given by

$$\int_0^\infty e^{-\lambda t}\mathbb{E}(X_t^\theta)dt = \frac{1}{\sigma^2} e^{-\frac{a}{\sigma^2}V_0 + (\frac{b}{\sigma^2} - \frac{\theta\rho}{\sigma})e^{V_0}}(J_1 + J_2),$$

with

$$\begin{aligned} J_1 &= 2\frac{\Gamma(a_2)}{\Gamma(b_2)} e^{-\frac{z_0}{2}} z_0^\eta U(a_2, b_2; z_0)(2\nu_2)^{-\theta - \frac{a}{\sigma^2}} I_1, \\ J_2 &= 2\frac{\Gamma(a_2)}{\Gamma(b_2)} e^{-\frac{z_0}{2}} z_0^\eta M(a_2, b_2; z_0)(2\nu_2)^{-\theta - \frac{a}{\sigma^2}} I_2, \end{aligned}$$

where

$$I_1 = \sum_{n=0}^{\infty} \frac{(a_2)_n}{(b_2)_n n!} i_n,$$

with i_n is given by

$$i_n = (-\delta(\theta))^{-\eta - \frac{a}{\sigma^2} - n} \gamma\left(\eta + \frac{a}{\sigma^2} + n, -\delta(\theta) z_0\right),$$

where γ denote the lower incomplete gamma function. Alternatively, i_n satisfies the following recurrence relation

$$\delta(\theta) i_{n+1} = z_0^{\eta + \frac{a}{\sigma^2} + n} e^{\delta(\theta) z_0} - \left(\eta + \frac{a}{\sigma^2} + n\right) i_n.$$

Furthermore

$$I_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Gamma(b_2 - 1 - n)}{\Gamma(a_2 - n)} j(-n) + \frac{\Gamma(1 - b_2 - n)}{\Gamma(a_2 + 1 - b_2 - n)} j(1 - b_2 - n) \right).$$

The function j is given by

$$j : t \mapsto \zeta^{\eta - \frac{a}{\sigma^2} + t} \Gamma\left(-\eta + \frac{a}{\sigma^2} - t, z_0 \zeta \right),$$

where $\Gamma(.,.)$ denote the upper incomplete gamma function and $\zeta = -\frac{1}{2} - \frac{\theta\rho\sigma - b}{2\nu_2\sigma^2}$ with

$$\begin{aligned}
a_2 &= \eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, \\
b_2 &= 1 + 2\eta, \\
\nu_1 &= \frac{(b - \theta\rho\sigma)}{\sigma^2}\left(\frac{a}{\sigma^2} + \frac{1}{2}\right), \\
\nu_2 &= \frac{1}{\sigma^2}\sqrt{(\theta\rho\sigma - b)^2 + \sigma^2\theta(1 - \theta)}, \\
z_0 &= 2\nu_2 e^{V_0}, \\
\eta^2 &= \frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}.
\end{aligned}$$

Note: For more details and for the proof of Proposition 6, Theorem 7 and Theorem 8 see [7]

2.2.3 Pricing Vanilla Option

To perform option pricing we will use the method of Mellin transform [8], where for a call option in the strike, it can be expressed in terms of moments, so for $\theta > 1$

$$\int_0^{\infty} \mathbb{E}(X_t - K)^+ K^{\theta - 2} dK = \frac{1}{\theta(\theta - 1)} \mathbb{E}(X_t^{\theta}).$$

Applying this to the 1-hypergeometric model and choosing λ and θ as in Theorem 8 and Laplace transforming in time leads to (see [7])

$$\int_0^{\infty} e^{-\lambda t} \int_0^{\infty} \mathbb{E}(X_t - K)^+ K^{\theta - 2} dK dt = \frac{1}{\theta(\theta - 1)} \underbrace{\int_0^{\infty} e^{-\lambda t} \mathbb{E}(X_t^{\theta}) dt}_{=:g(\theta,\lambda)},$$

which can already be calculated. Let $L(K,\lambda)$ denote the Laplace transforming in time of a call option with strike K, i.e.

$$L(K,\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{E}(X_t - K)^+ dt.$$

By Fubini's theorem there holds

$$\int_0^\infty L(K,\lambda)K^{\theta-2}dK = \frac{g(\theta,\lambda)}{\theta(\theta-1)}.$$

We need the next Lemma to invert the Mellin transform.

Lemma 9. *For $\theta \in (\theta^*, \theta_+)$ and λ as in Theorem 8 the function*

$$c \mapsto \frac{g(\theta+ic,\lambda)}{(\theta+ic)(\theta+ic-1)},$$

is $L^1(\mathbb{R})$.

Proof. The result follows immediately from

$$\left|\frac{g(\theta+ic,\lambda)}{(\theta+ic)(\theta+ic-1)}\right| \leq \frac{\int_0^\infty e^{-\lambda t}\mathbb{E}(|X_t^{\theta+ic}|)dt}{|(\theta+ic)(\theta+ic-1)|} \leq \frac{\int_0^\infty e^{-\lambda t}\mathbb{E}(X_t^{\theta})dt}{(\theta-1)^2+c^2},$$

and the fact that $\theta^* > 1$. □

Therefore we can obtain L by using Mellin's inversion formula

$$L(K,\lambda) = \int_{\theta+i\mathbb{R}} \frac{g(\tau,\lambda)}{\tau(\tau-1)}K^{-\tau+1}d\tau.$$

2.3 Vanilla Option Pricing Under α-Hypergeometric Model

In the current section, we will approximate option pricing using partial differential equation of α-hypergeometric model, in the case of deterministic volatility and the case of stochastic volatility concentrating on the first order expansion, where we going to focus on the 2-hypergeometric model in the both cases [54].
We rescale time in the volatility as $\xi > 0$, recall the α-hypergeometric model

$$\begin{aligned} dX_t &= e^{V_t}X_t dW_t^1, \\ dV_t &= \xi(a - be^{\alpha V_t})dt + \sqrt{\xi}\sigma dW_t^2, \end{aligned}$$

where $b, \alpha, \sigma > 0$, $a \in \mathbb{R}$ are constants, and W_t^1 , W_t^2 are Brownian motions with correlation ρ.

The price $P(t, X_t, V_t)$ of a vanilla option with payoff $h(X_T)$ and under the absence of arbitrage takes the form

$$P(t, X_t, V_t) = \mathbb{E}[h(X_T)/F_t],$$

where $(F_t)_{t\in[0,T]}$ is the filtration generated by $(W_t 1, W_t^2)_{t\in[0,T]}$ and the function $P(t, x, v)$ solves the PDE

$$\frac{\partial P}{\partial t} + \xi(a - be^{\alpha v_t})\frac{\partial P}{\partial v} + \frac{1}{2}x^2 e^{2v_t}\frac{\partial^2 P}{\partial x^2} + \sqrt{\xi}\rho\sigma x e^{v_t}\frac{\partial^2 P}{\partial x \partial v} + \xi\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial v^2} = 0, \tag{2.5}$$

with terminal condition $P(T, x, v) = h(x)$. We start by expanding $P(t, x, v)$ as

$$P(t, x, v) = P_0(t, x, v) + \xi P_1(t, x, v) + O(\xi), \tag{2.6}$$

by plugging the expansion 2.6 into the pricing PDE 2.5 we get system of equations

$$\frac{\partial P_n}{\partial t} + \mathcal{L}_0 P_n + \mathcal{L}_1 P_{n-1} + \mathcal{L}_2 P_{n-2} = 0, \qquad n \in \mathbb{N}, \tag{2.7}$$

with

$$\begin{cases} P_n & = 0 \qquad n \le -1, \\ P_0(T, x, v) & = h(x), \\ P_n(T, x, v) & = 0 \qquad n \ge 1. \end{cases}$$

In particular, operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\begin{aligned} \mathcal{L}_0 &= \xi(a - be^{\alpha v_t})\frac{\partial}{\partial v} + \frac{1}{2}x^2 e^{2v_t}\frac{\partial^2}{\partial x^2} \quad , \\ \mathcal{L}_1 &= \sqrt{\xi}\rho\sigma x e^{v_t}\frac{\partial^2}{\partial x \partial v} \quad , \\ \mathcal{L}_2 &= \frac{1}{2}\xi\sigma^2\frac{\partial^2}{\partial v^2} \quad . \end{aligned}$$

2.3.1 Deterministic Volatility

When $n = 0$, we have $\frac{\partial P_0}{\partial t} + \mathcal{L}_0 P_0 = 0$ and

$$\begin{aligned} dX_t^0 &= X_t^0 e^{V_t} dW_t^1 \quad , \\ dV_t^0 &= (a - be^{\alpha V_t})dt \quad . \end{aligned}$$

also the vanilla option price $P_0(t, X_t^0, V_t^0) = \mathbb{E}[h(X_T^0)/F_t]$ can be computed by the Black-Scholes formula as

$$P_0(t, X_t^0, V_t^0) = \mathbb{E}\left[(X_T^0 - K)^+/F_t\right] = \mathbb{E}\left[\left(X_t^0 e^{\left(N\gamma(t,V_t^0) - \frac{1}{2}\gamma^2(t,V_t^0)\right)} - K\right)^+ /F_t\right],$$

where $N \simeq N(0,1)$ is independent of F_t and $\gamma^2(t, V_t^0) = \int_t^T e^{2V_s^0} ds$, $t \in [0,T]$.
We note that in the α-hypergeometric model with $\sigma = 0$ the integral $\int_t^T e^{\alpha V_s^0} ds$ can be computed in closed form as (see [7], [16])

$$\begin{aligned}
\int_t^T e^{\alpha V_s^0} ds &= \frac{1}{\alpha b}\log\left(1 + \alpha b e^{\alpha V_t^0}\int_0^{T-t} e^{\alpha a s} ds\right), \\
&= \frac{1}{\alpha b}\log\left(1 + \alpha b e^{\alpha V_t^0}\frac{e^{\alpha a(T-t)} - 1}{\alpha a}\right),
\end{aligned}$$

this leads as to the following Proposition.

Proposition 10 ([54]). *In the 2-hypergeometric model with $\sigma = 0$ the European call price*

$$P_0(t, X_t^0, V_t^0) = \mathbb{E}\left[(X_T^0 - K)^+/F_t\right],$$

under the terminal condition $P_0(T, x, v) = (x - K)^+$ is given by

$$P_0(t, x, v) = x\Phi\left(d_+(t, x, v)\right) - K\Phi\left(d_-(t, x, v)\right),$$

where Φ is the standard Gaussian cumulative distribution function,

$$\begin{aligned}
d_\pm(t, x, v) &= \frac{1}{\gamma(t, v)}\left(\log(\frac{x}{K}) \mp \frac{\gamma^2(t, v)}{2}\right), \\
\gamma^2(t, x, v) &= \frac{1}{2b}\log\left(1 + 2be^{2v}\frac{e^{2a(T-t)} - 1}{2a}\right).
\end{aligned} \tag{2.8}$$

In the case of a put option the function $P_0(t, x, v)$ can be obtained as

$$P_0(t, x, v) = -x\Phi\left(-d_+(t, v, x)\right) + K\Phi\left(-d_-(t, x, v)\right), \qquad t \in [0,T].$$

2.3.2 First Order Expansion

When $n = 1$, equation 2.7 becomes

$$\frac{\partial P_1}{\partial t} + \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0,$$

with $P_1(T, x, v) = 0$.
Note that the approximation $(X_t, V_t)_{t\in[0,T]}$ does not lie within the class of 2-hypergeometric model

Proposition 11 ([54])**.** *The solution of* $\frac{\partial P_1}{\partial t} + \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$ *under terminal condition* $P_1(T, x, v) = 0$ *is given by*

$$P_1(t,x,v) = -\rho K \frac{\sigma}{2b} d_-(t,x,v) \Phi\left(d_-(t,x,v)\right) \frac{e^{-2b\gamma^2(t,v)} + 2b\gamma^2(t,v) - 1}{2b\gamma^2(t,v)}, \qquad t \in [0,T].$$

Proof. from the relation $\Phi(d_+(t,x,v)) = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{1}{2}(d_+(t,x,v))^2\right)} = \frac{K}{x}\Phi(d_-(t,x,v))$ and using the Feynman-Kac formula with locally Lipschitz coefficients as in e.g. Theorem 1 of Heath and Schweizer [31], we have

$$\begin{aligned} P_1(t, X_t^0, V_t^0) &= \int_t^T \mathbb{E}\left[\mathcal{L}_1 P_0(s, X_s^0, V_s^0 | F_t)\right] ds, \\ &= -\sigma \int_t^T \frac{\rho K e^{2V_s^0} \partial \gamma}{\gamma(s, V_s^0) \partial v}(s, V_s^0) \mathbb{E}\left[d_-(s, X_s^0, V_s^0) \Phi(d_-(s, X_s^0, V_s^0)) | F_t\right] ds, \\ &= -\sigma \rho K \frac{d_-(t, X_t^0, V_t^0)}{\gamma^2(t, V_t^0)} \Phi\left(d_-(t, X_t^0, V_t^0)\right) \int_t^T e^{2V_s^0} \gamma(s, V_s^0) \frac{\partial \gamma}{\partial v}(s, V_s^0) ds, \end{aligned}$$

by a standard computation based on the Gaussian distribution

$$d_-(t, X_t^0, V_t^0) \sim N\left(\frac{1}{\gamma(s, V_s^0)}\left(\log(\frac{X_t^0}{K}) - \frac{\gamma^2(t, V_t^0)}{2}\right), \frac{\gamma^2(t, V_t^0)}{\gamma^2(s, V_s^0)} - 1\right), \qquad s \in [t, T].$$

Finally, we note that from 2.8 we have

$$\begin{aligned} \int_t^T e^{2V_s^0} \gamma^2(s, V_s^0) \frac{\partial \gamma}{\partial V}(s, V_s^0) ds &= \frac{1}{2b} \int_t^T e^{2V_s^0} \left(1 - e^{-2b\gamma^2(s, V_s^0)}\right) ds, \\ &= \frac{1}{4b^2}\left(e^{-2b\gamma^2(t, V_t^0)} + 2b\gamma^2(t, V_t^0) - 1\right). \end{aligned}$$

□

Chapter 3

Uncertain Stochastic Volatility

In this chapter, we focus on the uncertain volatility model (UVM) developed by [3]. it attracted the attention of practitioners as it provides a worst-case pricing scenario for the seller. We study the UVM where the volatility is stochastic and bounded between two extremes values $\underline{\sigma}$ and $\overline{\sigma}$.
Under the risk-neutral measure $\mathbb{Q}$, the price process of the risky asset X_t is the solution of the following stochastic differential equation (SDE)

$$dX_t = rX_t dt + X_t \alpha_t dW_t^1, \tag{3.1}$$

where $r \in \mathbb{R}$ is a risk-free rate, α_t the volatility process such that $\underline{\sigma_t} \leq \alpha_t \leq \overline{\sigma_t}$ and W_t^1 is Brownian motion under the risk measure. We assume that the volatility bound itself is given by $\underline{\sigma_t} := \sigma_{min} F(V_t) \leq \alpha_t \leq \sigma_{max} F(V_t) := \overline{\sigma_t}$ for $0 \leq t \leq T$ and $\sigma_{min}, \sigma_{max} \in \mathbb{R}$ such that $0 < \sigma_{min} < 1 < \sigma_{max}$, where F is a positive increasing and differentiable function.
In our choosing model $F(V_t) = e^{V_t}$, we denote $\alpha_t = qe^{V_t}$ s.t $\sigma_{min} \leq q \leq \sigma_{max}$ for $0 \leq t \leq T$ then we obtain the following dynamic

$$dX_t = rX_t dt + X_t q e^{V_t} dW_t^1, \tag{3.2}$$

$$dV_t = (a - be^{\alpha V_t})dt + \sigma dW_t^2, \tag{3.3}$$

where W_t^1 and W_t^2 are Brownian motions with correlation ρ; $b, \alpha, \sigma > 0$ and $a \in \mathbb{R}$ are constants [47] .

3.1 Worst-Case Scenario Price

Our aim in this section is to derive worst-case pricing scenarios for the seller in the spirit of the work [16], without needing to calibrate the model exactly. To this end, we rescale time in the volatility equation (3.3) according to $t \mapsto \delta t$, which yields

$$dX_t = rX_t dt + X_t q e^{V_t} dW_t^1, \tag{3.4a}$$

$$dV_t = \delta(a - be^{\alpha V_t})dt + \sqrt{\delta}\sigma dW_t^2, \tag{3.4b}$$

and allows us to smoothly interpolate between an UVM and a fixed volatility model (cf. [24]). The parameter $\delta > 0$ symbolizes the reciprocal of the time-scale of the process V, and thus the standard UVM can be formally obtained by sending $\delta \to 0$, in which case $V_t = v$ and

$$dX_t^0 = rX_t^0 dt + qX_t^0 e^v dW_t^1. \tag{3.5}$$

Varying δ sheds some light on the importance of the stochastic volatility equation for the worst-case scenario: when the variation of the volatility is slow, the market price of the asset is not very volatile, so this price remains stable; in the opposite case, it may become too volatile and therefore more risky.

Let $\Theta = [\sigma_{\min}, \sigma_{\max}]$. For any $\delta > 0$, the worst-case scenario price at time $t < T$ is defined as

$$P^\delta := P^\delta(t; x, v) = \exp(-r(T-t)) \sup_{q \in \Theta} E_{(t;x,v)}[h(X_T^\delta)]. \tag{3.6}$$

If $\delta = 0$, we define

$$P^0 := P^0(t; x, v) = \exp(-r(T-t)) \sup_{q \in \Theta} E_{(t;x,v)}[h(X_T^0)]. \tag{3.7}$$

Where $E_{(t;x,v)}[\cdot]$ is the conditional expectation given $\mathcal{F}_t$ with $X_t^\delta = x$ and $V_t = v$.

3.1.1 Moment Bounds

Instead of confining ourselves to perturbations of Black-Scholes prices as in [23], we will work with general terminal payoff (neither convex, nor concave) as in [21]. In this case the Hessian of the resulting option prices is indefinite and we have to impose additional regularity conditions on the payoff function h to do some asymptotic analysis. Specifically, we suppose that the terminal payoff h is C^4 and gradient Lipschitz, and we impose the following polynomial growth conditions on the first four derivatives of h

$$\begin{cases} |h'(x)| \leq K_1, \\ |h''(x)| \leq K_2(1+|x|^m), \\ |h'''(x)| \leq K_3(1+|x|^n), & (K_i \text{ for } i \in \{1,2,3,4\}) m, n \text{ and } l \in \mathbb{N}, \\ |h^{(4)}(x)| \leq K_4(1+|x|^l). \end{cases} \tag{3.8}$$

Before we come to the convergence of P^δ as $\delta \to 0$, the next two Propositions show that the processes X_t and V_t have uniformly bounded moments of any order.

Proposition 12. *Let* $0 \leq \delta \leq 1$, *for* $t \leq T$. *The process* V_t *has uniformly bounded moments of any order*

$$\mathbb{E}_{(t,x,v)}\left[\int_t^T |V_s|^k ds\right] \leq \mathbb{E}_{(0,v)}\left[\int_0^T |V_s|^k ds\right] \leq C_k(T,v),$$

where $C_k(T,v)$ *independent of* δ.

Proof. See Lemma 4.9 in [23] also Lemma 3 in chapter one. □

Lemma 13. *For* $\eta \in \mathbb{R}$ *independent of* $0 \leq \delta \leq \delta_0$, *for some sufficiently small* $\delta_0 > 0$, *and* $t \leq T$, *the moment generating function of the integrated* α*-hypergeometric process*

$$M_v^\delta(\eta) := \mathbb{E}_{(t,v)}[e^{\eta \int_0^t V_s ds}], \qquad \text{for } \eta \in \mathbb{R},$$

is uniformly bounded, that is $|M_v^\delta(\eta)| \leq N(T,v,\eta) < \infty$, *where* $N(T,v,\eta)$ *is independent of* t.

Proof. Following the reasoning of [40, Sec. 5], we have an explicit form of the moment generating function of the integrated α-hypergeometric process

$$M_v^\delta(\eta) = \Psi(\eta,t)e^{-v\Xi(\eta,t)},$$

where

$$\begin{aligned} \Psi(\eta,t) &= \left(\frac{\bar{b}e^{\delta\frac{t}{2}}}{\bar{b}\cosh(\bar{b}\frac{t}{2}) + \delta\sinh(\bar{b}\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}, \\ \Xi(\eta,t) &= \left(\frac{2\eta\sinh(\bar{b}\frac{t}{2})}{\bar{b}\cosh(\bar{b}\frac{t}{2}) + \delta\sinh(\bar{b}\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}, \end{aligned}$$

and

$$\bar{b} = \sqrt{\hat{b}^2 - 2\eta\hat{\sigma}^2} = \sqrt{\delta^2 - 2\eta\delta\sigma^2}.$$

In the following, we are going to show that $|M_v^\delta(\eta)| \leq N(T, v, \eta) < \infty$, where $N(T, v, \eta)$ is independent of δ and t. To this end, we distinguish two cases

- If $\delta^2 - 2\eta\delta\sigma^2 \geq 0$, we have $\bar{b} \geq 0$ and

$$\begin{aligned}
\Psi(\eta, t) &\leq \left(\frac{\bar{b}e^{\delta\frac{t}{2}}}{\bar{b}\cosh(\bar{b}\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}, && \delta\sinh(\bar{b}\tfrac{t}{2}) \geq 0,\\
&\leq \left(e^{\delta\frac{t}{2}}\right)^{\frac{2}{\sigma^2}}, && \cosh(\bar{b}\tfrac{t}{2}) \geq 1,\\
&\leq \left(e^{\frac{T}{2}}\right)^{\frac{2}{\sigma^2}}.
\end{aligned}$$

  Since $\Xi(\eta, t) \geq 0$, we have $e^{-v\Xi(\eta,t)} \leq 1$. Therefore

$$M_v^\delta(\eta) = \Psi(\eta, t)e^{-v\Xi(\eta,t)} \leq \left(e^{\frac{T}{2}}\right)^{\frac{2}{\sigma^2}}.$$

- If $\delta^2 - 2\eta\delta\sigma^2 < 0$, let $\vartheta = \sqrt{2\eta\delta\sigma^2 - \delta^2}$ which is positive. Then

$$M_v^\delta(\eta) = \psi(\eta, t)e^{-v\Xi(\eta,t)},$$

$$\begin{aligned}
&= \left(\frac{i\vartheta e^{\delta\frac{t}{2}}}{i\vartheta\cosh(i\vartheta\frac{t}{2}) + \delta\sinh(i\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}} e^{-v\left(\frac{2\eta\sinh(i\vartheta\frac{t}{2})}{i\vartheta\cosh(i\vartheta\frac{t}{2})+\delta\sinh(i\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}},\\
&= \left(\frac{i\vartheta e^{\delta\frac{t}{2}}}{i\vartheta\cos(\vartheta\frac{t}{2}) + i\delta\sin(\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}} e^{-v\left(\frac{2i\eta\sin(\vartheta\frac{t}{2})}{i\vartheta\cos(\vartheta\frac{t}{2})+i\delta\sin(\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}},\\
&= \left(\frac{\vartheta e^{\delta\frac{t}{2}}}{\vartheta\cos(\vartheta\frac{t}{2}) + \delta\sin(\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}} e^{-v\left(\frac{2\eta\sin(\vartheta\frac{t}{2})}{\vartheta\cos(\vartheta\frac{t}{2})+\delta\sin(\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}}.
\end{aligned}$$

  Thus, for sufficiently small ϑ, since

$$\left(\frac{2\eta\sin(\vartheta\frac{t}{2})}{\vartheta\cos(\vartheta\frac{t}{2}) + \delta\sin(\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}} \geq 0,$$

we have

$$
\begin{aligned}
M_v^\delta(\eta) &\le \left(\frac{\vartheta e^{\delta\frac{t}{2}}}{\vartheta\cos(\vartheta\frac{t}{2})+\delta\sin(\vartheta\frac{t}{2})}\right)^{\frac{2}{\sigma^2}}, \\
&= \left(\frac{\vartheta e^{\delta\frac{t}{2}}}{\vartheta(1+O(\vartheta^2t^2))+\delta(\frac{\vartheta t}{2}+O(\vartheta^3t^3))}\right)^{\frac{2}{\sigma^2}}, \\
&= \left(\frac{e^{\delta\frac{t}{2}}}{1+\frac{\delta t}{2}+O(\vartheta^2t^2)}\right)^{\frac{2}{\sigma^2}}. \qquad .
\end{aligned}
$$

As a consequence, there exists ϑ_0 independent of t, such that for $\vartheta < \vartheta_0$, $M_v^\delta(\eta) \le \left(\frac{e^{\frac{T}{2}}}{1+\frac{T}{2}}\right)^{\frac{2}{\sigma^2}}$.

This concludes the proof. □

Proposition 14. *Let $\delta \ge 0$ be sufficiently small and for $t \le T$. Then the process X_t has uniformly bounded moments of arbitrary order.*

Proof. Let X_t, V_t satisfy (3.4), with $q_t \in [\sigma_{\min}, \sigma_{\max}]$. Then, for each finite $n \in \mathbb{N}$,

$$
\begin{aligned}
X_t^n &= x^n \exp\left(nrt - \frac{n}{2}\int_0^t (q_s e^{V_s})^2 ds + n\int_0^t q_s e^{V_s} dW_s^1\right), \\
&= x^n \exp\left(nrt + \frac{n^2-n}{2}\int_0^t (q_s e^{V_s})^2 ds\right)\exp\left(\frac{-n^2}{2}\int_0^t (q_s e^{V_s})^2 ds + n\int_0^t q_s e^{V_s} dW_s^1\right), \\
&\le x^n \exp\left(nrt + \frac{n^2-n}{2}\int_0^t \sigma_{\max}^2 e^{2V_s} ds\right)\Lambda_t,
\end{aligned}
$$

where in the last step we assume Novikov's condition which implies that

$$
\Lambda_t = \exp\left(\frac{-n^2}{2}\int_0^t (q_s e^{V_s})^2 ds + n\int_0^t q_s e^{V_s} dW_s^1\right),
$$

is a martingale.

Using Proposition 12, we find

$$
\mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{1}{2}\int_0^t (nqe^{V_s})^2 ds\right)\right]
$$

$$
\begin{aligned}
&\leq \mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{n^2u^2}{2}\int_0^t e^{2V_s}ds\right)\right],\\
&= \mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{n^2\sigma_{\max}^2}{2}\int_0^t \left(1+2V_s+O((2V_s)^2)\right)ds\right)\right],\\
&= \mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{n^2\sigma_{\max}^2}{2}\left[\int_0^t ds+2\int_0^t V_s ds+\int_0^t O((2V_s)^2)ds\right]\right)\right],\\
&= \mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{n^2\sigma_{\max}^2}{2}(t+C)+2\frac{n^2\sigma_{\max}^2}{2}\int_0^t V_s ds\right)\right],\\
&= \mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{n^2\sigma_{\max}^2}{2}(t+C)\right).\exp\left((n^2\sigma_{\max}^2)\int_0^t V_s ds\right)\right],\\
&= \exp\left(\frac{n^2\sigma_{\max}^2}{2}(t+C)\right)\mathbb{E}_{(0,x,v)}\left[\exp\left((n^2\sigma_{\max}^2)\int_0^t V_s ds\right)\right],\\
&= \exp\left(\frac{n^2\sigma_{\max}^2}{2}(t+C)\right)M_v^{\delta}(n^2\sigma_{\max}^2),\\
&< \infty.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}_{(0,x,v)}[X_t^n] &\leq x^n\exp(nrt)\mathbb{E}_{(0,x,v)}\left[\exp\left(\frac{(n^2-n)\sigma_{\max}^2}{2}\int_0^t e^{2V_s}ds\right)\right],\\
&= x^n\exp(nrt)\exp\left(\frac{(n^2-n)\sigma_{\max}^2}{2}(t+C)\right)M_v^{\delta}\left((n^2-n)\sigma_{\max}^2\right),\\
&\leq x^n\exp(nrT)\exp\left(\frac{(n^2-n)\sigma_{\max}^2}{2}(T+C)\right)N\left(T,v,(n^2-n)\sigma_{\max}^2\right) := L,
\end{aligned}
$$

where the upper bound L is independent of δ and t.

Therefore,

$$
\mathbb{E}_{(t,x,v)}\left[\int_t^T |X_s|^k ds\right] \leq \mathbb{E}_{(0,x,v)}\left[\int_0^T |X_s|^k ds\right] \leq N_k(T,x,v),
$$

where $N_k(T,x,v)$ may depend on (k,T,x,v) but not on δ. □

3.1.2 Convergence of the Payoff

As a consequence of the previous results, we have the following convergence result for the asset process.

Proposition 15. *Assume there exists $C_0 > 0$, independent of δ, such that X^δ, X^0 being the solution of the SDEs (3.4a) and (3.5) satisfy*

$$E_{(t;x,v)}(X_T^\delta - X_T^0)^2 \le C_0\delta\,.$$

Proof. Since X_t^δ, X_t^0 solve (3.4a), (3.5), we have

$$X_T^\delta = x + \int_t^T rX_s^\delta ds + \int_t^T qe^{V_s}X_s^\delta dW_s^1,$$

and

$$X_T^0 = x + \int_t^T rX_s^0 ds + \int_t^T qe^{v}X_s^0 dW_s^1\,,$$

which can be combined to give

$$\begin{aligned} X_T^\delta - X_T^0 &= \int_t^T r(X_s^\delta - X_s^0)ds + \int_t^T q(e^{V_s}X_s^\delta - e^v X_s^0)dW_s^1, \\ &= \int_t^T r(X_s^\delta - X_s^0)ds + \int_t^T qe^v(X_s^\delta - X_s^0)dW_s^1 + \int_t^T q(e^{V_s} - e^v)X_s^\delta dW_s^1\,. \end{aligned}$$

Now let $Y_s = X_s^\delta - X_s^0$, then $Y_t = 0$ and

$$Y_T = \int_t^T rY_s ds + \int_t^T qe^v Y_s dW_s^1 + \int_t^T q(e^{V_s} - e^v)X_s^\delta dW_s^1.$$

Thus,

$$\begin{aligned} E_{(t;x,v)}[Y_T^2] &\le 3E_{(t,x,v)}\left[\left(\int_t^T rY_s ds\right)^2 + \left(\int_t^T qe^v Y_s dW_s^1\right)^2 + \left(\int_t^T q(e^{V_s} - e^v)X_s^\delta dW_s^1\right)^2\right], \\ &\le \int_t^T (3Tr^2 + 3\sigma_{\max}^2 e^{2v})E_{(t,x,v)}[Y_s^2]ds + \underbrace{3\sigma_{\max}^2 \int_t^T E_{(t;x,v)}\left[(e^{V_s} - e^v)^2(X_s^\delta)^2\right]ds}_{R(\delta)}\,. \end{aligned}$$

We have seen before that X_t and V_t have uniformly bounded moments for δ sufficiently small. We can therefore show that $|R(\delta)| \le C\delta$ for C independent of δ. Setting $q = \sigma_{\max}$ and using Gronwall's inequality, the previous inequality can be recast as

$$f(T) \le \int_t^T \lambda f(s)ds + C\delta \le \delta \int_t^T C\lambda e^{\lambda(T-s)}ds + C\delta,$$

where $f(T) = E_{(t;x,v)}(Y_T^2)$ and $\lambda = 3Tr^2 + 3\sigma_{\max}^2 e^{2v} > 0$. As a consequence,

$$E_{(t;x,v)}(X_T^\delta - X_T^0)^2 = E_{(t;x,v)}Y_T^2 = f(T) \le C_0\delta\,.$$

□

Theorem 16. *The function P^δ uniformly converges to P^0 with rate $\sqrt{\delta}$ as $\delta \to 0$, where the convergence is uniform on any compact subset of* $[0,T]\times\mathbb{R}\times\mathbb{R}^+$.

Proof. Due to the Lipschitz continuity of h, the Cauchy-Schwartz inequality and Proposition 15, we get

$$\begin{aligned}
|P^\delta - P^0| &= \exp(-r(T-t))\left|\sup_{q\in\Theta} E_{(t;x,v)}[h(X_T^\delta)] - \sup_{q\in\Theta} E_{(t;x,v)}[h(X_T^0)]\right|,\\
&\le \exp(-r(T-t))\sup_{q\in\Theta}\left|E_{(t;x,v)}[h(X_T^\delta)] - E_{(t;x,v)}[h(X_T^0)]\right|,\\
&\le \exp(-r(T-t))\sup_{q\in\Theta} E_{(t;x,v)}\left|h(X_T^\delta) - h(X_T^0)\right|,\\
&\le K_0\exp(-r(T-t))\sup_{q\in\Theta} E_{(t;x,v)}\left|X_T^\delta - X_T^0\right|,\\
&\le K_0\exp(-r(T-t))\sup_{q\in\Theta}\left[E_{(t;x,v)}(X_T^\delta - X_T^0)^2\right]^{1/2}.
\end{aligned}$$

This entails

$$|P^\delta - P^0| \le C_1\sqrt{\delta},$$

and concludes the proof. □

3.2 Pricing G-PDE

The worst-case scenario price P^δ is the solution to the following Hamilton-Jacobi-Bellman (HJB) equation with terminal condition $P^\delta(T;x,v) = h(x)$ (see [43, 44])

$$\begin{aligned}
-\partial_t P^\delta &= r\left(x\partial_x P^\delta - P^\delta\right) + \sup_{q\in\Theta}\left\{\frac{1}{2}x^2q^2e^{2v}\partial^2_{xx}P^\delta + \sqrt{\delta}qxe^v\sigma\rho\partial^2_{xv}P^\delta\right\} \\
&+ \delta\left(\frac{1}{2}\sigma^2\partial^2_{vv}P^\delta + (a-be^{\alpha v})\partial_v P^\delta\right).
\end{aligned} \tag{3.9}$$

Throughout the rest of the section, we set $r=0$, i.e. we assume that the return of the asset is zero, but the return of the option depends on the volatility. In other words, even though the financial asset has no return, the option can have it.

Leading Order Term P_0

To approximate the value function P^δ, we use the regular perturbation expansion

$$P^\delta = P_0 + \sqrt{\delta} P_1 + \delta P_2 + \ldots, \tag{3.10}$$

where P_0 the leading order term and $P_1 := P_1(t, x, v)$ the first correction for the approximation of the worst-case scenario price P^δ. Substituting (3.10) in (3.9), and using Theorem 16, the leading order term P_0 is found to be the solution to

$$-\partial_t P_0 = \sup_{q \in \Theta} \left\{ \frac{1}{2} q^2 e^{2v} x^2 \partial^2_{xx} P_0 \right\}, \quad P_0(T; x, v) = h(x). \tag{3.11}$$

3.2.1 Convergence of the Second Partial Derivative

The gamma $\partial^2_{xx} P^\delta$ represents the convexity of the price of an option according to the price of the underlying asset. It indicates whether the price of the option tends to move faster or slower than the price of the underlying asset. Using the fact that $q \in [\sigma_{\min}, \sigma_{\max}]$, and the regularity results for uniformly parabolic equations which are referenced in [15],[28], we conclude that (3.9) is uniformly parabolic.

Proposition 17. *As $\delta \to 0$, the second partial derivative $\partial^2_{xx} P^\delta$ converges uniformly to $\partial^2_{xx} P_0$ on any compact subset of $[0, T] \times \mathbb{R} \times \mathbb{R}^+$ and with rate $\sqrt{\delta}$.*

Proof. The function $h \in C^4$ is gradient Lipschitz and satisfies polynomial growth conditions in its first four derivatives. By [26, Thm. 5.2.5], we conclude

- $P^\delta(t, ., .) \in C_p^{1,2,2}$ for δ fixed .
- $\partial_x P^\delta(t, ., .)$ and $\partial^2_{xx} P^\delta(t, ., .)$ are uniformly bounded in δ.

The assertion thus follows from Theorem 16. □

3.2.2 Optimal Controls

Following [21], we define $S^0_{t,v}$ to be the zero level set of $\partial^2_{xx} P_0$ and the set $A^\delta_{t,v}$ to be the set on which $\partial^2_{xx} P^\delta$ and $\partial^2_{xx} P_0$ have different signs, i.e.

$$S^0_{t,v} := \{x = x(t, v) \in \mathbb{R}^+ | \partial^2_{xx} P_0(t; x, v) = 0\},$$

and

$$A^{\delta}_{t,v} := \{x = x(t,v) | \partial^2_{xx} P^{\delta}(t;x,v) > 0, \partial^2_{xx} P_0(t;x,v) < 0\}. \tag{3.12}$$

Lemma 18. *Call*

$$q^{*,\delta}(t;x,v) := \arg\max_{q\in\Theta}\left\{\frac{1}{2}q^2 e^{2v} x^2 \partial^2_{xx} P^{\delta} + \sqrt{\delta}(q\rho\sigma e^{v} x \partial^2_{xv} P^{\delta})\right\}, \tag{3.13}$$

for $x \notin S^0_{t,v}$ and $\delta > 0$ sufficiently small, and

$$q^{*,0}(t;x,v) := \arg\max_{q\in\Theta}\left\{\frac{1}{2}q^2 e^{2v} x^2 \partial^2_{xx} P^{0}\right\}, \tag{3.14}$$

for $\delta = 0$. Moreover, let (3.13) and (3.14) denote the optimal controls in the G-PDE (3.9) for P^{δ} and in the G-PDE (3.11) for P_0, respectively. Then the limiting optimal control as $\delta \to 0$ is given by

$$q^{*,\delta}(t;x,v) = \begin{cases} \sigma_{\max}, & \partial^2_{xx} P^{\delta} \geq 0, \\ \sigma_{\min}, & \partial^2_{xx} P^{\delta} < 0, \end{cases} \tag{3.15}$$

and

$$q^{*,0}(t;x,v) = \begin{cases} \sigma_{\max}, & \partial^2_{xx} P_0 \geq 0, \\ \sigma_{\min}, & \partial^2_{xx} P_0 < 0. \end{cases} \tag{3.16}$$

Proof. Let

$$f(q) := \frac{1}{2}q^2 e^{2v} x^2 \partial^2_{xx} P^{\delta} + \sqrt{\delta}(q\rho\sigma e^{v} x \partial^2_{xv} P^{\delta}),$$

and suppose that the maximiser $\hat{q}^{*,\delta}$ is in the interior of the interval $[\sigma_{\min}, \sigma_{\max}]$. Then, for $x \notin S^0_{t,v}$, we have

$$\hat{q}^{*,\delta} = \frac{-\rho\sqrt{\delta}\sigma\partial^2_{xv} P^{\delta}}{x e^{v} \partial^2_{xx} P^{\delta}},$$

for the maximiser of $f(q)$. But since $f(\hat{q}^{*,\delta}) \to 0$ as $\delta \to 0$, the maximiser must be on the boundary whenever δ is sufficiently small. In this case, since the sign of $\partial^2_{xx} P^{\delta}$ determines the sign of the coefficient of the q^2 term in $f(q)$, we have $q^{*,\delta} \to q^{*,0}$ pointwise on $S^0_{t,v}$ where, for any sufficiently small $\delta \geq 0$, the maximiser can be represented by

$$q^{*,\delta} = \sigma_{\max}\mathbf{1}_{\{\partial^2_{xx} P^{\delta} \geq 0\}} + \sigma_{\min}\mathbf{1}_{\{\partial^2_{xx} P^{\delta} < 0\}}.$$

□

Lemma 18 allows us to rewrite the G-HJB equation (3.9) as

$$-\partial_t P^\delta = \frac{1}{2}(q^{*,\delta})^2 e^{2v} x^2 \partial^2_{xx} P^\delta + \sqrt{\delta}(q^{*,\delta}\rho\sigma e^v x \partial^2_{xv} P^\delta) + \delta(\frac{1}{2}\sigma^2\partial^2_{vv}P^\delta + (a - be^{\alpha v})\partial_v P^\delta), \tag{3.17}$$

with terminal condition $P^\delta(T; x, v) = h(x)$ and with $q^{*,\delta}$ as given above.

3.2.3 First-Order Corrector for the Limit Payoff

We will now derive a corrector result for the difference $P^\delta - P^0$. To this end, recall that P_1, the first order correction term of P^δ, is the solution to the linear equation

$$-\partial_t P_1 = \frac{1}{2}(q^{*,0})^2 e^{2v} x^2 \partial^2_{xx} P_1 + q^{*,0}\rho\sigma e^v x \partial^2_{xv} P_0, \quad P_1(T, x, v) = 0, \tag{3.18}$$

where $q^{*,0}$ is given by (3.16). Further recall that vanna $\partial^2_{xv}P^\delta$ is a second order derivative of the option, once to the underlying asset price and once to volatility. It is the sensitivity of the option delta with respect to change in volatility, or, alternatively, it is the sensitivity of vega $\partial^2_v P^\delta$ with respect to the underlying asset price. For more details see section 4.2.4 in [23].

In the following we will exploit results from [22] and [23] to show that, under the regularity conditions imposed on the derivatives of h, the pointwise approximation error $|P^\delta - P_0 - \sqrt{\delta}P_1|$ is indeed of order $O(\delta)$.

Theorem 19. $\forall (t; x, v) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$, $\exists C > 0$, *such that*

$$|E^\delta(t; x, v)| := |P^\delta(t; x, v) - P_0(t; x, v) - \sqrt{\delta}P_1(t; x, v)| \leq C\delta,$$

where C may depend on $(t; x, v)$ but not on δ.

Proof. Adopting the arguments of Secs. 1.9.3 and 4.1.2 in [23], we define the following linear parabolic differential operator

$$\begin{aligned} \mathcal{L}^\delta(q) :=& \partial_t + \frac{1}{2}q^2 e^{2v} x^2 \partial^2_{xx} + \sqrt{\delta} q \rho e^v x \partial^2_{xv} + \delta(\frac{1}{2}\sigma^2\partial^2_{vv} + (a - be^{\alpha v})\partial_v), \\ =& \mathcal{L}_0(q) + \sqrt{\delta}\mathcal{L}_1(q) + \delta\mathcal{L}_2, \end{aligned} \tag{3.19}$$

where $\mathcal{L}_0(q)$ contains the time derivative and the Black-Scholes operator, $\mathcal{L}_1(q)$ contains the mixed derivative due to the covariation between X_t and V_t, and $\delta\mathcal{L}_2$ is the infinitesimal generator of the volatility process V_t.

We can recast equation (3.17) as

$$\begin{aligned}\mathcal{L}^{\delta}(q^{*,\delta})P^{\delta} &= 0,\\ P^{\delta}(t;x,v) &= h(x).\end{aligned} \tag{3.20}$$

Equivalently, equation (3.11) reads

$$\begin{aligned}\mathcal{L}_0(q^{*,0})P_0 &= 0,\\ P_0(T;x,v) &= h(x),\end{aligned} \tag{3.21}$$

and (3.18) can be expressed by

$$\begin{aligned}\mathcal{L}_0(q^{*,0})P_1 + \mathcal{L}_1(q^{*,0})P_0 &= 0,\\ P_1(T,x,v) &= h(x).\end{aligned} \tag{3.22}$$

Now, applying the operator $\mathcal{L}^{\delta}(q^{*,\delta})$ to the error term $E^{\delta} = P^{\delta} - P_0 - \sqrt{\delta}P_1$, we obtain

$$\begin{aligned}
\mathcal{L}^{\delta}(q^{*,\delta})E^{\delta} &= \mathcal{L}^{\delta}(q^{*,\delta})(P^{\delta} - P_0 - \sqrt{\delta}P_1),\\
&= -(\mathcal{L}_0(q^{*,\delta}) + \sqrt{\delta}\mathcal{L}_1(q^{*,\delta}) + \delta\mathcal{L}_2 q^{*,\delta}))(P_0 + \sqrt{\delta}P_1),\\
&= -\underbrace{\sqrt{\delta}\mathcal{L}_0(q^{*,\delta})P_1 + \sqrt{\delta}\mathcal{L}_1(q^{*,\delta})P_0}_{=0} - \delta\mathcal{L}_2(q^{*,\delta})P_0 + \delta\mathcal{L}_1(q^{*,\delta})P_1 + \delta^{3/2}\mathcal{L}_2(q^{*,\delta})P_1,\\
&= \frac{1}{2}[(q^{*,\delta})^2 - (q^{*,0})^2]e^{2v}x^2\partial^2_{xx}P_0\\
&\quad - \sqrt{\delta}\left[\rho((q^{*,\delta}) - q^{*,0})e^{v}x\partial^2_{xv}P_0 + \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right)e^{2v}x^2\partial^2_{xx}P_1\right]\\
&\quad -\delta\left[\rho(q^{*,\delta})e^{v}x\partial^2_{xv}P_1 + \frac{1}{2}\sigma^2\partial^2_{vv}P_0 + (a - be^{\alpha v})\partial_v P_0\right]\\
&\quad -\delta^{\frac{3}{2}}\left[\frac{1}{2}\sigma^2\partial^2_{vv}P_1 + (a - be^{\alpha v})\partial_v P_1\right].
\end{aligned}$$

Using the terminal condition

$$E^{\delta}(T;x,v) = P^{\delta}(T;x,v) - P_0(T;x,v) - \sqrt{\delta}P_1(T;x,v) = 0,$$

and the continuity of the solution to the parabolic equation (3.18), we conclude that $|E^{\delta}(t;x,v)| = O(\delta)$. □

3.2.4 Feynman-Kac Representation of the Error Term

Now recall that the asset price in the worst-case scenario is governed by (3.4a) with $r = 0$ and $q = q^{*,\delta}$

$$dX_t^{*,\delta} = q_t^{*,\delta} e^{V_t} X_t^{*,\delta} dW_t^1, \tag{3.23}$$

where, by Lemma 18, the optimal control $(q_t) = (q^{*,\delta})$ is explicitly given for sufficiently small δ. (It is straightforward to establish the existence and the uniqueness of the solution of (3.23) $X_t^{*,\delta}$).

Existence and Uniqueness of $X_t^{*,\delta}$

For the existence and uniqueness of the worst case scenario price process, we consider the transformation $Y_t^{*,\delta} = \log X_t^{*,\delta}$ for any $t < \tau^\xi$ and $\xi > 0$, where

$$\begin{aligned} \tau^\xi &:= \inf\{t > 0 | X_t^{*,\delta} = \xi \quad or \quad X_t^{*,\delta} = \frac{1}{\xi}\}, \\ &= \inf\{t > 0 | Y_t^{*,\delta} = \log \xi \quad or \quad Y_t^{*,\delta} = -\log \xi\}. \end{aligned}$$

By applying Ito's formula on $Y_t^{*,\delta}$ we will obtain the following SDE

$$dY_t^{*,\delta} = -\frac{1}{2}(q^{*,\delta})^2 e^{2V_t} dt + q^{*,\delta} e^{V_t} dW_t^1.$$

In order to show (24) has a unique solution, it suffices to prove that for any $T > 0$

$$\lim_{\xi \to 0} \mathbb{Q}(\tau^\xi < T) = 0.$$

$\forall t \in [0, T]$,

$$Y_t^{*,\delta} = \int_0^t -\frac{1}{2}(q^{*,\delta})^2 e^{2V_s} ds + \int_0^t q^{*,\delta} e^{V_s} dW_s^1,$$

then

$$\begin{aligned} \mathbb{Q}(\sup_{t\in[0,T]} |Y_t^{*,\delta}| > |\log \xi|) &\leq \mathbb{Q}\left(\sup_{t\in[0,T]} \left[\int_0^t \frac{1}{2}\sigma_{max}^2 e^{2V_s} ds + \left|\int_0^t q^{*,\delta} e^{V_s} dW_s^1\right|\right] > |\log \xi|\right), \\ &\leq \mathbb{Q}\left(\frac{1}{2}\sigma_{max}^2 \int_0^T e^{2V_s} ds + \sup_{t\in[0,T]} \left|\int_0^t q^{*,\delta} e^{V_s} dW_s^1\right| > |\log \xi|\right), \\ &\leq \mathbb{Q}\left(\frac{1}{2}\sigma_{max}^2 \int_0^T e^{2V_s} ds > \frac{|\log \xi|}{2}\right) + \mathbb{Q}\left(\sup_{t\in[0,T]} \left|\int_0^t q^{*,\delta} e^{V_s} dW_s^1\right| > \frac{|\log \xi|}{2}\right), \\ &=: \mathcal{A} + \mathcal{B}. \end{aligned}$$

By the Markov inequality, we have

$$\mathcal{A} \le \frac{\sigma_{max}^2 \mathbb{E} \int_0^T e^{2V_s} ds}{|\log \xi|} \le \frac{\sigma_{max}^2 T C(T,v)}{|\log \xi|},$$

using Doob's martingale inequality, we have

$$\mathcal{B} \le \frac{\mathbb{E}(\int_0^t q^{*,\delta} e^{V_s} dW_s^1)^2}{\left(\frac{\log \xi}{2}\right)^2} \le \frac{\int_0^t \mathbb{E}\{(q^{*,\delta})^2 e^{2V_s}\} ds}{\left(\frac{\log \xi}{2}\right)^2} \le \frac{\sigma_{max}^2 \int_0^T \mathbb{E}\, e^{2V_s} ds}{\left(\frac{\log \xi}{2}\right)^2} \le \frac{\sigma_{max}^2 T C(T,v)}{\left(\frac{\log \xi}{2}\right)^2}.$$

Therefore,

$$\lim_{\xi \to 0} \mathcal{A} = \lim_{\xi \to 0} \mathcal{B} = 0.$$

Finally, for $T > 0$

$$\lim_{\xi \to 0} \mathbb{Q}(\tau^{\xi} < T) = \lim_{\xi \to 0} \mathbb{Q}(\sup_{t \in [0,T]} |Y_t^{*,\delta}| > |\log \xi|) = 0.$$

Probabilistic Representation of $E^{\delta}(t,x,v)$

We can apply the Feynman-Kac formula to get probabilistic representation of $E^{\delta}(t,x,v)$, namely,

$$E^{\delta}(t,x,v) = I_0 + \delta^{\frac{1}{2}} I_1 + \delta I_2 + \delta^{\frac{3}{2}} I_3,$$

where

$$I_0 = \mathbb{E}_{(t,x,v)}\left[\int_t^T \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right) e^{2V_s} (X_s^{*,\delta})^2 \partial_{xx}^2 P_0(s, X_s^{*,\delta}, V_s) ds\right],$$

$$\begin{aligned} I_1 = \mathbb{E}_{(t,x,v)}\Bigg[\int_t^T & (q^{*,\delta} - q^{*,0}) \rho\sigma e^{V_s} X_s^{*,\delta} \partial_{xv}^2 P_0(s, X_s^{*,\delta}, V_s) \\ & + \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right) e^{2V_s} (X_s^{*,\delta})^2 \partial_{xx}^2 P_1(s, X_s^{*,\delta}, V_s) ds\Bigg], \end{aligned}$$

$$\begin{aligned} I_2 = \mathbb{E}_{(t,x,v)}\Bigg[\int_t^T & q^{*,\delta} \rho\sigma e^{V_s} X_s^{*,\delta} \partial_{xv}^2 P_1(s, X_s^{*,\delta}, V_s) + \frac{1}{2}\sigma^2 \partial_{vv}^2 P_0(s, X_s^{*,\delta}, V_s) \\ & + (a - b e^{\alpha V_s}) \partial_v P_0(s, X_s^{*,\delta}, V_s) ds\Bigg], \end{aligned}$$

$$I_3 = \mathbb{E}_{(t,x,v)}\left[\int_t^T \frac{1}{2}\sigma^2\partial^2_{vv}P_1(s,X^{*,\delta}_s,V_s) + (a-be^{\alpha V_s})\partial_v P_1(s,X^{*,\delta}_s,V_s)ds\right].$$

Noting that

$$\begin{array}{rcl} \{q^{*,\delta}{\neq}q^{*,0}\} &=& \mathcal{A}^\delta_{t,v}, \\ q^{*,\delta}-q^{*,0} &=& (\sigma_{\max}-\sigma_{\min})(\mathbf{1}_{\{\partial^2_{xx}P^\delta\geq 0\}}-\mathbf{1}_{\{\partial^2_{xx}P_0\geq 0\}}), \\ \text{and} \qquad (q^{*,\delta})^2-(q^{*,0})^2 &=& (\sigma^2_{\max}-\sigma^2_{\min})(\mathbf{1}_{\{\partial^2_{xx}P^\delta\geq 0\}}-\mathbf{1}_{\{\partial^2_{xx}P_0\geq 0\}}). \end{array}$$

The next theorem shows that I_0 , I_1 are indeed of order $O(\delta)$ and $O(\sqrt{\delta})$.

Theorem 20. *There exist constants $M_0, M_1 > 0$ depending on (t,x,v), but not on δ, such that*

$$|I_0| \leq M_0\delta, \quad \textit{and} \quad |I_1| \leq M_1\sqrt{\delta}.$$

Proof. **Step 1**

$\mathcal{A}^\delta_{s,v}$ being compact, there exist a constant C_0 such that

$$|\partial^2_{xx}P_0(s,X^{*,\delta}_s,V_s)| \leq C_0\sqrt{\delta}, \text{ for } X^{*,\delta}_s \in \mathcal{A}^\delta_{s,v}.$$

Then, since $0 < \sigma_{min} \leq q^{*,\delta}, q^{*,0} \leq \sigma_{max}$, we have

$$\begin{aligned} |I_0| &\leq \mathbb{E}_{(t,x,v)}\left[\int_t^T \frac{1}{2}|\Big((q^{*,\delta})^2-(q^{*,0})^2\Big)e^{2V_s}(X^{*,\delta}_s)^2|\partial^2_{xx}P_0(s,X^{*,\delta}_s,V_s)ds\right], \\ &\leq \frac{\sigma^2_{max}}{2\sigma^2_{min}}C_0\sqrt{\delta}\mathbb{E}_{(t,x,v)}\left[\int_t^T \mathbf{1}_{\{X^{*,\delta}_s\in\mathcal{A}^\delta_{s,v}\}}(q^{*,\delta})^2e^{2V_s}(X^{*,\delta}_s)^2ds\right]. \end{aligned} \tag{3.24}$$

In order to show that I_0 is of order $O(\delta)$, it suffices to show that there exists a constant C_1 such that

$$\mathbb{E}_{(t,x,v)}\left[\int_t^T \mathbf{1}_{\{X^{*,\delta}_s\in\mathcal{A}^\delta_{s,v}\}}\zeta^2_s ds\right] \leq C_1\sqrt{\delta},$$

where $\zeta_s := q^{*,\delta}e^{V_s}X^{*,\delta}_s$ and $dX^{*,\delta}_s = \zeta_s dW^1_s$ by (24). Define the stopping time

$$\tau(\upsilon) := \inf\{s > t; \langle X^{*,\delta}\rangle_s > \upsilon\},$$

where

$$\langle X^{*,\delta}\rangle_s = \int_t^s \zeta^2(X^{*,\delta}_u)du.$$

We know that $X^{*,\delta}_{\tau(\upsilon)} = B_\upsilon$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}^B_\upsilon, \mathbb{Q}^B_\upsilon)$.

From the definition of $\tau(\upsilon)$ given above, we have

$$\int_t^{\tau(\upsilon)} \zeta^2(X^{*,\delta}_s) ds = \upsilon,$$

which tells us that the inverse function of $\tau(\upsilon)$ is

$$\tau^{-1}(T) = \int_t^T \zeta^2(X^{*,\delta}_s) ds. \tag{3.25}$$

Next use the substitution $s = \tau(\upsilon)$ and for any $i \in [1, m(\upsilon)]$, we have

$$\begin{aligned}
\int_t^T \mathbf{1}_{\{|X^{*,\delta}_s - x_i| < C\sqrt{\delta}\}} \zeta^2(X^{*,\delta}_s) ds &= \int_t^{\tau^{-1}(T)} \mathbf{1}_{\{|X^{*,\delta}_{\tau(\upsilon)} - x_i < C\sqrt{\delta}\}} \zeta^2(X^{*,\delta}_{\tau(\upsilon)}) d\tau(\upsilon), \\
&= \int_t^{\tau^{-1}(T)} \mathbf{1}_{\{|X^{*,\delta}_{\tau(\upsilon)} - x_i < C\sqrt{\delta}\}} \zeta^2(X^{*,\delta}_{\tau(\upsilon)}) \frac{1}{\zeta^2(X^{*,\delta}_{\tau(\upsilon)})} d\upsilon, \\
&= \int_t^{\tau^{-1}(T)} \mathbf{1}_{\{|X^{*,\delta}_{\tau(\upsilon)} - x_i < C\sqrt{\delta}\}} d\upsilon, \\
&= \int_t^{\tau^{-1}(T)} \mathbf{1}_{\{|B_\upsilon - x_i < C\sqrt{\delta}\}} d\upsilon.
\end{aligned} \tag{3.26}$$

Note that on the set $\{|B_\upsilon - x_i| < C\sqrt{\delta}\}$, we have $(X^{*,\delta}_s)^2 \le (x_i + C\sqrt{\delta})^2 \le D$, where D is a positive constant, and then by (3.25) we have

$$\tau^{-1}(T) = \int_t^T (q^{*,\delta} e^{V_s} X^{*,\delta}_s)^2 ds \le D\sigma^2_{max} T \sup_{t \le s \le T} e^{2V_s}. \tag{3.27}$$

Then from (3.27) and (3.26), by decomposing in $\{\sup_{t\le s\le T} e^{2V_s} \le M\}$ and $\{\sup_{t\le s\le T} e^{2V_s} > M\}$ for any $M > e^{2v}$, we obtain

$$\mathbb{E}_{(t,x,v)}\left[\int_t^{\tau^{-1}(T)} \mathbf{1}_{\{|B_\upsilon - x_i < C\sqrt{\delta}\}} d\upsilon\right] \sim O(\sqrt{\delta}). \tag{3.28}$$

Step 2

With the help of assumption 2.12 in [21] , we have

$$
\begin{aligned}
|I_1| &= \mathbb{E}_{(t,x,v)}\bigg[\int_t^T |q^{*,\delta} - q^{*,0}|\rho\sigma e^{V_s}X_s^{*,\delta}|\partial^2_{xv}P_0(s,X_s^{*,\delta},V_s)| \\
&\quad + \frac{1}{2}|(q^{*,\delta})^2 - (q^{*,0})^2|e^{2V_s}(X_s^{*,\delta})^2|\partial^2_{xx}P_1(s,X_s^{*,\delta},V_s)|ds\bigg], \\
&\le \frac{\rho\sigma_{max}}{\sigma^2_{min}}\mathbb{E}_{(t,x,v)}\bigg[\int_t^T \mathbf{1}_{\{X_s^{*,\delta}\in\mathcal{A}^\delta_{s,v}\}}(q^{*,\delta})^2 e^{2V_s}X_s^{*,\delta}a_{11}(1+(X_s^{*,\delta})^{b_{11}} + V_s^{c_{11}})ds\bigg] \\
&\quad + \frac{\rho\sigma_{max}}{\sigma^2_{min}}\mathbb{E}_{(t,x,v)}\bigg[\int_t^T \mathbf{1}_{\{X_s^{*,\delta}\in\mathcal{A}^\delta_{s,v}\}}(q^{*,\delta})^2 e^{2V_s}X_s^{*,\delta}\bar{a}_{20}(1+(X_s^{*,\delta})^{\bar{b}_{20}} + V_s^{\bar{c}_{20}})ds\bigg].
\end{aligned}
$$

Using the same techniques in Step 1, the result that $X_s^{*,\delta}$ and V_s have finite moments uniformly in δ, and $X_s^{*,\delta} \le C(X_s^{*,\delta})^2$ on $\{X_s^{*,\delta} \in \mathcal{A}^\delta_{s,v}\}$, we can deduce that I_1 is of order $O(\sqrt{\delta})$. □

Proof of Uniform Boundedness of I_2 and I_3 on δ.

Because that V_t , X_t have uniformly bounded moments, by using the Cauchy-Schwarz inequality and with the help of Assumption 2.12 in[21] We are going to prove that I_2 and I_3 are uniformly bounded in δ.

First recall that

$$
\begin{aligned}
I_2 &= \mathbb{E}_{(t,x,v)}\bigg[\int_t^T \rho\sigma(q^{*,\delta})e^{V_s}X_s^{*,\delta}\partial^2_{xv}P_1(s,X_s^{*,\delta},V_s) \\
&\quad + \frac{1}{2}\sigma^2\partial^2_{vv}P_0(s,X_s^{*,\delta},V_s) + (a - be^{\alpha V_s})\partial_v P_0(s,X_s^{*,\delta},V_s)ds\bigg], \\
&= I_2^{(1)} + I_2^{(2)} + I_2^{(3)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
I_2^{(1)} &\le \mathbb{E}_{(t,x,v)}\bigg[\int_t^T \rho\sigma\sigma_{max}e^{V_s}X_s^{*,\delta}|\partial^2_{xv}P_1(s,X_s^{*,\delta},V_s)|ds\bigg], \\
&\le \rho\sigma\sigma_{max}\mathbb{E}^{1/2}_{(t,x,v)}\bigg[\int_t^T \left(e^{V_s}X_s^{*,\delta}\right)^2 ds\bigg].\mathbb{E}^{1/2}_{(t,x,v)}\bigg[\int_t^T \left(\partial^2_{xv}P_1(s,X_s^{*,\delta},V_s)\right)^2 ds\bigg], \\
&\le \rho\sigma\sigma_{max}\mathbb{E}^{1/4}_{(t,x,v)}\bigg[\int_t^T (e^{V_s})^4 ds\bigg].\mathbb{E}^{1/4}_{(t,x,v)}\bigg[\int_t^T (X_s^{*,\delta})^4 ds\bigg].\bar{a}^2_{11}\mathbb{E}^{1/2}_{(t,x,v)}\bigg[\int_t^T \left(1+|X_s^{*,\delta}|^{\bar{b}_{11}} + |V_s|^{\bar{c}_{11}}\right)^2 ds\bigg], \\
&\le \rho\sigma\sigma_{max}(C_4(T,v))^{1/4}.(N_4(T,x,v))^{1/4}.\bar{A}_{11}[C_{2\bar{b}_{11}}(T,v) + N_{2\bar{c}_{11}}(T,x,v)]^{1/2}.
\end{aligned}
$$

$$
\begin{aligned}
I_2^{(2)} &\leq \frac{1}{2}\sigma^2(T-t)^{1/2}.\mathbb{E}^{1/2}_{(t,x,v)}\left[\int_t^T \left(\partial^2_{vv}P_0(s,X_s^{*,\delta},V_s)\right)^2 ds\right],\\
&\leq \frac{1}{2}\sigma^2(T-t)^{1/2}.A_{02}[C_{2b_{02}}(T,v)+N_{2c_{02}}(T,x,v)]^{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2^{(3)} &\leq \mathbb{E}^{1/2}(t,x,v)\left[\int_t^T (a-be^{\alpha V_s})^2 ds.\mathbb{E}^{1/2}_{(t,x,v)}\int_t^T \left(\partial_v P_0(s,X_s^{*,\delta},V_s)\right)^2 ds\right],\\
&\leq \mathbb{E}^{1/2}_{(t,x,v)}\left[\int_t^T a^2+b^2e^{2\alpha V_s}ds\right].\mathbb{E}^{1/2}_{(t,x,v)}\left[\int_t^T \left(\partial_v P_0(s,X_s^{*,\delta},V_s)\right)^2 ds\right],\\
&\leq \frac{1}{2}\left(C_2(T,v)+a^2(T-t)\right)^{1/2}.A_{01}[C_{2b_{01}}(T,v)+N_{2c_{01}}(T,x,v)]^{1/2},
\end{aligned}
$$

where A_{01}, $\bar{A}_{11}$ and A_{02} are positive constants.

Next recall that

$$
\begin{aligned}
I_3 &= \mathbb{E}_{(t,x,v)}\left[\int_t^T \frac{1}{2}\sigma^2\partial^2_{vv}P_1(s,X_s^{*,\delta},V_s)+(a-be^{\alpha V_s})\partial_v P_1(s,X_s^{*,\delta},V_s)ds\right],\\
I_3 &= I_3^{(1)}+I_3^{(2)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
I_3^{(1)} &\leq \frac{1}{2}\sigma^2(T-t)^{1/2}.\mathbb{E}^{1/2}_{(t,x,v)}\left[\int_t^T \left(\partial^2_{vv}P_1(s,X_s^{*,\delta},V_s)\right)^2 ds\right],\\
&\leq \frac{1}{2}\sigma^2(T-t)^{1/2}.\bar{A}_{02}[C_{2\bar{b}_{02}}(T,v)+N_{2\bar{c}_{02}}(T,x,v)]^{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
I_3^{(2)} &\leq \mathbb{E}^{1/2}_{(t,x,v)}\left[\int_t^T a^2+b^2e^{2\alpha V_s}ds\right].\mathbb{E}^{1/2}_{(t,x,v)}\left[\int_t^T \left(\partial_v P_1(s,X_s^{*,\delta},V_s)\right)^2 ds\right],\\
&\leq [a^2(T-t)+C_2(T,v)]^{1/2}.\bar{A}_{01}[C_{2\bar{b}_{01}}(T,v)+N_{2\bar{c}_{01}}(T,x,v)]^{1/2},
\end{aligned}
$$

where $\bar{A}_{01}$, $\bar{A}_{02}$ are positive constants.
We can see that

$$
E^{\delta}(t,x,v) = I_0+\delta^{\frac{1}{2}}I_1+\delta I_2+\delta^{\frac{3}{2}}I_3,
$$

is of order $O(\delta)$.

3.3 Second-Order BSDE Representation of the Worst-Case Scenario

Backward stochastic differential equations (BSDEs) introduced for the first time in Bismut [9] in the linear case, and by Pardoux and Peng [48] in the general case where they become a popular field of research. The theory has found many applications as stochastic control [52], theoretical economics [49], and mathematical finance [18]. On a filtered probability space $(\Omega, F, (F_t)_{t\in[0,T]}, \mathbb{P})$ a solution to BSDE consists of a pair of adapting processes (Y, Z) taking values in $\mathbb{R}^n$ and $\mathbb{R}^{d\times n}$, such that

$$\begin{aligned} dY_t &= f(t, Y_t, Z_t)dt + Z_t dW_t, \quad t \in [0, T] \\ Y_T &= \xi, \end{aligned}$$

where the generator f from $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{d\times n}$ to $\mathbb{R}^n$ is a progressively measurable function, T is a finite time horizon, $(W_t)_{t\in[0,T]}$ a d-dimensional Brownian motion, and ξ an F_T-measurable random variable is terminal condition that the solution is required to satisfy (see [51][58]).

The BSDE is referred to as a forward-backward stochastic differential equation (FBDE), if the randomness in the generator f and the terminal condition ξ is coming from the state of a forward SDE [2, 45]. If the solution of BSDE enter the dynamics of the FSDE then the FBSDEs are called coupled, and uncoupled if it does not, then the solution to the BSDE could be linked to the solution of a semilinear and quasi-linear parabolic PDE by means of generalized Feyman-kac formula. This link opened the way to probabilistic numerical methods for solving this PDEs (For more details see [49] [60]).

However, PDEs corresponding to standard FBSDEs cannot be nonlinear in the second-order derivatives because second-order term arise only linearly through It$\hat{o}$'s formula from the quadratic variation of the underlying state process.

Tauzi [13] introduce FBSDE with second-order dependence in the generator f, they called them second-order backward SDE (2BSDE) and they show how they are related to fully non-linear parabolic PDEs .

We recall the definition of 2BSDE, and we will explain how it is linked to our G-HJB equation; for details, we refer to [13].

Definition 3. *Let $(t, x) \in [0, T) \times \mathbb{R}^d$, $(X_s^{t,x})_{s\in[t,T]}$ a diffusion process and $(Y_s, Z_s, \Gamma_s, A_s)_{s\in[t,T]}$*

a quadruple of $\mathbb{F}^{t,T}$*-progressively measurable processes taking values in* $\mathbb{R}$, $\mathbb{R}^d$, $\mathcal{S}^d$ *and* $\mathbb{R}^d$, *respectively. The quadruple* (Y,Z,Γ,A) *is called a solution to the second order backward stochastic differential equation* (2BSDE) *corresponding to* $(X^{t,x}, f, g)$ *if*

$$
\begin{aligned}
dY_s &= f(s, X_s^{t,x}, Y_s, Z_s, \Gamma_s)\,ds + Z_s' \circ dX_s^{t,x}, \quad s \in [t,T), && (3.29)\\
dZ_s &= A_s\,ds + \Gamma_s\,dX_s^{t,x}, \quad s \in [t,T), && (3.30)\\
Y_T &= g\left(X_T^{t,x}\right), && (3.31)
\end{aligned}
$$

where $Z_s' \circ dX_s^{t,x}$ *denotes Fisk–Stratonovich integration, which is related to Itô integration by*

$$Z_s' \circ dX_s^{t,x} = Z_s'\,dX_s^{t,x} + \frac{1}{2}\,d\left\langle Z, X^{t,x}{}_s\right\rangle = Z_s'\,dX_s^{t,x} + \frac{1}{2}\,\mathrm{Tr}[\Gamma_s \sigma(X_s^{t,x})\sigma(X_s^{t,x})']\,ds.$$

The last definition furnishes a fundamental relation between 2BSDE like (3.29)-(3.31) and fully nonlinear parabolic PDEs. To understand this relation, let $f : [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ be continuous functions. Further assume that $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a continuous function with the properties

$$u_t, Du, D^2u, \mathcal{L}Du \in C^0([0,T) \times \mathbb{R}^d),$$

that solves the PDE

$$-u_t(t,x) + f\left(t, x, u(t,x), Du(t,x), D^2u(t,x)\right) = 0 \quad \text{on } [0,T) \times \mathbb{R}^d, \tag{3.32}$$

with terminal condition

$$u(T,x) = g(x), \quad x \in \mathbb{R}^d. \tag{3.33}$$

Then, it follows directly from Itô's formula that for each pair $(t,x) \in [0,T) \times \mathbb{R}^d$, the processes

$$
\begin{aligned}
Y_s &= u\left(s, X_s^{t,x}\right), \quad s \in [t,T],\\
Z_s &= Du\left(s, X_s^{t,x}\right), \quad s \in [t,T],\\
\Gamma_s &= D^2u\left(s, X_s^{t,x}\right), \quad s \in [t,T],\\
A_s &= \mathcal{L}Du\left(s, X_s^{t,x}\right), \quad s \in [t,T],
\end{aligned}
$$

solve the 2BSDE corresponding to $(X^{t,x}, f, g)$. Conversely, the first component of the solution of the 2BSDE (3.29) at the initial time is a solution of the fully nonlinear PDE (3.32) satisfies $Y_t = u(t,x)$. Note that the representation of (3.32) by a 2BSDE is not unique, even though its solution is (cf. [29]).

The representation of fully nonlinear parabolic PDEs, such as (3.17), allows to solve them numerically by solving the corresponding 2BSDE, e.g. by using the techniques described in [6].

3.3.1 2BSDE Representation of the Payoff

Here we specifically use the link between our G-HJB equation and 2BSDEs to improve the convergence rate of the convergence $P^\delta \to P^0$. To this end we write the 2BSDE for P^δ (resp. P^0) as follows: for all $s \in [t,T)$ it holds that

$$dY_s^{\delta;t,x} = f^\delta(s, \tilde{X}_s^{\delta;t,x}, Y_s^{\delta;t,x}, Z_s^{\delta;t,x}, \Gamma_s^{\delta;t,x})\,ds + (Z^{\delta;t,x})'_s \circ d\tilde{X}_s^{\delta;t,x}, \tag{3.34}$$

$$dZ_s^{\delta;t,x} = A_s^\delta\,ds + \Gamma_s^\delta\,d\tilde{X}_s^{\delta;t,x}, \tag{3.35}$$

$$Y_T^{\delta;t,x} = h\left(\tilde{X}_T^{\delta;t,x}\right), \tag{3.36}$$

where $\tilde{X}$ is the solution to the SDE

$$d(X_t^\delta, V_t) = d\tilde{X}_t = d\tilde{W}_t, \quad d\tilde{W}_t = d(W_t^1, W_t^2), \quad \tilde{X}_0 = \tilde{x}.$$

Similarly,

$$dY_s^{0;t,x} = f^0(s, X_s^{0;t,x}, Y_s^{0;t,x}, Z_s^{0;t,x}, \Gamma_s^0)\,ds + (Z^{0;t,x})'_s \circ dX_s^{0;t,x}, \tag{3.37}$$

$$dZ_s^{0;t,x} = A_s^0\,ds + \Gamma_s^0\,dX_s^{0;t,x}, \tag{3.38}$$

$$Y_T^{0;t,x} = h\left(X_T^{0;t,x}\right), \tag{3.39}$$

where X_t^0 is the solution to

$$dX_t^0 = dW_t^1, \quad X_0 = x.$$

Here h denotes the payoff function (specified below), and

$$f^0(s,x,y,z,S) = -\frac{1}{2}x^0 e^{2v}|\bar{\sigma}(S_{1,1})|^2 S_{1,1}$$

$$f^\delta(s,\tilde{x},y,z,S) = -\frac{1}{2}\tilde{x}^\delta e^{2v}|\bar{\sigma}(S_{1,1})|^2 S_{1,1} - 2\sqrt{\delta}\tilde{x}^\delta e^v \sigma\rho|\bar{\sigma}(S_{1,2})|S_{1,2} - \delta\left(\frac{1}{2}\sigma^2 S_{2,2} + (a - be^{\alpha v})z_2\right),$$

where,

$$\bar{\sigma} = \begin{cases} \sigma_{\max} & x \geq 0 \\ \sigma_{\min} & x < 0 \end{cases}. \tag{3.40}$$

Note that the nonlinear diffusion coefficient has been moved to the drift terms (or: drivers) f^0 and f^δ, which is why the SDE dynamics is trivial. Then from the link between G-PDEs and 2BSDEs we have $Y_t^{0;t,x} = P^0(t,x)$ and $Y_t^{\delta;t,x} = P^\delta(t,x)$.
We will now use this link to revisit the convergence result for $P^\delta \to P^0$.

Theorem 21 ([47])**.** *P^δ converges to P^0 as $\delta \to 0$, uniformly on compact sets and at rate δ.*

Proof. We have

$$\begin{aligned}
Y_t^{\delta;t,x} &= h(\tilde{X}_T^{\delta;t,x}) + \int_t^T f^\delta(r, \tilde{X}_r^{\delta;s,x}, Y_r^{\delta;s,x}, Z_r^{\delta;s,x}, \Gamma_r^{\delta;s,x})dr \\
&- \int_t^T (Z^{\delta;s,x})'_r \circ d\tilde{X}_r^{\delta;s,x}, \\
(Z^{\delta;s,x})'_r \circ d\tilde{X}_r^{\delta;s,x} &= (Z^{\delta;s,x})'_r d\tilde{X}_r^{\delta;s,x} + \frac{1}{2}\mathrm{Tr}[\Gamma_r^\delta \sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})']dr, \\
(Z^{\delta;s,x})'_r d\tilde{X}_r^{\delta;s,x} &= (Z_1^{\delta;t,x})'_r dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2,
\end{aligned}$$

and thus

$$\begin{aligned}
Y_t^{\delta;t,x} = h(\tilde{X}_T^{\delta;t,x}) + \int_t^T f^\delta(r, \tilde{X}_r^{\delta;s,x}, Y_r^{\delta;s,x}, Z_r^{\delta;s,x}, \Gamma_r^{\delta;s,x})dr - \int_t^T ((Z_1^{\delta;s,x})'_r dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2) \\
- \int_t^T \frac{1}{2}\mathrm{Tr}[\Gamma_r^\delta \sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})']dr.
\end{aligned}$$

$$\begin{aligned}
Y_t^{0;t,x} &= h(X^{0;t,x}) + \int_t^T f^0(r, X_r^{0;s,x}, Y_r^{0;s,x}, Z_r^{0;s,x}, \Gamma_r^{0;s,x})dr \\
&- \int_t^T (Z^{0;s,x})'_r \circ dX_r^{0;s,x}, \\
(Z^{0;s,x})'_r \circ d\tilde{X}_r^{0;s,x} &= (Z^{0;s,x})'_r d\tilde{X}_r^{0;s,x} + \frac{1}{2}\mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})']dr.
\end{aligned}$$

Calling $\tilde{Z}_r^{s,x} = (Z_r^{0;s,x}, 0)$

$$(Z^{0;s,x})'_r d\tilde{X}_r^{0;s,x} = (\tilde{Z}_s)' d\tilde{X}_r^{\delta;s,x} = (Z^{0;s,x})'_r dW_r^1 + 0,$$

we obtain

$$\begin{aligned}
Y_t^{0;t,x} = h(X_T^{0;t,x}) + \int_t^T f^0(r, X_r^{0;s,x}, Y_r^{0;s,x}, Z_r^{0;s,x}, \Gamma_r^{0;s,x})dr - \int_t^T (Z^{0;s,x})'_r dW_r^1 \\
- \int_t^T \frac{1}{2}\mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})']dr.
\end{aligned}$$

Now let $y_t = Y_t^{\delta;t,x} - Y_t^{0;t,x}$. Then

$$\begin{aligned}
y_t =& h(\tilde{X}_T^{\delta;t,x}) - h(X_T^{0;t,x}) + \int_t^T f^\delta(r, \tilde{X}_r^{\delta;s,x}, Y_r^{\delta;s,x}, Z_r^{\delta;s,x}, \Gamma_r^{\delta;s,x}) \\
&- f^0(r, X_r^{0;s,x}, Y_r^{0;s,x}, Z_r^{0;s,x}, \Gamma_r^{0;s,x})dr - \int_t^T (((Z_1^{\delta;s,x})'_r dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2) \\
&- (Z^{0;s,x})'_r dW_r^1) - \mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})'])) \\
=& h(\tilde{X}_T^{\delta;t,x}) - h(X_T^{0;t,x}) + \int_t^T f^\delta(r, \tilde{X}_r^{\delta;s,x}, Y_r^{\delta;s,x}, Z_r^{\delta;s,x}, \Gamma_r^{\delta;s,x}) \\
&- f^0(r, X_r^{0;s,x}, Y_r^{0;s,x}, Z_r^{0;s,x}, \Gamma_r^{0;s,x})dr - \int_t^T (((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2) \\
&- \int_t^T \frac{1}{2}(\mathrm{Tr}[\Gamma_r^\delta \sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})'] - \mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})'])dr,
\end{aligned}$$

where

$$\begin{aligned}
& f^\delta(r, \tilde{X}_r^{\delta;s,x}, Y_r^{\delta;s,x}, Z_r^{\delta;s,x}, \Gamma_r^{\delta;s,x}) - f^0(r, X_r^{0;s,x}, Y_r^{0;s,x}, Z_r^{0;s,x}, \Gamma_r^{0;s,x}) \\
&= -\frac{1}{2}(\tilde{x}^\delta e^{2V_t} - x^0 e^{2v})|\sigma(\Gamma_{11})|^2\Gamma_{11} - 2\sqrt{\delta}\sigma\rho\tilde{x}^\delta e^{V_t}|\sigma(\Gamma_{12})|\Gamma_{12} - \delta(\frac{1}{2}\sigma^2\Gamma_{22} + (a + be^{\alpha V_t})z_2^\delta).
\end{aligned}$$

Applying Itô's formula to $e^{\alpha t}|y_t|^2$ for some $\alpha > 0$ then yields

$$\begin{aligned}
d(e^{\alpha t}|y_t|^2) =& \alpha e^{\alpha s}|y_s|^2 ds \\
&- 2e^{\alpha s}|y_s|\{f^\delta(s, \tilde{X}_s^{\delta;t,x}, Y_s^{\delta;t,x}, Z_s^{\delta;t,x}, \Gamma_s^{\delta;t,x}) - f^0(s, X_s^{0;t,x}, Y_s^{0;t,x}, Z_s^{0;t,x}, \Gamma_s^{0;t,x})\}ds \\
&+ 2e^{\alpha s}|y_s|\{((Z_1^{\delta;t,x})'_s - (Z^{0;t,x})'_s)dW_s^1 + (Z_2^{\delta;t,x})'_s dW_s^2\} \\
&+ e^{\alpha s}(\mathrm{Tr}[\Gamma_s^\delta \sigma(\tilde{X}_s^{\delta;t,x})\sigma(\tilde{X}_s^{\delta;t,x})'] - \mathrm{Tr}[\Gamma_s^0 \sigma(\tilde{X}_s^{0;t,x})\sigma(\tilde{X}_s^{0;t,x})'])ds \\
&+ e^{\alpha s}\{\|((Z_1^{\delta;t,x})'_s - (Z^{0;t,x})'_t)\|^2 + \|(Z_2^{\delta;t,x})'_s\|^2\}ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& e^{\alpha t}|y_t|^2 + \int_t^T e^{\alpha r}\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 - \|(Z_2^{\delta;s,x})'_r\|^2\}dr \\
&\quad + \int_t^T e^{\alpha r}(\mathrm{Tr}[\Gamma_r^\delta \sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})'] - \mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})'])dr, \\
&= h(\tilde{X}_T^{\delta;t,x}) - h(X_T^{0;t,x}) + \int_t^T e^{\alpha r}(-\alpha)|y_s|^2 dr \\
&\quad + \int_t^T 2|y_s|\{f^\delta(r, \tilde{X}_r^{\delta;s,x}, Y_r^{\delta;s,x}, Z_r^{\delta;s,x}, \Gamma_r^{\delta;s,x}) - f^0(r, X_r^{0;s,x}, Y_r^{0;s,x}, Z_r^{0;s,x}, \Gamma_r^{0;s,x})\})dr \\
&\quad - \int_t^T 2e^{\alpha r}|y_s|\{((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2\}.
\end{aligned}$$

Since for all $\varepsilon > 0$, we have $2ab \le a^2/\varepsilon + \varepsilon b^2$, it follows that

$$\begin{aligned}
e^{\alpha t}|y_t|^2 &+ \int_t^T e^{\alpha r}\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 - \|(Z_2^{\delta;s,x})'_r\|^2\}dr \\
&+ \int_t^T e^{\alpha r}(\mathrm{Tr}[\Gamma_r^\delta \sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})'] - \mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})'])dr, \\
\le & h(\tilde{X}_T^{\delta;t,x}) - h(X_T^{0;t,x}) + \int_t^T e^{\alpha r}(-\alpha|y_s|^2 dr \\
&+ \int_t^T (|ys|^2/\varepsilon + \varepsilon\{f^\delta(r,\tilde{X}_r^{\delta;s,x},Y_r^{\delta;s,x},Z_r^{\delta;s,x},\Gamma_r^{\delta;s,x}) \\
&- f^0(r,X_r^{0;s,x},Y_r^{0;s,x},Z_r^{0;s,x},\Gamma_r^{0;s,x})\}^2)dr \\
&- \int_t^T 2e^{\alpha r}|y_s|\{((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2\}.
\end{aligned}$$

Therefore, setting $\alpha = \frac{1}{\varepsilon}$, we conclude

$$\begin{aligned}
e^{\alpha t}|y_t|^2 &+ \int_t^T e^{\alpha r}\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 + \|(Z_2^{\delta;s,x})'_r\|^2\}dr \\
&+ \int_t^T e^{\alpha r}(\mathrm{Tr}[\Gamma_r^\delta \sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})'] - \mathrm{Tr}[\Gamma_r^0 \sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})'])dr \\
\le & h(\tilde{X}_T^{\delta;t,x}) - h(X_T^{0;t,x}) \\
&+ \varepsilon \int_t^T \{f^\delta(r,\tilde{X}_r^{\delta;s,x},Y_r^{\delta;s,x},Z_r^{\delta;s,x},\Gamma_r^{\delta;s,x}) - f^0(r,X_r^{0;s,x},Y_r^{0;s,x},Z_r^{0;s,x},\Gamma_r^{0;s,x})\}^2)dr \\
&- \int_t^T 2e^{\alpha r}|y_s|\{((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)dW_r^1 + (Z_2^{\delta;s,x})'_r dW_r^2\}.
\end{aligned} \tag{3.41}$$

Because X_t and V_t have finite moments of any order, the imposed regularity condition on h, together with [26, Thm. 5.2.2 , thm. 5.2.5], Theorem 16, Proposition 15, and Proposition 17 in this thesis, imply

$$\mathbb{E}(h(\tilde{X}_T^{\delta;t,x}) - h(X_T^{0;t,x})) \le C\delta,$$

and

$$\mathbb{E}(\{f^\delta(r,\tilde{X}_r^{\delta;s,x},Y_r^{\delta;s,x},Z_r^{\delta;s,x},\Gamma_r^{\delta;s,x}) - f^0(r,X_r^{0;s,x},Y_r^{0;s,x},Z_r^{0;s,x},\Gamma_r^{0;s,x})\}^2) \le C_0\delta.$$

Hence

$$\mathbb{E}\left[\sup_{t\le s\le T} e^{\alpha t}|y_t|^2\right]$$
$$\le C\delta + C_0\varepsilon\delta + C_1\mathbb{E}\left[\left(\int_t^T e^{2\alpha r}|y_s|^2\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 + \|(Z_2^{\delta;s,x})'_r\|^2\}dr\right)^{\frac{1}{2}}\right],$$
$$\le C\delta + C_0\varepsilon\delta + C_1\mathbb{E}\left[\sup_{t\le s\le T} e^{\alpha t/2}|y_t|\left(\int_t^T e^{\alpha r}\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 + \|(Z_2^{\delta;s,x})'_r\|^2\}dr\right)^{\frac{1}{2}}\right],$$

which together with the inequality $ab \le a^2/2 + b^2/2$ yields

$$\mathbb{E}\left[\sup_{t\le s\le T} e^{\alpha t}|y_t|^2\right] \le C\delta + C_0\varepsilon\delta + \frac{1}{2}\mathbb{E}\left[\sup_{t\le s\le T} e^{\alpha t}|y_t|^2\right]$$
$$+\frac{C_1^2}{2}\mathbb{E}\left[\int_t^T e^{\alpha r}\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 + \|(Z_2^{\delta;s,x})'_r\|^2\}dr\right].$$

As a consequence of the inequality (3.41), we thus obtain

$$\mathbb{E}\left[\sup_{t\le s\le T} e^{\alpha t}|y_t|^2 + \int_t^T e^{\alpha r}\{\|((Z_1^{\delta;s,x})'_r - (Z^{0;s,x})'_r)\|^2 + \|(Z_2^{\delta;s,x})'_r\|^2\}dr\right.$$
$$\left.+2\int_t^T e^{\alpha r}(\mathrm{Tr}[\Gamma_r^\delta\sigma(\tilde{X}_r^{\delta;s,x})\sigma(\tilde{X}_r^{\delta;s,x})'] - \mathrm{Tr}[\Gamma_r^0\sigma(\tilde{X}_r^{0;s,x})\sigma(\tilde{X}_r^{0;s,x})'])dr\right]$$
$$\le C\delta + C_0\varepsilon\delta + C_1^2,$$

which entails the final result

$$\mathbb{E}\left[\sup_{t\le s\le T} e^{\alpha t}|y_t|^2\right] \le \delta\tilde{C}_\varepsilon,$$

for some $\tilde{C}_\varepsilon > 0$ independent of δ. □

3.3.2 Numerical Illustration

We conclude with a numerical demonstration of the theoretical results to confirm that $|P^\delta - P^0| = \mathcal{O}(\delta)$. To this end, note that the valuation of financial derivatives based on our UV model requires solving the G-HJB equation (3.9), which is typically not analytically solvable.

In low dimension, we can implement a finite difference scheme; here we follow a different route and take advantage of the link between G-PDE and 2BSDE. To be specific the payoff function is chosen as

$$h(x) = (x-90)^+ - 2(x-100)^+ + (x-110)^+ .$$

We consider the following parameters

$$\tilde{x} = (\tilde{x}_0, k_0) = (100, -1)\,,\ \sigma_{\min} = 0.1, \sigma_{\max} = 0.2\,,\ \alpha = 2\,,$$

$$T = 0.15\,,\ a = 0.6\,,\ b = 0.5\,,\ \rho = 0.5\,.$$

For these parameters, we compute the difference between P^δ and P^0, the solutions of the G-PDE (3.9) and (3.11), using the deep learning 2BSDE solver introduced by Beck et al. [6]. More specifically, we numerically solve the 2BSDEs (3.34)-(3.36) and (3.37)-(3.39) with the Python code provided in [6].

The result is shown in Table 3.1 and Figure 3.1. Neglecting the error invoked by the numerical approximation of the deep neural network, which is difficult to assess, the numerical calculation confirms that $|P^\delta - P^0| \simeq O(\delta^{0.7})$, which is in agreement with the predictions of Theorem 16 and Theorem 21.

δ	0.5	0.2	0.001
$error(\delta)$	1.2	0.6	0.02

Table 3.1. The error $\varepsilon^{0,x}(\delta) = P^{\delta}(0,x) - P^{0}(0,x)$ for $\tilde{x} = (100, -1)$.

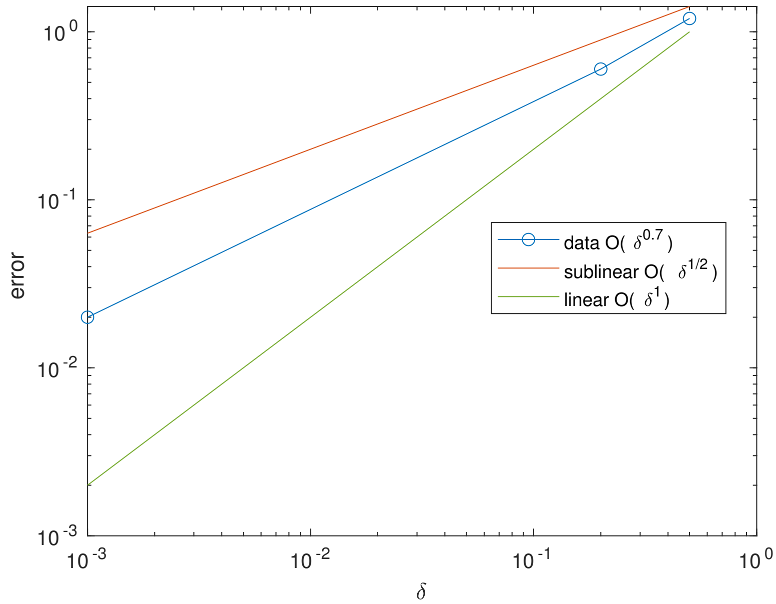

Figure 3.1. The error $\varepsilon^{0,x}(\delta) = P^{\delta}(0,x) - P^{0}(0,x)$ in doubly logarithmic scale; the slope is roughly 0.7.

Chapter 4

Hedging Strategies

In this chapter, we analyse different implications of the stochastic behavior of asset prices volatilities for option hedging purposes. We assuming a stochastic volatility environment, we study the accuracy of Black and Scholes implied volatility-based hedging. More precisely, we analyse the hedging ratios biases and investigate different hedging schemes in a dynamic setting. Also, we look at how the perturbation analysis helps with the risk management problem of hedging a derivative position (see [23]).

4.1 General Thoughts on Hedging

A stochastic volatility option pricing model is a special case of the two-state financial model, with two sources of risk.

$$\begin{bmatrix} dX(t) \\ d\sigma(t) \end{bmatrix} = \begin{bmatrix} \mu X(t) \\ \mu_2(t, X(t), \sigma(t)) \end{bmatrix} dt + \begin{bmatrix} \sigma(t)X(t) & \sigma_{1,2}(t, X(t), \sigma(t)) \\ \sigma_{2,1}(t, X(t), \sigma(t)) & \sigma_{2,2}(t, X(t), \sigma(t)) \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix},$$

with $\mathbb{E}[W_1(t)W_2(t)] = \rho t$.

Option price depends on several parameters, such as the underlying stock price $X(t)$ and its volatility $\sigma(t)$, the risk-free interest rate $r(t)$, and the time to maturity T.The sensitivity of the price of an option to the variations of its parameters is measured by using partial derivatives. These quantities, known as Greeks.

The Greeks of option prices which considered by the practitioners as following

$$
\begin{aligned}
\Delta &= \frac{\partial P}{\partial x} : \text{Delta is the rate of change of the portfolio value with respect to the asset price.}\\
\Gamma &= \frac{\partial^2 P}{\partial x^2} : \text{ Gamma is the rate of change of delta with respect to the asset price.}\\
\mathcal{V} &= \frac{\partial P}{\partial \sigma} : \text{ Vega is the rate of change of the portfolio value with respect to the asset's volatility.}\\
R &= \frac{\partial P}{\partial r} : \text{ Rho is the derivative of the option value with respect to the risk free interest rate.}\\
\Theta &= \frac{\partial P}{\partial t} : \text{ measures the sensitivity of the value of the derivative to the passage of time.}
\end{aligned}
$$

play a crucial role in trading and managing portfolios of options [30].

To hedging the exposure of the portfolio to market risk, practitioners use the delta, gamma and vega measures to quantify the different aspects of this inherent risk. They try to immune their option portfolio from the small changes in the price of the underlying asset (delta/gamma hedging) and its volatility (sigma hedging), so we need for very accurate computing of Greeks.

With the close-to-close historic volatility, the hedging ratios calculated by Black-Scholes model fail to realize a well-hedged position [55] [56]. To improve the hedging properties of the Black-Scholes model, usually we use the Black-Scholes implied volatility. However, a various biases in option hedging strategies may be produced by using Black -Scholes implied volatilities in conjunction with the Black-Scholes computed Greeks, in the existence of stochastic volatility.

4.1.1 Hedging Ratios Biases

According to Renault and Touzi (1996) [55], we define the delta hedging bias as the difference between the Black-Scholes implied volatility-based delta and the stochastic volatility model's one

$$\Delta_t^{BS}(x,\sigma^i) - \Delta_t^{SV}(x,\sigma).$$

Renault and Touzi proved that, provided we have $\rho = 0$, we verify $\forall x \geq 0$ and $\sigma > 0$

$$
\begin{aligned}
\Delta^{BS}(x,\sigma^i(x,\sigma)) &\leq \Delta^{SV}(x,\sigma),\\
\Delta^{BS}(-x,\sigma^i(-x,\sigma)) &\geq \Delta^{SV}(-x,\sigma),\\
\Delta^{BS}(0,\sigma^i(0,\sigma)) &= \Delta^{SV}(0,\sigma).
\end{aligned}
$$

For an in-the-money (out-of-the-money) option, the use of Black-Scholes implied volatility leads to an underhedged (overhedged) position.

Proposition 22. *$\forall x$ and $\forall \sigma > 0$ we have*

$$\begin{array}{lllllll} \rho & \to & -1 & \Longrightarrow & \Delta^{BS}(x, \sigma^i(x,\sigma)) & \leq & \Delta^{SV}(x,\sigma), \\ \rho & \to & +1 & \Longrightarrow & \Delta^{BS}(x, \sigma^i(x,\sigma)) & \geq & \Delta^{SV}(x,\sigma). \end{array}$$

Whatever the moneyness of the option, the Black-Scholes implied volatility-based delta hedging leads systematically to an underhedged position when ρ strongly negative. On the other hand, for ρ strongly positive, the use of Black-Scholes implicit volatility leads systematically too to an overhedged position [55].
In order to keep the portfolio delta neutral, gamma, which measures the rate of change in the delta with respect to changes in the underlying asset price, reflects the need of relatively frequent adjustments in this portfolio. Delta changes slowly and rebalancing to maintain a neutral portfolio can be performed relatively less frequently, if gamma is small. Further, if gamma is large, delta is highly sensitive to the price of underlying asset and good management of options portfolio requires an active delta hedging [30].

$$\begin{array}{lllllllll} \rho & = & 0 & \Longrightarrow & \Gamma^{BS}(x,\sigma^i) & \leq & \Gamma^{SV}(x,\sigma) & \text{for} & x \to 0, \\ \rho & \to & -1 & \Longrightarrow & \Gamma^{BS}(x,\sigma^i) & \leq & \Gamma^{SV}(x,\sigma) & \text{for} & x < 0, \\ \rho & \to & +1 & \Longrightarrow & \Gamma^{BS}(x,\sigma^i) & \leq & \Gamma^{SV}(x,\sigma) & \text{for} & x > 0. \end{array}$$

With a view to make a portfolio gamma neutral, we need a new $-\frac{\Gamma}{\Gamma_0}$ position in a traded option, where Γ and Γ_0 are respectively the gamma of the portfolio and of the traded option. As in the pure delta hedging case, the accurate computation of Greeks is crucial [55].
Such in the gamma case, taking a new $-\frac{\mathcal{V}}{\mathcal{V}_0}$ position in a traded option, allow us to make a portfolio immune to changes in volatility of the underlying asset price, where $\mathcal{V}$ is the vegas of the portfolio and $\mathcal{V}_0$ the vegas of a traded option. Next, in order to keep delta-neutrality of the portfolio we must readjust our position in the underlying asset. This type of strategy is called delta-sigma hedging. Bajeux and Rochet(1992) [4] have proved that the hedging problem can be solved through a delta-sigma hedging strategy, in a stochastic volatility context with $\rho = 0$.

4.1.2 Option Hedging Strategies in Stochastic Volatility Environment

A financial institution that sells an option faces the problem of managing its market risk. For example, the hedging problem for a financial institution that writes at time t_0 a European call option of price $C(t_0)$ consists of producing a wealth of $\max[X(T)-K,0]$ at the maturity time T.
In the Black-Scholes world, where volatilities of asset prices are constant, pure delta hedging suffices to solve the hedging problem [30]. A short position in an option is hedged with a time-varying long position in the underlying stock. At any given time, the long positions are readjusted to equal the delta of the option position. When the hedge is rebalanced continuously, the actualized cost of this strategy is exactly equal to the price $C(t_0)$ of the option : the net hedge cost is zero [55].

4.1.3 Pure Delta Hedging

Take for example, a continuously rebalanced hedge portfolio composed of a short position in one European call option and a long position in α underlying stocks

$$\Pi(t) = -C(t) + \alpha X(t). \tag{4.1}$$

The construction of this portfolio is financed by a loan constant risk-free interest rate r. The instantaneous change in the value of the hedge portfolio Π is given by

$$R(\Pi(t)) = -r\Pi(t)dt + d\Pi(t), \tag{4.2}$$

with

$$\begin{aligned} d\Pi(t) &= D\Pi dt + \Pi_s(t,X(t),\sigma(t))\sigma(t)X(t)dW_1(t) \\ &+ \Pi_\sigma(t,X(t),\sigma(t))\sigma_{2,2}(t,X(t),\sigma(t))dW_2(t), \end{aligned} \tag{4.3}$$

where D is the Dynkin operator.
We have

$$D\Pi = \Pi + \mu X(t)\Pi_s + \mu_2\Pi_\sigma + \frac{1}{2}\sigma^2(t)X^2(t)\Pi_{xx} + \frac{1}{2}\sigma_{2,2}^2\Pi_{\sigma\sigma} + \rho\sigma(t)X(t)\sigma_{2,2}\Pi_{x\sigma}.$$

If

$$\alpha = C_s = \frac{\partial C}{\partial x}(t,X(t),\sigma(t)), \tag{4.4}$$

the portfolio Π is delta neutral at time t.
To estimate α, we use the Black-Scholes model and the Black-Scholes implied volatility. This leads to

$$\alpha^{BS} = \Delta^{BS} = \Delta^{BS}\left(t, x(t), \sigma^i(t, x(t), \sigma(t))\right). \tag{4.5}$$

So we obtain

$$d\Pi^{BS}(t) = [-DC(t) + \Delta^{BS}\mu X(t)]dt + [\Delta^{BS} - C_X]\sigma(t)X(t)dW_1(t) + C_\sigma\sigma_{2,2}dW_2(t). \tag{4.6}$$

In a stochastic volatility world $C_X = \Delta^{SV}$.
Noting that $HB = \Delta^{BS} - \Delta^{SV}$ gives

$$d\Pi^{BS}(t) = [-DC(t) + \Delta^{BS}\mu X(t)]dt + HB\sigma(t)X(t)dW_1(t) + C_\sigma\sigma_{2,2}dW_2(t). \tag{4.7}$$

The instantaneous change in value of the Black and Scholes implied volatility-based hedge portfolio has two stochastic components. The first arises from the delta hedging bias. The second arises from the fact that the volatility is not hedged at all. The instantaneous variance of $d\Pi^{BS}$ is

$$\frac{var[d\Pi^{BS}(t)|F_t]}{dt} = HB^2\sigma^2(t)X^2(t) + C_\sigma^2\sigma_{2,2}^2 + 2\rho HB\sigma(t)X(t)C_\sigma\sigma_{2,2}. \tag{4.8}$$

Now consider the delta neutral hedge portfolio based on the stochastic volatility option pricing model. In this case, we have

$$\alpha^{SV} = \Delta^{SV} = C_X(t, X(t), \sigma(t)), \tag{4.9}$$

and the instantaneous variance of the change in value of the hedge portfolio is

$$\frac{var[d\Pi^{SV}(t)|F_t]}{dt} = C_\sigma^2\sigma_{2,2}^2. \tag{4.10}$$

Hedging positions of financial institutions are exposed to significant risks, if there is a fail to hedge against stochastic volatility. Delta-sigma hedging able to substantially reduce this risk.

4.1.4 Delta-Sigma Hedging

Consider a continuously rebalanced hedge portfolio consisting of a short position in one European call option, a position in α_2 units of the underlying asset and a position in α_1 units of any other exchange-traded option on the same asset

$$P(t) = -C^1(t) + \alpha_1 C^2(t) + \alpha_2 X(t). \tag{4.11}$$

The setting up of this portfolio is financed by a loan at the constant risk-free interest rate r. The portfolio Π is delta and vega neutral if

$$\begin{cases} \alpha_1 &= \frac{C^1_\sigma}{C^2_\sigma}, \\ \alpha_2 &= C^1_x - \alpha_1 C^2_X. \end{cases} \tag{4.12}$$

Using the Black-Scholes model and the Black-Scholes implied volatility to estimate α_1 and α_2 leads to

$$\begin{cases} \alpha_1^{BS} &= \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}, \\ \alpha_2^{BS} &= \Delta_1^{BS} - \alpha_1^{BS}\Delta_2^{BS}. \end{cases} \tag{4.13}$$

In this case we obtain

$$\begin{aligned} d\Pi^{BS}(t) &= [-DC^1 + \alpha_1^{BS} DC^2 + \alpha_2^{BS}\mu X(t)]dt \\ &\quad +[-C^1_x + \alpha_1^{BS} C^2_x + \alpha_2^{BS}]\sigma(t)X(t)dW_1(t) \\ &\quad +[-C^1_\sigma + \alpha_1^{BS} C^2_\sigma]\sigma_{2,2}dW_2(t). \end{aligned} \tag{4.14}$$

In a stochastic volatility world $C^i_x = \Delta_i^{SV}$ and $C^i_\sigma = \mathcal{V}_i^{SV}$, $i = 1, 2$. Noting that $HB_1 = \Delta_1^{BS} - \Delta_1^{SV}$ and $HB_2 = \Delta_2^{BS} - \Delta_2^{SV}$ gives

$$\begin{aligned} d\Pi^{BS}(t) &= \left[-DC^1 + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}} DC^2 + \left(\Delta_1^{BS} - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\Delta_2^{BS}\right)\mu X(t)\right]dt \\ &\quad + \left[HB_1 - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}} HB_2\right]\sigma(t)X(t)dW_1(t) \\ &\quad + \left[-\mathcal{V}_1^{SV} + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{V}_2^{SV}\right]\sigma_{2,2}dW_2(t). \end{aligned} \tag{4.15}$$

The instantaneous change in the value of the Black and Scholes implied volatility-based hedge portfolio has two stochastic components which arise from the delta and vega hedging biases [56]. The instantaneous variance of $d\Pi^{BS}$ is

$$\begin{aligned} \frac{var[d\Pi^{BS}(t)|F_t]}{dt} &= \left[HB_1 - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}} HB_2\right]^2 \sigma^2(t)X^2(t) + \left[-\mathcal{V}_1^{SV} + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{V}_2^{SV}\right]^2 \sigma^2_{2,2} \\ &\quad +2\rho\left[HB_1 - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}} HB_2\right]\left[-\mathcal{V}_1^{SV} + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{V}_2^{SV}\right]\sigma(t)X(t)\sigma_{2,2}. \end{aligned} \tag{4.16}$$

Now consider the delta-sigma hedging portfolio based on the stochastic volatility option pricing model. In this case, we have

$$\begin{cases} \alpha_1^{SV} &= \frac{\mathcal{V}_1^{VS}}{\mathcal{V}_2^{VS}}, \\ \alpha_2^{SV} &= \Delta_1^{SV} - \alpha_1^{SV}\Delta_2^{SV}. \end{cases} \tag{4.17}$$

If the hedge is rebalanced continuously, the instantaneous variance in the value of this portfolio is zero : the stochastic volatility model based delta-sigma scheme solves the hedging problem.

4.2 Hedging by Perturbation Analysis

In this section, we look at how the perturbation analysis helps with the risk management problem of hedging a derivative position (see [23]).

4.2.1 The Strategy and its Cost Under the α-Hypergeometric Model

According to chapter 3, the dynamic process of the α-hypergeometric model given by

$$\begin{cases} dX_t = rX_t dt + X_t e^{V_t} dW_t^1, \\ dV_t = (a - be^{V_t})dt + \sigma dW_t^2. \end{cases} \tag{4.18}$$

Noting that, $P_{BS}(t,x) = P_0(t,x,v)$ the leading-order term in our price approximation drived in the chapter 3

Such a strategy replication the derivative at maturity T since $P_0(T, X_T, V_T) = h(X_T)$ but it is not self-financing. The portfolio has

$$a_t = \frac{\partial P_0}{\partial x}(t, X_t, V_t), \tag{4.19}$$

stocks,

$$b_t = e^{-rt}\left[P_0(t, X_t, V_t) - X_t \frac{\partial P_0}{\partial x}(t, X_t, V_t)\right], \tag{4.20}$$

bounds at time t. So that its value is

$$a_t X_t + b_t e^{rt} = P_0(t, X_t, V_t). \tag{4.21}$$

The infinitesimal change of $P_0(t, X_t, V_t)$, by using Ito's formula, is given as

$$\begin{aligned} dP_0(t, X_t, V_t) &= \left(\frac{\partial P_0}{\partial t} + \frac{1}{2}e^{2V_t}X_t^2\frac{\partial^2 P_0}{\partial x^2} + \sqrt{\delta}\rho\sigma e^{V_t}X_t\frac{\partial^2 P_0}{\partial x \partial v} + \frac{\delta}{2}\sigma^2\frac{\partial^2 P_0}{\partial v^2}\right)dt \\ &+ a_t dX_t + \frac{\partial P_0}{\partial v}dV_t. \end{aligned} \tag{4.22}$$

Where we have dropped the argument (t, X_t, V_t) in the derivatives of P_0. The infinitesimal change due to the market (the self-financing part) is given by

$$a_t dX_t + rb_t e^{rt} dt, \tag{4.23}$$

and thus the infinitesimal P& L (positive or negative)induced by the strategy is given cost by the difference

$$\begin{aligned}
& dP_0(t, X_t, V_t) - a_t dX_t - rb_t e^{rt} dt \\
= \; & \frac{1}{2}\left(e^{2v_t} - \bar{\sigma}^2(V_t) X_t^2 \frac{\partial^2 P_0}{\partial x^2}\right) dt + \sqrt{\delta}\left(\rho\sigma e^{V_t} X_t \frac{\partial^2 P_0}{\partial x \partial v} + \frac{\sqrt{\delta}}{2}\sigma^2 \frac{\partial^2 P_0}{\partial v^2}\right) dt + \frac{\partial P_0}{\partial v} dV_t, \\
= \; & \frac{1}{2}\left(e^{2V_t} - \bar{\sigma}^2\right) X_t^2 \frac{\partial^2 P_0}{\partial x^2} dt + \sqrt{\delta}\rho\sigma e^{V_t} X_t \frac{\partial^2 P_0}{\partial x \partial v} dt + \sqrt{\delta}\sigma \frac{\partial P_0}{\partial v} dW_t^2 + O(\delta),
\end{aligned} \tag{4.24}$$

where we have used the Black-Scholes equation satisfied by P_0, and then identified the terms of order at most $\sqrt{\delta}$. Noting that $\bar{\sigma} = \bar{\sigma}(v)$ is the local effective volatility estimated from historical returns data.

The corresponding cumulative P& L up to time t, and discounted to time 0, is

$$\begin{aligned}
E_0(t) \; = \; & \frac{1}{2}\int_0^t e^{-rs}\left(e^{2V_s} - \bar{\sigma}^2(V_s)\right) X_t^2 \frac{\partial^2 P_0}{\partial x^2} ds \\
+ \; & \sqrt{\delta}\rho \int_0^t e^{-rs}\sigma e^{V_s} X_s \frac{\partial^2 P_0}{\partial x \partial v} ds \\
+ \; & \sqrt{\delta} \int_0^t e^{-rs}\sigma \frac{\partial P_0}{\partial v} dW_s^2 + O(\delta).
\end{aligned} \tag{4.25}$$

4.2.2 Approximation of the Cost

Due to the volatility factor V_t, the cost $E_0^\delta(t)$ is given in the second line of 4.25 by

$$E_0^\delta(t) = \sqrt{\delta}\rho \int_0^t \sigma e^{v_t} X_t \frac{\partial P_0}{\partial x \partial v} dS + \sqrt{\delta} \int_0^t e^{-rs}\sigma \frac{\partial P_0}{\partial v} dW_s^{(2)} + O(\delta). \tag{4.26}$$

This cost can be writhen

$$E_0^\delta(t) = \sqrt{\delta}(B_t^\delta + M_t^\delta) + O(\delta). \tag{4.27}$$

Where we define the bias

$$\beta_t^\delta = \rho \int_0^t e^{-rs}\sigma e^{v_t} X_s \frac{\partial^2 P_0}{\partial x \partial y}(s, X_s, V_s) ds, \tag{4.28}$$

and the zero-mean martingale

$$M_t^\delta = \int_0^t e^{-rs}\sigma\frac{\partial P_0}{\partial v}(s, X_s, V_s)dW_s^2. \tag{4.29}$$

The cost E_0 induced by a Black-Scholes hedging strategy, given in 4.25, can written

$$E_0 = \sqrt{\delta}(B_t^\delta + M_t^\delta) + O(\delta). \tag{4.30}$$

4.2.3 Mean Self-Financing Hedgin Strategy

To remove the biases in the cumulative cost underlined in the calculation above, we propose to use the perturbation method. we will correct the Hedging strategy in the following way: manage the portfolio made of

$$a_t = \frac{\partial(P_{BS} + Q_{0,1}^\delta)(t, X_t, V_t)}{\partial x}(t, X_t, V_t),$$

shares of the risky asset and

$$b_t = e^{-rt}\left[(P_{BS} + Q_{0,1}^\delta)(t, X_t, V_t) - X_t\frac{\partial(P_{BS} + Q_{0,1}^2)}{\partial x}(t, X_t, V_t)\right], \tag{4.31}$$

shares of the riskless asset, where $P_{BS}(t, X_t) = P_0(t, X_t, V_t)$ in computed at $\bar{\sigma} = \bar{\sigma}(v)$ and $Q_{0,1}^\delta(tx, v)$ solve the following partial differential equation,

$$\mathcal{L}_{BS}(\bar{\sigma}(v))Q_{0,1}^\delta = -2V_1^\delta(v)x\frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma}),$$

with zero terminal conditions $Q_{0,1}^\delta(T, x, v) = 0$.
$Q_{0,1}^\delta$ is given explicitly by

$$Q_{0,1}^\delta = (T-t)V_1^\delta(v)x\frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma}). \tag{4.32}$$

Notice that indeed $Q_{0,1}^\delta$ is small of order $\sqrt{\delta}$.
The hedging ratio a_t is now given by

$$\frac{\partial P_{BS}}{\partial x} + (T-t)V_1^\delta(v)\frac{\partial}{\partial x}(x\frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma})),$$

which is the usual Black-Sholes delta corrected by a combination of Greeks up to fourth-order derivatives [23].

With this new hedging ratio, the infinitesimal cost of the hedging strategy. denoted by $dE_1^Q(t)$, is given by

$$dE_1^Q = d(P_{BS} + Q_{0,1}^{\delta})(t, X_t, V_t) - a_t dX_t - rb_t e^{rt} dt.$$

With the new hedging strategy, we repeat the previous calculation, we find

$$\begin{aligned} dE_1^Q(t) &= \frac{1}{2}(e^{2v_t} - \bar{\sigma}^2(V_t))X_t^2 \frac{\partial^2 (P_{BS} - Q_{0,1}^{\delta})}{\partial x^2} dt + \sqrt{\delta}\rho\sigma e^{v_t} X_t \frac{\partial^2 P_{BS}}{\partial x \partial v} dt \\ &+ \sqrt{\delta}\sigma \frac{\partial P_{BS}}{\partial v} dW_t^2 + \mathcal{P}_{BS}((\bar{\sigma}))(Q_{0,1}^{\delta})dt + O(\delta). \end{aligned}$$

The cumulative cost discounted at time zero is given by

$$\begin{aligned} E_1^Q(t) &= \frac{1}{2}\int_0^t e^{-rs}(e^{2v_s} - \bar{\sigma}^2(v_s))X_s^2 \frac{\partial (P_{BS} + Q_{0,1}^{\delta})}{\partial x^2} ds \\ &+ \sqrt{\delta}\rho \int_0^t e^{-rs} \sigma e^{v_t} X_s \frac{\partial^2 P_{BS}}{\partial x \partial v} ds + \sqrt{\delta} \int_0^t e^{-rs} \sigma \frac{\partial P_{BS}}{\partial v} dW_s^2 \\ &- \int_0^t e^{-rs} V_1^{\delta}(v) X_s \frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma}) ds + O(\delta), \end{aligned} \tag{4.33}$$

where

$$V_1^{\delta} = \frac{\sqrt{\delta}\rho\sigma}{2} < f > \bar{\sigma}(v).$$

So

$$E_1^Q(t) = \sqrt{\delta}(B_t^{\delta} + M_t^{\delta}) - 2\int_0^t e^{-rs} V_1^{\delta}(v_s) X_s \frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma}) ds + O(\delta), \tag{4.34}$$

we then observe that, with this choice of corrections in the hedging ratio and the definition of V_1^{δ}, we have

$$\begin{aligned} \sqrt{\delta} B_t^{\delta} - 2\int_0^t e^{-rs} V_1^{\delta}(v_t) X_s \frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma}) ds &= \rho \sqrt{\delta} \int_0^t e^{-rs} \sigma \{ e^{v_t} - < f > (v_t) \bar{\sigma}(v_s) \} \\ &\times \left[x \frac{\partial}{\partial x}(\frac{\partial P_{BS}}{\partial \sigma}) \right] (s, X_s, V_s) ds. \end{aligned}$$

In the end, all the terms in the corrected cumulative cost 4.33 are of order (δ) except the remaining mean-zero martingale term form 4.30 so that

$$E_1^Q(t) = \sqrt{\delta} M_1^{\delta} + O(\delta).$$

Doing so we have removed the systematic biases in 4.30 and therefore we have reduced the variance of this small "nonhedgable" part of the risk due to stochastic volatility. The strategy is now mean self-financing to order δ [23].

Conclusions and Perspectives

In this chapter, we summarize the obtained results throughout the thesis. Subsequently, we propose some potential problems regarding the future research topics.

Conclusions

This thesis has dealt with α-hypergeometric stochastic models with uncertain volatility (UV). The idea is to connect the UV model with a nonlinear expectation framework to derive a worst-case price scenario, avoiding the complicated and numerically expensive model calibration step. We have studied the asymptotic behavior of the worst-case scenario option prices in the case when the time scale at which the stochastic volatility process varies tends to infinity (i.e. when the volatility process becomes infinitely slow).

As we have shown, the limit model is an accurate simplified description of the UV model in the regime of the slow variable of the uncertain volatility bounds. The method presented here can also be applied for other models such as the Heston model.

We have illustrated our results by a numerical example. The numerical solution of our problem is based on the known link of fully nonlinear second order partial differential equations that describe the worst-case price scenario and second-order backward stochastic differential equations (2BSDEs). We should emphasize that the numerical algorithm we use for solving 2BSDEs even works when the terminal cost that determines the payoff is non-differentiable. Although this paper is only giving a proof of concept, we expect that the ideas can be applied also in the case of UV models when, for example, there is only partial information from the market.

Perspectives

As we have noticed throughout the thesis, pricing options are calibrated in many methods, In the coming studies, we will propose models with fractional or multi-fractional stochastic process. We suggest multiscale stochastic volatility models (Vol-of-Vol models) and analysis them with the First-order perturbation theory using a combination of singular and regular perturbation techniques to derive approximations for the option prices.

Bibliography

[1] Y. Aït-Sahalia and R. Kimmel, "Maximum likelihood estimationof stochastic volatility models," *Journal of Financial Economics*, vol. 83, pp. 413–452, 2007.

[2] F. Antonelli, "Backward forward stochastic differential equations," Ph.D. dissertation, Purdue University, 1993.

[3] M. Avellaneda, A. Levy, and A. Parás, "Pricing and hedging derivative securities in markets with uncertain volatilities," *Applied Mathematical Finance*, vol. 2, pp. 73–88, 1995.

[4] I. Bajeux-Besnainou and J.-C. Rochet, "Dynamic spanning: Are options an appropriate instrument? 1," *Mathematical Finance*, vol. 6, pp. 1–16, 1996.

[5] E. Barker and R. Stalley, *The politics*. Oxford Paperbacks, 1998.

[6] C. Beck, E. Weinan, and A. Jentzen, "Machine learning approximation algorithms for high-dimensional fully nonlinear partial differential equations and second-order backward stochastic differential equations," *Journal of Nonlinear Science*, vol. 29, pp. 1563–1619, 2019.

[7] M. Bernkopf, "Analysis of the α-hypergeometric stochastic volatility model," Ph.D. dissertation, Wien, 2016.

[8] J. Bertrand, P. Bertrand, and J.-P. Ovarlez, "The mellin transform," 1995.

[9] J.-M. Bismut, "Conjugate convex functions in optimal stochastic control," *Journal of Mathematical Analysis and Applications*, vol. 44, pp. 384–404, 1973.

[10] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, pp. 637–654, 1973.

[11] H. Brezis, A. Ambrosetti, T. A. Bahri, F. Browder, L. Caffarelli, L. C. Evans, M. Giaquinta, D. Kinderlehrer, S. Klainerman, R. Kohn *et al.*, "Progress in nonlinear differential equations and their applications," vol. 65, 2005.

[12] Z. Chen, "Pricing and hedging exotic options in stochastic volatility models," Ph.D. dissertation, The London School of Economics and Political Science (LSE), 2013.

[13] P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir, "Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs," *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, vol. 60, pp. 1081–1110, 2007.

[14] R. Cont, "Model uncertainty and its impact on the pricing of derivative instruments," *Mathematical Finance*, vol. 16, pp. 519–547, 2006.

[15] M. G. Crandall, M. Kocan, and A. Święch, "Lp-theory for fully nonlinear uniformly parabolic equations: Parabolic equations," *Communications in Partial Differential Equations*, vol. 25, pp. 1997–2053, 2000.

[16] J. Da Fonseca and C. Martini, "The α-hypergeometric stochastic volatility model," *Stochastic Processes and their Applications*, vol. 126, pp. 1472–1502, 2016.

[17] P. Devolder, J. Janssen, and R. Manca, *Basic stochastic processes*. John Wiley & Sons, 2015.

[18] N. El Karoui, S. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, pp. 1–71, 1997.

[19] Y. Fan, "The PDEs and numerical scheme for derivatives under uncertainty volatility," *Mathematical Problems in Engineering*, vol. 2019, 2019.

[20] G. Fiorentini, A. Leon, and G. Rubio, "Estimation and empirical performance of Heston's stochastic volatility model: the case of a thinly traded market," *Journal of Empirical Finance*, vol. 9, pp. 225–255, 2002.

[21] J.-P. Fouque and N. Ning, "Uncertain volatility models with stochastic bounds," *SIAM Journal on Financial Mathematics*, vol. 9, pp. 1175–1207, 2018.

[22] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna, "Singular perturbations in option pricing," *SIAM Journal on Applied Mathematics*, vol. 63, pp. 1648–1665, 2003.

[23] ——, *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*. Cambridge University Press, 2011.

[24] J.-P. Fouque and B. Ren, "Approximation for option prices under uncertain volatility," *SIAM Journal on Financial Mathematics*, vol. 5, pp. 360–383, 2014.

[25] J. Gatheral, *The Volatility Surface: A Practitioner's Guide*. Wiley Finance, 2006.

[26] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear partial differential equations: Asymptotic behavior of solutions and self-similar solutions*. Springer Science et Business Media, 2010.

[27] M. Gilli and E. Schumann, "Calibrating option pricing models with heuristics," *Studies in Computational Intelligence*, vol. 380, p. 9, 2017.

[28] J. Guyon and P. Henry-Labordere, *Nonlinear option pricing*. CRC Press, 2013.

[29] C. Hartmann, O. Kebiri, L. Neureither, and L. Richter, "Variational approach to rare event simulation using least-squares regression," *Chaos*, vol. 29, p. 063107, 2019.

[30] R. J. Hauser and J. S. Eales, "Option hedging strategies," *North Central Journal of Agricultural Economics*, vol. 9, pp. 123–134, 1987.

[31] D. Heath and M. Schweizer, "Martingales versus PDEs in finance: an equivalence result with examples," *Journal of Applied Probability*, vol. 37, pp. 947–957, 2000.

[32] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The Review of Financial Studies*, vol. 6, pp. 327–343, 1993.

[33] M. Hu, S. Ji, S. Peng, and Y. Song, "Backward stochastic differential equations driven by G-Brownian motion," *Stochastic Processes and their Applications*, vol. 124, pp. 759–784, 2014.

[34] ——, “Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion,” *Stochastic Processes and their Applications*, vol. 124, pp. 1170–1195, 2014.

[35] J. Hull and A. White, “The pricing of options on assets with stochastic volatilities,” *The Journal of Finance*, vol. 42, pp. 281–300, 1987.

[36] A. Hurn, K. Lindsay, and A. McClelland, “Estimating the parameters of stochastic volatility models using option price data,” *Journal of Business and Economic Statistics*, vol. 33, pp. 579–594, 2015.

[37] M. G. Kendall, “A. stuart the advanced theory of statistics,” 1977.

[38] Y.-K. Kwok, *Mathematical models of financial derivatives*. Springer, 2008.

[39] S. S. Lee and J. Hannig, “Detecting jumps from Lévy jump diffusion processes,” *Journal of Financial Economics*, vol. 96, pp. 271–290, 2010.

[40] T. Lepage, S. Lawi, P. Tupper, and D. Bryant, “Continuous and tractable models for the variation of evolutionary rates,” *Mathematical Biosciences*, vol. 199, pp. 216–233, 2006.

[41] A. L. Lewis, *Option Valuation Under Stochastic Volatility: With Mathematica Code*. Finance Press, 2000.

[42] ——, *Option valuation under stochastic volatility II*. Finance Press, 2009.

[43] P.-L. Lions, “Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations part 2: viscosity solutions and uniqueness,” *Communications in Partial Differential Equations*, vol. 8, pp. 1229–1276, 1983.

[44] ——, “Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations part i: the dynamic programming principle and application,” *Communications in Partial Differential Equations*, vol. 8, pp. 1101–1174, 1983.

[45] J. Ma, P. Protter, and J. Yong, “Solving forward-backward stochastic differential equations explicitly-a four step scheme,” *Probability Theory and Related Fields*, vol. 98, pp. 339–359, 1994.

[46] A. Matoussi, D. Possamaï, and C. Zhou, "Robust utility maximization in nondominated models with 2BSDE: the Uncertain volatility model," *Mathematical Finance*, vol. 25, pp. 258–287, 2015.

[47] Z. Mezdoud, C. Hartmann, M. R. Remita, and O. KEBIRI, "α-hypergeometric uncertain volatility models and their connection to 2bsdes," *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, vol. 16, pp. 263–288, 2021.

[48] E. Pardoux and S. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, pp. 55–61, 1990.

[49] E. Pardoux and A. Rascanu, *Stochastic differential equations, Backward SDEs, Partial differential equations*. Springer, 2014.

[50] S. Peng, *Nonlinear expectations and stochastic calculus under uncertainty: with robust CLT and G-Brownian motion*. Springer Nature, 2019.

[51] N. Perkowski, "Backward stochastic differential equations: an introduction," *Tanári jegyzet*, p. 8, 2011.

[52] H. Pham, *Continuous-time stochastic control and optimization with financial applications*. Springer Science & Business Media, 2009.

[53] D. Possamaï and C. Zhou, "Second order backward stochastic differential equations with quadratic growth," *Stochastic Processes and their Applications*, vol. 123, pp. 3770–3799, 2013.

[54] N. Privault and Q. She, "Option pricing and implied volatilities in a 2-hypergeometric stochastic volatility model," *Applied Mathematics Letters*, vol. 53, pp. 77–84, 2016.

[55] E. Renault and N. Touzi, "Option hedging and implied volatilities in a stochastic volatility model 1," *Mathematical Finance*, vol. 6, pp. 279–302, 1996.

[56] K. R. Sircar, "Hedging under stochastic volatility." World Scientific, 2001.

[57] J. Vorbrink, "Financial markets with volatility uncertainty," *Journal of Mathematical Economics*, vol. 53, pp. 64–78, 2014.

[58] J. Zhang, *Backward stochastic differential equations*. Springer, 2017.

[59] Q. Zhang, J. Han, and M. Gao, “Option price with stochastic volatility for both fast and slow mean-reverting regimes,” *Comptes Rendus Mathematique*, vol. 351, pp. 411–414, 2013.

[60] W. Zhao, Y. Li, and Y. Fu, “Second-order schemes for solving decoupled forward backward stochastic differential equations,” *Science China Mathematics*, vol. 57, pp. 665–686, 2014.