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Periodicity and positivity of solutions for certain delay functional differential equations by the fixed point technique

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A Doctoral Thesis, By Guerfi Abderrahim
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ملخــــــــص

هذا العمل مخصص لدراسة دورية وإيجابية واستقرار الحلول لبعض أصناف المعادلات وجمل المعادلات الدالية ذات تأخير. الطريقة المستخدمة هنا هي نظريات النقطة الثابتة لإثبات النتائج المرجوة. تعتمد فكرة هذه الطريقة على تحويل المعادلة المدروسة إلى معادلة تكاملية مكافئة ومن ثم إظهار وجود وتفرد واستقرار الحلول الدورية والحلول الدورية الموجبة باستخدام نظريات النقطة الثابتة.

الكلمـات المفتاحيـة: النقطة الثابتة، الوجود، الوحدانية، الدورية، الإيجابية، المعـادلات وجمل المعادلات التفاضلية ذات تأخير، جمل معـادلات الفروق ذات تأخير، جمل المعادلات الديناميكية ذات تأخير، المعـادلات التفاضلية الكسرية.

Abstract

This work is devoted to the study of the periodicity, positivity and stability of solutions for some classes of delay functional equations and systems. The method used here is the fixed point theorems for proving the desired results. The idea of this method is based on the converting of the considered equation into an equivalent integral equation and then show the existence, uniqueness and stability of periodic and positive periodic solutions by using the fixed point theorems.

Keywords: Fixed point, existence, uniqueness, periodicity, positivity, delay differential equations and systems, delay difference systems, delay dynamic systems, fractional differential equations.

Mathematics Subject Classification: 26A33, 34A08, 34B15, 34B18, 34K20, 34K30, 34K40, 39A12, 39A23, 45N05, 47H10.

Résumé

Ce travail est consacré à l'étude de la périodicité, de la positivité et de la stabilité des solutions pour certaines classes d'équations et de systèmes fonctionnels à retard. La méthode utilisée ici est celle des théorèmes du point fixe pour prouver les résultats souhaités. L'idée de cette méthode est basée sur la conversion de l'équation considérée en une équation intégrale équivalente puis de montrer l'existence, l'unicité et la stabilité des solutions périodiques et solutions périodiques positives en utilisant les théorèmes du point fixe.

Mots-clés: Point fixe, existence, unicité, périodicité, positivité, equations et systèmes différentielles à retard, systèmes aux différences à retard, systèmes dynamiques à retard, équations différentielles fractionnaires.

Mathematics Subject Classification: 26A33, 34A08, 34B15, 34B18, 34K20, 34K30, 34K40, 39A12, 39A23, 45N05, 47H10.

Contents

Introduction

Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis. The origins of the theory, which date to the later part of the nineteenth century, rest in the use of successive approximations establish the existence and uniqueness of solutions, particularly to differential equations. In recent years a number of excellent books, monographs and surveys by distinguished authors about fixed point theory have appeared. Fixed point theory concerns itself with a very simple and basic mathematical setting. A point is often called fixed point when it remains invariant, irrespective of the type of transformation it undergoes. Many mathematicians like Banach, Brouwer, Schauder, Krasnoselskii, Burton and Dhage contributed for this theory, see [48], [56], [58], [111], [117] and the references cited therein.

Delay differential equations are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Delay differential equations are also called time-delay systems, systems with aftereffect or dead-time. Delay differential equations were initially introduced in the 18th century by Laplace and Condorect. However, the rapid development of the theory and applications of those equations did not come until after the second world war, and continues till today. Delay differential equations are often more realistic in describing natural phenomena compared to those without delay. They model many natural phenomena and appear in many fields such as physics, chemistry, biology, dynamics of populations, medicine, etc.

Mathematical models employing delay differential equations turn out to be useful especially in the situation, where the description of investigated systems depends not only on the position of a system in the current time, but also in the past. In such cases the use of ordinary differential equations turns out to be insufficient. The presence of delayed time argument in the investigated system may frequently influence properties of solutions.

For these reasons, this type of equations was given a great importance in the work of many researchers. There has been recently many activities concerning the existence, uniqueness, stability, periodicity and positivity of solutions for delay differential equations.

But it is often difficult to prove the existence of such solutions because there is no specific way to solve this kind of problems. Where some researchers used the theory of differential equations while others used the fixed point theory, etc.

Recently, the study of the existence and qualitative properties of periodic solutions for various kinds of delay functional equations, especially for differential, difference and dynamic equations with delays has attracted much attention. For related results, we refer the reader to [1]–[39], [42]–[62], [64]–[79], [82]–[90], [92]–[110], [112]–[116], [118]–[125] and the references cited therein. There are many methods for obtaining the existence and uniqueness of periodic and positive periodic solutions. For example, Lyapunov method, Fourier analysis method, fixed point theory.

We have interested in the use of the fixed point theory to problems of periodicity, positivity and stability for delay functional equations. We have studied and contributed to it and have obtained interesting results. In this thesis we present a collection of results to some problems of delay functional equations and systems of delay functional equations by using fixed point theory.

A brief description of the organization of the thesis follows.

Chapter 1 summarizes some concepts, definitions and results which are mostly relevant to the undergraduate curriculum and are thus assumed as basically known, or have specific roots in rather distant areas and have rather auxiliary character with respect to the purpose of this study.

In Chapter 2, we investigate the existence of periodic or nonnegative periodic solutions for the totally nonlinear neutral differential equation with infinite delay

$$\frac{d}{dt}x\left(t\right)=-a\left(t\right)h\left(x\left(t-\tau\left(t\right)\right)\right)+\frac{d}{dt}Q\left(t,x\left(t-g\left(t\right)\right)\right)+\int_{-\infty}^{t}D\left(t,s\right)f\left(x\left(s\right)\right)ds.$$

In the process we convert the given neutral differential equation into an equivalent integral equation. Then we employ Krasnoselskii-Burton's fixed point theorem to prove the existence of periodic or nonnegative periodic solutions. Two examples are provided to illustrate the obtained results. The results presented in this chapter are accepted in Proyecciones (2022), see [72].

Chapter 3 studies the existence of periodic solutions of the first order totally nonlinear iterative differential equation

$$\begin{aligned}\frac{d}{dt}x\left(t\right)=&-a\left(t\right)h\left(x\left(t\right)\right)+\frac{d}{dt}g\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right)\\&+f\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right).\end{aligned}$$

The equivalent integral equation of the given equation defines a fixed point mapping written as a sum of a large contraction and a compact map. The main results assert the existence of periodic solutions by making use of Krasnoselskii-Burton's fixed point technique. The results presented in this chapter are published in Bulletin of the International Mathematical Virtual Institute (2022), see [73].

Sufficient conditions in Chapter 4 are presented for the existence of periodic solutions of the second order totally nonlinear iterative differential equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)h(x(t))\\&=\frac{d}{dt}g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right).\end{aligned}$$

The equivalent integral equation of the given equation defines a fixed point mapping written as a sum of a large contraction and a compact map. The main tool used here is Krasnoselskii-Burton's fixed point technique. The results presented in this chapter are published in The Journal of Analysis (2021), see [74].

In Chapter 5, we prove the existence of periodic and nonnegative periodic solutions for the third-order nonlinear delay differential equation with periodic coefficients

$$\frac{d^3}{dt^3}x(t)+p(t)\frac{d^2}{dt^2}x(t)+q(t)\frac{d}{dt}x(t)+r(t)h(x(t))=f(t,x(t),x(t-\tau(t))).$$

The technique employed to show our results depends on Green's function and Krasnoselskii-Burton's fixed point theorem. The results presented in this chapter are published in MESA (2021), see [75].

In Chapter 6, we study the periodicity and stability of solutions for the neutral differential system

$$\begin{aligned}&\frac{d}{dt}u(t)-q\frac{d}{dt}u(t-r)\\&=P(t)+A(t)u(t)+A(t)qu(t-r)-bf(u(t))+bqf(u(t-r)).\end{aligned}$$

In the process we use the fundamental matrix solution to convert the given differential system into an equivalent integral system. Then we employ Krasnoselskii's fixed point theorem to show the existence and stability of periodic solutions of this neutral differential system. Our results extend and complement some earlier publications. The results presented in this chapter are published in Proceedings of the Institute of Mathematics and Mechanics (2020), see [78].

In Chapter 7, we prove the existence and uniqueness of mild solutions for the initial value problem of the nonlinear hybrid first order Caputo fractional integro-differential equation

$$\begin{cases}{}^{C}D_{0^+}^{\alpha}\left(\frac{u(t)-f(t,u(t))}{p(t)+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s,u(s))ds}\right)=h(t,u(t)),\ t\in[0,T],\\u(0)=f(0,u(0))+p(0)\theta.\end{cases}$$

The main tool employed here is the Krasnoselskii and Banach fixed point theorems. An example is also given to illustrate the main results. In addition, the case of the Higher order Caputo fractional integro-differential equations is studied. The results presented in this chapter are published in Results in Nonlinear Analysis (2021), see [76].

In Chapter 8, we use Krasnoselskii's fixed point theorem to establish new results on the existence of periodic solutions for the almost linear Volterra integro-dynamic system on periodic time scales of the form

$$\begin{cases} x^{\Delta}(t) = a(t)\,p(x(t)) + \int_{-\infty}^{t} C(t,s)\,h(y(s))\,\Delta s + e(t), \\ y^{\Delta}(t) = b(t)\,q(y(t)) + \int_{-\infty}^{t} D(t,s)\,g(x(s))\,\Delta s + f(t). \end{cases}$$

The results presented in this chapter are published in Malaya Journal of Matematik (2020), see [79].

In Chapter 9, We use Krasnoselskii's fixed point theorem to show that the neutral nonlinear summation-difference system with infinite delay

$$\Delta x(n) = P(n) + A(n)\,x(n-\tau(n)) + \Delta Q(n, x(n-g(n))) + \sum_{k=-\infty}^{n} D(n,k)\,f(x(k)),$$

has a periodic solution. We also use the contraction mapping principle to show that the periodic solution is unique. An example is given to illustrate our results. The results presented in this chapter are published in Rocky Mountain Journal of Mathematics (2021), see [77].

We conclude this thesis with a general conclusion, as well as the perspectives of our future research.

Chapter 1

Preliminaries

In this chapter we shall introduce the basic concepts, notations, and elementary results that are used throughout the thesis like functional analysis, the basic concepts of fixed point theorems and delay differential equations which are necessary for the construction of this thesis. Moreover, the results in this chapter may be found in most standard books (see for examples [48], [50], [63], [80], [81], [111], [117]). We begin this chapter by recalling a well-known concept in functional analysis.

1.1 Fundamental concepts

Definition 1.1 A metric space is couple (X, d) where X is a set and d is a metric on X, that is a function $d : X \times X \to \mathbb{R}^+$ such that
1) $d(x, y) \geq 0$ (non negativity),
2) $d(x, y) = 0$ if and only if $x = y$ (identity),
3) $d(x, y) = d(y, x)$ (symmetry),
4) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition 1.2 A metric space (X, d) in which every Cauchy sequence converges (has a limit in X) is called complete.

Theorem 1.1 (Ascoli-Arzela) *Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. We consider a subset $\mathbb{M}$ of $C(X, Y)$ the set of continuous functions from X to Y endowed with the distance*

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Suppose we have
i) the subset $\mathbb{M}$ is equicontinuous, i.e.

$$\forall x, x' \in X,\ \forall \varepsilon > 0,\ \exists \eta > 0,\ d_X(x, x') < \eta \Rightarrow \forall f \in \mathbb{M},\ d_Y(f(x), f(x')) < \varepsilon,$$

ii) for every $x \in X$, the set $\{f(x),\ f \in \mathbb{M}\}$ is of compact closure.
Then, $\mathbb{M}$ is relatively compact, i.e. its closure is a compact set.

Definition 1.3 Let consider a vector space $\mathbf{E}$ on $\mathbb{R}$. A mapping $N : \mathbf{E} \to \mathbb{R}^+$ is a seminorm on $\mathbf{E}$ if and only if the two following assertions are satisfied
a) $N(x+y) \leq N(x) + N(y)$,
b) for every $\lambda \in \mathbb{R},\ N(\lambda x) = |\lambda| N(x)$.
A norm is a seminorm with the additional property $N(x) = 0$ if and only if $x = 0$.

Definition 1.4 Let $\mathbf{E}$ be a vector space and let N be a norm on $\mathbf{E}$. The pair $(\mathbf{E}, N)$ is called a normed space.

Proposition 1.1 *Let $(\mathbf{E}, \|.\|)$ be a normed space. The map $\mathbf{E}\times\mathbf{E} \to \mathbb{R}^+,\ (x,y) \mapsto \|x-y\|$ is a distance on $\mathbf{E}$, called the distance associated to the norm $\|.\|$.*

Definition 1.5 Let $(\mathbf{E}, \|.\|)$ be a normed space. $(\mathbf{E}, \|.\|)$ is a Banach space if and only if the metric space $(\mathbf{E}, d)$ where d is the distance associated to the norm $\|.\|$, i.e. $d(x,y) = \|x-y\|$, is a complete space.

Example 1.1 Given $a, b \in \mathbb{R}$, with $a < b$ and $n \in \mathbb{N}$, we consider the space

$$C([a,b], \mathbb{R}^n) = \{u : [a,b] \to \mathbb{R}^n,\ u \text{ is continuous at each } x \in [a,b]\}.$$

endowed with the norm

$$\|u\|_\infty = \sup_{x\in[a,b]} |u(x)| < +\infty,$$

is a Banach space.

Definition 1.6 Let $\mathbf{E}$ and F be Banach spaces. A linear map $\mathcal{A} : \mathbf{E} \to F$ is compact if, for any bounded sequence $\{x_n\}$ in $\mathbf{E}$, the sequence $\{\mathcal{A}x_n\}$ in F contains a convergent subsequence.

Definition 1.7 Let $\mathbf{E}$ and F be two Banach spaces. A linear map $\mathcal{A} : \mathbf{E} \to F$ is said to be compact if and only if for every bounded subset $\mathbb{M}$, the set $\mathcal{A}(\mathbb{M})$ is relatively compact.

1.2 Fixed point theorems

Let $\mathcal{A}$ be a mapping of a set X into itself. An element $u \in X$ is said to be a fixed point of the mapping $\mathcal{A}$ if $\mathcal{A}u = u$.

Theorem 1.2 (Banach's fixed point theorem [111]) *Let $(\mathbb{S}, d)$ be a complete metric space. If $\mathcal{A} : \mathbb{S} \to \mathbb{S}$ is a contraction mapping, i.e., there is a constant $\alpha < 1$ such that for each pair $\phi_1, \phi_2 \in \mathbb{S}$, we have $d(\mathcal{A}\phi_1, \mathcal{A}\phi_2) \leq \alpha d(\phi_1, \phi_2)$, then there is a unique point $\phi \in \mathbb{S}$, with $\mathcal{A}\phi = \phi$.*

Theorem 1.3 (Schauder's fixed point theorem [111]) *Let $\mathbb{M}$ be a nonempty bounded closed convex subset of a Banach space X. Let $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ be continuous and compact. Then $\mathcal{A}$ has a fixed point.*

Theorem 1.4 (Krasnoselskii's fixed point theorem [111]) *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{S}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{S}$ such that*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$, $\forall x, y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is continuous and $\mathcal{A}\mathbb{M}$ is contained in a compact set,

(iii) $\mathcal{B}$ is a contraction with constant $\alpha < 1$.

Then there is a $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

Remark 1.1 Note that if $\mathcal{A} = 0$, the theorem becomes the theorem of Banach. If $\mathcal{B} = 0$, then the theorem is not other than the theorem of Schauder.

Definition 1.8 ([48]) Let $(\mathbb{M}, d)$ be a metric space and suppose that $\mathcal{B} : \mathbb{M} \to \mathbb{M}$. $\mathcal{B}$ is said to be a large contraction, if for $\varphi, \psi \in \mathbb{M}$, with $\varphi \neq \psi$, we have $d(B\varphi, B\psi) \leq d(\varphi, \psi)$ and if $\forall \epsilon > 0$, $\exists \delta < 1$ such that

$$[\varphi, \psi \in \mathbb{M},\ d(\varphi, \psi) \geq \epsilon] \Rightarrow d(\mathcal{B}\varphi, \mathcal{B}\psi) \leq \delta d(\varphi, \psi).$$

Theorem 1.5 ([2]) *Let $\|.\|$ be the supremum norm, $\mathbb{M} = \{\varphi \in C(\mathbb{R}, \mathbb{R}) : \|\varphi\| \leq L\}$, where L is a positive constant. Suppose that h is satisfying the following conditions*

(H1) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

(H2) the function h is strictly increasing on $[-L, L]$,

(H3) $\sup_{t \in (-L,L)} h'(t) \leq 1$.

Then the mapping H define by $H(\varphi) = \varphi - h(\varphi)$ is a large contraction on the set $\mathbb{M}$.

Example 1.2 If $(H\varphi)(t) = \varphi(t) - \varphi^3(t)$, then H is a large contraction on the set

$$\mathbb{M} = \left\{\varphi \in C(\mathbb{R}, \mathbb{R}) : \|\varphi\| \leq \sqrt{3}/3\right\}.$$

Theorem 1.6 ([48]) *Let $(\mathbb{M}, d)$ be a complete metric space and $\mathcal{B}$ a large contraction. Suppose there is an $x \in \mathbb{M}$ and an $L > 0$, such that $d(x, \mathcal{B}^n x) \leq L$ for all $n \geq 1$. Then $\mathcal{B}$ has a unique fixed point in $\mathbb{M}$.*

Burton studied the theorem of Krasnoselskii and observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated the below.

Theorem 1.7 (Krasnoselskii-Burton [48]) *Let $\mathbb{M}$ be a bounded closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) $\mathcal{A}$ is continuous and compact,

(ii) $\mathcal{B}$ is large contraction,

(iii) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$.

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

1.3 Retarded functional differential equations

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence.

If $[a,b] = [-r,0]$ we let $C = C([-r,0], \mathbb{R}^n)$ and designate the norm of an element ψ in C by $\|\psi\| = \sup_{-r \leq s \leq 0} |\psi(s)|$.

If $t_0 \in \mathbb{R}$, $\sigma \geq 0$ and $x \in C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + \sigma]$, we let $x_t \in C$ be defined by $x_t(s) = x(t+s)$, $-r \leq s \leq 0$. If Ω is a subset of $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is a given function, we say that the relation

$$x'(t) = f(t, x_t), \tag{1.1}$$

is a *retarded functional differential equation* on Ω. A function x is said to be a solution of (1.1) on $[t_0 - r, t_0 + \sigma)$ if there are $t_0 \in \mathbb{R}$ and $\sigma > 0$ such that $x \in C([t_0 - r, t_0 + \sigma), \mathbb{R}^n)$, $(t, x_t) \in \Omega$ and x satisfies (1.1) for $t \in [t_0, t_0 + \sigma)$.

For given $t_0 \in \mathbb{R}$, $\psi \in C$, we say $x(t, t_0, \psi)$ is a solution of (1.1) with initial value ψ at t_0 or simply a solution through (t_0, ψ) if there is an $\sigma > 0$ such that $x(t, t_0, \psi)$ is a solution of (1.1) on $[t_0 - r, t_0 + \sigma)$ and $x_{t_0}(t, t_0, \psi) = \psi$.

Equation (1.1) is a very general type of equation and includes the ordinary differential equation $(r = 0)$

$$x'(t) = f(t, x(t)).$$

Definition 1.9 ([50]) Suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of equation (1.1) is said to be stable if for any $t_0 \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\psi\| \leq \delta$ implies $|x(t, t_0, \psi)| \leq \varepsilon$ for $t \geq t_0$. The solution $x = 0$ of (1.1) is said to be uniformly stable if the number δ in definition is independent of t_0.

Definition 1.10 ([50]) The solution $x = 0$ of (1.1) is said to be asymptotically stable if it is stable and there is a $\delta_1 = \delta_1(t_0) > 0$ such that $\|\psi\| \leq \delta_1$ implies $|x(t, t_0, \psi)| \to 0$ as $t \to \infty$. The solution $x = 0$ of (1.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is $\delta_1 > 0$ such that for every $\eta > 0$ there is a $c(\eta) > 0$ such that $\|\psi\| \leq \delta_1$ implies $|x(t, t_0, \psi)| \leq \eta$ for $t \geq t_0 + c(\eta)$ for every $t_0 \in \mathbb{R}$.

Lemma 1.1 ([80]) *If $t_0 \in \mathbb{R}$, $\psi \in C$ are given, and $f(t, \psi)$ is continuous, then finding a solution of (1.1) through (t_0, ψ) is equivalent to solving the integral equation*

$$\begin{aligned} x_{t_0} &= \psi, \\ x(t) &= \psi(0) + \int_{t_0}^{t} f(s, x_s) ds, \ t \geq t_0. \end{aligned}$$

Theorem 1.8 (Existence, [80]) *Suppose Ω is an open subset in $\mathbb{R} \times C$ and $f \in C(\Omega, \mathbb{R}^n)$. If $(t_0, \psi) \in \Omega$, then there is a solution of (1.1) passing through (t_0, ψ).*

Definition 1.11 We say $f(t,\psi)$ is Lipschitz in ψ in a compact set Ω of $\mathbb{R}\times C$ if there is a constant $k>0$ such that, for any $(t,\psi_i)\in\Omega$, $i=1,2$

$$|f(t,\psi_1)-f(t,\psi_2)|\leq k\,\|\psi_1-\psi_2\|\,.$$

Theorem 1.9 (Existence and uniqueness, [80]) *Suppose Ω is an open subset in $\mathbb{R}\times C$, $f:\Omega\to\mathbb{R}^n$ is continuous and $f(t,\psi)$ is Lipschitz in ψ in each compact set in Ω . If $(t_0,\psi)\in\Omega$, then there is a unique solution of (1.1) through (t_0,ψ).*

Definition 1.12 ([80]) Suppose that $\Omega\subseteq\mathbb{R}\times C$ is open, $f:\Omega\to\mathbb{R}^n$ and $G:\Omega\to\mathbb{R}^n$ are given continuous functions with G atomic at zero. The relation

$$\frac{d}{dt}G(t,x_t)=f(t,x_t), \tag{1.2}$$

is called the neutral functional differential equation.

Definition 1.13 ([80]) A function x is said to be a solution of the (1.2) on $[t_0-r,t_0+\sigma)$ if there are $t_0\in\mathbb{R}$, $\sigma>0$, such that

$$x\in C([t_0-r,t_0+\sigma)\,,\mathbb{R}^n),\;(t,x_t)\in\Omega,\;t\in[t_0,t_0+\sigma)\,,$$

$G(t,x_t)$ is continuously differentiable and satisfies (1.2) on $[t_0,t_0+\sigma)$. For a given $(t_0,\psi)\in\Omega$, we say $x(t,t_0,\psi)$ is a solution of (1.2) with initial value ψ at t_0 or simply a solution through (t_0,ψ) if there is an $\sigma>0$ such that $x(t,t_0,\psi)$ is a solution of (1.2) on $[t_0,t_0+\sigma)$ and $x_{t_0}=\psi$ on $[t_0-r,t_0]$.

Theorem 1.10 (Existence [80]) *If Ω is an open set in $\mathbb{R}\times C$ and $(t_0,\psi)\in\Omega$, Then there exists a solution of (1.2) through (t_0,ψ).*

Theorem 1.11 (Uniqueness [80]) *If $\Omega\subseteq\mathbb{R}\times C$ is open and $f:\Omega\to\mathbb{R}^n$ and $G:\Omega\to\mathbb{R}^n$ are Lipschitz in ψ on compact sets of Ω, for any $(t_0,\psi)\in\Omega$, there exists a unique solution of (1.2) through (t_0,ψ).*

Chapter 2

Existence of periodic or nonnegative periodic solutions for totally nonlinear neutral differential equations with infinite delay

Keywords. Krasnoselskii-Burton's fixed point, Large contraction, Periodic solutions, Nonnegative periodic solutions, Infinite delay.

This chapter essentially corresponds to the paper [72],

A. Guerfi, A. Ardjouni, Existence of periodic or nonnegative periodic solutions for totally nonlinear neutral differential equations with infinite delay, Proyecciones, Accepted.

In this chapter, we present sufficient conditions for the existence of periodic or nonnegative periodic solutions of the totally nonlinear neutral differential equation with infinite delay

$$\frac{d}{dt}x(t) = -a(t)\,h(x(t-\tau(t))) + \frac{d}{dt}Q(t, x(t-g(t))) + \int_{-\infty}^{t} D(t,s)\,f(x(s))\,ds, \quad (2.1)$$

where a is a positive continuous function. The functions $h, f : \mathbb{R} \to \mathbb{R}$ are continuous, $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the Caratheodory condition. The main purpose of this chapter is to use Krasnoselskii-Burton's fixed point theorem (see [48]) to prove the existence of periodic or nonnegative periodic solutions for (2.1). During the process we employ the variation of parameter formula and the integration by parts to transform (2.1) into an equivalent integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use Krasnoselskii-Burton's fixed point theorem, to prove the existence of periodic or nonnegative periodic solutions. Two examples are given to illustrate the obtained results.

2.1 Inversion of the equation

For $T>0$ define

$$P_T=\left\{\varphi\in C\left(\mathbb{R},\mathbb{R}\right),\ \varphi\left(t+T\right)=\varphi\left(t\right)\right\},$$

where $C\left(\mathbb{R},\mathbb{R}\right)$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

In this paper we assume that

$$\begin{aligned} a(t-T) &= a(t),\ D(t-T,s-T)=D(t,s),\\ \tau(t-T) &= \tau(t)\geq\tau^{*}>0,\ g(t-T)=g(t)\geq g^{*}>0. \end{aligned} \tag{2.2}$$

with τ and g are continuously differentiable functions, τ^{*} and $g^{*}$are positive constants, a is a positive function and

$$1-e^{-\int_{t-T}^{t}a(k)dk}\equiv\frac{1}{\eta}\neq 0. \tag{2.3}$$

The function $Q(t,x)$ is periodic in t of period T, that is

$$Q(t+T,x)=Q(t,x). \tag{2.4}$$

Also, there is a positive constant E such that,

$$\int_{-\infty}^{t}\left|D\left(t,s\right)\right|ds\leq E<\infty. \tag{2.5}$$

The following lemma is fundamental to our results.

Lemma 2.1 *Suppose (2.2)-(2.4) hold. If $x\in P_T$, then x is a solution of (2.1) if and only if*

$$\begin{aligned} x\left(t\right)=&\ \eta\int_{t-T}^{t}a\left(u\right)H\left(x\left(u\right)\right)e^{-\int_{u}^{t}a(k)dk}du+Q\left(t,x\left(t-g\left(t\right)\right)\right)\\ &+\int_{t-\tau(t)}^{t}a\left(u\right)h\left(x\left(u\right)\right)du-\eta\int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u}a\left(s\right)h\left(x\left(s\right)\right)ds\right]a\left(u\right)e^{-\int_{u}^{t}a(k)dk}du\\ &+\eta\int_{t-T}^{t}b\left(u\right)h\left(x\left(u-\tau\left(u\right)\right)\right)e^{-\int_{u}^{t}a(k)dk}du\\ &+\eta\int_{t-T}^{t}\left[-a\left(u\right)Q\left(u,x\left(u-g\left(u\right)\right)\right)+\int_{-\infty}^{u}D\left(u,s\right)f\left(x\left(s\right)\right)ds\right]e^{-\int_{u}^{t}a(k)dk}du. \end{aligned} \tag{2.6}$$

where

$$H\left(x\right)=x-h\left(x\right), \tag{2.7}$$

and

$$b\left(u\right)=\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)-a\left(u\right).$$

Proof. Let $x \in P_T$ be a solution of (2.1). Rewrite (2.1) as

$$\begin{aligned}
&\frac{d}{dt}\left[x(t) - Q(t, x(t - g(t)))\right] + a(t)\left[x(t) - Q(t, x(t - g(t)))\right] \\
&= a(t)\left[x(t) - Q(t, x(t - g(t)))\right] - a(t) h(x(t)) + a(t) h(x(t)) \\
&- a(t) h(x(t - \tau(t))) + \int_{-\infty}^{t} D(t,s) f(x(s))\, ds \\
&= a(t)\left[x(t) - h(x(t))\right] + \frac{d}{dt}\int_{t-\tau(t)}^{t} a(s) h(x(s))\, ds \\
&+ \left[(1 - \tau'(t)) a(t - \tau(t)) - a(t)\right] h(x(t - \tau(t))) \\
&- a(t) Q(t, x(t - g(t))) + \int_{-\infty}^{t} D(t,s) f(x(s))\, ds.
\end{aligned}$$

Multiply both sides of the above equation by $\exp\left(\int_0^t a(k)\, dk\right)$ and then integrate from $t - T$ to t, we get

$$\begin{aligned}
&\int_{t-T}^{t} \left[\left[x(u) - Q(u, x(u - g(u)))\right] e^{\int_0^u a(k)dk}\right]' du \\
&= \int_{t-T}^{t} a(u)\left[x(u) - h(x(u))\right] e^{\int_0^u a(k)dk} du \\
&+ \int_{t-T}^{t} \left[\frac{d}{du}\int_{u-\tau(u)}^{u} a(s) h(x(s))\, ds\right] e^{\int_0^u a(k)dk} du \\
&+ \int_{t-T}^{t} b(u) h(x(u - \tau(u))) e^{\int_0^u a(k)dk} du \\
&+ \int_{t-T}^{t} \left[-a(u) Q(u, x(u - g(u))) + \int_{-\infty}^{u} D(u,s) f(x(s))\, ds\right] e^{\int_0^u a(k)dk} du,
\end{aligned}$$

with $b(u) = (1 - \tau'(u)) a(u - \tau(u)) - a(u)$. As a consequence, we have

$$\begin{aligned}
&\left[x(t) - Q(t, x(t - g(t)))\right] e^{\int_0^t a(k)dk} \\
&- \left[x(t - T) - Q(t - T, x(t - T - g(t - T)))\right] e^{\int_0^{t-T} a(k)dk} \\
&= \int_{t-T}^{t} a(u)\left[x(u) - h(x(u))\right] e^{\int_0^u a(k)dk} du \\
&+ \int_{t-T}^{t} \left[\frac{d}{du}\int_{u-\tau(u)}^{u} a(s) h(x(s))\, ds\right] e^{\int_0^u a(k)dk} du \\
&+ \int_{t-T}^{t} b(u) h(x(u - \tau(u))) e^{\int_0^u a(k)dk} du \\
&+ \int_{t-T}^{t} \left[-a(u) Q(u, x(u - g(u))) + \int_{-\infty}^{u} D(u,s) f(x(s))\, ds\right] e^{\int_0^u a(k)dk} du.
\end{aligned}$$

By dividing both sides of the above equation by $\exp\left(\int_0^t a(u)\, du\right)$ and using the fact that

$x(t) = x(t-T)$, we obtain

$$\begin{aligned}
& x(t) - Q(t, x(t-g(t))) \\
&= \eta \int_{t-T}^{t} a(u)\left[x(u) - h(x(u))\right] e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} \left[\frac{d}{du} \int_{u-\tau(u)}^{u} a(s) h(x(s)) ds\right] e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} b(u) h(x(u-\tau(u))) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} \left[-a(u) Q(u, x(u-g(u))) + \int_{-\infty}^{u} D(u,s) f(x(s)) ds\right] e^{-\int_u^t a(k)dk} du. \qquad (2.8)
\end{aligned}$$

Integration by parts the second integral in the above expression, we get

$$\begin{aligned}
& \int_{t-T}^{t} \left[\frac{d}{du} \int_{u-\tau(u)}^{u} a(s) h(x(s)) ds\right] e^{-\int_u^t a(k)dk} du \\
&= \left[\int_{u-\tau(u)}^{u} a(s) h(x(s)) ds e^{-\int_u^t a(k)dk}\right]_{t-T}^{t} \\
&- \int_{t-T}^{t} \left[\int_{u-\tau(u)}^{u} a(s) h(x(s)) ds\right] a(u) e^{-\int_u^t a(k)dk} du \\
&= \left[\int_{t-\tau(t)}^{t} a(s) h(x(s)) ds - \int_{t-T-\tau(t)}^{t-T} a(s) h(x(s)) ds e^{-\int_{t-T}^t a(k)dk}\right] \\
&- \int_{t-T}^{t} \left[\int_{u-\tau(u)}^{u} a(s) h(x(s)) ds\right] a(u) e^{-\int_u^t a(k)dk} du \\
&= -\int_{t-T}^{t} \left[\int_{u-\tau(u)}^{u} a(s) h(x(s)) ds\right] a(u) e^{-\int_u^t a(k)dk} du + \frac{1}{\eta} \int_{t-\tau(t)}^{t} a(u) h(x(u)) du. \qquad (2.9)
\end{aligned}$$

Then substituting the result of (2.9) into (2.8) to obtain (2.6). The converse implication is easily obtained and the proof is complete. ■

Definition 2.1 The map $\mathcal{P} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Caratheodory conditions with respect to $L^1[0,T]$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \longmapsto \mathcal{P}(t,z)$ is Lebesgue measurable.

(ii) For almost all $t \in [0,T]$, the mapping $z \longmapsto \mathcal{P}(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0,T], \mathbb{R})$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|\mathcal{P}(t,z)| \leq \alpha_r(t)$.

2.2 Existence of periodic solutions

To apply Theorem 1.7 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a completely continuous and the other is a large contraction. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and

$$\mathbb{M} = \{\varphi \in P_T,\ \|\varphi\| \leq L,\ |\varphi(t_2) - \varphi(t_1)| \leq K|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\}, \tag{2.10}$$

with $L \in (0, 1]$ and $K > 0$. $\mathbb{M}$ is a closed convex and bounded subset of P_T.

Define a mapping $\mathcal{S} : P_T \to P_T$ by

$$\begin{aligned}(\mathcal{S}\varphi)(t) &= \eta \int_{t-T}^{t} a(u) H(\varphi(u)) e^{-\int_u^t a(k)dk} du + Q(t, \varphi(t - g(t))) \\ &+ \int_{t-\tau(t)}^{t} a(u) h(\varphi(u)) du - \eta \int_{t-T}^{t} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds\right] a(u) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^{t} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^{t} \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u, s) f(\varphi(s)) ds\right] e^{-\int_u^t a(k)dk} du.\end{aligned} \tag{2.11}$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}, \mathcal{B} : P_T \to P_T$ are given by

$$\begin{aligned}(\mathcal{A}\varphi)(t) &= Q(t, \varphi(t - g(t))) + \int_{t-\tau(t)}^{t} a(u) h(\varphi(u)) du \\ &- \eta \int_{t-T}^{t} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds\right] a(u) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^{t} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k)dk} du \\ &+ \eta \int_{t-T}^{t} \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u, s) f(\varphi(s)) ds\right] e^{-\int_u^t a(k)dk} du,\end{aligned} \tag{2.12}$$

and

$$(\mathcal{B}\varphi)(t) = \eta \int_{t-T}^{t} a(u) H(\varphi(u)) e^{-\int_u^t a(k)dk} du. \tag{2.13}$$

We will assume that the following conditions hold.

(H4) $a \in L^1[0, T]$ is bounded.

(H5) h, f, Q are locally Lipschitz continuous, then for $t \geq 0$ and $x, y \in \mathbb{M}$ there exists constants $E_1, E_2, E_3 > 0$, such that

$$\begin{aligned}|h(x) - h(y)| &\leq E_1 \|x - y\|, \\ |f(x) - f(y)| &\leq E_2 \|x - y\|, \\ |Q(t, x) - Q(t, y)| &\leq E_3 \|x - y\|.\end{aligned}$$

2.2. Existence of periodic solutions

(H6) Q satisfies Caratheodory condition with respect to $L^1[0,T]$.

(H7) There exist positive periodic functions $q_1, q_2 \in L^1[0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1(t)|x| + q_2(t).$$

(H8) The function $Q(t,x)$ is also supposed locally Lipschitz in t, i.e, there exists $K_Q > 0$ such that

$$|Q(t_2,x) - Q(t_1,x)| \leq K_Q |t_2 - t_1|.$$

Now, we need the following assumptions

$$\beta_1\beta_2(E_1L + |h(0)|) \leq \frac{\gamma_1}{2}L, \tag{2.14}$$

where $\beta_1 = \max_{t\in[0,T]}|\tau(t)|$and $\beta_2 = \max_{t\in[0,T]}\{a(t)\}$,

$$q_1(t)L + q_2(t) \leq \frac{\gamma_2}{2}L, \tag{2.15}$$

$$|b(u)|(E_1L + |h(0)|) \leq \gamma_3 a(u)L, \tag{2.16}$$

$$TE\eta\beta_3(E_2L + |f(0)|) \leq \gamma_4 L, \tag{2.17}$$

where $\beta_3 = \max_{u\in[t-T,t]}\left\{e^{-\int_u^t a(k)dk}\right\}$,

$$J[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4] \leq 1. \tag{2.18}$$

where γ_1, γ_2, γ_3, γ_4 and J are positive constants with $J \geq 3$. Also, suppose that there are constants $k_1, k_2, k_3 > 0$ such that for $0 \leq t_1 < t_2$

$$|\tau(t_2) - \tau(t_1)| \leq k_1|t_2 - t_1|, \tag{2.19}$$

$$|g(t_2) - g(t_1)| \leq k_2|t_2 - t_1|, \tag{2.20}$$

$$\int_{t_1}^{t_2} a(u)\,du \leq k_3|t_2 - t_1|, \tag{2.21}$$

and

$$\begin{aligned} &K_Q + (1+k_2)E_3K + 2\gamma_4\beta_2\beta_3L + \Big[(2+k_1)E_1 + (1+4\eta)\gamma_3 \\ &+ \left(\eta + \frac{1}{2}\right)\gamma_2 + \gamma_4 + \left(\eta + \frac{1}{2}\right)\gamma_1\Big]k_3L \leq \frac{K}{J}. \end{aligned} \tag{2.22}$$

Lemma 2.2 *For $\mathcal{A}$ defined in (2.12), suppose that (2.2)–(2.5), (2.14)–(2.22) and (H4)–(H8) hold. Then $\mathcal{A}:\mathbb{M}\to\mathbb{M}$.*

Proof. Let $\mathcal{A}$ be defined by (2.12). First by (2.2) and (2.4), a change of variable in (2.12) shows that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. That is, if $\varphi \in P_T$ then $\mathcal{A}\varphi$ is periodic with period T. For having $\mathcal{A}\varphi \in \mathbb{M}$ we will prove that $\|\mathcal{A}\varphi\| \leq L$ and $|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. By (H5) we have

$$|h(x)| \leq E_1|x| + |h(0)| \text{ and } |f(x)| \leq E_2|x| + |f(0)|.$$

Then, let $\varphi \in \mathbb{M}$, by (2.14)–(2.18) and (H4)–(H7) we have

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\leq |Q(t, \varphi(t - g(t)))| + \int_{t-\tau(t)}^{t} a(u)|h(\varphi(u))|\,du \\
&+ \eta \int_{t-T}^{t} a(u) \int_{u-\tau(u)}^{u} a(s)|h(\varphi(s))|\,ds e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} |b(u)||h(\varphi(u - \tau(u)))| e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} \left[a(u)|Q(u, \varphi(u - g(u)))| + \int_{-\infty}^{u} |D(u,s)||f(\varphi(s))|\,ds\right] e^{-\int_u^t a(k)dk} du \\
&\leq q_1(t)|\varphi(t - g(t))| + q_2(t) + \beta_1\beta_2(E_1 L + |h(0)|) \\
&\times \left(1 + \eta \int_{t-T}^{t} a(u) e^{-\int_u^t a(k)dk} du\right) + \eta \int_{t-T}^{t} |b(u)|(E_1 L + |h(0)|) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} a(u)[q_1(u)|\varphi(u - g(u))| + q_2(u)] e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} E(E_2 L + |f(0)|) e^{-\int_u^t a(k)dk} du \\
&\leq \frac{\gamma_2}{2}L + \gamma_1 L + \gamma_3 L + \frac{\gamma_2}{2}L + \gamma_4 L \leq \frac{L}{J} \leq L.
\end{aligned}$$

Let $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, we get

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\leq |Q(t_2, \varphi(t_2 - g(t_2))) - Q(t_1, \varphi(t_1 - g(t_1)))| \\
&+ \left|\int_{t_2-\tau(t_2)}^{t_2} a(u) h(\varphi(u))\,du - \int_{t_1-\tau(t_1)}^{t_1} a(u) h(\varphi(u))\,du\right| \\
&+ \eta \left|\int_{t_2-T}^{t_2} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s))\,ds\right] a(u) e^{-\int_u^{t_2} a(k)dk} du\right. \\
&\left. - \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s))\,ds\right] a(u) e^{-\int_u^{t_1} a(k)dk} du\right| \\
&+ \eta \left|\int_{t_2-T}^{t_2} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_2} a(k)dk} du - \int_{t_1-T}^{t_1} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^{t_1} a(k)dk} du\right| \\
&+ \eta \left|\int_{t_2-T}^{t_2} \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u,s) f(\varphi(s))\,ds\right] e^{-\int_u^{t_2} a(k)dk} du\right. \\
&\left. - \int_{t_1-T}^{t_1} \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u,s) f(\varphi(s))\,ds\right] e^{-\int_u^{t_1} a(k)dk} du\right|. \qquad (2.23)
\end{aligned}$$

By hypotheses (H5) and (2.19)–(2.21), we obtain

$$\begin{aligned}
&\left|\int_{t_2-\tau(t_2)}^{t_2} a(u)\, h(\varphi(u))\, du - \int_{t_1-\tau(t_1)}^{t_1} a(u)\, h(\varphi(u))\, du\right| \\
&\leq E_1 L\left(\int_{t_1}^{t_2} a(u)\, du + \int_{t_1-\tau(t_1)}^{t_2-\tau(t_2)} a(u)\, du\right) \\
&\leq E_1 L k_3 |t_2 - t_1| + E_1 L k_3 (1 + k_1) |t_2 - t_1| \\
&= (2E_1 L k_3 + E_1 L k_3 k_1) |t_2 - t_1|, \qquad (2.24)
\end{aligned}$$

and

$$\begin{aligned}
&|Q(t_2, \varphi(t_2 - g(t_2))) - Q(t_1, \varphi(t_1 - g(t_1)))| \\
&\leq K_Q |t_2 - t_1| + E_3 K |(t_2 - t_1) - (g(t_2) - g(t_1))| \\
&\leq (K_Q + E_3 K + E_3 K k_2) |t_2 - t_1|, \qquad (2.25)
\end{aligned}$$

where K is the Lipschitz constant of φ. By the hypotheses (H5), (2.16) and (2.21), we get

$$\begin{aligned}
&\eta\left|\int_{t_2-T}^{t_2} b(u)\, h(\varphi(u-\tau(u)))\, e^{-\int_u^{t_2} a(k)dk} du \right.\\
&\left. - \int_{t_1-T}^{t_1} b(u)\, h(\varphi(u-\tau(u)))\, e^{-\int_u^{t_1} a(k)dk} du\right| \\
&\leq \eta\left|\int_{t_1}^{t_2} b(u)\, h(\varphi(u-\tau(u)))\, e^{-\int_u^{t_2} a(k)dk} du\right| \\
&+ \eta\left|\int_{t_1-T}^{t_1} b(u)\, h(\varphi(u-\tau(u)))\left(e^{-\int_u^{t_2} a(k)dk} - e^{-\int_u^{t_1} a(k)dk}\right) du\right| \\
&+ \eta\left|\int_{t_1-T}^{t_2-T} b(u)\, h(\varphi(u-\tau(u)))\, e^{-\int_u^{t_2} a(k)dk} du\right| \\
&\leq 2\eta\left|\int_{t_1}^{t_2} b(u)\, h(\varphi(u-\tau(u)))\, e^{-\int_u^{t_2} a(k)dk} du\right| \\
&+ \eta\left|\int_{t_1-T}^{t_1} b(u)\, h(\varphi(u-\tau(u)))\, e^{-\int_u^{t_1} a(k)dk}\left(e^{-\int_{t_1}^{t_2} a(k)dk} - 1\right) du\right| \\
&\leq 2\eta(E_1 L + |h(0)|)\int_{t_1}^{t_2} |b(u)|\, e^{-\int_u^{t_2} a(k)dk} du \\
&+ \eta\gamma_3 L\left|e^{-\int_{t_1}^{t_2} a(k)dk} - 1\right|\int_{t_1-T}^{t_1} a(u)\, e^{-\int_u^{t_1} a(k)dk} du.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\eta\left|\int_{t_2-T}^{t_2} b(u)\,h(\varphi(u-\tau(u)))\,e^{-\int_u^{t_2} a(k)dk}du-\int_{t_1-T}^{t_1} b(u)\,h(\varphi(u-\tau(u)))\,e^{-\int_u^{t1} a(k)dk}du\right|\\
&\leq \gamma_3 L\int_{t_1}^{t_2} a(u)\,du+2\eta\,(E_1L+|h(0)|)\int_{t_1}^{t_2} d\left(\int_{t_1}^{u}|b(r)|\,dr\right)e^{-\int_u^{t_2} a(k)dk}du\\
&=\gamma_3 L\int_{t_1}^{t_2} a(u)\,du+2\eta\,(E_1L+|h(0)|)\left[\int_{t_1}^{u}|b(r)|\,dre^{-\int_u^{t_2} a(k)dk}\right]_{t_1}^{t_2}\\
&+2\eta\,(E_1L+|h(0)|)\int_{t_1}^{t_2}\left(\int_{t_1}^{u}|b(r)|\,dr\right)a(u)\,e^{-\int_u^{t_2} a(k)dk}du\\
&\leq \gamma_3 L\int_{t_1}^{t_2} a(u)\,du+2\eta\,(E_1L+|h(0)|)\int_{t_1}^{t_2}|b(u)|\,du\left(1+\int_{t_1}^{t_2} a(u)\,e^{-\int_u^{t_2} a(k)dk}du\right)\\
&\leq \gamma_3 L\int_{t_1}^{t_2} a(u)\,du+4\eta\int_{t_1}^{t_2}|b(u)|\,(E_1L+|h(0)|)\,du\\
&\leq \gamma_3 L\int_{t_1}^{t_2} a(u)\,du+4\eta\gamma_3 L\int_{t_1}^{t_2} a(u)\,du\leq(1+4\eta)\,\gamma_3Lk_3\,|t_2-t_1|\,. \qquad (2.26)
\end{aligned}$$

In the same way, by (2.15)–(2.17) and (2.21), we have

$$\begin{aligned}
&\eta\left|\int_{t_2-T}^{t_2}\left[-a(u)\,Q(u,\varphi(u-g(u)))+\int_{-\infty}^{u} D(u,s)\,f(\varphi(s))\,ds\right]e^{-\int_u^{t_2} a(k)dk}du\right.\\
&\left.-\int_{t_1-T}^{t_1}\left[-a(u)\,Q(u,\varphi(u-g(u)))+\int_{-\infty}^{u} D(u,s)\,f(\varphi(s))\,ds\right]e^{-\int_u^{t_1} a(k)dk}du\right|\\
&\leq \eta\left|\int_{t_1}^{t_2}\left[-a(u)\,Q(u,\varphi(u-g(u)))+\int_{-\infty}^{u} D(u,s)\,f(\varphi(s))\,ds\right]e^{-\int_u^{t_2} a(k)dk}du\right|\\
&+\eta\left|\int_{t_1-T}^{t_1}\left[-a(u)\,Q(u,\varphi(u-g(u)))+\int_{-\infty}^{u} D(u,s)\,f(\varphi(s))\,ds\right]\right.\\
&\left.\times\left(e^{-\int_u^{t_2} a(k)dk}-e^{-\int_u^{t_1} a(k)dk}\right)du\right|\\
&+\eta\left|\int_{t_1-T}^{t_2-T}\left[-a(u)\,Q(u,\varphi(u-g(u)))+\int_{-\infty}^{u} D(u,s)\,f(\varphi(s))\,ds\right]e^{-\int_u^{t_2} a(k)dk}du\right|\\
&\leq 2\eta\int_{t_1}^{t_2}\left[a(u)\,\frac{\gamma_2}{2}L+(E_2L+|f(0)|)\int_{-\infty}^{u}|D(u,s)|\,ds\right]e^{-\int_u^{t_2} a(k)dk}du\\
&+\eta\left|e^{-\int_{t_1}^{t_2} a(k)dk}-1\right|\left|\int_{t_1-T}^{t_1}\left[a(u)\,\frac{\gamma_2}{2}L+(E_2L+|f(0)|)\int_{-\infty}^{u}|D(u,s)|\,ds\right]e^{-\int_u^{t_1} a(k)dk}du\right|\\
&\leq \eta\gamma_2L\int_{t_1}^{t_2} a(u)\,du+2\gamma_4L\beta_2\beta_3\,|t_2-t_1|+\left[\frac{\gamma_2}{2}L+\gamma_4L\right]\int_{t_1}^{t_2} a(u)\,du\\
&\leq\left[\left[\left(\eta+\frac{1}{2}\right)\gamma_2+\gamma_4\right]k_3+2\gamma_4\beta_2\beta_3\right]L\,|t_2-t_1|\,. \qquad (2.27)
\end{aligned}$$

and

$$\begin{aligned}
&\eta \left| \int_{t_2-T}^{t_2} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k)dk} du \right. \\
&\left. - \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_1} a(k)dk} du \right| \\
&\leq \eta \left| \int_{t_1}^{t_2} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k)dk} du \right| \\
&+ \eta \left| \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) \left(e^{-\int_u^{t_2} a(k)dk} - e^{-\int_u^{t_1} a(k)dk} \right) du \right| \\
&+ \eta \left| \int_{t_1-T}^{t_2-T} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k)dk} du \right| \\
&\leq 2\eta \left| \int_{t_1}^{t_2} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_2} a(k)dk} du \right| \\
&+ \eta \left| \int_{t_1-T}^{t_1} \left[\int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds \right] a(u) e^{-\int_u^{t_1} a(k)dk} \left(e^{-\int_{t_1}^{t_2} a(k)dk} - 1 \right) du \right| \\
&\leq 2\eta \frac{\gamma_1}{2} L \int_{t_1}^{t_2} a(u) e^{-\int_u^{t_2} a(k)dk} du + \eta \left| e^{-\int_{t_1}^{t_2} a(k)dk} - 1 \right| \frac{\gamma_1}{2} L \int_{t_1-T}^{t_1} a(u) e^{-\int_u^{t_1} a(k)dk} du \\
&\leq \eta\gamma_1 L \int_{t_1}^{t_2} a(u) du + \frac{\gamma_1}{2} L \int_{t_1}^{t_2} a(u) du \leq \left[\eta + \frac{1}{2} \right] \gamma_1 L k_3 |t_2 - t_1| . \qquad (2.28)
\end{aligned}$$

Thus, by substituting (2.24)–(2.28) in (2.23), we obtain

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\leq (2E_1 L k_3 + E_1 L k_3 k_1) |t_2 - t_1| + (K_Q + E_3 K + E_3 K k_2) |t_2 - t_1| \\
&+ (1 + 4\eta) \gamma_3 L k_3 |t_2 - t_1| + \left[\left[\left(\eta + \frac{1}{2} \right) \gamma_2 + \gamma_4 \right] k_3 + 2\gamma_4 \beta_2 \beta_3 \right] L |t_2 - t_1| \\
&+ \left[\eta + \frac{1}{2} \right] \gamma_1 L k_3 |t_2 - t_1| \\
&\leq \frac{K}{3} |t_2 - t_1| \leq K |t_2 - t_1| .
\end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}$. ■

Lemma 2.3 *For $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ defined in (2.12), suppose that (2.2)–(2.5), (2.14)–(2.22) and (H4)–(H8) hold. Then $\mathcal{A}$ is completely continuous.*

Proof. Since $\mathbb{M}$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, T]$ we can apply the Arzela-Ascoli theorem to confirm that $\mathbb{M}$ is a compact subset from this space. Also, since any continuous operator maps compact sets into compact sets, then to prove that $\mathcal{A}$ is a compact operator it's suffices to prove that it is continuous.

We prove that $\mathcal{A}$ is continuous in the supremum norm, let $\varphi_n \in \mathbb{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&\left|\left(\mathcal{A}\varphi_n\right)(t)-\left(\mathcal{A}\varphi\right)(t)\right| \\
&\leq \left|Q\left(t,\varphi_n\left(t-g\left(t\right)\right)\right)-Q\left(t,\varphi\left(t-g\left(t\right)\right)\right)\right| \\
&+\int_{t-\tau(t)}^{t} a\left(u\right)\left|h\left(\varphi_n\left(u\right)\right)-h\left(\varphi\left(u\right)\right)\right| du \\
&+\eta\int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u} a\left(s\right)\left|h\left(\varphi_n\left(s\right)\right)-h\left(\varphi\left(s\right)\right)\right| ds\right] a\left(u\right)e^{-\int_u^t a(k)dk}du \\
&+\eta\int_{t-T}^{t}\left|b\left(u\right)\right|\left|h\left(\varphi_n\left(u-\tau\left(u\right)\right)\right)-h\left(\varphi\left(u-\tau\left(u\right)\right)\right)\right| e^{-\int_u^t a(k)dk}du \\
&+\eta\int_{t-T}^{t}\left[a\left(u\right)\left|Q\left(u,\varphi_n\left(u-g\left(u\right)\right)\right)-Q\left(u,\varphi\left(u-g\left(u\right)\right)\right)\right|\right. \\
&\left.+\int_{-\infty}^{u}\left|D\left(u,s\right)\right|\left|f\left(\varphi_n\left(s\right)\right)-f\left(\varphi\left(s\right)\right)\right| ds\right] e^{-\int_u^t a(k)dk}du.
\end{aligned}$$

By the dominated convergence theorem, $\lim_{n\to\infty}\left|\left(\mathcal{A}\varphi_n\right)(t)-\left(\mathcal{A}\varphi\right)(t)\right| = 0$. Then $\mathcal{A}$ is continuous. Therefore, $\mathcal{A}$ is compact. ■

The next result shows the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions. Assume that

$$\max\left\{\left|H\left(-L\right)\right|,\left|H\left(L\right)\right|\right\} \leq \frac{\left(J-1\right)L}{J}, \tag{2.29}$$

and

$$\left[2\eta+1\right]Lk_3 \leq K. \tag{2.30}$$

Lemma 2.4 *Let $\mathcal{B}$ be defined by (2.13), suppose (2.21), (2.29), (2.30) and all conditions of Theorem 1.5 hold. Then $\mathcal{B}:\mathbb{M}\to\mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (2.13). Obviously, $\mathcal{B}$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. For having $\mathcal{B}\varphi\in\mathbb{M}$ we will show that $\|\mathcal{B}\varphi\|\leq L$ and

$$\left|\left(\mathcal{B}\varphi\right)(t_2)-\left(\mathcal{B}\varphi\right)(t_1)\right| \leq K\left|t_2-t_1\right|,\ \forall t_1,t_2\in\left[0,T\right].$$

Let $\varphi\in\mathbb{M}$ by (2.29), we get

$$\begin{aligned}
\left|\left(\mathcal{B}\varphi\right)(t)\right| &\leq \eta\int_{t-T}^{t} a\left(u\right)\max\left\{\left|H\left(-L\right)\right|,\left|H\left(L\right)\right|\right\} e^{-\int_u^t a(k)dk}du \\
&\leq \frac{\left(J-1\right)L}{J} \leq L.
\end{aligned}$$

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, by (2.21), (2.29), (2.30), we have

$$
\begin{aligned}
&\left|(\mathcal{B}\varphi)(t_1) - (\mathcal{B}\varphi)(t_2)\right| \\
&\leq \eta \left| \int_{t_2-T}^{t_2} a(u) H(\varphi(u)) e^{-\int_u^{t_2} a(k)dk} du - \int_{t_1-T}^{t_1} a(u) H(\varphi(u)) e^{-\int_u^{t_1} a(k)dk} du \right| \\
&\leq \eta \left| \int_{t_1}^{t_2} a(u) H(\varphi(u)) e^{-\int_u^{t_2} a(k)dk} du \right| \\
&+ \eta \left| \int_{t_1-T}^{t_1} a(u) H(\varphi(u)) \left(e^{-\int_u^{t_2} a(k)dk} - e^{-\int_u^{t_1} a(k)dk} \right) du \right| \\
&+ \eta \left| \int_{t_1-T}^{t_2-T} a(u) H(\varphi(u)) e^{-\int_u^{t_2} a(k)dk} du \right| \\
&\leq 2\eta \int_{t_1}^{t_2} a(u) \left| H(\varphi(u)) \right| e^{-\int_u^{t_2} a(k)dk} du \\
&+ \eta \left| e^{-\int_{t_1}^{t_2} a(k)dk} - 1 \right| \int_{t_1-T}^{t_1} a(u) \left| H(\varphi(u)) \right| e^{-\int_u^{t_1} a(k)dk} du \\
&\leq 2\frac{(J-1)}{J} L\eta \int_{t_1}^{t_2} a(u)\, du + \frac{(J-1)}{J} L \int_{t_1}^{t_2} a(u)\, du \\
&\leq [2\eta + 1] \frac{(J-1)}{J} L k_3 \left| t_2 - t_1 \right| \\
&\leq \frac{(J-1)}{J} K \left| t_2 - t_1 \right| \leq K \left| t_2 - t_1 \right|.
\end{aligned}
$$

which implies $\mathcal{B} : \mathbb{M} \to \mathbb{M}$.

By Theorem 1.5, H is large contraction on $\mathbb{M}$, then for any $\varphi, \psi \in \mathbb{M}$ with $\varphi \neq \psi$, we get

$$\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \|\varphi - \psi\|.$$

Now, let $\varepsilon \in (0,1)$ be given and let $\varphi, \psi \in \mathbb{M}$, with $\|\varphi - \psi\| \geq \varepsilon$ from the proof of Theorem 1.5, we have found a $\delta \in (0,1)$, such that

$$\left|(H\varphi)(t) - (H\psi)(t)\right| \leq \delta \|\varphi - \psi\|.$$

Thus,

$$
\begin{aligned}
\left|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)\right| &\leq \left| \eta \int_{t-T}^{t} a(u) \left[H(\varphi(u)) - H(\psi(u)) \right] e^{-\int_u^t a(k)dk} du \right| \\
&\leq \delta \|\varphi - \psi\| \eta \int_{t-T}^{t} a(u) e^{-\int_u^t a(k)dk} du \\
&\leq \delta \|\varphi - \psi\|.
\end{aligned}
$$

The proof is complete. ■

Theorem 2.1 *Suppose the hypotheses of Lemmas 2.2–2.4 hold. Let* $\mathbb{M}$ *defined by (2.10), Then (2.1) has a* T*-periodic solution in* $\mathbb{M}$.

Proof. By Lemmas 2.2 and 2.3 $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 2.4, the mapping $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq L$ and $|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (2.14)–(2.18) and (2.29), we get

$$\begin{aligned} \|\mathcal{A}\varphi + \mathcal{B}\psi\| &\leq [\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4] L + \frac{(J-1)L}{J} \\ &\leq \frac{L}{J} + \frac{(J-1)L}{J} = L. \end{aligned}$$

Now, let $\varphi, \psi \in \mathbb{M}$ and $t_1, t_2 \in [0, T]$. By (2.14)–(2.22), (2.29) and (2.30), we have

$$\begin{aligned} &|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \\ &\leq |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\psi)(t_2) - (\mathcal{B}\psi)(t_1)| \\ &\leq \frac{K}{J}|t_2 - t_1| + \frac{(J-1)K}{J}|t_2 - t_1| \\ &= K|t_2 - t_1|. \end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 2.1 this fixed point is a solution of (2.1). Hence (2.1) has a T-periodic solution. ■

Example 2.1 Consider the following nonlinear neutral differential equation

$$\frac{d}{dt}[x(t) - Q(t, x(t - g(t)))] = -a(t) h(x(t - \tau(t))) + \int_{-\infty}^{t} D(t,s) f(x(s)) ds, \quad (2.31)$$

where

$$\begin{aligned} T &= 2\pi,\ a(t) = 2,\ \tau(t) = \frac{10^{-2}}{\sqrt{3}},\ g(t) = 2 \times 10^{-2} e^{-t},\ h(x) = x^3, \\ Q(t,x) &= 10^{-4}\sin(x),\ D(t,s) = e^{s-t},\ f(x) = x^2. \end{aligned}$$

Then (2.31) has a 2π-periodic solution.

Proof. We have $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\left[-\sqrt{3}/3, \sqrt{3}/3\right]$, differentiable on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$, strictly increasing on $\left[-\sqrt{3}/3, \sqrt{3}/3\right]$ and $\sup_{t \in \left(-\sqrt{3}/3, \sqrt{3}/3\right)} h'(t) \leq 1$. By Theorem 1.5, the mapping $H(x) = x - x^3$ is a large contraction on the set

$$\mathbb{M} = \left\{\varphi \in P_{2\pi},\ \|\varphi\| \leq \sqrt{3}/3,\ |\varphi(t_2) - \varphi(t_1)| \leq 100|t_2 - t_1|,\ \forall t_1, t_2 \in [0, 2\pi]\right\},$$

where $L = \sqrt{3}/3$ and $K = 100$. Doing straightforward computations, we obtain

$$\begin{aligned} E &= 1,\ \beta_1 = \frac{10^{-2}}{\sqrt{3}},\ \beta_2 = 2,\ \beta_3 = e^{-4\pi},\ E_1 = 1,\ E_2 = 2\sqrt{3}/3,\ E_3 = 10^{-4}, \\ q_1(t) &= 10^{-4},\ q_2(t) = 0,\ \eta = \left(1 - e^{-4\pi}\right)^{-1},\ \gamma_1 = \frac{4}{\sqrt{3}}10^{-2},\ \gamma_2 = 2 \times 10^{-4}, \\ \gamma_3 &= 0,\ \gamma_4 = 4\pi\left(1 - e^{-4\pi}\right)^{-1} e^{-4\pi},\ J \in [3, 42],\ k_1 = 0,\ k_2 = 2 \times 10^{-2},\ k_3 = 2. \end{aligned}$$

All hypotheses of Theorem 2.1 are fulfilled and so (2.31) has a 2π-periodic solution belonging to $\mathbb{M}$. ■

2.3 Existence of nonnegative periodic solutions

In this section we obtain the existence of a nonnegative periodic solution of (2.1). By applying Theorem 1.7, we need to define a closed, convex, and bounded subset $\mathcal{M}$ of P_T. So, let

$$\mathcal{M} = \{\varphi \in P_T : 0 \leq \varphi \leq L,\ \ |\varphi(t_2) - \varphi(t_1)| \leq K |t_2 - t_1|,\ \forall t_1, t_2 \in [0,T]\}, \tag{2.32}$$

where L and K are positive constants. To simplify notation, we let

$$F(t, x(t)) = \int_{t-\tau(t)}^{t} a(u)\, h(x(u))\, du, \tag{2.33}$$

and

$$m = \min_{u \in [t-T,t]} e^{-\int_u^t a(k)dk},\ M = \max_{u \in [t-T,t]} e^{-\int_u^t a(k)dk}. \tag{2.34}$$

It is easy to see that for all $(t,u) \in [0,2T]^2$,

$$m \leq e^{-\int_u^t a(k)dk} \leq M. \tag{2.35}$$

Then we obtain the existence of a nonnegative periodic solution of (2.1) by considering the two cases

(1) $F(t, x(t)) \geq 0,\ \forall t \in [0,T],\ x \in \mathcal{M}$.

(2) $F(t, x(t)) \leq 0,\ \forall t \in [0,T],\ x \in \mathcal{M}$.

In the case one, we assume for all $t \in [0,T]$, $x \in \mathcal{M}$, that there exist positive constants c_1 and c_2 such that

$$0 \leq Q(t, x(t)) \leq c_1 L, \tag{2.36}$$

$$0 \leq F(t, x(t)) \leq c_2 L, \tag{2.37}$$

$$c_1 + c_2 < 1, \tag{2.38}$$

$$0 \leq -a(u) F(t, x(t)) + b(t) h(x(t)) - a(t) Q(t, x(t)) + \int_{-\infty}^{t} D(t,s) f(x(s))\, ds, \tag{2.39}$$

$$\begin{aligned} &- a(u) F(t, x(t)) + b(t) h(x(t)) + a(t) H(x(t)) \\ &- a(t) Q(t, x(t)) + \int_{-\infty}^{t} D(t,s) f(x(s))\, ds \ \leq \ \frac{L(1 - c_1 - c_2)}{M\eta T}. \end{aligned} \tag{2.40}$$

Lemma 2.5 *Let $\mathcal{A}$, $\mathcal{B}$ given by (2.12), (2.13), respectively, assume (2.36)–(2.40) hold. Then $\mathcal{A}, \mathcal{B} : \mathcal{M} \to \mathcal{M}$.*

Proof. For having $\mathcal{A}\varphi, \mathcal{B}\varphi \in \mathcal{M}$ we show that $0 \leq \mathcal{A}\varphi, \mathcal{B}\varphi \leq L$ and $\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \leq K\left|t_2 - t_1\right|$, $\left|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)\right| \leq K\left|t_2 - t_1\right|$, $\forall t_1, t_2 \in [0, T]$. Let $\mathcal{A}$ defined by (2.12). So, for any $\varphi \in \mathcal{M}$, we have

$$
\begin{aligned}
&0 \leq (\mathcal{A}\varphi)(t) \\
&\leq Q(t, \varphi(t - g(t))) + F(t, \varphi(t)) - \eta \int_{t-T}^{t} F(t, \varphi(u))\, a(u)\, e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} b(u)\, h(\varphi(u - \tau(u)))\, e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} \left[-a(u)\, Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u, s)\, f(\varphi(s))\, ds\right] e^{-\int_u^t a(k)dk} du \\
&\leq \eta \int_{t-T}^{t} M \frac{L(1 - c_1 - c_2)}{M\eta T} du + c_1 L + c_2 L = L.
\end{aligned}
$$

From Lemma 2.2, we see that

$$
\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \leq \frac{K}{J}\left|t_2 - t_1\right| \leq K\left|t_2 - t_1\right|.
$$

That is $\mathcal{A}\varphi \in \mathcal{M}$.

Now, let $\mathcal{B}$ defined by (2.13). So, for any $\varphi \in \mathcal{M}$, we have

$$
\begin{aligned}
0 &\leq (\mathcal{B}\varphi)(t) \\
&\leq \eta \int_{t-T}^{t} M \frac{L(1 - c_1 - c_2)}{M\eta T} du \leq \eta M T \frac{L}{M\eta T} = L,
\end{aligned}
$$

and from Lemma 2.4, we see that

$$
\left|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)\right| \leq \frac{(J-1)K}{J}\left|t_2 - t_1\right| \leq K\left|t_2 - t_1\right|
$$

That is $\mathcal{B}\varphi \in \mathcal{M}$. ■

Theorem 2.2 *Suppose the hypotheses of Lemmas 2.3–2.5 hold. Then (2.1) has a non-negative T-periodic solution x in the subset $\mathcal{M}$.*

Proof. By Lemma 2.3, $\mathcal{A}$ is completely continuous. Also, from Lemma 2.4, the mapping $\mathcal{B}$ is a large contraction. By Lemma 2.5, $\mathcal{A}, \mathcal{B} : \mathcal{M} \to \mathcal{M}$. Next, we show that if $\varphi, \psi \in \mathcal{M}$, we have $0 \leq \mathcal{A}\varphi + \mathcal{B}\psi \leq L$ and $\left|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)\right| \leq K\left|t_2 - t_1\right|$, $\forall t_1, t_2 \in [0, T]$.

Let $\varphi, \psi \in \mathcal{M}$ with $0 \leq \varphi, \psi \leq L$. By (2.36)–(2.40), we get

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t) \\
&= \eta \int_{t-T}^{t} a(u) H(\psi(u)) e^{-\int_u^t a(k)dk} du + Q(t, \varphi(t - g(t))) \\
&+ F(t, \varphi(t)) - \eta \int_{t-T}^{t} F(t, \varphi(u)) a(u) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u,s) f(\varphi(s)) ds \right] e^{-\int_u^t a(k)dk} du \\
&\leq \eta \int_{t-T}^{t} M \frac{L(1 - c_1 - c_2)}{M\eta T} du + c_1 L + c_2 L = L.
\end{aligned}$$

On the other hand, we have

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t) \geq 0.$$

Now, let $\varphi, \psi \in \mathcal{M}$ and $t_1, t_2 \in [0, T]$. By Lemmas 2.2, 2.4, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \\
&\leq |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\psi)(t_2) - (\mathcal{B}\psi)(t_1)| \\
&\leq \frac{K}{J} |t_2 - t_1| + \frac{(J-1)K}{J} |t_2 - t_1| \\
&\leq K |t_2 - t_1|.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 1.7 this fixed point is a solution of (2.1) and the proof is complete. ■

Example 2.2 Consider the following equation

$$\frac{d}{dt}[x(t) - Q(t, x(t - g(t)))] = -a(t) h(x(t - \tau(t))) + \int_{-\infty}^{t} D(t,s) f(x(s)) ds, \quad (2.41)$$

where

$$T = 2\pi,\ a(t) = \frac{10^{-2}}{4},\ \tau(t) = 2\pi,\ h(x) = x^3,\ Q(t,x) = 10^{-4}x,$$
$$F(t, x(t)) = \frac{10^{-2}}{4} \int_{t-2\pi}^{t} x^3(u) du,\ D(t,s) = e^{s-t},\ f(x) = 10^{-4}\left(x + \frac{\pi^4}{4}\right).$$

Then (2.41) has a nonnegative 2π-periodic solution.

Proof. By Example 2.1, the mapping $H(x) = x - x^3$ is a large contraction on the set

$$\mathcal{M} = \left\{ \varphi \in P_{2\pi},\ 0 \leq \varphi \leq \sqrt{3}/3,\ \ |\varphi(t_2) - \varphi(t_1)| \leq 100 |t_2 - t_1|,\ \forall t_1, t_2 \in [0, T] \right\}.$$

A simple calculation yields

$$\begin{aligned} F(t,x(t)) &= \frac{10^{-2}}{4}\int_{t-2\pi}^{t} x^3(u)\,du = \frac{1}{4}\int_{0}^{2\pi} x^3(u)\,du = \frac{10^{-2}}{4}\left[\frac{x^4}{4}\right]_0^{2\pi} = 10^{-2}\pi^4 \geq 0, \\ m &= e^{-\frac{10^{-2}}{2}\pi},\ M=1,\ \eta = \left(1-e^{-\frac{10^{-2}}{2}\pi}\right)^{-1},\ c_1 = 10^{-4},\ c_2 = \frac{10^{-2}}{6}\pi. \end{aligned}$$

Then for $x \in \left[0, \sqrt{3}/3\right]$ we have

$$0 \leq -a(t)F(t,x(t)) + b(t)h(x(t)) - a(t)Q(t,x(t)) + \int_{-\infty}^{t} D(t,s)f(x(s))\,ds.$$

On the other hand, we have

$$\begin{aligned} &-a(t)F(t,x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ &-a(t)Q(t,x(t)) + \int_{-\infty}^{t} D(t,s)f(x(s))\,ds \\ &\leq 1.006\times 10^{-3} < 1.425\times 10^{-3} \simeq \frac{L(1-c_1-c_2)}{M\eta T}. \end{aligned}$$

All conditions of Theorem 2.2 hold and so (2.41) has a nonnegative 2π-periodic solution belonging to $\mathcal{M}$. ■

In the case two, we substitute conditions (2.37)–(2.40) with the following conditions, respectively. We assume that there exist a negative constant c_3 such that

$$c_3 L \leq F(t,x(t)) \leq 0, \tag{2.42}$$

$$-c_3 + c_1 < 1, \tag{2.43}$$

$$\begin{aligned} \frac{-c_3 L}{m\eta T} &\leq -a(u)F(t,x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ &\quad - a(t)Q(t,x(t)) + \int_{-\infty}^{t} D(t,s)f(x(s))\,ds, \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} &-a(u)F(t,x(t)) + b(t)h(x(t)) + a(t)H(x(t)) \\ &-a(t)Q(t,x(t)) + \int_{-\infty}^{t} D(t,s)f(x(s))\,ds \leq \frac{L(1-c_1)}{M\eta T}. \end{aligned} \tag{2.45}$$

Theorem 2.3 *Suppose (2.36), (2.42)–(2.45) and the hypotheses of Lemmas 2.2–2.4 hold. Then (2.1) has a nonnegative T-periodic solution x in the subset $\mathcal{M}$.*

Proof. By Lemma 2.3, $\mathcal{A}$ is completely continuous. Also, from Lemma 2.4, the mapping $\mathcal{B}$ is a large contraction. It is easy to show as in Lemma 2.5, $\mathcal{A},\mathcal{B}:\mathcal{M}\to\mathcal{M}$. Next, we show that if $\varphi,\psi\in\mathcal{M}$, we have $0\leq \mathcal{A}\varphi+\mathcal{B}\psi \leq L$ and $|(\mathcal{A}\varphi+\mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi+\mathcal{B}\psi)(t_1)| \leq$

$K\left|t_2 - t_1\right|$, $\forall t_1, t_2 \in [0,T]$. Let $\varphi, \psi \in \mathcal{M}$ with $0 \leq \varphi, \psi \leq L$. By (2.36) and (2.42)–(2.45) we get

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t) \\
&= \eta \int_{t-T}^{t} a(u) H(\psi(u)) e^{-\int_u^t a(k)dk} du + Q(t, \varphi(t - g(t))) \\
&+ F(t, \varphi(t)) - \eta \int_{t-T}^{t} F(t, \varphi(u)) a(u) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} b(u) h(\varphi(u - \tau(u))) e^{-\int_u^t a(k)dk} du \\
&+ \eta \int_{t-T}^{t} \left[-a(u) Q(u, \varphi(u - g(u))) + \int_{-\infty}^{u} D(u,s) f(\varphi(s)) ds \right] e^{-\int_u^t a(k)dk} du \\
&\leq \eta \int_{t-T}^{t} M \frac{L(1-c_1)}{M\eta T} du + c_1 L = L.
\end{aligned}$$

On the other hand, we have

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t) \geq \eta \int_{t-T}^{t} m \frac{-c_3 L}{m\eta T} du + c_3 L = 0.$$

Now, let $\varphi, \psi \in \mathcal{M}$ and $t_1, t_2 \in [0,T]$. By Lemmas 2.2 and 2.4, we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)\right| \\
&\leq \left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| + \left|(\mathcal{B}\psi)(t_2) - (\mathcal{B}\psi)(t_1)\right| \\
&\leq \frac{K}{J}\left|t_2 - t_1\right| + \frac{(J-1)K}{J}\left|t_2 - t_1\right| \\
&= K\left|t_2 - t_1\right|.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 1.7 this fixed point is a solution of (2.1) and the proof is complete. ■

Chapter 3

Periodic solutions for first order totally nonlinear iterative differential equations

Keywords. Krasnoselskii-Burton's fixed point, large contraction, iterative differential equations, periodic solutions.

This chapter present a very recent published work [73],

A. Guerfi, A. Ardjouni, Periodic solutions for totally nonlinear iterative differential equations, Bull. Int. Math. Virtual Inst. 12(1) (2022), 69–82.

In this chapter, we consider the following first order totally nonlinear iterative differential equation

$$\frac{d}{dt}x(t) = -a(t)h(x(t)) + \frac{d}{dt}g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \tag{3.1}$$

where $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t))$, ..., $x^{[n]}(t) = x^{[n-1]}(x(t))$ and a is a continuous real-valued function. The functions $h : \mathbb{R} \to \mathbb{R}$, $g, f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are continuous. Our purpose here is to use Krasnoselskii-Burton's fixed point technique to prove the existence of periodic solutions for (3.1). During the process we use the variation of parameter formula and the integration by parts to transform (3.1) into an equivalent integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use Krasnoselskii-Burton's fixed point theorem, to prove the existence of a periodic solution. The obtained results in this work extend the main results in [42].

3.1 Preliminaries and inversion of the equation

For $T > 0$, define

$$P_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\},$$

where $C(\mathbb{R},\mathbb{R})$ denoted the set of all real valued continuous functions map $\mathbb{R}$ into $\mathbb{R}$. Then P_T is a Banach space with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)| .$$

For $L, K > 0$, define the set

$$P_T(L,K) = \{x \in P_T,\ \|x\| \leq L,\ \ |x(t_2) - x(t_1)| \leq K|t_2 - t_1| \text{ for all } t_1, t_2 \in \mathbb{R}\},$$

which is a closed convex and bounded subset of P_T.

We assume that

$$a(t+T) = a(t), \quad \int_0^T a(t)\,dt > 0. \tag{3.2}$$

The functions $f(t, x_1, x_2, ..., x_n)$ and $g(t, x_1, x_2, ..., x_n)$ are supposed periodic in t with period T and globally Lipschitz in x_1, x_2, ..., x_n, i.e,

$$\begin{aligned} f(t+T, x_1, ..., x_n) &= f(t, x_1, ..., x_n), \\ g(t+T, x_1, ..., x_n) &= g(t, x_1, ..., x_n), \end{aligned} \tag{3.3}$$

and there exist n positive constants k_1, k_2, ..., k_n and n positive constants c_1, c_2, ..., c_n such that

$$|f(t, x_1, ..., x_n) - f(t, y_1, ..., y_n)| \leq \sum_{i=1}^{n} k_i |x_i - y_i|, \tag{3.4}$$

and

$$|g(t, x_1, ..., x_n) - g(t, y_1, ..., y_n)| \leq \sum_{i=1}^{n} c_i |x_i - y_i|. \tag{3.5}$$

The function $g(t, x_1, ..., x_n)$ is also supposed globally Lipschitz in t, i.e, there exists a positive constant K_g such that

$$|g(t_2, x_1, ..., x_n) - g(t_1, x_1, ..., x_n)| \leq K_g |t_2 - t_1|. \tag{3.6}$$

The following lemma is essential for our results.

Lemma 3.1 *Suppose (3.2) and (3.3) hold. If $x \in P_T(L,K)$, then x is a solution of (3.1) if and only if*

$$\begin{aligned} x(t) = & \int_t^{t+T} G(t,s)\,a(s)\,H(x(s))\,ds \\ & + \int_t^{t+T} \left\{ f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right. \\ & \left. - a(s)\,g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right\} G(t,s)\,ds \\ & + g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \end{aligned} \tag{3.7}$$

where

$$G(t,s)=\frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right)-1}, \tag{3.8}$$

and

$$H(x)=x-h(x). \tag{3.9}$$

Proof. Let $x\in P_T(L,K)$ be a solution of (3.1). Rewrite (3.1) as

$$\begin{aligned}&\frac{d}{dt}x(t)+a(t)x(t)-\frac{d}{dt}g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\\&=a(t)H(x(t))+f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{d}{dt}\left\{\left[x(t)-g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\right]\exp\left(\int_0^t a(u)\,du\right)\right\}\\&=\left\{a(t)H(x(t))-a(t)g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\right.\\&\left.+f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\right\}\exp\left(\int_0^t a(u)\,du\right).\end{aligned}$$

The integration from t to $t+T$ gives

$$\begin{aligned}&\int_t^{t+T}\frac{d}{ds}\left\{\left[x(s)-g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]\exp\left(\int_0^s a(u)\,du\right)\right\}ds\\&=\int_t^{t+T}\left\{a(s)H(x(s))-a(s)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right.\\&\left.+f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right\}\exp\left(\int_0^s a(u)\,du\right)ds.\end{aligned}$$

Since

$$\begin{aligned}&\int_t^{t+T}\frac{d}{ds}\left\{\left[x(s)-g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]\exp\left(\int_0^s a(u)\,du\right)\right\}ds\\&=\left\{x(t)-g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\right\}\\&\times\exp\left(\int_0^t a(u)\,du\right)\left[\exp\left(\int_t^{t+T}a(u)\,du\right)-1\right],\end{aligned}$$

then

$$\begin{aligned}x(t)&=g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\\&+\int_t^{t+T}\left\{a(s)H(x(s))-a(s)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right.\\&\left.+f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right\}\frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_t^{t+T}a(u)\,du\right)-1}ds.\end{aligned}$$

The proof is completed. ■

3.1. Preliminaries and inversion of the equation

Lemma 3.2 *Green function G satisfies the following properties*

$$G(t+T,s+T)=G(t,s),$$

and

$$\alpha=\frac{\exp\left(-\int_0^T a(u)\,du\right)}{\left|\exp\left(\int_0^T a(u)\,du\right)-1\right|}\leq |G(t,s)|\leq \frac{\exp\left(\int_0^T a(u)\,du\right)}{\left|\exp\left(\int_0^T a(u)\,du\right)-1\right|}=\beta.$$

Lemma 3.3 ([121]) *For any $\varphi,\psi\in P_T(L,K)$, we have*

$$\left\|\varphi^{[m]}-\psi^{[m]}\right\|\leq \sum_{j=0}^{m-1}K^j\left\|\varphi-\psi\right\|,\ m=1,2,....$$

Lemma 3.4 ([120]) *It holds*

$$\begin{aligned}&P_T(L,K)\\&=\{x\in P_T,\ \|x\|\leq L,\ |x(t_2)-x(t_1)|\leq K|t_2-t_1|\ \text{for all } t_1,t_2\in[0,T]\}.\end{aligned}$$

3.2 Existence of periodic solutions

To apply the Theorem 1.7 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a completely continuous and the other is a large contraction. So, we let $(\mathbb{B},\|.\|)=(P_T,\|.\|)$ and

$$\begin{aligned}\mathbb{M}&=P_T(L,K)\\&=\{\varphi\in P_T,\ \|\varphi\|\leq L,\ |\varphi(t_2)-\varphi(t_1)|\leq K|t_2-t_1|\ \text{for all } t_1,t_2\in[0,T]\},\end{aligned}\tag{3.10}$$

with $L,K>0$. Define a mapping $\mathcal{S}:\mathbb{M}\to P_T$ by

$$\begin{aligned}(\mathcal{S}\varphi)(t)=&\int_t^{t+T}G(t,s)\,a(s)\,H(\varphi(s))\,ds\\&+\int_t^{t+T}\left\{f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right.\\&\left.-a(s)\,g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right\}G(t,s)\,ds\\&+g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right).\end{aligned}\tag{3.11}$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi=\mathcal{A}\varphi+\mathcal{B}\varphi,$$

where $\mathcal{A},\mathcal{B}:\mathbb{M}\to P_T$ are given by

$$\begin{aligned}(\mathcal{A}\varphi)(t)=&\int_t^{t+T}\left\{f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right.\\&\left.-a(s)\,g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right\}G(t,s)\,ds\\&+g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right),\end{aligned}\tag{3.12}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t,s)\, a(s)\, H(\varphi(s))\, ds. \tag{3.13}$$

To simplify notations, we introduce the following constants

$$\sigma = \max_{t\in[0,T]} |a(t)|, \ \rho_1 = \max_{t\in[0,T]} |f(t,0,0,...,0)|, \ \rho_2 = \max_{t\in[0,T]} |g(t,0,0,...,0)|. \tag{3.14}$$

We need the following assumptions

$$J\left[\beta T(\rho_1+\sigma\rho_2)+\rho_2+L\sum_{i=1}^{n}[c_i+\beta T(k_i+\sigma c_i)]\sum_{j=0}^{i-1}K^j\right] \le L, \tag{3.15}$$

and

$$\begin{aligned} &J\left((2\beta+T\alpha\|a\|)(\rho_1+\sigma\rho_2)+K_g\right.\\ &\left.+\sum_{i=1}^{n}[(2\beta+T\alpha\|a\|)L(k_i+\sigma c_i)+Kc_i]\sum_{j=0}^{i-1}K^j\right) \le K, \end{aligned} \tag{3.16}$$

where J is a positive constant with $J \ge 3$.

Lemma 3.5 *For $\mathcal{A}$ defined in (3.12), suppose that (3.2)–(3.6) and (3.14)–(3.16) hold. Then $\mathcal{A}:\mathbb{M}\to\mathbb{M}$.*

Proof. Let $\varphi\in\mathbb{M}$. For having $\mathcal{A}\varphi\in\mathbb{M}$ we will show that $\mathcal{A}\varphi\in P_T$, $\|\mathcal{A}\varphi\|\le L$ and $|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)|\le K|t_2-t_1|$ for all $t_1,t_2\in[0,T]$. First, it is easy to prove that $(\mathcal{A}\varphi)(t+T)=(\mathcal{A}\varphi)(t)$. That is, if $\varphi\in P_T$ then $\mathcal{A}\varphi\in P_T$. By (3.14), we get

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| \le \beta &\int_t^{t+T}\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\ &+\beta\sigma\int_t^{t+T}\left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\ &+\left|g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right)\right|, \end{aligned}$$

and in view of conditions (3.5), (3.6) and Lemma 3.3, we obtain

$$\begin{aligned} &\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\\ &\le \left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)-f(s,0,0,...,0)\right|+|f(s,0,0,...,0)|\\ &\le \rho_1+\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\|\varphi\|\\ &\le \rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\left|g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)\right| \\
&\leq \left|g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)-g\left(s,0,0,...,0\right)\right|+\left|g\left(s,0,0,...,0\right)\right| \\
&\leq \rho_2+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j\left\|\varphi\right\| \\
&\leq \rho_2+L\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j. \qquad (3.18)
\end{aligned}$$

Thus, it follows from (3.17) and (3.18) that

$$\begin{aligned}
\left|\left(\mathcal{A}\varphi\right)(t)\right| &\leq \beta T\left(\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\right) \\
&\quad+\left(\beta\sigma T+1\right)\left(\rho_2+L\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j\right) \\
&=\beta T\left(\rho_1+\sigma\rho_2\right)+\rho_2+L\sum_{i=1}^{n}\left[c_i+\beta T\left(k_i+\sigma c_i\right)\right]\sum_{j=0}^{i-1}K^j.
\end{aligned}$$

Therefore, from (3.15), we get

$$\left\|\mathcal{A}\varphi\right\|\leq\frac{L}{J}\leq L.$$

Let $t_1,t_2\in[0,T]$ with $t_1<t_2$, we obtain

$$\begin{aligned}
&\left|\left(\mathcal{A}\varphi\right)(t_2)-\left(\mathcal{A}\varphi\right)(t_1)\right| \\
&\leq\left|\int_{t_2}^{t_2+T}f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_2,s\right)ds\right. \\
&\left.-\int_{t_1}^{t_1+T}f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_1,s\right)ds\right| \\
&+\left|\int_{t_2}^{t_2+T}a\left(s\right)g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_2,s\right)ds\right. \\
&\left.-\int_{t_1}^{t_1+T}a\left(s\right)g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_1,s\right)ds\right| \\
&+\left|g\left(t_2,\varphi\left(t_2\right),\varphi^{[2]}\left(t_2\right),...,\varphi^{[n]}\left(t_2\right)\right)-g\left(t_1,\varphi\left(t_1\right),\varphi^{[2]}\left(t_1\right),...,\varphi^{[n]}\left(t_1\right)\right)\right|.
\end{aligned}$$

But,

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_2,s\right)ds\right.\\
&\left.-\int_{t_1}^{t_1+T} f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_1,s\right)ds\right|\\
&\leq\left|\int_{t_2}^{t_1} G\left(t_2,s\right)f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right.\\
&\left.+\int_{t_1+T}^{t_2+T} G\left(t_2,s\right)f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right|\\
&+\left|\int_{t_1}^{t_1+T}\left[G\left(t_2,s\right)-G\left(t_1,s\right)\right]f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right|\\
&\leq\int_{t_2}^{t_1}\left|G\left(t_2,s\right)\right|\left|f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)\right|ds\\
&+\int_{t_1+T}^{t_2+T}\left|G\left(t_2,s\right)\right|\left|f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)\right|ds\\
&+\frac{1}{\left|\exp\left(\int_0^T a\left(u\right)du\right)-1\right|}\int_{t_1}^{t_1+T}\left|f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)\right|\\
&\times\left|\exp\left(\int_{t_2}^{s} a\left(u\right)du\right)-\exp\left(\int_{t_1}^{s} a\left(u\right)du\right)\right|ds,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_1}^{t_1+T}\left|\exp\left(\int_{t_2}^{s} a\left(u\right)du\right)-\exp\left(\int_{t_1}^{s} a\left(u\right)du\right)\right|ds\\
&=\int_{t_1}^{t_1+T}\exp\left(\int_{t_2}^{s} a\left(u\right)du\right)\left|1-\exp\left(\int_{t_1}^{t_2} a\left(u\right)du\right)\right|ds\\
&\leq T\left\|a\right\|\left|t_2-t_1\right|\exp\left(-\int_0^T a\left(u\right)du\right),
\end{aligned}$$

so,

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_2,s\right)ds\right.\\
&\left.-\int_{t_1}^{t_1+T} f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)G\left(t_1,s\right)ds\right|\\
&\leq 2\beta\left|t_2-t_1\right|\left(\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\right)+T\alpha\left\|a\right\|\left|t_2-t_1\right|\left(\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\right)\\
&\leq\left|t_2-t_1\right|\left(\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\right)\left(2\beta+T\alpha\left\|a\right\|\right). \qquad (3.19)
\end{aligned}$$

3.2. Existence of periodic solutions

Similarly, we get

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_2,s)\, ds\right.\\
&\left.-\int_{t_1}^{t_1+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_1,s)\, ds\right|\\
&\leq \left|\int_{t_2}^{t_1} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_2,s)\, ds\right.\\
&\left.+\int_{t_1+T}^{t_2+T} a(s)\, g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) G(t_2,s)\, ds\right|\\
&+\left|\int_{t_1}^{t_1+T} a(s)\left[G(t_2,s)-G(t_1,s)\right] g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds\right|\\
&\leq \int_{t_2}^{t_1} |a(s)|\, |G(t_2,s)| \left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\
&+\int_{t_1+T}^{t_2+T} |a(s)|\, |G(t_2,s)| \left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right| ds\\
&+\frac{1}{\left|\exp\left(\int_0^T a(u)\, du\right)-1\right|}\int_{t_1}^{t_1+T} |a(s)| \left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\\
&\times\left|\exp\left(\int_{t_2}^{s} a(u)\, du\right)-\exp\left(\int_{t_1}^{s} a(u)\, du\right)\right| ds\\
&\leq |t_2-t_1|\, \sigma\left(\rho_2+L\sum_{i=1}^{n} c_i\sum_{j=0}^{i-1} K^j\right)(2\beta+T\alpha\, \|a\|). \qquad (3.20)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right|\\
&=\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)\right.\\
&\left.+\,g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right|\\
&\leq\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)\right|\\
&+\left|g\left(t_1,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right|.
\end{aligned}$$

By (3.4)–(3.6) and Lemma 3.3, we get

$$\begin{aligned}
&\left|g\left(t_2,\varphi(t_2),\varphi^{[2]}(t_2),...,\varphi^{[n]}(t_2)\right)-g\left(t_1,\varphi(t_1),\varphi^{[2]}(t_1),...,\varphi^{[n]}(t_1)\right)\right|\\
&\leq K_g\, |t_2-t_1|+\sum_{i=1}^{n} c_i\left\|\varphi^{[i]}(t_2)-\varphi^{[i]}(t_1)\right\|\\
&\leq\left(K_g+\sum_{i=1}^{n} c_i\sum_{j=0}^{i-1} K^{j+1}\right)|t_2-t_1|. \qquad (3.21)
\end{aligned}$$

3.2. Existence of periodic solutions

Thus, it follows from (3.19)–(3.21) and (3.16) that

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \\
&\leq \left((2\beta + T\alpha \|a\|) \left(\rho_1 + \sigma\rho_2 + L\sum_{i=1}^{n} (k_i + \sigma c_i) \sum_{j=0}^{i-1} K^j \right) \right. \\
&\left. + \left(K_g + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} K^{j+1} \right) \right) |t_2 - t_1| .
\end{aligned}$$

Therefore,

$$\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \leq \frac{K}{J} |t_2 - t_1| \leq K |t_2 - t_1| .$$

Consequently, $\mathcal{A} : \mathbb{M} \to \mathbb{M}$. ■

Lemma 3.6 *Suppose that conditions (3.2)–(3.6) and (3.14)–(3.16) hold. Then the operator* $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ *given by (3.12), is continuous and compact.*

Proof. Since $\mathbb{M}$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, T]$ we can apply the Ascoli-Arzela theorem to confirm that $\mathbb{M}$ is a compact subset from this space. Also, and since any continuous operator maps compact sets into compact sets, then to prove that $\mathcal{A}$ is a compact operator it's suffices to prove that it is continuous. For $\varphi, \psi \in \mathbb{M}$, we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)\right| \\
&\leq \int_t^{t+T} |G(t,s)| \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right. \\
&\left. - f\left(s, \psi(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right) \right| ds \\
&+ \int_t^{t+T} |a(s)| |G(t,s)| \left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right. \\
&\left. - g\left(s, \psi(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right) \right| ds \\
&+ \left| g\left(t, \varphi(t), \varphi^{[2]}(t), ..., \varphi^{[n]}(t)\right) - g\left(t, \psi(t), \psi^{[2]}(t), ..., \psi^{[n]}(t)\right) \right| .
\end{aligned}$$

In view of conditions (3.5) and (3.6) and notations (3.14), we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)\right| \\
&\leq \beta T \sum_{i=1}^{n} k_i \left\| \varphi^{[i]} - \psi^{[i]} \right\| + (\beta\sigma T + 1) \sum_{i=1}^{n} c_i \left\| \varphi^{[i]} - \psi^{[i]} \right\| .
\end{aligned}$$

From Lemma 3.3, it follows that

$$\begin{aligned}
&\left|\left(\mathcal{A}\varphi\right)(t)-\left(\mathcal{A}\psi\right)(t)\right| \\
&\leq \beta T\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\left\|\varphi-\psi\right\| \\
&+\left(\beta\sigma T+1\right)\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j\left\|\varphi-\psi\right\| \\
&=\sum_{i=1}^{n}\left(\beta Tk_i+\left(\beta\sigma T+1\right)c_i\right)\sum_{j=0}^{i-1}K^j\left\|\varphi-\psi\right\|.
\end{aligned}$$

which proves that the operator $\mathcal{A}$ is continuous. Therefore, $\mathcal{A}$ is compact and continuous. ■

The next result proves the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions. Assume that

$$\beta\sigma T\leq 1, \tag{3.22}$$

$$\max\left(\left|H\left(-L\right)\right|,\left|H\left(L\right)\right|\right)\leq\frac{(J-1)}{J}L, \tag{3.23}$$

and

$$\left(2\beta+T\alpha\left\|a\right\|\right)\sigma L\leq K. \tag{3.24}$$

Lemma 3.7 *Let $\mathcal{B}$ be defined by (3.13), suppose (3.2), (3.22), (3.23), (3.24) and all conditions of Theorem 1.5 hold. Then $\mathcal{B}:\mathbb{M}\to\mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (3.13). For having $\mathcal{B}\varphi\in\mathbb{M}$ we will show that $\left\|\mathcal{B}\varphi\right\|\leq L$ and $\left|\left(\mathcal{B}\varphi\right)(t_2)-\left(\mathcal{B}\varphi\right)(t_1)\right|\leq K\left|t_2-t_1\right|$ for all $t_1,t_2\in[0,T]$. First, it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. That is, if $\varphi\in P_T$ then $\mathcal{B}\varphi\in P_T$. Let $\varphi\in\mathbb{M}$, by (3.23), we obtain

$$\begin{aligned}
\left|\left(\mathcal{B}\varphi\right)(t)\right| &\leq \int_t^{t+T}\left|G\left(t,s\right)\right|\left|a\left(s\right)\right|\left|H\left(\varphi\left(s\right)\right)\right|ds \\
&\leq \beta\sigma T\max\left\{\left|H\left(-L\right)\right|,\left|H\left(L\right)\right|\right\} \\
&\leq \frac{\left(J-1\right)L}{J}\leq L.
\end{aligned}$$

Then, for any $\varphi\in\mathbb{M}$, we have

$$\left\|\mathcal{B}\varphi\right\|\leq L.$$

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, by (3.22)–(3.24), we get

$$\begin{aligned}
&\left|(\mathcal{B}\varphi)(t_1) - (\mathcal{B}\varphi)(t_2)\right| \\
&\leq \left|\int_{t_2}^{t_2+T} G(t_2, s)\, a(s)\, H(\varphi(s))\, ds - \int_{t_1}^{t_1+T} G(t_1, s)\, a(s)\, H(\varphi(s))\, ds\right| \\
&\leq \left|\int_{t_2}^{t_1} G(t_2, s)\, a(s)\, H(\varphi(s))\, ds + \int_{t_1+T}^{t_2+T} G(t_2, s)\, a(s)\, H(\varphi(s))\, ds\right| \\
&+ \left|\int_{t_1}^{t_1+T} [G(t_2, s) - G(t_1, s)]\, a(s)\, H(\varphi(s))\, ds\right| \\
&\leq \int_{t_2}^{t_1} |G(t_2, s)|\, |a(s)|\, |H(\varphi(s))|\, ds + \int_{t_1+T}^{t_2+T} |G(t_2, s)|\, |a(s)|\, |H(\varphi(s))|\, ds \\
&+ \frac{1}{\left|\exp\left(\int_0^T a(u)\, du\right) - 1\right|} \int_{t_1}^{t_1+T} |a(s)|\, |H(\varphi(s))| \\
&\times \left|\exp\left(\int_{t_2}^{s} a(u)\, du\right) - \exp\left(\int_{t_1}^{s} a(u)\, du\right)\right| ds \\
&\leq 2\beta\sigma\left(\frac{(J-1)L}{J}\right)|t_2 - t_1| + \frac{1}{\left|\exp\left(\int_0^T a(u)\, du\right) - 1\right|} \int_{t_1}^{t_1+T} |a(s)|\, |H(\varphi(s))| \\
&\times \exp\left(\int_{t_2}^{s} a(u)\, du\right)\left|1 - \exp\left(\int_{t_1}^{t_2} a(u)\, du\right)\right| ds \\
&\leq 2\beta\sigma\frac{(J-1)L}{J}|t_2 - t_1| + T\alpha\|a\|\,\sigma\frac{(J-1)L}{J}|t_2 - t_1| \\
&= (2\beta + T\alpha\|a\|)\,\sigma\frac{(J-1)L}{J}|t_2 - t_1|.
\end{aligned}$$

Then

$$\left|(\mathcal{B}\varphi)(t_1) - (\mathcal{B}\varphi)(t_2)\right| \leq \frac{(J-1)K}{J}|t_2 - t_1| \leq K|t_2 - t_1|.$$

Therefore, $\mathcal{B} : \mathbb{M} \to \mathbb{M}$.

It remains to prove that $\mathcal{B}$ is a large contraction. By Theorem 1.5, H is a large contraction on $\mathbb{M}$, then for any $\varphi, \psi \in \mathbb{M}$, with $\varphi \neq \psi$ we get

$$\begin{aligned}
&\left|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)\right| \\
&\leq \left|\int_{t}^{t+T} G(t, s)\, a(s)\, [H(\varphi(s)) - H(\psi(s))]\, ds\right| \\
&\leq \beta\sigma T\|\varphi - \psi\| \leq \|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \|\varphi - \psi\|$. Now, let $\varepsilon \in (0, 1)$ be given and let $\varphi, \psi \in \mathbb{M}$, with $\|\varphi - \psi\| \geq \varepsilon$ from the proof of Theorem 1.5, we have found a $\delta \in (0, 1)$, such that

$$\left|(H\varphi)(t) - (H\psi)(t)\right| \leq \delta\|\varphi - \psi\|.$$

Thus,

$$\begin{aligned}
&|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \\
&\leq \left| \int_t^{t+T} G(t,s)\, a(s) \left[H(\varphi(s)) - H(\psi(s)) \right] ds \right| \\
&\leq \beta\sigma T\delta \|\varphi - \psi\| \leq \delta \|\varphi - \psi\| .
\end{aligned}$$

The proof is complete. ■

Theorem 3.1 *Suppose the hypothesis of Lemmas 3.5–3.7 hold. Let* $\mathbb{M}$ *defined by (3.10), then (3.1) has a* T*-periodic solution in* $\mathbb{M}$.

Proof. By Lemmas 3.5 and 3.6 $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 3.7, the mapping $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction. Next, we prove that if $\varphi, \psi \in \mathbb{M}$, we have $\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq L$ and $|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \leq K|t_2 - t_1|$ for all $t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (3.15) and (3.23), we have

$$\begin{aligned}
&\|\mathcal{A}\varphi + \mathcal{B}\phi\| \\
&\leq \beta T(\rho_1 + \sigma\rho_2) + \rho_2 + L \sum_{i=1}^{n} [c_i + \beta T(k_i + \sigma c_i)] \sum_{j=0}^{i-1} K^j + \frac{(J-1)L}{J} \\
&\leq \frac{L}{J} + \frac{(J-1)L}{J} = L.
\end{aligned}$$

Now, let $\varphi, \psi \in \mathbb{M}$ and $t_1, t_2 \in [0, T]$. By (3.16) and (3.24), we get

$$\begin{aligned}
&|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \\
&\leq |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\psi)(t_2) - (\mathcal{B}\psi)(t_1)| \\
&\leq \frac{K}{J}|t_2 - t_1| + \frac{(J-1)K}{J}|t_2 - t_1| \\
&\leq K|t_2 - t_1| .
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 3.1, this fixed point is a solution of (3.1). Hence (3.1) has a T-periodic solution. ■

Chapter 4

Periodic solutions for second order totally nonlinear iterative differential equations

Keywords. Krasnoselskii-Burton's fixed point, large contraction, iterative differential equations, periodic solutions, Green's function.

This chapter present a very recent published work [74],

A. Guerfi, A. Ardjouni, Periodic solutions for second order totally nonlinear iterative differential equations, The Journal of Analysis, https://doi.org/10.1007/s41478-021-00347-0.

In this chapter, we consider the following second order totally nonlinear iterative differential equation

$$\begin{aligned}&\frac{d^2}{dt^2}x\left(t\right)+p\left(t\right)\frac{d}{dt}x\left(t\right)+q\left(t\right)h\left(x\left(t\right)\right)\\&=\frac{d}{dt}g\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right)+f\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right),\end{aligned}\tag{4.1}$$

where $x^{[1]}\left(t\right)=x\left(t\right)$, $x^{[2]}\left(t\right)=x\left(x\left(t\right)\right)$, ..., $x^{[n]}\left(t\right)=x^{[n-1]}\left(x\left(t\right)\right)$, p and q are positive continuous real-valued functions. The functions $h:\mathbb{R}\to\mathbb{R}$ and $f,g:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are continuous with respect to their arguments. Our purpose here is to use Krasnoselskii-Burton's fixed point theorem to prove the existence of periodic solutions for (4.1). Clearly, the present problem is totally nonlinear so that the variation of parameters cannot be applied directly. Then, we resort to the idea of adding and subtracting of terms. As noted by Burton [48], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we use the variation of parameter formula and the integration by parts to transform (4.1) into an equivalent integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use Krasnoselskii-Burton's fixed point theorem, to prove the existence of periodic solutions.

4.1 Preliminaries and inversion of the equation

For $T>0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T,\|.\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

For $L,K>0$, define the set

$$P_T(L,K)=\{x\in P_T,\ \|x\|\le L,\ |x(t_2)-x(t_1)|\le K|t_2-t_1|,\ \forall t_1,t_2\in\mathbb{R}\},$$

which is a closed convex and bounded subset of P_T.

We assume that p and q are two continuous real-valued functions such that

$$p(t+T)=p(t),\ q(t+T)=q(t), \tag{4.2}$$

and

$$\int_0^T p(s)\,ds>0,\ \int_0^T q(s)\,ds>0. \tag{4.3}$$

The functions $f(t,x_1,x_2,...,x_n)$ and $g(t,x_1,x_2,...,x_n)$ are supposed to be periodic in t with period T and globally Lipschitz in x_1, x_2, ..., x_n, i.e,

$$\begin{aligned} f(t+T,x_1,...,x_n) &= f(t,x_1,...,x_n),\\ g(t+T,x_1,...,x_n) &= g(t,x_1,...,x_n), \end{aligned} \tag{4.4}$$

and there exist n positive constants k_1, k_2, ..., k_n and n positive constants c_1, c_2, ..., c_n such that

$$|f(t,x_1,...,x_n)-f(t,y_1,...,y_n)|\le\sum_{i=1}^n k_i|x_i-y_i|, \tag{4.5}$$

and

$$|g(t,x_1,...,x_n)-g(t,y_1,...,y_n)|\le\sum_{i=1}^n c_i|x_i-y_i|. \tag{4.6}$$

Lemma 4.1 ([92]) *Suppose that (4.2) and (4.3) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1T}\ge 1, \tag{4.7}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right)-1}q(s)\,ds\right|,$$

and

$$Q_1=\left(1+\exp\left(\int_0^T p(u)\,du\right)\right)^2R_1^2.$$

Then there are continuous and T-periodic functions a and b such that $b(t) > 0$, $\int_0^T a(u)du > 0$, and

$$a(t) + b(t) = p(t), \ \frac{d}{dt}b(t) + a(t)b(t) = q(t) \text{ for all } t \in \mathbb{R}.$$

Lemma 4.2 ([114]) *Suppose the conditions of Lemma 4.1 hold and $\phi \in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)\,ds,$$

where

$$G(t,s) = \frac{\int_t^s \exp\left[\int_t^u b(v)\,dv + \int_u^s a(v)\,dv\right] du}{\left[\exp\left(\int_0^T a(u)\,du\right) - 1\right]\left[\exp\left(\int_0^T b(u)\,du\right) - 1\right]} + \frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\,dv + \int_u^{s+T} a(v)\,dv\right] du}{\left[\exp\left(\int_0^T a(u)\,du\right) - 1\right]\left[\exp\left(\int_0^T b(u)\,du\right) - 1\right]}. \tag{4.8}$$

Corollary 4.1 ([114]) *Green's function G satisfies the following properties*

$$\begin{aligned} G(t,t+T) &= G(t,t), \ G(t+T,s+T) = G(t,t), \\ \frac{\partial}{\partial s}G(t,s) &= a(s)G(t,s) - \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right) - 1}, \\ \frac{\partial}{\partial t}G(t,s) &= -b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right) - 1}. \end{aligned} \tag{4.9}$$

Lemma 4.3 *Suppose (4.2)–(4.4) and (4.7) hold. If $x \in P_T(L,K)$, then x is a solution of (4.1) if and only if*

$$\begin{aligned} x(t) = &\int_t^{t+T} G(t,s)q(s)H(x(s))\,ds \\ &+ \int_t^{t+T} \left\{[E(t,s) - a(s)G(t,s)]\,g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right. \\ &\left.+G(t,s)f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right)\right\} ds, \end{aligned} \tag{4.10}$$

where

$$E(t,s) = \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right) - 1}, \tag{4.11}$$

and

$$H(x) = x - h(x). \tag{4.12}$$

Proof. Let $x \in P_T(L,K)$ be a solution of (4.1). Rewrite (4.1) as

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)\\&=q(t)H(x(t))+\frac{d}{dt}g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right).\end{aligned}$$

From Lemma 4.2, we get

$$\begin{aligned}x(t)=&\int_t^{t+T}G(t,s)q(s)H(x(s))ds\\&+\int_t^{t+T}G(t,s)\left\{\frac{d}{ds}g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right.\\&\left.+f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right\}ds.\end{aligned}$$

Performing an integration by parts, we obtain

$$\begin{aligned}&\int_t^{t+T}G(t,s)\frac{d}{ds}g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds\\&=\left[G(t,s)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]_t^{t+T}\\&-\int_t^{t+T}\left(\frac{d}{ds}G(t,s)\right)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds.\end{aligned}$$

Since

$$\left[G(t,s)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]_t^{t+T}=0,$$

from (4.9), we get

$$\begin{aligned}&\int_t^{t+T}G(t,s)\frac{d}{ds}g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds\\&=\int_t^{t+T}\left[E(t,s)-a(s)G(t,s)\right]g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds.\end{aligned}$$

Consequently,

$$\begin{aligned}x(t)=&\int_t^{t+T}G(t,s)q(s)H(x(s))ds\\&+\int_t^{t+T}\left\{\left[E(t,s)-a(s)G(t,s)\right]g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right.\\&\left.+G(t,s)f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right\}ds.\end{aligned}$$

The proof is completed. ■

Lemma 4.4 ([114]) *Let* $A=\int_0^T p(u)\,du$ *and* $B=T^2\exp\left(\frac{1}{T}\int_0^T\ln(q(u))\,du\right)$. *If*

$$A^2\geq 4B, \tag{4.13}$$

then

$$\min\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \geq \frac{1}{2}\left(A-\sqrt{A^2-4B}\right) := l$$

and

$$\max\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \leq \frac{1}{2}\left(A+\sqrt{A^2-4B}\right) := m.$$

Corollary 4.2 ([114]) *Functions G and E satisfy*

$$\frac{T}{(e^m-1)^2} \leq G(t,s) \leq \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l-1)^2}, \quad |E(t,s)| \leq \frac{e^m}{e^l-1}.$$

Lemma 4.5 ([121]) *For any $\varphi,\psi \in P_T(L,K)$, we have*

$$\left\|\varphi^{[m]}-\psi^{[m]}\right\| \leq \sum_{j=0}^{m-1} K^j \left\|\varphi-\psi\right\|, \ m=1,2,....$$

Lemma 4.6 ([120]) *It holds*

$$P_T(L,K) = \left\{x \in P_T, \ \|x\| \leq L, \ \ |x(t_2)-x(t_1)| \leq K|t_2-t_1|, \ \forall t_1,t_2 \in [0,T]\right\}.$$

4.2 Existence of periodic solutions

To apply the Theorem 1.7 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a completely continuous and the other is a large contraction. So, we let $(\mathbb{B},\|.\|) = (P_T,\|.\|)$ and

$$\mathbb{M} = P_T(L,K) = \left\{\varphi \in P_T, \ \|\varphi\| \leq L, \ \ |\varphi(t_2)-\varphi(t_1)| \leq K|t_2-t_1|, \ \forall t_1,t_2 \in [0,T]\right\}, \tag{4.14}$$

with $L,K>0$. Define a mapping $\mathcal{S}:\mathbb{M}\to P_T$ by

$$\begin{aligned}(\mathcal{S}\varphi)(t) &= \int_t^{t+T} G(t,s)\,q(s)\,H(\varphi(s))\,ds \\ &+ \int_t^{t+T} \left\{[E(t,s)-a(s)G(t,s)]\,g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right. \\ &\left. +G(t,s)\,f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right\}ds. \end{aligned} \tag{4.15}$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A},\mathcal{B}:\mathbb{M}\to P_T$ are given by

$$\begin{aligned}(\mathcal{A}\varphi)(t) &= \int_t^{t+T} \left\{[E(t,s)-a(s)G(t,s)]\,g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right. \\ &\left. +G(t,s)\,f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right\}ds, \end{aligned} \tag{4.16}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t,s)\, q(s)\, H(\varphi(s))\, ds. \tag{4.17}$$

To simplify notations, we introduce the following constants

$$\begin{aligned}
\alpha &= \frac{T\exp\left(\int_0^T p(u)\,du\right)}{\left(e^l-1\right)^2},\ \beta = \frac{e^m}{e^l-1},\ \gamma = \exp\left(\int_0^T b(v)\,dv\right),\\
\theta &= \frac{1}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]},\\
\lambda_1 &= \max_{t\in[0,T]}|a(t)|,\ \lambda_2 = \max_{t\in[0,T]}|b(t)|,\ \sigma = \max_{t\in[0,T]}|q(t)|\\
\rho_1 &= \max_{t\in[0,T]}|f(t,0,0,\ldots,0)|,\ \rho_2 = \max_{t\in[0,T]}|g(t,0,0,\ldots,0)|,\\
\zeta_1 &= \rho_1 + L\sum_{i=1}^{n} k_i \sum_{j=0}^{j=i-1} K^j,\ \zeta_2 = \rho_2 + L\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} K^j.
\end{aligned} \tag{4.18}$$

Lemma 4.7 ([43]) *For any $t_1, t_2 \in [0,T]$,*

$$\int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\, ds \le Te^{2m}\theta\left[T\lambda_2\gamma\left(2e^{2m}+1\right)+e^m+1\right]|t_2-t_1|.$$

Also, we need the following assumptions

$$JT\left[(\beta+\alpha\lambda_1)\zeta_1 + \alpha\zeta_2\right] \le L, \tag{4.19}$$

and

$$\begin{aligned}
&J\left(\left(2\alpha + Te^{2m}\theta\left[T\lambda_2\gamma\left(2e^{2m}+1\right)+e^m+1\right]\right)(\lambda_1\zeta_2+\zeta_1)\right.\\
&\left.+(2\beta+T\lambda_2\beta)\zeta_2\right) \le K,
\end{aligned} \tag{4.20}$$

where J is a positive constant with $J \ge 3$.

Lemma 4.8 *For $\mathcal{A}$ defined in (4.16), suppose that (4.2)–(4.7), (4.19) and (4.20) hold. Then $\mathcal{A} : \mathbb{M} \to \mathbb{M}$.*

Proof. Let $\varphi \in \mathbb{M}$. For having $\mathcal{A}\varphi \in \mathbb{M}$ we show that $\mathcal{A}\varphi \in P_T$, $\|\mathcal{A}\varphi\| \le L$ and $|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \le K|t_2-t_1|$, $\forall t_1, t_2 \in [0,T]$. First it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. That is, if $\varphi \in P_T$ then $\mathcal{A}\varphi \in P_T$. By Corollary 4.2 and notations (4.18), we get

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| \le (\beta+\alpha\lambda_1)&\int_t^{t+T}\left|g\left(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right| ds\\
+\alpha&\int_t^{t+T}\left|f\left(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right| ds.
\end{aligned}$$

From conditions (4.5), (4.6) and Lemma 4.5, we obtain

$$\begin{aligned}
&\left|f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)\right| \\
&\leq \left|f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)-f\left(s,0,0,...,0\right)\right|+\left|f\left(s,0,0,...,0\right)\right| \\
&\leq \rho_1+\sum_{i=1}^{n}k_i\sum_{j=0}^{j=i-1}K^j\left\|\varphi\right\|\leq \rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{j=i-1}K^j=\zeta_1,
\end{aligned}$$

and

$$\begin{aligned}
&\left|g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)\right| \\
&\leq \left|g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)-g\left(s,0,0,...,0\right)\right|+\left|g\left(s,0,0,...,0\right)\right| \\
&\leq \rho_2+\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}K^j\left\|\varphi\right\|\leq \rho_2+L\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}K^j=\zeta_2.
\end{aligned}$$

So

$$\left|\left(\mathcal{A}\varphi\right)(t)\right|\leq T\left(\beta+\alpha\lambda_1\right)\zeta_1+T\alpha\zeta_2.$$

Therefore, from (4.19), we have

$$\left\|\mathcal{A}\varphi\right\|\leq \frac{L}{J}\leq L.$$

Let $t_1,t_2\in[0,T]$ with $t_1<t_2$, we get

$$\begin{aligned}
&\left|\left(\mathcal{A}\varphi\right)(t_2)-\left(\mathcal{A}\varphi\right)(t_1)\right| \\
&\leq \left|\int_{t_2}^{t_2+T}E\left(t_2,s\right)g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right. \\
&\left.-\int_{t_1}^{t_1+T}E\left(t_1,s\right)g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right| \\
&+\left|\int_{t_2}^{t_2+T}a\left(s\right)G\left(t_2,s\right)g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right. \\
&\left.-\int_{t_1}^{t_1+T}a\left(s\right)G\left(t_1,s\right)g\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right| \\
&+\left|\int_{t_2}^{t_2+T}G\left(t_2,s\right)f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right. \\
&\left.-\int_{t_1}^{t_1+T}G\left(t_1,s\right)f\left(s,\varphi\left(s\right),\varphi^{[2]}\left(s\right),...,\varphi^{[n]}\left(s\right)\right)ds\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} E\left(t_2, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right.\\
&\left.-\int_{t_1}^{t_1+T} E\left(t_1, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right|\\
&\leq \int_{t_2}^{t_1} \left|E\left(t_2, s\right)\right| \left|g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right)\right| ds\\
&+\int_{t_1+T}^{t_2+T} \left|E\left(t_2, s\right)\right| \left|g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right)\right| ds\\
&+\int_{t_1}^{t_1+T} \left|E\left(t_2, s\right)-E\left(t_1, s\right)\right| \left|g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right)\right| ds\\
&\leq \left(2\beta+T\lambda_2\beta\right)\zeta_2\left|t_2-t_1\right|. \qquad (4.21)
\end{aligned}$$

Also

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} a\left(s\right) G\left(t_2, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right.\\
&\left.-\int_{t_1}^{t_1+T} a\left(s\right) G\left(t_1, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right|\\
&\leq \left|\int_{t_2}^{t_1} a\left(s\right) G\left(t_2, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right|\\
&+\left|\int_{t_1+T}^{t_2+T} a\left(s\right) G\left(t_2, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right|\\
&+\left|\int_{t_1}^{t_1+T} a\left(s\right) \left[G\left(t_2, s\right)-G\left(t_1, s\right)\right] g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right|.
\end{aligned}$$

From Lemma 4.7, notations (4.18) and conditions (4.5), (4.6), we have

$$\begin{aligned}
&\left|\int_{t_2}^{t_2+T} a\left(s\right) G\left(t_2, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right.\\
&\left.-\int_{t_1}^{t_1+T} a\left(s\right) G\left(t_1, s\right) g\left(s, \varphi\left(s\right), \varphi^{[2]}\left(s\right), ..., \varphi^{[n]}\left(s\right)\right) ds\right|\\
&\leq \lambda_1\zeta_2\left(2\alpha+Te^{2m}\theta\left[T\lambda_2\gamma\left(2e^{2m}+1\right)+e^m+1\right]\right)\left|t_2-t_1\right|. \qquad (4.22)
\end{aligned}$$

We get also

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2,s) f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right. \\
&\left. - \int_{t_1}^{t_1+T} G(t_1,s) f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right| \\
&\leq \left| \int_{t_2}^{t_1} G(t_2,s) f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right| \\
&+ \left| \int_{t_1+T}^{t_2+T} G(t_2,s) f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right| \\
&+ \left| \int_{t_1}^{t_1+T} \left[G(t_2,s) - G(t_1,s)\right] f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right|.
\end{aligned}$$

From Lemma 4.7, notations (4.18) and conditions (4.5), (4.6), we have

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2,s) f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right. \\
&\left. - \int_{t_1}^{t_1+T} G(t_1,s) f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right) ds \right| \\
&\leq \zeta_1 \left(2\alpha + Te^{2m}\theta\left[T\lambda_2\gamma\left(2e^{2m}+1\right) + e^m + 1\right]\right) |t_2 - t_1|
\end{aligned} \tag{4.23}$$

Thus, it follows from (4.21), (4.22) and (4.23) that

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\leq \left(\left(2\alpha + Te^{2m}\theta\left[T\lambda_2\gamma\left(2e^{2m}+1\right) + e^m + 1\right]\right)(\lambda_1\zeta_2 + \zeta_1)\right. \\
&\left. + (2\beta + T\lambda_2\beta)\zeta_2\right) |t_2 - t_1|
\end{aligned}$$

From (4.20), we obtain

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq \frac{K}{J} |t_2 - t_1| \leq K |t_2 - t_1|,$$

which implies that $\mathcal{A} : \mathbb{M} \to \mathbb{M}$. ■

Lemma 4.9 *Suppose that conditions (4.2)–(4.7), (4.19) and (4.20) hold. Then the operator $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ given by (4.16), is continuous and compact.*

Proof. Since $\mathbb{M}$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, T]$ we can apply the Ascoli-Arzela theorem to confirm that $\mathbb{M}$ is a compact subset from this space. Also, and since any continuous operator maps compact sets into compact sets, then to prove that $\mathcal{A}$ is a compact operator it's suffices to show that it is continuous. For $\varphi, \psi \in \mathbb{M}$, we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)\right| \\
&\leq \int_t^{t+T} |E(t,s)| \left|g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - g\left(s, \psi(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right)\right| ds \\
&+ \int_t^{t+T} |a(s)| |G(t,s)| \left|g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right)\right. \\
&\left.-g\left(s, \psi(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right)\right| ds \\
&+ \int_t^{t+T} |G(t,s)| \left|f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \psi(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right)\right| ds.
\end{aligned}$$

By (4.5) and (4.6), Corollary 4.2, and notations (4.18), we get

$$\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)\right| \leq (\beta + \alpha\lambda_1) T \sum_{i=1}^{n} c_i \left\|\varphi^{[i]} - \psi^{[i]}\right\| + \alpha T \sum_{i=1}^{n} k_i \left\|\varphi^{[i]} - \psi^{[i]}\right\|.$$

From Lemma 4.5, it follows that

$$\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)\right| \leq T \sum_{i=1}^{n} \left((\beta + \alpha\lambda_1) c_i + \alpha k_i\right) \sum_{j=0}^{j=i-1} K^j \left\|\varphi - \psi\right\|.$$

which proves that the operator A is continuous. Therefore, $\mathcal{A}$ is compact and continuous. ■

The next result proves the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions. Assume that

$$\alpha\sigma T \leq 1, \tag{4.24}$$

$$\max\left(|H(-L)|, |H(L)|\right) \leq \frac{(J-1)}{J} L, \tag{4.25}$$

and

$$\sigma L \left[2\alpha + T e^{2m}\theta \left[T\lambda_2\gamma\left(2e^{2m}+1\right) + e^m + 1\right]\right] \leq K. \tag{4.26}$$

Lemma 4.10 *Let $\mathcal{B}$ be defined by (4.17), suppose (4.2), (4.3), (4.7), (4.24)–(4.26) and all conditions of Theorem 1.5 hold. Then $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (4.17). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to prove that $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. For having $\mathcal{B}\varphi \in \mathbb{M}$ we will prove that $\|\mathcal{B}\varphi\| \leq L$ and $\left|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)\right| \leq K |t_2 - t_1|, \forall t_1, t_2 \in [0, T]$. Let $\varphi \in \mathbb{M}$, by (4.24) and (4.25) we get

$$\begin{aligned}
\left|(\mathcal{B}\varphi)(t)\right| &\leq \int_t^{t+T} |G(t,s)| |q(s)| |H(\varphi(s))| ds \\
&\leq \alpha\sigma T \max\left\{|H(-L)|, |H(L)|\right\} \leq \frac{(J-1)L}{J} \leq L.
\end{aligned}$$

Then, for any $\varphi \in \mathbb{M}$, we have

$$\|\mathcal{B}\varphi\| \le L.$$

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, by (4.24), (4.25) and Lemma 4.7 we obtain

$$\begin{aligned}
&|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)| \\
&\le \left| \int_{t_2}^{t_2+T} G(t_2, s)\, q(s)\, H(\varphi(s))\, ds - \int_{t_1}^{t_1+T} G(t_1, s)\, q(s)\, H(\varphi(s))\, ds \right| \\
&\le \int_{t_2}^{t_1} |G(t_2, s)|\, |q(s)|\, |H(\varphi(s))|\, ds + \int_{t_1+T}^{t_2+T} |G(t_2, s)|\, |q(s)|\, |H(\varphi(s))|\, ds \\
&+ \int_{t_1}^{t_1+T} |G(t_2, s) - G(t_1, s)|\, |q(s)|\, |H(\varphi(s))|\, ds \\
&\le \left[2\alpha\sigma \frac{(J-1)L}{J} + \sigma \frac{(J-1)L}{J} T e^{2m} \theta \left[T \lambda_2 \gamma \left(2e^{2m} + 1 \right) + e^m + 1 \right] \right] |t_2 - t_1|.
\end{aligned}$$

From (4.26), we get

$$|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)| \le \frac{(J-1)K}{J} |t_2 - t_1|.$$

Consequently, $\mathcal{B} : \mathbb{M} \to \mathbb{M}$.

It remains to prove that $\mathcal{B}$ is large contraction. By Theorem 1.5 H is large contraction on $\mathbb{M}$, then for any $\varphi, \psi \in \mathbb{M}$, with $\varphi \ne \psi$ we get

$$\begin{aligned}
|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| &\le \left| \int_t^{t+T} G(t, s)\, q(s) \left[H(\varphi(s)) - H(\psi(s)) \right] ds \right| \\
&\le \alpha\sigma T \|\varphi - \psi\| \le \|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \le \|\varphi - \psi\|$. Now, let $\varepsilon \in (0, 1)$ be given and let $\varphi, \psi \in \mathbb{M}$, with $\|\varphi - \psi\| \ge \varepsilon$, from the proof of Theorem 1.5, we have found a $\delta \in (0, 1)$, such that

$$|(H\varphi)(t) - (H\psi)(t)| \le \delta \|\varphi - \psi\|.$$

Thus,

$$\begin{aligned}
|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| &\le \left| \int_t^{t+T} G(t, s)\, q(s) \left[H(\varphi(s)) - H(\psi(s)) \right] ds \right| \\
&\le \alpha\sigma T \delta \|\varphi - \psi\| \le \delta \|\varphi - \psi\|.
\end{aligned}$$

The proof is complete. ■

Theorem 4.1 *Suppose the hypotheses of Lemmas 4.8–4.10 hold. Let $\mathbb{M}$ defined by (4.14), Then (4.1) has a T-periodic solution in $\mathbb{M}$.*

Proof. By Lemmas 4.8, 4.9 $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 4.10, the mapping $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction. Next, we prove that if $\varphi, \psi \in \mathbb{M}$, we have $\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq L$ and $|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (4.19), (4.24), (4.25) and notations (4.18), we get

$$\begin{aligned}\|\mathcal{A}\varphi + \mathcal{B}\phi\| &\leq T[(\beta + \alpha\lambda_1)\zeta_1 + \alpha\zeta_2] + \frac{(J-1)L}{J} \\ &\leq \frac{L}{J} + \frac{(J-1)L}{J} = L.\end{aligned}$$

Now, let $\varphi, \psi \in \mathbb{M}$ and $t_1, t_2 \in [0, T]$. By (4.20), (4.26) and Lemma 4.7, we obtain

$$\begin{aligned}&|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \\ &\leq |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\psi)(t_2) - (\mathcal{B}\psi)(t_1)| \\ &\leq \frac{K}{J}|t_2 - t_1| + \frac{(J-1)K}{J}|t_2 - t_1| \\ &\leq K|t_2 - t_1|.\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii–Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 4.3 this fixed point is a solution of (4.1). Hence (4.1) has a T-periodic solution. ■

Chapter 5

Study of the existence of periodic and nonnegative periodic solutions for third order nonlinear differential equations

Keywords. Krasnoselskii-Burton's fixed point, large contraction, periodic solutions, nonnegative periodic solutions, Green's function.

This chapter has been extracted from the research paper [75],

A. Guerfi, A. Ardjouni, Study of the existence of periodic and nonnegative periodic solutions for third order nonlinear differential equations, MESA 12(3) (2021), 883–893.

In this chapter, we concentrate on the existence of periodic and nonnegative periodic solutions for the third-order nonlinear delay differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)h(x(t)) = f(t, x(t), x(t-\tau(t))), \tag{5.1}$$

where p, q, r are continuous functions. The functions $h : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions in their respective arguments, $\tau : \mathbb{R} \to \mathbb{R}^+$ is a continuous function. To show the existence of periodic and nonnegative periodic solutions, we transform (5.1) into an equivalent integral equation and then use Krasnoselskii-Burton's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a large contraction and the other is compact.

5.1 Preliminaries and inversion of the equation

In this section, we give the assumptions as follows that will be used in the main results.

(H1) There exist two differentiable positive T-periodic functions a_1, a_2 and a positive real constant ρ such that

$$\begin{cases} a_1(t) + \rho = p(t), \\ a_1'(t) + a_2(t) + \rho a_1(t) = q(t), \\ a_2'(t) + \rho a_2(t) = r(t). \end{cases}$$

(H2) $p, q, r \in C(\mathbb{R}, \mathbb{R}^+)$ are T-periodic functions with $\tau(t) \geq \tau^* > 0$, and

$$\int_0^T p(s)\, ds > \rho T \text{ and } \int_0^T q(s)\, ds > 0.$$

(H3) The function $f(t,x,y)$ is continuous T-periodic in t and globally Lipshitz continuous in x and y. That is

$$f(t+T,x,y) = f(t,x,y),$$

and there are positive constants k_1 and k_2 such that

$$|f(t,x,y) - f(t,z,w)| \leq k_1 |x - z| + k_2 |y - w|.$$

For $T > 0$, let P_T be the set of all continuous functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Now, we consider the equation

$$x'''(t) + p(t)\, x''(t) + q(t)\, x'(t) + r(t)\, x(t) = e(t), \tag{5.2}$$

where e is a continuous T-periodic function. Obviously, by the condition (H1), the above equation can be transformed into the following system

$$\begin{cases} y'(t) + \rho y(t) = e(t), \\ x''(t) + a_1(t)\, x'(t) + a_2(t)\, x(t) = y(t). \end{cases}$$

Lemma 5.1 ([25]) *If $y, e \in P_T$, then y is a solution of the equation*

$$y'(t) + \rho y(t) = e(t),$$

if and only if

$$y(t) = \int_t^{t+T} G_1(t,s)\, e(s)\, ds, \tag{5.3}$$

where

$$G_1(t,s) = \frac{\exp(\rho(s-t))}{\exp(\rho T) - 1}. \tag{5.4}$$

Corollary 5.1 ([104]) *Green's function G_1 satisfies the following properties*

$$G_1(t+T,s+T)=G_1(t,s)\,,\ G_1(t,t+T)=G_1(t,t)\exp(\rho T)\,,$$
$$G_1(t+T,s)=G_1(t,s)\exp(-\rho T)\,,\ G_1(t,s+T)=G_1(t,s)\exp(\rho T)\,,$$
$$\frac{\partial}{\partial t}G_1(t,s)=-\rho G_1(t,s)\,,\ \frac{\partial}{\partial s}G_1(t,s)=\rho G_1(t,s)\,,$$

and

$$m_1\leq G_1(t,s)\leq M_1,$$

where

$$m_1=\frac{1}{\exp(\rho T)-1},\ M_1=\frac{\exp(\rho T)}{\exp(\rho T)-1}.$$

Lemma 5.2 ([92]) *Suppose that (H1), (H2) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T a_1(v)\,dv\right)-1\right]}{Q_1T}\geq 1, \tag{5.5}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_t^s a_1(v)\,dv\right)}{\exp\left(\int_0^T a_1(v)\,dv\right)-1}a_2(s)\,ds\right|,$$

and

$$Q_1=\left(1+\exp\left(\int_0^T a_1(v)\,dv\right)\right)^2R_1^2.$$

Then, there are continuous and T-periodic functions a and b such that

$$b(t)>0,\ \int_0^T a(v)dv>0,$$

and

$$a(t)+b(t)=a_1(t),\ \frac{d}{dt}b(t)+a(t)b(t)=a_2(t)\ \text{for all } t\in\mathbb{R}.$$

Lemma 5.3 ([114]) *Suppose the conditions of Lemma 5.2 hold and $y\in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t)+a_1(t)\frac{d}{dt}x(t)+a_2(t)\,x(t)=y(t)\,,$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t)=\int_t^{t+T}G_2(t,s)\,y(s)\,ds, \tag{5.6}$$

where

$$G_2(t,s)=\frac{\int_t^s\exp\left[\int_t^v b(u)\,du+\int_v^s a(u)\,du\right]dv}{\left[\exp\left(\int_0^T a(v)\,dv\right)-1\right]\left[\exp\left(\int_0^T b(v)\,dv\right)-1\right]}$$
$$+\frac{\int_s^{t+T}\exp\left[\int_t^v b(u)\,du+\int_v^{s+T}a(u)\,du\right]dv}{\left[\exp\left(\int_0^T a(v)\,dv\right)-1\right]\left[\exp\left(\int_0^T b(v)\,dv\right)-1\right]}. \tag{5.7}$$

Corollary 5.2 ([104]) *Green's function G_2 satisfies the following properties*

$$G_2(t+T,s+T) = G_2(t,s), \; G_2(t,t+T) = G_2(t,t),$$
$$G_2(t+T,s) = \exp\left(-\int_0^T b(v)\,dv\right)\left[G_2(t,s) + \int_t^{t+T} E(t,u)\,F(u,s)\,du\right],$$
$$\frac{\partial}{\partial t}G_2(t,s) = -b(t)\,G_2(t,s) + F(t,s),$$
$$\frac{\partial}{\partial s}G_2(t,s) = a(t)\,G_2(t,s) - E(t,s),$$

where

$$E(t,s) = \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}, \; F(t,s) = \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1}.$$

Lemma 5.4 ([114]) *Let $A = \int_0^T a_1(v)\,dv$ and $B = T^2\exp\left(\frac{1}{T}\int_0^T \ln(a_2(v))\,dv\right)$. If*

$$A^2 \geq 4B, \tag{5.8}$$

then

$$\min\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2-4B}\right) := l,$$

and

$$\max\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2-4B}\right) := L.$$

Corollary 5.3 ([104]) *Functions G_2, E and F satisfy*

$$m_2 \leq G_2(t,s) \leq M_2, \; E(t,s) \leq \frac{e^L}{e^L-1}, \; F(t,s) \leq e^L,$$

where

$$m_2 = \frac{T}{\left(e^L-1\right)^2} \text{ and } M_2 = \frac{T\exp\left(\int_0^T a_1(v)\,dv\right)}{\left(e^l-1\right)^2}.$$

Lemma 5.5 ([52]) *Suppose the conditions of Lemma 5.2 hold and $e \in P_T$. Then the equation*

$$x'''(t) + p(t)\,x''(t) + q(t)\,x'(t) + r(t)\,x(t) = e(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\,e(s)\,ds, \tag{5.9}$$

where

$$G(t,s) = \int_t^{t+T} G_2(t,\sigma)\,G_1(\sigma,s)\,d\sigma. \tag{5.10}$$

5.1. Preliminaries and inversion of the equation

Corollary 5.4 ([104]) *Green's function G satisfies the following properties*

$$G(t+T, s+T) = G(t,s)\,,\ \ G(t,t+T) = G(t,t)\exp(\rho T)\,,$$
$$\frac{\partial}{\partial t}G(t,s) = (\exp(-\rho T)-1)\,G_1(t,t)\,G_2(t,s) - b(t)\,G(t,s) + \int_t^{t+T} F(t,\sigma)\,G_1(\sigma,s)\,d\sigma,$$
$$\frac{\partial}{\partial s}G(t,s) = \rho G(t,s)\,,$$

and

$$m \le G(t,s) \le M,$$

where

$$m = \frac{T^2}{\left(e^L-1\right)^2(\exp(\rho T)-1)} \text{ and } M = \frac{T^2\left(\rho T + \exp\left(\int_0^T a(v)\,dv\right)\right)}{\left(e^l-1\right)^2(\exp(\rho T)-1)}.$$

Lemma 5.6 *Suppose (H1)–(H3) and (5.5) hold. The function $x \in P_T$ is a solution of (5.1) if and only if*

$$x(t) = \int_t^{t+T} r(s)\,H(x(s))\,G(t,s)\,ds + \int_t^{t+T} f(s,x(s),x(s-\tau(s)))\,G(t,s)\,ds, \quad (5.11)$$

where

$$H(x) = x - h(x). \quad (5.12)$$

Proof. Let $x \in P_T$ be a solution of (5.1). Rewrite (5.1) as

$$x'''(t) + p(t)\,x''(t) + q(t)\,x'(t) + r(t)\,x(t)$$
$$= r(t)\,H(x(t)) + f(t,x(t),x(t-\tau(t)))\,.$$

From Lemma 5.5, we have

$$x(t) = \int_t^{t+T} G(t,s)\left[r(s)\,H(x(s)) + f(s,x(s),x(s-\tau(s)))\right]ds.$$

The proof is completed. ∎

5.2 Existence of periodic solutions

In this section, we will study the existence of T-periodic solutions of (5.1). To apply Theorem 1.7 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a compact and the other is a large contraction. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and

$$\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le N\}\,, \quad (5.13)$$

with $N>0$. Define a mapping $\mathcal{S}:\mathbb{M}\rightarrow P_T$ by

$$(\mathcal{S}\varphi)(t)=\int_t^{t+T} r(s)\,H(\varphi(s))\,G(t,s)\,ds \\ +\int_t^{t+T} f(s,\varphi(s),\varphi(s-\tau(s)))\,G(t,s)\,ds,$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi=\mathcal{A}\varphi+\mathcal{B}\varphi,$$

where $\mathcal{A},\mathcal{B}:\mathbb{M}\rightarrow P_T$ are given by

$$(\mathcal{A}\varphi)(t)=\int_t^{t+T} f(s,\varphi(s),\varphi(s-\tau(s)))\,G(t,s)\,ds, \tag{5.14}$$

and

$$(\mathcal{B}\varphi)(t)=\int_t^{t+T} r(s)\,H(\varphi(s))\,G(t,s)\,ds. \tag{5.15}$$

To simplify notations, we introduce the following constants

$$\beta=\max_{t\in[0,T]}|b(t)|,\ \theta=\max_{t\in[0,T]}|r(t)|,\ \mu=\max_{t\in[0,T]}|f(t,0,0)|. \tag{5.16}$$

We need the following assumptions

$$\theta MT\leq 1, \tag{5.17}$$

$$JMT\left[(k_1+k_2)N+\mu\right]\leq N, \tag{5.18}$$

$$\max\left(|H(-N)|,|H(N)|\right)\leq\frac{(J-1)}{J}N, \tag{5.19}$$

where J is a positive constant with $J\geq 3$.

Lemma 5.7 *Suppose (H1)–(H3), (5.5), (5.8) and (5.18) hold. Then the operator* $\mathcal{A}:\mathbb{M}\rightarrow\mathbb{M}$ *is compact.*

Proof. Let $\mathcal{A}$ defined by (5.14). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T)=(\mathcal{A}\varphi)(t)$. Observe that in view of (H3) we get

$$\begin{aligned}|f(t,x,y)|&\leq|f(t,x,y)-f(t,0,0)+f(t,0,0)|\\&\leq|f(t,x,y)-f(t,0,0)|+|f(t,0,0)|\\&\leq k_1\|x\|+k_2\|y\|+\mu,\end{aligned}$$

So, for any $\varphi\in\mathbb{M}$, we have

$$\begin{aligned}|(\mathcal{A}\varphi)(t)|&\leq\int_t^{t+T}|f(s,\varphi(s),\varphi(s-\tau(s)))|\,|G(t,s)|\,ds\\&\leq M\int_t^{t+T}\left[(k_1+k_2)N+\mu\right]ds\\&\leq MT\left[(k_1+k_2)N+\mu\right]\leq\frac{N}{J}\leq N.\end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}$.

To see that $\mathcal{A}$ is continuous, we let $\varphi, \psi \in \mathbb{M}$, Given $\varepsilon > 0$, take $\xi = \varepsilon/\eta$ with $\eta = MT(k_1 + k_2)$ where k_1 and k_2 are given by (H3). Now, for $\|\varphi - \psi\| \leq \xi$, we have

$$\begin{aligned}\|\mathcal{A}\varphi - \mathcal{A}\psi\| &\leq M \int_t^{t+T} (k_1 + k_2) \|\varphi - \psi\| \, ds \\ &\leq \eta \|\varphi - \psi\| < \varepsilon.\end{aligned}$$

This proves that $\mathcal{A}$ is continuous.

To prove that the image of $\mathcal{A}$ is contained in a compact set. Let $\varphi_n \in \mathbb{M}$, where n is a positive integer. Then, as above, we see that

$$\|\mathcal{A}\varphi_n\| \leq N.$$

Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and prove that it is uniformly bounded. By using (H1), (H2) and (H3) we get by taking the derivative in (5.14) that

$$\begin{aligned}&\frac{d}{dt}(\mathcal{A}\varphi_n)(t) = f(t, \varphi_n(t), \varphi_n(t - \tau(t))) G(t,t)(\exp(\rho T) - 1) \\ &+ \int_t^{t+T} \left[(\exp(-\rho T) - 1) G_1(t,t) G_2(t,s) - b(t) G(t,s) + \int_t^{t+T} F(t,\sigma) G_1(\sigma,s) \, d\sigma\right] \\ &\times f(s, \varphi_n(s), \varphi_n(s - \tau(s))) \, ds.\end{aligned}$$

Consequently, by invoking (H3) and (5.16), we obtain

$$\begin{aligned}\left|\frac{d}{dt}(\mathcal{A}\varphi_n)(t)\right| &\leq [(k_1 + k_2) N + \mu] M (\exp(\rho T) - 1) \\ &+ \left[(\exp(-\rho T) - 1) M_1 M_2 + \beta M + M_1 T e^L\right]((k_1 + k_2) N + \mu) T \\ &\leq D,\end{aligned}$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact subset of $\mathbb{M}$. ■

Lemma 5.8 *For $\mathcal{B}$ be defined in (5.15), suppose (H1), (H2), (5.5), (5.17), (5.19) and all conditions of Theorem 1.5 hold. Then $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (5.15). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to prove that $(\mathcal{B}\varphi)(t + T) = (\mathcal{B}\varphi)(t)$. So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}|(\mathcal{B}\varphi)(t)| &\leq \int_t^{t+T} |r(s)| |H(\varphi(s))| |G(t,s)| \, ds \\ &\leq \theta M T \max\{|H(-N)|, |H(N)|\} \leq \frac{(J-1)N}{J} \leq N,\end{aligned}$$

by (5.17) and (5.19). Then, for any $\varphi \in \mathbb{M}$, we get

$$\|\mathcal{B}\varphi\| \leq N.$$

Thus $\mathcal{B}\varphi \in \mathbb{M}$. Consequently, we have $\mathcal{B} : \mathbb{M} \to \mathbb{M}$.

It remains to prove that $\mathcal{B}$ is a large contraction. By Theorem 1.5 H is large contraction on $\mathbb{M}$, then for any $\varphi, \psi \in \mathbb{M}$, with $\varphi \neq \psi$ we have

$$\begin{aligned} |(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| &\leq \left| \int_t^{t+T} G(t,s) r(s) [H(\varphi(s)) - H(\psi(s))] ds \right| \\ &\leq \theta M T \|\varphi - \psi\| \leq \|\varphi - \psi\|. \end{aligned}$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \|\varphi - \psi\|$. Now, let $\varepsilon \in (0,1)$ be given and let $\varphi, \psi \in \mathbb{M}$, with $\|\varphi - \psi\| \geq \varepsilon$, from the proof of Theorem 1.5, we have found a $\delta \in (0,1)$, such that

$$|(H\varphi)(t) - (H\psi)(t)| \leq \delta \|\varphi - \psi\|.$$

Thus,

$$\begin{aligned} |(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| &\leq \left| \int_t^{t+T} G(t,s) r(s) [H(\varphi(s)) - H(\psi(s))] ds \right| \\ &\leq \theta M T \delta \|\varphi - \psi\| \leq \delta \|\varphi - \psi\|. \end{aligned}$$

So,

$$\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \delta \|\varphi - \psi\|.$$

The proof is complete. ∎

Theorem 5.1 *Let $\mathbb{M}$ defined by (5.13), β, θ, μ be given by (5.16). Suppose (H1)–(H3), (5.5), (5.8), (5.17)–(5.19) and all conditions of Theorem 1.5 hold. Then (5.1) has a T-periodic solution in $\mathbb{M}$.*

Proof. By Lemmas 5.7, the mapping $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is compact and continuous. Also, from Lemma 5.8, the mapping $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction. Moreover, if $\varphi, \psi \in \mathbb{M}$, we see that

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq \|\mathcal{A}\varphi\| + \|\mathcal{B}\psi\| \leq \frac{N}{J} + \frac{(J-1)N}{J} = N.$$

Thus $\mathcal{A}\varphi + \mathcal{B}\psi \in \mathbb{M}$.

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 5.6 this fixed point is a solution of (5.1). Hence (5.1) has a T-periodic solution. ∎

5.3 Existence of nonnegative periodic solutions

This section is concerned with the existence of a nonnegative T-periodic solution of (5.1). Again, we arrive at our results by using Theorem 1.7. Since we are looking for the existence of nonnegative T-periodic solutions, some of the conditions in previous sections will have to be modified accordingly. For a positive constant N we define the set

$$\mathbb{M} = \{\varphi \in P_T : 0 \leq \varphi \leq N\}, \tag{5.20}$$

which is a closed convex and bounded subset of the Banach space P_T.

We assume that for all $t \in [0, T]$, $x, y \in \mathbb{M}$

$$0 \leq r(t) H(x) + f(t, x, y) \leq \frac{N}{MT}. \tag{5.21}$$

Lemma 5.9 *Let $\mathcal{A}$ and $\mathcal{B}$ given by (5.14) and (5.15) respectively. Assume (H1)–(H3), (5.5), (5.21) hold. Then $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{M}$.*

Proof. Let $\mathcal{A}$ defined by (5.14). So, for any $\varphi \in \mathbb{M}$, by (5.21) we have

$$\begin{aligned} 0 \leq (\mathcal{A}\varphi)(t) &\leq \int_t^{t+T} [r(s) H(\varphi(s)) + f(s, \varphi(s), \varphi(s - \tau(s)))] G(t, s) ds \\ &\leq \int_t^{t+T} \frac{N}{MT} M ds = N. \end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}$.

Now, let $\mathcal{B}$ defined by (5.15). So, for any $\varphi \in \mathbb{M}$, by (5.21) we have

$$\begin{aligned} 0 \leq (\mathcal{B}\varphi)(t) &\leq \int_t^{t+T} [r(s) H(\varphi(s)) + f(s, \varphi(s), \varphi(s - \tau(s)))] G(t, s) ds \\ &\leq \int_t^{t+T} \frac{N}{MT} M ds = N. \end{aligned}$$

That is $\mathcal{B}\varphi \in \mathbb{M}$. ∎

Theorem 5.2 *Suppose the hypotheses of Lemmas 5.7, 5.8 and 5.9 hold. Then (5.1) has a nonnegative T-periodic solution x in the subset $\mathbb{M}$.*

Proof. By Lemma 5.7, $\mathcal{A}$ is compact and continuous. Also, from Lemma 5.8, the mapping $\mathcal{B}$ is a large contraction. By Lemma 5.9, $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we get $0 \leq \mathcal{A}\varphi + \mathcal{B}\psi \leq N$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \leq \varphi, \psi \leq N$. By (5.21), we obtain

$$\begin{aligned} &(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t) \\ &= \int_t^{t+T} G(t, s) [r(s) H(\psi(s)) + f(s, \varphi(s), \varphi(s - \tau(s)))] ds \\ &\leq \int_t^{t+T} \frac{N}{MT} M ds = N. \end{aligned}$$

On the other hand,

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t) \geq 0.$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 5.6, this fixed point is a nonnegative T-periodic solution of (5.1) and the proof is complete. ■

Chapter 6

Investigation of the periodicity and stability in the neutral differential systems by using Krasnoselskii's fixed point theorem

Keywords. Periodic solutions, stability, Krasnoselskii's fixed point theorem, neutral differential systems.

This chapter has been extracted from the research paper [78],

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In this Chapter, we are interested on the existence and asymptotic stability of periodic solutions of the following neutral differential system

$$\begin{aligned} &\frac{d}{dt}u(t) - q\frac{d}{dt}u(t-r) \\ &= P(t) + A(t)u(t) + A(t)qu(t-r) - bf(u(t)) + bqf(u(t-r)), \end{aligned} \tag{6.1}$$

where $b > 0$, $|q| < 1$, $r > 0$ and A is nonsingular $n \times n$ matrix with continuous real-valued functions as its elements. The functions $P : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable.

In the analysis we use the fundamental matrix solution coupled with Floquet theory to invert the differential system (6.1) into an integral system. Then, we employ Krasnoselskii's fixed point theorem to show the existence and asymptotic stability of periodic solutions of the system (6.1). The obtained integral system is the sum of two mappings, one is a compact operator and the other is a contraction. The results obtained here extend some results of the work of Ding and Li [62].

6.1 Existence of periodic solutions

In this section, $C^1(\mathbb{R},\mathbb{R}^n)$ and $C(\mathbb{R},\mathbb{R}^n)$ denote the set of all continuously differentiable functions and all continuous functions $\phi:\mathbb{R}\to\mathbb{R}^n$ respectively. For $T>0$, $C_T=\{\phi\in C(\mathbb{R},\mathbb{R}^n),\ \phi(t+T)=\phi(t)\}$ is a Banach space with the supremum norm

$$\|\phi\|_0=\sup_{t\in\mathbb{R}}|\phi(t)|=\sup_{t\in[0,T]}|\phi(t)|,$$

where $|.|$ denotes the infinity norm for $x\in\mathbb{R}^n$ and $C_T^1=C^1(\mathbb{R},\mathbb{R}^n)\cap C_T$ is a Banach space with the norm $\|\phi\|_1=\|\phi\|_0+\|\phi'\|_0$ in a period interval. Also, if A is an $n\times n$ real matrix, then we define the norm of A by $|A|=\max_{1\le i\le n}\sum_{j=1}^n|a_{ij}|$.

For a sufficiently small positive L, (6.1) can be transformed as

$$\begin{aligned}&\frac{d}{dt}v(t)-q\frac{d}{dt}v(t-\tau)\\&=LP_1(t)+LA_1(t)v(t)+LA_1(t)qv(t-\tau)-Lbf(v(t))+Lbqf(v(t-\tau)),\end{aligned}\tag{6.2}$$

where $v(t)=u(Lt)$, $\tau=\frac{r}{L}$, $P_1(t)=P(Lt)$ and $A_1(t)=A(Lt)$.

First we make the following definition.

Definition 6.1 If the matrix A_1 is periodic of period $\omega=\frac{T}{L}$, then the linear system

$$y'(t)=LA_1(t)y(t),\tag{6.3}$$

is said to be noncritical with respect to ω if it has no periodic solution of period ω except the trivial solution $y=0$.

Throughout this paper it is assumed that system (6.3) is noncritical. Next we state some known results [54] about system (6.3). Let K represent the fundamental matrix of (6.3) with $K(0)=I$, where I is the $n\times n$ identity matrix. Then

(a) $\det K(t)\neq 0$.

(b) There exists a constant matrix B such that $K(t+\omega)=K(t)e^{B\omega}$, by Floquet theory.

(c) System (6.3) is noncritical if and only if $\det(I-K(\omega))\neq 0$.

Lemma 6.1 *If the matrix LA_1 is periodic of period ω and $h\in C_\omega$, then the linear system*

$$x'(t)=LA_1(t)x(t)+h(t),\tag{6.4}$$

has a unique ω-periodic solution

$$x(t)=K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)h(s)\,ds.$$

Proof. Since $K(t)K^{-1}(t) = I$, it follows that

$$\begin{aligned} 0 &= \frac{d}{dt}\left(K(t)K^{-1}(t)\right) = \frac{d}{dt}\left(K(t)\right)K^{-1}(t) + K(t)\frac{d}{dt}\left(K^{-1}(t)\right) \\ &= \left(LA_1(t)\,K(t)\right)K^{-1}(t) + K(t)\frac{d}{dt}\left(K^{-1}(t)\right) \\ &= LA_1(t) + K(t)\frac{d}{dt}\left(K^{-1}(t)\right). \end{aligned}$$

This implies

$$\frac{d}{dt}\left(K^{-1}(t)\right) = -K^{-1}(t)LA_1(t). \tag{6.5}$$

If x is a solution of (6.4) with $x(0) = x_0$, then

$$\begin{aligned} \frac{d}{dt}\left[K^{-1}(t)x(t)\right] &= \frac{d}{dt}\left(K^{-1}(t)\right)x(t) + K^{-1}(t)\frac{d}{dt}x(t) \\ &= -K^{-1}(t)LA_1(t)\,x(t) + K^{-1}(t)\left[LA_1(t)\,x(t) + h(t)\right] \\ &= K^{-1}(t)h(t), \end{aligned}$$

by (6.5). An integration of the above equation from 0 to t yields

$$x(t) = K(t)x(0) + K(t)\int_0^t K^{-1}(s)h(s)\,ds. \tag{6.6}$$

Since $x(\omega) = x_0 = x(0)$, we get

$$x(0) = \left(I - K(\omega)\right)^{-1}\int_0^\omega K(\omega)K^{-1}(s)h(s)\,ds. \tag{6.7}$$

A substitution of (6.7) into (6.6) yields

$$\begin{aligned} x(t) = {} & K(t)\left(I - K(\omega)\right)^{-1}\int_0^\omega K(\omega)K^{-1}(s)h(s)\,ds \\ & + K(t)\int_0^t K^{-1}(s)h(s)\,ds. \end{aligned} \tag{6.8}$$

Since

$$\left(I - K(\omega)\right)^{-1} = \left(K(\omega)\left(K^{-1}(\omega) - I\right)\right)^{-1} = \left(K^{-1}(\omega) - I\right)^{-1}K^{-1}(\omega),$$

(6.8) becomes

$$\begin{aligned} x(t) &= K(t)\left(K^{-1}(\omega) - I\right)^{-1}\int_0^\omega K^{-1}(s)h(s)\,ds + K(t)\int_0^t K^{-1}(s)h(s)\,ds \\ &= K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_0^\omega K^{-1}(s)h(s)\,ds \right. \\ &\left. +K^{-1}(\omega)\int_0^t K^{-1}(s)h(s)\,ds - \int_0^t K^{-1}(s)h(s)\,ds\right\}. \\ &= K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_t^\omega K^{-1}(s)h(s)\,ds \right. \\ &\left. +K^{-1}(\omega)\int_0^t K^{-1}(s)h(s)\,ds\right\} \end{aligned}$$

By letting $s = \mu - \omega$, the above expression implies

$$x(t) = K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_t^{\omega} K^{-1}(s)h(s)\,ds \right.$$
$$\left. +K^{-1}(\omega)\int_{\omega}^{t+\omega} K^{-1}(\mu-\omega)h(\mu-\omega)\,d\mu\right\}. \tag{6.9}$$

By (b) we have $K(t-\omega) = K(t)e^{-B\omega}$and $K(\omega) = e^{B\omega}$. Hence,

$$K^{-1}(\omega)K^{-1}(\mu-\omega) = K^{-1}(\mu).$$

Consequently, (6.9) becomes

$$x(t) = K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_t^{\omega} K^{-1}(s)h(s)\,ds + \int_{\omega}^{t+\omega} K^{-1}(s)h(s)\,ds\right\}.$$

■

By applying Lemma 6.1 and Theorem 1.4, we obtain in this section the existence of periodic solutions of (6.1).

Theorem 6.1 *Suppose that* $f \in C^1(\mathbb{R}^n)$ *and* $P_1, A_1 \in C^1_{\omega}$. *If there exists a constant* $H > 0$ *such that*

$$\frac{\sup\limits_{|u|\leq H} |f(u)|}{H} < \frac{1}{(1+(1+L\|A_1\|)c\omega)Lb}, \tag{6.10}$$

and that

$$|q| < \frac{1-(1+(1+L\|A_1\|)c\omega)Lb\frac{\sup_{|u|\leq H}|f(u)|}{H}}{1+2\|A_1\|(1+(1+L\|A_1\|)c\omega)L+(1+(1+L\|A_1\|)c\omega)Lb\frac{\sup_{|u|\leq H}|f(u)|}{H}}, \tag{6.11}$$

and

$$\|P_1\|_0 \leq \frac{(1-|q|)H}{(1+(1+L\|A_1\|)c\omega)L} - 2\|A_1\||q|H - b(1+|q|)\sup_{|u|\leq H}|f(u)|, \tag{6.12}$$

where $\|A_1\| = \sup\limits_{t\in[0,\omega]} |A_1(t)|$ *and*

$$c = \sup_{t\in[0,\omega]}\left(\sup_{t\leq s\leq t+\omega} |[K(s)(K^{-1}(\omega)-I)K^{-1}(t)]^{-1}|\right).$$

Then (6.1) has a T*-periodic solution.*

Proof. According to the condition (6.12), we get

$$\begin{aligned}
&(1+(1+L\|A_1\|)c\omega)L\|P_1\|_0+[1+(1+(1+L\|A_1\|)c\omega)2L\|A_1\|]|q|H \\
&+(1+(1+L\|A_1\|)c\omega)Lb(1+|q|)\sup_{|u|\le H}|f(u)| \\
&\le(1+(1+L\|A_1\|)c\omega)L \\
&\times\left\{\frac{(1-|q|)H}{(1+(1+L\|A_1\|)c\omega)L}-2\|A_1\||q|H-b(1+|q|)\sup_{|u|\le H}|f(u)|\right\} \\
&+[1+(1+(1+L\|A_1\|)c\omega)2L\|A_1\|]|q|H \\
&+(1+(1+L\|A_1\|)c\omega)Lb(1+|q|)\sup_{|u|\le H}|f(u)| \\
&=H.
\end{aligned}\tag{6.13}$$

We need to prove that (6.2) has a ω-periodic solution. Let

$$\mathbb{S}=\left\{\phi\in C^1_\omega,\ \|\phi\|_1=\|\phi\|_0+\|\phi'\|_0<+\infty\right\},$$

and

$$\mathbb{M}=\{\phi\in\mathbb{S},\ \|\phi\|_1\le H\},$$

then $\mathbb{M}$ is a bounded closed convex set of the Banach space $\mathbb{S}$.

Consider the system

$$\begin{aligned}
\frac{d}{dt}v(t)&=LA_1(t)v(t)+LP_1(t)+LA_1(t)qv(t-\tau) \\
&\quad-Lbf(v(t))+Lbqf(v(t-\tau))+q\frac{d}{dt}v(t-\tau).
\end{aligned}$$

According to Lemma 6.1, this equation has a unique ω-periodic solution

$$\begin{aligned}
v(t)=K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}&K^{-1}(s)\left[LP_1(s)+LA_1(s)qv(s-\tau)\right. \\
&\left.-Lbf(v(s))+Lbqf(v(s-\tau))+q\frac{\partial}{\partial s}v(s-\tau)\right]ds,
\end{aligned}$$

Performing an integration by part and the fact that $v(t+\omega-\tau)=v(t-\tau)$, we obtain

$$\begin{aligned}
&K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)q\frac{\partial}{\partial s}v(s-\tau)ds \\
&=K(t)\left(K^{-1}(\omega)-I\right)^{-1}\left\{\left[K^{-1}(t+\omega)-K^{-1}(t)\right]qv(t-\tau)\right. \\
&\left.-\int_t^{t+\omega}\frac{\partial}{\partial s}\left[K^{-1}(s)\right]qv(s-\tau)ds\right\}
\end{aligned}\tag{6.14}$$

Noting that $K^{-1}(t+\omega)=e^{-B\omega}K^{-1}(t)$, we have

$$\begin{aligned}
K^{-1}(t+\omega)-K^{-1}(t)&=e^{-B\omega}K^{-1}(t)-K^{-1}(t) \\
&=\left(K^{-1}(\omega)-I\right)K^{-1}(t).
\end{aligned}\tag{6.15}$$

Since

$$\frac{d}{dt}K^{-1}(t) = -K^{-1}(t)LA_1(t), \tag{6.16}$$

then, a substitution of (6.15) and (6.16) into (6.14) yields

$$\begin{aligned} &K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)q\frac{\partial}{\partial s}v(s-\tau)\,ds \\ &= qv(t-\tau)+K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)LA_1(s)\,qv(s-\tau)\,ds. \end{aligned}$$

Therefore

$$\begin{aligned} v(t) = qv(t-\tau) + K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\,[LP_1(s) \\ +2LA_1(s)\,qv(s-\tau) - Lbf(v(s)) + Lbqf(v(s-\tau))]\,ds. \end{aligned}$$

Define the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\begin{aligned} (\mathcal{A}\varphi)(t) = K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\,[LP_1(s) \\ +2LA_1(s)\,q\varphi(s-\tau) - Lbf(\varphi(s)) + Lbqf(\varphi(s-\tau))]\,ds, \end{aligned}$$

and

$$(\mathcal{B}\varphi)(t) = q\varphi(t-\tau).$$

In order to prove (6.2) has a ω-periodic solution, we shall make sure that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Theorem 1.4.

For all $x, y \in \mathbb{M}$, we have

$$x(t+\omega) = x(t),\ \ y(t+\omega) = y(t) \text{ and } \|x\|_1 \le H,\ \|y\|_1 \le H.$$

Now let us discuss $\mathcal{A}x + \mathcal{B}y$. We have

$$\begin{aligned} (\mathcal{A}x)(t+\omega) = K(t+\omega)\left(K^{-1}(\omega)-I\right)^{-1}\int_{t+\omega}^{t+2\omega}K^{-1}(s)\,[LP_1(s) \\ +2LA_1(s)\,qx(s-\tau) - Lbf(x(s)) + Lbqf(x(s-\tau))]\,ds. \end{aligned}$$

By letting $s = \mu + \omega$, the above expression implies

$$\begin{aligned} (\mathcal{A}x)(t+\omega) = K(t+\omega)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(\mu+\omega)\,[LP_1(\mu+\omega) \\ +2LA_1(\mu+\omega)\,qx(\mu+\omega-\tau) \\ -Lbf(x(\mu+\omega)) + Lbqf(x(\mu+\omega-\tau))]\,d\mu. \end{aligned}$$

By (b) we have

$$K(t+\omega) = K(t)e^{B\omega} \text{ and } K(\omega) = e^{B\omega}.$$

Hence

$$K(t+\omega)\left(K^{-1}(\omega)-I\right)^{-1}K^{-1}(\mu+\omega)$$
$$=K(t)\left(K^{-1}(\omega)-I\right)^{-1}K^{-1}(\mu).$$

Consequently, the above expression implies

$$(\mathcal{A}x)(t+\omega)=K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\,[LP_1(s)$$
$$+2LA_1(s)\,qx(s-\tau)-Lbf(x(s))+Lbqf(x(s-\tau))]\,ds$$
$$=(\mathcal{A}x)(t),$$

and

$$(\mathcal{B}y)(t+\omega)=qy(t+\omega-\tau)$$
$$=qy(t-\tau)=(\mathcal{B}y)(t),$$

therefore

$$(\mathcal{A}x+\mathcal{B}y)(t+\omega)=(\mathcal{A}x+\mathcal{B}y)(t).$$

Meanwhile, we get

$$(\mathcal{A}x)'(t)=K'(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\,[LP_1(s)$$
$$+2LA_1(s)\,qx(s-\tau)-Lbf(x(s))+Lbqf(x(s-\tau))]\,ds$$
$$+K(t)\left(K^{-1}(\omega)-I\right)^{-1}\left[K^{-1}(t+\omega)-K^{-1}(t)\right][LP_1(t)$$
$$+2LA_1(t)\,qx(t-\tau)-Lbf(x(t))+Lbqf(x(t-\tau))]. \tag{6.17}$$

Since

$$K'(t)=LA_1(t)\,K(t), \tag{6.18}$$

and noting that $K^{-1}(t+\omega)=e^{-B\omega}K^{-1}(t)$, we have

$$K^{-1}(t+\omega)-K^{-1}(t)=e^{-B\omega}K^{-1}(t)-K^{-1}(t)$$
$$=\left(K^{-1}(\omega)-I\right)K^{-1}(t). \tag{6.19}$$

A substitution of (6.18) and (6.19) into (6.17) yields

$$(\mathcal{A}x)'(t)=LA_1(t)\,(\mathcal{A}x)(t)+LP_1(t)+2LA_1(t)\,qx(t-\tau)$$
$$-Lbf(x(t))+Lbqf(x(t-\tau)).$$

Thus,

$$\begin{aligned}
\|\mathcal{A}x\|_1 &= \|\mathcal{A}x\|_0 + \left\|(\mathcal{A}x)'\right\|_0 \\
&= \sup_{t\in[0,\omega]} \left| K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega} K^{-1}(s)\left[LP_1(s)\right.\right. \\
&\left.\left. +2LA_1(s)\,qx(s-\tau) - Lbf(x(s)) + Lbqf(x(s-\tau))\right] ds \right| \\
&+ \sup_{t\in[0,\omega]} \left| LA_1(t)\,K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega} K^{-1}(s)\left[LP_1(s)\right.\right. \\
&\left. +2LA_1(s)\,qx(s-\tau) - Lbf(x(s)) + Lbqf(x(s-\tau))\right] ds \\
&\left. +LP_1(t) + 2LA_1(t)\,qx(t-\tau) - Lbf(x(t)) + Lbqf(x(t-\tau)) \right| \\
&\le c\omega\left[2L\|A_1\|\,|q|\,H + Lb(1+|q|)\sup_{|u|\le H}|f(u)| + L\|P_1\|_0\right] \\
&+ (1+L\|A_1\|\,c\omega)\left[2L\|A_1\|\,|q|\,H + Lb(1+|q|)\sup_{|u|\le H}|f(u)| + L\|P_1\|_0\right] \\
&= (1+(1+L\|A_1\|)\,c\omega)\left[2L\|A_1\|\,|q|\,H + Lb(1+|q|)\sup_{|u|\le H}|f(u)| + L\|P_1\|_0\right],
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{B}y\|_1 &= \|\mathcal{B}y\|_0 + \left\|(\mathcal{B}y)'\right\|_0 \le |q|\,\|y\|_0 + |q|\,\|y'\|_0 = |q|\,\|y\|_1 \\
&\le |q|\,H.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathcal{A}x + \mathcal{B}y\|_1 \\
&\le \|\mathcal{A}x\|_1 + \|\mathcal{B}y\|_1 \\
&\le (1+(1+L\|A_1\|)\,c\omega)\left[2L\,|A_1|\,|q|\,H\right. \\
&\left. +Lb(1+|q|)\sup_{|u|\le H}|f(u)| + L\|P_1\|_0\right] + |q|\,H \\
&= [1+(1+(1+L\|A_1\|)\,c\omega)\,2L\|A_1\|]\,|q|\,H + (1+(1+L\|A_1\|)\,c\omega)\,L\|P_1\|_0 \\
&+ (1+(1+L\|A_1\|)\,c\omega)\,Lb(1+|q|)\sup_{|u|\le H}|f(u)|.
\end{aligned}$$

By (6.13), $\|\mathcal{A}x + \mathcal{B}y\|_1 \le H$. Accordingly, $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$.

For all $x \in \mathbb{M}$, $\|\mathcal{A}x\|_0 \le H$, $\left\|(\mathcal{A}x)'\right\|_0 \le H$. According to Ascoli Arzela lemma, the subset $\mathcal{A}\mathbb{M}$ of C_ω is a precompact set, therefore for all subsequence $\{x_n\}$ of $\mathbb{M}$, there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\mathcal{A}x_{n_k} \to x_0 \in C_\omega$ as $k \to +\infty$.

Meanwhile, we get

$$\begin{aligned}(\mathcal{A}x)''(t) &= LA_1'(t)(\mathcal{A}x)(t) + L^2A_1^2(t)(\mathcal{A}x)(t) + LA_1(t)[LP_1(t) \\ &\quad +2LA_1(t)qx(t-\tau) - Lbf(x(t)) + Lbqf(x(t-\tau))] \\ &\quad + [LP_1'(t) + 2Lq[A_1'(t)x(t-\tau) + A_1(t)x'(t-\tau)] \\ &\quad -Lbf'(x(t))x'(t) + Lbqf'(x(t-\tau))x'(t-\tau)].\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{t\in[0,\omega]}\left|(\mathcal{A}x)''(t)\right| \leq \left(L\|A_1\| + \left(L\|A_1'\| + L^2\|A_1\|^2\right)c\omega\right)&[2L\|A_1\||q|H \\ &\left.+Lb(1+|q|)\sup_{|u|\leq H}|f(u)| + L\|P_1\|_0\right] \\ &+ [2L(\|A_1\| + \|A_1'\|)|q|H \\ &\left.+LbH(1+|q|)\sup_{|u|\leq H}|f(u)| + L\|P_1'\|_0\right].\end{aligned}$$

Therefore there is a constant $H_1 > 0$ such that

$$\sup_{t\in[0,\omega]}\left|(\mathcal{A}x)''(t)\right| \leq H_1 \text{ and } \left\{(\mathcal{A}x)' : x\in\mathbb{M}\right\} \subset C_\omega.$$

According to Ascoli Arzela lemma, $\{x_{n_k}\}$ has a subsequence, for simplicity, written as $\{x_{n_k}\}$, such that $(\mathcal{A}x_{n_k})' \to z_0 \in C_\omega$. Since $\frac{d}{dt}$ is a closed operator, $z_0 = (x_0)'$. Hence, $x_0 \in C_\omega^1$ and $\{\mathcal{A}x_n\}$ is contained in a compact set. Then, $\mathcal{A}$ is a compact operator.

Suppose that $\{x_n\} \in \mathbb{M}$, $x \in \mathbb{S}$, $x_n \to x$, then $\|x_n - x\|_0 \to 0$ and $\|x_n' - x'\|_0 \to 0$ as $n \to +\infty$. And we get

$$\begin{aligned}&\|\mathcal{A}x_n - \mathcal{A}x\|_0 \\ &= \sup_{t\in[0,\omega]}\left|K(t)\left(K^{-1}(\omega) - I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\right. \\ &\times [2LA_1(s)q(x_n(s-\tau) - x(s-\tau)) - Lb(f(x_n(s)) - f(x(s))) \\ &+Lbq(f(x_n(s-\tau)) - f(x(s-\tau)))]ds| \\ &\leq \omega c\left[2L\|A_1\||q|\|x_n - x\| + Lb(1+|q|)\sup_{t\in[0,\omega]}|f(x_n(t)) - f(x(t))|\right],\end{aligned}$$

and

$$\begin{aligned}
&\left\|(\mathcal{A}x_n)' - (\mathcal{A}x)'\right\|_0 \\
&= \sup_{t\in[0,\omega]} |LA_1(t)((\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)) \\
&+ 2LA_1(t)q(x_n(t-\tau) - x(t-\tau)) - Lb(f(x_n(t)) - f(x(t))) \\
&+ Lbq(f(x_n(t-\tau)) - f(x(t-\tau)))]| \\
&\leq (1 + L\|A_1\|\omega c)[2L\|A_1\||q|\|x_n - x\| \\
&\left. + Lb(1+|q|)\sup_{t\in[0,\omega]}|f(x_n(t)) - f(x(t))|\right].
\end{aligned}$$

When $\|x_n - x\|_1 \to 0$ as $n \to +\infty$, $|x_n(t) - x(t)| \to 0$ for $t \in [0,\omega]$ uniformly. And since f is continuous, $\|\mathcal{A}x_n - \mathcal{A}x\|_0 \to 0$, $\left\|(\mathcal{A}x_n)' - (\mathcal{A}x)'\right\|_0 \to 0$. Consequently, $\mathcal{A}$ is continuous.

For all $x, y \in \mathbb{M}$, $\|\mathcal{B}x - \mathcal{B}y\|_1 \leq |q|\|x-y\|_1$ and $|q| < 1$, therefore $\mathcal{B}$ is a contraction operator.

Thus, the conditions of Theorem 1.4 are satisfied and there is a $\phi \in \mathbb{M}$ such that $\phi = \mathcal{A}\phi + \mathcal{B}\phi$. It is a ω-periodic solution for (6.2). Since $v(t) = u(Lt)$, $P_1(t) = P(Lt)$ and $A_1(t) = A(Lt)$, then (6.1) has a T-periodic solution. ■

Example 6.1 Consider the following neutral differential system

$$\begin{aligned}
&\frac{d}{dt}u(t) - q\frac{d}{dt}u(t-r) \\
&= P(t) + A(t)u(t) + A(t)qu(t-r) - bf(u(t)) + bqf(u(t-r)),
\end{aligned} \tag{6.20}$$

where $T = 2\pi$, $b = 1$, $q = \frac{1}{80}$, $r = 2$, $A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $P(t) = \begin{pmatrix} 0 \\ 0.01\cos(t) \end{pmatrix}$ and $f(u(t)) = \begin{pmatrix} 0 \\ \sin(u(t)) \end{pmatrix}$. For $L = 0.25$, (6.20) can be transformed as

$$\begin{aligned}
&\frac{d}{dt}v(t) - q\frac{d}{dt}v(t-\tau) \\
&= LP_1(t) + LA_1(t)v(t) + LA_1(t)qv(t-\tau) - Lbf(v(t)) + Lbqf(v(t-\tau)),
\end{aligned}$$

where $v(t) = u(0.25t)$, $\omega = 8\pi$, $\tau = 8$, $P_1(t) = \begin{pmatrix} 0 \\ 0.01\cos(0.25t) \end{pmatrix}$ and $A_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Since the matrix A_1 has eigenvalues with non-zero real parts, the system $\frac{d}{dt}v(t) = LA_1(t)v(t)$ is noncritical. Let $H = 30$, then all conditions of Theorem 6.1 are satisfied and hence (6.20) has a 2π-periodic solution.

6.2 Asymptotic stability of periodic solutions

This section concerned with the asymptotic stability of periodic solutions. When the conditions of Theorem 6.1 are satisfied, there is a T-periodic solution u^* for (6.1). Let $v(t) = u(t) - u^*(t)$, then (6.1) is transformed as

$$\begin{aligned} v'(t) - qv'(t-r) &= A(t)v(t) + A(t)qv(t-r) - b\left[f(v(t)+u^*(t)) - f(u^*(t))\right] \\ &\quad + bq\left[f(v(t-r)+u^*(t-r)) - f(u^*(t-r))\right]. \end{aligned} \tag{6.21}$$

Obviously, (6.21) has the zero solution. Now we use Krasnoselskii's fixed point theorem to prove the zero solution for (6.21) is asymptotically stable. We set $\mathbb{S}$ as the Banach space of bounded continuous function $\phi : [-r,\infty) \to \mathbb{R}^n$ with the supremum norm $\|.\|$. Also, Given the initial function ψ, denote the norm of ψ by $\|\psi\| = \sup_{t\in[-r,0]} |\psi(t)|$, which should not cause confusion with the same symbol for the norm in $\mathbb{S}$.

Proposition 6.1 ([54], Proposition 2.14) *If $t \to \Phi(t)$ is a fundamental matrix solution for the system*

$$y'(t) = A(t)y(t), \tag{6.22}$$

defined on an open interval J, then $\Phi(t,r) := \Phi(t)\Phi^{-1}(r)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman-Kolmogorov identities

$$\Phi(r,r) = I,\ \Phi(t,s)\Phi(s,r) = \Phi(t,r),$$

and the identities

$$\Phi(t,s)^{-1} = \Phi(s,t),\ \frac{\partial\Phi(t,s)}{\partial s} = -\Phi(t,s)A(s).$$

Theorem 6.2 *If all conditions of Theorem 6.1 are satisfied, f satisfies the locally Lipschitz condition. Further assume that*

$$\Phi(t) \to 0 \text{ as } t \to \infty,$$

and there exists $Q > H$ such that

$$\sup_{|u|\le H+Q} |f(u)| + \sup_{|u|\le H} |f(u)| < \frac{Q}{\lambda b}, \tag{6.23}$$

and that

$$|q| < \frac{Q - \lambda b\left(\sup_{|u|\le H+Q} |f(u)| + \sup_{|u|\le H} |f(u)|\right)}{(1+2\lambda\|A\|)Q + \lambda b\left(\sup_{|u|\le H+Q} |f(u)| + \sup_{|u|\le H} |f(u)|\right)}, \tag{6.24}$$

and

$$\|\psi\| \leq \frac{(1-(1+2\lambda\|A\|)|q|)Q-\lambda b(1+|q|)\left(\sup\limits_{|u|\leq H+Q}|f(u)|+\sup\limits_{|u|\leq H}|f(u)|\right)}{\theta(1+|q|)}, \quad (6.25)$$

where $\theta=\sup_{t\geq 0}|\Phi(t,0)|$ *and* $\lambda=\sup\limits_{t\geq 0}\left|\int_0^t\Phi(t,s)ds\right|$. *Then the solution of (6.21)* $v(t)\to 0$ *as* $t\to\infty$.

Proof. According to the conditions (6.23), (6.24) and (6.25), we have

$$\begin{aligned}&(1+2\lambda\|A\|)|q|Q+\theta(1+|q|)\|\psi\|\\&+\lambda b(1+|q|)\left(\sup_{|u|\leq H+Q}|f(u)|+\sup_{|u|\leq H}|f(u)|\right)\leq Q.\end{aligned} \quad (6.26)$$

Given the initial function ψ, there exists a unique solution v for (6.21). Let

$$\mathbb{M}_\psi=\{\phi\in\mathbb{S},\ \|\phi\|\leq Q,\ \phi(t)=\psi(t)\ \text{if}\ t\in[-r,0],\ |\phi(t)|\to 0\ \text{as}\ t\to\infty\},$$

then $\mathbb{M}_\psi$ is a bounded convex closed set of $\mathbb{S}$.

Let v be a solution of (6.21). We write (6.21) as

$$\begin{aligned}&\frac{d}{dt}\{v(t)-qv(t-r)\}\\&=A(t)v(t)+A(t)qv(t-r)-b[f(v(t)+u^*(t))-f(u^*(t))]\\&+bq[f(v(t-r)+u^*(t-r))-f(u^*(t-r))]\end{aligned}$$

Since $\varPhi$ is a fundamental matrix solution for the system (6.22). We have

$$\begin{aligned}&\frac{d}{dt}\left\{\varPhi^{-1}(t)(v(t)-qv(t-r))\right\}\\&=\left\{\frac{d}{dt}\varPhi^{-1}(t)\right\}(v(t)-qv(t-r))+\varPhi^{-1}(t)\frac{d}{dt}\{(v(t)-qv(t-r))\}.\end{aligned}$$

By the Proposition 6.1, it follows that

$$\frac{d}{dt}\varPhi^{-1}(t)=-\varPhi^{-1}(t)A(t).$$

Then

$$\begin{aligned}&\frac{d}{dt}\left\{\varPhi^{-1}(t)(v(t)-qv(t-r))\right\}\\&=-\varPhi^{-1}(t)A(t)(v(t)-qv(t-r))+\varPhi^{-1}(t)\{A(t)v(t)\\&+A(t)qv(t-r)-b[f(v(t)+u^*(t))-f(u^*(t))]\\&+bq[f(v(t-r)+u^*(t-r))-f(u^*(t-r))]\}\\&=\varPhi^{-1}(t)\{2A(t)qv(t-r)-b[f(v(t)+u^*(t))-f(u^*(t))]\\&+bq[f(v(t-r)+u^*(t-r))-f(u^*(t-r))]\}.\end{aligned}$$

An integration of the above equation from 0 to t yields

$$\begin{aligned}
&\varPhi^{-1}(t)\left(v\left(t\right)-qv\left(t-r\right)\right)-\varPhi^{-1}(0)\left(v\left(0\right)-qv\left(0-r\right)\right)\\
&=\int_{0}^{t}\varPhi^{-1}(s)\left\{2A\left(s\right)qv\left(s-r\right)-b\left[f\left(v\left(s\right)+u^{*}\left(s\right)\right)-f\left(u^{*}\left(s\right)\right)\right]\right.\\
&\left.+bq\left[f\left(v\left(s-r\right)+u^{*}\left(s-r\right)\right)-f\left(u^{*}\left(s-r\right)\right)\right]\right\}ds. \qquad (6.27)
\end{aligned}$$

(6.27) can be expressed by

$$\begin{aligned}
v\left(t\right)&=\Phi(t,0)\left(v\left(0\right)-qv\left(0-r\right)\right)+qv\left(t-r\right)\\
&+\int_{0}^{t}\Phi(t,s)\left\{2A\left(s\right)qv\left(s-r\right)-b\left[f\left(v\left(s\right)+u^{*}\left(s\right)\right)-f\left(u^{*}\left(s\right)\right)\right]\right.\\
&\left.+bq\left[f\left(v\left(s-r\right)+u^{*}\left(s-r\right)\right)-f\left(u^{*}\left(s-r\right)\right)\right]\right\}ds,
\end{aligned}$$

then we have

$$\begin{aligned}
v\left(t\right)&=\Phi(t,0)\left(\psi\left(0\right)-q\psi\left(0-r\right)\right)+qv\left(t-r\right)\\
&+\int_{0}^{t}\Phi(t,s)\left\{2A\left(s\right)qv\left(s-r\right)-b\left[f\left(v\left(s\right)+u^{*}\left(s\right)\right)-f\left(u^{*}\left(s\right)\right)\right]\right.\\
&\left.+bq\left[f\left(v\left(s-r\right)+u^{*}\left(s-r\right)\right)-f\left(u^{*}\left(s-r\right)\right)\right]\right\}ds.
\end{aligned}$$

For all $\phi\in\mathbb{M}_{\psi}$, define the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\left(\mathcal{A}\phi\right)\left(t\right)=\begin{cases}0,\ t\in\left[-r,0\right],\\ \int_{0}^{t}\Phi(t,s)\left\{2A\left(s\right)q\phi\left(s-r\right)-b\left[f\left(\phi\left(s\right)+u^{*}\left(s\right)\right)-f\left(u^{*}\left(s\right)\right)\right]\right.\\ \left.+bq\left[f\left(\phi\left(s-r\right)+u^{*}\left(s-r\right)\right)-f\left(u^{*}\left(s-r\right)\right)\right]\right\}ds,\ t\geq 0,\end{cases}$$

and

$$\left(\mathcal{B}\phi\right)\left(t\right)=\begin{cases}\psi\left(t\right),\ t\in\left[-r,0\right],\\ \Phi(t,0)\left(\psi\left(0\right)-q\psi\left(-r\right)\right)+q\phi\left(t-r\right),\ t\geq 0.\end{cases}$$

(i) For all $x,y\in\mathbb{M}_{\psi}$, $x\left(t\right)\to 0$ and $y\left(t\right)\to 0$ as $t\to\infty$, then $\left(\mathcal{B}y\right)\left(t\right)\to 0$ and

$$\begin{aligned}
&\lim_{t\to\infty}\left(\mathcal{A}x\right)\left(t\right)\\
&=\lim_{t\to\infty}\left\{\varPhi\left(t\right)\int_{0}^{t}\varPhi^{-1}(s)\left\{2A\left(s\right)qx\left(s-r\right)-b\left[f\left(x\left(s\right)+u^{*}\left(s\right)\right)-f\left(u^{*}\left(s\right)\right)\right]\right.\right.\\
&\left.\left.+bq\left[f\left(x\left(s-r\right)+u^{*}\left(s-r\right)\right)-f\left(u^{*}\left(s-r\right)\right)\right]\right\}ds\right\}\\
&=0,
\end{aligned}$$

therefore $\lim_{t\to\infty}(\mathcal{A}x+\mathcal{B}y)(t)=0$. And

$$\begin{aligned}\|\mathcal{A}x\| &= \sup_{t\geq 0}\left|\int_0^t \Phi(t,s)\left\{2A(s)\,qx(s-r)-b\left[f(x(s)+u^*(s))-f(u^*(s))\right]\right.\right. \\ &\quad \left.\left. +bq\left[f(x(s-r)+u^*(s-r))-f(u^*(s-r))\right]\right\}ds\right| \\ &\leq \left\{2\|A\|\,|q|\sup_{t\geq -r}|x(t)|+b\left[\sup_{|u|\leq H+Q}|f(u)|+\sup_{|u|\leq H}|f(u)|\right]\right. \\ &\quad \left. +b\,|q|\left[\sup_{|u|\leq H+Q}|f(u)|+\sup_{|u|\leq H}|f(u)|\right]\right\}\sup_{t\geq 0}\left|\int_0^t\Phi(t,s)ds\right| \\ &\leq \lambda\left[2\|A\|\,|q|\,Q+b(1+|q|)\left(\sup_{|u|\leq H+Q}|f(u)|+\sup_{|u|\leq H}|f(u)|\right)\right],\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{B}y\| &= \sup_{t\geq -r}|(\mathcal{B}y)(t)| \\ &= \max\left\{\|\psi\|,\sup_{t\geq 0}|\Phi(t,0)(\psi(0)-q\psi(-r))+qy(t-r)|\right\} \\ &\leq \theta(1+|q|)\|\psi\|+\sup_{t\geq 0}|qy(t-r)| \\ &\leq \theta(1+|q|)\|\psi\|+|q|\,Q.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\mathcal{A}x+\mathcal{B}y\| \\ &\leq \|\mathcal{A}x\|+\|\mathcal{B}y\| \\ &\leq \lambda\left[2\|A\|\,|q|\,Q+b(1+|q|)\left(\sup_{|u|\leq H+Q}|f(u)|+\sup_{|u|\leq H}|f(u)|\right)\right] \\ &+\theta(1+|q|)\|\psi\|+|q|\,Q \\ &=(1+2\lambda\|A\|)\,|q|\,Q+\theta(1+|q|)\|\psi\| \\ &+\lambda b(1+|q|)\left(\sup_{|u|\leq H+Q}|f(u)|+\sup_{|u|\leq H}|f(u)|\right).\end{aligned}$$

According to the condition (6.26), $\|\mathcal{A}x+\mathcal{B}y\|\leq Q$. Thus, $\mathcal{A}x+\mathcal{B}y\in\mathbb{M}_\psi$.

(ii) For all $x\in\mathbb{M}_\psi$, $\|x\|\leq Q$. And

$$\left|(\mathcal{A}x)'(t)\right|=0,\ t\in[-r,0],$$

and

$$
\begin{aligned}
&\left|(\mathcal{A}x)'(t)\right| \\
&= \Big| A(t)\int_0^t \Phi(t,s)\left\{2A(s)\,qx(s-r) - b\left[f(x(s)+u^*(s)) - f(u^*(s))\right]\right. \\
&\left. +bq\left[f(x(s-r)+u^*(s-r)) - f(u^*(s-r))\right]\right\} ds \\
&+ \left\{2A(t)\,qx(t-r) - b\left[f(x(t)+u^*(t)) - f(u^*(t))\right]\right. \\
&\left. +bq\left[f(x(t-r)+u^*(t-r)) - f(u^*(t-r))\right]\right\}\Big| \\
&\le (1+\lambda\|A\|)\left[2\|A\|\,|q|\,Q + b(1+|q|)\left(\sup_{|u|\le H+Q}|f(u)| + \sup_{|u|\le H}|f(u)|\right)\right],
\end{aligned}
$$

here, the derivative of $(\mathcal{A}x)'(t)$ at zero means the left hand side derivative when $t \le 0$ and the right hand side derivative when $t \ge 0$. One can see $\left|(\mathcal{A}x)'(t)\right|$ is bounded for all $x \in \mathbb{M}_\psi$ and $\mathcal{A}\mathbb{M}_\psi$ is a precompact set of $\mathbb{S}$. Therefore $\mathcal{A}$ is compact.

Suppose $\{x_n\} \subset \mathbb{M}_\psi$, $x \in \mathbb{S}$, $x_n \to x$ as $n \to \infty$, then $|x_n(t) - x(t)| \to \infty$ uniformly for $t \ge -r$ as $n \to \infty$. Since

$$
\begin{aligned}
&\|\mathcal{A}x_n - \mathcal{A}x\| \\
&= \sup_{t\ge 0}\Big|\int_0^t \Phi(t,s)\left\{2A(s)\,q\left(x_n(s-r) - x(s-r)\right)\right. \\
&- b\left[f(x_n(s)+u^*(s)) - f(x(s)+u^*(s))\right] \\
&\left. +bq\left[f(x_n(s-r)+u^*(s-r)) - f(x(s-r)+u^*(s-r))\right]\right\} ds\Big| \\
&\le \lambda\Big[2\|A\|\,|q|\,\|x_n - x\| + b(1+|q|) \\
&\times \sup_{t\ge -r}|f(x_n(t)+u^*(t)) - f(x(t)+u^*(t))|\Big],
\end{aligned}
$$

and f is continuous, therefore$\|\mathcal{A}x_n - \mathcal{A}x\| \to 0$ as $n \to \infty$ and $\mathcal{A}$ is continuous.

(iii) For all $x, y \in \mathbb{M}_\psi$,

$$\|\mathcal{B}x - \mathcal{B}y\| = \sup_{t\ge 0}|qx(t-r) - qy(t-r)| \le |q|\,\|x-y\|,$$

and $|q| < 1$, therefore $\mathcal{B}$ is a contraction operator.

According to Krasnoselskii's fixed point theorem, there is a $\phi \in \mathbb{M}_\psi$ such that $(\mathcal{A}+\mathcal{B})\,\phi = \phi$ and ϕ is a solution for (6.21). Because the solution through ψ for the equation is unique, the solution $v(t) = \phi(t) \to 0$ as $t \to \infty$. ■

When f satisfies the locally Lipschitz condition, H in Theorem 6.1 and Q in Theorem 6.2 exists, there is a constant $R > 0$ such that

$$|f(v(t)+u^*(t)) - f(v(t))| < R\,|v(t)|.$$

Since ϕ satisfies

$$\begin{aligned}\phi(t) &= \Phi(t,0)(\psi(0) - q\psi(-r)) + q\phi(t-r) \\ &+ \int_0^t \Phi(t,s)\{2A(s)\, q\phi(s-r) - b[f(\phi(s) + u^*(s)) - f(u^*(s))] \\ &+ bq[f(\phi(s-r) + u^*(s-r)) - f(u^*(s-r))]\}ds,\end{aligned}$$

then

$$\|\phi\| \leq \theta(1+|q|)\|\psi\| + |q|\,\|\phi\| + \lambda[2\|A\|\,|q|\,\|\phi\| + b(1+|q|)\, R\,\|\phi\|],$$

that is

$$[1 - |q| - \lambda(2\|A\|\,|q| + b(1+|q|)\, R)]\,\|\phi\| \leq \theta(1+|q|)\,\|\psi\|\,.$$

Then there clearly exists a $\delta > 0$ for each $\epsilon > 0$ such that $|\phi(t)| < \epsilon$ for all $t \geq -r$ if $\|\psi\| < \delta$. Thus we have the following theorem.

Theorem 6.3 *If R satisfies*

$$1 - |q| - \lambda(2\|A\|\,|q| + b(1+|q|))\, R > 0.$$

Then the zero solution for (6.21) is stable.

Chapter 7

Existence and uniqueness of mild solutions for nonlinear hybrid Caputo fractional integro-differential equations via fixed point theorems

Keywords. Hybrid fractional integro-differential equations, fixed point theorems, existence, uniqueness.

This chapter has been extracted from the research paper [76],

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7.1 Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of fractional differential equations have received the attention of many authors, see [9], [10], [37], [56], [60], [82], [90], [93], [113], [123], [125] and the references therein.

Hybrid differential equations involve the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. This class of equations arises from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [56], [57], [59], [61], [113], [122].

Recently, Dhage [57] discussed the following first order hybrid differential equation with mixed perturbations of the second type

$$\begin{cases} \frac{d}{dt}\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right] = g\left(t, u\left(t\right)\right), \ t \in \left[t_0, t_0 + a\right], \\ u\left(t_0\right) = x_0 \in \mathbb{R}, \end{cases}$$

where $[t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, $f : [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ and $k, g : [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. He developed the theory of hybrid differential equations with mixed perturbations of the second type and provided some original and interesting results.

Zhao et al. [125] discussed the following boundary value problem of nonlinear fractional differential equations with mixed perturbations of the second type

$$\begin{cases} {}^C D^{\alpha}_{0^+}\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right] = g\left(t, u\left(t\right)\right), \ t \in J = \left[0, T\right], \\ a\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right]_{t=0} + b\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right]_{t=T} = c, \end{cases}$$

where $0 < \alpha \leq 1$, ${}^C D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ and $k, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, a, b and c are real constants with $a + b \neq 0$. They established an existence theorem for the boundary value problem under mixed Lipschitz and Carathéodory conditions by using the fixed point theorem in Banach algebra due to Dhage.

In [9], Ardjouni and Djoudi studied the existence and approximation of solutions for the following initial value problem of nonlinear hybrid Caputo fractional integro-differential equations

$$\begin{cases} {}^C D^{\alpha}_{0^+}\left(\frac{u(t)}{p(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s))ds}\right) = f\left(t, u\left(t\right)\right), \ t \in J = \left[0, a\right], \\ u\left(0\right) = p\left(0\right)\theta, \end{cases}$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\theta \in \mathbb{R}$, $g, f : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $p : J \to \mathbb{R}$ is a continuous function. By using the Dhage iteration principle, the authors obtained the existence and approximation of solutions under weaker partially continuity and partially compactness type conditions.

In this chapter, we discuss the existence and uniqueness of mild solutions for the following initial value problem of nonlinear hybrid first order Caputo fractional integro-differential equations

$$\begin{cases} {}^C D^{\alpha}_{0^+}\left(\frac{u(t)-f(t,u(t))}{p(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s))ds}\right) = h\left(t, u\left(t\right)\right), \ t \in \left[0, T\right], \\ u\left(0\right) = f\left(0, u\left(0\right)\right) + p\left(0\right)\theta, \end{cases} \tag{7.1}$$

where ${}^C D^{\alpha}_{0^+}$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\theta \in \mathbb{R}$, $p : [0, T] \to \mathbb{R}$ and $f, g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $p(t) + I^{\beta}_{0^+} g(t, u(t)) \neq 0$. To show the existence and uniqueness of mild solutions, we transform

(7.1) into an integral equation and then use the Krasnoselskii and Banach fixed point theorems. Also, we provide an example to illustrate our obtained results. Finally, we study the Higher order Caputo fractional integro-differential equations.

7.2 Preliminaries

Let $C([0,T],\mathbb{R})$ be the Banach space of all real-valued continuous functions defined on the compact interval $[0,T]$, endowed with the norm

$$\|u\| = \sup_{t\in[0,T]} |u(t)|.$$

$L^1([0,T],\mathbb{R})$ denotes the space of Lebesgue integrable functions on $[0,T]$ equipped with the norm $\|.\|_{L^1}$ defined by

$$\|u\|_{L^1} = \int_0^T |u(s)|\,ds.$$

We consider the following set of assumptions.

(A_1) There exists a constant $K_f > 0$ such that

$$|f(t,u) - f(t,v)| \leq K_f |u - v|$$

for all $t \in [0,T]$ and $u, v \in \mathbb{R}$.

(A_2) There exist functions $H, G \in L^1([0,T],\mathbb{R}_+)$ such that

$$|h(t,u)| \leq H(t) \text{ and } |g(t,u)| \leq G(t),\ t \in [0,T].$$

(A_3) There exists a constant $K_p > 0$ such that

$$|p(t_2) - p(t_1)| \leq K_p |t_2 - t_1| \text{ for all } t_1, t_2 \in [0,T].$$

(A_4) There exist constants $K_h, K_g > 0$ such that

$$|h(t,u) - h(t,v)| \leq K_h |u - v| \text{ and } |g(t,u) - g(t,v)| \leq K_g |u - v|$$

for all $t \in [0,T]$ and $u, v \in \mathbb{R}$.

We introduce some basic definitions and necessary lemmas related to fractional calculus and fixed point theorems that will be used throughout this chapter.

Definition 7.1 ([90]) The left sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : [0,T] \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds,$$

where Γ denotes the gamma function.

Definition 7.2 ([90]) Let $n-1<\alpha<n$. The left sided Riemann-Liouville fractional derivative of order α of a function $u:[0,T]\to\mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}u\left(t\right)=\frac{d^n}{dt^n}I_{0^+}^{n-\alpha}u\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^n}{dt^n}\int_0^t\left(t-s\right)^{n-\alpha-1}u\left(s\right)ds,\ t>0,$$

provided the right side integral is pointwise defined on $[0,T]$. In particular, if $0<\alpha<1$, then

$$D_{0^+}^{\alpha}u\left(t\right)=\frac{d}{dt}I_{0^+}^{1-\alpha}u\left(t\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\int_0^t\frac{u\left(s\right)}{\left(t-s\right)^{\alpha}}ds,\ t>0.$$

Definition 7.3 ([90]) Let $n-1<\alpha<n$. The left sided Caputo fractional derivative of order $\alpha>0$ of a function $u\in C^n([0,T],\mathbb{R})$ is given by

$${}^{C}D_{0^+}^{\alpha}x\left(t\right)=I_{0^+}^{n-\alpha}x^{(n)}\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_0^t\left(t-s\right)^{n-\alpha-1}x^{(n)}\left(s\right)ds,\ t>0.$$

In particular, if $0<\alpha<1$, then

$${}^{C}D_{0^+}^{\alpha}u\left(t\right)=I_{0^+}^{1-\alpha}u'\left(t\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\int_0^t\frac{u'\left(s\right)}{\left(t-s\right)^{\alpha}}ds,\ t>0.$$

Moreover, the Caputo derivative of a constant is equal to zero.

Lemma 7.1 ([90]) *Let $\alpha>0$ and $u\in C^n\left([0,T],\mathbb{R}\right)$. Then*
1) ${}^{C}D_{0^+}^{\alpha}I_{0^+}^{\alpha}u\left(t\right)=u\left(t\right)$.
2) $I_{0^+}^{\alpha}{}^{C}D_{0^+}^{\alpha}u\left(t\right)=u\left(t\right)-\sum\limits_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}t^k$.
In particular, when $\alpha\in\left(0,1\right)$, $I_{0^+}^{\alpha}{}^{C}D_{0^+}^{\alpha}u\left(t\right)=u\left(t\right)-u\left(0\right)$.

From the definition of the Caputo derivative, we can obtain the following lemma.

Lemma 7.2 ([90]) *Let $n-1<\alpha<n$ and $u\in C^n\left([0,T],\mathbb{R}\right)$. Then*

$$I_{0^+}^{\alpha}{}^{C}D_{0^+}^{\alpha}u\left(t\right)=u(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1},$$

for some $c_k\in\mathbb{R}$, $k=0,1,2,...,n-1$.
In particular, when $\alpha\in\left(0,1\right)$, $I_{0^+}^{\alpha}{}^{C}D_{0^+}^{\alpha}u\left(t\right)=u(t)+c_0$.

7.3 First order Caputo fractional integro-differential equations

In this section, we discuss the existence and uniqueness results for the initial value problems (7.1).

Let us start by defining what we mean by a mild solution of the problem (7.1).

Definition 7.4 A function $u \in C([0,T],\mathbb{R})$ is said to be a mild solution of the problem (7.1) if u satisfies the corresponding integral equation of (7.1).

For the existence and uniqueness of solutions for the problem (7.1), we need the following lemma.

Lemma 7.3 *$u \in C([0,T],\mathbb{R})$ is a mild solution of (7.1) if u satisfies*

$$\begin{aligned} u(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s))\,ds\right) \\ &\times\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,u(s))\,ds + \theta\right) + f(t,u(t)). \end{aligned} \tag{7.2}$$

Proof. Let u be a solution of the problem (7.1). Applying the Riemann-Liouville fractional integral I_{0+}^{α} on both sides of (7.1), by Lemma 7.2, then we obtain

$$\frac{u(t) - f(t,u(t))}{p(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s))\,ds} = I_{0+}^{\alpha} h(t,u(t)) + c,$$

for some $c \in \mathbb{R}$. So, we get

$$\begin{aligned} u(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s))\,ds\right) \\ &\times\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,u(s))\,ds + c\right) + f(t,u(t)). \end{aligned} \tag{7.3}$$

Substituting $t = 0$ in the above equality, we have

$$u(0) = p(0)\,c + f(0,u(0)).$$

The condition $u(0) = f(0,u(0)) + p(0)\,\theta$ implies that

$$c = \theta. \tag{7.4}$$

Substituting (7.4) in (7.3) we get the integral equation (7.2). ■

Now we will give the following existence and uniqueness theorems for the initial value problem (7.1).

Theorem 7.1 *Assume that hypotheses (A_1)-(A_3) hold. Furthermore, if*

$$K_f < 1, \tag{7.5}$$

then the initial value problem (7.1) has a mild solution defined on $[0,T]$.

Proof. Set $\mathbb{B} = C([0,T],\mathbb{R})$ and define a subset $\mathbb{M}$ of $\mathbb{B}$ by

$$\mathbb{M} = \{u \in \mathbb{B},\ \|u\| \le N\},$$

where

$$N = K_f N + F_0 + \left(K_p T + |p(0)| + \frac{T^{\beta} \|G\|_{L^1}}{\Gamma(\beta+1)}\right) \left(\frac{T^{\alpha} \|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right),$$

with $F_0 = \sup_{t\in[0,T]} |f(t,0)|$. Clearly, $\mathbb{M}$ is a closed, convex and bounded subset of the Banach space $\mathbb{B}$.

Define two operators $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{B}$ by

$$\begin{aligned}(\mathcal{A}u)(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s))\, ds\right) \\ &\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, u(s))\, ds + \theta\right),\ t \in [0,T], \end{aligned} \tag{7.6}$$

and

$$(\mathcal{B}u)(t) = f(t, u(t)),\ t \in [0,T]. \tag{7.7}$$

Now, (7.2) is equivalent to the operator equation

$$u(t) = (\mathcal{A}u)(t) + (\mathcal{B}u)(t),\ t \in [0,T].$$

We shall use Krasnoselskii's fixed point theorem to prove there exists at least one fixed point of the operator $\mathcal{A} + \mathcal{B}$ in $\mathbb{M}$. The proof will be given in several steps.

Step 1. We prove that $\mathcal{B}$ is a contraction with constant $K_f < 1$. Let $u, v \in \mathbb{M}$. Then by (A_1), we get

$$\begin{aligned}|(\mathcal{B}u)(t) - (\mathcal{B}v)(t)| &= |f(t,u(t)) - f(t,v(t))| \le K_f |u(t) - v(t)| \\ &\le K_f \|u - v\|\end{aligned}$$

for all $t \in [0,T]$. Taking supremum over t, then we have

$$\|\mathcal{B}u - \mathcal{B}v\| \le K_f \|u - v\|$$

for all $u, v \in \mathbb{M}$. Thus, by (7.5), $\mathcal{B}$ is a contraction operator on $\mathbb{M}$ with constant $K_f < 1$.

Step 2. We prove $\mathcal{A}$ is a compact operator on $\mathbb{M}$ into $\mathbb{B}$. It is enough to prove that $\mathcal{A}(\mathbb{M})$ is a uniformly bounded and equicontinuous set in $\mathbb{B}$. On the one hand, let $u \in \mathbb{M}$ be arbitrary. Then by (A_2), we get

$$\begin{aligned}|(\mathcal{A}u)(t)| &\le \left(|p(t)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |g(s,u(s))|\, ds\right) \\ &\quad\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s,u(s))|\, ds + |\theta|\right) \\ &\le \left(K_p t + |p(0)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |G(s)|\, ds\right) \\ &\quad\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |H(s)|\, ds + |\theta|\right) \\ &\le \left(K_p T + |p(0)| + \frac{T^{\beta} \|G\|_{L^1}}{\Gamma(\beta+1)}\right) \left(\frac{T^{\alpha} \|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right)\end{aligned}$$

for all $t \in [0, T]$. Taking supremum over t, we obtain

$$\|\mathcal{A}u\| \leq \left(K_p T + |p(0)| + \frac{T^{\beta} \|G\|_{L^1}}{\Gamma(\beta+1)}\right)\left(\frac{T^{\alpha} \|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right)$$

for all $u \in \mathbb{M}$. This shows that $\mathcal{A}(\mathbb{M})$ is uniformly bounded on $\mathbb{M}$.

On the other hand, let $t_1, t_2 \in [0, T]$ be arbitrary with $t_1 < t_2$. Then for any $u \in \mathbb{M}$, we get

$$\begin{aligned}
&|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| \\
&= \left| \left(p(t_2) + \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} g(s, u(s))\, ds\right) \right. \\
&\times \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} h(s, u(s))\, ds + \theta\right) \\
&- \left(p(t_1) + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} g(s, u(s))\, ds\right) \\
&\left. \times \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s, u(s))\, ds + \theta\right) \right| \\
&\leq \left(|p(t_2)| + \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} |g(s, u(s))|\, ds\right) \\
&\times \left(\frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} h(s, u(s))\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s, u(s))\, ds \right| \right) \\
&+ \left(|p(t_2) - p(t_1)| + \frac{1}{\Gamma(\beta)} \left| \int_0^{t_2} (t_2 - s)^{\beta-1} g(s, u(s))\, ds \right.\right. \\
&\left.\left. - \int_0^{t_1} (t_1 - s)^{\beta-1} g(s, u(s))\, ds \right| \right) \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} |h(s, u(s))|\, ds + |\theta|\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| \\
&\leq \left(|p(t_2)| + \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} G(s)\, ds\right) \frac{T^{\alpha}}{\Gamma(\alpha+1)} \left| \int_{t_1}^{t_2} H(s)\, ds \right| \\
&+ \left(K_p |t_2 - t_1| + \frac{T^{\beta}}{\Gamma(\beta+1)} \left| \int_{t_1}^{t_2} G(s)\, ds \right| \right) \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} H(s)\, ds + |\theta|\right) \\
&\leq \left(|p(t_2)| + \frac{T^{\beta} \|G\|_{L^1}}{\Gamma(\beta+1)}\right) \frac{T^{\alpha}}{\Gamma(\alpha+1)} \left| \int_{t_1}^{t_2} H(s)\, ds \right| \\
&+ \left(K_p |t_2 - t_1| + \frac{T^{\beta}}{\Gamma(\beta+1)} \left| \int_{t_1}^{t_2} G(s)\, ds \right| \right) \left(\frac{T^{\alpha} \|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right) \\
&= \left(|p(t_2)| + \frac{T^{\beta} \|G\|_{L^1}}{\Gamma(\beta+1)}\right) \frac{T^{\alpha}}{\Gamma(\alpha+1)} |\rho(t_2) - \rho(t_1)| \\
&+ \left(\frac{T^{\alpha} \|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right) \left(K_p |t_2 - t_1| + \frac{T^{\beta}}{\Gamma(\beta+1)} |\sigma(t_2) - \sigma(t_1)|\right),
\end{aligned}$$

where $\rho(t) = \int_0^t G(s)\, ds$ and $\sigma(t) = \int_0^t H(s)\, ds$. Since the functions ρ and σ are continuous on compact $[0,T]$, they are uniformly continuous. Hence, for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|t_2 - t_1| < \delta \Longrightarrow |(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| < \varepsilon$$

for all $t_1, t_2 \in [0,T]$ and $u \in \mathbb{M}$. This shows that $\mathcal{A}(\mathbb{M})$ is an equicontinuous set in $\mathbb{B}$. Now the set $\mathcal{A}(\mathbb{M})$ is uniformly bounded and equicontinuous set in $\mathbb{B}$, so it is a relatively compact by Arzela-Ascoli theorem. Thus, $\mathcal{A}$ is a compact operator on $\mathbb{M}$.

Step 3. We prove $\mathcal{A}$ is a continuous operator on $\mathbb{M}$ into $\mathbb{B}$. Let $\{u_n\}$ be a sequence in $\mathbb{M}$ converging to a point $u \in \mathbb{M}$. Then by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty} (\mathcal{A}u_n)(t) &= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \lim_{n\to\infty} g(s, u_n(s))\, ds\right)\\ &\quad\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lim_{n\to\infty} h(s, u_n(s))\, ds + \theta\right)\\ &= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s))\, ds\right)\\ &\quad\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, u(s))\, ds + \theta\right)\\ &= (\mathcal{A}u)(t)\end{aligned}$$

for all $t \in [0,T]$. This shows that $\{\mathcal{A}u_n\}$ converges to $\mathcal{A}u$ pointwise on $[0,T]$. Moreover, the sequence $\{\mathcal{A}u_n\}$ is equicontinuous by a similar proof of Step 2. Therefore $\{\mathcal{A}u_n\}$ converges uniformly to $\mathcal{A}u$ and hence $\mathcal{A}$ is a continuous operator on $\mathbb{M}$.

Step 4. We show $\mathcal{A}u + \mathcal{B}v \in \mathbb{M}$ for all $u, v \in \mathbb{M}$. For any $u, v \in \mathbb{M}$ and $t \in [0,T]$, we have

$$\begin{aligned}&|(\mathcal{A}u)(t) + (\mathcal{B}v)(t)|\\ &\le \left(|p(t)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |g(s, u(s))|\, ds\right)\\ &\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s, u(s))|\, ds + |\theta|\right) + |f(t, v(t))|\\ &\le \left(K_p t + |p(0)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} G(s)\, ds\right)\\ &\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H(s)\, ds + |\theta|\right) + |f(t, v(t)) - f(t,0)| + |f(t,0)|\\ &\le \left(K_p T + |p(0)| + \frac{T^\beta \|G\|_{L^1}}{\Gamma(\beta+1)}\right) \left(\frac{T^\alpha \|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right) + K_f \|v\| + F_0 \le N.\end{aligned}$$

This shows that $\mathcal{A}u + \mathcal{B}v \in \mathbb{M}$ for all $u, v \in \mathbb{M}$.

7.3. First order Caputo fractional integro-differential equations

Thus, all the conditions of Theorem 1.4 are satisfied and hence the operator equation $\mathcal{A}z + \mathcal{B}z = z$ has a solution in $\mathbb{M}$. Therefore, the initial value problem (7.1) has a mild solution defined on $[0, T]$. ■

Theorem 7.2 *Assume that (A_1)-(A_4) are satisfied and*

$$\left[\left(K_pT + |p(0)| + \frac{T^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)\frac{T^{\alpha}K_h}{\Gamma(\alpha+1)} + \left(\frac{T^{\alpha}\|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right)\frac{T^{\beta}K_g}{\Gamma(\beta+1)} + K_f\right] = \lambda < 1. \tag{7.8}$$

Then the initial value problem (7.1) has a unique mild solution defined on $[0, T]$.

Proof. From Theorem 7.1, it follows that the initial value problem (7.1) has a mild solution in $\mathbb{M}$. Hence, we need only to prove that the operator $\mathcal{A} + \mathcal{B}$ is a contraction on $\mathbb{M}$. In fact, for any $u, v \in \mathbb{M}$, we have

$$\begin{aligned}
&|((\mathcal{A}+\mathcal{B})u)(t) - ((\mathcal{A}+\mathcal{B})v)(t)| \\
&\leq \left(|p(t)| + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|g(s,u(s))|\,ds\right) \\
&\times \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s,u(s)) - h(s,v(s))|\,ds\right) \\
&+ \left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|g(s,u(s)) - g(s,v(s))|\,ds\right) \\
&\times \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s,v(s))|\,ds + |\theta|\right) \\
&+ |f(t,u(t)) - f(t,v(t))| \\
&\leq \left[\left(K_pT + |p(0)| + \frac{T^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)\frac{T^{\alpha}K_h}{\Gamma(\alpha+1)}\right. \\
&\left. + \left(\frac{T^{\alpha}\|H\|_{L^1}}{\Gamma(\alpha+1)} + |\theta|\right)\frac{T^{\beta}K_g}{\Gamma(\beta+1)} + K_f\right]\|u-v\|.
\end{aligned}$$

Thus,

$$\|(\mathcal{A}+\mathcal{B})u - (\mathcal{A}+\mathcal{B})v\| \leq \lambda\|u-v\|.$$

Hence, the operator $\mathcal{A} + \mathcal{B}$ is a contraction mapping by (7.8). Therefore, by Banach's fixed point theorem, the initial value problem (7.1) has a unique mild solution in $\mathbb{M}$. ■

Example 7.1 Let us consider the following initial value problem

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}\left(\frac{u(t) - \frac{1}{8}\sin u(t)}{\pi + \sin t + \frac{1/9}{\Gamma(1/3)}\int_0^t (t-s)^{-2/3}\sin u(s)ds}\right) = \frac{1}{7}\cos u(t),\ t \in [0,1], \\ u(0) = \frac{1}{8}\sin u(0) + \pi, \end{cases} \tag{7.9}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $T = 1$, $\theta = 1$, $f(t, u(t)) = \frac{1}{8}\sin u(t)$, $p(t) = \pi + \sin t$, $g(t, u(t)) = \frac{1}{9}\sin u(t)$, $h(t, u(t)) = \frac{1}{7}\cos u(t)$. Let $K_f = \frac{1}{8}$, $K_p = 1$, $G(t) = \frac{1}{9}$, $H(t) = \frac{1}{7}$. Then hypotheses (A_1)-(A_3) hold. Since

$$K_f = \frac{1}{8} < 1,$$

hence (7.5) holds. Therefore, by Theorem 7.1, the initial value problem (7.9) has a mild solution. Also, we have

$$K_g = \frac{1}{9},\ K_h = \frac{1}{7} \text{ and } \lambda \simeq 0.957 < 1,$$

then (A_4) and (7.8) hold. So, by Theorem 7.2, (7.9) has a unique mild solution.

7.4 Higher order fractional integro-differential equations

The method in Section 3 can be extended to the following initial value problem of nonlinear hybrid higher order Caputo fractional integro-differential equations

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}\left(\frac{u(t)-f(t,u(t))}{p(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s,u(s))ds}\right) = h(t, u(t)),\ t \in [0, T], \\ \left(\frac{u(t)-f(t,u(t))}{p(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s,u(s))ds}\right)^{(k)}\Bigg|_{t=0} = \theta_k,\ k = 0, ..., n-1, \end{cases} \tag{7.10}$$

where $\alpha \in (n-1, n)$, $\beta \in (n-1, n)$, $\theta_k \in \mathbb{R}$, $p : [0, T] \to \mathbb{R}$ and $f, g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $p(t) + I_{0^+}^{\beta} g(t, u(t)) \neq 0$.

Lemma 7.4 *$u \in C([0, T], \mathbb{R})$ is a mild solution of (7.10) if u satisfies*

$$\begin{aligned} u(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, u(s))\, ds\right) \\ &\times \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s, u(s))\, ds + \sum_{k=0}^{n-1}\frac{\theta_k}{k!}t^k\right) + f(t, u(t)). \end{aligned} \tag{7.11}$$

The proof is similar to that of Lemma 7.3 and hence, we omit it.

Theorem 7.3 *Suppose that hypotheses (A_1)-(A_3) and (7.5) hold. Then (7.10) has a mild solution.*

The proof is similar to that of Theorem 7.1 and hence, we omit it.

Theorem 7.4 *Suppose that (A_1)-(A_4) are satisfied and*

$$\begin{aligned} &\left[\left(K_p T + |p(0)| + \frac{T^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)\frac{T^{\alpha}K_h}{\Gamma(\alpha+1)}\right. \\ &\left. + \left(\frac{T^{\alpha}\|H\|_{L^1}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1}\frac{|\theta_k|}{k!}T^k\right)\frac{T^{\beta}K_g}{\Gamma(\beta+1)} + K_f\right] = \Lambda < 1. \end{aligned} \tag{7.12}$$

Then (7.10) has a unique mild solution.

The proof is similar to that of Theorem 7.2 and hence, we omit it.

7.4. Higher order fractional integro-differential equations

Chapter 8

Periodic solutions of almost linear Volterra integro-dynamic systems

Keywords. Volterra integro-dynamic systems, time scales, Krasnoselskii's fixed point theorem, periodic solutions.

This chapter has been extracted from the research paper [79],

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8.1 Introduction

Delay dynamic equations arise from a variety of applications including in various fields of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of delay dynamic equations have received the attention of many authors, see [1], [3], [19], [26], [40], [41], [45], [46], [47], [71], [91], [106] and the references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this chapter, we consider the following almost linear Volterra integro-dynamic system on time scales

$$\begin{cases} x^{\Delta}(t) = a(t)\, p(x(t)) + \int_{-\infty}^{t} C(t,s)\, h(y(s))\, \Delta s + e(t), \\ y^{\Delta}(t) = b(t)\, q(y(t)) + \int_{-\infty}^{t} D(t,s)\, g(x(s))\, \Delta s + f(t), \end{cases} \tag{8.1}$$

where a, b, e and f are rd-continuous functions, p, q, f and g are continuous functions. We assume that there exist constants P, Q, H, G and positive constants P^*, Q^*, H^*, G^* such that

$$|p(x) - Px| \le P^*, \;\; |q(x) - Qx| \le Q^*, \tag{8.2}$$

and

$$|h(x) - Hx| \le H^*, \;\; |g(x) - Gx| \le G^*. \tag{8.3}$$

To show the existence of periodic solutions of (8.1), we transform (8.1) into an integral system and then use Krasnoselskii's fixed point theorem. The obtained integral system is the sum of two mappings, one is a contraction and the other is compact. Our results generalize previous results due to Raffoul [106], from the one dimension to the two dimensions.

8.2 Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1], [3], [19], [26], [40], [41], [45], [46], [47], [71], [91], [106] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [40, 41, 91] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Kaufmann and Raffoul [88]. The following two definitions are borrowed from [88].

Definition 8.1 We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.

Example 8.1 The following time scales are periodic.
1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih],\ h > 0$ has period $\omega = 2h$.
2) $\mathbb{T} = h\mathbb{Z}$ has period $\omega = h$.
3) $\mathbb{T} = \mathbb{R}$.
4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $\omega = 1$.

Remark 8.1 ([88]) All periodic time scales are unbounded above and below.

Definition 8.2 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 8.2 ([88]) If $\mathbb{T}$ is a periodic time scale with period ω, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.

Definition 8.3 ([40]) A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 8.4 ([40]) For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of f on $\mathbb{T}^k$.

Definition 8.5 ([40]) A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \ \forall t \in \mathbb{T}\}.$$

Definition 8.6 ([40]) Let $p \in \mathcal{R}$, then the generalized exponential function e_p is defined as the unique solution of the initial value problem

$$x^{\Delta}(t) = p(t)x(t), \ x(s) = 1, \ \text{where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(v)}(p(v))\Delta v\right), \text{ for all } s, t \in \mathbb{T},$$

with

$$\xi_\mu(p) = \begin{cases} \frac{\log(1+\mu p)}{\mu} & \text{if } \mu \neq 0, \\ p & \text{if } \mu = 0, \end{cases}$$

where log is the principal logarithm function.

Lemma 8.1 ([40]) *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$,

(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Lemma 8.2 ([1]) *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \le \exp\left(\int_s^t p(v)\Delta v\right), \ \forall t \in \mathbb{T}.$$

8.3 Periodic Solutions

Let $\mathbb{T}$ be a periodic time scale with period ω. Let $T > 0$ be fixed, and if $\mathbb{T} \neq \mathbb{R}$, then $T = n\omega$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \le t \le b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly. Let P_T be the set of all continuous scalar functions, periodic of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{T}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

In this section we investigate the existence of a periodic solution of (8.1) using Krasnoselskii's fixed point theorem.

The next lemma is essential to our next results. Its proof can be found in [88].

Lemma 8.3 *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this section we assume that for all $(t,s) \in \mathbb{T}\times\mathbb{T}$,

$$\sup_{t\in\mathbb{T}} \int_{-\infty}^{t} |C(t,s)|\,\Delta s < \infty, \ \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} |D(t,s)|\,\Delta s < \infty. \tag{8.4}$$

We assume $a, b \in \mathcal{R}^+$ with $e_{\ominus(Pa)}(t, t-T) \neq 1$ and $e_{\ominus(Qb)}(t,t-T) \neq 1$. Suppose that

$$\begin{aligned} a(t+T) &= a(t),\ b(t+T) = b(t),\ e(t+T) = e(t),\ f(t+T) = f(t), \\ C(t+T, s+T) &= C(t,s),\ D(t+T,s+T) = D(t,s). \end{aligned} \tag{8.5}$$

Let $\mathbb{P}_T = P_T \times P_T$, then $\mathbb{P}_T$ is a Banach space when endowed with the maximum norm

$$\|(x,y)\| = \max\left\{\sup_{t\in[0,T]} |x(t)|, \sup_{t\in[0,T]} |y(t)|\right\}.$$

For any positive constant m the set

$$\mathbb{M} = \{(x,y) \in \mathbb{P}_T : \|(x,y)\| \le m\}. \tag{8.6}$$

is a bounded closed convex subset of $\mathbb{P}_T$.

Lemma 8.4 *If $(x, y) \in \mathbb{P}_T$, then (x, y) is a solution of (8.1) if and only if*

$$x(t) = \eta_1 \int_{t-T}^{t} \left[Pa\,(u)\, x^\sigma(u) + a\,(u)\, p\,(x(u)) + k\,(u)\right] e_{\ominus(Pa)}(t, u)\Delta u, \tag{8.7}$$

and

$$y(t) = \eta_2 \int_{t-T}^{t} \left[Qb\,(u)\, y^\sigma(u) + b\,(u)\, q\,(y(u)) + l\,(u)\right] e_{\ominus(Qa)}(t, u)\Delta u, \tag{8.8}$$

where

$$\eta_1 = \left[1 - e_{\ominus(Pa)}(T, 0)\right]^{-1}, \ \eta_2 = \left[1 - e_{\ominus(Qa)}(T, 0)\right]^{-1},$$

$$k\,(t) = e\,(t) + \int_{-\infty}^{t} C\,(t, s)\,[h\,(y\,(s)) - Hy\,(s)]\,\Delta s + \int_{-\infty}^{t} C\,(t, s)\, Hy\,(s)\,\Delta s,$$

and

$$l\,(t) = f\,(t) + \int_{-\infty}^{t} D\,(t, s)\,[g\,(x\,(s)) - Gx\,(s)]\,\Delta s + \int_{-\infty}^{t} D\,(t, s)\, Gx\,(s)\,\Delta s.$$

Proof. For convenience we put the first equation in (8.1) in the form

$$\begin{aligned} & x^\Delta\,(t) + Pa\,(t)\, x^\sigma(t) \\ & = Pa\,(t)\, x^\sigma(t) + a\,(t)\, p\,(x\,(t)) + e\,(t) \\ & + \int_{-\infty}^{t} C\,(t, s)\,[h\,(y\,(s)) - Hy\,(s)]\,\Delta s + \int_{-\infty}^{t} C\,(t, s)\, Hy\,(s)\,\Delta s. \end{aligned} \tag{8.9}$$

Let

$$k\,(t) = e\,(t) + \int_{-\infty}^{t} C\,(t, s)\,[h\,(y\,(s)) - Hy\,(s)]\,\Delta s + \int_{-\infty}^{t} C\,(t, s)\, Hy\,(s)\,\Delta s.$$

Then we may write (8.9) as

$$x^\Delta\,(t) + Pa\,(t)\, x^\sigma(t) = Pa\,(t)\, x^\sigma(t) + a\,(t)\, p\,(x\,(t)) + k\,(t)\,. \tag{8.10}$$

Let $x \in P_T$ and assume (8.5). Multiply both sides of (8.10) by $e_{Pa}(t, 0)$ and then integrate both sides from $t - T$ to t to obtain

$$\begin{aligned} & e_{Pa}(t, 0)x(t) - e_{Pa}(t - T, 0)x(t - T) \\ & = \int_{t-T}^{t} \left[Pa\,(u)\, x^\sigma(u) + a\,(u)\, p\,(x\,(u)) + k\,(u)\right] e_{Pa}(u, 0)\Delta u. \end{aligned}$$

Divide both sides of the above equation by $e_{Pa}(t, 0)$ and use the fact that $x(t-T) = x(t)$ to obtain

$$\begin{aligned} & x(t)\left[1 - e_{\ominus(Pa)}(t, t - T)\right] \\ & = \int_{t-T}^{t} \left[Pa\,(u)\, x^\sigma(u) + a\,(u)\, p\,(x\,(u)) + k\,(u)\right] e_{\ominus(Pa)}(t, u)\Delta u, \end{aligned}$$

where we have used Lemma 8.1 to simplify the exponentials. Since every step is reversible, the converse holds. The proof of (8.8) is similar and hence we omit it. ■

Define mappings $\mathcal{A}$ and $\mathcal{B}$ from $\mathbb{M}$ into $\mathbb{P}_T$ as follows. For $(\varphi_1, \varphi_2) \in \mathbb{M}$,

$$\mathcal{A}(\varphi_1, \varphi_2)(t) = (\mathcal{A}_1(\varphi_1, \varphi_2)(t), \mathcal{A}_2(\varphi_1, \varphi_2)(t)),$$

such that

$$\mathcal{A}_1(\varphi_1, \varphi_2)(t) = \eta_1 \left\{ \int_{t-T}^{t} a(u) \left[p(\varphi_1(u)) + P\varphi_1^{\sigma}(u) \right] e_{\ominus(Pa)}(t,u)\Delta u \right. \\ \left. + \int_{t-T}^{t} \int_{-\infty}^{u} C(t,s) \left[h(\varphi_2(s)) - H\varphi_2(s) \right] \Delta s e_{\ominus(Pa)}(t,u)\Delta u \right\},$$

$$\mathcal{A}_2(\varphi_1, \varphi_2)(t) = \eta_2 \left\{ \int_{t-T}^{t} b(u) \left[q(\varphi_2(u)) + Q\varphi_2^{\sigma}(u) \right] e_{\ominus(Qb)}(t,u)\Delta u \right. \\ \left. + \int_{t-T}^{t} \int_{-\infty}^{u} D(t,s) \left[g(\varphi_1(s)) - G\varphi_1(s) \right] \Delta s e_{\ominus(Qb)}(t,u)\Delta u \right\},$$

and for $(\psi_1, \psi_2) \in \mathbb{M}$,

$$\mathcal{B}(\psi_1, \psi_2)(t) = (\mathcal{B}_1(\psi_1, \psi_2)(t), \mathcal{B}_2(\psi_1, \psi_2)(t)),$$

such that

$$\mathcal{B}_1(\psi_1, \psi_2)(t) = \eta_1 \left\{ \int_{t-T}^{t} \int_{-\infty}^{u} C(u,s) H\psi_2(s) \Delta s e_{\ominus(Pa)}(t,u)\Delta u \right. \\ \left. + \int_{t-T}^{t} e(u) e_{\ominus(Pa)}(t,u)\Delta u \right\}.$$

$$\mathcal{B}_2(\psi_1, \psi_2)(t) = \eta_2 \left\{ \int_{t-T}^{t} \int_{-\infty}^{u} D(u,s) G\psi_1(s) \Delta s e_{\ominus(Qb)}(t,u)\Delta u \right. \\ \left. + \int_{t-T}^{t} f(u) e_{\ominus(Qb)}(t,u)\Delta u \right\}.$$

It can be easily verified that both $\mathcal{A}(\varphi_1, \varphi_2)$ and $\mathcal{B}(\psi_1, \psi_2)$ are T-periodic and continuous. Assume

$$|\eta_1| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} \int_{-\infty}^{u} |C(u,s)| \, |H| \, \Delta s e_{\ominus(Pa)}(t,u)\Delta u \le \alpha_1 < 1, \tag{8.11}$$

$$|\eta_2| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} \int_{-\infty}^{u} |D(u,s)| \, |G| \, \Delta s e_{\ominus(Qb)}(t,u)\Delta u \le \alpha_2 < 1, \tag{8.12}$$

$$|\eta_1| \sup_{t\in\mathbb{T}} \left\{ \int_{t-T}^{t} |a(u)| \, P^* e_{\ominus(Pa)}(t,u)\Delta u \right. \\ \left. + \int_{t-T}^{t} \int_{-\infty}^{u} |C(t,s)| \, H^* \Delta s e_{\ominus(Pa)}(t,u)\Delta u \right\} \\ \le \beta_1 < \infty, \tag{8.13}$$

and

$$|\eta_2| \sup_{t\in\mathbb{T}} \left\{ \int_{t-T}^{t} |b(u)| \, Q^* e_{\ominus(Qb)}(t,u)\Delta u \right.$$
$$\left. + \int_{t-T}^{t} \int_{-\infty}^{u} |D(t,s)| \, G^* \Delta s e_{\ominus(Qb)}(t,u)\Delta u \right\}$$
$$\leq \beta_2 < \infty. \tag{8.14}$$

Choose the constant m of (8.6) satisfying

$$|\eta_1| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} |e(u)| \, e_{\ominus(Pa)}(t,u)\Delta u + \alpha_1 m + \beta_1 \leq m, \tag{8.15}$$

and

$$|\eta_2| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} |f(u)| \, e_{\ominus(Qb)}(t,u)\Delta u + \alpha_2 m + \beta_2 \leq m. \tag{8.16}$$

Lemma 8.5 *Assume (8.4), (8.5) and (8.11)-(8.16) hold. Then $\mathcal{B}$ is a contraction from $\mathbb{M}$ into $\mathbb{M}$.*

Proof. For $(\psi_1, \psi_2) \in \mathbb{M}$,

$$|\mathcal{B}_1(\psi_1,\psi_2)(t)| \leq m\,|\eta_1| \int_{t-T}^{t} \int_{-\infty}^{u} |C(u,s)| \, |H| \, \Delta s e_{\ominus(Pa)}(t,u)\Delta u$$
$$+ |\eta_1| \int_{t-T}^{t} |e(u)| \, e_{\ominus(Pa)}(t,u)\Delta u$$
$$\leq |\eta_1| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} |e(u)| \, e_{\ominus(Pa)}(t,u)\Delta u + \alpha_1 m \leq m,$$

and

$$|\mathcal{B}_2(\psi_1,\psi_2)(t)| = m\,|\eta_2| \int_{t-T}^{t} \int_{-\infty}^{u} |D(u,s)| \, |G| \, \Delta s e_{\ominus(Qb)}(t,u)\Delta u$$
$$+ |\eta_2| \int_{t-T}^{t} |f(u)| \, e_{\ominus(Qb)}(t,u)\Delta u$$
$$\leq |\eta_2| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} |f(u)| \, e_{\ominus(Qb)}(t,u)\Delta u + \alpha_2 m \leq m,$$

then

$$\|\mathcal{B}(\psi_1,\psi_2)\| \leq m.$$

For $(\phi_1,\phi_2), (\psi_1,\psi_2) \in \mathbb{M}$, we obtain

$$|\mathcal{B}_1(\phi_1,\phi_2)(t) - \mathcal{B}_1(\psi_1,\psi_2)(t)|$$
$$\leq |\eta_1| \int_{t-T}^{t} \int_{-\infty}^{u} |C(u,s)| \, |H| \, |\phi_2(s) - \psi_2(s)| \, \Delta s e_{\ominus(Pa)}(t,u)\Delta u$$
$$\leq \alpha_1 \|(\phi_1,\phi_2) - (\psi_1,\psi_2)\|,$$

and in a similar way one can easily show that

$$|\mathcal{B}_2(\phi_1,\phi_2)(t) - \mathcal{B}_2(\psi_1,\psi_2)(t)| \le \alpha_2 \|(\phi_1,\phi_2)-(\psi_1,\psi_2)\|.$$

Therefore

$$\|\mathcal{B}(\phi_1,\phi_2)(t) - \mathcal{B}(\psi_1,\psi_2)(t)\| \le \alpha \|(\phi_1,\phi_2)-(\psi_1,\psi_2)\|.$$

where $\alpha = \max\{\alpha_1,\alpha_2\} < 1$. This proves that $\mathcal{B}$ is a contraction mapping from $\mathbb{M}$ into $\mathbb{M}$. ■

Lemma 8.6 *Assume (8.2), (8.3), (8.4), (8.5) and (8.13)-(8.16). Then $\mathcal{A}$ from $\mathbb{M}$ into $\mathbb{M}$ is continuous, and $\mathcal{A}\mathbb{M}$ is contained in a compact subset of $\mathbb{P}_T$.*

Proof. For any $(\varphi_1,\varphi_2)\in\mathbb{M}$, it follows from (8.2) and (8.3) that

$$\begin{aligned}
&|\mathcal{A}_1(\varphi_1,\varphi_2)(t)| \\
&\le |\eta_1| \left\{ \int_{t-T}^{t} |a(u)|\,|p(\varphi_1(u)) + P\varphi_1^{\sigma}(u)|\, e_{\ominus(Pa)}(t,u)\Delta u \right. \\
&\left. + \int_{t-T}^{t}\int_{-\infty}^{u} |C(t,s)|\,|h(\varphi_2(s)) - H\varphi_2(s)|\,\Delta s e_{\ominus(Pa)}(t,u)\Delta u \right\} \\
&\le |\eta_1| \left\{ \int_{t-T}^{t} |a(u)|\, P^* e_{\ominus(Pa)}(t,u)\Delta u \right. \\
&\left. + \int_{t-T}^{t}\int_{-\infty}^{u} |C(t,s)|\, H^* \Delta s e_{\ominus(Pa)}(t,u)\Delta u \right\},
\end{aligned}$$

Using (8.13) and (8.15), we get

$$|\mathcal{A}_1(\varphi_1,\varphi_2)(t)| \le \beta_1 \le m.$$

and in a similar way we have

$$|\mathcal{A}_2(\varphi_1,\varphi_2)(t)| \le \beta_2 \le m.$$

Therefore

$$\|\mathcal{A}(\varphi_1,\varphi_2)\| \le m. \tag{8.17}$$

So, $\mathcal{A}$ maps $\mathbb{M}$ into $\mathbb{M}$, and the set $\{\mathcal{A}(\phi_1,\phi_2)\}$ for $(\phi_1,\phi_2)\in\mathbb{M}$ is uniformly bounded. To show that $\mathcal{A}$ is a continuous we let $\{(\phi_1^n,\phi_2^n)\}$ be any sequence of functions in $\mathbb{M}$ with $\|(\phi_1^n,\phi_2^n)-(\phi_1,\phi_2)\|\to 0$ as $n\to\infty$. Since $\mathbb{M}$ is closed, we have $(\phi_1,\phi_2)\in\mathbb{M}$. Then by the definition of $\mathcal{A}$ we have

$$\begin{aligned}
&\|\mathcal{A}(\phi_1^n,\phi_2^n) - \mathcal{A}(\phi_1,\phi_2)\| \\
&= \max\left\{ \sup_{t\in[0,T]} |\mathcal{A}_1(\phi_1^n,\phi_2^n)(t) - \mathcal{A}_1(\phi_1,\phi_2)(t)|, \right. \\
&\left. \sup_{t\in[0,T]} |\mathcal{A}_2(\phi_1^n,\phi_2^n)(t) - \mathcal{A}_2(\phi_1,\phi_2)(t)| \right\},
\end{aligned}$$

in which

$$\begin{aligned}
&\left|\mathcal{A}_1\left(\phi_1^n, \phi_2^n\right)(t)-\mathcal{A}_1\left(\phi_1, \phi_2\right)(t)\right| \\
&=\left|\eta_1\left\{\int_{t-T}^t a(u)\left[p\left(\phi_1^n(u)\right)+P \phi_1^{n \sigma}(u)\right] e_{\ominus(P a)}(t, u) \Delta u\right.\right. \\
&-\int_{t-T}^t a(u)\left[p\left(\phi_1(u)\right)+P \phi_1^\sigma(u)\right] e_{\ominus(P a)}(t, u) \Delta u \\
&+\int_{t-T}^t \int_{-\infty}^t C(t, s)\left[h\left(\phi_2^n(s)\right)-H \phi_2^n(s)\right] \Delta s e_{\ominus(P a)}(t, u) \Delta u \\
&\left.\left.-\int_{t-T}^t \int_{-\infty}^t C(t, s)\left[h\left(\phi_2(s)\right)-H \phi_2(s)\right] \Delta s e_{\ominus(P a)}(t, u) \Delta u\right\}\right| \\
&\leq\left|\eta_1\right|\left\{\int_{t-T}^t|a(u)|\left[\left|p\left(\phi_1^n(u)\right)-p\left(\phi_1(u)\right)\right|\right.\right. \\
&\left.+\left|P \phi_1^{n \sigma}(u)-P \phi_1^\sigma(u)\right|\right] e_{\ominus(P a)}(t, u) \Delta u \\
&+\int_{t-T}^t \int_{-\infty}^t|C(t, s)|\left[\left|h\left(\phi_2^n(s)\right)-h\left(\phi_2(s)\right)\right|\right. \\
&\left.\left.+\left|H \phi_2^n(s)-H \phi_2(s)\right|\right] \Delta s e_{\ominus(P a)}(t, u) \Delta u\right\} .
\end{aligned}$$

The continuity of p and h along with the Lebesgue dominated convergence theorem implies that

$$\sup_{t \in[0, T]}\left|\mathcal{A}_1\left(\phi_1^n, \phi_2^n\right)(t)-\mathcal{A}_1\left(\phi_1, \phi_2\right)(t)\right| \rightarrow 0 \text{ as } n \rightarrow \infty .$$

By a similar argument one can easily argue that

$$\sup_{t \in[0, T]}\left|\mathcal{A}_2\left(\phi_1^n, \phi_2^n\right)(t)-\mathcal{A}_2\left(\phi_1, \phi_2\right)(t)\right| \rightarrow 0 \text{ as } n \rightarrow \infty .$$

Thus,

$$\left\|\mathcal{A}\left(\phi_1^n, \phi_2^n\right)-\mathcal{A}\left(\phi_1, \phi_2\right)\right\| \rightarrow 0 \text{ as } n \rightarrow \infty .$$

This proves that $\mathcal{A}$ is a continuous mapping.

It is trivial to show that for all $(\phi_1, \phi_2) \in \mathbb{M}$, there exist constants $L_1, L_2>0$ such that $\left|\mathcal{A}_1\left(\phi_1, \phi_2\right)^\Delta(t)\right| \leq L_1$ and $\left|\mathcal{A}_2\left(\phi_1, \phi_2\right)^\Delta(t)\right| \leq L_2$. This means $\left|\mathcal{A}\left(\phi_1, \phi_2\right)^\Delta(t)\right| \leq L$ where $L=\max\{L_1, L_2\}$. Therefore that the set $\{\mathcal{A}(\phi_1, \phi_2)\}$ for $(\phi_1, \phi_2) \in \mathbb{M}$ is equicontinuous. Hence, by the Arzela-Ascoli theorem, $A\mathbb{M}$ is contained in a compact subset of $\mathbb{P}_T$. ∎

Theorem 8.1 *Suppose the assumptions of Lemmas 8.5 and 8.6 hold. Then (8.1) has a continuous T-periodic solution.*

Proof. For $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \mathbb{M}$, we get

$$\begin{aligned}
&|\mathcal{A}_1(\varphi_1, \varphi_2)(t) + \mathcal{B}_1(\psi_1, \psi_2)(t)| \\
&= \left| \eta_1 \left\{ \int_{t-T}^{t} a(u)\left[p(\varphi_1(u)) + P\varphi_1^\sigma(u)\right] e_{\ominus(Pa)}(t,u)\Delta u \right.\right. \\
&\left. + \int_{t-T}^{t} \int_{-\infty}^{u} C(t,s)\left[h(\varphi_2(s)) - H\varphi_2(s)\right] \Delta s e_{\ominus(Pa)}(t,u)\Delta u \right\} \\
&+ \eta_1 \left\{ \int_{t-T}^{t} \int_{-\infty}^{u} C(u,s) H\psi_2(s) \Delta s e_{\ominus(Pa)}(t,u)\Delta u \right. \\
&\left.\left. + \int_{t-T}^{t} e(u) e_{\ominus(Pa)}(t,u)\Delta u \right\} \right| \\
&\leq |\eta_1| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} |e(u)| e_{\ominus(Pa)}(t,u)\Delta u + \alpha_1 m + \beta_1 \\
&\leq m.
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{A}_2(\varphi_1, \varphi_2)(t) + \mathcal{B}_2(\psi_1, \psi_2)(t)| \\
&\leq |\eta_1| \sup_{t\in\mathbb{T}} \int_{t-T}^{t} |f(u)| e_{\ominus(Qb)}(t,u)\Delta u + \alpha_2 m + \beta_2 \\
&\leq m.
\end{aligned}$$

This implies that

$$\|\mathcal{A}(\varphi_1, \varphi_2) + \mathcal{B}(\psi_1, \psi_2)\| \leq m,$$

which proves that $\mathcal{A}(\varphi_1, \varphi_2) + \mathcal{B}(\psi_1, \psi_2) \in \mathbb{M}$.

Therefore, by Krasnoselskii's theorem there exists a function (x, y) in $\mathbb{M}$ such that

$$(x, y) = \mathcal{A}(x, y) + \mathcal{B}(x, y).$$

This proves that (8.1) has a continuous T-periodic solution (x, y). ■

Chapter 9

Existence and uniqueness of periodic solutions in neutral nonlinear summation-difference systems with infinite delay

Keywords. Krasnoselskii's theorem, Contraction, Neutral difference equation, Periodic solution, Fundamental matrix solution.

This chapter has been extracted from the research paper [77],

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9.1 Introduction

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential and difference equations, see the references [11], [99], [109], [112] and references therein. In this chapter, we study the existence and uniqueness of periodic solutions of the nonlinear neutral summation-difference system with infinite delay

$$\begin{aligned}\Delta x(n) = P(n) + A(n)\,x(n-\tau(n)) \\ + \Delta Q(n, x(n-g(n))) + \sum_{k=-\infty}^{n} D(n,k)\, f(x(k)),\end{aligned} \tag{9.1}$$

where A and D are $N \times N$ sequence matrices on $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$, respectively, $P : \mathbb{Z} \to \mathbb{R}^N$ is a sequence vector, $\tau, g : \mathbb{Z} \to \mathbb{Z}^+$ are scalar sequences and the functions $f : \mathbb{R}^N \to \mathbb{R}^N$

and $Q : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous in x. The sets $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the integers and the nonnegative integers, respectively. For more details on the calculus of difference equations, we refer the reader to [66] and [89]. In this analysis we use the fundamental matrix solution of $\Delta x(n) = A(n)x(n)$ to invert the system (9.1). Then we employ Krasnoselskii's fixed point theorem to show the existence of periodic solutions of system (9.1). The obtained mapping is the sum of two mappings, one is a compact operator and the other is a contraction. Also, transforming system (9.1) to a fixed point problem enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

9.2 Preliminaries

For the definitions of the different notions used throughout this chapter we refer, for example [66], [89], [99], [112]. For $T > 1$ define

$$C_T = \left\{\phi \in C\left(\mathbb{Z}, \mathbb{R}^N\right),\ \phi(n+T) = \phi(n)\right\},$$

where $C\left(\mathbb{Z}, \mathbb{R}^N\right)$ is the space of all N-vector sequences. Then C_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{n\in\mathbb{Z}} |x(n)| = \max_{n\in[0,T-1]\cap\mathbb{Z}} |x(n)|.$$

Note that C_T is equivalent to the Euclidean space $\mathbb{R}^{NT}$, where $|.|$ denotes the infinity norm for $x \in \mathbb{R}^N$. Also, if A is an $N \times N$ real matrix, then we define the norm of A by

$$|A| = \max_{1\le i\le N} \sum_{j=1}^{N} |a_{ij}|.$$

Definition 9.1 If the matrix B is periodic of period T, then the linear system

$$y(n+1) = B(n)y(n), \tag{9.2}$$

is said to be noncritical with respect to T, if it has no periodic solution of period T except the trivial solution $y = 0$.

In this chapter we assume that

$$\begin{aligned} A(n+T) &= A(n),\ D(n+T,k+T) = D(n,k), \\ \tau(n+T) &= \tau(n) \ge \tau^* > 0,\ g(n+T) = g(n) \ge g^* > 0, \end{aligned} \tag{9.3}$$

where τ^* and g^* are constants. For $n \in \mathbb{Z}$, $x, y \in \mathbb{R}^N$, the function $Q(n,x)$ is periodic in n of period T, that is

$$Q(n+T,x) = Q(n,x). \tag{9.4}$$

The functions Q and f are globally Lipschitz continuous. That is there are positive constants k_1 and k_2 such that

$$|Q(n,x) - Q(n,y)| \leq k_1 |x-y|, \tag{9.5}$$

$$|f(x) - f(y)| \leq k_2 |x-y|, \ f(0) = 0. \tag{9.6}$$

Also, there is a positive constant k_3 such that

$$\sum_{k=-\infty}^{n} |D(n,k)| \leq k_3 < \infty. \tag{9.7}$$

Throughout this chapter it is assumed that the matrix $B(n) = I + A(n)$ is nonsingular and the system (9.2) is noncritical, where I is the $N \times N$ identity matrix. Also, if x is a sequence, then the forward operator $\mathbb{E}$ is defined as $\mathbb{E}x(n) = x(n+1)$. Now, we state some known results about system (9.2). Let K represent the fundamental matrix of (9.2) with $K(0) = I$, then

a) $\det K(n) \neq 0$.

b) $K(n+T) = B(n)K(n)$ and $K^{-1}(n+T) = K^{-1}(n)B^{-1}(n)$.

c) System (9.2) is noncritical if and only if $\det(I - K(T)) \neq 0$.

d) There exists a nonsingular matrix L such that

$$K(n+T) = B(n)K(n)L^T, \ K^{-1}(n+T) = L^{-T}K^{-1}(n).$$

The following lemma is fundamental to our results.

Lemma 9.1 *Suppose (9.3) and (9.4) hold. If $x \in C_T$, then x is a solution of the equation (9.1) if and only if*

$$\begin{aligned} x(n) &= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \\ &+ \sum_{s=n}^{n+T-1} \Theta(n,s) \Bigg[P(s) + A(s) \Bigg(Q(s, x(s-g(s))) - \sum_{t=s-\tau(s)}^{s-1} A(t)x(t) \Bigg) \\ &+ U(s)x(s-\tau(s)) + \sum_{k=-\infty}^{s} D(s,k) f(x(k)) \Bigg], \end{aligned} \tag{9.8}$$

where

$$\Theta(n,s) = K(n) \left(K(T)^{-1} - I \right)^{-1} K^{-1}(s) \left(I - A(s) B^{-1}(s) \right),$$

and

$$U(s) = A(s) - (1 - \Delta\tau(s)) A(s - \tau(s)).$$

Proof. Let $x \in C_T$ be a solution of (9.1) and K is a fundamental matrix of solutions for (9.2). Rewrite the equation (9.1) as

$$\begin{aligned}
\Delta x(n) &= P(n) + A(n)x(n) - A(n)x(n) + A(n)\,x(n-\tau(n)) \\
&\quad + \Delta Q(n, x(n-g(n))) + \sum_{k=-\infty}^{n} D(n,k)\, f(x(k)) \\
&= P(n) + A(n)x(n) - \Delta_n \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \\
&\quad + [A(n) - (1-\Delta\tau(n))\, A(n-\tau(n))]\, x(n-\tau(n)) \\
&\quad + \Delta Q(n, x(n-g(n))) + \sum_{k=-\infty}^{n} D(n,k)\, f(x(k)).
\end{aligned}$$

We put $A(n) - (1-\Delta\tau(n))\, A(n-\tau(n)) = U(n)$, we obtain

$$\begin{aligned}
&\Delta \left[x(n) - Q(n, x(n-g(n))) + \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \right] \\
&= P(n) + A(n) \left[x(n) - Q(n, x(n-g(n))) + \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \right] \\
&+ A(n) \left[Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \right] \\
&+ U(n)x(n-\tau(n)) + \sum_{k=-\infty}^{n} D(n,k)\, f(x(k)).
\end{aligned}$$

Since $K(n)K^{-1}(n) = I$, it follows that

$$\begin{aligned}
0 &= \Delta\left[K(n)K^{-1}(n)\right] = \Delta K(n)\mathbb{E}K^{-1}(n) + K(n)\Delta K^{-1}(n) \\
&= A(n)\, K(n)K^{-1}(n)B^{-1}(n) + K(n)\Delta K^{-1}(n) \\
&= A(n)\, B^{-1}(n) + K(n)\Delta K^{-1}(n).
\end{aligned}$$

This implies

$$\Delta K^{-1}(n) = -K^{-1}(n)A(n)\, B^{-1}(n). \tag{9.9}$$

If x is a solution of (9.1) with $x(0) = x_0$, then

$$\begin{aligned}
&\Delta\left[K^{-1}(n)\left(x\left(n\right)-Q\left(n,x\left(n-g\left(n\right)\right)\right)+\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\right]\\
&=\Delta K^{-1}(n)\mathbb{E}\left[x\left(n\right)-Q\left(n,x\left(n-g\left(n\right)\right)\right)+\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right]\\
&+K^{-1}(n)\Delta\left[x\left(n\right)-Q\left(n,x\left(n-g\left(n\right)\right)\right)+\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&\Delta\left[K^{-1}(n)\left(x\left(n\right)-Q\left(n,x\left(n-g\left(n\right)\right)\right)+\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\right]\\
&=-K^{-1}(n)A\left(n\right)B^{-1}(n)\\
&\times\left[P\left(n\right)+B(n)\left(x\left(n\right)-Q\left(n,x\left(n-g\left(n\right)\right)\right)+\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\right.\\
&+A(n)\left(Q\left(n,x\left(n-g\left(n\right)\right)\right)-\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\\
&\left.+U(n)x\left(n-\tau\left(n\right)\right)+\sum_{k=-\infty}^{n}D\left(n,k\right)f\left(x\left(k\right)\right)\right]\\
&+K^{-1}(n)A(n)\left(x\left(n\right)-Q\left(n,x\left(n-g\left(n\right)\right)\right)+\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\\
&+K^{-1}(n)\left[P\left(n\right)+A(n)\left(Q\left(n,x\left(n-g\left(n\right)\right)\right)-\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\right.\\
&\left.+U(n)x\left(n-\tau\left(n\right)\right)+\sum_{k=-\infty}^{n}D\left(n,k\right)f\left(x\left(k\right)\right)\right]\\
&=K^{-1}(n)\left(I-A\left(n\right)B^{-1}(n)\right)\\
&\times\left[P\left(n\right)+A(n)\left(Q\left(n,x\left(n-g\left(n\right)\right)\right)-\sum_{t=n-\tau(n)}^{n-1}A(t)x(t)\right)\right.\\
&\left.+U(n)x\left(n-\tau\left(n\right)\right)+\sum_{k=-\infty}^{n}D\left(n,k\right)f\left(x\left(k\right)\right)\right].
\end{aligned}$$

Summing of the above equation from 0 to $n-1$ yields

$$\begin{aligned} x(n) &= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \\ &+ K(n)\left(x(0) - Q(0, x(0-g(0))) + \sum_{t=-\tau(0)}^{-1} A(t)x(t)\right) \\ &+ K(n)\sum_{s=0}^{n-1} K^{-1}(s)\left(I - A(s)B^{-1}(s)\right)[P(s) \\ &+ A(s)\left(Q(s, x(s-g(s))) - \sum_{t=s-\tau(s)}^{s-1} A(t)x(t)\right) \\ &+ U(s)x(s-\tau(s)) + \sum_{k=-\infty}^{s} D(s,k) f(x(k))\Bigg]. \end{aligned} \tag{9.10}$$

For the sake of simplicity, we let

$$H(s) = \left(I - A(s)B^{-1}(s)\right)\Bigg[P(s) + A(s)\left(Q(s, x(s-g(s))) - \sum_{t=s-\tau(s)}^{s-1} A(t)x(t)\right) \\ + U(s)x(s-\tau(s)) + \sum_{k=-\infty}^{s} D(s,k) f(x(k))\Bigg].$$

Since $x(T) = x_0 = x(0)$, using (9.10) we get

$$\begin{aligned} &x(0) - Q(0, x(0-\tau(0))) + \sum_{t=-\tau(0)}^{-1} A(t)x(t) \\ &= (I - K(T))^{-1} \sum_{s=0}^{T-1} K(T)K^{-1}(s)H(s). \end{aligned} \tag{9.11}$$

A substitution of (9.11) into (9.10) yields

$$\begin{aligned} x(n) &= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \\ &+ K(n)(I - K(T))^{-1} \sum_{s=0}^{T-1} K(T)K^{-1}(s)H(s) \\ &+ \sum_{s=0}^{n-1} K(n)K^{-1}(s)H(s). \end{aligned} \tag{9.12}$$

It remains to show that expression (9.12) is equivalent to equation (9.8). Since

$$(I - K(T))^{-1} = \left(K(T)\left(K(T)^{-1} - I\right)\right)^{-1} = \left(K(T)^{-1} - I\right)^{-1} K^{-1}(T),$$

then the equations (9.12) becomes

$$\begin{aligned}
x(n) &= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \\
&+ K(n)\left(K(T)^{-1} - I\right)^{-1} \sum_{s=0}^{T-1} K^{-1}(s)H(s) + \sum_{s=0}^{n-1} K(n)K^{-1}(s)H(s) \\
&= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) + K(n)\left(K(T)^{-1} - I\right)^{-1} \\
&\times \left\{ \sum_{s=0}^{T-1} K^{-1}(s)H(s) + \sum_{s=0}^{n-1} K(T)^{-1}K^{-1}(s)H(s) - \sum_{s=0}^{n-1} K^{-1}(s)H(s) \right\} \\
&= Q(n, x(n-g(n))) + K(n)\left(K(T)^{-1} - I\right)^{-1} \\
&\times \left\{ -\sum_{s=T}^{n-1} K^{-1}(s)H(s) + \sum_{s=0}^{n-1} K(T)^{-1}K^{-1}(s)H(s) \right\}.
\end{aligned}$$

By letting $s = u - T$ in the third term on the right side of the above expression, we end up with

$$\begin{aligned}
x(n) &= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) + K(n)\left(K(T)^{-1} - I\right)^{-1} \\
&\times \left\{ -\sum_{s=T}^{n-1} K^{-1}(s)H(s) + \sum_{u=T}^{T+n-1} K(T)^{-1}K^{-1}(u-T)H(u-T) \right\}. \quad (9.13)
\end{aligned}$$

By (d) we have $K(n-T) = K(n)L^{-T}$ and $K(T) = L^T$. Hence,

$$K^{-1}(T)K^{-1}(u-T) = K^{-1}(u).$$

Moreover, since $H(u-T) = H(u)$ then the expression (9.13) becomes

$$\begin{aligned}
x(n) &= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) + K(n)\left(K(T)^{-1} - I\right)^{-1} \\
&\times \left\{ -\sum_{s=T}^{n-1} K^{-1}(s)H(s) + \sum_{s=T}^{n+T-1} K^{-1}(s)H(s) \right\} \\
&= Q(n, x(n-g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)x(t) \\
&+ K(n)\left(K(T)^{-1} - I\right)^{-1} \sum_{s=n}^{n+T-1} K^{-1}(s)H(s).
\end{aligned}$$

The converse implication is easily obtained and the proof is complete. ■

9.2. Preliminaries

9.3 Existence and uniqueness of periodic solutions

By applying Theorems 1.4 and 1.2, we obtain in this section the existence and the uniqueness of the periodic solution of (9.1). So, let a Banach space $(C_T, \|.\|)$, a closed bounded convex subset of C_T,

$$\mathbb{M} = \{\varphi \in C_T,\ \|\varphi\| \leq J\}, \tag{9.14}$$

with $J > 0$, and by the Lemma 9.1, let a mapping $\mathcal{F}$ given by

$$\begin{aligned}(\mathcal{F}\varphi)(n) &= Q(n, \varphi(n - g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)\varphi(t) \\ &+ \sum_{s=n}^{n+T-1} \Theta(n,s)\left[P(s) + A(s)\left(Q(s, \varphi(s - g(s))) - \sum_{t=s-\tau(s)}^{s-1} A(t)\varphi(t)\right)\right. \\ &\left. + U(s)\varphi(s - \tau(s)) + \sum_{k=-\infty}^{s} D(s,k) f(\varphi(k))\right].\end{aligned} \tag{9.15}$$

Therefore, we express equation (9.15) as

$$\mathcal{F}\varphi = \mathcal{R}\varphi + \mathcal{S}\varphi,$$

where $\mathcal{R}$ and $\mathcal{S}$ are given by

$$\begin{aligned}(\mathcal{R}\varphi)(n) &= \sum_{s=n}^{n+T-1} \Theta(n,s)\left[P(s) + A(s)\left(Q(s, \varphi(s - g(s))) - \sum_{t=s-\tau(s)}^{s-1} A(t)\varphi(t)\right)\right. \\ &\left. + U(s)\varphi(s - \tau(s)) + \sum_{k=-\infty}^{s} D(s,k) f(\varphi(k))\right],\end{aligned} \tag{9.16}$$

and

$$(\mathcal{S}\varphi)(n) = Q(n, \varphi(n - g(n))) - \sum_{t=n-\tau(n)}^{n-1} A(t)\varphi(t). \tag{9.17}$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 1.4. So that, since $\varphi \in C_T$, (9.3) and (9.4) hold, we have for $\varphi \in \mathbb{M}$

$$(\mathcal{R}\varphi)(n+T) = (\mathcal{R}\varphi)(n) \text{ and } \mathcal{R}\varphi \in C\left(\mathbb{Z}, \mathbb{R}^N\right) \Longrightarrow \mathcal{R}(\mathbb{M}) \subset C_T, \tag{9.18}$$

and

$$(\mathcal{S}\varphi)(n+T) = (\mathcal{S}\varphi)(n) \text{ and } \mathcal{S}\varphi \in C\left(\mathbb{Z}, \mathbb{R}^N\right) \Longrightarrow \mathcal{S}(\mathbb{M}) \subset C_T. \tag{9.19}$$

The next lemma plays an important role in the compactness of $\mathcal{R}$.

Lemma 9.2 *Suppose (9.3)–(9.7) hold. If $\mathcal{R}$ is defined by (9.16), then $\mathcal{R}$ is continuous and the image of $\mathcal{R}$ is contained in a compact set.*

Proof. Let $\varphi_{\mathcal{N}} \in \mathbb{M}$ where $\mathcal{N}$ is a positive integer such that $\varphi_{\mathcal{N}} \to \varphi$ as $\mathcal{N} \to \infty$. Then

$$\begin{aligned}
&\left|(\mathcal{R}\varphi_{\mathcal{N}})(n) - (\mathcal{R}\varphi)(n)\right| \\
&\leq \sum_{s=n}^{n+T-1} |\Theta(n,s)| \left[|A(s)| \left(\sum_{t=s-\tau(s)}^{s-1} |A(t)| \left|\varphi_{\mathcal{N}}(t) - \varphi(t)\right| \right.\right. \\
&+ \left|Q(s, \varphi_{\mathcal{N}}(s - g(s))) - Q(s, \varphi(s - g(s)))\right| \\
&+ |U(s)| \left|\varphi_{\mathcal{N}}(s - \tau(s)) - \varphi(s - \tau(s))\right| \\
&\left.+ \sum_{k=-\infty}^{s} |D(s,k)| \left|f(\varphi_{\mathcal{N}}(k)) - f(\varphi(k))\right| \right].
\end{aligned}$$

Since Q and f are continuous, the Dominated Convergence Theorem implies,

$$\lim_{\mathcal{N}\to\infty} \left|(\mathcal{R}\varphi_{\mathcal{N}})(n) - (\mathcal{R}\varphi)(n)\right| = 0,$$

then $\mathcal{R}$ is continuous. Next, we show that the image of $\mathcal{R}$ is contained in a compact set, let $\mathbb{M}$ defined by (9.14), by (9.5) and (9.6), we obtain

$$\begin{aligned}
|Q(n,x)| &\leq |Q(n,x) - Q(n,0) + Q(n,0)| \\
&\leq k_1 |x| + |Q(n,0)|,
\end{aligned}$$

and

$$|f(x)| \leq |f(x) - f(0) + f(0)| \leq k_2 |x|.$$

Let $\varphi \in \mathbb{M}$, then by (9.16) we obtain

$$\begin{aligned}
\|\mathcal{R}\varphi\| &\leq c \sum_{s=0}^{T-1} \left[\alpha + |A| (k_1 J + \gamma + \beta |A| J) + |U| J + k_3 k_2 J\right] \\
&\leq cT \left[\alpha + |A| (k_1 J + \gamma + \beta |A| J) + |U| J + k_3 k_2 J\right],
\end{aligned}$$

where

$$\alpha = \sup_{n\in[0,T-1]\cap\mathbb{Z}} |P(n)|, \ \beta = \sup_{n\in[0,T-1]\cap\mathbb{Z}} |\tau(n)|, \ \gamma = \sup_{n\in[0,T-1]\cap\mathbb{Z}} |Q(n,0)|,$$

and

$$c = \sup_{n\in[0,T-1]\cap\mathbb{Z}} \left(\sup_{s\in[n,n+T-1]\cap\mathbb{Z}} |\Theta(n,s)| \right).$$

Second, we show that $\mathcal{R}$ maps bounded subsets into compact sets. As $\mathbb{M}$ is bounded and $\mathcal{R}$ is continuous, then $\mathcal{R}(\mathbb{M})$ is a subset of $\mathbb{R}^{NT}$ which is bounded. Thus $\mathcal{R}(\mathbb{M})$ is contained in a compact subset of $\mathbb{M}$. Therefore $\mathcal{R}$ is continuous in $\mathbb{M}$ and $\mathcal{R}(\mathbb{M})$ is contained in a compact subset of $\mathbb{M}$. ■

9.3. Existence and uniqueness of periodic solutions

Lemma 9.3 *Suppose (9.3)–(9.5) hold and*

$$k_1 + \beta |A| < 1. \tag{9.20}$$

If $\mathcal{S}$ is defined by (9.17), then $\mathcal{S}$ is a contraction.

Proof. Let $\mathcal{S}$ be defined by (9.17). Then for $\varphi_1, \varphi_2 \in \mathbb{M}$, we have by (9.5)

$$\begin{aligned}
&|(\mathcal{S}\varphi_1)(n) - (\mathcal{S}\varphi_2)(n)| \\
&= \Big|Q(n, \varphi_1(n - g(n))) - Q(n, \varphi_2(n - g(n))) \\
&+ \sum_{t=n-\tau(n)}^{n-1} A(t)\varphi_1(t) - \sum_{t=n-\tau(n)}^{n-1} A(t)\varphi_2(t)\Big| \\
&\le (k_1 + \beta |A|) \|\varphi_1 - \varphi_2\| .
\end{aligned}$$

Hence $\mathcal{S}$ is contraction by (9.20). ■

Theorem 9.1 *Suppose that the assumptions of the Lemmas 9.2 and 9.3 hold. If there exists a constant $J > 0$ defined in $\mathbb{M}$ such that*

$$cT[\alpha + |A|(k_1 J + \gamma + \beta |A| J) + |U| J + k_3 k_2 J] + k_1 J + \gamma + \beta |A| J \le J. \tag{9.21}$$

Then (9.1) has a T-periodic solution in the subset $\mathbb{M}$.

Proof. By Lemma 9.2, $\mathcal{R} : \mathbb{M} \to C_T$ is continuous and $\mathcal{R}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 9.3, the mapping $\mathcal{S} : \mathbb{M} \to C_T$ is a contraction. Next, we show that if $\varphi, \phi \in \mathbb{M}$, we have$\|\mathcal{R}\varphi + \mathcal{S}\phi\| \le J$. Let $\varphi, \phi \in \mathbb{M}$ with $\|\varphi\|, \|\phi\| \le J$. Then

$$\begin{aligned}
&\|\mathcal{R}\varphi + \mathcal{S}\phi\| \\
&\le cT[\alpha + |A|(k_1 J + \gamma + \beta |A| J) + |U| J + k_3 k_2 J] + k_1 J + \gamma + \beta |A| J \\
&\le J.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{R}z + \mathcal{S}z$. By Lemma 9.1 this fixed point is a solution of (9.1). Hence (9.1) has a T-periodic solution. ■

Theorem 9.2 *Suppose the assumptions of Lemma 9.1 hold. If*

$$cT[|A|(k_1 + \beta |A|) + |U| + k_3 k_2] + k_1 + \beta |A| < 1, \tag{9.22}$$

then (9.1) has a unique T-periodic solution.

Proof. Let the mapping $\mathcal{F}$ be given by (9.15). For $\varphi_1, \varphi_2 \in C_T$, we have

$$\begin{aligned}
&\left|\left(\mathcal{F}\varphi_1\right)(n) - \left(\mathcal{F}\varphi_2\right)(n)\right| \\
&\leq \left| Q\left(n, \varphi_1\left(n - g\left(n\right)\right)\right) - Q\left(n, \varphi_2\left(n - g\left(n\right)\right)\right) \right. \\
&\left. + \sum_{t=n-\tau(n)}^{n-1} A(t)\varphi_2(t) - \sum_{t=n-\tau(n)}^{n-1} A(t)\varphi_1(t) \right| \\
&+ \sum_{s=n}^{n+T-1} \left|\Theta\left(n,s\right)\right| \left|A\left(s\right)\right| \left[\sum_{t=s-\tau(s)}^{s-1} \left|A(t)\right| \left|\varphi_1(t) - \varphi_2(t)\right| \right. \\
&+ \left|Q\left(s, \varphi_1\left(s - g\left(s\right)\right)\right) - Q\left(s, \varphi_2\left(s - g\left(s\right)\right)\right)\right|\Big] \\
&+ \sum_{s=n}^{n+T-1} \left|\Theta\left(n,s\right)\right| \Big[\left|U(s)\right| \left|\varphi_1\left(s - \tau\left(s\right)\right) - \varphi_2\left(s - \tau\left(s\right)\right)\right| \\
&\left. + \sum_{k=-\infty}^{s} \left|D\left(s,k\right)\right| \left|f\left(\varphi_1\left(k\right)\right) - f\left(\varphi_2\left(k\right)\right)\right| \right] \\
&\leq \left[k_1 + \beta \left|A\right| + cT\left[\left|A\right|\left(\beta\left|A\right| + k_1\right) + \left|U\right| + k_3k_2\right]\right] \left\|\varphi_1 - \varphi_2\right\|.
\end{aligned}$$

Since (9.22) holds, the contraction mapping principle completes the proof. ∎

Corollary 9.1 *Suppose (9.3) and (9.4) hold. Let* $\mathbb{M}$ *defined by (9.14). Suppose there are positive constants* k_1^*, k_2^* *and* k_3^*, *such that for* $x, y \in \mathbb{M}$ *and* $n \in \mathbb{Z}$ *we have*

$$\left|Q\left(n, x\left(n - g\left(n\right)\right)\right) - Q\left(n, y\left(n - g\left(n\right)\right)\right)\right| \leq k_1^* \left\|x - y\right\| \;\; \text{and } k_1^* < 1,$$

$$\left|f\left(x\left(n\right)\right) - f\left(y\left(n\right)\right)\right| \leq k_2^* \left\|x - y\right\|, \; f\left(0\right) = 0,$$
$$\sum_{k=-\infty}^{n} \left|D\left(n,k\right)\right| \leq k_3^* < \infty,$$

and

$$cT\left[\alpha + \left|A\right|\left(k_1^*J + \gamma + \beta\left|A\right|J\right) + \left|U\right|J + k_3^*k_2^*J\right] + k_1^*J + \gamma + \beta\left|A\right|J \leq J.$$

If $\left\|\mathcal{F}\varphi\right\| \leq J$, *for* $\varphi \in \mathbb{M}$, *then (9.1) has a* T*-periodic solution in* $\mathbb{M}$. *Moreover, if*

$$cT\left[\left|A\right|\left(k_1^* + \beta\left|A\right|\right) + \left|U\right| + k_3^*k_2^*\right] + k_1^* + \beta\left|A\right| < 1,$$

then (9.1) has a unique T*-periodic solution in* $\mathbb{M}$.

Proof. Let the mapping $\mathcal{F}$ defined by (9.15). Then the proof follow immediately from Theorem 9.1 and Theorem 9.2. ∎

Example 9.1 Consider the 2-dimensional nonlinear neutral summation-difference system

$$\Delta\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_4 \sin(n) \end{pmatrix} + \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & -\lambda_1 \end{pmatrix}\begin{pmatrix} x_1(n-\tau(n)) \\ x_2(n-\tau(n)) \end{pmatrix}$$
$$+\Delta\begin{pmatrix} 0 \\ \lambda_2 \sin(n)\, x_1^2(n-g(n)) \end{pmatrix}$$
$$+\sum_{k=-\infty}^{n}\begin{pmatrix} 0 & 0 \\ 0 & \lambda_3 2^{k-n} \end{pmatrix}\begin{pmatrix} 0 \\ x_1^2(k) \end{pmatrix}, \tag{9.23}$$

where

$$P(n) = \begin{pmatrix} 0 \\ \lambda_4 \sin(n) \end{pmatrix},\ A(n) = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & -\lambda_1 \end{pmatrix},$$
$$Q(n, x(n-g(n))) = \begin{pmatrix} 0 \\ \lambda_2 \sin(n)\, x_1^2(n-g(n)) \end{pmatrix},$$

and

$$D(n,k) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_3 2^{k-n} \end{pmatrix},\ f(x(k)) = \begin{pmatrix} 0 \\ x_1^2(k) \end{pmatrix}.$$

Let $\tau(n) = \beta \in \mathbb{Z}^+$, $g : \mathbb{Z} \to \mathbb{Z}^+$ is a nonnegative sequence and 2π-periodic. Since the matrix $B = I + A$ has eigenvalues with non-zero real parts, the system $x(n+1) = Bx(n)$ is noncritical. So, let a Banach space $(C_{2\pi}, \|.\|)$,

$$C_{2\pi} = \left\{\phi \in C\left(\mathbb{Z}, \mathbb{R}^2\right),\ \phi(n+2\pi) = \phi(n)\right\},$$

a closed bounded convex subset of $C_{2\pi}$,

$$\mathbb{M} = \{\varphi \in C_{2\pi},\ \|\varphi\| \le J\}.$$

Let $\varphi = (\varphi_1, \varphi_2)$, $\phi = (\phi_1, \phi_2)$. Then for $\varphi, \phi \in \mathbb{M}$ we have

$$|Q(n, x(n-g(n))) - Q(n, y(n-g(n)))| \le 2\lambda_2 J \|x-y\|,$$

$$|f(x(n)) - f(y(n))| \le 2J\|x-y\|,\ f(0) = 0,$$

and

$$\sum_{k=-\infty}^{n} |D(n,k)| = \sum_{k=-\infty}^{n} \lambda_3 2^{k-n} = 2\lambda_3 < \infty.$$

Hence $k_1^* = 2\lambda_2 J$, $k_2^* = 2J$, $k_3^* = 2\lambda_3$, $\alpha = \lambda_4$, $\gamma = 0$ and

$$U(n) = A(n) - (1 - \Delta\tau(n))\, A(n-\tau(n)) = 0,\ |A| = 2\lambda_1.$$

Consequently

$$cT\left[\lambda_4 + \lambda_1\left(2\lambda_2 J^2 + \beta\lambda_1 J\right) + 4\lambda_3 J^2\right] + 2\lambda_2 J^2 + 2\beta\lambda_1 J \le J,$$

for all λ_i, $1 \leq i \leq 4$ small enough. Then (9.23) has a 2π-periodic solution, by Corollary 9.1. Moreover,

$$cT\left[\lambda_1\left(2\lambda_2 J + \beta\lambda_1\right) + 4\lambda_3 J\right] + 2\lambda_2 J + 2\beta\lambda_1 < 1,$$

is satisfied for λ_i, $1 \leq i \leq 3$ small enough. Then (9.23) has a unique 2π-periodic solution, by Corollary 9.1.

Conclusion

As we have seen in the present thesis, we used the technique of fixed point to study the existence, uniqueness, periodicity, positivity and stability of solutions for a class of nonlinear delay functional equations and systems, because its have been of great interest recently. We have reached new results from which we can proceed in the future.

The main aspect of the future work is to take other problems of functional equations and systems with or without delay with different conditions and study it theoretical and numerical.

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