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Sur une classe d'équations différentielles fractionnaire à retards

Option
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Dedication

I dedicate my thesis to:

My parents for their love and support throughout my life,

thanks to your encouragement and sacrifices, my dreams are finally realized.

To my husband, my wonderful children Meriem and Ayoub

and my respected husband's father.

To all my family, and my friends.

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In the name of Allah, the most Gracious, most Merciful.

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الملخص

تتعلق هذه الأطروحة بدراسة بعض مسائل القيم الحدية غير الخطية على مجال لانهائي وتتضمن مشتقات كسرية من نوع ريمان-ليوفيل. يتعلق الجزء الأول بمشكلة القيم الحدية للمعادلات التفاضلية الكسرية ذات التأخيرات المتغيرة. يتم الحصول على وجود الحلول وتفردها واستقرارها من خلال نظريات النقطة الثابتة مثل نظرية بناخ و لوري شودر . و في الجزء الثاني تمت دراسة مشكلة القيم الحدية للمعادلات التفاضلية الكسرية باستخدام ف- لابلاسيان. يتم تحديد نتائج الوجود بواسطة نظرية النقطة الثابتة لكراسنوسيسكي ، بالإضافة إلى ذلك ، تم إعطاء بعض الأمثلة التوضيحية

الكلمات المفتاحية: المعادلات التفاضلية الكسرية ، مسألة القيم الحدية ، مشتق ريمان-ليوفيل الكسري ، الوجود ، الوحدانية ، استقرار الحلول ، المعادلات التفاضلية بتأخر، عامل ف- لابلاسيان ، المجال الغير منتهي، نظرية النقطة الثابتة لوري شودر، نظرية النقطة الثابتة لبناخ ، نظرية النقطة الثابتة لكراسنوسيسكي .

Résumé

Cette thèse concerne l'étude de quelques problèmes aux limites non linéaires sur des intervalles infinis et comportant les derivées fractionnaires de type Riemann-Liouville. La première partie concerne un problème aux limites pour des équations différentielles fractionnaires avec des retards variables. L'existence, l'unicité et la stabilité des solutions sont obtenues via les théorèmes de point fixe, comme le théorème de contraction de Banach et l'alternative non linéaire de Leray-Schauder. Dans la deuxième partie, un problème aux limites pour les équations différentielles fractionnaires avec l'opérateur p-Laplacien est étudié. Les principaux résultats d'existence sont établis par le théorème du point fixe de Krasnoselskii. En outre, quelques exemples illustratifs sont donnés.

Mots clés: Equations différentielles fractionnaires, Problème aux limites, Dérivée fractionnaire de Riemann-Liouville, Existence, Unicité, Stabilité des solutions, Équation différentielle à retard, Intervalle infini, Opérateur p-Laplacien, Théorème du point fixe, Théorème de Krasnoselskii, Principe de contraction de Banach, Leray-Schauder nonlinéaire alternative.

Abstract

This thesis deals wit the study of some nonlinear Riemann-Liouville fractional boundary value problems on the half-line. The first part concerns a boundary value problem for fractional differential equations with variable delays. The existence, uniqueness and stability of solutions are obtained via certain fixed point theorems, such as the Banach's contraction theorem, the nonlinear alternative Leray-Schauder. In the second part, we deal with a boundary value problem for fractional differential equations with the p-Laplacian operator. The main existence results are established by the help of Krasnoselskii fixed point theorem. Furthermore, some illustrative examples are given.

Keywords: Fractional differential equations, Boundary value problem, Riemann-Liouville fractional derivative, Existence, Uniqueness, Stability of solutions, Delay differential equations, Infinite interval, p-Laplacian operator, Fixed point theorem, Krasnoselskii fixed point, Banach contraction principle, Leray-Schauder nonlinear alternative.

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Introduction

Fractional integrals and derivatives can be seen as generalization of classical calculus. Although fractional calculus has a history of over three hundred years, and dates back to the mathematicians Leibniz and Euler, it was only recently developed. Over the past three decades, the subject has grown exponentially and many researchers are actively working on this subject, mainly due to its various applications in many fields of science such as physics, mechanics, chemistry, engineering, see [43, 64, 46, 66, 66, 52, 60, 71] and the references therein.

Liouville was the first person to try to solve fractional differential equations, then, some books played a considerable role in understanding the subject and gave the applications of fractional differential equations and methods to solve them, such as the books of Miller and Ross, Oldham and Spanier, Podlubny, Samko, Kilbas, Marichev,...

Fractional calculus studies have reached a significant and appropriate level for modern mathematics in the past decades, and since then efficient and reliable techniques for solving modeled problems with fractional integral and differential operators have been established.

Over the years, various definitions corresponding to the idea of an integral or a derivative of non integer order have been used, but the Riemann-Liouville definition of integrals and derivatives of fractional order remains the most common popular in the world of fractional calculus, moreover, most of the other definitions of fractional calculus are largely variations of that of Riemann-Liouville, it is this version that will be mainly discussed in this thesis.

Boundary value problems over infinite intervals often appear in applied mathematics and physics and thus the existence of solutions for such problems has become an important area of investigation and many papers focus on the existence solutions for boundary value problems on unbounded intervals, see [28, 49, 12, 46, 18, 23, 47, 51, 50, 27].

In [40], the authors studied a Riemann-Liouville fractional boundary value problem at resonance and on an infinite interval:

$$D_{0^+}^{\alpha} x(t) = f(t, x(t)), \; t > 0$$

$$I_{0^+}^{2-\alpha} x(0) = 0, \; D_{0^+}^{\alpha-1} x(\infty) = D_{0^+}^{\alpha-1} x(0).$$

Where $1 < \alpha < 2$, $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville derivative and $I_{0^+}^{2-\alpha}$ denotes the Riemann-Liouville integral. Under some conditions on the nonlinear term f, the authors proved the existence of maximal and minimal positive solutions uppon the upper and lower solutions method and a fixed point theorem for an increasing operator.

Differential equations involving p-Laplacian operator have been studied in several papers and attracted more attention since they have various applications in different fields of sciences such as fluid flow in a porous medium, elasticity, electrorheological fluid,...see [7, 11, 17, 35, 51, 57, 70, 79].

Leibenson [48] applied the p-Laplacian differential equation for the first time to model the turbulent flow in a porous medium, then he proved the existence of solutions of the p-Laplacian differential equation

$$(\phi_p(u'(t)))' = f(t, u(t)), \; 0 \leq t \leq 1,$$

where ϕ_p is the p-Laplacian operator defined by $\phi_p(s) = |s|^{p-2} s$, $p > 1$.

In [35], the authors proved the existence of positive solutions by using some fixed point theorems, for a Riemann-Liouville fractional boundary value problem containing the p-Laplacian operator:

$$D_{0^+}^{\beta} \left(\phi_p \left(D_{0^+}^{\alpha} x(t)\right)\right) = f(t, x(t)), \; 0 < t < 1$$

$$\begin{aligned} x(0) &= D_{0^+}^{\gamma} x(0) = 0, \\ D_{0^+}^{\gamma} x(1) &= \sum_{i=1}^{m-2} a_i D_{0^+}^{\gamma} x(\eta_i) \\ \phi_p\left(D_{0^+}^{\alpha} x(1)\right) &= \sum_{i=1}^{m-2} b_i \phi_p\left(D_{0^+}^{\alpha} x(\eta_i)\right), \end{aligned}$$

where $1 < \alpha, \beta \leq 2, \quad 0 < \gamma \leq 1,\ 0 < a_i, b_i, \eta_i < 1,\ i = 1, ...m-2,\ f \in C\left([0,1] \times \mathbb{R}_+, \mathbb{R}_+\right),$ f is singular at $x = 0$.

The stability of solutions for fractional differential equations is an important topic to study. Note that fractional derivatives are nonlocal and have a singular kernel and therefore the analysis of the stability of fractional differential equations is more complex than ordinary differential equations. The most used methods are those of Lyapunov direct or indirect, for some papers dealing with the stability of solutions for fractional differential equations, we refer to [14, 15, 20, 52, 55].

In [20], the stability of a Caputo fractional initial value problem is discussed by Krasnoselskii's fixed point theorem in a weighted Banach space:

$$\begin{aligned} {}^{C}D_{0^+}^{\alpha} x(t) &= f(t, x(t)), \ t > 0, \quad 1 < \alpha < 2, \\ x(0) &= x_0, \quad x'(0) = x_1. \end{aligned}$$

In [55], thanks to Caputo type fractional comparison principle and a fractional differential inequality, the authors investigated the stability and instability of a class of Caputo fractional differential equations

$$\begin{aligned} {}^{C}D_{t_0^+}^{\alpha} x(t) &= f(t, x(t)), \ t > t_0, \quad 0 < \alpha < 1, \\ x(t_0) &= x_0. \end{aligned}$$

A differential equation of delay is a differential equation where the temporal derivatives at the current instant depend on the solution and possibly on its derivatives at the preceding instants. Instead of a simple initial condition, an initial history function must be specified. In many current models, the history is a constant, but we regularly encounter nonconstant history functions.

The differential equations involving delays are useful for analysis and prediction in various domains of sciences as in population dynamics, epidemiology, immunology, physiology..., see [1, 4, 9, 10, 30, 74, 75, 76].

In In [1], the existence and uniqueness of solutions for a delay Riemann-Liouville fractional boundary value problem is investigated by some fixed point theorems,

$$D_{0^+}^{\alpha}x\left(t\right)=f\left(t,x\left(t\right),x\left(t-\tau\right)\right),\ t\in\left[0,T\right],\quad 0<\alpha<1,$$

$$x\left(t\right)=\varphi\left(t\right),\quad t\in\left[-\tau,0\right].$$

The aim of this thesis is to discuss the existence, uniqueness and stability for some fractional nonlinear boundary value problems on infinite intervals, by application of fixed point theory. Let us give the review of each chapter of the thesis.

In Chapter 1, we expose its properties. We provide some basic properties for fractional integrals and derivatives. We give some useful fixed point theorems, then we end this chapter by citing the concept of stability of solutions.

Chapter 2, focuses on the existence, uniqueness and stability of solutions for a nonlinear Riemann-Liouville fractional boundary value problem with variable delays:

$$\begin{cases} D_{0^+}^{\alpha}u\left(t\right)-q(t)f(t,u(t-\theta_1(t)),u(t-\theta_2(t))=0,\quad t>0,\\ \\ u\left(t\right)=\varphi(t),\quad t\in\left[-\tau,0\right],\\ \\ u''(0)=0,\qquad \lim_{t\to\infty}D_{0^+}^{\alpha-1}u(t)=\Gamma(\alpha)u(0),\end{cases}$$

where $2\leq\alpha<3$, $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, the functions $\theta_i\in C\left(\mathbb{R}_+,\mathbb{R}_+\right)$ such $\lim_{t\to\infty}(t-\theta_i(t))=+\infty$, $i=1,2$, $\tau=-\min_{0\leq i\leq 2}\min_{t\geq 0}(t-\theta_i(t))$. We assume that $q:[0,\infty)\to[0,\infty)$, $f\in C\left(\mathbb{R}_+\times\mathbb{R}^2,\mathbb{R}\right)$ and φ is a continuous function on the interval $[-\tau,0]$.

We investigate the existence of solutions, by the nonlinear alternative of Leray Schauder and the uniqueness of solution by the Banach contraction principle and finally we discuss the uniform stability of the solution.

The results of this chapter are submitted for publication:

F. Fenizri, A. Guezane-Lakoud, R. Khaldi, Stability of solutions to fractional differential equations with time delays.

In Chapter 3, we study the existence of solutions for a fractional boundary value problem with the p-Laplacian operateur, by using Krasnoselskii fixed point theorem:

$$\begin{cases} D_{0^+}^{\delta}(\phi_p(D_{0^+}^{\alpha}u\,(t))) + q(t)f(t,u(t),D_{0^+}^{\alpha-1}u(t)) = 0,\ t>0 \\ \\ u(0)=0, \quad D_{0^+}^{\alpha-1}u(\infty) = \int_0^\infty g(s)u(s)ds, \\ \\ D_{0^+}^{\alpha-2}u(0) = \int_0^\infty h(s)u(s)ds, \quad D_{0^+}^{\delta-1}(\phi_p(D_{0^+}^{\alpha}u\,(0))) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+,}^{\delta}$ are the standard Riemann-Liouville fractional derivatives, $2<\alpha\leq 3$, $0<\delta\leq 1$, $\phi_p(s)=|s|^{p-2}\,s$, $p>1$, $q:[0,\infty)\rightarrow\mathbb{R}$ and $f:[0,\infty)\times\mathbb{R}^2\rightarrow\mathbb{R}$ are continuous functions and $g,h\in L^1\left([0,\infty)\right)$ are nonnegative functions. The results of this chapter are published in:

F. Fenizri, A. Guezane-Lakoud, R. Khaldi, Existence of solutions for integral boundary value problems with p-Laplacian operator on infinite interval, J. Nonlinear Funct. Anal., 2021 (2021), Article ID 13, 1–13.

CHAPTER 1

Preliminaries

In this chapter, we introduce some important functions that are used in fractional calculus such the Gamma function that acts as a generalized factorial. We give some necessary concepts on the theory of fractional calculus, namely the integral and the derivative of Riemann-Liouville and the derivative of Caputo. We quote their basic properties including the rules of their composition. We can find more information in [30, 60, 11, 12, 29, 53, 56, 62]. We expose some important fixed point theorems as well as the concept of stability.

1.1 Function spaces

1.1.1 Integral function spaces

The spaces L^p are spaces of measurable functions that are p-integrables in the sense of Lebesgue.

Definition 1.1.1 *[65] Let $I = (a, b)$, where $-\infty \leq a < b \leq +\infty$, be a finite or infinite interval in $\mathbb{R}$. We denote by $L^p(I)$ $(1 \leq p < \infty)$. the set of those Lebesgue real valued measurable functions f on I for which*

$$\|f\|_p = \left(\int\limits_I |f(s)|^p \, ds \right)^{\frac{1}{p}} < \infty$$

If $p = \infty$ the space $L^p(I)$ consists of all measurable functions with a finite norm

$$\|f\|_\infty = ess \sup_{t \in J} |f(t)|$$

1.1.2 Space of absolutely continuous functions

Let $I = [a, b])$, be a bounded interval.

Definition 1.1.2 *[65] We denote by $AC[a, b]$ the space of functions f which are absolutely continuous on $[a, b]$. It coincides with the space of primitives of Lebesgue* *summable functions:*

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int\limits_a^x \varphi(t)dt : (\varphi(t) \in L(a,b)),$$

Definition 1.1.3 *[65] We denote by $C^n(I)$ the space of functions f which are n times continuously differentiable on I with respect to the norm*

$$\|f\|_{C^n} = \sum_{k=0}^{n} \max_{x \in I} \left| f^k(x) \right|.$$

In particular, for $n = 0$, $C^0(I) = C(I)$ is the space of continuous functions f on I with respect to the norm

$$\|f\| = \max_{x \in I} |f(x)|$$

1.2 Gamma function

Here we give some information about the gamma function, which plays an important role in the theory of fractional order differentiation and in the theory of fractional differential equations.

Definition 1.2.1 *[60] The Gamma function $\Gamma(.)$ is defined by the integral*

$$\Gamma(z) = \int\limits_0^{+\infty} e^{-t} t^{z-1} dt, \quad \forall z \in \mathbb{R}_+^*$$

and possesses the following basic properties

$$\Gamma(z+1) = z\Gamma(z), \ \mathrm{Re}(z) > 0,$$

for any integer $n \geq 0$, we have

$$\Gamma(n+1) = n!,$$

and

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

A limit definition of the Gamma function is given by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\dots(z+n)}, \ Re(z) > 0.$$

1.3 Fractional integrals and fractional derivatives

Definition 1.3.1 *[65] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function g is defined by*

$$\left(I_{a^+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)\, ds, \ t > a$$

provided that the right side is pointwise defined on $(a, +\infty)$.

Properties [65] *Let $\alpha, \beta > 0$, then the following relations hold*

$$I_{a^+}^{\alpha} (x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (t-a)^{\alpha+\beta-1},$$

$$I_{a^+}^{1} (x-a)^{\beta}(t) = \frac{1}{\beta+1} (t-a)^{\beta+1}.$$

In particular, if $\alpha > 0$ and $k \in \mathbb{N}$, then

$$\left(D^k I_{a^+}^{\alpha} g\right)(x) = I_{a^+}^{\alpha-k} g(x).$$

If $\alpha > 0$ and $\beta > 0$, then

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} g(x) = I_{a^+}^{\alpha+\beta} g(x)$$

Definition 1.3.2 *[65]The Riemann-Liouville fractional derivative of order $\alpha \in$*

$\mathbb{R}^+$ *of a function* g *is defined by*

$$\begin{aligned} D_{a^+}^{\alpha} g\,(t) &:= \left(\frac{d}{dx}\right)^n (I_{a^+}^{n-\alpha} g)(x) \\ &= \frac{1}{\Gamma\,(n-\alpha)} \left(\frac{d}{ds}\right)^n \int\limits_a^t (t-s)^{n-\alpha-1}\, g\,(s)\, dt,\ \ t > a, \end{aligned}$$

where $n = [\alpha] + 1$, $[\alpha]$ *is the integer part of* α.

Properties. [65] *Let* $\alpha, \beta > 0$ *and* $n = [\alpha] + 1$, *then the following relation holds:*

$$\left(D_{a^+}^{\alpha}\,(x-a)^{\beta-1}\right)(t) = \frac{\Gamma\,(\beta)}{\Gamma\,(\beta-\alpha)}\,(t-a)^{\beta-\alpha-1}\,.$$

On the other hand, for $k = 1, 2, ..., n$, *we have*

$$\left(D_{a^+}^{\alpha}\,(x-a)^{\alpha-k}\right)(t) = 0.$$

In particular, the Riemann-Liouville fractional derivative of a constant is in general not equal to zero, in fact

$$\left(D_{a^+}^{\alpha} 1\right)(x) = \frac{(x-a)^{-\alpha}}{\Gamma\,(1-\alpha)},\ \ 0 < \alpha < 1.$$

Lemma 1.3.1 *[65]Let* $\alpha > 0$, *then the fractional differential equation*

$$D_{0^+}^{\alpha} g\,(t) = 0,$$

has

$$\begin{aligned} g\,(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n}, \\ c_i &\in \mathbb{R}, i = 1, 2, ..., n \end{aligned}$$

as solution.

Lemma 1.3.2 *[65] Let* $\alpha > 0$, $n = [\alpha] + 1$. *If* $g \in L^1\,[a, b]$ *and* $g_{n-\alpha} \in AC^n\,[a, b]$, *then the equality*

$$\left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} g\right)(t) = g(t) - \sum_{j=1}^{n} \frac{g_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha - j}$$

holds almost every where on $[a,b]$*. In particular for* $0 < \alpha < 1$*, we have*

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} g(t) = g(t) - \frac{g_{1-\alpha}(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1},$$

where $g_{n-\alpha} = I_{a^+}^{n-\alpha} g$ *and* $g_{1-\alpha} = I_{a^+}^{1-\alpha} g$*.*

Lemma 1.3.3 *[65] Let* $\beta > \alpha > 0$*, then we have*

$$D_{a^+}^{\beta} I_{a^+}^{\alpha} g(x) = I_{a^+}^{\alpha-\beta} g(x).$$

1.4 Fixed point theorems

Fixed point theory is an important subject with a large number of applications in various fields of mathematics. The fixed point theorems concern a function f satisfying $f(x) = x$ under some conditions on f. Depending on whether the conditions are imposed on the function or the set, different fixed point theorems are given, we quote the following that can be found in [65].

Theorem 1.4.1 *(Banach contraction principle) Let* T *be a contraction on a Banach space* X*. Then* T *has a unique fixed point.*

Theorem 1.4.2 *(Leray-Schauder nonlinear alternative) Let* C *be a convex subset of a Banach space,* U *be an open subset of* C *with* $0 \in U$*. Let* $N : \bar{U} \to C$ *be a completely continuous mapping. Then either*

(i) N *has a fixed point in* $\bar{U}$*, or*

(ii) There is an $x \in \partial U$ *and* $\lambda \in (0,1)$ *with* $x = \lambda N x$*.*

Theorem 1.4.3 *(Krasnoselskii fixed point theorem) Let* Ω *be a closed bounded and convex nonempty subset of a Banach space* X*. Suppose that* A *and* T *map* Ω *into* X *such that*

(i) A is continuous and compact.
(ii) T is a contraction mapping.
(iii) $Ax + Ty \in \Omega$ for all $x, y \in \Omega$.
Then there exists $x \in \Omega$ with $x = Ax + Tx$.

The criteria for compactness for sets in the space of continuous functions $C([a, b])$ is the following.

Theorem 1.4.4 *(Arzela-Ascoli theorem) A set $\Omega \subset C([a, b])$ is relatively compact in $C([a, b])$ iff the functions in Ω are uniformly bounded and equicontinuous on $[a,\ b]$.*

We recall that a family Ω of continuous functions is uniformly bounded if there exists $M > 0$ such that

$$\|f\| = \max_{x\in[a,b]} |f(x)| \leq M, \qquad \forall f \in \Omega.$$

The family Ω is equicontinuous on $[a, b]$, if $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\forall t_1,\, t_2 \in [a, b]$ and $\forall f \in \Omega$, we have

$$|t_1 - t_2| < \eta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon.$$

As the Arzela-Ascoli theorem is no longer valid in the case of a non-compact interval, the following compactness criteria will be useful [14].

Theorem 1.4.5 *Let $C_\infty = \{y \in C([0, +\infty)), \lim_{t\to\infty} y(t) \text{ exists}\}$ equipped with the norm $\|y\|_\infty = \sup\limits_{t\in[0,+\infty)} |y(t)|$. Let $F \subset C_\infty$. Then F is relatively compact if the following conditions hold :*

(1) *F is bounded in C_∞.*

(2) *The functions belonging to F are equicontinuous on any compact subinterval of $[0, \infty)$.*

(3) *The functions from F are equiconvergent at $+\infty$.*

1.5 Stability of solutions

The stability of nonlinear fractional differential equations is studied by different methods such as Mittag-Leffler stability, Ulam stability, Lypunov method ... Since it is difficult to apply these methods when the fractional order is higher to one, much work is devoted to finding another efficient method to study the stability of nonlinear fractional differential equations.

Let X be a Banach space, $g : [t_0, \infty) \times X \to X$ be a function and consider the differential equation

$$\frac{dx}{dt} = g\left(t, x\right), t_0 \leq t < \infty. \tag{1.1}$$

with $g\left(t, 0\right) = 0$.

Definition 1.5.1 *[5] A solution x of equation (1.1) is said to be stable if for every $\varepsilon > 0$ there exists $\delta = \delta\left(\varepsilon, t_0\right) > 0$ such that for every $x_0 \in X$, the inequality $\|x_0\| < \delta$ implies $\|x\left(t, t_0, x_0\right)\| < \varepsilon$, for all $t \geq t_0$.*

A solution x of equation (1.1) is said to be uniformly stable if the constant δ can be chosen independly of t_0, i.e. $\delta = \delta\left(\varepsilon\right)$

A solution x of equation (1.1) is said to be asymptotically stable if it's stable and for any $t \geq t_0$, there exists $\delta = \delta\left(t_0\right) > 0$ such that $\|x_0\| < \delta$ implies $lim_{t\to+\infty} x\left(t, t_0, x_0\right) = 0$.

CHAPTER 2

Stability of solutions for nonlinear fractional differential equations with variable delays

2.1 Introduction

Fractional differential equations involving delays are becoming an active area of research and therefore attracting increasing interest, this is due to their applications to a variety of problems in science and engineering, such as population dynamics, epidemiology, immunology, physiology, we refer to [1, 4, 9, 10, 30, 74, 75, 76] for some interesting articles.

When the course of the process at a certain point in time depends on its past history, the delays may be related to the duration of certain previous processes such as the duration of the infectious period, the time between the infection of a cell and the production of new viruses.

In this chapter, we are interested in investigating the following nonlinear fractional boundary value problem with variable delays:

$$(P)\begin{cases} D_{0^+}^{\alpha} u\,(t) + q(t) f(t, u(t-\theta_1(t)), u(t-\theta_2(t)) = 0, \quad 2 \leq \alpha < 3,\ t > 0, \\ u\,(t) = \varphi(t), \quad t \in [-\tau, 0]\,, \\ u''(0) = 0, \lim_{t\to\infty} D_{0^+}^{\alpha-1} u(t) = \Gamma(\alpha) u(0), \end{cases}$$

where $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, the functions $\theta_i : [0, \infty) \to [0, \infty)$ are continuous for all $i = 1, 2$, such $\lim_{t\to\infty}(t - \theta_i(t)) = +\infty$, $\tau = -\min_{0\leq i\leq 2} \min_{t\geq 0}(t - \theta_i(t))$, $q : [0, \infty) \to [0, \infty)$, the function f is continuous on $[0, \infty) \times \mathbb{R}^2$and φ is a continuous function on the interval $[-\tau, 0]$.

By means of the nonlinear alternative of Leray Schauder and the Banach contraction principle, we prove the existence and uniqueness results then we analyze the stability of the solution.

Recently, several works have appeared dealing with the existence of solutions for fractional differential equations with delay, [1, 9, 43, 75, 76].

In [43], the authors discussed the existence of solutions for the fractional Riemann-Liouville differential equations with a constant delay,

$$\begin{aligned} D_{0^+}^{\alpha} u\,(t) &= u(t) + f(t, u(t), u(t-\tau)), \quad 0 < \alpha < 1, \quad t \in (0,1) \\ u\,(t) &= \varphi(t), \quad t \in [-\tau, 0]\,, \end{aligned}$$

subject to the boundary conditions

$$\begin{aligned} u(0) &= \lim_{t \to 0^+} t^{1-\alpha} u(t) = u(1), \\ D_{0^+}^{1-\alpha} u(t)_{t=0} &= c\Gamma(\alpha). \end{aligned}$$

In [77], by Leray-Schauder nonlinear alternative, sufficient conditions on the nonlinear term, that guarantee the existence of solutions to the following fractional boundary value problem over an unbounded interval are established,

$$\begin{aligned} D_{0^+}^{\alpha} u\,(t) + f(t, u(t)) &= 0, \quad \alpha \in (1,2)\,, \; t \in (0, \infty) \\ u\,(0) &= 0, \quad \lim_{t \to \infty} D_{0^+}^{\alpha-1} u(t) = \beta u(\xi), \quad 0 < \xi < \infty. \end{aligned}$$

Where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative and f is a continuous function.

In [75], the author established the stability of solutions for a nonlinear fractional differential equation on an infinite interval with constant delays and subject to a Riemann-Liouville fractional integral boundary condition:

$$\begin{aligned} D_{0^+}^{\alpha} u\,(t) &= \sum_{j=1}^{n} a_j(t) f(t, u(t), u(t-\tau_j) = 0, \quad 0 < \alpha < 1, \; 0 < t < \infty \\ u\,(t) &= \varphi(t), \; t < 0 \\ I_{0^+}^{\alpha-1} u(t)_{|t=0} &= 0, \quad \lim_{t \longrightarrow 0^-} \varphi(t) = 0, \end{aligned}$$

here $f:\mathbb{R}^{+}\times\mathbb{R}^{2}\longrightarrow\mathbb{R}$ is a continuous function, a_j and φ are given continuous functions, $\tau_j\geq 0,\ j=1,2,...,n$ are constants.

2.2 Existence and uniqueness of solution

First, we prove that the corresponding linear problem has a unique solution, then we transform the problem (P) into a fixed point problem.

Lemma 2.2.1 *The following linear fractional boundary value problem*

$$D_{0^+}^{\alpha}u\left(t\right)=-e(t),\qquad 2\leq\alpha<3,\ t>0$$

$$u\left(t\right)=\varphi(t),\quad t\in[-\tau,0]$$

$$u''(0)=0,\lim_{t\to\infty}D_{0^+}^{\alpha-1}u(t)=\Gamma(\alpha)u(0)$$

has a unique solution given by

$$u(t)=\begin{cases}\varphi(0)t^{\alpha-1}+\int\limits_{0}^{\infty}G(t,s)e(s)ds, & t>0\\ \\ \varphi(t), & t\in[-\tau,0]\end{cases}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t<\infty\\ t^{\alpha-1}, & 0\leq t\leq s<\infty.\end{cases}$$

Proof Applying the integral operator $I_{0^+}^{\alpha}$ to the equation $D_{0^+}^{\alpha}u\left(t\right)=-e(t)$,then using Lemma (1.3.1), it yields,

$$u\left(t\right)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}-I_{0^+}^{\alpha}e(t).\tag{2.1}$$

The conditions $u(0) = \varphi(0)$ and $u''(0) = 0$, imply $c_2 = c_3 = 0$ and the boundary condition $\lim_{t\to\infty} D_{0^+}^{\alpha-1}u(t) = \Gamma(\alpha)u(0)$, gives

$$c_1 = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^\infty e(s)ds.$$

Replacing the constants c_1, c_2 and c_3 by their values in (2.1), then the solution can be written as

$$u(t) = \varphi(0)t^{\alpha-1} + \int_0^\infty G(t,s)e(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t < \infty \\ t^{\alpha-1}, & 0 \le t \le s < \infty. \end{cases}$$

■

Lemma 2.2.2 *The function G is continuous, nonnegative and satisfies*

$$\frac{G(t,s)}{1+t^{\alpha-1}} \le \frac{1}{\Gamma(\alpha)}, \quad \textit{for all} \quad s,t \ge 0$$

Proof The proof is easy, then we omit it. ■

Let $(X, \|.\|)$ be the Banach space

$$X = \left\{ u \in C[-\tau,\infty) : \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t^{\alpha-1}} < \infty \right\}$$

with respect to the norm

$$\|u\|_X = \|u\|_0 + \|u\|_\infty,$$

where

$$\|u\|_0 = \max_{t\in[-\tau,0]} |u(t)|, \quad \|u\|_\infty^0 = \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t^{\alpha-1}}$$

The following compactness criteria will be useful as we are on an infinite interval, see [14, 67].

Lemma 2.2.3 *Let $Z \subseteq X$ be a bounded set. Then Z is relatively compact in X if* *the following condition hold:*

i) The functions belonging to Z are equicontinuous on any compact subinterval of $[-\tau,\infty)$.

ii) The functions from Z are equiconvergent at $+\infty$, i.e. given $\varepsilon>0$,there exists a constant $l=l\left(\varepsilon\right)>0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}}-\frac{u(t_2)}{1+t_2^{\alpha-1}}\right|<\varepsilon,\ t_1,t_2\geq l,\ \ u\in Z.$$

Denote by T the operator

$$\begin{aligned} T \quad &: \quad X\rightarrow X \\ Tu(t) \quad &= \quad \begin{cases} \varphi(0)t^{\alpha-1}+\int\limits_0^{\infty}G(t,s)q(s)f(s,u(s-\theta_1(s)),u(s-\theta_2(s))ds,\ t>0, \\ \qquad\qquad \varphi(t),\quad t\in\left[-\tau,0\right], \end{cases} \end{aligned}$$

We therefore transform the problem (P) into a fixed point problem, i.e. u is a solution for the problem (P) if and only if u is a fixed point for the operator T, i.e. $Tu=u$.

Let us make the assumptions necessary to solve the problem (P).

(H_1) *There exist two nonnegative functions $L_1,L_2\in L^1\left(0,\infty\right)$ such that*

$$\begin{aligned} &\left|f(t,(1+t^{\alpha-1})x_1,(1+t^{\alpha-1})y_1)-f(t,(1+t^{\alpha-1})x_2,(1+t^{\alpha-1})y_2)\right|, \\ &\qquad\leq L_1(t)\left|x_1-x_2\right|+L_2(t)\left|y_1-y_2\right|, \end{aligned} \tag{2.2}$$

for all $x_1,y_1,x_2,y_2\in\mathbb{R}$, $t>0$ *and*

$$C=\max\left(\int\limits_0^{\infty}q(s)L_1(s)ds,\int\limits_0^{\infty}q(s)L_2(s)ds\right)<\frac{\Gamma(\alpha)}{2}. \tag{2.3}$$

(H_2) *There exist $t_i>0$, such that $t-\theta_i(t)<0$, if $0\leq t\leq t_i$, $t-\theta_i(t)\geq0$, if $t>t_i$, $i=1,2$.*

(H_3) *The function f is continuous and there exist nonnegative functions a, b and $c \in L^1(0,\infty)$ such that*

$$|f(t,u,v)| \leq a(t)\,|u| + b(t)\,|v| + c(t)$$

and

$$\begin{aligned} \int_0^\infty q(t)\left(1+t^{\alpha-1}\right)(a(t)+b(t))\,dt &< \infty, \\ \int_0^\infty q(t)\,(a(t)+b(t))\,dt &< \infty, \\ \int_0^\infty q(t)c(t)dt &< \infty. \end{aligned}$$

Theorem 2.2.1 *Assume that assumptions (H_1) and (H_2) hold, then the nonlinear fractional boundary value problem (P) has a unique solution in X.*

Proof To prove the uniqueness of the solution, we shall apply the Banach contraction principle. Let $u, v \in X$, we have

$$\|Tu - Tv\|_0 = \max_{t\in[-\tau,0]} |Tu(t) - Tv(t)| = 0. \tag{2.4}$$

Now, let $t > 0$, then it yields

$$\left|\frac{Tu(t)-Tv(t)}{1+t^{\alpha-1}}\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)\,|f(s,u(s-\theta_1(s)),u(s-\theta_2(s))$$

$$-f(s,v(s-\theta_1(s)),v(s-\theta_2(s))|\,ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int\limits_0^\infty q(s)\left|f(s, \frac{(1+s^{\alpha-1})\,u(s-\theta_1(s))}{1+s^{\alpha-1}}, \frac{(1+s^{\alpha-1})\,u(s-\theta_2(s))}{1+s^{\alpha-1}})\right.$$

$$\left. -\ f(s, \frac{(1+s^{\alpha-1})\,v(s-\theta_1(s))}{1+s^{\alpha-1}}, \frac{(1+s^{\alpha-1})\,v(s-\theta_2(s))}{1+s^{\alpha-1}}\right| ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int\limits_0^\infty q(s)L_1(s)\left|\frac{u(s-\theta_1(s))-v(s-\theta_1(s))}{1+s^{\alpha-1}}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_0^\infty q(s)L_2(s)\left|\frac{u(s-\theta_2(s))-v(s-\theta_2(s))}{1+s^{\alpha-1}}\right| ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int\limits_0^{t_1} q(s)L_1(s)\left|\frac{u(s-\theta_1(s))-v(s-\theta_1(s))}{1+s^{\alpha-1}}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_1}^\infty q(s)L_1(s)\left|\frac{u(s-\theta_1(s))-v(s-\theta_1(s))}{1+s^{\alpha-1}}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_0^{t_2} q(s)L_2(s)\left|\frac{u(s-\theta_2(s))-v(s-\theta_2(s))}{1+s^{\alpha-1}}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_2}^\infty q(s)L_2(s)\left|\frac{u(s-\theta_2(s))-v(s-\theta_2(s))}{1+s^{\alpha-1}}\right| ds$$

$$\leq \frac{2C}{\Gamma(\alpha)}\left(\|u-v\|_0+\|u-v\|_\infty\right), \tag{2.5}$$

consequently,

$$\|Tu - Tv\|_X \leq \frac{2C}{\Gamma(\alpha)} \|u - v\|_X .$$

Taking (2.4) and (2.5) into account, we obtain

$$\|Tu - Tv\|_X \leq \frac{2C}{\Gamma(\alpha)} \|u - v\|_X .$$

Thanks to (2.3), we conclude that T is a contraction then by Banach contraction principle we deduce that the operator T has a unique fixed point in X which is the unique solution of the problem (P). ■

The properties of the operator T are given in the following.

Theorem 2.2.2 *Assume that (H_2) and (H_3) hold. Then the operator T is completely continuous.*

Proof We prove it in four steps.

Step 1: We show that T is continuous. Let $u_n \to u$ as $n \to \infty$ in X, we have

$$\frac{u_n(t)}{1+t^{\alpha-1}} \to \frac{u(t)}{1+t^{\alpha-1}},\ n \to \infty.$$

If $-\tau \leq t \leq 0$, then

$$\|Tu_n - Tu\|_0 = \max_{t\in[-\tau,0]} |Tu_n(t) - Tu(t)| = 0.$$

If $t > 0$, then the continuity of T follows from the continuity of f, in fact we have

$$\left|\frac{Tu_n(t) - Tu(t)}{1+t^{\alpha-1}}\right| \leq \frac{1}{\Gamma(\alpha)} \int\limits_0^{\infty} q(s) \left| f(s, u_n(s-\theta_1(s)), u_n(s-\theta_2(s)) \right.$$
$$\left. - f(s, u(s-\theta_1(s)), u(s-\theta_2(s)) \right| ds \to 0,\ n \to \infty,$$

thus

$$\|Tu_n - Tu\|_X \to 0,\ n \to \infty,$$

hence, T is continuous.

Step 2: The operator T is uniformly bounded. In fact, let $L > 0$ and $\Delta = \{u \in X, \|u\|_X < L\}$ be any bounded subset of X. Let $u \in \Delta$, then

$$\|Tu\|_0 = \max_{t\in[-\tau,0]} |Tu(t)| = \|\varphi\|_0 < \infty.$$

$$\begin{aligned}
\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| &\leq \left|\varphi(0)\frac{t^{\alpha-1}}{1+t^{\alpha-1}} + \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} q(s) f(s, u(s-\theta_1(s)), u(s-\theta_2(s)) ds\right| \\
&\leq |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s) \left| f(s, u(s-\theta_1(s)), u(s-\theta_2(s))\right| ds \\
&\leq |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s) \left(a(s) \left|u(s-\theta_1(s))\right| + b(s) \left|u(s-\theta_2(s))\right| + c(s)\right) ds \\
&\leq |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)a(s) \left|u(s-\theta_1(s))\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^\infty b(s) \left|u(s-\theta_2(s))\right| ds + \frac{1}{\Gamma(\alpha)} \int_0^\infty c(s) ds \\
&\leq |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} q(s)a(s) \left|u(s-\theta_1(s))\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^\infty q(s)a(s) \left|u(s-\theta_1(s))\right| ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} q(s)b(s) \left|u(s-\theta_2(s))\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^\infty q(s)b(s) \left|u(s-\theta_2(s))\right| ds + \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s) ds \\
&\leq |\varphi(0)| + \frac{\|u\|_0}{\Gamma(\alpha)} \int_0^{t_1} q(s)a(s) ds + \frac{\|u\|_\infty}{\Gamma(\alpha)} \int_{t_1}^\infty q(s)a(s) \left(1+s^{\alpha-1}\right) ds \\
&+ \frac{\|u\|_0}{\Gamma(\alpha)} \int_0^{t_2} q(s)b(s) ds + \frac{\|u\|_\infty}{\Gamma(\alpha)} \int_{t_2}^\infty q(s)b(s) \left(1+s^{\alpha-1}\right) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s) ds
\end{aligned}$$

$$\le |\varphi(0)| + \frac{\|u\|_0}{\Gamma(\alpha)} \int_0^\infty q(s)a(s)ds + \frac{\|u\|_\infty}{\Gamma(\alpha)} \int_0^\infty q(s)a(s)\left(1 + s^{\alpha-1}\right) ds$$
$$+ \frac{\|u\|_0}{\Gamma(\alpha)} \int_0^\infty q(s)b(s)ds + \frac{\|u\|_\infty}{\Gamma(\alpha)} \int_0^\infty q(s)b(s)\left(1 + s^{\alpha-1}\right) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s)ds$$

$$\begin{aligned} \le \quad & |\varphi(0)| + \frac{L}{\Gamma(\alpha)} \int_0^\infty q(s)\left(a(s) + b(s)\right) ds \\ & + \frac{L}{\Gamma(\alpha)} \int_0^\infty q(s)\left(a(s) + b(s)\right)\left(1 + s^{\alpha-1}\right) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s)ds < \infty. \end{aligned}$$

Hence, $T(\Delta)$ is uniformly bounded.

Step 3. We shall prove that $T(\Delta)$ is equicontinuous on any compact subinterval of $[-\tau, \infty)$.

Case 1. Let $S > 0$, $t_1, t_2 \in [0, S]$ and $u \in \Delta$. Assume that $t_2 > t_1$, then

$$\left| \frac{Tu(t_2)}{1 + t_2^{\alpha-1}} - \frac{Tu(t_1)}{1 + t_1^{\alpha-1}} \right| \quad \le \quad \left| \varphi(0) \left(\frac{t_2^{\alpha-1}}{1 + t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1 + t_1^{\alpha-1}} \right) \right|$$

$$+ \int_0^\infty \left| \frac{G(t_2, s)}{1 + t_2^{\alpha-1}} - \frac{G(t_1, s)}{1 + t_1^{\alpha-1}} \right| |q(s)f(s, u(s - \theta_1(s)), u(s - \theta_2(s))| \, ds$$

$$\leq \left|\varphi(0)\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\right|$$

$$+\int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$+\int_0^\infty \left|\frac{G(t_1,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$\leq \left|\varphi(0)\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\right|$$

$$+\int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$+\int_0^\infty \left|\frac{G(t_1,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$\leq \left|\varphi(0)\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\right|$$

$$+\int_0^{t_1}\left|\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})-((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{\left(1+t_2^{\alpha-1}\right)\Gamma(\alpha)}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$+\int_{t_1}^{t_2}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$+\int_{t_2}^{\infty}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\left(1+t_2^{\alpha-1}\right)\Gamma(\alpha)}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$+\int_0^{\infty}\left|G(t_1,s)\left(\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right)\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$\leq \left|\varphi(0)\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|$$

$$+ \int_0^{t_1} \left|\frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) - ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})}{\left(1+t_2^{\alpha-1}\right)\Gamma(\alpha)}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0 + \left(1+s^{\alpha-1}\right)\|u\|_\infty\right) + c(s)\right] ds$$

$$+ \frac{3S^{\alpha-1}}{\left(1+t_2^{\alpha-1}\right)\Gamma(\alpha)} \int_{t_1}^{t_2} |q(s)\left((a(s)+b(s))\|u\|_0\right.$$

$$\left. + (a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty + c(s)\right)| \, ds$$

$$+ \left|\frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\left(1+t_2^{\alpha-1}\right)\Gamma(\alpha)}\right| \int_{t_2}^{\infty} |q(s)\left((a(s)+b(s))\|u\|_0\right.$$

$$\left. + (a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty + c(s)\right)| \, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \left|\frac{t_1^{\alpha-1} - t_2^{\alpha-1}}{\left(1+t_2^{\alpha-1}\right)\left(1+t_1^{\alpha-1}\right)}\right|$$

$$\times \int_0^{\infty} |q(s)\left((a(s)+b(s))\|u\|_0\right.$$

$$\left. + (a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty + c(s)\right)| \, ds$$

$$\to 0, \text{ as } t_1 \to t_2$$

Case 2. Let $-\tau \leq t_1 < t_2 \leq 0$, then

$$\max_{t_1,t_2\in[-\tau,0]} |Tu(t_2) - Tu(t_1)| \quad = \quad \max_{t_1,t_2\in[-\tau,0]} |\varphi(t_2) - \varphi(t_1)| \to 0, \text{ as } t_1 \to t_2$$

Case 3. Let $-\tau \leq t_1 < 0 < t_2 < \infty$, hence

$$\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| = \left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - Tu(0) + Tu(0) - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$\leq \left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - Tu(0)\right| + \left|Tu(0) - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$\leq \left|\varphi(0)\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right| + \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - G(0,s)\right|$$

$$\times\, q(s)\left[(a(s)+b(s))\left(\|u\|_0 + \left(1+s^{\alpha-1}\right)\|u\|_\infty\right) + c(s)\right] ds$$

$$+ \left|\varphi(0) - \frac{\varphi(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$\leq \left|\varphi(0)\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right| + \int_0^{t_2} \left|\frac{t_2^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - s^{\alpha-1}\right|$$

$$\times\, q(s)\left[(a(s)+b(s))\left(\|u\|_0 + \left(1+s^{\alpha-1}\right)\|u\|_\infty\right) + c(s)\right] ds + \int_{t_2}^{\infty} \left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|$$

$$\times\, q(s)\left[(a(s)+b(s))\left(\|u\|_0 + \left(1+s^{\alpha-1}\right)\|u\|_\infty\right) + c(s)\right] ds$$

$$+ \left|\varphi(0) - \frac{\varphi(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$\to 0, \text{ as } t_1 \to 0^-, t_2 \to 0^+.$$

Consequently, $T(\Delta)$ is equicontinuous on any compact subinterval of $[-\tau, \infty)$.

Step 4. We show that T is equiconvergent at ∞. In view of assumption (H_3), it yields for any $u \in \Delta$,

$$\int_0^\infty q(s)\left|f(s, u(s-\theta_1(s)), u(s-\theta_2(s))\right| ds$$

$$\leq \int_0^\infty q(s)\left((a(s)+b(s))\|u\|_0 + (a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty + c(s)\right) ds$$

$$\leq L\int_0^\infty q(s)\left(a(s)+b(s)\right) ds + L\int_0^\infty q\left(s\right)\left(a(s)+b(s)\right)\left(1+s^{\alpha-1}\right) ds + \int_0^\infty q(s)c(s)ds$$

$$< \infty.$$

Now, since $\lim_{t\to\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}=1$, then there exists $T_1>0$ such that , for any $t_2>t_1>T_1$,

$$\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \le \left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-1\right|+\left|1-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|<\varepsilon. \tag{2.6}$$

Moreover, we have $\lim_{t\to\infty}\frac{(t-N)^{\alpha-1}}{1+t^{\alpha-1}}=1$ for all $N>0$. Thus there exists T_2 such that, for any $t_2>t_1>T_2$ and $0\le s\le N$,

$$\begin{aligned}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right| &\le \left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-1\right|+\left|1-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \\ &\le \left|1-\frac{(t_2-N)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|+\left|1-\frac{(t_1-N)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|<\varepsilon.\end{aligned} \tag{2.7}$$

Choose $M>\max\{T_1,T_2\}$, then for any $u\in\Delta$, $t_2>t_1>M$ and $t_1\to t_2$, by (2.6)– (2.7) we obtain

$$\begin{aligned}\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \le& \left|\varphi(0)\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\right| \\ &+\int_0^\infty\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| |q(s)f(s,u(s-\theta_1(s)),u(s-\theta_2(s))|\,ds \\ \le& \left|\varphi(0)\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\right| \\ &+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}+\frac{(t_1-s)^{\alpha-1})}{1+t_1^{\alpha-1}}\right| \\ &\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|$$

$$\times q(s)\left[(a(s)+b(s))\left(\|u\|_0+\left(1+s^{\alpha-1}\right)\|u\|_\infty\right)+c(s)\right]ds$$

$$\leq \varphi(0)\varepsilon++\frac{2\varepsilon}{\Gamma(\alpha)}\int_{0}^{t_1}|q(s)\left((a(s)+b(s))\|u\|_0\right.$$

$$\left.+(a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty+c(s)\right)\Big|\,ds$$

$$+\frac{\varepsilon}{\Gamma(\alpha)}\int_{t_1}^{t_2}|q(s)\left((a(s)+b(s))\|u\|_0\right.$$

$$\left.+(a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty+c(s)\right)\Big|\,ds$$

$$+\frac{\varepsilon}{\Gamma(\alpha)}\int_{t_2}^{\infty}|q(s)\left((a(s)+b(s))\|u\|_0\right.$$

$$\left.+(a(s)+b(s))\left(1+s^{\alpha-1}\right)\|u\|_\infty+c(s)\right)\Big|\,ds$$

$$\leq \left(\varphi(0) + \frac{6}{\Gamma(\alpha)}\right)\varepsilon \int_0^\infty |q(s)\left((a(s)+b(s)\right)\|u\|_0$$

$$+\left(a(s)+b(s)\right)\left(1+s^{\alpha-1}\right)\|u\|_\infty + c(s)\big)\big|\,ds$$

$$\to 0 \text{ as } t_1, t_2 > M.$$

Hence T is equiconvergent at ∞. Finally by Lemma 2.2.3 we conclude that T is completely continuous. ■

Now we give an existence result.

Theorem 2.2.3 *Assume that* (H_2) *and* (H_3) *hold. Then boundary value problem* (P) *has at least one solution.*

Proof We will prove that all the assumptions of the Leray-Schauder nonlinear alternative are satisfied. Set

$$U = \left\{u \in X : \|u\|_X < \eta\right\},$$

where

$$2\|\varphi\|_0 + \frac{\eta}{\Gamma(\alpha)}\int_0^\infty q(s)\left(a(s)+b(s)\right)\left[1+\left(1+s^{\alpha-1}\right)\right]ds + \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)c(s)ds < \eta. \tag{2.8}$$

Assume that there exists $u \in \partial U$ with $u = \lambda T u$ and $\lambda \in (0,1)$, then

$$\|u\|_0 = \|\lambda T u\|_0 = \max_{t\in[-\tau,0]}|\lambda T u(t)| \leq \max_{t\in[-\tau,0]}|Tu(t)| = \max_{t\in[-\tau,0]}|\varphi(t)| = \|\varphi\|_0,$$

$$\|u\|_\infty = \|\lambda T u\|_\infty = \sup_{t\in[0,\infty)}\left|\frac{\lambda T u(t)}{1+t^{\alpha-1}}\right| \leq \sup_{t\in[0,\infty)}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right|$$

$$\leq \sup_{t\in[0,\infty)}\left|\varphi(0)\frac{t^{\alpha-1}}{1+t^{\alpha-1}} + \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}q(s)f(s,u(s-\theta_1(s)),u(s-\theta_2(s))ds\right|$$

$$\leq |\varphi(0)| + \frac{\|u\|_0}{\Gamma(\alpha)} \int_0^\infty q(s)\,(a(s)+b(s))\,ds + \frac{\|u\|_\infty}{\Gamma(\alpha)} \int_0^\infty q(s)\,(a(s)+b(s))\left(1+s^{\alpha-1}\right) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s)ds$$

$$\leq \|\varphi\|_0 + \frac{\eta}{\Gamma(\alpha)} \int_0^\infty q(s)\,(a(s)+b(s))\,ds + \frac{\eta}{\Gamma(\alpha)} \int_0^\infty q(s)\,(a(s)+b(s))\left(1+s^{\alpha-1}\right) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s)ds$$

Now taking into account (2.8) and the fact that $u \in \partial U$, it yields

$$\begin{aligned} \eta &= \|u\|_X = \|u\|_0 + \|u\|_\infty \\ &\leq 2\|\varphi\|_0 + \frac{\eta}{\Gamma(\alpha)} \int_0^\infty q(s)\,(a(s)+b(s))\left[1+\left(1+s^{\alpha-1}\right)\right] ds + \frac{1}{\Gamma(\alpha)} \int_0^\infty q(s)c(s)ds \\ &< \eta, \end{aligned}$$

which is impossible and then we conclude by the nonlinear alternative of Leray-Schauder that the operator T has a fixed point in $\overline{U}$ and then the problem (P) has at least one solution satisfying $\|u\|_X \leq \eta$. ■

2.3 Stability of solutions

In this section, we analyze the stability of the solution for the nonlinear fractional boundary value problem (P). Let $\tilde{u}$ be a solution of the following fractional boundary value problem

$$(\tilde{P})\begin{cases} D_{0^+}^{\alpha}\tilde{u}\,(t) - q(t)f(t,\tilde{u}(t-\theta_1(t)),\check{u}(t-\theta_2(t)) = 0, & 2 \leq \alpha < 3, t > 0 \\ \tilde{u}\,(t) = \tilde{\varphi}(t), & t \in [-\tau, 0] \\ \tilde{u}''(0) = 0, \lim_{t\to\infty} D^{\alpha-1}\tilde{u}(t) = \Gamma(\alpha)\tilde{u}(0). \end{cases}$$

By a stable solution we mean the following:

Definition 2.3.1 *The solution of the fractional boundary value problem (P) is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any two solutions u and $\tilde{u}$ of the problems (P) and $(\tilde{P})$ respectively, one has $\|\varphi - \tilde{\varphi}\|_0 \leq \delta$, then $\|u - \tilde{u}\|_X < \varepsilon$.*

Theorem 2.3.1 *Under the assumptions (H_1) and (H_2), the unique solution of the fractional boundary value problem (P) is uniformly stable.*

Proof Let u be the unique solution of the problem (P) and $\tilde{u}$ be the unique solution of the problem $(\tilde{P})$. Then we get

$$\|u - \tilde{u}\|_0 = \max_{t\in[-\tau,0]} |u(t) - \tilde{u}(t)| = \max_{t\in[-\tau,0]} |\varphi(t) - \tilde{\varphi}(t)| = \|\varphi - \tilde{\varphi}\|_0 . \tag{2.9}$$

Now, let $t > 0$, then we have

$$\begin{aligned} \left|\frac{u(t) - \tilde{u}(t)}{1+t^{\alpha-1}}\right| &\leq \frac{t^{\alpha-1}}{1+t^{\alpha-1}} |\varphi(0) - \tilde{\varphi}(0)| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\infty} q(s) \left| f(s, u(s-\theta_1(s)), u(s-\theta_2(s)) \right. \\ &\quad \left. - f(s, \tilde{u}(s-\theta_1(s)), \tilde{u}(s-\theta_2(s)) \right| ds \\ &\leq \frac{t^{\alpha-1}}{1+t^{\alpha-1}} |\varphi(0) - \tilde{\varphi}(0)| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\infty} q(s) L_1(s) \left| \frac{u(s-\theta_1(s)) - \tilde{u}(s-\theta_1(s))}{1+s^{\alpha-1}} \right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\infty} q(s) L_2(s) \left| \frac{u(s-\theta_2(s)) - \tilde{u}(s-\theta_2(s))}{1+s^{\alpha-1}} \right| ds. \end{aligned}$$

By following a reasoning similar to that of the proof of theorem 2.2.1, we obtain

$$\|u-\tilde{u}\|_{\infty} \leq \left(1+\frac{2C}{\Gamma(\alpha)}\right)\|\varphi-\tilde{\varphi}\|_0 + \frac{2C}{\Gamma(\alpha)}\|u-\tilde{u}\|_{\infty},$$

thus

$$\|u-\tilde{u}\|_{\infty} \leq \left(\frac{\Gamma(\alpha)+2C}{\Gamma(\alpha)-2C}\right)\|\varphi-\tilde{\varphi}\|_0 . \tag{2.10}$$

From (2.9) and (2.10), we get

$$\|u-\tilde{u}\|_X \leq \left(\frac{2\Gamma(\alpha)}{\Gamma(\alpha)-2C}\right)\|\varphi-\tilde{\varphi}\|_0 .$$

Hence, for $\epsilon > 0$, there exists $\delta = \left(\frac{2\Gamma(\alpha)}{\Gamma(\alpha)-2C}\right)^{-1}\varepsilon$ such that if $\|\varphi-\tilde{\varphi}\|_0 < \delta$, then $\|u-\tilde{u}\|_X < \epsilon$, that proves the uniformly stability of the unique solution. ■

Now we give some numerical examples.

2.4 Examples

Example 2.4.1 *Consider the fractional boundary value problem (P) where*

$$\alpha = \frac{12}{5},\ f(t,x,y) = \frac{e^{-t}}{6}\left(x+ty-\frac{1}{1+x^2}\right),\ \varphi(t) = t^2,$$
$$q(t) = \frac{1}{1+t^{\alpha-1}},\ \theta_1(t) = \frac{t}{2}+\frac{1}{2},\ \theta_2(t) = \frac{2t}{3}+\frac{1}{3},\ \tau = \frac{1}{2}.$$

Then the assumptions (H_1) and (H_2) hold. In fact, if we choose

$$L_1(t) = \frac{e^{-t}}{6}\left(1+t^{\alpha-1}+\left(1+t^{\alpha-1}\right)^2\right),\ L_2(t) = \frac{te^{-t}\left(1+t^{\alpha-1}\right)}{6},$$

then the assumption (H_1) holds with $C = 0,54036$.

Choosing $t_1 = t_2 = 1$, then $t-\theta_i(t) < 0$, for $0 \leq t \leq 1$, and $t-\theta_i(t) \geq 0$, if $t > 1, i = 1,2$. Hence the hypothesis (H_2) is satisfied. By Theorems 2.2.1 and 2.3.1,we deduce that the problem (P) has a unique solution which is uniformly stable in X.

Example 2.4.2 *Consider the fractional boundary value problem (P) where*

$$\alpha=\frac{5}{2},\ f(t,x,y)=\frac{e^{-t}}{10}\left(\frac{x}{1+t}+ty\right),\ \varphi(t)=t^2,$$
$$q(t)=\frac{1}{1+t^{\frac{3}{2}}},\ \theta_1(t)=1,\ \theta_2(t)=\frac{t}{2}+\frac{1}{2},\ \tau=1$$

Let us check the hypotheses of Theorem 2.2.1. In fact the hypothesis (H_1) *holds if we choose*

$$L_1(t)=\frac{e^{-t}}{10}\left(\frac{1+t^{\frac{3}{2}}}{1+t}\right),\qquad L_2(t)=\frac{te^{-t}\left(1+t^{\frac{3}{2}}\right)}{10}$$

and then

$$\int_0^\infty q(s)L_1(s)ds=\int_0^\infty \frac{1}{1+s}\frac{e^{-s}}{10}ds=5.963\,5\times10^{-2},$$
$$\int_0^\infty q(s)L_2(s)ds=\int_0^\infty \frac{se^{-s}}{10}ds=0.1,\qquad \Gamma(\frac{5}{2})=1.329\,3,$$

$$C=0,1<0,66465.$$

Furthermore, there exist $t_1=t_2=1$*, such that* $t-\theta_i(t)<0$*, for* $0\le t\le1$ *and* $t-\theta_i(t)\ge0$*, if* $t>1, i=1,2$*. Then the hypothesis* (H_2) *is satisfied. We deduce by Theorems 2.2.1 and 2.3.1 that the problem* (P) *has a unique solution that is uniformly stable in* X*.*

Example 2.4.3 *Consider the fractional boundary value problem (P) where*

$$\alpha=\frac{12}{5},\ f(t,x,y)=\frac{\ln(1+|y|)}{30\,(1+t^2)}+\frac{\sqrt{|xy|}}{30\exp\left\{\sqrt{t}\right\}}+\left(\frac{1+t^{\frac{7}{5}}}{30\,(1+t^2)}\right),$$

$$\varphi(t)=t^2,\ q(t)=\frac{1}{1+t^{\frac{7}{5}}},\ \theta_1(t)=\frac{t}{2}+\frac{1}{2},$$

$$\theta_2(t)=\frac{2t}{3}+\frac{1}{3},\ \tau=\frac{1}{2}.$$

By computation, we get

$$|f(t,x,y)| \leq \frac{|x|\, e^{-\sqrt{t}}}{60} + \left(\frac{1}{30\,(1+t^2)} + \frac{e^{-\sqrt{t}}}{60} \right) |y| + \left(\frac{1+t^{\frac{7}{5}}}{30\,(1+t^2)} \right),$$

then, we can set

$$\begin{aligned} a(t) &= \frac{e^{-\sqrt{t}}}{60}, \\ b(t) &= \left(\frac{1}{30\,(1+t^2)} + \frac{e^{-\sqrt{t}}}{60} \right), \\ c(t) &= \left(\frac{1+t^{\frac{7}{5}}}{30\,(1+t^2)} \right), \end{aligned}$$

and by calculation, we obtain

$$\int_0^\infty q(t) \left(1+t^{\alpha-1}\right) (a(t)+b(t))\, dt$$

$$= \int_0^\infty \frac{1}{1+t^{\frac{7}{5}}} \left(1+t^{\frac{7}{5}}\right) \left(\frac{e^{-\sqrt{t}}}{60} + \frac{1}{30\,(1+t^2)} + \frac{e^{-\sqrt{t}}}{60} \right) dt = 0.119\,03,$$

$$\int_0^\infty q(t)\, (a(t)+b(t))\, dt$$

$$= \int_0^\infty \frac{1}{1+t^{\frac{7}{5}}} \left(\frac{1}{60 \exp\left\{\sqrt{t}\right\}} + \frac{1}{30\,(1+t^2)} + \frac{1}{60 \exp\left\{\sqrt{t}\right\}} \right) dt$$

$$= 4.728\,4 \times 10^{-2},$$

$$\int_0^\infty q(t)c(t)dt = \int_0^\infty \frac{1}{1+t^{\frac{7}{5}}}\left(\frac{1+t^{\frac{7}{5}}}{30\left(1+t^2\right)}\right)dt = 5.2360\times 10^{-2},$$

$$\Gamma\left(\frac{12}{5}\right) = 1.2422, \|\varphi\|_0 = \frac{1}{4}.$$

Moreover, the inequality (2.8) is satisfied if $\eta > 2.6465$. Hence, from Theorem 2.2.3, we conclude that problem (P) has at least one solution u such that $\|u\|_X \leq \eta$.

Example 2.4.4 *Consider the fractional boundary value problem (P) with*

$$\begin{aligned}
\alpha &= \frac{5}{2},\ f(t,x,y) = \frac{|y|}{40\,(1+t)} + \frac{e^{-t}\sqrt{|xy|}}{40} + \left(\frac{1}{40\left(1+t^{\frac{3}{2}}\right)(1+t)}\right),\\
\varphi(t) &= t^2,\ q(t) = e^{-t},\ \theta_1(t) = 1,\ \theta_2(t) = \frac{t}{2}+\frac{1}{2},\ \tau = 1.
\end{aligned}$$

Then by computation it yields

$$|f(t,x,y)| \leq \frac{e^{-t}}{80}|x| + \left(\frac{1}{40\,(1+t)} + \frac{e^{-t}}{80}\right)|y| + \left(\frac{1}{40\left(1+t^{\frac{3}{2}}\right)(1+t)}\right),$$

$$a(t) = \frac{e^{-t}}{80},\quad b(t) = \left(\frac{1}{40\,(1+t)} + \frac{e^{-t}}{80}\right),$$

$$c(t) = \left(\frac{1}{40\left(1+t^{\frac{3}{2}}\right)(1+t)}\right),$$

$$\begin{aligned}
&\int_0^\infty q(t)\left(1+t^{\alpha-1}\right)(a(t)+b(t))\,dt\\
&= \int_0^\infty \exp\{-t\}\left(1+t^{\frac{3}{2}}\right)\left(\frac{2e^{-t}}{80} + \frac{1}{40\,(1+t)}\right)dt = 0.04471,
\end{aligned}$$

$$\int_{0}^{\infty} q(t)\left(a(t)+b(t)\right)dt = \int_{0}^{\infty} e^{-t}\left(\frac{2e^{-t}}{80}+\frac{1}{40\left(1+t\right)}\right)dt = 2.740\,9\times 10^{-2},$$

$$\int_{0}^{\infty} q(t)c(t)dt = \int_{0}^{\infty} e^{-t}\frac{1}{40\left(1+t^{\frac{3}{2}}\right)\left(1+t\right)}dt = 1.062\,4\times 10^{-2}.$$

Finally, if we choose $\eta > 2.123\,2$ *then the inequality (2.8) holds and then we conclude by Theorem 2.2.3 that the problem* (P) *has at least one solution* u *such that* $\|u\|_X \leq \eta$.

CHAPTER 3

Existence of solutions for integral boundary value problems with p-Laplacian operator on infinite interval

3.1 Introduction

Differential equations with p-Laplacian operator appear in the modeling of several problems in science and engineering. It is well known that differential equations with p-Laplacian operators are often used to simulate practical problems such as tides caused by celestial gravity and elastic deformation of beams [17, 79].

To solve fractional differential equations with a p-Laplacian operator, several methods are used such as the fixed point theory, upper and lower solutions method, coincidence degree theory, iterative method, ... [57, 51, 17, 11, 79, 7]

In [57], the authors discussed the existence of solutions for the following p-Laplacian fractional boundary value problem involving left and right fractional derivatives,

$$\begin{cases} -{}^{C}D^{\beta}_{1^-}\left(\phi_p\left(D^{\alpha}_{0^+}u\,(t)\right)\right)+f\,(t,u\,(t))=0,\ 0\le t\le 1,\\ u\,(0)=u'\,(0)=0,\ D^{\alpha}_{0^+}u(1)=0,\end{cases}$$

where $1<\alpha<2$, $0<\beta<1$, ${}^{C}D^{\beta}_{1^-}$ represents the right Caputo derivative of order β, defined as

$${}^{C}D^{\beta}_{1-}f\,(t)=\frac{(-1)^n}{\Gamma\,(n-\beta)}\int_t^1 (s-t)^{n-\beta-1}\,f^{(n)}\,(s)\,ds,\qquad t<1.$$

$D^{\alpha}_{0^+}$ denotes the left Riemann-Liouville derivative of order α and $f\in C([0,1]\times\mathbb{R},\mathbb{R})$. Using the lower and upper solutions method and Schauder's fixed point theorem, they proved the existence results.

In [35], a multipoint Riemann-Liouville fractional boundary value problem is studied,

$$D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u\,(t)))=f(t,u(t)),\qquad 0<t<1$$

$$u(0)=0, D^{\gamma}_{0^+}u(1)=\textstyle\sum_{i=1}^{m-2}a_iD^{\gamma}_{0^+}u(\xi_i),$$

$$D^{\alpha}_{0^+}u(0)=0,\ \phi_p(D^{\alpha}_{0^+}u\,(1)))=\textstyle\sum_{i=1}^{m-2}b_i\phi_p\left(D^{\alpha}_{0^+}u(\eta_i)\right),$$

where $1<\alpha,\beta\le 2$, $0<\gamma\le 1$, $0<\xi_i,\eta_i<1$, $i=1,2...m-2$, the function f is nonnegative and may be singular at $u = 0$. Thanks to a fixed point theorem for mixed monotone operators, the existence of positive solutions is proved.

This chapter focuses on the study of a fractional differential equation over an infinite interval and involving fractional derivatives of Riemann-Liouville type and the p-Laplacian operator:

$$\begin{cases} D_{0^+}^{\delta}(\phi_p(D_{0^+}^{\alpha}u\,(t))) + a(t)f(t,u(t),D_{0^+}^{\alpha-1}u(t)) = 0,\ t>0 \\ u(0)=0,\ D_{0^+}^{\alpha-1}u(+\infty) = \int_0^{+\infty} g(s)u(s)ds, \\ D_{0^+}^{\alpha-2}u(0) = \int_0^{+\infty} h(s)u(s)ds,\ I_{0^+}^{1-\delta}(\phi_p(D_{0^+}^{\alpha}u\,(0))) = 0, \end{cases} \tag{P}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\delta}$ denote the Riemann-Liouville fractional derivatives, $2 < \alpha \leq 3$, $0 < \delta \leq 1$, $\phi_p(s) = |s|^{p-2}\,s$, $p > 1$. The functions $a : [0,+\infty) \to [0,+\infty)$, $f : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and $g, h \in L_1\,[0,+\infty)$ are nonnegative.

To prove the existence of solutions for the nonlinear fractional boundary value problem (P), we use Krasnoselskii fixed point theorem and the boundedness of the Riemann-Liouville fractional operator.

3.2 Existence of solutions

To solve the problem (P), we need the following theorem on the boundedness of Riemann-Liouville fractional integral operator.

Theorem 3.2.1 *[64] Let $1 \leq \mu, \nu \leq +\infty$ and $0 < \delta < 1$. If $1 < \mu < \frac{1}{\delta}$ and $\nu = \frac{\mu}{1-\delta\mu}$, then the operator $I_{0^+}^{\delta}$ is bounded from $L_\mu\,(0,+\infty)$ to $L_\nu\,(0,+\infty)$:*

$$\left(\int_0^{+\infty} \left|I_{0^+}^{\delta}a(r)\right|^{\nu} dr\right)^{\frac{1}{\nu}} \leq k\left(\int_0^{+\infty} |a(r)|^{\mu}\, dr\right)^{\frac{1}{\mu}},$$

where the constant $k = \frac{\Gamma\left(\frac{1}{\mu}-\delta\right)}{\Gamma\left(\frac{1}{\mu}\right)} > 0$.

Throughout this paper, we assume the following conditions.

(C1) For all $\rho > 0$*, there exists* $M_\rho > 0$ *such that*

$$\left|f\left(t, \left(1+t^{\alpha-1}\right)x, y\right)\right| \leq M_\rho, \text{ for all } t > 0,\ x, y \in [-\rho, \rho].$$

(C2) The function a *is not identical null on any closed subinterval of* $[0, +\infty)$ *and*

$$\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r)dr\right) ds < +\infty.$$

If $a \in L_\mu [0, +\infty)$, where $\mu = \frac{q-1}{1+\delta(q-1)}$ such that $0 < \mu < \frac{1}{\delta}$, then condition (C2) is satisfied. Indeed, taking $\nu = q - 1 = \frac{\mu}{1-\delta\mu}$ in Theorem 3.2.1, we obtain

$$\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r)dr\right) ds = \int_0^{+\infty} \left|I_{0^+}^{\delta} a(r)\right|^{q-1} dr$$

$$\leq k^{q-1} \left(\int_0^{+\infty} |a(r)|^{\mu} dr\right)^{\frac{q-1}{\mu}} < +\infty.$$

Set $(X, \|.\|)$ the Banach space [67],

$$\begin{aligned} X &= \Big\{u : u \in C\left(\mathbb{R}^+, \mathbb{R}\right),\ D_{0^+}^{\alpha-1}u \in C\left(\mathbb{R}^+, \mathbb{R}\right), \\ &\quad \sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty, \quad \sup_{t\in\mathbb{R}^+} \left|D_{0^+}^{\alpha-1}u(t)\right| < +\infty\Big\}, \end{aligned}$$

with the norm

$$\|u\| = \max\left\{\sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^{\alpha-1}}, \quad \sup_{t\in\mathbb{R}^+} \left|D_{0^+}^{\alpha-1}u(t)\right|\right\}.$$

The following compactness criteria will be useful as we are on infinite intervals [14, 67]

Lemma 3.2.1 *Let $Z \subseteq X$ be bounded set. Then Z is relatively compact in X if the following condition hold:*

i) The functions belonging to Z are equicontinuous on any compact subinterval of $[0, \infty)$, i.e. for any $u \in Z$, $\frac{u(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}u(t)$ are equicontinuous on any compact interval of $\mathbb{R}^+$.

ii) The functions from Z are equiconvergent at $+\infty$. i.e. given $\varepsilon > 0$,there exists a constant $l = l(\varepsilon) > 0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon \text{ and } \left|D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)\right| < \varepsilon,$$

for any $t_1, t_2 \geq l, u \in Z$.

Lemma 3.2.2 *Assume $e \in L^1(\mathbb{R}^+)$. The linear boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha}u(t) + e(t) = 0, \ t > 0 \\ u(0) = 0, \ D_{0^+}^{\alpha-1}u(+\infty) = \int_0^{+\infty} g(s)u(s)ds, \\ D_{0^+}^{\alpha-2}u(0) = \int_0^{+\infty} h(s)u(s)ds, \end{cases} \tag{3.1}$$

has a unique solution u given by

$$u(t) = \int_0^{+\infty} k(t,s)e(s)ds + \int_0^{+\infty} G(t,s)u(s)ds,$$

where

$$G(t,s) = \frac{g(s)t^{\alpha-1} + (\alpha-1)h(s)t^{\alpha-2}}{\Gamma(\alpha)}, \tag{3.2}$$

$$k(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t < +\infty \\ t^{\alpha-1}, & 0 \leq t \leq s < +\infty. \end{cases} \tag{3.3}$$

Proof Applying the integral operator $I_{0^+}^{\alpha}$ to the differential equation in (3.1), then using Lemma 1.3.1, we get

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I_{0^+}^{\alpha}e(t). \tag{3.4}$$

The initial condition $u(0) = 0$, implies $c_3 = 0$ and by the integral conditions $D_{0^+}^{\alpha-1}u(+\infty) = \int_0^{+\infty} g(s)u(s)ds$ and $D_{0^+}^{\alpha-2}u(0) = \int_0^{+\infty} h(s)u(s)ds$, it yields

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} e(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} g(s)u(s)ds, \\ c_2 &= \frac{1}{\Gamma(\alpha-1)}\int_0^{+\infty} h(s)u(s)ds. \end{aligned}$$

Now substituting the constants c_i, $i = 1, 2, 3$ in (3.4), we obtain

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^{+\infty} e(s)ds + \int_0^{+\infty} g(s)u(s)ds\right] \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^{+\infty} h(s)u(s)ds \\ = &\int_0^{+\infty} k(t,s)e(s)ds + \int_0^{+\infty} G(t,s)u(s)ds, \end{aligned}$$

where $k(t,s)$ and $G(t,s)$ are given (3.3) and (3.2) respectively. ■

Lemma 3.2.3 *The function u is a solution of the boundary value problem (P) if and only if u satisfies the integral equation*

$$\begin{aligned} u(t) \;=\; &\int_0^{+\infty} k(t,s)\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds \\ &+ \int_0^{+\infty} G(t,s)u(s)ds, \end{aligned}$$

where $k(t,s)$ and $G(t,s)$ are defined in (3.3) and (3.2) respectively.

Proof Applying the fractional Riemann-Liouville integral operator $I_{0^+}^{\delta}$ to the differential equation in (P), then using Lemma 1.3.1, we get

$$\phi_p\left(D_{0^+}^{\alpha}u(t)\right) = ct^{\delta-1} - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} a(s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds. \tag{3.5}$$

By the boundary condition $I_{0^+}^{1-\delta}(\phi_p(D_{0^+}^{\alpha}u\,(0))) = 0$ we obtain $c = 0$ and then applying $\phi_p^{-1} = \phi_q$ to the equation (3.5), it yields

$$D_{0^+}^{\alpha}u(t) = -\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} a(s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right). \tag{3.6}$$

Now it suffices to apply Lemma 3.2.2, to the fractional differential equation (3.6) to complete the proof. ■

Lemma 3.2.4 *The functions $k(t,s)$ and $G(t,s)$ are nonnegative and satisfy:*

1) $0 \leq \frac{k(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}$ *for all* $(t,s) \in (0,+\infty)\times[0,+\infty)$.

2) $0 \leq \frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}\left(g(s)+(\alpha-1)h(s)\right)$ *for all* $(t,s) \in (0,+\infty)\times[0,+\infty)$.

Define the operators A and $T : X \to X$ as

$$Au(t) = \int_0^{+\infty} G(t,s)u(s)ds,$$

and

$$Tu(t) = \int_0^{+\infty} k(t,s)\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds.$$

By computations, we get

$$
\begin{aligned}
D_{0^+}^{\alpha-1} Au(t) &= \int_0^{+\infty} g(s)u(s)ds, \\
D_{0^+}^{\alpha-1} Tu(t) &= \int_t^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) f(r, u(r), D_{0^+}^{\alpha-1} u(r) dr \right) ds.
\end{aligned}
$$

Theorem 3.2.2 *If conditions (C1) and (C2) hold, then the operator T is completely continuous.*

Proof First, we shall prove that T is continuous. Let (u_n) be a sequence in X that converges to u as n tends to ∞. Then by the Lebesgue dominated convergence theorem and the continuity of f, we get

$$
\int_0^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) f(r, u_n(r), D_{0^+}^{\alpha-1} u_n(r) dr \right) ds
$$

$$
\underset{n \to +\infty}{\longrightarrow} \int_0^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) f(r, u(r), D_{0^+}^{\alpha-1} u(r) dr \right) ds.
$$

Thus,

$$
\left| \frac{Tu_n(t) - Tu(t)}{1 + t^{\alpha-1}} \right| =
$$

$$
\left| \int_0^{+\infty} \frac{k(t,s)}{1+t^{\alpha-1}} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) f(r, u_n(r), D_{0^+}^{\alpha-1} u_n(r) dr \right) ds \right.
$$

$$
\left. - \int_0^{+\infty} \frac{k(t,s)}{1+t^{\alpha-1}} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) f(r, u(r), D_{0^+}^{\alpha-1} u(r) dr \right) ds \right|
$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\left|\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\right)\right.\right.$$
$$\left.\left.-\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|\right)$$
$$\to 0,\ n\to+\infty.$$

Moreover, we have

$$\left|D_{0^+}^{\alpha-1}Tu_n(t)-D_{0^+}^{\alpha-1}Tu(t)\right| =$$
$$\left|\int_t^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\right)ds\right.$$
$$\left.-\int_t^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|$$
$$\leq \int_0^{+\infty}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\right)\right.$$
$$\left.-\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds$$
$$\to 0, n\to+\infty.$$

So, T is continuous.

Second, we shall show that T is a compact operator. Let B be a nonempty bounded closed subset of X and let

$$M=\sup_{t\in[0,+\infty)}\left\{\left|f(t,u(t),D_{0^+}^{\alpha-1}u(t))\right|,u\in B\right\}.$$

Taking condition (C2) into account, we get for any $u \in B$,

$$\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| = \left|\int_0^{+\infty} \frac{k(t,s)}{1+t^{\alpha-1}}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right| \tag{3.7}$$

$$\leq \frac{M^{q-1}}{\Gamma(\alpha)}\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)dr\right)ds < +\infty$$

and

$$\left|D_{0^+}^{\alpha-1}Tu(t)\right| = \left|\int_t^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|$$

$$\leq \int_0^{+\infty}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|$$

$$\leq M^{q-1}\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)dr\right)ds < +\infty,$$

thus

$$\|Tu\| \leq M^{q-1}\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r)dr\right)ds < +\infty,$$

and so, $T(B)$ is bounded.

Furthermore, let $l = [0, T_0]$ be a compact interval for $T_0 > 0$. From the continuity of $\frac{k(t,s)}{1+t^{\alpha-1}}$ and $\frac{t^{\alpha-1}}{1+t^{\alpha-1}}$, $t, s \in l$, we have, for any $\varepsilon > 0$, there exists a constant $0 < \delta_1 < \varepsilon$ such that for all $t_1, t_2, s_1, s_2 \in l$ and $t_1 < t_2$, as $|t_1 - t_2| < \delta_1$, $|s_1 - s_2| < \delta_1$, we have

$$\left|\frac{k(t_1,s_1)}{1+t_1^{\alpha-1}} - \frac{k(t_2,s_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon, \left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right| < \varepsilon.$$

Therefore, we have

$$
\begin{aligned}
\left|\frac{Tu(t_1)}{1+t_1^{\alpha-1}}-\frac{Tu(t_2)}{1+t_2^{\alpha-1}}\right| \leq{}& \int_0^{t_2}\left|\frac{k(t_1,s)}{1+t_1^{\alpha-1}}-\frac{k(t_2,s)}{1+t_2^{\alpha-1}}\right| \times \\
&\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds \\
&+\int_{t_2}^{+\infty}\left|\frac{k(t_1,s)}{1+t_1^{\alpha-1}}-\frac{k(t_2,s)}{1+t_2^{\alpha-1}}\right| \times \\
&\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds \\
&\leq \varepsilon\int_0^{t_2}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right| \\
&\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds \\
&\leq \varepsilon\int_0^{t_2}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds \\
&+\frac{\varepsilon}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds \\
&\leq 2\varepsilon\int_0^{+\infty}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds,
\end{aligned}
$$

and

$$\left|D_{0^+}^{\alpha-1}Tu(t_1) - D_{0^+}^{\alpha-1}Tu(t_2)\right|$$

$$= \left|\int_{t_1}^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r) f(r, u(r), D_{0^+}^{\alpha-1}u(r) dr\right) ds\right.$$

$$\left. - \int_{t_2}^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r) f(r, u(r), D_{0^+}^{\alpha-1}u(r) dr\right) ds\right|$$

$$\leq M^{q-1}\int_{t_1}^{t_2} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r) dr\right) ds$$

$$\to 0 \text{ as } t_1 \to t_2,$$

hence $\frac{Tu(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}Tu(t)$ are equicontinuous.

Third, we shall show that T is equiconvergent at $+\infty$. Let $u \in B$, then

$$\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r) \left|f(r, u(r), D_{0^+}^{\alpha-1}u(r)\right| dr\right) ds$$

$$\leq M^{q-1}\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} a(r) dr\right) ds < +\infty,$$

consequently, for $\varepsilon > 0$, there exists a constant $L > 0$ such that

$$\int_L^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} (r) \left|f(r, u(r), D_{0^+}^{\alpha-1}u(r)\right| dr\right) ds < \varepsilon.$$

Now, since $\lim_{t\to+\infty} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} = 1$, then there exists a constant $T_1 > 0$ such that for any $t_1, t_2 \geq T_1$

$$\left| \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} \right| < \varepsilon.$$

From $\lim_{t\to+\infty} \frac{K(t,L)}{1+t^{\alpha-1}} = 0$, and the inequalities $0 \leq \frac{K(t,s)}{1+t^{\alpha-1}} \leq \frac{K(t,L)}{1+t^{\alpha-1}}$ for $0 \leq s \leq L$, there exists a constant $T_2 > L$ such that for any $t_1, t_2 \geq T_2$ and $0 \leq s \leq L$, we have

$$\left| \frac{k(t_2,s)}{1+t_1^{\alpha-1}} - \frac{k(t_2,s)}{1+t_2^{\alpha-1}} \right| < \varepsilon.$$

Set $T_3 > \max\{T_1, T_2\}$, then for any $t_1, t_2 \geq T_3$ and by using Lemma 3.2.4, we get

$$\begin{aligned}
\left| \frac{Tu(t_1)}{1+t_1^{\alpha-1}} - \frac{Tu(t_2)}{1+t_2^{\alpha-1}} \right| \leq & \int_0^{+\infty} \left| \frac{k(t_1,s)}{1+t_1^{\alpha-1}} - \frac{k(t_2,s)}{1+t_2^{\alpha-1}} \right| \\
& \times \phi_q\left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1}u(r)) \right| dr \right) ds \\
& \leq \int_0^L \left| \frac{k(t_2,s)}{1+t_1^{\alpha-1}} - \frac{k(t_2,s)}{1+t_2^{\alpha-1}} \right| \\
& \times \phi_q\left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1}u(r)) \right| dr \right) ds \\
& + \int_L^{+\infty} \left| \frac{k(t_2,s)}{1+t_1^{\alpha-1}} - \frac{k(t_2,s)}{1+t_2^{\alpha-1}} \right| \\
& \times \phi_q\left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1}u(r)) \right| dr \right) ds
\end{aligned}$$

$$\leq \varepsilon \int_0^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1} u(r)) \right| dr \right) ds$$

$$+ \frac{2}{\Gamma(\alpha)} \int_L^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1} u(r)) \right| dr \right) ds$$

$$\leq \varepsilon \int_0^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1} u(r)) \right| dr \right) ds$$

$$+ \frac{2}{\Gamma(\alpha)} \varepsilon \leq \left(M^{q-1} \int_0^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) dr \right) ds + \frac{2}{\Gamma(\alpha)} \right) \varepsilon,$$

moreover, we have

$$\left| D_{0^+}^{\alpha-1} Tu(t_1) - D_{0^+}^{\alpha-1} Tu(t_2) \right|$$

$$\leq \int_{t_1}^{t_2} \phi_q \left(\tfrac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) \left| f(r, u(r), D_{0^+}^{\alpha-1} u(r)) \right| dr \right) ds < \varepsilon.$$

Finally thanks to Lemma 3.2.1, $T(B)$ is relatively compact and then T is completely continuous. ■

Next we give an existence result. Denote

$$\lambda_1 = \int_0^{+\infty} g(s) \left(1 + s^{\alpha-1}\right) ds, \ \lambda_2 = \int_0^{+\infty} h(s) \left(1 + s^{\alpha-1}\right) ds.$$

Theorem 3.2.3 *Suppose that the conditions (C1)-(C2) hold and the following condition is satisfied*

(C3) We have

$$0 < \lambda_1 + (\alpha - 1) \lambda_2 < \Gamma(\alpha), \quad 0 < \lambda_1 < 1$$

and there exists a constant $R > 0$ such that

$$M^{q-1} \int_0^{+\infty} \phi_q \left(\frac{1}{\Gamma(\delta)} \int_0^s (s-r)^{\delta-1} a(r) dr \right) ds \leq (1-\lambda_1) R. \tag{3.8}$$

Then the boundary value problem (P) has at least one solution.

Proof We shall prove that all conditions of Krasnoselskii fixed point theorem are satisfied.

We know that T is completely continuous from Theorem 3.2.2.

We claim that A is a contraction mapping. In fact, let $u, v \in X$, we obtain

$$\begin{aligned} \left| \frac{Au(t) - Av(t)}{1+t^{\alpha-1}} \right| &\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (g(s) + (\alpha-1) h(s)) \left(1+s^{\alpha-1}\right) \left| \frac{u(s)-v(s)}{1+s^{\alpha-1}} \right| ds \\ &\leq \frac{\lambda_1 + (\alpha-1)\lambda_2}{\Gamma(\alpha)} \|u-v\| \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \left| D_{0^+}^{\alpha-1} Au(t) - D_{0^+}^{\alpha-1} Av(t) \right| &\leq \int_0^{+\infty} g(s) \left(1+s^{\alpha-1}\right) \left| \frac{u(s)-v(s)}{1+s^{\alpha-1}} \right| ds \\ &\leq \lambda_1 \|u-v\|. \end{aligned}$$

By the condition (C3), we get $0 < \frac{\lambda_1+(\alpha-1)\lambda_2}{\Gamma(\alpha)} < 1$ and then A is a contraction mapping.

Denote $B_R = \{u \in X : \|u\| \leq R\}$ a nonempty bounded closed convex subset of X. Let $u, v \in B_R$, we claim that $Au + Tv \in B_R$. In fact, using (3.7), (3.8) and (3.9), we obtain

$$\left| \frac{Au(t) + Tv(t)}{1+t^{\alpha-1}} \right| \leq \frac{\lambda_1 + (\alpha-1)\lambda_2}{\Gamma(\alpha)} R < R$$

On the other hand we have

$$\left|D_{0^+}^{\alpha-1}Au(t)+D_{0^+}^{\alpha-1}Tv(t)\right| = \left|\int_0^{+\infty} g(s)x(s)ds\right.$$

$$\left. + \int_t^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} q(r)f(r,v(r),D_{0^+}^{\alpha-1}v(r))dr\right) ds\right|$$

$$\leq \int_0^{+\infty} g(s)\left(1+s^{\alpha-1}\right)\frac{|u(s)|}{1+s^{\alpha-1}}ds$$

$$+ M^{q-1}\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} q(r)dr\right) ds$$

$$\leq \lambda_1 R + M^{q-1}\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1} q(r)dr\right) ds$$

$$\leq \lambda_1 R + (1-\lambda_1)R \leq R.$$

Therefore, $\|Au+Tv\| \leq R$, which implies that $Au+Tv \in B_R$.

Hence the problem (P) has at least one solution in X. ■

3.3 Examples

Example 3.3.1 *Consider the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\frac{1}{2}}(\phi_p(D_{0^+}^{\frac{5}{2}}u(t))) + e^{-t}\frac{\sqrt{|u(t)|D^{\frac{3}{2}}u(t)}}{\left(1+t^{\frac{3}{2}}\right)} = 0, & t>0 \\ u(0)=0, D^{\frac{3}{2}}u(+\infty) = \int_0^{+\infty}\frac{e^{-t}}{5\left(1+t^{\frac{3}{2}}\right)}u(s)ds, \\ D^{\frac{1}{2}}u(0) = \int_0^{+\infty}\frac{1}{15(1+t)^2\left(1+t^{\frac{3}{2}}\right)}u(s)ds,\ I_{0^+}^{\frac{1}{2}}(\phi_p(D_{0^+}^{\frac{3}{2}}u(0))) = 0, \end{cases}$$

where

$$p=\frac{4}{3},\ \alpha=\frac{5}{2},\ \delta=\frac{1}{2},\ a(t)=e^{-t}.$$

Let us check that all conditions (Ci), $i=1,2,3$ are satisfied.

(C1) Let $x,y\in[-\rho,\rho]$, then

$$\left|f(t,\left(1+t^{\frac{3}{2}}\right)x,y)\right| = \frac{\sqrt{\left(1+t^{\frac{3}{2}}\right)|x|y}}{\left(1+t^{\frac{3}{2}}\right)} \le M_\rho = \rho^{\frac{3}{2}}.$$

(C2) Using Theorem 3.2.1, with $q=4$, $a(t)=e^{-t}$, $\delta=\frac{1}{2}$ and $\nu=3$, we obtain $\mu=\frac{6}{5}$, $1<\mu<\frac{1}{\delta}$. Then, we get

$$\begin{aligned} \int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{-\frac{1}{2}}a(r)dr\right)ds &= \int_0^{+\infty}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds \\ &\le k^3\left(\int_0^{+\infty}|a(s)|^{\frac{6}{5}}ds\right)^{\frac{5}{2}} \\ &= k^3\left(\int_0^{+\infty}e^{-\frac{6}{5}s}ds\right)^{\frac{5}{2}} = \left(\frac{5}{6}\right)^{\frac{5}{2}}k^3 < +\infty. \end{aligned}$$

where

$$k=\frac{\Gamma\left(\frac{1}{\mu}-\delta\right)}{\Gamma\left(\frac{1}{\mu}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} \simeq 2.3733.$$

(C3) By computation we obtain

$$\lambda_1 = \int_0^{+\infty} g(s)\left(1+s^{\alpha-1}\right)ds = \int_0^{+\infty} \frac{e^{-s}}{5\left(1+s^{\frac{3}{2}}\right)}\left(1+s^{\frac{3}{2}}\right)ds = \frac{1}{5} = 0.2$$

and

$$\lambda_2 = \int_0^{+\infty} h(s)\left(1+s^{\alpha-1}\right)ds = \int_0^{+\infty} \frac{1}{15\left(1+s\right)^2\left(1+s^{\frac{3}{2}}\right)}\left(1+s^{\frac{3}{2}}\right)ds = \frac{1}{15}.$$

Hence

$$0 < \lambda_2 + (\alpha-1)\lambda_2 = 0.3 < \Gamma\left(\frac{5}{2}\right) = 1.329\,3$$

Choosing $\rho = \left(\frac{1}{4}\right)^{\frac{2}{3}}$, *it yields*

$$M_\rho^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)dr\right)ds \leq \rho^{\frac{9}{2}}\left(\frac{5}{6}\right)^{\frac{5}{2}}k^3$$

$$\leq \left(\frac{1}{4}\right)^3\left(\frac{5}{6}\right)^{\frac{5}{2}}(2.4)^3 \simeq 0.136\,93,$$

and

$$(1-\lambda_1)R = 0.2,$$

then

$$M^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)dr\right)ds < (1-\lambda_1)R.$$

Thanks to Theorem 3.2.3, the problem (P) has at least one solution in X.

Example 3.3.2 *Consider the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\frac{1}{2}}(\phi_p(D_{0^+}^{\frac{7}{3}}u(t))) + \frac{1}{1+t^{\frac{4}{3}}}e^{-t}\frac{|u(t)|\left(D^{\frac{4}{3}}u(t)\right)^2}{\left(1+t^{\frac{4}{3}}\right)} = 0, & 0 < t < +\infty \\ u(0) = 0, D^{\frac{4}{3}}u(+\infty) = \int_0^{+\infty} \frac{0.1}{7\left(1+s^{\frac{4}{3}}\right)(1+s^2)}u(s)ds, & \\ D^{\frac{1}{3}}u(0) = \int_0^{+\infty} \frac{0.1}{20\left(1+s^{\frac{4}{3}}\right)(1+s^3)^{\frac{1}{2}}}u(s)ds,\ I_{0^+}^{\frac{1}{2}}(\phi_p(D_{0^+}^{\frac{7}{3}}u(0))) = 0, & \end{cases}$$

where

$$p = \frac{5}{4},\ \alpha = \frac{7}{3},\ \delta = \frac{1}{2},\ a(t) = \frac{1}{1+t^{\frac{4}{3}}}.$$

Conditions (C1)-(C3) hold. Indeed,

(C1) If $x, y \in [-\rho, \rho]$, then

$$\left|f(t, \left(1+t^{\frac{4}{3}}\right)x, y)\right| = e^{-t}\frac{\left(1+t^{\frac{4}{3}}\right)|x|\,y^2}{\left(1+t^{\frac{4}{3}}\right)} \le M_\rho = \rho^3.$$

(C2) Applying Theorem 3.2.1 with

$$q = 5,\ a(t) = \frac{1}{1+t^{\frac{4}{3}}},\ \delta = \frac{1}{2},\ \nu = 4,$$

we obtain $\mu = \frac{4}{3}$ thus $1 < \mu < \frac{1}{\delta}$. Moreover, we have

$$\int_0^{+\infty} \phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{-\frac{1}{2}}a(r)dr\right)ds = \int_0^{+\infty}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^4 ds$$

$$\le k^4\left(\int_0^{+\infty}|a(s)|^{\frac{4}{3}}ds\right)^3$$

$$= k^4\left(\int_0^{+\infty}\left(\frac{1}{1+s^{\frac{4}{3}}}\right)^{\frac{4}{3}}ds\right)^3 = 3.8948 \times k^4 < +\infty,$$

here

$$k=\frac{\Gamma\left(\frac{1}{\mu}-\delta\right)}{\Gamma\left(\frac{1}{\mu}\right)}=\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}=2.958\,7.$$

(C3) By computations it yields,

$$\lambda_1=\int_0^{+\infty} g(s)\left(1+s^{\alpha-1}\right)ds=0.0224\,40$$

and

$$\lambda_2=\int_0^{+\infty} h(s)\left(1+s^{\alpha-1}\right)ds=0.0140\,22.$$

Taking $\rho=\left(\frac{1}{3}\right)^{\frac{7}{12}}$, *then*

$$\begin{aligned} M_\rho^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)dr\right)ds &\leq \rho^{12}\times 3.894\,8\times k^4 \\ &\leq \left(\frac{1}{3}\right)^7\times 3.894\,8\times(2.958\,7)^4\simeq 0.136\,47, \\ (1-\lambda_1)R &= 0.258\,53. \end{aligned}$$

Consequently,

$$M^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)dr\right)ds<(1-\lambda_1)R.$$

We conclude by Theorem 3.2.3, that the problem (P) has at least one solution in X.

Conclusion

This thesis consists of two main parts. The first part is devoted to the study of the existence, uniqueness and stability of the solutions for a class of nonlocal fractional boundary value problems on infinite intervals with variable delays. The results are obtained via some fixed point theorems such as the Banach contraction principle and the Leray-Schauder nonlinear alternative.

The second part deals with the existence of solutions for Riemann-Liouville fractional boundary value problems on an infinite interval involving the p-Laplacian and integral boundary conditions. Thanks to Krasnoselskii fixed point theorem and a Corduneanu compactness criteria, the existence results are established.

Future studies could address to boundary value problems containing other types of fractional derivatives, such Antagan-Baleanu fractional derivatives, Liouville-Grunwald fractional derivative, Hadamard fractional derivative, Hille-Tamarkin fractional derivative, Riesz fractional derivative, Marchaud fractional derivative, Hilfer fractional derivative, Liouville-Sonine-Caputo fractional derivative, ...

In addition, one can deal with similar problems by using other methods to prove the existence of solutions such as the method of lower and upper solutions and numerical methods and then other conditions have to be introduced.

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