وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR -ANNABA
UNIVERSITY
UNIVERSITÉ BADJI
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ANNABA

جامعة باجي مختار
- عنابة -

Faculté de Sciences

Année : 2021-2022

Département de Mathématiques

Laboratoire de Modélisation Mathématiques et Simulation Numérique

THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat en Mathématiques

Étude de systèmes d'équations différentielles fractionnaires dégénérées

Option : Équations différentielles et Applications

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To the beloved souls of my grand parents Dewadia, Zoubida, Ali.

To all women who struggle everyday to make this world a better place.

"Mothers, scientists, engineers, teachers, doctors, housewives."

God bless you all!

ملخّص

شهد المجال الرياضياتي المتعلق بدراسة المعادلات التفاضلية الكسرية المنحلة تطورا ملحوظا خلال العقد الأخير وذلك لتطبيقاتها المتنوعة في شتى الميادين المتعلقة بالعلوم والهندسة. تتناول هذه الأطروحة دراسة وجود ووحدانية حل مسألة تفاضلية لمشتقة ذات رتب كسرية من نوع كابوتو منحلّة في فضاء بناخ كيفي بالاعتماد على عمليات جبرية بسيطة وخاصة مقلوب درازين. كما تتناول المطروحة أيضا مناقشة حل لنفس المسألة بمشتقة كسرية مختلفة من نوع المشتقات الكسرية الموافقة، بالإضافة إلى جزء يناقش انتظام هذا النوع من المعادلات باستعمال مقاربة جديدة. كما سنقدم أمثلة توضيحية لإثبات صحة النتائج المحصل عليها.

كلمات مفتاحية: معادلات تفاضلية كسرية منحلة, مقلوب درازين, الوجود و الوحدانية, مشتقة كابوتو, المشتقة الكسرية الموافقة .

Abstract

Recently, the research area of singular fractional systems, known also as "degenerate" or "differential-algebraic" fractional systems, has attracted many mathematicians and physicists. The first application of such systems have arisen in modeling systems of science and engineering, such as electrical networks, economics, optimization problems, analysis of control systems, constrained mechanics, aircraft and robot dynamics, biology. The aim of the present thesis is to establish a rigorous analytical study on fractional singular initial value problems on a Banach space by studying the existence and uniqueness of the solution. Moreover, a new approach to regularity using Drazin inverses is presented. We also use a suitable method to write down the explicit formula to the solution of the given Cauchy problem using decompositions and canonical forms. In addition, we illustrate our results with some numerical examples. Finally, concluding remarks are given together with future perspectives.

Keyword: singular equations, Drazin inverse, Caputo's fractional Cauchy Problem, fractional conformable derivatives, Banach space, existence and uniqueness.

Résumé

Au cours de ces derniéres annèes l'ètude des problèmes fractionnaires singuliers est un sujet de recherche très actif, vu ses applications nombreuses dans la modélisation des différentes systèmes en science et ingénierie notamment les systèmes de contrôles, l'économie, la mécanique des fluides et même la sociologie. Dans cette thèse, on étudie d'une façon analytique un système de Cauchy fractionnaire singulier généralisé dans un espace de Banach, en traitant l'existence et l'unicité du problème présenté sous deux types de dérivées fractionnaires différentes: la dérivée fractionnaire de Caputo et la nouvelle dérivée fractionnaire conforme avec des exemples numériques pour rendre concret nos résultats théoriques. En plus, une nouvelle approche de la régularité au sens de l'inverse de Drazin est proposée. Finalement, on conclut avec des remarques et des perspectives.
Mots clés: problème de Cauchy fractionnaire singulier, existence et unicité, espace de Banach, inverse de Drazin, dérivée conforme.

Acknowledgments

No words are enough to express my deepest gratitude and sincerest appreciation to my supervisor **Prof. Said Mazouzi**, who has been such a great mentor throughout the making of this project, his constant guidance and persistent help are what made the realization of this research thesis possible. I would especially thank him for his kindness and encouragement. " *Thank you Professor! I am truly grateful*".
I would like to thank **Prof. Peter Kunkel**, my co-supervisor, who has supported me in every way possible during my stay in Germany, he taught me a lot, never hesitated once to answer my naive questions. "*Thank you Professor for believing in me. I could not think of a better way to spend an internship!*".
My special regards also goes to **Prof. Fatma Zohra Nouri** who has accepted to be the head of the defense jury, along with **Prof. Messaoud Maouni** and **Dr. Mohamed Amine Kerker** as jury members, I am thankful for your devoted time and effort.
I wish to acknowledge the unconditional support and motivation I got from my parents **Kamla Mehanaoui** and **Gharib Aouri**, they are the reason I am here and I am forever grateful for them.

My appreciation also goes to my sisters: **Asma**, **Atika**, and brother **Mohamed Assem**, my friends, who showed their support and love throughout this journey.

Contents

Introduction

Fractional Calculus is a branch of mathematics as old as Classical Calculus, its origin began with an innocent question asked by Marquise De l'Hopital to the mathematician Gottfried W. Leibniz in 1695 concerning the new notation of the n^{th}-derivation introduced by the latter, namely $\frac{d^n}{dx^n}$. He asked him whether it was suitable to use the non-integer value $n = \frac{1}{2}$. Since it was too difficult to give a rigorous mathematical answer at that time, he simply answered him in a prescient way, "This is an apparent paradox from which one day useful consequences will be drawn...". In fact, this naive question is now a revealed reality that has been deeply explored and exploited by a variety of researchers in the context of Fractional Calculus [3, 11]. Due to the non-local character of fractional derivative, several real-world processes and natural phenomena are mathematically modeled by different types of fractional derivatives, including Caputo's derivative and conformable derivative, for more details see [8, 14, 22, 13].

As mentioned earlier, fractional differential equations occur in the modeling of many systems in physics and engineering. The following example was originally proposed by Nutting [19]. It explains the need for the use of fractional order derivatives in systems of mechanics. We want to describe the behavior of certain materials under the influence of external forces. Usually, we use the laws of Hooke and Newton, the relation we are interested in is the relation between the stress $\sigma(t)$ and the strain $\epsilon(t)$. Now, if we are dealing with viscous fluids, then Newton's law holds

$$\sigma(t) = \theta D^1 \epsilon(t),$$

where θ is the viscosity of the material. However, in the case of modeling the stress-strain relationship for elastic solids, we use Hooke's law

$$\sigma(t) = ED^0\epsilon(t) = E\epsilon(t).$$

Now consider such an experiment with $\epsilon(t) = t$ for $t \in [0, T]$ and $T \geq 0$. It follows that the stress behaves as follows

$$\sigma(t) = Et,$$

if it is an elastic solid, and

$$\sigma(t) = \theta = const,$$

for a viscous fluid.

We can then summarize these equations as follows.

$$\gamma_k = \frac{\sigma(t)}{\epsilon(t)} t^k. \tag{1}$$

Obviously, the case $k = 0$ corresponds to Hooke's law for solids and $k = 1$ to Newton's law for liquids. So we can say that for materials called **viscoelastic** that exhibit behavior somewhere between pure viscous fluid and pure elastic solid, such as polymers or even some types of biological tissues, at least under certain temperature and pressure conditions, relation (1) is observed for $0 < k < 1$.

In view of all these properties, it is reasonable to suppose that it is possible to express the relation between stress and strain for such a viscoelastic material by means of an equation of the form

$$\sigma(t) = \mu D^k\epsilon(t),$$

where μ is a material constant and $k \in (0, 1)$, which is called Nutting's law.

On the other hand, ordinary differential equations are a universal tool to describe and model physical processes, but there are physical systems in which we are forced to add more constraints, which are usually written in terms of algebraic equations and the system is considered as **singular**. Therefore, singular equations are considered to be a more natural way to model a variety of systems in science and engineering, see [6, 7, 15, 21, 23, 26] for more details.

Nevertheless, recently the combination of the above topics **Fractional Calculus** and **Singular Systems** has been an active area of research, although it can be really difficult to obtain analytical solutions for singular fractional differential initial value problems, because of their complexity and also, because it is sometimes difficult to separate the algebraic equations from the differential equations in order to identify the analytical solution.
In [15], a rigorous study of ordinary singular differential systems with constant coefficient matrix and time-varying matrix was presented, while in [27], the solution and stability of a homogeneous singular system under Riemman-Liouville fractional derivative was discussed. Meanwhile, in [20] the author used projections on Banach spaces to give the solution to a linear and a non linear degenerate Caputo fractional Cauchy problem. In this manuscript, we are interested in the analytical study of two kinds of fractional singular initial value problems, we mention previous stability results, and discuss the existence and uniqueness of the solution with respect to the regularity of the given problem. Moreover, we use generalized inverses and canonical forms to obtain the solution explicit formula. We refer the reader to recent research works on differential fractional singular systems by authors E. Shishkina & S. Sitnik [23], Y. Zhao [26], S. Bu & G. Cai [5], M. Plekhanova [20], and references therein.
The structure of this thesis is as follows: We begin by recalling basic notions and definitions from the theory of both fractional and singular systems. In Chapter2 we discuss the existence and uniqueness of a singular system of Caputo type; we derive an explicit formula of the solution by using the Drazin inverse and canonical representations. Two numerical examples are shown to illustrate the results, and a new concept of regularity is introduced at the end of this chapter. In Chapter 3, we introduce a new type of fractional singular problems by using the new conformable fractional derivative introduced in [13].

Then, we discuss the existence and uniqueness of solution to these systems, using a similar approach as in Chapter 2. Furthermore, In chapter 4, a numerical application is presented to compare both solution formulas given in Chapter 2 and Chapter 3.
Finally, we finish our thesis with some concluding remarks and some perspectives for future investigations in fractional singular differential systems and their applications.

Chapter 1

Preliminaries

In this chapter, we briefly state known definitions and theorems from the field of Fractional Calculus, such as the one of systems of differential algebraic equations in order to guarantee the coherence of this manuscript. We first start by presenting the functional spaces we needed throughout our work. Then, we give an overview on the theory of Fractional Calculus especially **Caputo fractional derivatives** and **Conformable fractional derivative**. Finally, we turn our attention to the theory of **Differential Algebraic equations.**

1.1 Functional spaces

Let $(X, \|.\|)$ be a complex Banach space. We denote by $\mathscr{B}(X)$ the Banach space of linear bounded operators from X into itself endowed with the norm $\|A\|_{op} = \sup\{\|Ax\| : \|x\| = 1\}$, for every $A \in \mathscr{B}(X)$.

Definition 1.1.1 *A function $f : \mathbb{R}^+ \to X$ is said to be absolutely continuous, if for any compact interval $J \subset \mathbb{R}^+$ and, for any $\varepsilon > 0$, there exists a positive real number $\delta > 0$, such that*

$$\sum_{k=1}^{n} \|f(b_k) - f(a_k)\|_X < \varepsilon,$$

for any finite set of mutually disjoint intervals $[a_k, b_k] \subset J,\ k = 1, 2, \ldots, n$*, such that*

$$\sum_{k=1}^{n}(b_k - a_k) < \delta.$$

Definition 1.1.2 *The vector space of all absolutely continuous functions on* $\mathbb{R}^+$ *taking their values in* X *is denoted by* $AC(\mathbb{R}^+; X)$*. Moreover, we shall use the following generalization: If* $n \in \mathbb{N}^* := \{1, 2, 3, ...\}$*, then*

$$AC^n\left(\mathbb{R}^+; X\right) = \left\{f : \mathbb{R}^+ \to X : f \in C^{n-1}\left(\mathbb{R}^+; X\right) \text{ and } f^{(n-1)} \in AC\left(\mathbb{R}^+; X\right)\right\}.$$

In particular, we have $AC^1(\mathbb{R}^+; X) := AC(\mathbb{R}^+; X)$*.*

Remark 1.1.1 *We notice that a function* $f : \mathbb{R}^+ \to X$ *is absolutely continuous if and only if there are* $\varphi \in L^1(\mathbb{R}^+; X)$ *and a constant vector* $c \in X$ *such that*

$$f(x) = c + \int_0^x \varphi(t)dt,\ x \geq 0,$$

and from which we get $f'(x) = \varphi(x)$*, a.e.* $x \geq 0$*.*

1.2 Fractional Calculus

Some definitions for the fractional derivatives were driven over the years [14, 8, 22], in this manuscript, we restrict our attention to the use of Caputo fractional derivatives and conformable fractional derivative. In this section, we introduce some basic definitions and properties of the fractional integrals and fractional derivatives which are further used in this manuscript.

Definition 1.2.1 *[14] The Gamma function* $\Gamma(z)$ *is defined by*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt, Re(z) > 0,$$

where $t^{z-1} = e^{(z-1)log(t)}$. *This integral is convergent for all complex* $z \in \mathbb{C} : Re(z) > 0$. *For this function the reduction formula*

$$\Gamma(z+1) = z\Gamma(z), Re(z) > 0$$

holds. In particular, if $z = n \in \mathbb{N}$, *then*

$$\Gamma(n+1) = n!, n \in \mathbb{N}$$

with $0! = 1$.

Definition 1.2.2 *[14]***(Mittag-Leffler function)** *Let* $\alpha > 0$. *The function* E_α *defined for* $z \in \mathbb{C}$ *by*

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}$$

is called the Mittag-Leffler function of order α. *This function has been introduced by Mittag-Leffler and one can immediately notice that*

$$E_1(z) = exp(z)$$

which is the well known exponential function.

Let us consider some of the starting points ideas of the theory of fractional calculus for a better understanding of this theory, this has began with the fractional integral which is a generalization of repeated integration. Thus if f is locally integrable on $(0,\infty)$, then the n-fold iterated integral is given by

$$D^{-n}f(t) = \int_0^t \int_a^{\sigma_1} \cdots \int_0^{\sigma_{n-1}} f(\sigma_n)\, \mathrm{d}\sigma_n \cdots \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds,$$

for $t > 0$ and $n \in \mathbb{N}$. Based on this integral an immediate generalization for $\alpha \in \mathbb{C}$, where $Re(\alpha) > 0$, called the Riemman-Liouville fractional integral. Let us first introduce the Fractional Liouville-Riemman integral

Definition 1.2.3 *[14] Let $\alpha \in \mathbb{C}$, and let N be given by $N = [\alpha] + 1$ where $Re(\alpha) > 0$, for $f \in L^1(\mathbb{R}+, X)$, the Riemman-Liouville fractional integral is defined by*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, t > 0.$$

1.2.1 Caputo Fractional derivative

Definition 1.2.4 *[14] Let $\alpha \in \mathbb{R}_+^*$, and let N be given by $N = [\alpha] + 1$ if $\alpha \notin \mathbb{N}_0$. The Caputo fractional derivative of a function $f \in AC^N(\mathbb{R}^+, X)$ of order α, ${}^C D_0^\alpha f(t)$ exists, and it is given by*

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s) ds.$$

In particular, when $0 < \alpha < 1$, then

$${}^C D_0^\alpha f(t) = D_0^\alpha (f(t) - f(0)),$$

where D_0^α denotes the Riemman-Liouville fractional derivative.
If $\alpha = N \in \mathbb{N}$ and the usual derivative $f^{(N)}(t)$ of order N exists, then ${}^C D_0^\alpha f(t)$ is represented by

$${}^C D_0^N f(t) = f^{(N)}(t).$$

Property 1.2.1 *[14] For $\alpha \in (0,1)$ and $f \in AC(\mathbb{R}_+, X)$,*

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds$$

and

$${}^C D_0^\alpha (J_0^\alpha f(t)) = f(t), \forall t \geq 0$$

Remark 1.2.1 *we point out that the Caputo's fractional derivative of order α on a Banach space is well defined whenever $f \in AC^N(\mathbb{R}_+; X)$.*

If we define the convolution product $\varphi * \psi$ of two functions φ and ψ by

$$\varphi * \psi(t) = \int_0^t \varphi(s)\psi(t-s)\,ds,\ t > 0,$$

We have these two useful relations:

- ${}^C\mathscr{D}^{\alpha}_{0^+} J^{\alpha}_{0^+} f = f$, *whenever* $J^{\alpha}_{0^+} f \in AC^N(\mathbb{R}_+; X)$,
- $J^{\alpha C}_{0^+} D^{\alpha}_{0^+} f(t) = f(t) - \sum_{k=0}^{N-1} \frac{f^{(k)}(0^+)}{k!} t^k$, *for* $f \in AC^N(\mathbb{R}^+; X)$.

  We shall denote throughout this thesis the left-sided Caputo's fractional derivative of order α *of* $f : \mathbb{R}^+ \to X$, *initiated at* 0, *by* $D^{\alpha}_{0^+} f(t)$ *instead of* ${}^C D^{\alpha}_{0^+} f(t)$.

- *All the integrals presented in this work are taken in Bochner's sense, that is, a strongly measurable function* $f : (0,b) \to X$ *is Bochner integrable if* $||f||$ *is Lebesgue integrable over (0,b).*

The Laplace transform

As known in the theory of Ordinary Differential Equations(ODEs), Laplace transform is a very simple and efficient way to solve a linear ODE, let us state without proof some facts about the application of Laplace transform to Caputo's fractional derivative initiated at the origin. We have

Definition 1.2.5 *[14] Let* $f : \mathbb{R}^+ \to X$ *be piecewise continuous on every finite interval* $[0, T]$, $T > 0$, *and if there exist positive constants* M *and* a *such that* $||f(t)|| \le M e^{at}$, $t \ge 0$, *then the Laplace transform of* $f(t)$ *is defined by*

$$F(p) = \mathscr{L}(f)(p) = \int_0^{+\infty} e^{-ps} f(s) ds, Re(p) > a.$$

The inverse Laplace transform is formally given by

$$\frac{f(t^+) + f(t^-)}{2} = \mathscr{L}^{-1}(F)(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tp} F(p) dp,\ t > 0,$$

where the integral is carried out along the line $c+iy, -\infty < y < +\infty$, with $c > a$.

Let $\alpha > 0$ and $N = [\alpha] + 1$, if α is non integer and $N = \alpha$, if α is integer, then the Laplace transform of the Caputo's fractional derivative $\mathscr{D}^{\alpha}_{0^+} g$ of a vector-valued function $g \in C^N(\mathbb{R}^+; X)$ such that $g^{(N)} \in L^1(0, T; X)$, for every $T > 0$ and

$$\left| g^{(N)}(x) \right| \leq M e^{ax}, \text{ for every } x > T > 0,$$

for some constants $M > 0$ and $a > 0$, is given by

$$(\mathscr{L}\mathscr{D}^{\alpha}_{0^+} g)(p) = p^{\alpha}(\mathscr{L}g)(p) - \sum_{k=0}^{N-1} g^{(k)}(0) p^{\alpha-k-1}, Re(p) > a. \tag{1.1}$$

1.2.2 Conformable Fractional derivative

As stated previously fractional derivatives previously defined using the Riemann-Liouville integral cannot satisfy some of the basic properties that usual derivatives have, such as the product rule and chain rule. However, in [13] the authors introduced a new well behaved simple fractional derivative called "a conformable fractional derivative" (CFD), defined by:

Definition 1.2.6 *[13] Given a function $f\colon [0, +\infty) \to \mathbb{R}$. Then the so called conformable fractional derivative of f of order $\alpha \in (0, 1]$ is defined by*

$$T^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{\alpha-1}) - f(t)}{\epsilon}$$

for all $t > 0$. If f is α-differentiable in some $(0, b)$, $b > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then we put

$$f^{(\alpha)}(0) := \lim_{t \to 0^+} f^{(\alpha)}(t).$$

If the conformable fractional derivative of f of order α exists, then we simply say that f is α-differentiable.

Definition 1.2.7 *[13] Let $\alpha \in (0, 1]$, the conformable integral of order α starting at the origin* *of a function f of order α is defined by*

$$I_0^\alpha f(t) = \int_0^t s^{\alpha-1} f(s) ds.$$

We have the following results:

Lemma 1.2.1 *[13] Let f be a continuous function on $[0, +\infty[$, then we have*

$$T^\alpha I_0^\alpha f(t) = f(t), t > 0$$

Theorem 1.2.1 *If a function $f : [0, +\infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0, \alpha \in [0, 1)$, then f is continuous at t_0.*

Proof Since

$$f(t_0 + \epsilon t_0^{1-\alpha}) - f(t_0) = \frac{f(t_0 + \epsilon t_0^{1-\alpha}) - f(t_0)}{\epsilon} \epsilon.$$

Then,

$$\lim_{\epsilon \to 0} [f(t_0 + \epsilon t_0^{1-\alpha}) - f(t_0)] = \lim_{\epsilon \to 0} \frac{f(t_0 + \epsilon t_0^{1-\alpha}) - f(t_0)}{\epsilon} \lim_{\epsilon \to 0} \epsilon.$$

Let $h = \epsilon t_0^{1-\alpha}$. Then,

$$\lim_{h \to 0} [f(t_0 + h) - f(t_0)] = f^{(\alpha)}(t_0) = 0$$

which implies that

$$\lim_{h \to 0} f(t_0 + h) = f(t_0).$$

Hence, f is continuous at t_0.

Furthermore, we have the following theorem

Theorem 1.2.2 *[13] Let $\alpha \in (0, 1]$, f, g be α-differentiable at a point $t > 0$. Then*

- $T^\alpha(af + bg)(t) = aT^\alpha(f)(t) + bT^\alpha(g)(t)$, *for all a, $b \in \mathbb{R}$,*
- $T^\alpha(t^p) = pt^{p-\alpha}$, *for every $p \in \mathbb{R}$,*

- $T^{\alpha}(\lambda) = 0$, *for every constant* $\lambda \in \mathbb{R}$,

- $T^{\alpha}(fg)(t) = fT^{\alpha}g(t) + gT^{\alpha}f(t)$,

- $T^{\alpha}(\frac{f}{g})(t) = \dfrac{gT^{\alpha}f(t) - fT^{\alpha}g(t)}{g(t)^2}$, *with* $g(t) \neq 0$,

- *In addition, if f is differentiable at t, then* $T^{\alpha}f(t) = t^{1-\alpha}f'(t)$.

The following theorem deals with the conformable fractional derivative's chain rule,

Theorem 1.2.3 *[13](chain rule). Assume* $f, g : (0,\infty) \to (0,\infty)$ *be α-differentiable functions, where* $0 < \alpha \leq 1$. *Let* $h(t) = f(g(t))$, *then* $h(t)$ *is α-differentiable and for all* $t \in (0,\infty)$ *such that* $g(t) \neq 0$, *we have*

$$T^{\alpha}(h(t)) = T^{\alpha}(f(g(t))T^{\alpha}g(t)g^{\alpha-1}(t).$$

If t=0, we have

$$T^{\alpha}(h(t)) = \lim_{t\to 0} T^{\alpha}(f(g(t))T^{\alpha}g(t)g^{\alpha-1}g(t).$$

Proof By setting $u = t + \epsilon t^{1-\alpha}$ in the definition and using continuity of g we see that

$$T^{\alpha}h(t) = \lim_{u\to t} \frac{f(g(u)) - f(g(t))}{(u-t)} t^{1-\alpha}$$

$$= \lim_{u\to t} \frac{f(g(u)) - f(g(t))}{(g(u)-g(t))} \Delta \lim_{u\to t} \frac{g(u)-g(t)}{u-t} t^{1-\alpha}$$

$$= \lim_{g(u)\to g(t)} \frac{f(g(u)) - f(g(t))}{(g(u)-g(t))} \Delta g(t)^{1-\alpha} \Delta T^{\alpha}g(t) \Delta g^{\alpha-1}(t) = (T^{\alpha}f)(g(t)) \Delta (T^{\alpha}g)(t) \Delta g^{\alpha-1}(t).$$

The CFD attracted a lot of scientists and researches for its main advantages such as

- It satisfies all the concepts and rules of an ordinary derivative such as: quotient, product and chain rule while the other fractional definitions fail to meet these rules.

- It can be extended to solve exactly and numerically fractional differential equations and systems easily and efficiently.
- It generalizes well-known transforms such as Laplace transforms and are used as tools for solving some singular fractional differential equations.
- It enables new comparisons of CFD and other previous fractional definitions in many applications.

Here are some conformable fractional derivative of certain functions:

- $T^{\alpha}(e^{cx}) = cx^{1-\alpha}e^{cx}$, for any constant $c \in \mathbb{R}$,
- $T^{\alpha}(t^{p}) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
- $T^{\alpha}(1) = 0$,
- $T^{\alpha}(\frac{1}{\alpha}t^{\alpha}) = 1$.

1.3 Differential Algebraic Equations

1.3.1 What is a differential algebraic equation?

Initial value problems as well as boundary value problems, are usually written in the form of an explicit differential equation system

$$x' = f(t, x),$$

subject to some initial or boundary conditions. A more general form could be an implicit ODE of the form

$$F(t, y, y') = 0,$$

where the Jacobian matrix is assumed to be nonsingular for all values in an appropriate domain.

However, we may encounter another form of an explicit ODE

$$x' = f(t, x, z), \tag{1.2}$$

$$0 = g(t, x, z). \tag{1.3}$$

with some appropriate algebraic constraints This kind of ODE for $x(t)$ depends on additional algebraic variables $z(t)$, therefore the solution is forced in addition to satisfy the algebraic constraints, and the jacobian matrix is no longer nonsingular. Such an ODE is called **a differential algebraic equations (DAE)**, **singular**, **degenerate**, **implicit** equations. In order to be solved systems of the form (1.2)-(1.3) were usually transformed into ordinary differential equations

$$x' = g(t, x)$$

via analytical transformations. One way to achieve this is to explicitly solve the constraint equations analytically in order to reduce the given differential-algebraic equation to an ordinary differential equation with fewer variables. Nevertheless, this approach heavily relies on either transformations by hand or symbolic computation software which are both not feasible for medium or large scale systems.
Another way to treat this kind of equations is the algebraic approach via canonical forms and generalized inverses, which allows us to study them in a systematic way besides being very efficient in numerical discretization methods.

1.3.2 Drazin inverse operator

As we mentioned previously, solving DAEs is not trivial, one can face real difficulties when trying to solve a system of DAEs for the reason that it contains two parts, a differential and an algebraic part. To show that, we consider the following linear system:

$$Ax = b,$$

where $A \in \mathscr{B}(X)$ is a singular operator, so the system has no solution or has multiple ones. However, what if we can find a suitable operator E such that above system admits $x = Eb$ as a solution?. In fact, such an operator E exists and is called the generalized inverse of A which is reduced to the regular inverse A^{-1} when A is nonsingular. Although, several classes of generalized inverses are given, namely, the Moore-Penrose inverse, the {i, j, k} inverses, we are interested in the Drazin inverse operator defined as follows:

Definition 1.3.1 *[24] The index of an operator $E \in \mathscr{B}(X)$, denoted $indE$, is the least non-negative integer m such that* $\ker E^m = \ker E^{m+1}$ *and* $\mathscr{R}(E^m) = \mathscr{R}\left(E^{m+1}\right)$.

Definition 1.3.2 *[24] Let $E \in \mathscr{B}(X)$, $indE = m < \infty$, and $\mathscr{R}(E^m)$ is closed, then the unique operator $E^D \in \mathscr{B}(X)$ satisfying*

$$E^D E E^D = E^D,$$

$$E^D E = E E^D,$$

$$E^D E^{m+1} = E^m,$$

is called the Drazin inverse of E.

Remark 1.3.1 *If E is invertible, then $E^D = E^{-1}$ and $indE = 0$; we set $indE = 1$, if $E = 0$.*

Now, having defined the Drazin inverse and the index of an operator, it is time to argue about the existence theorem of the Drazin inverse of an operator.

Theorem 1.3.1 *[24] Let $E \in \mathscr{B}(X)$, if $ind(E) = m < \infty$, then the Drazin inverse exists and $E^D \in \mathscr{L}(X)$. Moreover, if $\mathscr{R}(E^m)$ is closed, then $E^D \in \mathscr{B}(X)$.*

Theorem 1.3.2 *[24] if the Drazin inverse of $E \in \mathscr{B}(X)$ exists, then it is unique.*

Definition 1.3.3 *Let $E \in \mathscr{B}(X)$, $ind(E) = m$, and $\mathscr{R}(E^m)$ be closed, we call the product EE^DE the core part of E, denoted by C. Let $N = E - C$, then*

$$E = C + N$$

is the core-nilpotent decomposition of E. It follows that N is the nilpotent operator with index m, since

$$N^m = (E - EE^D E)^m = E^m(I - E^D E) = 0,$$

and

$$N = E^l(I - EE^D) \neq 0, l < m.$$

Theorem 1.3.3 *[24] Let $E \in \mathscr{B}(X)$, $ind(E) = m$, and $\mathscr{R}(E^m)$ be closed, then*

- $ind(E^D) = ind(C) = 1$, *when* $ind(E) \geq 1$, *when* $ind(E) = 0$;
- $NC = CN = 0$;
- $NE^D = E^D N = 0$;
- $CEE^D = EE^D C = C$;
- $E = C$ *if and only if* $ind(E) \leq 1$;
- $((E)^D)^D = C$;
- $E^D = C^D$;
- $(E^D)^p = (E^p)^D$, *where p is an arbitrary positive integer;*
- $(E^D)^* = (E^*)^D$.

We have the following Proposition,

Proposition 1.3.1 *Let $A, L \in \mathscr{B}(X)$ such that L is bijective and $LA = AL$. Then*

$$\ker(LA) = \ker(A) \text{ and } \mathscr{R}(LA) = \mathscr{R}(A).$$

Moreover, if $indA = m < \infty$, then

$$ind(LA) = indA. \tag{1.4}$$

Proof Let $x \in \ker(LA)$, then $L(Ax) = 0$; hence $Ax = 0$, that is $x \in \ker A$. Conversely, if $x \in \ker A$, then $Ax = 0$ implying that $LAx = 0$; hence $x \in \ker(LA)$. Therefore, we obtain $\ker(LA) = \ker A$.

To prove the relation (1.4), let $z \in \mathscr{R}(LA)$, there is $x \in X : z = LAx = A(Lx)$. It follows that $z \in \mathscr{R}(A)$. Conversely, if $z \in \mathscr{R}(A)$, then there exists $x \in X$: $z = Ax = LA\left(L^{-1}x\right)$, and so, $z \in \mathscr{R}(LA)$. Therefore, $\mathscr{R}(LA) = \mathscr{R}(A)$.

Suppose that $indA = m$, then m is the least integer number for which we have $\ker A^m = \ker A^{m+1}$ and $\mathscr{R}(A^m) = \mathscr{R}\left(A^{m+1}\right)$. Since L^m and L^{m+1} are bijective we can apply the previous assertion of this Proposition to $\{L^m, A^m\}$ and $\left\{L^{m+1}, A^{m+1}\right\}$ to get

$$\begin{aligned} \ker(LA)^m &= \ker A^m = \ker A^{m+1} = \ker(LA)^{m+1}, \\ \mathscr{R}(LA)^m &= \mathscr{R}(A^m) = \mathscr{R}\left(A^{m+1}\right) = \mathscr{R}(LA)^{m+1}. \end{aligned}$$

We conclude that

$$ind(LA) = indA = m.$$

Chapter 2

Singular Caputo Fractional initial values problems

Let $\mathbb{R}^+ := [0, \infty)$ be the set of nonnegative real numbers, α a positive non integer number and let $N = [\alpha] + 1$, where $[\alpha]$ is the integral part of α, $(X, \|.\|)$ is a complex Banach space. We denote by $\mathscr{B}(X)$ the Banach space of linear bounded operators from X into itself endowed with the norm $\|A\|_{op} = \sup\{\|Ax\| : \|x\| = 1\}$, for every $A \in \mathscr{B}(X)$. We are interested in solving explicitly the following singular fractional differential initial value problem with respect to Caputo's fractional derivative in the unknown vector function $x(t) : \mathbb{R}^+ \to X$, namely

$$E\mathscr{D}_{0_+}^{\alpha} x(t) = Ax(t) + f(t), t > 0, \tag{2.1}$$

subject to the initial conditions

$$x^{(k)}(0) = v_k, k = 0, 1, \ldots, N-1, \tag{2.2}$$

where $E, A \in \mathscr{B}(X)$, so that $\ker E \neq \{0\}$ (and possibly $\ker A \neq \{0\}$), and $\mathscr{D}_{0_+}^{\alpha}$ denotes the (left sided) Caputo's fractional derivative of order $\alpha > 0$ initiated at 0, $(v_0,\ v_1, ..., v_N\ )$ are known vectors in X and f is a given absolutely continuous function defined on $\mathbb{R}^+$.

Unlike the projection operators approach used in the works [10, 20], we shall express the solution to problem (2.1)-(2.2) in terms of Mittag-Leffler functions and Drazin inverses [12, 24] of the operators A and E, when $AE = EA$. In particular, if these operators are non singular, we obtain the explicit solution of a regular fractional initial value problem. The technique used in our investigation consists of decoupling the operator E into the sum of two operators, one of them is nilpotent, so that the given problem (2.1)-(2.2) is equivalent to a certain couple of manageable subproblems.

Finally, at the end of this chapter and in order to investigate general singular fractional initial value problems, when the operators A and E do not necessarily commute we introduce a new notion of regularity that allows solving this type of problems.

2.1 Explicit solution to a singular fractional Caputo type Cauchy problem

Let us first state and solve explicitly some fractional differential equation with a nilpotent operator coefficient. The obtained solution is unique and there is no initial value imposed. We have

Lemma 2.1.1 *[2] Let $B,\ \mathbf{N},\ L \in \mathscr{B}(X)$ such that B is invertible, $\mathbf{N}$ a nilpotent operator of index (of nilpotency) $m \in \mathbb{N}^*$ so that $BL\mathbf{N} = L\mathbf{N}B$. Then, for any function $f : \mathbb{R}^+ \to X$ such that*

$$\left(B^{-1}\right)^{k+1}(L\mathbf{N})^k\left(\mathscr{D}_{0^+}^{\alpha}\right)^k f \in AC\left(\mathbb{R}^+;X\right),\ \textit{for } k = 0, 1, \ldots, m-1,$$

the fractional differential equation

$$L\mathbf{N}\mathscr{D}_{0^+}^{\alpha}\xi(t) = B\xi(t) + f(t),\ t > 0, \tag{2.3}$$

has a unique solution given by

$$\xi(t) = -\sum_{k=0}^{m-1} \left(B^{-1}\right)^{k+1} (L\mathbf{N})^k \left(\mathscr{D}_{0^+}^{\alpha}\right)^k f(t), \ t > 0. \tag{2.4}$$

Proof By applying B^{-1} to the both sides of the first equation of (2.3) we find

$$B^{-1}L\mathbf{N}\mathscr{D}_{0^+}^{\alpha}\xi(t) = \xi(t) + B^{-1}f(t). \tag{2.5}$$

It is worth to notice that the assumption $BL\mathbf{N} = L\mathbf{N}B$ implies that $B(L\mathbf{N})^k = (L\mathbf{N})^k B$, for $k = 1, 2, \ldots, m-1$, and so $(L\mathbf{N})^k B^{-1} = B^{-1}(L\mathbf{N})^k$, for $k = 1, 2, \ldots, m-1$. Setting $Q = B^{-1}L\mathbf{N}\mathscr{D}_{0^+}^{\alpha}$, we get for every $k = 1, 2, \ldots, m-1$,

$$\begin{aligned} Q^k B^{-1} &= \left(B^{-1}L\mathbf{N}\mathscr{D}_{0^+}^{\alpha}\right)^k B^{-1} = \left(B^{-1}\right)^k (L\mathbf{N})^k B^{-1} \left(\mathscr{D}_{0^+}^{\alpha}\right)^k \\ &= \left(B^{-1}\right)^{k+1} (L\mathbf{N})^k \left(\mathscr{D}_{0^+}^{\alpha}\right)^k. \end{aligned}$$

By expressing equation (2.5) in term of Q we obtain

$$Q\xi(t) = \xi(t) + B^{-1}f(t). \tag{2.6}$$

Next, we apply the operators Q^k, $k = 1, 2, \ldots, m-1$, to equation (2.6) we get respectively

$$\begin{aligned} Q^2\xi(t) &= Q\xi(t) + QB^{-1}f(t) \\ &= \xi(t) + B^{-1}f(t) + QB^{-1}f(t), \end{aligned}$$

$$\begin{aligned} Q^3\xi(t) &= Q\xi(t) + QB^{-1}f(t) + Q^2B^{-1}f(t) \\ &= \xi(t) + B^{-1}f(t) + QB^{-1}f(t) + Q^2B^{-1}f(t), \end{aligned}$$

$$\vdots$$

$$Q^m \xi(t) = 0 = \xi(t) + \sum_{k=0}^{m-1} Q^k B^{-1} f(t).$$

So that, the unique solution to the fractional differential equation (2.3) is given by

$$\begin{aligned} \xi(t) &= -\sum_{k=0}^{m-1} Q^k B^{-1} f(t) \\ &= -\sum_{k=0}^{m-1} \left(B^{-1}\right)^{k+1} (L\mathbf{N})^k \left(\mathscr{D}_{0^+}^{\alpha}\right)^k f(t),\ t \geq 0, \end{aligned}$$

which completes the Lemma's proof.

Our next step is to establish an equivalence between the fractional differential equation (2.1) and a couple of appropriate fractional differential equations. We have

Proposition 2.1.1 *Let $E,\ A \in \mathscr{B}(X)$ with* $\ker E \neq \{0\}$. *We assume the Drazin inverse E^D exists and $EA = AE$. Then, equation (2.1) is equivalent to the fractional differential system*

$$\begin{cases} \mathbf{C}\mathscr{D}_{0_+}^{\alpha} y(t) = Ay(t) + f_1(t), \\ \mathbf{N}\mathscr{D}_{0_+}^{\alpha} z(t) = Az(t) + f_2(t),\ t \geq 0, \end{cases} \tag{2.7}$$

where $\mathbf{C} = EE^D E$, $\mathbf{N} = E - \mathbf{C}$, and

$$\begin{aligned} y(t) &= E^D E x(t), & z(t) = \left(I - E^D E\right) x(t), \\ f_1(t) &= E^D E f(t), & f_2(t) = \left(I - E^D E\right) f(t). \end{aligned}$$

Moreover, the function $y(t) = E^D E x(t)$ is a solution to the first equation of (2.7), if and only if, it satisfies the regular fractional differential equation

$$\mathscr{D}_{0_+}^{\alpha} y(t) = E^D A y(t) + E^D f_1(t),\quad t \geq 0. \tag{2.8}$$

Proof It is worth to notice that we have $\left(E^D E\right)^2 = E^D E$, so that, by applying the operator

$E^D E$ to both sides of equation (2.1) we obtain the following fractional differential equation

$$\begin{aligned}\left(E^D E\right)^2 E\mathscr{D}_{0_+}^{\alpha} x(t) &= EE^D E\mathscr{D}_{0_+}^{\alpha}\left(E^D E\right) x(t)\\ &= \mathbf{C}\mathscr{D}_{0_+}^{\alpha} y(t)\\ &= E^D EAx(t) + E^D Ef(t)\\ &= Ay(t) + f_1(t),\end{aligned}$$

which is a solution to equation $(2.7)_1$.

Likewise, noticing that $\left(I - E^D E\right)^2 = \left(I - E^D E\right)$, and applying the operator $\left(I - E^D E\right)$ to both sides of equation (2.1) we get

$$\begin{aligned}\left(I - E^D E\right)^2 E\mathscr{D}_{0_+}^{\alpha} x(t) &= E\left(I - E^D E\right)\mathscr{D}_{0_+}^{\alpha}\left(I - E^D E\right) x(t)\\ &= \mathbf{N}\mathscr{D}_{0_+}^{\alpha} z(t)\\ &= \left(I - E^D E\right) Ax(t) + \left(I - E^D E\right) f(t)\\ &= Az(t) + f_2(t).\end{aligned}$$

Therefore, $z(t)$ satisfies second equation from (2.7).

Conversely, if $(y(t), z(t))$ satisfies system (2.7), then, thanks to the linearity of the fractional derivative, the function $x(t) = y(t) + z(t)$ satisfies

$$\begin{aligned}E\mathscr{D}_{0_+}^{\alpha} x(t) &= E\mathscr{D}_{0_+}^{\alpha}(y(t) + z(t)) = E\mathscr{D}_{0_+}^{\alpha} y(t) + E\mathscr{D}_{0_+}^{\alpha} z(t)\\ &= A(y(t) + z(t)) + f_1(t) + f_2(t)\\ &= Ax(t) + f(t).\end{aligned}$$

To establish the last assertion we notice that $y(t) = E^D Ex(t)$ is already a solution to the first equation of (2.7), and we have

$$\begin{aligned}
\mathscr{D}_{0_+}^{\alpha} y(t) &= \mathscr{D}_{0_+}^{\alpha} \left(E^D E\right) x(t) = E^D E \mathscr{D}_{0_+}^{\alpha} x(t) \\
&= E^D \left(Ax(t) + f(t)\right) \\
&= E^D \left[A\left(E^D E\right) x(t) + E^D E f(t)\right] \\
&= E^D A y(t) + E^D f_1(t), \quad t \geq 0.
\end{aligned}$$

Conversely, multiplying (2.8) by **C** we obtain

$$\begin{aligned}
\mathbf{C}\mathscr{D}_{0_+}^{\alpha} y(t) &= E^D E A y(t) + \mathbf{C} E^D f(t) \\
&= E^D E A\left[E^D E x(t)\right] + f_1(t) \\
&= A y(t) + f_1(t).
\end{aligned}$$

Let us now state and prove another important result which is

Proposition 2.1.2 *Let E, $A \in \mathscr{B}(X)$ such that $EA = AE$ and E^D, A^D exist. Then the following assertions are equivalent*

a)

$$\ker E^D \cap \ker A^D = \{0\}. \tag{2.9}$$

b)

$$A^D A\left(I - E^D E\right) = I - E^D E. \tag{2.10}$$

Proof **a)⇒b)**: Suppose that $\ker E^D \cap \ker A^D = \{0\}$ and set

$$B = A^D A\left(I - E^D E\right) - \left(I - E^D E\right).$$

Applying the operators A^D and E^D to the latter equation we get respectively

$$\begin{aligned} A^D B &= A^D A^D A\big(I - E^D E\big) - A^D\left(I - E^D E\right) \\ &= A^D\left(I - E^D E\right) - A^D\left(I - E^D E\right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} E^D B &= E^D A^D A\big(I - E^D E\big) - E^D\left(I - E^D E\right) \\ &= A^D A\big(E^D - E^D E^D E\big) - \big(E^D - E^D E^D E\big) \\ &= 0. \end{aligned}$$

Hence, for any $x \in X$, we have

$$A^D (Bx) = E^D (Bx) = 0,$$

that is

$$Bx \in \ker E^D \cap \ker A^D = \{0\}.$$

It follows that $Bx = 0$, for every $x \in X$, and accordingly (2.10) holds.

b)⇒a): Suppose that (2.10) holds. Let $x \in \ker E^D \cap \ker A^D$, then

$$A^D x = E^D x = 0.$$

It follows that

$$\begin{aligned} \big(I - E^D E\big)x &= x - EE^D x = x \\ &= A\big(I - E^D E\big)A^D x = 0. \end{aligned}$$

Hence $x = 0$; therefore $\ker E^D \cap \ker A^D = \{0\}$.

Remark 2.1.1 *i) It is not hard to check the following inclusion by using the property* $E^D = E^D E^D E$ *and* $A^D = A^D A^D A$*,*

$$\ker E \cap \ker A \subset \ker E^D \cap \ker A^D.$$

ii) If $E,\ A \in \mathscr{B}(X)$ *commute,* E^D*,* A^D *exist, with* $indE = m < \infty$ *and* $indA = k < \infty$*, and*

$$\ker E^m \cap \ker A^k = \{0\},$$

then the relation (2.10) holds. Indeed, applying respectively E^m *and* A^k *to the operator* B *we obtain*

$$\begin{aligned} E^m B &= E^m A^D A\left(I - E^D E\right) - E^m\left(I - E^D E\right) \\ &= A^D\left(E^m - E^D E^{m+1}\right) - A^D\left(E^m - E^D E^{m+1}\right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} A^k B &= A^k A^D A\left(I - E^D E\right) - A^k\left(I - E^D E\right) \\ &= A^k\left(I - E^D E\right) - A^k\left(I - E^D E\right) \\ &= 0. \end{aligned}$$

Reasoning as above we conclude that (2.10) holds.

Before tackling the general singular fractional differential equation we would like to investigate the homogeneous one, we have

Theorem 2.1.1 *Let* $E,\ A \in \mathscr{B}(X)$ *with* $\ker\ E \neq \{0\}$ *so that* $EA = AE$ *and* $indE = m$*. We assume that* E *and* A *have bounded Drazin inverses* E^D *and* A^D *obeying condition (2.9). Then, the general solution of*

$$E\mathscr{D}^{\alpha}_{0_+} x(t) = Ax(t), t > 0, \tag{2.11}$$

is given by

$$x(t)=\sum_{k=0}^{N-1} t^k \mathscr{E}_{\alpha,k+1}\left(t^\alpha E^D A\right) E^D E b_k, t \geq 0,$$

for some constant vectors $b_0,\ b_1,\dots,b_{N-1}\in X$*, where*

$$\mathscr{E}_{\alpha,\beta}(t)=\sum_{k\geq 0}\frac{1}{\Gamma(\alpha k+\beta)}t^k \tag{2.12}$$

is the Mittag-Leffler function of two parameters $\alpha,\beta>0$*.*

Proof Let us define $y(t)=E^D E x(t)$, then

$$\mathscr{D}^\alpha_{0_+} y(t)=E^D E \mathscr{D}^\alpha_{0_+} x(t)=E^D A y(t),\quad t\geq 0. \tag{2.13}$$

By applying Laplace transform to the equation (2.13), we obtain by virtue of the linearity of $\mathscr{L}$,

$$\begin{aligned}\mathscr{L}(D^\alpha_{0^+} y)(p) &= p^\alpha(\mathscr{L} y)(p)-\sum_{k=0}^{N-1} y^{(k)}(0)\, p^{\alpha-k-1}\\ &= \mathscr{L}(E^D A y)(p)=E^D A\, \mathscr{L}(y)(p).\end{aligned}$$

Setting $Y(p)=(\mathscr{L} y)(p)$, we infer

$$\left(p^\alpha I-E^D A\right) Y(p)=\sum_{k=0}^{N-1} p^{\alpha-k-1} y^{(k)}(0).$$

If $|p|>\left\|E^D A\right\|_{op}^{1/\alpha}$, then we get

$$\left(p^\alpha I-E^D A\right)^{-1}=\sum_{j\geq 0} p^{-\alpha(j+1)}\left(E^D A\right)^j.$$

It follows that

$$\begin{aligned} Y(p) &= \left(p^{\alpha} I - E^{D} A\right)^{-1} \sum_{k=0}^{N-1} p^{\alpha-k-1} y^{(k)}(0) && (2.14)\\ &= \mathscr{L}\left(\sum_{k=0}^{N-1} t^{k} \mathscr{E}_{\alpha,k+1}\left(t^{\alpha} E^{D} A\right) y^{(k)}(0)\right). && (2.15) \end{aligned}$$

To simplify the solution's expression we shall put throughout

$$T_{\alpha,\beta}(t) := t^{\beta-1} \mathscr{E}_{\alpha,\beta}\left(t^{\alpha} E^{D} A\right). \tag{2.16}$$

Therefore,

$$y(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t)\, y^{(k)}(0). \tag{2.17}$$

We notice that for any constant vectors $b_0, b_1, \ldots, b_{N-1} \in X$ the function

$$y(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t)\, b_k,$$

satisfies the following

$$\begin{aligned} \mathbf{C}\mathscr{D}_{0_+}^{\alpha} y(t) &= \mathbf{C} \sum_{k=0}^{N-1} \mathscr{D}_{0_+}^{\alpha} T_{\alpha,k+1}(t)\, b_k \\ &= E E^{D} A \sum_{k=0}^{N-1} T_{\alpha,k+1}(t)\, b_k = A y(t). \end{aligned}$$

Hence $y(t)$ is a solution to the homogeneous equation associated with first equation from (2.7).

Let us now obtain the closed form of the general solution to the equation (2.11).

Consider the following homogeneous equation associated with the second equation of (2.7)

$$\mathbf{N}\mathscr{D}_{0_+}^{\alpha} z(t) = A z(t), \; t > 0 \tag{2.18}$$

Applying $\mathbf{N}^{m-1}$ to both sides of (2.18) we infer

$$\mathbf{N}^m \mathscr{D}_{0_+}^{\alpha} z(t) = 0 = A\mathbf{N}^{m-1} z(t).$$

It follows that $A^D A \mathbf{N}^{m-1} z(t) = 0$, and thanks to the assumption (2.9) and Proposition 2.1.2 we get

$$\begin{aligned} 0 &= A^D A\left(I - E^D E\right)\mathbf{N}^{m-1} z(t) \\ &= \left(I - E^D E\right)\mathbf{N}^{m-1} z(t) = \mathbf{N}^{m-1} z(t), \end{aligned}$$

by assuming of course that $m-1>0$. Hence, $\mathbf{N}^{m-1} z(t) = 0$, and continuing in this manner we arrive at the final result $\mathbf{N} z(t) = 0$, which in turn implies that

$$\mathbf{N}\mathscr{D}_{0_+}^{\alpha} z(t) = \mathscr{D}_{0_+}^{\alpha} \mathbf{N} z(t) = 0 = Az(t).$$

Finally, since $\left(I - E^D E\right) y(t) = E^D E z(t) = 0$, then $\left(I - E^D E\right) z(t) = z(t)$. It follows by virtue of Proposition 2.1.2 that

$$\begin{aligned} A^D A\left(I - E^D E\right) z(t) &= A^D \left(I - E^D E\right) A z(t) = 0 \\ &= \left(I - E^D E\right) z(t) = z(t). \end{aligned}$$

Therefore, the unique solution to the differential equation (2.18) is the null one. Accordingly, the general solution to the singular fractional differential equation (2.18) is

$$x(t) = y(t) = E^D E y(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) E^D E b_k,\ t \geq 0,$$

for some constant vectors $b_0,\ b_1, \ldots, b_{N-1} \in X$.

We are now in the position to establish the existence and uniqueness of the solution to the singular fractional differential initial value problem (2.1)-(2.2). We have

Theorem 2.1.2 *Let E, $A \in \mathscr{B}(X)$ with* $\ker E \neq \{0\}$ *so that $AE = EA$ and $indE = m$. We assume that E and A possess bounded Drazin inverses E^D and A^D and both satisfy condition (2.9). Let $f \in C^N(\mathbb{R}^+;X)$ so that $T_{\alpha,\alpha} * f$ is integrable, the composite Caputo's fractional derivative $(\mathscr{D}^{\alpha}_{0^+})^i f(t)$, $t > 0$, exists for every $i = 1, \ldots, m-1$, $\lim_{t\to 0^+}(\mathscr{D}^{\alpha}_{0^+})^i f^{(j)}(t)$ exists for every $i = 1, \ldots, m-1$ and $j = 0, 1, \ldots, N-1$. If the initial conditions satisfy*

$$\begin{aligned} v_j &= E^D E b_j - (I - E^D E)\sum_{i=0}^{m-1}(A^D E)^i A^D (\mathscr{D}^{\alpha}_{0^+})^i f^{(j)}(0^+), \\ \text{for } j &= 0, 1, \ldots, N-1, \end{aligned} \tag{2.19}$$

for some constant vectors b_j, $j = 0, 1, \ldots, N-1$, then the unique solution $x(t)$ to problem (2.1)-(2.2) has the closed form

$$\begin{aligned} x(t) &= \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) E^D E b_k + \int_0^t T_{\alpha,\alpha}(s) E^D f(t-s)\, ds \\ &\quad -(I - E^D E)\sum_{i=0}^{m-1}(A^D E)^i A^D (\mathscr{D}^{\alpha}_{0^+})^i f(t), \ t \geq 0. \end{aligned}$$

Proof By applying the Laplace transform $\mathscr{L}$ to the equation (2.8), we obtain by virtue of the linearity of $\mathscr{L}$,

$$\begin{aligned} \mathscr{L}(D^{\alpha}_{0^+} y)(p) &= p^{\alpha}(\mathscr{L} y)(p) - \sum_{k=0}^{n-1} y^{(k)}(0) p^{\alpha-k-1} \\ &= E^D A\, \mathscr{L}(y)(p) + E^D \mathscr{L}(f)(p). \end{aligned}$$

Setting $Y(p) = (\mathscr{L} y)(p)$ and $F(p) = \mathscr{L}(f)(p)$, the latter equation becomes

$$\left(p^{\alpha} I - E^D A\right) Y(p) = \sum_{k=0}^{n-1} p^{\alpha-k-1} y^{(k)}(0) + E^D F(p).$$

Assuming that $|p| > \left\|E^D A\right\|_{op}^{1/\alpha}$ we obtain

$$\left(p^{\alpha} I - E^D A\right)^{-1} = \sum_{j\geq 0} p^{-\alpha(j+1)} \left(E^D A\right)^j.$$

It follows that

$$
\begin{aligned}
Y(p) &= \left(p^{\alpha} I - E^{D} A\right)^{-1} \sum_{k=0}^{n-1} p^{\alpha-k-1} y^{(k)}(0) + \left(p^{\alpha} I - E^{D} A\right)^{-1} E^{D} F(p) \\
&= \mathscr{L}\left(\sum_{k=0}^{n-1} T_{\alpha,k+1}(t)\, y^{(k)}(0) + T_{\alpha,\alpha}(t) * E^{D} f(t)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(t) &= \sum_{k=0}^{n-1} T_{\alpha,k+1}(t)\, y^{(k)}(0) + T_{\alpha,\alpha}(t) * E^{D} f(t) \\
&= \sum_{k=0}^{n-1} T_{\alpha,k+1}(t)\, E^{D} E b_{k} + \int_{0}^{t} T_{\alpha,\alpha}(s)\, E^{D} f(t-s)\, ds,
\end{aligned}
$$

for some constant vectors b_j, $j = 0, 1, \ldots, N-1$,.

Next, to solve explicitly the nilpotent fractional differential equation from (2.7) we apply the operator A^D to both sides of the equation and by virtue of Proposition 2.1.2, we get

$$
\begin{aligned}
A^{D} \mathbf{N} \mathscr{D}_{0_{+}}^{\alpha} z(t) &= A^{D} A z(t) + A^{D}\left(I - E^{D} E\right) f(t) \\
&= \left(I - E^{D} E\right) x(t) + A^{D}\left(I - E^{D} E\right) f(t) \\
&= z(t) + A^{D}\left(I - E^{D} E\right) f(t),
\end{aligned}
$$

Next, we apply Lemma 2.1.1, for $B = I$ and $L = A^D$, one gets the unique solution of the latter equation which is

$$
z(t) = -\left(I - E^{D} E\right) \sum_{i=0}^{m-1} \left(A^{D} \mathbf{N}\right)^{i} \left(\mathscr{D}_{0^{+}}^{\alpha}\right)^{i} A^{D} f(t), \; t \geq 0.
$$

Since we have $\mathbf{N} = E\left(I - E^{D} E\right)$ and $\left(I - E^{D} E\right)^{i} = \left(I - E^{D} E\right)$, for $i = 1, 2, \ldots, m-1$, then

$$
z(t) = -\left(I - E^{D} E\right) \sum_{i=0}^{m-1} \left(A^{D} E\right)^{i} \left(\mathscr{D}_{0^{+}}^{\alpha}\right)^{i} A^{D} f(t).
$$

Summing up the solutions of the above subproblems $y(t)$ and $z(t)$ we obtain the unique solution to the singular fractional differential initial value problem (2.1)-(2.2), that is

$$\begin{aligned} x(t) &= y(t)+z(t) \\ &= \sum_{k=0}^{N-1} T_{\alpha,k+1}(t)E^D E b_k + \int_0^t T_{\alpha,\alpha}(s)E^D f(t-s)\,ds \\ &\quad -\left(I-E^D E\right)\sum_{i=0}^{m-1}\left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f(t),\ t\geq 0. \end{aligned}$$

Let us now check the given initial values. Using the derivation rule regarding integrals depending upon a certain real parameter, we get, for $j=1,\ldots,N-1$, the following

$$\begin{aligned} x^{(j)}(t) &= \sum_{k=0}^{j-1}\sum_{m\geq 1}\frac{\alpha m}{\Gamma(\alpha m+k-j+1)}t^{\alpha m+k-j}\left(E^D A\right)^m E^D E b_k \\ &\quad +\sum_{k=j}^{N-1}\sum_{m\geq 0}\frac{\alpha m}{\Gamma(\alpha m+k-j+1)}t^{\alpha m+k-j}\left(E^D A\right)^m E^D E b_k \\ &\quad +\sum_{k=0}^{j-1} T_{\alpha,\alpha-k}(t)E^D f^{(j-k-1)}(0)+\int_0^t T_{\alpha,\alpha}(s)E^D f^{(j)}(t-s)\,ds \\ &\quad -\left(I-E^D E\right)\sum_{i=0}^{m-1}\left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f^{(j)}(t) \end{aligned}$$

so that, by letting $t\to 0^+$, we obtain

$$\begin{aligned} x^{(j)}(0) &= v_j = E^D E b_j - \left(I-E^D E\right)\sum_{i=0}^{m-1}\left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f^{(j)}(0) \\ &\quad j=0,1,\ldots,N-1. \end{aligned}$$

Regarding the uniqueness of the solution (under assumption (2.19)), it suffices to cope with the homogeneous problem whose solution is identically zero, and accordingly the uniqueness follows.

Remark 2.1.2 *We point out that if $f\equiv 0$, then the compatibility assumption (2.19) reduces merely to $v_j = E^D E v_j$, for $j=0,1,\ldots,N-1$. Moreover, if E is nonsingular, then $E^D E = I$, and once again, assumption (2.19) becomes $v_j = E^D E v_j$, for $j=0,1,\ldots,N-1$. Whence, we* *obtain as a unique solution in such a case as expected the function*

$$x(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) v_k + \int_0^t T_{\alpha,\alpha}(s) E^{-1} f(t-s)\, ds, \;, \; t \geq 0.$$

2.2 Stability Concepts

In order to guarantee that problem (2.1) is stable, we state the following theorem,

Theorem 2.2.1 *[16] Consider system* (2.1) *and its reduced form* (2.7) *with inherent ODE*

$${}^{c}D_{0+}^{\alpha} y(t) = Ay(t)$$

- *If the inherent ODE is stable and $||A|| \leq c$ holds with some constant $c > 0$ for all $t > 0$, then* (2.1) *is stable in the sense that $||y(t,0)|| < L$ on $[0, \infty)$, for some positive constant L.*
- *If the inherent ODE is asymptotically stable and $||A|| \leq c$ holds for some constant $c > 0$ for all $t \in I$, then* (2.1) *is asymptotically stable in the sense that $y(t,0) \to 0$ as $t \to \infty$.*

Theorem 2.2.2 *[5] Consider the fractional differential initial value problem*

$${}^{C}D_{0+}^{\alpha} y(t) = Ay(t),$$

$$y(0) = v_0.$$

Where $y(t)$ is a $C^n(\mathbb{R}_+, X)$ function and A is a bounded linear operator on X.

- *The trivial solution of the fractional differential equation* (2.1) *is said to be stable if $||A|| < c$, where c is a positive constant and for every $\epsilon > 0$ there exists a $\sigma = \sigma(\epsilon)$ such that for any initial condition $||v_0|| < \sigma$, the solution $y(t)$ of the system* (2.1) *satisfies the inequality $||y(t)|| < \epsilon$ for all $t > 0$.*

- *The trivial solution of the system* (2.1) *is said to be asymptotically stable if it is stable and furthermore* $\lim_{t \to +\infty} y(t) = 0$.

2.3 Illustrating examples

In order to illustrate the obtained results we consider the following examples.

Example 2.3.1 *Consider the following singular fractional differential initial value problem in* $\mathbb{R}^4$*:*

$$\begin{cases} E\mathscr{D}_{0^+}^{4/3} x(t) = Ax(t) + f(t), t > 0, \\ x_0 = \begin{pmatrix} -1, & 1, & 0, & 1 \end{pmatrix}^T, \\ x_1 = \begin{pmatrix} 1, & 1, & 1, & 1 \end{pmatrix}^T, \end{cases} \tag{2.20}$$

where $E,\ A \in \mathbb{R}^{4\times 4}$ *are as follows*

$$E = \frac{1}{12}\begin{pmatrix} 10 & -1 & 4 & 5 \\ 5 & -2 & -1 & 4 \\ 4 & 5 & 10 & -1 \\ -1 & 4 & 5 & -2 \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

and $f(t) = \begin{pmatrix} t^2, & t, & 0, & -t \end{pmatrix}^T$ *and* $D_{0^+}^{4/3}$ *is Caputo derivative of order* $\alpha = \dfrac{4}{3}$*, We notice that* E *and* A *are singular matrices whose Drazin inverses are*

$$E^D = \begin{pmatrix} 1 & -2 & 1 & -2 \\ -2 & 7 & -2 & 7 \\ 1 & -2 & 1 & -2 \\ -2 & 7 & -2 & 7 \end{pmatrix}, \qquad A^D = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Hence, using our previous results the explicit representation of the solution is given by

$$\begin{aligned} x(t) &= T_{\frac{4}{3},1}(t)E^D E x_0 + T_{\frac{4}{3},2}(t)E^D E x_1 + \int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)E^D F(t-s)ds \\ &\quad -(I-EE^D)\sum_{i=0}^{1}(A^D E)^i A^D (\mathscr{D}_{0^+}^{4/3})^i f(t), \end{aligned}$$

with

$$-(I-EE^D)\sum_{i=0}^{1}(A^D E)^i A^D (\mathscr{D}_{0^+}^{4/3})^i f(t) = 0.$$

Therefore, the closed form of the solution to the given problem is

$$x(t) = \begin{pmatrix} T_{\frac{4}{3},1}(t) - \frac{1}{2}T_{\frac{4}{3},2}(t) + \int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2 ds \\ T_{\frac{4}{3},1}(t) + T_{\frac{4}{3},2}(t) - 2\int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2 ds \\ T_{\frac{4}{3},1}(t) - \frac{1}{2}T_{\frac{4}{3},2}(t) + \int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2 ds \\ T_{\frac{4}{3},1}(t) + T_{\frac{4}{3},2}(t) - 2\int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2 ds \end{pmatrix}, t \geq 0.$$

Our next example deals with a singular fractional differential initial value problem in an infinite dimensional space, namely the Banach space

$$l^2 = \left\{ x = (x_n)_{n\geq 1} \subset \mathbb{R} : \sum_{n\geq 1} |x_n|^2 < \infty \right\},$$

endowed with the norm $\|x\| = \left(\sum\limits_{n\geq 1} |x_n|^2 \right)^{1/2}$. We have

Example 2.3.2 *Consider the following singular fractional differential initial value problem*

$$\begin{cases} E\mathscr{D}_{0_+}^{2/3} x(t) = Ax(t) + f(t), \ t > 0, \\ x(0) = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots) \in l^2, \end{cases} \tag{2.21}$$

where, $f(t) = (\frac{1}{n}\sin nt)_{n\geq 1}$, $E, A \in \mathscr{L}(l^2)$ are projection operators, defined respectively by

$$Ex = \big(x_1, x_2, 0, 0, x_5, x_6, 0, 0, x_9, x_{10}, 0, 0, x_{13}, x_{14}, 0, 0, \ldots\big) \tag{2.22}$$

and

$$Ax = \left(x_1, 0, x_3, x_4, x_5, 0, x_7, x_8, x_9, 0, x_{11}, x_{12}, x_{13}, 0, ...\right). \tag{2.23}$$

Taking into account the projection properties, we get at once

$$E^D = E,\ A^D = A,$$

and

$$A^D A(I - E^D E) = I - E^D E.$$

Hence condition (2.10) in Proposition 2.1.2 is satisfied, and so, according to Theorem 2.1.2, we obtain

$$\begin{aligned} x(t) \quad = \quad & T_{\frac{2}{3},1}(t)Ex(0) + \int_0^t T_{\frac{2}{3},\frac{2}{3}}(s)Ef(t-s)ds \\ & -(I-E)Af(t) - (I-E)EA\mathscr{D}_{0^+}^{2/3} f(t). \end{aligned}$$

We notice that

$$-(I-E)EA\mathscr{D}_{0^+}^{2/3} f(t) = 0,\ t > 0.$$

It follows that the closed form of the given singular fractional differential initial value problem is

$$x(t) = (x_n)_{n\geq 1},$$

where

$$x_n(t) = \begin{cases} -\frac{1}{n}\sin nt,\ \textit{if } n \in J = \{4k-1, 4k, \textit{for } k = 1, 2, ...\}, \\ \frac{1}{n}T_{\frac{2}{3},1}(t) + \frac{1}{n}\int_0^t T_{\frac{2}{3},\frac{2}{3}}(s)\sin n(t-s)ds,\ \textit{if}\ \ n \in \mathbb{N}^*\backslash J. \end{cases}$$

2.4 Drazin regular problems

We shall revisit along this section the problem (2.1)-(2.2) under some regularity between the given operators A^D and E^D. We set the following

Definition 2.4.1 *For any given two operators E, $A \in \mathscr{B}(X)$, the operator pair (E,A) is called Drazin regular if there is some $\lambda \in \mathbb{C}$ such that $\lambda E^D - A^D$ is invertible. The system (2.1) is called Drazin regular, if (E,A) is Drazin regular. If (E,A) is a Drazin regular pair of bounded linear operators and there $\lambda \in \mathbb{C}$ such that $\lambda E^D - A^D$ is invertible, then the following notations will be used*

$$\begin{aligned} I_\lambda &= \left(\lambda E^D - A^D\right)^{-1},\ E_\lambda = I_\lambda E,\ A_\lambda = I_\lambda A, \\ f_\lambda &= I_\lambda f,\ T^\lambda_{\alpha,\beta}(t) = t^{\beta-1}\mathscr{E}_{\alpha,\beta}\left(t^\alpha E^D_\lambda A_\lambda\right). \end{aligned} \tag{2.24}$$

We have the following Proposition,

Proposition 2.4.1 *Let (E,A) be a Drazin regular pair of bounded linear operators and let $\lambda \in \mathbb{C}$ such that $\lambda E^D - A^D$ is invertible with inverse I_λ. If B, $C \in \mathscr{B}(X)$ commute with E and A and they have bounded Drazin inverses B^D and C^D, then*

$$B_\lambda C_\lambda = C_\lambda B_\lambda,\ B^D_\lambda C_\lambda = C_\lambda B^D_\lambda,$$

where $B_\lambda = I_\lambda B$ and $C_\lambda = I_\lambda C$. Moreover, the operators $B^D_\lambda B_\lambda$, $B^D_\lambda C_\lambda$, $B_\lambda C^D_\lambda$ are independent of the parameter λ. In particular, if $B = E$ and $C = A$, then the operators $E^D_\lambda E_\lambda$, $E^D_\lambda A_\lambda$, $E_\lambda A^D_\lambda$ as well as $ind E_\lambda$ are independent of λ.

Proof We have

$$B_\lambda C_\lambda = I_\lambda B I_\lambda C = I_\lambda B C I_\lambda = I_\lambda C B I_\lambda = C_\lambda B_\lambda.$$

Let $\alpha \in \mathbb{C}$ such that $\alpha I - C_\lambda$ is invertible, then

$$[(\alpha I - C_\lambda)B_\lambda]^D = [B_\lambda(\alpha I - C_\lambda)]^D = (\alpha I - C_\lambda)^{-1} B^D_\lambda,$$

giving

$$(\alpha I - C_\lambda) B_\lambda^D = B_\lambda^D (\alpha I - C_\lambda).$$

It follows that $B_\lambda^D C_\lambda = C_\lambda B_\lambda^D$.

To show the remaining relations, it suffices to establish that the operator $B_\lambda^D I_\lambda$ is independent of λ. Indeed, let $\mu \in \mathbb{C}$ such that I_μ is invertible, then

$$\begin{aligned} B_\lambda^D I_\lambda &= \left[I_\lambda I_\mu^{-1} I_\mu B \right]^D I_\lambda \\ &= \left[B_\mu^D \left(I_\mu I_\lambda^{-1} \right) \right] I_\lambda = B_\mu^D I_\mu. \end{aligned}$$

So, we have for instance,

$$\begin{aligned} B_\lambda^D B_\lambda &= B_\lambda^D I_\lambda B = B_\mu^D I_\mu B = B_\mu^D B_\mu \\ B_\lambda^D C_\lambda &= B_\lambda^D I_\lambda C = B_\mu^D I_\mu C = B_\mu^D C_\mu \\ B_\lambda C_\lambda^D &= C_\lambda^D B_\lambda = C_\mu^D B_\mu \text{ (by interchanging the roles of } C \text{ and } B). \end{aligned}$$

Finally, to prove that $ind E_\lambda$ is independent of λ, we notice that I_λ is bijective and it commutes with E. So, applying the second assertion of Proposition 1.3.1 regarding the index we obtain

$$ind E_\lambda = ind (I_\lambda E) = ind E.$$

We have the following Lemma,

Lemma 2.4.1 *Let E, $A \in \mathscr{B}(X)$ be a commuting pair of Drazin regular operators having bounded Drazin inverses E^D and A^D. Let $\lambda \in \mathbb{C}$ be such that $\lambda E^D - A^D$ is invertible, then*

$$\begin{aligned} \ker E_\lambda^D \cap \ker A_\lambda^D &= \ker E^D \cap \ker A^D \\ &= \ker E_\lambda \cap \ker A_\lambda = \ker E \cap \ker A = \{0\}. \end{aligned}$$

Proof It follows at once from the fact that $\lambda E^D - A^D$ is invertible that $\ker E^D \cap \ker A^D = \{0\}$, and since

$$\ker E_\lambda \cap \ker A_\lambda = \ker E \cap \ker A \subset \ker E^D \cap \ker A^D = \{0\},$$

then

$$\ker E^D \cap \ker A^D = \ker E_\lambda \cap \ker A_\lambda = \ker E \cap \ker A = \{0\}.$$

On the other hand, we have $E_\lambda = I_\lambda E = E I_\lambda$ and $A_\lambda = I_\lambda A = A I_\lambda$; it follows that

$$E_\lambda^D = (E I_\lambda)^D = I_\lambda^{-1} E^D \text{ and } A_\lambda^D = (A I_\lambda)^D = I_\lambda^{-1} A^D,$$

from which we get

$$\ker E_\lambda^D \cap \ker A_\lambda^D = \ker E^D \cap \ker A^D = \{0\}.$$

Our last result is the following Theorem:

Theorem 2.4.1 *Given a commuting pair of Drazin regular operators A, $E \in \mathscr{B}(X)$ having bounded Drazin inverses E^D and A^D. Let $\lambda \in \mathbb{C}$ be such that $\lambda E^D - A^D$ is invertible and let $ind E_\lambda = m$. If $f \in AC^N(\mathbb{R}^+; X)$, then the general solution to the degenerate fractional differential equation (2.1) is given by*

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1} T_{\alpha,k+1}^\lambda(t) E_\lambda^D E_\lambda b_k + \int_0^t T_{\alpha,\alpha}^\lambda(s) E_\lambda^D f_\lambda(t-s)\,ds \\ &\quad - \left(I - E_\lambda^D A_\lambda\right) \sum_{i=0}^{m-1} \left(A_\lambda^D E_\lambda\right)^i \left(\mathscr{D}_{0^+}^\alpha\right)^i A_\lambda^D f_\lambda(t),\ t \geq 0, \end{aligned} \tag{2.25}$$

for some constant vectors $b_0, b_1, \ldots, b_{N-1} \in X$.
Moreover, $x(t)$ satisfies the initial conditions (2.2), if and only if, $\{v_j\}_{j=0}^{N-1}$ are of the form

$$\begin{aligned} v_j &= E_\lambda^D E_\lambda b_j - (I - E_\lambda^D E_\lambda) \sum_{i=0}^{m-1} (A_\lambda^D E_\lambda)^i A_\lambda^D (\mathscr{D}_{0^+}^\alpha)^i f_\lambda^{(j)}(0), \\ \textit{for } j &= 0, 1, \ldots, N-1, \end{aligned}$$

for some constant vectors $b_0, b_1, \ldots, b_{N-1} \in X$*; the solution is therefore unique.*

Proof We first note that

$$A_\lambda E_\lambda = I_\lambda A I_\lambda E = I_\lambda I_\lambda AE = I_\lambda I_\lambda EA = I_\lambda E I_\lambda A = E_\lambda A_\lambda.$$

Next, applying the operator I_λ to both sides of the equation (2.1) we get

$$E_\lambda \mathscr{D}^{\alpha}_{0^+} x(t) = A_\lambda x(t) + f_\lambda(t), \; t > 0. \tag{2.26}$$

Thanks to Lemma 2.4.1, we have $\ker E^D_\lambda \cap \ker A^D_\lambda = \{0\}$, so condition (2.9) is satisfied. On the other hand, we have $f_\lambda \in AC^N(\mathbb{R}^+; X)$ and $ind E_\lambda = m$. It suffices to apply Theorem 2.1.2 to get the closed form of the solution (2.25) to problem (2.26). The proof is now complete.

Chapter 3

Singular Conformable Cauchy problems

3.1 Introduction

As discussed earlier, the theory of fractional calculus has known a remarkable evolution in the last decades. However, despite all the work done in fractional calculus a tremendous work is still needed in order to overcome some issues, like the physical interpretation of the fractional derivative, the chain rule, the product rule. Nevertheless, recently Khalil et al.[13] introduced a new definition of fractional derivative, called "fractional conformable derivative". Unlike other fractional operators, this one provides more useful proprieties (chain rule, product rule) and so far it is the closest one to the ordinary derivative.

Throughout this chapter, we will combine two trending topics besides conformable fractional differential equations, we will be dealing with **singular** systems, which are famously known to be the most natural way to model real world phenomena, and used to describe plenty of models in science and engineering, see [7, 9, 15, 21].

Let $0 < \alpha < 1$ and $N = [\alpha] + 1$. Suppose that the state vector function $y(t) : [0, +\infty[\to \mathbb{R}^n$, satisfies the following singular fractional differential equation

$$\begin{cases} ET^{\alpha} y(t) - Ay(t) = f(t), t > 0, \\ y(0) = v_0, \end{cases} \tag{3.1}$$

Let $E,A \in \mathbb{C}^{n\times n}$, with $det(E) = 0$ (and possibly $\ker A \neq \{0\}$), f and $v_0 \in \mathbb{R}^n$ and T^α denotes the conformable fractional derivative of order α, f is a given $\alpha-$ differentiable function defined on $\mathbb{R}^+$. In [2], the authors gave the solution to such a fractional (Caputo's type) singular equation using generalized inverses, while in [25] the author expressed the solution of a fractional singular homogenous equation using the canonical forms of algebraic differential equations presented in [15].
In this section, we shall give the solution formula to problem (3.1) using Drazin inverse but before, we shall tackle the regularity of the problem

Definition 3.1.1 *[15] Let $E,A \in \mathbb{C}^{n\times n}$, $\lambda \in \mathbb{C}$. The matrix pair (E,A) is called regular if the so-called characteristic polynomial defined by*

$$p(\lambda) = det(\lambda E - A)$$

is not the zero polynomial.
If (E,A) is a regular pair, then system (3.1) is said to be regular.

Definition 3.1.2 *Let $A \in \mathbb{C}^{n\times n}$. The index of nilpotency of A is the least nonnegative integer k such that $A^k = 0$ and $A^{k-1} \neq 0$.*

Definition 3.1.3 *The Drazin inverse of A is the unique matrix A^D which satisfies*

$$A^D A A^D = A^D, AA^D = A^D A, A^{k+1}A^D = A^D.$$

Theorem 3.1.1 *[15] Consider matrices $E,A \in \mathbb{C}^{n\times n}$ with $EA = AE$. Then we have*

$$EA^D = A^D E,$$
$$E^D A = AE^D,$$
$$E^D A^D = A^D E^D.$$

Theorem 3.1.2 *[15] Let $E \in C^{n\times n}$ with $k = indE$. There is one and only one decomposition*

$$E = C + N,$$

with the properties

$$CN = NC = 0, N^k = 0, N^{k-1} \neq 0.$$

In particular, the following statements hold:

$$NC^D = 0, C^D N = 0,$$

$$E^D = C^D,$$

$$C^D E = E^D E$$

$$CC^D C = C,$$

$$C = EE^D E, N = E(I - E^D E).$$

3.2 Main results

In order to solve problem (3.1), we start by the following proposition:

Proposition 3.2.1 *Let $E, A \in \mathbb{C}^{n\times n}$ with E a* ***singular*** *matrix, and $EA = AE$. Then, the system (3.1) is equivalent to*

$$\begin{cases} \mathbf{C}T^{\alpha}x(t) = Ax(t) + g(t), \\ \mathbf{N}T^{\alpha}z(t) = Az(t) + h(t), t \geq 0, \end{cases} \tag{3.2}$$

where $\mathbf{C} = EE^D E$, $\mathbf{N} = E - \mathbf{C}$, and

$$\begin{aligned} x(t) &= E^D E y(t), \quad z(t) = \left(I - E^D E\right) y(t), \\ g(t) &= E^D E f(t), \quad h(t) = \left(I - E^D E\right) f(t). \end{aligned}$$

Furthermore, the first equation in (3.2) *is equivalent to the following*

$$T^{\alpha}x(t) = E^D Ax(t) + E^D g(t), \quad t \geq 0. \tag{3.3}$$

Proof It is clear that

$$y(t) = E^D E y(t) - E^D E y(t) + I y(t) = x(t) + z(t).$$

Using decomposition in proposition(3.1.2) and following the same steps from the proof of [15] we obtain the equivalence between (3.2) and (3.1).
Next, instead of solving (3.1) explicitly we solve separately the equations of system (3.2), and in order to do so, we shall state and establish the following lemma:

Lemma 3.2.1 *Let $N \in \mathbb{C}^{n\times n}$, a nilpotent matrix of order $p > 0$ ($N^p = 0$), A an invertible matrix such that $NA^{-1} = A^{-1}N$, h a C^{∞}-function, then the solution of the following fractional differential equation*

$$NT^{\alpha}(z(t)) = Az(t) + h(t) \tag{3.4}$$

is given by

$$z(t) = -\sum_{i=0}^{p-1}(NA^{-1}T^{\alpha})^i A^{-1}h(t), t > 0$$

As an immediate consequence, the solution to the second equation of (3.4) *is*

$$z(t) = -(I - E^D E)\sum_{i=0}^{p-1}(NA^{-1}T^{\alpha})^i A^{-1}f(t).$$

Proof Multiplying (3.4) by A^{-1}, we get

$$A^{-1}NT^{\alpha}z(t) = z(t) + A^{-1}h(t)$$

setting $Q_1 = A^{-1}NT^{\alpha}$, equation (3.4) become

$$Q_1 z(t) = z(t) + A^{-1}h(t), \tag{3.5}$$

applying Q_1 k-times to (3.5), we find

$$0 = z(t) + \sum_{i=0}^{k-1} Q^i A^{-1} h(t).$$

Thus

$$z(t) = -(I - E^D E) \sum_{i=0}^{k-1} (A^{-1} N T^\alpha)^i A^{-1} f(t).$$

Our main result is the following:

Theorem 3.2.1 *Let* $E, A \in \mathbb{C}^{n\times n}$ *with* $det E = 0$ *and* $AE = EA$, $ind E = p$. *We assume that system* (3.1) *is regular, then the homogeneous solution to*

$$\begin{cases} T^\alpha y(t) = E^D A y(t) + E^D f(t), t > 0, \\ x(0) = v_0, \end{cases} \tag{3.6}$$

is given by

$$x_h(t) = v_0 e^{\int_0^t s^{\alpha-1} E^D A ds}.$$

Furthermore the particular solution for (3.6) *is given by*

$$x_p(t) = e^{\frac{t^\alpha}{\alpha} E^D A} \int_0^t e^{-\frac{s^\alpha}{\alpha} E^D A} s^{\alpha-1} E^D g(s) ds, t > 0.$$

Finally, the general solution of equation (3.6) *is expressed as follows*

$$x(t) = x_h(t) + x_p = v_0 e^{\int_0^t s^{\alpha-1} E^D A ds} + e^{\frac{t^\alpha}{\alpha} E^D A} \int_0^t e^{-\frac{s^\alpha}{\alpha} E^D A} s^{\alpha-1} E^D g(s) ds, t \geq 0.$$

Proof Using the last property from **Theorem 2.4**, system (3.6) becomes

$$\begin{cases} E t^{1-\alpha} x'(t) = A x(t) + g(t), t > 0, \\ x(0) = v_0, \end{cases} \tag{3.7}$$

which is equivalent to the following system

$$\begin{cases} x'(t) = t^{\alpha-1}E^D A x(t) + E^D t^{\alpha-1} g(t), t > 0, \\ x(0) = v_0, \end{cases} \tag{3.8}$$

Therefore, the homogeneous solution to system (3.6) is given by

$$x_h(t) = v_0 e^{\int_0^t s^{\alpha-1} E^D A ds}.$$

Let us now verify that x_p is a particular solution to system (3.6).

$$x_p(t) = e^{\frac{t^\alpha}{\alpha}E^D A} \int_0^t e^{-\frac{s^\alpha}{\alpha}E^D A} s^{\alpha-1} E^D g(s) ds.$$

Indeed, we have

$$\begin{aligned} x_p'(t) &= t^{\alpha-1}E^D A e^{\frac{t^\alpha}{\alpha}E^D A} \int_0^t e^{-\frac{s^\alpha}{\alpha}E^D A} s^{\alpha-1} E^D g(s) ds \\ &+ e^{\frac{t^\alpha}{\alpha}E^D A} \frac{d}{dt} \int_0^t e^{-\frac{s^\alpha}{\alpha}E^D A} s^{\alpha-1} E^D g(s) ds \\ &= t^{\alpha-1}E^D A e^{\frac{t^\alpha}{\alpha}E^D A} \int_0^t e^{-\frac{s^\alpha}{\alpha}E^D A} s^{\alpha-1} E^D g(s) ds \\ &+ e^{\frac{t^\alpha}{\alpha}E^D A}[e^{-\frac{t^\alpha}{\alpha}E^D A} t^{\alpha-1} E^D g(t)] \\ &= t^{\alpha-1}E^D A e^{\frac{t^\alpha}{\alpha}E^D A} \int_0^t e^{-\frac{s^\alpha}{\alpha}E^D A} s^{\alpha-1} E^D g(s) ds + t^{\alpha-1}E^D g(t) \\ &= t^{\alpha-1}E^D A x_p(t) + t^{\alpha-1} E^D g(t). \end{aligned}$$

This shows that x_p is effectively a particular solution to system (3.6).

Theorem 3.2.2 *Let $E, A \in \mathbb{C}^{n\times n}$ with $detE = 0$ and $AE = EA$, $indE$=p. We assume that system* (3.1) *is regular. Let f be a α-differentiable function. If the initial conditions satisfy*

$$v_0 \;=\; E^D E v - (I - E^D E) \sum_{i=0}^{p-1} (A^{-1}N)^i A^{-1} (T^\alpha)^i f(0), \tag{3.9}$$

for some constant vector $v \in \mathbb{R}^n$, then the unique solution $y(t)$ to problem (3.1) is given by

$$\begin{aligned} y(t) &= x(t)+z(t) \\ &= ve^{\int_0^t s^{\alpha-1}E^DAds} + e^{\frac{t^\alpha}{\alpha}E^DA}\int_0^t e^{-\frac{s^\alpha}{\alpha}E^DA}s^{\alpha-1}E^Df(s)ds \\ &\quad -(I-E^DE)\sum_{i=0}^{p-1}(NA^{-1}T^\alpha)^iA^{-1}f(t), t\geq 0. \end{aligned}$$

3.3 A worked example

In this section, we use Matlab to show the difference between the representation of solution to a singular equation under the Caputo derivative and the fractional conformable one. Consider the following singular fractional differential initial value problem in $\mathbb{R}^2$:

$$\begin{cases} E\mathscr{D}_{0^+}^{1/2}y(t) = Ay(t)+f(t), t>0, \\ y_0 = \begin{pmatrix} -1, & \frac{5}{6} \end{pmatrix}^T, \end{cases} \tag{3.10}$$

where $E, A \in \mathbb{R}^{2\times 2}$ are as follows

$$E = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix},$$

and $f(t) = \begin{pmatrix} t^2, & -t \end{pmatrix}^T$, $D_{0^+}^{1/2}$ is Caputo derivative of order $\alpha = \frac{1}{2}$.
We can easily verify that E is a singular matrix with ind E=1, whereas A is nonsingular, with the following inverses

$$E^D = \frac{1}{49}\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}, \qquad A^{-1} = A^D = \begin{pmatrix} \frac{1}{4} & \frac{-1}{2} \\ \frac{1}{8} & \frac{1}{4} \end{pmatrix}.$$

Hence, the explicit representation of the solution is given by

$$\begin{aligned} y(t) &= T_{\frac{1}{2},1}(t)E^D E y_0 + \int_0^t T_{\frac{1}{2},\frac{1}{2}}(s)E^D f(t-s)ds \\ &\quad -(I-EE^D)A^{-1}(\mathscr{D}_{0^+}^{1/2})^0 f(t), \end{aligned}$$

with

$$-(I-EE^D)A^{-1}(\mathscr{D}_{0^+}^{1/2})^0 f(t) = 0$$

Thus, $y(t)$ the vector solution is given by

$$y(t) = \begin{pmatrix} \frac{-1}{7} T_{\frac{1}{2},1}(t) + \frac{1}{49}\int_0^t T_{\frac{1}{2},\frac{1}{2}}(s)[3(t-s)^2 - 2(t-s)]ds \\ \frac{-2}{7} T_{\frac{1}{2},1}(t) + \frac{2}{49}\int_0^t T_{\frac{1}{2},\frac{1}{2}}(s)[3(t-s)^2 - 2(t-s)]ds \end{pmatrix}, t \geq 0.$$

Using ode45, we get the following solution graph of the equivalent ordinary singular differential equation to system (3.10)

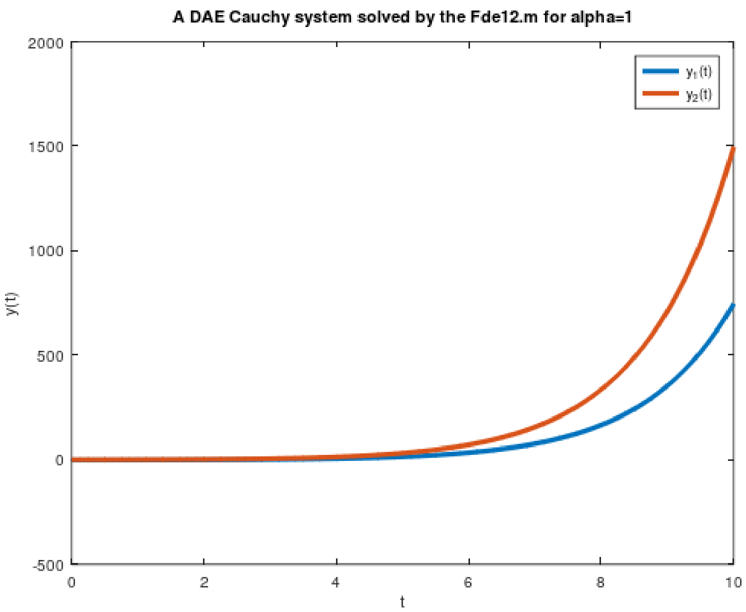

Figure 3.1: Solution to a DAE for $\alpha = 1$

Using the following Matlab code **fde12.m**,

```
> h=2^(-3);
> t0=0;
> tfinal=10;
> y0=[-1;5/6];
> alpha=0.5;
> fdefun=@(t,y)[(4/49)*y(1)+(16/49)*y(2)+(3/49)*t^2-(2/49)*t;(8/49)*y(1)+(32/49)*y(2)+(6/49)*t^2-(4/49)*t];
> [t,y_fde12]= fde12(alpha,fdefun,t0,tfinal,y0,h);
> plot(t,y_fde12(1,:),'LineWidth',2,t,y_fde12(2,:),'Linewidth',2);
> title('Caputo fractional DAE Cauchy system solved by the Fde12.m for alpha=0.5')
> |
```

Figure 3.2: The fde12.m code used to solve the DAE

we get the solution graph of system (3.10) under the Caputo derivative,

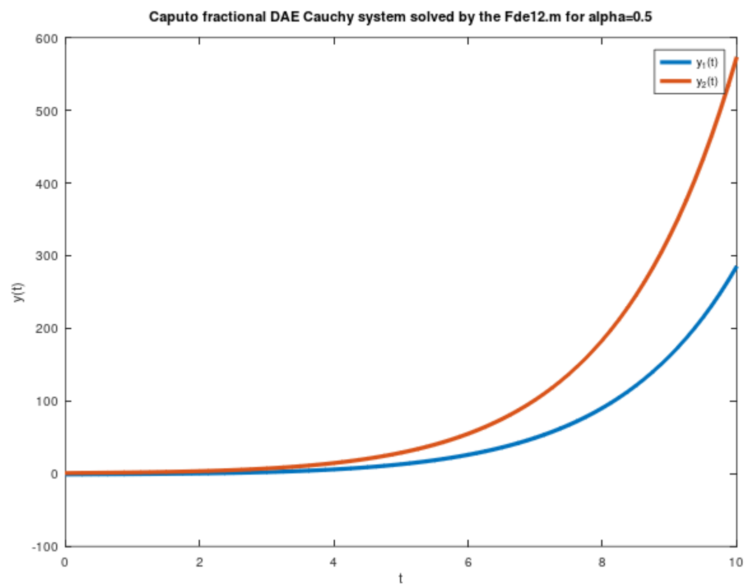

Figure 3.3: Solution to a Caputo fractional DAE for $\alpha = 0.5$.

Now, the solution formula of a singular conformable fractional initial value problem using the same previous data and formula (3.10), is given by

$$y(t) = \left(\frac{8}{49} \begin{pmatrix} e^{\sqrt{t}} & e^{4\sqrt{t}} \\ e^{2\sqrt{t}} & e^{8\sqrt{t}} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{32}{49^3} \begin{pmatrix} e^{\sqrt{t}} & 4e^{\sqrt{t}} \\ 2e^{\sqrt{t}} & 8e^{\sqrt{t}} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-2s^{\frac{1}{2}}} & 4e^{-2s^{\frac{1}{2}}} \\ 2e^{-2s^{\frac{1}{2}}} & 8e^{-2s^{\frac{1}{2}}} \end{pmatrix} \begin{pmatrix} 3s^{\frac{3}{2}} + 2s^{\frac{1}{2}} \\ 6s^{\frac{3}{2}} + 4s^{\frac{1}{2}} \end{pmatrix} ds \right), t > 0$$

Plotting the vector solution $y(t)$, we get

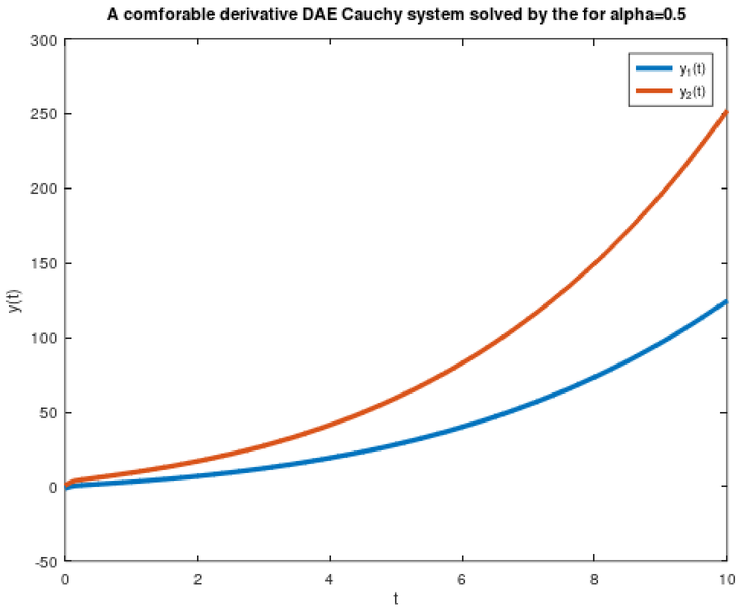

Figure 3.4: Solution of a conformable fractional DAE for $\alpha = 0.5$.

Comments

Since we have used three types of derivatives, namely the ordinary one, Caputo's fractional and the conformable derivatives, then we have three expressions of the solutions. The first figure corresponds to the graph of the solution to system (3.10) with the ordinary derivative, the second figure is the graph of the solution to the same system using Caputo fractional derivative. While the third graph is the graph of the solution of the corresponding conformable singular system.

Conclusion

This research thesis was mainly dedicated to the analytical study of singular fractional Cauchy problems on a Banach space, our goal was to establish the existence and uniqueness of solution to such problems and find an accessible way to represent their solution using simple algebraic techniques as decoupling method, canonical forms and Drazin inverse.
We have applied our results to two different types of fractional derivatives: Caputo fractional derivative and the conformable fractional derivative, and as we obtained the solution formulas to the studied problems under these two derivatives, we also illustrated our theoretical results with numerical examples. Our future perspectives are to provide a numerical study to the systems studied above and maybe to more complicated ones, as the non-linear case and time-varying case which can be a real challenging topic.

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