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THESIS

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Study of Some Inverse Problems for the Biharmonic

Option: Mathematics

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Preface

This thesis was prepared in partial fulfilment of the requirements for acquiring the PhD degree at the University of Badji Mokhtar Annaba (UBMA). The work was carried out between Sept 2017 and Mars 2021 at the Section for Inverse problems and Integral equations method at the Department of Applied Mathematics and Computer Sciences, Laboratory of Applied Mathematics (LMA), under the supervision of the Professor Hacene Saker.

During my studies I have co-authored the following papers:

- A. Hadj and H. Saker. *On an inverse problem of Identifying an unknown boundary for the Biharmonic.* Submitted.

- A. Hadj and H. Saker. *A new regularized Trefftz method for solving an inverse problem of the Laplace equation.* Submitted.

- A. Hadj and H. Saker. *Integral equations method for solving a Biharmonic inverse problem in detection of Robin coefficients*, App Numerical Math, v 160, (2021), pp 436-450. https://doi.org/10.1016/j.apnum.2020.10.005.

Abstract

The aim of this thesis, is to study some types of inverse problems for Laplacian and Bi-Laplacian operators in the planar domain, which occurs in many engineering applications and describes various phenomena in the applied sciences.

In the first type, we establish a new regularized Trefftz method to solve an inverse problem for the Harmonic equation with Dirichlet-Neumann conditions given on an accessible part of an annulus.

In the second type, we will be interested in the Biharmonic equation to find an unknown boundary in a doubly connected domain from a mixed Cauchy data on a known part of the boundary.

The third type, addresses the Biharmonic equation to reconstruct Robin's coefficients on a non-accessible part of the boundary from partial Cauchy data on an accessible part of that boundary.

keywords. *Harmonic equation, Biharmonic equation, Inverse problems, Data completion, Nonlinear integral equation, Tikhonov regularization, Non-accessible boundary, Robin boundary condition, Cauchy problems, Ill-posed problems, New regularized Trefftz method.*

ملخص

الهدف من هذه الأطروحة هو دراسة بعض أنواع المشاكل العكسية للمؤثرين التوافقي و الثنائي التوافقي في مجال مستوي، والتي تحدث في العديد من التطبيقات الهندسية وتصف العديد من الظواهر في العلوم التطبيقية.

في النوع الأول، أنشأنا طريقة جديدة منتظمة لترافز Trefftz لحل مشكلة عكسية للمعادلة التوافقية مع شروط ديريتشليت - نيومان -Dirichlet Neumann المعطاة على جزء يمكن الوصول إليه من الحلقة.

في النوع الثاني، سنهتم بمعادلة الثنائي التوافقي لإيجاد حد غير معروف انطلاقا من بيانات كوشي Cauchy المختلطة على جزء معروف يمكن الوصول إليه من الحدود.

النوع الثالث يعالج معادلة الثنائي التوافقي لإعادة بناء معاملات روبن Robin على حدود لا يمكن الوصول إليها، انطلاقا بيانات كوشي Cauchy الجزئية على حد يمكن الوصول إليه.

الكلمات المفتاحية

Harmonic equation, Biharmonic equation, Inverse problems, Data completion, Nonlinear integral equation, Tikhonov regularization, Non-accessible boundary, Robin boundary condition, Cauchy problems, Ill-posed problems, New regularized Trefftz method.

Résumé

L'objectif de cette thèse est d'étudier certains types de problèmes inverses pour les opérateurs Laplacien et Bi-Laplacien dans un domaine planaire. Ces problèmes interviennent dans de nombreuses applications d'ingénierie et décrivent divers phénomènes dans les sciences appliquées.

Dans le premier type, nous mettrons en évidence une nouvelle méthode de Trefftz régularisée pour résoudre un problème inverse pour l'équation Harmonique avec des conditions de Dirichlet-Neumann données sur une partie accessible d'un anneau.

Dans le second type, nous nous intéresserons à l'équation Biharmonique pour trouver une frontière inconnue dans un domaine doublement connexe à partir de données mixtes de Cauchy sur une partie connue de la frontière.

Au troisième type, on s'intéressera à l'équation Biharmonique pour reconstruire des coefficients de Robin sur une frontière non accessible à partir de données de Cauchy partielles sur une frontière accessible.

mots clés. *Harmonic equation, Biharmonic equation, Inverse problems, Data completion, Nonlinear integral equation, Tikhonov regularization, Non-accessible boundary, Robin boundary condition, Cauchy problems, Ill-posed problems, New regularized Trefftz method.*

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Moreover, I would like to thank my professors and colleges in both the Department of Mathematics and Informatics at the University of Mustapha ben Boulaid Batna 2, and the Department of Mathematics at the University of Badji Mokhtar Annaba, each one bearing its name.

Last but not least, I am grateful to my parents, my wife, my son and all my family for all their love, encouragement and support along the way.

List of Symbols

$\mathbb{N}$: set of all positive natural numbers
$\mathbb{R}$: of all real numbers
$\mathbb{R}^n$: set of all real n-tuples
Ω : an open, nonempty subset of $\mathbb{R}^2$
$\partial\Omega$: the boundary of Ω
$\overline{\Omega}$: the closure of Ω
$B(r)$: open disk in $\mathbb{R}^n$ with center at 0 and radius $r > 0$
$\approx$: stands for is approximately equal to
$\bar{B}(r)$: closed disk in $\mathbb{R}^n$ with center at 0 and radius $r > 0$
X, Y : are sets
$L(X, Y)$: the set of all the bounded linear operators from X to Y
$L(X) :=$ $L(X, X)$
$D(A)$: the domain of the (linear) operator A
I : the identity operator in a Banach space X
I_n : the identity matrix in $\mathbb{R}^n$
$\delta_{i,j}$: the Kronecker symbol
$T : X \to Y$ mapping whose X is a domain of definition, Y is a set
$T(X) \subset Y$ range of T
∇ : the gradient
Δ : Laplacian operator
$\Delta\Delta = \Delta^2$: Bi-Laplacian operator
$(\cdot, \cdot)$: the usual Euclidian inner product
$\partial_n = \nabla \cdot n$: differentiation with respect to the outward unit normal n
$C(\Omega)$: set of continuous functions on Ω
$C^m(\Omega)$: set of m times continuously differentiable functions on Ω
$C^\infty(\Omega)$: set of functions that belong to $C^k(\Omega)$ for every k
$C_0^\infty(\Omega) \subset C^\infty(\Omega)$: with all of their derivatives, have compact support in Ω
$C^{m,\alpha}(\Omega) \subset C^m(\Omega)$: with m-th order derivatives are locally Hölder on Ω
$suppu$: support of a given function u
$C^\infty(\Omega)$: space of infinitely differentiable functions on Ω
$\mathcal{D}(\Omega)$: space of functions in $C^\infty(\Omega)$ having support in Ω
$\mathcal{S}(\mathbb{R}^n)$: schwartz class of rapidly decreasing C^∞ functions on $\mathbb{R}^n$

List of Abbreviations

TM	Trefftz Method
CTM	Collocation Trefftz Method
MTM	Modified Trefftz Method
MCTM	Modified Collocation Trefftz Method
NRTM	New Regularized Trefftz Method
NRCTM	New Regularized Collocation Trefftz Method
BIE	Boundary Integral Equation
BIEMs	Boundary Integral equation methods
IBIEMs	Indirect boundary integral equation methods
MFS	The method of fundamental solution
BVP	Boundary Value Problem
FDM	Finite Difference Method
FEM	Finite Element Method
BEM	Boundary Element Method
2D	Two-Dimensional

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General Introduction

Starting from the middle of the 20th century, the terms "inverse problems" and "ill-posed problems" have been introducing and surely gaining popularity in modern science [12, 23, 65]. Both causality and reversibility lead to a sort of several patterns in science, which can be summarized mathematically in two classes of problems: well-posed and ill-posed problems.

The direct problem can be described in the standard way:

$$\text{input (cause)} \to \text{process (model)} \to \text{output (effect)}$$

An inverse problem is a situation on which from experimental observations, one tries to determine the cause of a phenomenon. It's studies and applications began systematically in physics, geophysics, medicine, astronomy, and all other areas of knowledge where mathematical methods are used. The reason is that solutions to inverse problems describe important properties of media under study, such as density and velocity of wave propagation, elasticity parameters, conductivity, electric permittivity, magnetic permeability, and properties of inhomogeneities in inaccessible areas, etc.[76]. One can classify the inverse problems in two categories: problems that aim to determine the boundary conditions or unknown sources, and those related to the estimation of the parameters of the system.

The class of ill-posed problems was first identified by Jacques Hadamard (1902). As well, examples were treated later in the well-known courses of mathematics [12]. The necessity in studying ill-posed problems stems from one of the main problems in applied mathematics, gaining reliable computing results with due allowance for errors that inevitably occur in setting coefficients and parameters of a mathematical model used to perform computations. According to [62, 63, 65], a problem is ill-posed if at least one of the following three conditions is messed: the solution exists, the solution is unique, the solution is stable, i.e., arbitrarily small variations of coefficients, parameters, initial or boundary conditions give rise to arbitrarily small solution changes.

On the contrary of direct problems, which are generally well posed, the ill-posed problems, often specific to inverse problems render their mathematical resolution rather delicate, because the experimental measurements are not sufficient to determine the parameters of the model exactly, and because the numerical solution remains very sensitive to a slight perturbation of these often inaccurate measurements due to uncertainty errors. To reduce the sensibility of the solution with respect to the final measured data, it is necessary to add to the mathematical problem that models the physical phenomenon to invert a priory information or constraints on the solution.

We shall analyze the three conditions of a well-posed problem in the context of inverse problems. First of all, the fact that the solution of the inverse problem may not exist is not a great difficulty. It is generally possible to recover the existence by relaxing the notion of solution (see example 0.1.2). Second, the fact that the solution of the inverse problem is not unique is a slightly more serious problem. If a problem has several solutions, then we need a way to choose between them, and for this reason it is necessary to provide additional information (for examples see the chapter 2). The third problem is the lack of continuity, which is the most important one (see example 0.1.1). To solve this problem, several techniques, called regularization methods, have been developed [85], which aim to approximate the problem under study by a family of well-posed problems depending on regularization parameters.

The linear elliptic equations arise in several models describing various phenomena in the applied sciences, the harmonic and Poly-harmonic operators $\Delta^m, m \geq 2$, of order m are the prototype of an elliptic operator which play a crucial role in many areas of mathematics, physics and engineering. The classical 2D Harmonic and Biharmonic problems occur in several physical applications, such as: temperature distributions, potentials of electrostatic, magneto-static fields, velocity potentials of incompressible irrotational fluid flows, the electrostatic problems, in-compressible fluid, the deformation of thin plates, the motion of fluids, free boundary problems, non-linear elasticity and elastic bending beam,...(for more historical information, we refer to [2, 5, 7, 25, 31, 41, 66, 67, 87]).

The knowledge of appropriate boundary conditions over the boundary regarding the considered domain of the solution leads to direct problems managed by the harmonic and Biharmonic equation. However, numerous experimental situations do not belong to this category, because of physical difficulties or geometrical inaccessibility, as: (1) the boundary conditions are often incomplete; (2) in the form of under- and over-specified boundary conditions on different parts of the boundary; (3) the solution is prescribed at some internal points in the domain. These are an important class of inverse problems known to be generally ill-posed problems, i.e., the existence, uniqueness, and stability of their solutions are not always guaranteed. (For more details, we refer to [3, 5, 9, 10, 12, 23, 33]).

0.1 Examples of inverse and ill-posed problems.

Example 0.1.1 (calculus; summing Fourier series.)**.** The problem of summing a Fourier series consists in finding a function $f(x)$ from its Fourier coefficients.

We show that the problem of summing a Fourier series is unstable with respect to small variations in the Fourier coefficients in the l^2 metric if the variations of the sum are estimated in the $\mathbb{C}$ space. Let

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

and let the Fourier coefficients a_k of the function $f(x)$ have small perturbations: $\tilde{a}_k = a_k + \frac{\epsilon}{k}$. Set

$$\tilde{f}(x) = \sum_{k=1}^{\infty} \tilde{a}_k \cos kx$$

The difference between the coefficients of these series in the l^2 metric is

$$\|f - \tilde{f}\|_{l^2} = \left\{ \sum_{k=1}^{\infty} (a_k - \tilde{a}_k)^2 \right\}^{\frac{1}{2}} = \epsilon \left\{ \sum_{k=1}^{\infty} \frac{1}{k^2} \right\}^{\frac{1}{2}} = \epsilon \sqrt{\frac{\pi^2}{6}}$$

which vanishes as $\epsilon \longrightarrow 0$. However, the difference

$$f(x) - \tilde{f}(x) = \epsilon \sum_{k=1}^{\infty} \frac{1}{k} \cos kx$$

can be as large as desired because the series diverges for $x = 0$.

Thus, if the $\mathbb{C}$ metric is used to estimate variations in the sum of the series, then summation of the Fourier series is not stable.

Example 0.1.2 (For the Laplacian [1]). Let Ω be a doubly connect planar domain with boundary $\partial\Omega = \Gamma_m \cup \Gamma_c$. In a partial differential equation:

$$\begin{cases} \Delta u = 0, & in\ \Omega \\ u = u_0, & on\ \Gamma_m \\ \frac{\partial u}{\partial n} = u_1, & on\ \Gamma_m \end{cases} \tag{0.1.1}$$

and one of the following equations are satisfies

$$u = 0, \quad on\ \Gamma_c \tag{0.1.2}$$

Or

$$\frac{\partial u}{\partial n} = 0, \quad on\ \Gamma_c \tag{0.1.3}$$

We consider the inverse problem: find Γ_c from the knowledge of the accessible part Γ_m, and u_0, u_1 on Γ_m.

The existence of a solution to problem (0.1.1)-(0.1.2) cannot be guaranteed for arbitrary data u_0, u_1. Let $u_0 = 0$ and $u = 0$ on Γ_c, then $u = 0$ in $\partial\Omega$, from [1], we obtain $u = 0$ in Ω, and $u_1 = 0$. Therefore, equation (0.1.2) has no solution if $u_0 = 0$ and $u_1 \neq 0$, and the inverse problem turn to be ill-posed (for more detail see 2.1).

Example 0.1.3 (For the Bi-Laplacian). The Biharmonic equation in an open bounded domain $\Omega \in \mathbb{R}^2$, namely

$$\Delta^2 u(x) = 0, \qquad in\ \Omega \tag{0.1.4}$$

or, equivalently the system of equations

$$\Delta u(x) = v(x), \quad \Delta u(x) = v(x), \qquad in\ \Omega \tag{0.1.5}$$

is a well-known example of a mathematical model governing the interior 2D flow of viscous fluids at small Reynolds numbers, i.e., the Stokes flow, or the Kirchhoff theory of plates in elasticity. For example, in viscous fluids, functions u and v satisfying equations (0.1.5) are called the stream-function and the vorticity of the fluid flow, whilst for plate bending problems, they represent the deflection and the bending moment of the plate, respectively.

If u and its normal derivative $\partial_n u$, or u and v, or u and $\partial_n v$, are prescribed at all points of the boundary $\partial\Omega$, then u and v can be uniquely determined everywhere in the domain Ω. Moreover, the solution depends continuously on the input boundary data, i.e., the so-called direct problem is well posed [33].

However, many experimental situations where it is not always possible to measure the boundary conditions at all points on the boundary $\partial\Omega$. But, some other interior or boundary information may be given elsewhere. If a boundary portion $\Gamma_m \subset \partial\Omega$ is accessible to measurements, and the remaining boundary portion $\Gamma_c = \partial\Omega \setminus \Gamma_c$ is non-accessible to measurements. Then, the problem is known to be an inverse boundary value problem for the Biharmonic equation and it becomes ill-posed (for more information see [33]).

0.2 Thesis problem

The purpose of this thesis is to consider the analysis of certain inverse problems related to recover some missing informations for Harmonic and Biharmonic equation. We mainly focus on: $^{(1)}$the completion of boundary value; $^{(2)}$finding an unknown boundary; $^{(3)}$reconstructing of Robin's coefficients.

These problems describe methods based on quantified partial measures in one way or another, and search for ways to complete missing data. This naturally leads to mathematical models expressed by inverse problems, and offers the possibility to introduce many interesting mathematical techniques. For example, Tikhonov's regularization method, integral equation methods, Trefftz method, least squares method, conjugate gradient method, collocation methods, numerical interpolation.

0.3 Bibliographical note

Physicists refer to the movement of a point on a vibrating string as "harmonic motion". Such motion can be described using sine and cosine functions. In this context, sine and cosine functions are sometimes called harmonics. Harmonic and Biharmonic functions are the solutions of the Laplace and Bilaplace equations, respectively. They play a crucial role in many areas of mathematics, physics and engineering [2, 3, 9, 15, 25, 29, 31, 35, 39, 22, 45, 71, 87, 95].

Harmonic and Biharmonic problems are defined by their boundary conditions. Many types of boundary conditions are adopted for studies, such as: Dirichlet's problem, Neumann's problem, mixed or Dirichlet-Neumann's problems, Robin's boundary value problems, which are well-known boundary value problems for Harmonic and Biharmonic equations. Recently, other types of boundary value problems for the Biharmonic equation have been introduced, such as: the Navier and Riquier-Neumann's boundary conditions [26, 46, 48, 49].

There are a different methods for solving problems in applied science. One of the best adopted is the integral equations method, that reducing the dimensions, and instead of solving a problem in a defined region, we can solve a boundary integral equation [91]. This has several applications for a large class of direct boundary value problems and also for inverse problems (for more detail we refer to [17, 18, 19, 20, 21, 28, 43, 44, 51, 58, 60, 61]).

The Trefftz method (TM) was developed since 1926, it has been widely studied and applied to many engineering problems. The main idea of this method is to extend the numerical solution in terms of T-complete functions that satisfy the governing equation. The TM is less popular than other numerical methods such as: FDM, FEM, BEM, BIE, etc. Because, the system of linear equations resulting from the TM is an ill-posed problem, even for a well-posed boundary value problem, also for multi-connected domains, the conventional TM fails (for more details, see [78, 92]).

Many scientific researches devoted to the data completion problems. Some of which have been encountered through our bibliographical research, for example, in [1] it is proposed a method to determine the bottom of reservoir in an in-compressible fluid modeled by Laplace's equation. In [71] it is presented a method to detect the boundary of a crack in the plane static problems of elasticity. Detecting the corrosion of complex metal assemblies in aircraft structures, in [68] it is used a numerical method based on the fundamental solution to determine material loss on an inaccessible material. In [17, 19, 20] the authors proposed a method to recover shape and impedance function based on the integral equation method. For the problem of reconstruction an interior boundary curve from the knowledge of temperature and thermal flux on the exterior boundary curve see [58].

Recently, Young, Chen and Kao (2007) have proposed the modified collocation Trefftz method (MCTM), for solving the Laplace problems, that provide a very interest method which make converge the serie expansion of solution and decreased the condition number of discretization matrices, compared with the CTM, (see [4, 13, 15, 22]). There is a considerable scientific research in studying inverse problems using the MCTM, such as: detection of corrosion inside the pip, determine the robin coefficients, detect the cracks position inside the disc and boundary identification, etc. (for more information see [5, 7, 9, 10, 11, 30, 33, 36, 75, 78, 79, 80, 81, 82, 83, 84]).

Numerical calculation is an important aspect in the evaluation of studies. Several methods are available to solve inverse problems, e.g., Thikhonov method, quasi reversibility method, iterative Methods, the KMF method, the iterative solution of finite difference approximation, finite element treatment, iterative method of Kozlov, method of approximate by the fundamental solutions, the Trefftz collocation method, the modified collocation Trefftz method. [28, 42, 51, 58, 63, 85].

0.4 Thesis content

The remainder of this thesis is divided into four chapters which are organized as follows:

The first chapter contains a general notions and basic tools required in the development of our study.

The second chapter is composed of two parts addressing a geometrical inverse problem. In the first one, we present an inverse problem that was suggested in [1], which is to find an unknown boundary for the harmonic equation from the extra boundary conditions. In the remaining part, we discuss the extension of the first part to the Biharmonic equation.

In the third chapter, we address the data completion problem and here a new regularized Trefftz method will be presented and discussed in detail. A new solution scheme is constructed and an estimation of errors is obtained under data with and without noise, we consider the finite term truncation and the collocation method to obtain a linear equation system that can be solved by the conjugate gradient method to obtain the coefficients, and to complete the data on the whole boundary.

In the last chapter of the thesis, we are interested in the Biharmonic equation to recover Robin's coefficients on an inaccessible part of the boundary in a simply connected domain, from Riquier-Neumann measurements on an accessible part of the boundary. Here, the integral equation methods will be considered to derive a system of non-linear and ill-posed integral equations that can be solved by the Tikhonov regularization method. Thus, to complete the missing Cauchy data, and eventually to recover the Robin coefficients by the least squares method.

Chapter 1

Preliminaries and general concepts

In this introductory chapter, we recall some necessary background material from functional analysis and numerical analysis. This chapter can be long due to the numerous mathematical tools that are employed in our approach. So, we will recall all the tools required for our development based on renowned publications in the field [4, 9, 16, 20, 21, 22, 23, 24, 28, 38, 40, 41, 42, 43, 44, 45, 47, 55, 56, 59, 64, 66, 69, 70, 75, 79, 80, 88]. The proofs of the basic theorems, will be referred at the end of each theorem.

1.1 Initial concepts

Definition 1.1.1 (Harmonic and Biharmonic Functions)**.** A real function u defined on an open subset Ω of $\mathbb{R}^n$ is called Harmonic (respectively Biharmonic) if it is two times continuously differentiable (respectively four times continuously differentiable) and verifies:

$$\Delta u = 0, \quad \text{(respectively } \Delta^2 u = 0)$$

where $\Delta = \sum_{j=1}^{n} D_i^2$, $\Delta^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} D_i^2 D_j^2$, and $D_j^2 = \frac{\partial^2}{\partial x_j^2}$ denotes the second partial derivative, with respect to the j^{th} coordinate variable. The operators Δ, Δ^2 are called Laplace, Bi-Laplace, respectively.

Definition 1.1.2 (The Dirac's function)**.** The Dirac's function δ which is not exactly a function, was presented by the British physicist Paul Adrien-Maurice Dirac (1902-1984) as a device technique in the mathematical formulation of quantum mechanics. The Dirac function is set by the following properties

$$\delta(x) = \begin{cases} \infty, & if\ x = 0 \\ 0, & if\ x \neq 0 \end{cases}$$

and

$$\int_{\Omega} \delta(x) dx = 1, \ if\ 0 \in \Omega.$$

Definition 1.1.3 (The Fundamental Solution)**.** Technically, a fundamental solution for a linear differential operator $\mathcal{L}$ with constant coefficients which is defined on the distribution space $\mathcal{D}'(\mathbb{R}^n)$ is a distribution E satisfying:

$$\mathcal{L}E = \delta \quad in\ \mathcal{D}'(\mathbb{R}^n), \tag{1.1.1}$$

where δ is the Dirac's function, originally centered. The interest of the fundamental solution consists in the fact that if convolution makes sense, then for a given function f, the solution of the equation:

$$\mathcal{L}u = f \quad in\ \mathcal{D}'(\mathbb{R}^n), \tag{1.1.2}$$

is given by:

$$u = E * f. \tag{1.1.3}$$

The linearity of $\mathcal{L}$, as well as E being a fundamental solution, and δ is the neutral element of the convolution, yields:

$$\mathcal{L}u = \mathcal{L}(E * f) = \mathcal{L}E * f = \delta * f = f. \tag{1.1.4}$$

Definition 1.1.4 (Green's function)**.** In the case of the Green's function, the fundamental solution considers homogeneous boundary conditions, and the Dirac delta function is not centered at the origin, but at some fixed source point. Thus, a Green's function of a linear partial differential operator $\mathcal{L}_y$ of constant coefficients relative to y defined on $\mathcal{D}'(\mathbb{R}^n)$, with homogeneous boundary conditions, is a distribution G such that:

$$\mathcal{L}_y(G(x, y)) = \delta_x(y) \quad in\ \mathcal{D}'(\mathbb{R}^n), \tag{1.1.5}$$

where δ_x is the Dirac function with Dirac mass centered at the source point x, i.e., $\delta_x(y) = \delta(y - x)$. The Green's function also represents the impulse response of the operator $\mathcal{L}_y$ with respect to a source point x, which is the kernel of the inverse operator of $\mathcal{L}_y$ noted $\mathcal{L}_y^{-1}$ that corresponds to an integral operator where $G(x, y) = \mathcal{L}_y^{-1}(\delta_x(y))$. The Green's function differs as a fundamental solution, it is searched in a certain particular domain which satisfies certain boundary conditions, but for simplicity we consider here only $\Omega = \mathbb{R}^n$.

The solution of non-homogeneous differential boundary problem

$$\mathcal{L}_x(u(x)) = f(x) \quad in\ \mathcal{D}'(\mathbb{R}^n), \tag{1.1.6}$$

in the sense of convolution, it is given as:

$$u(x) = G(x, y) * f(y), \tag{1.1.7}$$

where G is the Green's function of the operator $\mathcal{L}_x$, which is symmetrical, i.e.,

$$G(x, y) = G(y, x). \tag{1.1.8}$$

as in the fundamental solution, we take

$$\mathcal{L}_x(u(x)) = \mathcal{L}_x(G(x, y) * f(y)) = \delta_x(y) * f(y) = f(x). \tag{1.1.9}$$

We observe that the Green's function of free space or all space, i.e., without boundary conditions, is related to the fundamental solution by the relation

$$G(x, y) = E(x - y) = E(y - x). \tag{1.1.10}$$

Recall that $|x|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. We now consider some examples of a Green's function in a free space as follow:

1. Laplace's equation satisfied in the sense of distributions

$$\Delta_y G(x,y) = \delta_x(y), \quad in\ \mathcal{D}'(\mathbb{R}^n),$$

which is given by

$$G(x,y) = \frac{1}{2\pi}\ln|x-y|, \quad n = 2, \tag{1.1.11}$$

2. Bi-laplace's equation satisfied in the sense of distributions

$$\Delta_y^2 G(x,y) = \delta_x(y), \quad in\ \mathcal{D}'(\mathbb{R}^n),$$

which is given by

$$G(x,y) = \frac{1}{8\pi}|x-y|^2\ln|x-y|, \quad n = 2. \tag{1.1.12}$$

Most of the basic properties of Harmonic and bi-harmonic functions can be deduced from the fundamental solution that is introduced in the following theorem:

Theorem 1.1.1. *The functions*

$$\begin{aligned} E_1(x,y) &= \frac{1}{2\pi}\ln|x-y|, && n = 2 \\ E_2(x,y) &= \frac{1}{8\pi}|x-y|^2\ln|x-y|, && n = 2 \end{aligned}$$

defined for all $x \neq y$ in $\mathbb{R}^2$ is called the fundamental solution of Laplace, Bilaplace's equation. For fixed $y \in \mathbb{R}^n$ it is Harmonic, bi-harmonic in $\mathbb{R}^2 \setminus y$, which satisfies

$$\begin{aligned} \Delta_x E_1(x,y) &= \delta(x-y), && in\ \mathbb{R}^2 \\ \Delta_x^2 E_2(x,y) &= \delta(x-y), && in\ \mathbb{R}^2 \end{aligned}$$

Proof. See [21, 42] □

Basic Properties

- Each Harmonic function can be considered as a bi-harmonic.
- Sums, translations, dilates and scalar multiples of Harmonic, Biharmonic functions are Harmonic, Biharmonic, respectively.
- Many basic properties of Harmonic functions follow from Green's identity, for u and v are C^2 functions on a neighborhood of $\overline{\Omega}$ we have that

$$\int_\Omega (u\Delta v - v\Delta u)d\Omega = \int_{\partial\Omega}(u\partial_n v - v\partial_n u)d\partial\Omega \tag{1.1.13}$$

- Many basic properties of bi-harmonic functions follow from Green's and second Green's identities, for u and v are C^4 functions on a neighborhood of $\overline{\Omega}$ we have that

$$\int_\Omega (u\Delta^2 v - v\Delta^2 u)d\Omega = \int_{\partial\Omega}(u\partial_n(\Delta v) - \Delta v\partial_n u + v\partial_n(\Delta u) - \Delta u\partial_n v)\, d\partial\Omega \tag{1.1.14}$$

Definition 1.1.5 (Simply Connected Domain)**.** A two-dimensional region Ω of the plane consisting of one connected piece is called simply-connected if it has this property: whenever a simple closed curve Γ lies entirely in Ω, then its interior also lies entirely in Ω. Except that, it is "double connected domain" or "multiply connected domain".

Theorem 1.1.2 (Local Maximum Principle)**.** *Suppose Ω is connected, u is real valued and Harmonic on Ω, and u has a local maximum in Ω. Then u is constant.*

Proof. See [45] □

Theorem 1.1.3 (Maximum-Minimum Principle)**.** *Suppose Ω is connected, u is real valued and Harmonic on Ω, and u has a maximum or a minimum in Ω. Then u is constant.ie., a Harmonic function on a domain cannot attain its maximum or its minimum unless it is constant.*

Proof. See [41, 45] □

Corollary 1.1.1. *Suppose Ω is bounded and u is a continuous real valued function on Ω that is Harmonic on Ω. Then u attains its maximum and minimum values over $\overline{\Omega}$ on $\partial\Omega$.*

Proof. See [45] □

Remark 1.1.1. The corollary (1.1.1) implies that on a bounded domain a Harmonic function is determined by its boundary values.

Theorem 1.1.4. *Suppose u is Harmonic on Ω and $a \in \Omega$. Then there exist Harmonic homogeneous polynomials p_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a)$$

Proof. See [45] □

Theorem 1.1.5. *Suppose Ω is connected, u is Harmonic in Ω, and $u = 0$ on a nonempty open subset of Ω. Then $u \equiv 0$ in Ω.*

Proof. Let ω be a non-empty open subset of Ω such as

$$\omega = \{x \in int(\Omega) \ : \ u(x) = 0\}$$

If $a \in \partial\omega \subset \Omega$ is a limit point of ω, from the corollary (1.1.1) then u attains its maximum values over $\overline{\omega}$ on $\partial\omega$, i.e., all derivatives of u vanish at a by continuity, implying that the power series of u at a is identically zero, therefore $a \in \omega$. Thus ω is closed in Ω. As consequence we must have $\omega = \Omega$ by connectivity of domain, giving $u \equiv 0$ in Ω. □

Lemma 1.1.1. *Suppose Ω is connected, u is Harmonic in Ω, and $\frac{\partial u}{\partial n} = 0$ on a nonempty open subset of Ω. Then $u \equiv Constant$ in Ω.*

Proof. Let ω be a nonempty open subset of Ω such as

$$\omega = \left\{ x \in int(\Omega) \ : \ \frac{\partial u}{\partial n} = 0 \right\}$$

If $a \in \partial\omega \subset \Omega$ is a limit point of ω, while u Harmonic then is real analytic. Then u attains its maximum values over $\overline{\omega}$ on $\partial\omega$, i.e., $\nabla u(a) = 0$ and $\frac{\partial u}{\partial n}(a) = 0$ thus $u(a) = constant$ by continuity, implying that the power series of $\frac{\partial u}{\partial n}(a)$ at a is identically zero and the power series of $u(a)$ at a is identically *constant*, therefore $a \in \omega$. Thus ω is closed non empty in Ω and u has a local maximum in Ω, as consequence from the theorem of Local Maximum Principle (1.1.2) we have that u is *constant* in it's domain of definition. □

Theorem 1.1.6 (Cauchy-Kovalevskaya)**.** *In a simply connected domain Ω. Suppose I is a real-analytic non-trivial arc of $\partial\Omega$. Then if f_j, for $j = 1, ..., 2N$, are real-analytic functions on I, there is a function u with $\Delta^N = 0$ in a (planar) neighborhood of I, having $\partial_n^{j-1} u|_I = f_j$ for $j = 1, ..., 2N$. The solution u is unique among the real-analytic functions.*

Proof. See [56] □

Theorem 1.1.7 (Holmgren)**.** *In a simply connected domain Ω. Suppose I is a real-analytic non-trivial arc of $\partial\Omega$. Then if u is smooth on a planar neighbourhood $\mathcal{O}$ of I and $\Delta^N u = 0$, $N \geq 1$ holds on $\mathcal{O} \cap \Omega$, with $\partial_n^{j-1} u|_I = 0$ for $j = 1, ..., 2N$, then $u \equiv 0$ on $\mathcal{O} \cap \Omega$, provided that the open set $\mathcal{O} \cap \Omega$ is connected.*

Proof. See [56] □

1.2 Basic facts from functional analysis

1.2.1 Normed spaces and Hilbert spaces

Definition 1.2.1 (Scalar Product, Pre-Hilbert Space)**.** Let X be a real linear space (vector space) over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A scalar product or inner product is a mapping

$$(\cdot, \cdot) : X \times X \longrightarrow \mathbb{R}$$

with the following properties:

(H1) $(x, x) \geq 0$, for all $x \in X$	(positivity)
(H2) $(x, x) = 0$ if and only if $x = 0$, for all $x \in X$	(definiteness)
(H3) $(x, y) = \overline{(y, x)}$, for all $x, y \in X$	(symmetry)
(H4) $(\alpha x + \beta y, z) = \alpha(y, z) + \beta(y, z)$, for all $x, y \in X$, and $\alpha, \beta \in \mathbb{K}$	(linearity)

A vector space X over $\mathbb{K}$ with inner product $(\cdot, \cdot)$ is called a pre-Hilbert space over $\mathbb{K}$.

Definition 1.2.2 (Norm)**.** Let X be a real linear space (vector space) over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A norm on X is a mapping

$$|| \cdot || : X \to \mathbb{R}$$

(N1) $\|x\| \geq 0$, for all $x \in X$ (positivity)
(N2) $\|x\| = 0$ if and only if $x = 0$, (definiteness)
(N3) $\|\alpha x\| = |\alpha|.\|x\|$, for all, $x \in X$, $\alpha \in \mathbb{K}$ (homogeneity)
(N4) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ (triangle inequality)

A vector space X over $\mathbb{K}$ equipped with norm $\|\cdot\|$ is called normed space over $\mathbb{K}$. For $X = \mathbb{R}^n$ we will also call the norm a vector norm. Some examples of norms on $\mathbb{R}^n$ are given by

$$\|x\|_1 = \sum_{j=1}^n |x_j| \quad , \qquad \|x\|_2 = \left(\sum_{j=1}^n |x_j|^2\right)^{\frac{1}{2}} \quad , \qquad \|x\|_\infty = \max_{j=1,\dots,n} |x_j|$$

for $x = (x_1, ..., x_n)^T$. The three norms are special cases of the norm

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}}$$

defined for any real number $1 \leq p \leq \infty$. For two elements x, y in a normed space $d(x, y) = \|x - y\|$ is called the distance between x and y.

Definition 1.2.3 (Convergence)**.** A sequence (x_n) of elements in a normed space X is called convergent if there exists an element $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e., if for every $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that $\|x_n - x\| < \epsilon$ for all $n \geq N(\epsilon)$. The element x is called the limit of the sequence (x_n), and we write

$$\lim_{n\to\infty} x_n = x$$

A sequence that does not converge is called divergent.

Theorem 1.2.1. *Let X be a pre-Hilbert space. The mapping:* $\|x\| : X \longrightarrow \mathbb{R}$ *defined by*

$$\|x\| = \sqrt{(x, x)}, \quad x \in X$$

is a norm, i.e., it has properties (N1), (N2), (N3) and (N4) of Definition 1.2.2. Furthermore,

(N5) $(x, y) \leq \|x\| \, \|y\|$*, for all* $x, y \in X$ *(Cauchy–Schwarz inequality)*
(N6) $\|x + y\|^2 \leq 2\|x\|^2 + 2\|x\|^2$*, for all* $x, y \in X$ *(binomial formula)*

Proof. See [4] □

Theorem 1.2.2. *(i) The limit of a convergent sequence is uniquely determined.*

(ii) Two norms $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *on a linear space X are equivalent if and only if there exist positive numbers c and C such that*

$$c\|x\|_a \leq \|x\|_b \leq C\|x\|_a$$

for all $x \in X$*. The limits with respect to the two norms coincide.*

(iii) On a finite-dimensional linear space all norms are equivalent.

(iv) Any bounded sequence in a finite-dimensional normed space X contains a convergent sub-sequence.

Proof. See [41, 42, 77] □

Example 1.2.1. In this example we list some of the most important pre-Hilbert and normed spaces.

(a) $\mathbb{R}^n$ is a pre-Hilbert space of dimension n over $\mathbb{R}^n$ with inner product

$$(x, y) = \sum_{i=1}^{n} x_i y_i$$

(b) Define the linear space ℓ^2 of (real-valued) sequences by

$$\ell^2 = \left\{ (x_k) \subset \mathbb{R} : \sum_{k=1}^{\infty} x_k^2 < \infty \right\}$$

and

$$(x, y) = \sum_{k=1}^{\infty} x_k y_k, \quad x = (x_k), y = (y_k) \in \ell^2$$

defines an inner product on ℓ^2. It is well-defined by the Cauchy-Schwarz inequality.

(c) The space $C[a, b]$ of (real or complex-valued) continuous functions on $[a, b]$ is a pre-Hilbert space over $\mathbb{R}$ with inner product

$$(x, y)_{L^2} = \int_a^b x(t)\overline{y(t)}dt, \quad x, y \in L^2([a, b])$$

The corresponding norm is called the Euclidean norm and is denoted by

$$||x||_{L^2} = \sqrt{(x, x)_{L^2}} = \int_a^b x(t)\overline{y(t)}dt, \quad x \in L^2([a, b])$$

(c) On the same vector space $C[a, b]$ as in example (b), we introduce a norm by

$$||x||_{L^\infty} = \max_{a \leq t \leq b} |x(t)|, \quad x \in L^2([a, b])$$

that we call the supremum norm.

(d) Let $m \in \mathbb{N}$. We define the spaces $C^m([a, b])$ and $C^{m,k}([a, b])$ as:

$$C^m([a, b]) = \left\{ \begin{matrix} x \in C([a, b]) : \text{ x is m times continuously differentiable} \\ \text{on [a,b] equiped with the norm } ||x||_{C^m} := \max_{0 \leq k \leq m} ||x^{(k)}||_\infty \end{matrix} \right\}$$

$$C^{m,k}([a, b]) = \left\{ \begin{matrix} x \in C^m([a, b]) : \text{ equiped with the norm} \\ ||x||_{C^{m,k}} := ||x||_{C^m} + \sup_{t \neq s} \dfrac{\left| x^{(m)}(t) - x^{(m)}(s) \right|}{|t - s|^k}. \end{matrix} \right\}$$

Definition 1.2.4 (Banach space, Hilbert space)**.** A normed space X over $\mathbb{K}$ is called complete or a Banach space if every Cauchy sequence converges in X. A complete pre-Hilbert space is called a Hilbert space.

Definition 1.2.5 (L^p Spaces)**.** Let X be a measured space and let μ be a positive, not necessarily finite, measure on X. For $0 < p < \infty$. The space $L^p(X)$ is a Banach space with the norm

$$||f||_{L^p(X)} = \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}$$

and for $p = \infty$ by

$$||f||_{L^\infty(X)} = ess.sup|f| = \inf\{c > 0 \ : \ |f(x)| \leq c, \ \forall x \in X\}$$

For $p = 2$ the space $L^2(X)$ is a Hilbert space with the inner product

$$(f, g)_{L^2(X),L^2(X)} = \int_X f(x)g(x)dx$$

and the norm induced

$$||f||_{L^2(X)} = \sqrt{(f, f)} = \left(\int_X |f(x)|^2 dx\right)^{\frac{1}{2}}$$

It is well known that Minkowski's (or the triangle) inequality , holds for all $f, g \in L^p(X)$, whenever $1 \leq p \leq \infty$ by

$$||f + g||_{L^p(X)} \leq ||f||_{L^p(X)} + ||g||_{L^p(X)}$$

Ḧolder's inequality says that for all $1 \leq p < \infty$ and all measurable functions $f \in L^p(X)$, $g \in L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$||fg||_{L^1(X)} \leq ||f||_{L^p(X)}||g||_{L^q(X)}$$

Definition 1.2.6 (Compactness)**.** A subset of a normed space is called relatively compact if it's closure is compact.

Definition 1.2.7 (dual space)**.** Let X be a normed vector space. A functional on X is a map from X to the scalars. The dual X' of X is the Banach space of bounded linear functional on X; that is, $X' = (X, Y)$, where Y is the Banach space of scalars with absolute value taken as norm.

Remark 1.2.1 (see [4])**.** The spaces $\mathbb{C}^n$ and $\mathbb{R}^n$ are Hilbert spaces with respect to their canonical inner products. The space $C[a, b]$ is not complete with respect to the inner product $(\cdot, \cdot)_{L^2}$. We denote the completion of the pre-Hilbert space $(C[a, b], (\cdot, \cdot)_{L^2})$ by $L^2(a, b)$.

Definition 1.2.8 (Orthonormal System)**.** A countable set of elements
$A = \{x_k : k = 1, 2, 3, ...\}$ is called an orthonormal system (ONS) if

$$(x_k, x_j) = \begin{cases} 0, & k \neq j \\ 1, & k = j \end{cases}$$

Definition 1.2.9. For any set $B \subset X$, let

$$span\ B := \left\{ \sum_{k=1}^{n} \alpha_k x_k : \alpha_k \in \mathbb{K}, x_k \in B, n \in \mathbb{N} \right\}$$

be the subspace of X spanned by B.

Theorem 1.2.3. *Let $B = \{x_k \ : \ k = 1, 2, ...\}$ be an orthonormal system. Then*

(a) If B is finite, i.e., $B = \{x_k : k = 1, 2, ..., n\}$, then for every $x \in X$ there exist uniquely determined coefficients $\alpha_k \in \mathbb{K}, k = 1, ..., n$, such that

$$\|x - \sum_{k=1}^{n} \alpha_k x_k\| \leq \|x - a\|, \textit{ for all } a \in span\ B.$$

The coefficients α_k are given by $\alpha_k = (a, x_k),\ for\ k = 1, ..., n.$

(b) For every $x \in X$, the following Bessel inequality holds:

$$\|x\|^2 \leq \sum_{k=1}^{\infty} |\,(x, x_k)\,|^2$$

and the series converges in X.

(c) B is complete if and only if span B is dense in X.

(d) B is complete if and only if every $x \in X$ has a (generalized) Fourier expansion of the form

$$x = \sum_{k=1}^{\infty} (x, x_k)\, x_k$$

where the convergence is understood in the norm of X. In this case, the Parseval equation holds in the following more general form:

$$(x, y) = \sum_{k=1}^{\infty} (x, x_k)\, \overline{(y, x_k)}$$

Proof. See [4, 77] □

Example 1.2.2 (Fourier series expansion [4, 52])**.** We denote by φ_k the trigonometric monomials

$$\varphi_k(t) = e^{ikt}$$

for $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then the set $\{\varphi_k : k \in \mathbb{Z}\}$ form a complete system of orthonormal functions in $L^2[0, 2\pi]$. For a function $x, y \in L^2[0, 2\pi]$ the series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt},$$

where

$$c_k = \frac{(x, \varphi_k)_{L^2}}{||\varphi_k||^2_{L^2}} = \frac{1}{2\pi} \int_0^{2\pi} x(t) e^{-ikt} dt$$

is called the Fourier series of x, its coefficients c_k, are called the Fourier coefficients of x. On $L^2[0, 2\pi]$, as usual, the mean square norm is introduced by the scalar product

$$(x, y)_{L^2[0,2\pi]} = \int_0^{2\pi} x(t)\overline{y}(t) dt.$$

If x is a periodic, piecewise continuous function with period T, then the associated trigonometric Fourier series is defined as an infinite series of the form

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\frac{2k\pi}{T} t) + \sum_{k=1}^{\infty} b_k \sin(\frac{2k\pi}{T} t)$$

where

$$a_0 = c_0 = \frac{1}{T} \int_0^T x(t) dt$$

and, for $k = 1, 2, 3, ...,$

$$a_k = c_k + c_{-k} = \frac{2}{T} \int_0^T x(t) \cos(\frac{2k\pi}{T} t) dt$$
$$b_k = (c_k - c_{-k}) i = \frac{2}{T} \int_0^T x(t) \sin(\frac{2k\pi}{T} t) dt$$

and the infinite series all converge,

$$\sum_{k=-\infty}^{\infty} c_k = c_0 + \sum_{k=1}^{\infty} c_{-k} + \sum_{k=1}^{\infty} c_k$$

In addition the two infinite series $\sum_{k=1}^{\infty} |a_k|^2$ and $\sum_{k=1}^{\infty} |b_k|^2$ are converges. Then $\sum_{k=1}^{\infty} a_k b_k$, is absolutely convergent with

$$|\sum_{k=1}^{\infty} a_k b_k| \leq \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |b_k|^2 \right)^{\frac{1}{2}} \quad \text{(Schwartz inequality)}$$

$$\sum_{k=1}^{\infty} |c_k|^2 ||\varphi_k||^2 \leq ||x||^2 \quad \text{(Bessel's inequality)}$$

In finite sequences of n real or complex numbers both inequality are verified.

1.2.2 Linear Bounded and Compact Operators

For this subsection, let X and Y always be normed spaces and $A : X \longrightarrow Y$ be a linear operator, i.e., verify the property (H4) in definition 1.2.1.

Definition 1.2.10 (Boundedness, Non-boundedness, Norm of A)**.** The linear operator A is called bounded if there exists $c > 0$ such that

$$\|Ax\| \le c\|x\| \quad \text{for all } x \in X.$$

The smallest of these constants is called the norm of A, i.e.,

$$\|A\|_{L(X,Y)} = \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

Any linear application T defined on a vector subspace $D(T) \subset X$ is called a non-bounded linear operator, to values in Y, $D(T)$ is the domain of T.

Definition 1.2.11 (Adjoint Operator)**.** Let $A : X \longrightarrow Y$ be a linear and bounded operator between Hilbert spaces. Then there exists one and only one linear bounded operator $A^* : Y \longrightarrow X$ with the property

$$(Ax, y) = (x, A^* y) \quad \text{for all } x \in X, y \in Y.$$

This operator $A^* : Y \longrightarrow X$ is called the adjoint operator of A. For $X = Y$, the operator A is called self-adjoint if $A^* = A$.

Theorem 1.2.4 ([4] A.18. p255)**.** *The following assertions are equivalent:*

(a) A is bounded.
(b) A is continuous at $x = 0$, i.e., $x_j \to 0$ implies that $Ax_j \to 0$.
(c) A is continuous for every $x \in X$.

The space $L(X, Y)$ of all linear bounded mappings from X to Y with the operator norm is a normed space, i.e., the operator norm has properties (N1), (N2), (N3) and (N4) of Definition 1.2.2 and the following: Let $B \in L(X, Y)$ and $A \in L(Y, Z)$ then $AB \in L(X, Z)$ and $\|AB\| \le \|A\|\,\|B\|$.

Definition 1.2.12 (Jordan measures)**.** A subset $G \subset \mathbb{R}^m$ is said to be measurable in Jordan's sense when its interior and exterior Jordan measurements coincide,

$$\lambda_*^J(G) = \sup_{E\subset G} mesure(E) = \inf_{E\subset G} mesure(E) = \lambda_J^*(G)$$

Theorem 1.2.5. *Let $G \subset \mathbb{R}^m$ be a nonempty compact and Jordan measurable set that coincides with the closure of its interior. Let $K : G \times G \to \mathbb{R}$ be a continuous function. Then the linear operator $T : C(G) \to C(G)$ defined by*

$$(T\varphi)(x) = \int_G K(x, y)\varphi(y)dy, \quad x \in G \tag{1.2.1}$$

is called an integral operator with continuous kernel K. It is a bounded linear operator with

$$\|T\|_\infty = \max_{x\in G} \int_G |K(x, y)|dy.$$

Can be extended to the integral operator $T : C(G) \to C(M)$ given by

$$(T\varphi)(x) = \int_G K(x, y)\varphi(y)dy, \quad x \in M \tag{1.2.2}$$

where $K : M \times G \to C$ is continuous, $M \subset \mathbb{R}^n$ is a compact set and n can be different from m.

Proof. See [42] □

Definition 1.2.13. We said weakly singular kernel, i.e., the kernel K is defined and continuous for all $x, y \in G \subset \mathbb{R}^m$, $x \neq y$, and there exist positive constants M and $\alpha \in [0, m]$ such that

$$|K(x, y)| \leq M|x - y|^{\alpha - m}, \qquad x, y \in G,\ x \neq y.$$

Definition 1.2.14 (Compact operator)**.** A linear operator $T : X \to Y$ from a normed space X into a normed space Y is called compact if it maps each bounded set in X into a relatively compact set in Y. We recall that a set $M \subset Y$is called relatively compact if every bounded sequence $(y_j) \subset M$ has an accumulation point in $\overline{M}$, i.e., if the closure $\overline{M}$ is compact.

Theorem 1.2.6. *(i) Let $T : X \to Y$ be a bounded linear operator with finite-dimensional range $T(X)$. Then T is compact.*

(ii) A compact linear operator $T : X \to Y$ cannot have a bounded inverse unless X has finite dimension.

(iii) Integral operators with continuous kernel are compact linear operators on $C(G)$.

(iv) Integral operators with continuous kernel are compact linear operators on $L^2(G)$.

(v) The integral operator with a weakly singular kernel is a compact operator on $C(G)$.

Proof. See [40, 42] □

Theorem 1.2.7 (Riesz.)**.** *Let X be a normed space and $A : X \longrightarrow X$ be a linear compact operator.*

(a) The null space $N(I - A) = \{x \in X : x = Ax\}$ is finite-dimensional and the range $(I - A)(X)$ is closed in X.

(b) If the homogeneous equation $x - Ax = 0$ admits only the trivial solution $x = 0$, then the in-homogeneous equation $x - Ax = y$ is uniquely solvable for every $y \in Y$ and the solution x depends continuously on y.

1.2.3 Boundary Integral operator

Integral operators are an important examples for our thesis. From the schwartz kernel theorem [57], for a given operator T on Ω, there is a single distribution $\mathcal{K} \in \mathcal{D}'(\Omega \times \Omega)$ such as

$$\langle Tu, v \rangle = \langle \mathcal{K}, u \otimes v \rangle, \quad \forall u, v \in \mathcal{D}(\Omega) \tag{1.2.3}$$

Definition 1.2.15. Given a function $\varphi \in C(\partial\Omega)$, the functions

$$u(x) = \int_{\partial\Omega} E_1(x, y)\varphi(y)ds(y), \qquad x \in \mathbb{R}^2 \setminus \partial\Omega, \tag{1.2.4}$$

$$v(x) = \int_{\partial\Omega} \frac{\partial E_1(x, y)}{\partial n(y)}\varphi(y)ds(y), \qquad x \in \mathbb{R}^2 \setminus \partial\Omega \tag{1.2.5}$$

are called, respectively, single-layer and double-layer potential with density φ. In two dimensions, occasionally, for obvious reasons we will call them logarithmic single-layer and logarithmic double-layer potential.

Theorem 1.2.8. *Let $\partial\Omega$ be of class C^2 and $\varphi \in C(\partial\Omega)$. Then the single-layer potential u with density φ is continuous throughout $\mathbb{R}^n$. On the boundary we have*

$$u(x) = \int_{\partial\Omega} E_1(x, y)\varphi(y)ds(y), \qquad x \in \partial\Omega,$$

where the integral exists as an improper integral.

Definition 1.2.16. From [21, 41] we define the integral operators A, B, B', S, K, K' from $C(\partial\Omega)$ to $C(\partial\Omega)$ as

$$\begin{aligned}
(A\varphi)(x) &= \int_{\partial\Omega} E_2(x, y)\varphi(y)ds(y), && x \in \partial\Omega \\
(B\varphi)(x) &= \int_{\partial\Omega} \frac{\partial E_2(x, y)}{\partial n(y)}\varphi(y)ds(y), && x \in \partial\Omega \\
(B'\varphi)(x) &= \int_{\partial\Omega} \frac{\partial E_2(x, y)}{\partial n(x)}\varphi(y)ds(y), && x \in \partial\Omega \\
(S\varphi)(x) &= \int_{\partial\Omega} E_1(x, y)\varphi(y)ds(y), && x \in \partial\Omega \\
(K\varphi)(x) &= \int_{\partial\Omega} \frac{\partial E_1(x, y)}{\partial n(y)}\varphi(y)ds(y), && x \in \partial\Omega \\
(K'\varphi)(x) &= \int_{\partial\Omega} \frac{\partial E_1(x, y)}{\partial n(x)}\varphi(y)ds(y), && x \in \partial\Omega
\end{aligned} \tag{1.2.6}$$

Lemma 1.2.1. *The boundary integral operators A, B, B', S are continuous.*

Proof. See [55]. □

Lemma 1.2.2. *The integral operators K and K' have weakly singular kernels and therefore are compact. In two dimensions for C^2 boundaries the kernels of K and K' actually turn out to be continuous.*

Proof. See [42]. □

Remark 1.2.2. For the double-layer potential with constant density we have

$$\int_{\partial\Omega} \frac{\partial E_1(x, y)}{\partial n_y} ds(y) = \begin{cases} 1, & x \in \Omega \\ \frac{1}{2}, & x \in \partial\Omega \\ 0, & x \in \mathbb{R}^2 \setminus \overline{\Omega} \end{cases}$$

Theorem 1.2.9 (Jump relation 1)**.** *For $\partial\Omega$ of class C^2, the double-layer potential v with continuous density φ can be continuously extended from Ω to $\overline{\Omega}$ and from $\mathbb{R}^n \setminus \overline{\Omega}$ to $\mathbb{R}^n \setminus \Omega$ with limiting values*

$$v(x)_{\pm} = \int_{\partial\Omega} \frac{\partial E_1(x, y)}{\partial n(y)}\varphi(y)ds(y) \pm \frac{1}{2}\varphi(x), \qquad x \in \partial\Omega$$

and where the integral exists as an improper integral.

Theorem 1.2.10 (Jump relation 2)**.** *Let $\partial\Omega$ be of class C^2. Then for the single-layer potential u with continuous density φ can be continuously extended from Ω to $\overline{\Omega}$ and from $\mathbb{R}^n \setminus \overline{\Omega}$ to $\mathbb{R}^n \setminus \Omega$ with limiting values*

$$\frac{\partial u_{\pm}}{\partial n}(x) = \int_{\partial\Omega} \frac{\partial E_1(x,y)}{\partial n(x)} \varphi(y) ds(y) \mp \frac{1}{2}\varphi(x), \qquad x \in \partial\Omega$$

where the integral exists as an improper integral, and, if $n = 2$, also satisfies

$$\int_{\partial\Omega} \varphi(y) ds(y) = 0$$

Proof. See [42] □

Lemma 1.2.3. *If $|x - x_0| \neq 1$ for a some $x_0 \in \Omega$, and for all $x \in \partial\Omega$ the operator S is injective.*

Proof. See [41]. □

Theorem 1.2.11 (Theorem 7.38. [40] and 3.16. [4])**.** *We assume there exist $x_0 \in \Omega$, such that $|x - x_0| \neq 1$ for all $x \in \partial\Omega$. Then the single-layer operator $S : C(\partial\Omega) \to C(\partial\Omega)$ is injective.*

Theorem 1.2.12. *The operators $I - K$ and $I - K'$ have trivial null-spaces*

$$N(I - K) = N(I - K') = \{0\}$$

Proof. See [41] □

Lemma 1.2.4. *Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ simply or doubly-connected, and u is a smooth function on Ω then the following properties are verified*

(i) If $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$. therefore $u = 0$ in Ω.

(ii) If $\Delta(\Delta u) = 0$ in Ω and $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. therefore $u = 0$ in Ω.

(iii) If $\Delta(\Delta u) = 0$ in Ω and $u = \Delta u = 0$ on $\partial\Omega$. therefore $u = 0$ in Ω.

Proof. It is a consequence of theorem 1.1.5 to obtain the property (*i*). Take $\Delta u = v$ with $\Delta v = 0$, and we consider the problem

$$\begin{cases} \Delta u = v, & in\ \Omega \\ u = \frac{\partial u}{\partial n} = 0, & on\ \partial\Omega \end{cases} \tag{a}$$

Using the Green formula (1.1.13) and from theorem 1.2.11 then the problem (a) have a solution $u = 0$ in Ω that prove (*ii*). We consider the coupled system

$$\begin{cases} \Delta u = v, & in\ \Omega \\ u = 0, & on\ \partial\Omega \end{cases}, \qquad \begin{cases} \Delta v = 0, & in\ \Omega \\ v = 0, & on\ \partial\Omega \end{cases} \tag{b}$$

It is a consequence of property (i) applying in (b) to obtain $u = 0$ in Ω which prove (*iii*). □

1.2.4 Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set, $m \geq 2$, and $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. In studying boundary value problems, we shall need to make sense of the restriction $u|_\Gamma$ as an element of a Sobolev space on $\Gamma \subseteq \partial\Omega$ when u belongs to a Sobolev space on Ω.

Definition 1.2.17 ($W^{m,p}$ Spaces)**.** The Sobolev space $W^{m,p}$ is defined by

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \left| \begin{array}{l} \forall\, |\alpha| \leq m,\ \exists g_\alpha \in L^p(\Omega) \textit{ such that} \\ \int_{\partial\Omega} uD^\alpha\varphi = (-1)^{|\alpha|} \int_{\partial\Omega} g_\alpha\varphi,\ \forall\varphi \in C_c^\infty(\Omega) \end{array} \right. \right\}$$

where we use the standard multi-index notation $\alpha = (\alpha_1, \alpha_2..., \alpha_n)$ with $\alpha_i \geq 0$ an integer,

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \textit{and} \quad D^\alpha\varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}.....\partial x_n^{\alpha_n}}$$

We set $D^\alpha u = g_\alpha$. The space $W^{m,p}(\Omega)$ equipped with the norm

$$||u||_{W^{m,p}(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)} \right)^{\frac{1}{2}}$$

is a Banach space.

The space $H^m(\Omega) = W^{m,2}(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), |\alpha| \ \leq m\}$ equipped with the scalar product

$$(u, v)_{H^m} = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2}$$

is a Hilbert space.

Theorem 1.2.13 (Rellich's)**.** *Suppose that Ω is of class C^1 with $\partial\Omega$ bounded (or else $\Omega = \mathbb{R}^n_+$). Then there exists a linear extension operator*

$$P\ :\ W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^n), \qquad 1 \leq p \leq \infty$$

such that for all $u \in W^{m,p}(\Omega)$,

(i) $Pu_{|\Omega} = u$
(ii) $||Pu||_{L^p(\mathbb{R}^n)} \leq C||u||_{L^p(\Omega)}$
(iii) $||Pu||_{W^{m,p}(\mathbb{R}^n)} \leq C||u||_{W^{m,p}(\Omega)}$

where C depends only on Ω.

Proof. see [40] □

Definition 1.2.18 (The spaces $H^s(\Omega)$)**.** The same approach can be used for defining the Sobolev space $W^s(\Omega)$ for non-integer real positive s. Let $s = m + \sigma$, with $m \in \mathbb{N}$, and $0 < \sigma < 1$, and let us introduce the function space

$$C^s_*(\Omega) = \{u \in C^m(\Omega)\ :\ ||u||_{W^s(\Omega)} < \infty\}$$

where

$$\|u\|_{W^s(\Omega)} = \left\{ \|u\|^2_{W^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dxdy \right\}^{\frac{1}{2}}$$

As consequence of Strong extension property [21], we have that $H^s(\Omega) = W^s(\Omega)$ and the norms $\|u\|_{H^s(\Omega)}$, $\|u\|_{W^s(\Omega)}$ are equivalent. As consequence from (theorem 3.18 [88]) we denote that $for\ s \geq 0$ then $H^s(\mathbb{R}^n) = W^s(\mathbb{R}^n)$.

A second definition of the spaces $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ by using the Bessel potential of order s and $F[.]$ the Fourier transform (see [88]) we define $H^s(\mathbb{R}^n)$, as

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}(\mathbb{R}^n)' \ : \ (1 + |\xi|^2)^{\frac{s}{2}} |F[u]| \in L^2(\mathbb{R}^n) \right\}$$

and the associated norm:

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |F[u]|^2 d\xi \right)^{\frac{1}{2}}.$$

in addition, $H^{-s}(\mathbb{R}^n)$ is an isometric realization of the dual space of $H^s(\mathbb{R}^n)$, i.e.,

$$H^{-s}(\mathbb{R}^n) = H^s(\mathbb{R}^n)' \ , \ for\ s \in \mathbb{R}$$

and the following norm for $u \in H^{-s}(\mathbb{R}^n)$ is given by

$$\|u\|_{H^{-s}(\mathbb{R}^n)} = \sup_{\|v\|_{H^s(\mathbb{R}^n)} \neq 0} \frac{|(u, v)|}{\|v\|_{H^s(\mathbb{R}^n)}}$$

whereas we define the spaces $H^s(\Omega)$ of order s as

$$H^s(\Omega) = \{ u \in \mathcal{D}(\Omega)' \ : \ u = v|_\Omega \ \text{ for some } v \in H^s(\mathbb{R}^n) \} \tag{1.2.7}$$

and the induced norm satisfies

$$\|u\|_{H^s(\Omega)} = \sqrt{(u, u)_{H^s(\Omega)}} = \inf_{v \in H^s(\mathbb{R}^n)} \{ \|v\|_{H^s(\mathbb{R}^n)} \mid u = v|_\Omega \}$$

Definition 1.2.19. We also define two other Sobolev spaces on Ω

$$\tilde{H}^s(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ in } H^s(\mathbb{R}^n),$$
$$H^s_0(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ in } H^s(\Omega)$$

which we make into Hilbert spaces in the obvious way, by restriction of the inner products in $H^s(\mathbb{R}^n)$ and in $H^s(\Omega)$, respectively. The above definitions imply that

$$\tilde{H}^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^n) \mid spp\ u \subset \overline{\Omega} \right\}$$
$$H^s_0(\Omega) = \left\{ u \in H^s(\Omega) \mid spp\ u \subset \overline{\Omega} \right\}$$

respectively, with the following inclusion holds,

$$\tilde{H}^s(\Omega) \subseteq H^s_{\overline{\Omega}}(\Omega) \ , \quad \tilde{H}^s(\Omega) \subseteq H^s_0(\Omega)$$

and

$$H^s_{\overline{\Omega}}(\Omega) = \left\{ u \in H^s(\Omega) \ : \ supp\ u \subset \overline{\Omega} \right\}$$

in addition the spaces

$$\mathcal{D}(\overline{\Omega}) = \{ u \ : \ u = v|_\Omega \ \text{ for some } v \in \mathcal{D}(\mathbb{R}^n) \} \tag{1.2.8}$$

is dense in $H^s(\Omega)$ because $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Theorem 1.2.14 (Rellich's)**.** *Let $s, t \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the imbedding*

$$\begin{aligned} \tilde{H}^t(\Omega) &\hookrightarrow H^s(\mathbb{R}^n) \quad s < t. \\ \tilde{H}^t(\Omega) &\hookrightarrow \tilde{H}^s(\Omega) \quad s < t. \\ H^t(\Omega) &\hookrightarrow H^s(\Omega) \quad s < t. \end{aligned}$$

is bounded and compact, the symbol $\hookrightarrow$ will be used for compact imbedding.

Proof. See [21] □

Definition 1.2.20 (Lipschitz Domains)**.** The open set Ω is a Lipschitz domain if its boundary which denoted by $\partial\Omega = \overline{\Omega} \cap (\mathbb{R}^n \backslash \Omega)$ is compact and if there exist finite families $\left\{W_j\right\}$ and $\left\{\Omega_j\right\}$ having the following properties:

(i) The family W_j is a finite open cover of $\partial\Omega$, i.e., each W_j is an open subset of $\mathbb{R}^n$, and $\partial\Omega \subset \cup_j W_j$.

(ii) Each Ω_j can be transformed to a Lipschitz hypograph by a rigid motion.

(iii) The set Ω_j satisfies $W_j \cap \Omega = \Omega_j \cap W_j$ for each j.

Notice that if Ω is a Lipschitz hypograph [88], then

$$\partial\Omega = \left\{x \in \mathbb{R}^{n-1} \ : \ x_n = \xi(x') \text{ for all } x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\right\}$$

with ξ is Lipschitz, i.e., if there is a constant M such that

$$|\xi(x') - \xi(y')| \leq M|x' - y'| \text{ , for all } x', y' \in \mathbb{R}^{n-1}$$

Any Lipschitz domain Ω has a surface measure σ, and an outward unit normal n that exists σ-almost everywhere on Γ. If Ω is the Lipschitz hypograph, then (see theorem 3.34 [88])

$$d\sigma_x = \sqrt{1 + |\nabla\xi(x')|}dx' \ , \quad n(x) = \frac{(-\nabla\xi(x'), 1)}{\sqrt{1 + |\nabla\xi(x')|}} \ , \qquad for\ x \in \partial\Omega \tag{1.2.9}$$

Example 1.2.3. (a) Likewise, for $0 < k < 1$, we define a $C^{m,k}$ domain by adding the requirement that the $mth - order$ partial derivatives of be Holder-continuous with exponent k, i.e.,

$$|\partial^\alpha \xi(x') - \partial^\alpha \xi(y')| \leq M|x' - y'|^k \text{ , for all } x', y' \in \mathbb{R}^{n-1} \ and\ |\alpha| = m$$

A Lipschitz domain is the same thing as a $C^{0,1}$. If $\Gamma \in C^{0,1}$ the boundary is called a Lipschitz boundary with a strong Lipschitz property and Ω is called a strong Lipschitz domain [21].

(b) Any polygon in $\mathbb{R}^2$ is a Lipschitz domain.

(c) In the case of simply-connected domain, the ellipse is a Lipschitz domain.

(d) In the case of doubly connected domain the annulus is not a Lipschitz domain anymore. [21].

(f) If $\partial\Omega \in C^{\infty}$ then we say that $\partial\Omega$ is a smooth boundary.

Example 1.2.4 (Parametrization)**.** A parameterized planar curve is a path in the plane traced out by the point

$$z(t) = (z_1(t), z_2(t)), \qquad 0 \le t \le L \tag{1.2.10}$$

For a regular boundary $\partial\Omega$, the parameterized curve taken $z \in C^2(0, L)$, with $|z'(t)| \neq 0$, as the parameter t ranges over an interval $[0, L]$.

We define surface measure dt, and the exterior unit normal $n(t)$ that is orthogonal to the curve $\partial\Omega$ at $z(t)$ as:

$$d\sigma_x = \sqrt{z_1'(t)^2 + z_2'(t)^2 dt}\,, \quad n(t) = \frac{[z'(t)]^{\perp}}{\sqrt{z_1(t)^2 + z_2(t)^2}}\,, \quad for\ t \in [0, L] \tag{1.2.11}$$

with the notation $[z'(t)]^{\perp}$ is the vector orthogonal of $[z'(t)]$.

Theorem 1.2.15. *If Ω is a C^0 domain, then, $\mathcal{D}(\Omega)$ is dense in $H^s_{\overline{\Omega}}(\Omega)$ or in other words $\tilde{H}^s(\Omega)$ is dense in $H^s_{\overline{\Omega}}(\Omega)$ for $s \in \mathbb{R}$.*

Proof. see [88] □

Theorem 1.2.16. *Let Ω be a Lipschitz domain, then*

$$H^s(\Omega)' = \tilde{H}^{-s}(\Omega) \text{ and } \tilde{H}^s(\Omega)' = H^{-s}(\Omega) \text{ for all } s \in \mathbb{R}$$

For $s \geq 0$ define the spaces

$$\tilde{H}^s(\Omega) = \left\{u \in L^2(\Omega) \ : \ \tilde{u} \in H^s(\mathbb{R}^n)\right\} \subseteq H^s_0(\Omega)$$

where $\tilde{u}$ denotes the extension of u by zero:

$$\tilde{u}(x) = \begin{cases} u(x) & if\ x \in \Omega \\ 0 & if\ x \in \mathbb{R}^n \backslash \Omega \end{cases} \tag{1.2.12}$$

In fact,

$$\tilde{H}^s(\Omega) = H^s_0(\Omega) \text{ provided } s \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...\right\}$$

Proof. see theorem 3.30 and theorem 3.33 in [88]. □

Definition 1.2.21 (The Spaces $L^2(\Gamma)$)**.** We are introducing by definition of $L^2(\Gamma)$ be the completion of $C^0(\Gamma)$, the space of all continuous functions on Γ, with respect to the norm

$$||u||_{L^2(\Gamma)} = \left\{\int_{\Gamma} |u(x)|^2 ds_x\right\}^{\frac{1}{2}}$$

Definition 1.2.22 (The Trace Spaces $H^s(\Gamma)$)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\Gamma := \partial\Omega$, we define

$$H^s(\Gamma) = \left\{ \begin{array}{l} \{u|_{\Gamma} : u \in H^{s+\frac{1}{2}}(\mathbb{R}^n)\}, s > 0 \\ L^2(\Gamma), \quad s = 0 \\ (H^{-s}(\Gamma))' \ \text{(dual space)}, s < 0 \end{array} \right\}$$

If $s \geq 0$ and $\Gamma \subset \partial\Omega$ be an open subset of the boundary, from 1.2.19 then [19, 20],

$$\begin{aligned} H^s(\Gamma) &= \{u|_\Gamma : u \in H^s(\partial\Omega)\}, \\ \tilde{H}^s(\Gamma) &= \left\{u \in H^s(\Gamma) : Supp\ u \subset \overline{\Gamma}\right\}. \\ \tilde{H}^{-s}(\Gamma) &= H^{-s}_{\overline{\Gamma}}(\partial\Omega) = \left\{u \in H^{-s}(\partial\Omega) : supp\ u \subset \overline{\Gamma}\right\} \end{aligned}$$

with the norm

$$\|u\|_{H^s(\Gamma)} = \inf_{\{v \in H^s(\partial\Omega),\ v|_\Gamma = u\}} \left\{\|v\|_{H^s(\partial\Omega)}\right\}$$

We define by $H^{-s}(\Gamma)$ the dual space of $\tilde{H}^s(\Gamma)$, in addition the following inclusion are satisfies

$$\tilde{H}^s(\Gamma) \subset H^s(\Gamma), \quad for\ s \geq 0.$$
$$\tilde{H}^s(\Gamma) \subset H^s(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-s}(\Gamma) \subset H^{-s}(\Gamma), \quad for\ s > 0.$$

For $s < 0$, we can define the boundary spaces of negative orders $H^s(\Gamma)$ as the dual of $H^{-s}(\Gamma)$ with respect to the $L^2(\Gamma)$ scalar product; i.e. the completion of $L^2(\Gamma)$ with respect to the norm: [21]

$$\|u\|_{H^s(\Gamma)} = \sup_{\|\varphi\|_{H^{-s}(\Gamma)}=1} |(u, \varphi)|$$

Theorem 1.2.17 (The Trace Operator)**.** *Define the trace operator by*

$$\begin{aligned} \gamma\ :\ \mathcal{D}(\overline{\Omega}) &\longrightarrow \mathcal{D}(\Gamma) \\ u &\longrightarrow \gamma u = u|_\Gamma \end{aligned}$$

If Ω is a $C^{k-1,1}$ domain, and if $\frac{1}{2} < s < k$, then γ has a unique extension to a bounded linear operator

$$\gamma\ :\ H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$$

and this extension has a continuous right inverse. With respect to the norm

$$\|u\|_{H^{s-\frac{1}{2}}(\Gamma)} = \inf_{\gamma\tilde{u}=u} \|\tilde{u}\|_{H^s(\Omega)}$$

Proof. [88] □

Definition 1.2.23 (Trace Spaces on Curved Polygons in $\mathbb{R}^2$)**.** Let $\partial\Omega \in C^{0,1}$ be a curved polygon which is composed of m simple C^∞-arcs Γ_j, $j = 1, ..., m$ such that their closures $\overline{\Gamma}_j$ are C^∞. The curve Γ_{j+1} follows Γ_j according to the positive orientation. We denote by Z_j the vertex being the end point of Γ_j and the starting point of Γ_{j+1}.

For $s \in \mathbb{R}$ let $H^s(\Gamma_j)$ be the standard Sobolev spaces on the pieces Γ_j, $j = 1, ..., m$ which are defined as follows. Without loss of generality, we assume for each of the Γ_j we have a parametric representation

$$x = z_j(t)\ for\ t \in \Omega_j = [a_j, b_j] \subset \mathbb{R}$$

with $z_j(a_j) = Z_{j-1}$, $z_j(b_j) = Z_j$, $\Omega_j = (a_j, b_j)$, $j = 1, ..., m$, where $z_j \in C^\infty(\overline{\Omega})$, then we define the space (see [21] pp 186)

$$\tilde{H}^s(\Gamma_j) = \left\{\varphi \mid \varphi \circ z_j \in H^s(\Omega_j)\right\}$$

to be equipped with the norm

$$\|\varphi\|_{\tilde{H}^s(\Gamma_j)} := \|\varphi \circ z_j\|_{H^s(\Omega_j)}$$

where $\|\cdot\|_{H^s(\Omega_j)}$ is defined as in definition 1.2.18. Then $\tilde{H}^s(\Gamma_j)$ is a Hilbert space with inner product

$$(\varphi, \psi)_{\tilde{H}^s(\Gamma_j)} := \left(\varphi \circ z_j, \psi \circ z_j\right)_{H^s(\Omega_j)}$$

1.3 Regularization method

It's well known that many inverse problems can be formulated as operator equations [4] of the form

$$Tx = y \tag{1.3.1}$$

where T is a linear compact operator between Hilbert spaces X and Y. In what follows, the regularization parameter $\alpha = \alpha(\delta)$ is chosen a priori; that is, before we start to compute the regularized solution.

In practice, the right-hand side $y \in Y$ is never known exactly but only up to an error of, say, $\delta > 0$. Therefore, we assume that we know $\delta > 0$ and $y^\delta \in Y$ with

$$\|y^\delta - y\| \leq \delta \tag{1.3.2}$$

It is our aim to solve the perturbed equation

$$Tx^\delta = y^\delta \tag{1.3.3}$$

In general, (1.3.3) is not solvable because we cannot assume that the measured data y^δ is in the range $R(T)$ of T. Therefore, the best we can hope is to determine an approximation $x^\delta \in X$ to the exact solution x. In other words, it is our aim to construct a suitable bounded approximation $R : Y \to X$ of the (unbounded) inverse operator $T^{-1} : R(X) \to X$.

Definition 1.3.1. A regularization strategy is a family of linear and bounded operators

$$R_\alpha : Y \to X$$

such that

$$\lim_{\alpha \to 0} R_\alpha T x = x, \ \forall \ x \in X$$

A regularization strategy $\alpha(\delta)$ is called admissible if $\alpha(\delta) \to 0$ and

$$\sup_{x \in X} \left\{ \|R_{\alpha(\delta)} y^\delta - x\| : y^\delta \in Y \ , \ \|Tx^\delta - x\| \right\}$$

We define

$$x_\alpha = R_\alpha y^\delta$$

as an approximation of the solution x to the eq.(1.3.1).

Definition 1.3.2. Let X, Y be Hilbert spaces, $T : X \to Y$ be a compact linear operator, and $T^* : Y \to X$ be its adjoint. The non-negative square roots of the eigenvalues of the nonnegative self-adjoint compact operator $T^*T : X \to X$ are called singular values of T.

Theorem 1.3.1. *Let X, Y be Hilbert spaces, (μ_n) denote the sequence of the nonzero singular values of the compact linear operator T (with $T \neq 0$) repeated according to their multiplicity, i.e., according to the dimension of the nullspaces $N(\mu_n^2 I - T^*T)$. Then there exist orthonormal sequences (φ_n) in X and (ψ_n) in Y such that*

$$T\varphi_n = \mu_n\varphi_n, \qquad T^*\psi_n = \mu_n\psi_n.$$

for all $n \in \mathbb{N}$. For each $x \in X$ we have the singular value decomposition

$$x = \sum_{n=1}^{\infty}(x, \varphi_n)\varphi_n + Qx$$

with the orthogonal projection operator $Q : X \longrightarrow N(T)$ and

$$Tx = \sum_{n=1}^{\infty}\mu_n(x, \varphi_n)\psi_n$$

Each system $(\mu_n, \varphi_n, \psi_n)$, $n \in \mathbb{N}$, with these properties is called a singular system of T. When there are only finitely many singular values, this previous series degenerate into finite sums. (Note that for an injective operator T the orthonormal system $\varphi_n : n \in \mathbb{N}$ provided by the singular system is complete in X.)

Proof. [42] □

Theorem 1.3.2 (Picard)**.** *Let X, Y be Hilbert spaces, and $T : X \to Y$ be a compact linear operator with singular system $(\mu_n, \varphi_n, \psi_n)$. The equation of the first kind 1.3.1 is solvable if and only if y belongs to the orthogonal complement $N(T^*)^{\perp}$ and satisfies*

$$\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}|(y, \psi_n)|^2 < \infty$$

In this case a solution is given by

$$x = \sum_{n=1}^{\infty}\frac{1}{\mu_n}(y, \psi_n)\varphi_n \tag{1.3.4}$$

Proof. see [4]. □

Theorem 1.3.3. *Let X, Y be Hilbert spaces, and $T : X \longrightarrow Y$ be an injective compact linear operator with singular system $(\mu_n, \varphi_n, \psi_n)$, $n \in \mathbb{N}$ and let $q : (0, \infty) \times (0, \|T\|] \longrightarrow \mathbb{R}$ be a bounded function such that for each $\alpha > 0$ there exists a positive constant $c(\alpha)$ with the properties*

$$|q(\alpha, \mu)| \leq c(\alpha)\mu, \qquad 0 < \mu \leq \|T\| \tag{1.3.5}$$

$$\lim_{\alpha\to 0} q(\alpha, \mu) = 1, \qquad 0 < \mu \leq \|T\| \tag{1.3.6}$$

Then the bounded linear operators $R_\alpha : Y \longrightarrow X$, $\alpha > 0$, defined by

$$R_\alpha f = \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(\alpha, \mu_n)(f, \psi_n)\varphi_n, \qquad f \in Y \tag{1.3.7}$$

describe a regularization scheme with

$$\|R_\alpha\| \leq c(\alpha). \tag{1.3.8}$$

Proof. See [4, 42] □

1.3.1 Tikhonov regularization

The Tikhonov's method is one of the most important regularization strategies, as introduced independently by Phillips in 1962 and Tikhonov 1963 is obtained from (1.3.4) by multiplying $\frac{1}{\mu_n}$ by the damping factor

$$\frac{\mu_n^2}{\alpha + \mu_n^2}$$

where α is some positive regularization parameter.

Definition 1.3.3. A sequence (x_n) of elements from a Hilbert space X is called weakly convergent to an element $x \in X$ if

$$\lim_{n\to\infty} (\varphi, x_n) = (\varphi, x)$$

For a weakly convergent sequence we will write $x_n \rightharpoonup x$, $n \to \infty$. Note that norm convergence $x_n \to x$, $n \to \infty$, always implies weak convergence $x_n \rightharpoonup x$, $n \to \infty$.

Theorem 1.3.4. *Let X, Y be Hilbert spaces, and $T : X \to Y$ be a bounded linear operator and let $\alpha > 0$. Then for each $y \in Y$ there exists a unique $x_\alpha \in X$ such that*

$$\|Tx_\alpha - y\|^2 + \alpha\|x_\alpha\|^2 = \inf_{\varphi\in X}\left\{\|T\varphi - y\|^2 + \alpha\|\varphi\|^2\right\}$$

The minimizer x_α is given by the unique solution of the equation

$$\alpha x_\alpha + T^*Tx_\alpha = T^*y$$

and depends continuously on y .

Proof. See [4] □

Theorem 1.3.5. *Let X, Y be Hilbert spaces, and $T : X \to Y$ be an injective bounded linear operator. Then*

$$R_\alpha = (\alpha I + T^*T)^{-1}T^*$$

describes a regularization scheme with

$$\|R_\alpha\| \leq \frac{\|T\|}{\alpha}$$

Proof. See [42] □

Theorem 1.3.6 (Theorem 16.8. p327 [41])**.** *Let X, Y be Hilbert spaces, and $T : X \to Y$ be an injective bounded linear operator with dense range. Then*

$$\|Tx_\alpha - f\| \to 0, \qquad \alpha \to 0$$

for all $y \in Y$.

1.3.2 Least squares approximations

The term least squares describes a frequently used approach to solving over-determined or inexactly specified systems of equations in an approximate sense. Instead of solving the equations exactly, we seek only to minimize the sum of the squares of the residuals.

Definition 1.3.4 (Best Approximation)**.** Let $U \subset X$ be a subset of a normed space X and let $x \in X$. An element $\hat{u} \in U$ is called a best approximation to x with respect to U if

$$\|x - \hat{u}\| = \inf_{u \in U} \|x - u\| \tag{1.3.9}$$

i. e., if $\hat{u} \in U$ has smallest distance from x.

Theorem 1.3.7. *Let U be a finite-dimensional subspace of a normed space X. Then for every element in normed space X there exists a best approximation with respect to U.*

Proof. See [42] □

Corollary 1.3.1. *Let U be a finite-dimensional linear subspace of a preHilbert space H with basis $u_1, ..., u_n$. The linear combination*

$$\hat{u} = \sum_{k=1}^{n} \alpha_k u_k \tag{1.3.10}$$

is the best approximation to $x \in X$ with respect to U if and only if the coefficients $\alpha_1, ..., \alpha_n$ satisfy the normal equations

$$\sum_{k=1}^{n} \alpha_k (u_k, u_j) = (x, u_j), \qquad j = 1, ..., n. \tag{1.3.11}$$

Proof. See [42] □

Definition 1.3.5 (Least Squares Problem)**.** Let X, Y be Hilbert spaces, with $T \in L(X, Y)$ and $y \in Y$. We call the linear least squares problem associated with the equation (1.3.1), the following minimization problem

$$\min_{x \in X} \|y - Tx\| \tag{1.3.12}$$

The equation system

$$T^* T x = T^* y \tag{1.3.13}$$

is called a system of normal equations for the least squares problem.

Theorem 1.3.8. *Let X, Y be Hilbert spaces, $T \in L(X, Y)$, and $y \in Y$, an element $\hat{x} \in X$ is a solution to the least square problem (1.3.12), if and only if, $\hat{x}$ is a solution to the normal equation (1.3.13).*

Proof. Let $\hat{x} \in X$ a solution to the normal equation, for all $x \in X$ we have

$$0 = y - Tx = y - T\hat{x} + T(\hat{x} - x) \tag{$*$}$$

The normal equation involves that the two terms of the equation $(*)$ are orthogonal, indeed

$$\begin{aligned} \langle y - Tx, T(\hat{x} - x)\rangle &= \langle T^*(y - Tx), \hat{x} - x\rangle \\ &= 0 \end{aligned}$$

and

$$\|y - Tx\|^2 = \|y - T\hat{x}\|^2 + \|T(\hat{x} - x)\|^2$$

therefore

$$\|y - T\hat{x}\|^2 \leq \|y - Tx\|^2, \quad \forall x \in \mathbb{R}^n$$

Then x resolve the least square problem. □

Lemma 1.3.1 (Existence and uniqueness)**.** *Let X, Y be Hilbert spaces, $T \in L(X, Y)$, and $y \in Y$. The following properties are satisfied*

(i) The normal equation (1.3.13) accepts a solution if and only if $y \in R(T) + R(T)^{\perp}$.

(ii) The solution of the problem (1.3.13) is unique if and only if T is injective.

Proof. (i) Let $\hat{x}$ a solution to the normal equation then

$$\begin{aligned} &T^*(T\hat{x} - y) = 0 \Rightarrow T\hat{x} - y \in N(T^*) = R(T)^{\perp} \\ &\textit{and} \\ &y = T\hat{x} + (y - T\hat{x}) \in R(T) + R(T)^{\perp} \end{aligned}$$

Inverting, let $y = y_1 + y_2$, with $y_1 \in R(T)$ and $y_2 \in R(T)^{\perp} = N(T^*)$, then there exists $\hat{x} \in X$ such that

$$\begin{aligned} T\hat{x} = y_1 &\Rightarrow T^*T\hat{x} = T^*y_1 \\ &\Rightarrow T^*T\hat{x} = T^*y_1 + T^*y_2 = T^*y \end{aligned}$$

Therefore $\hat{x}$ solve the normal equation.

(ii) For $x \in N(T^*T) \Leftrightarrow x \in N(T)$, then T^*T and T both are injective.

□

Definition 1.3.6 (Curve Fitting)**.** A very common source of least squares problems is curve fitting. Let t be the independent variable and let $y(t)$ denote an real unknown function of $t \in \mathbb{R}$ that we want to approximate. Assume there are m observations, i.e. values of y measured at specified values of t.

$$y_i = y(t_i), \quad i = 1, ..., m \tag{1.3.14}$$

The idea is to model $y(t)$ by a linear combination of n basis functions,

$$y(t) \approx \sum_{j=1}^{n} \omega_j \phi_j(t) \tag{1.3.15}$$

The design matrix A is a rectangular matrix of order m-by-n with elements $a_{i,j} = \phi_j(t_i)$. The design matrix usually has more rows than columns. In matrix-vector notation the model is:

$$A\omega \approx y \tag{1.3.16}$$

The basis functions $\phi_j(t)$ can be nonlinear functions of t, but the unknown parameters, ω_j , appear in the model linearly. For examples [74]

(a) Gaussian basis example: The means and variances appear non-linearly:

$$\phi_j(t) = e^{-\left(\frac{t-\mu_j}{\sigma_j}\right)^2}$$
$$y(t) \approx \sum_{j=1}^{n} \omega_j e^{-\left(\frac{t-\mu_j}{\sigma_j}\right)^2}$$

(b) Polynomials: The coefficients ω_j appear linearly:

$$\begin{aligned} \phi_j(t) &= t^{n-j}, \quad j = 1, ..., n \\ y(t) &\approx \sum_{j=1}^{n} \omega_j t^{n-j} \end{aligned}$$

Remark 1.3.1. The residuals are the differences between the observations and the model,

$$r_i = y_i - \sum_{j=1}^{n} \omega_j \phi_j(t), \qquad i = 1, ..., m,$$

or, in matrix-vector notation,

$$r = y - A\omega$$

We want to find ω that make the residuals as small as possible. Least squares sens method consist to minimize the squares sum of the residuals (research on the best approximation of y).

$$||r||^2 = \sum_{i=1}^{n} r_i^2$$

1.4 Numerical methods

1.4.1 Trefftz method (TM) - Modified Trefftz method (MTM)

The Trefftz method (TM) was first proposed in 1926 . The method can be classified as a boundary-type solution procedure. The main idea of the TM is to use particular solutions as the admissible functions, which satisfy the partial differential equation (PDE) exactly. The numerical effort is required only to approximate the boundary conditions [79].

For example, the general solution, for the two-dimensional Laplace equation in simply-connected planar domain can express by the series form [75]

$$u(r,\theta) = c_0 + \sum_{n=1}^{\infty} c_n r^n \cos(n\theta) + d_n r^n \sin(n\theta) \tag{1.4.1}$$

In the conventional Trefftz method, the numerical solution for the two-dimensional Laplace equation in simply-connected planar domain is expressed by linear summation of the following bases [80, 81, 82].

$$\{1, r^n \cos(n\theta), r^n \sin(n\theta), n = 1, 2...\} \tag{1.4.2}$$

Recently, Liu [10, 83, 84] had modified the T-complete functions in (1.4.2) by considering the characteristic length of the computational domain is $R_1 = \max_{0\leq\theta\leq 2\pi} r(\theta)$ to stabilize the numerical scheme,

$$\left\{1, \left(\frac{r}{R_1}\right)^n \cos(n\theta), \left(\frac{r}{R_1}\right)^n \sin(n\theta), n = 1, 2...\right\} \tag{1.4.3}$$

Hence, the following admissible functions with finite terms truncation m can be used to obtain the Trefftz method (TM) and the modified Trefftz method (MTM), respectively as follows:

$$c_0 + \sum_{n=1}^{m} c_n r^n \cos(n\theta) + d_n r^n \sin(n\theta) \tag{1.4.4}$$

$$c_0 + \sum_{n=1}^{m} c_n \left(\frac{r}{R_1}\right)^n \cos(n\theta) + d_n \left(\frac{r}{R_1}\right)^n \sin(n\theta) \tag{1.4.5}$$

Definition 1.4.1 (Collocation Method)**.** The collocation method has a great advantage to apply on different geometric shapes, and the simplicity for computer programming. In order to applying the collocation method for approximately solving the equations (1.4.4) and (1.4.5) we seek an approximate solution from a finite-dimensional subspace by requiring that the equations (1.4.4) and (1.4.5) be satisfied at only a finite number of so-called collocation points.

We choose $m + 1$ points $0 \leq \theta_0 < \theta_1, ..., < \theta_m \leq 2\pi$ such that the interpolation at these grid points with respect to the subspace X_n is uniquely solvable. For example define θ_i are the equidistant collocated points on $[0, 2\pi]$ given by

$$\theta_i = ih, \qquad for\ i = 0, ..., m, \qquad and, \qquad h = \frac{2\pi}{m+1}. \tag{1.4.6}$$

In both Eq. (1.4.4) and (1.4.5) respectively, for each one there are $2m + 1$ unknown coefficients, can be obtained by imposing the different collocated points (1.4.6) with $[r_0(\theta_i), \theta_i]$, for $i = 1, ..., 2m + 1$, then the collocation Trefftz method (CTM) ia given by

$$c_0 + \sum_{n=1}^{m} c_n r_0(\theta_i)^n \cos(n\theta_i) + d_n r_0(\theta_i)^n \sin(n\theta_i) = u(r_0(\theta_i), \theta_i) \tag{1.4.7}$$

and the modified collocation Trefftz method (MCTM) ia given by

$$c_0 + \sum_{n=1}^{m} c_n \left(\frac{r_0(\theta_i)}{R_1}\right)^n \cos(n\theta_i) + d_n \left(\frac{r_0(\theta_i)}{R_1}\right)^n \sin(n\theta_i) = u(r_0(\theta_i), \theta_i) \tag{1.4.8}$$

We obtain a linear equations systems with dimensions $n = 2m + 1$ can be solved for example by the conjugate gradient method to obtain the coefficients c_n, d_n, for $n \in \mathbb{N}$.

1.4.2 The conjugate gradient method

The classical conjugate gradient method is restricted to solving linear systems $Ax = b$ with matrix A symmetric and positive definite. But recently, many generalizations have been given for solving linear systems that are not symmetric and positive definite. We consider an approach to handling non-symmetric linear systems is to convert them to symmetric linear systems and to then use the original conjugate gradient method [28].

We consider the normal equation $A^T Ax = A^T y$ such that The matrix of coefficients $B = A^T A$ is symmetric and positive definite, and therefore, the conjugate gradient method is applicable.

Let x_0 be an initial guess for the solution $x^* = B^{-1}A^T y$. Define $r_0 = A^T y - Bx_0$ and $s_0 = r_0$. For $k > 0$, define

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k, \qquad \alpha_k = \frac{\|r_k\|^2}{\langle Bs_k, s_k\rangle} \\ r_{k+1} &= A^T y - Bx_{k+1} \\ s_{k+1} &= r_{k+1} + \beta_k s_k, \qquad \beta_k = \frac{\|r_{k+1}\|^2}{\|r_k\|^2} \end{aligned} \tag{1.4.9}$$

with the norm and the inner product are both in L^2.

Theorem 1.4.1. *Let B be a self-adjoint matrix. Assume $C = I - B$ is a symmetric positive definite operator. Let x_k be generated by the conjugate gradient iteration (1.4.9). Then $x_k \to x^*$ superlinearly:*

$$\|x^* - x_k\| \leq (c_k)^k \|x^* - x_0\|, \qquad k \geq 0 \tag{1.4.10}$$

with $c_k \to 0$, as $k \to \infty$.

1.4.3 Numerical integration

Numerical integration formula, or quadrature formula, are methods for the approximate evaluation of definite integrals.

Definition 1.4.2. In general, a quadrature formula is a numerical method for approximating an integral of the form

$$Q(f) = \int_a^b f(x)dx \tag{1.4.11}$$

of a continuous function f over the interval $[a, b]$ with $a < b$ by a weighted sum

$$Q_n(f) = \sum_{j=0}^{n} a_j^n f(x_j^n)$$

with quadrature points $x_0^n, ..., x_n^n \in [a, b]$ and real quadrature weights $a_0^n, ..., a_n^n$.

1.4.4 Trigonometric interpolation

One important method of constructing degenerate kernels that approximate a given continuous kernel is the interpolation, for which we recall the following theorems.

Definition 1.4.3. For $n \in \mathbb{N}$ we denote by T_n the linear space of trigonometric polynomials

$$q(t) = \sum_{m=0}^{n} a_n \cos mt + \sum_{m=1}^{n} b_m \sin mt \tag{1.4.12}$$

with real (or complex) coefficients $a_0, ..., a_n$ and $b_0, ..., b_n$. A trigonometric polynomial $q \in T_n$ is said to be of degree n if $|a_n| + |b_n| > 0$.

Theorem 1.4.2. *Given* $2n + 1$ *distinct points* $t_0, ..., t_{2n} \in [0, 2\pi)$ *and* $2n + 1$ *values* $y_0, ..., y_{2n} \in \mathbb{R}$, *there exists a uniquely determined trigonometric polynomial* $q_n \in T_n$ *with the property*

$$q_n(t_j) = y_j, \quad j = 0, ..., 2n.$$

Proof. [42] □

Theorem 1.4.3. *Let* $t_j = \frac{j\pi}{n}$, $j = 0, ..., 2n - 1$, *be an equidistant subdivision of the interval* $[0, 2\pi]$ *with an even number of grid points. Then, given the values* $y_0, ..., y_{2n-1}$, *there exists a unique trigonometric polynomial of the form*

$$q_n(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} [\alpha_k \cos kt + \beta_k \sin kt] + \frac{\alpha_n}{2} \cos nt \tag{1.4.13}$$

satisfying the interpolation property

$$q_n(t_j) = y_j, \quad j = 0, ..., 2n - 1$$

.

It's coefficients are given by

$$\alpha_k = \frac{1}{n} \sum_{k=0}^{2n-1} \alpha_k \cos kt, \quad k = 0, ..., n,$$
$$\beta_k = \frac{1}{n} \sum_{k=0}^{2n-1} \alpha_k \sin kt, \quad k = 0, ..., n - 1,$$

Proof. See [41, 42] □

Remark 1.4.1. As consequence from theorem 1.4.3. The Lagrange basis for the trigonometric interpolation has the form (see [41])

$$L_j(t) = \frac{1}{2n} \left\{ 1 + 2 \sum_{k=1}^{n-1} \cos k(t - t_j) + \sin n(t - t_j) \right\} \tag{1.4.14}$$

for $t \in [0, 2\pi]$ and $j = 0, ..., 2n - 1$.

1.4.5 The Nyström method

The Nyström method or quadrature method, was originally introduced to handle approximations based on numerical integration of the integral operator, the resulting solution is found first at the set of quadrature node points, and then it is extended to all points in Ω by means of a special, and generally quite accurate, interpolation formula. The numerical method is much simpler to implement on a computer, but the error analysis is more sophisticated.

We choose a convergent sequence Q_n of quadrature formula for the integral in definition 1.4.2. With quadrature points $x_0^n, ..., x_n^n \in [a, b]$ and real quadrature weights $a_0^n, ..., a_n^n$. We approximate the integral operator

$$(T\varphi)(x) = \int_a^b K(x, y)\varphi(y)dy, \quad x \in [a, b];$$

with continuous kernel K by a sequence of numerical integration operators

$$(T_n\varphi)(x) = \sum_{j=0}^{n} a_j K(x, y_j)\varphi(y_j), \quad x \in [a, b];$$

i.e., we apply the quadrature formula for $g = K(x, .)\varphi$. Then the solution to the integral equation of the second kind is approximated by the solution of

$$\varphi_n - T_n\varphi_n = f$$

which reduces to solving a finite-dimensional linear system.

Theorem 1.4.4. *Let φ_n be a solution of*

$$\varphi_n(x) - \sum_{i=1}^{n} K_n(x, y_i)\varphi_n(y_i) = f(x), \quad s \in [a, b].$$

Then the values $\varphi_j^n = \varphi_n(x_j)$, $j = 1, ..., n$, at the quadrature points satisfy the linear system

$$\varphi_j^n - \sum_{i=1}^{n} K_n(x, y_i)\varphi_i^n = f(x_j), \quad j = 1, ..., n.$$

Theorem 1.4.5. *Assume the quadrature formulas (Q_n) are convergent. Then the sequence (T_n) is collectively compact and pointwise convergent (i.e., $T_n\varphi \longrightarrow T\varphi$, $n \to \infty$, for all $\varphi \in C([a, b])$), but not norm convergent.*

Corollary 1.4.1. *For a uniquely solvable integral equation of the second kind with a continuous kernel and a continuous right-hand side, the Nyström method with a convergent sequence of quadrature formulas is uniformly convergent.*

Remark 1.4.2 (see [41])**.** We will now describe the application of Nyström's method for the approximate solution of integral equations of the second kind with weakly singular kernels of the form

$$(T\varphi)(x) = \int_a^b \mathcal{H}(|x - y|)K(x, y)\varphi(y)dy, \quad x \in \Omega;$$

$\mathcal{H}$ is continuous and satisfies $|\mathcal{H}(t)| \leq Mt^{\alpha-m}$ for all $t > 0$ and some positive constants M and α. Then we approximate the weakly singular integral operator by a sequence of numerical integration operators

$$(T_n\varphi)(x) = \sum_{j=1}^{n} a_j K(x, y_j)\varphi(y_j)$$

We confine ourselves to a special case by considering a weakly singular operator with a logarithmic singularity

$$(T\varphi)(t) = \int_0^{2\pi} \ln(4\sin^2\frac{t-\tau}{2})K(t, \tau)\varphi(\tau)d\tau, \quad 0 \leq t \leq 2\pi; \tag{1.4.15}$$

in the space $C(0, 2\pi) \subset C(\mathbb{R})$ of 2π-periodic continuous functions. The kernel function K is assumed to be continuous and 2π-periodic with respect to both variables. We construct numerical quadratures for the improper integral

$$(Q\varphi)(t) = \int_0^{2\pi} \ln(4\sin^2 \frac{t-\tau}{2})\varphi(\tau)d\tau, \quad 0 \le t \le 2\pi; \tag{1.4.16}$$

Using the Lagrange basis 1.4.14 we obtain that [41]

$$(Q_n\varphi)(t) = \sum_{j=0}^{2n-1} a_j R_j^n(t)\varphi(t_j), \quad 0 \le t \le 2\pi; \tag{1.4.17}$$

with the quadrature weights, for $j = 0, ..., 2n-1$.

$$R_j^n(t) = -\frac{1}{n}\left\{\sum_{k=1}^{n-1} \frac{1}{k}\cos k(t-t_j) + \frac{1}{2n}\cos n(t-t_j)\right\}, \tag{1.4.18}$$

and

$$\sum_{j=0}^{2n-1} |R_j^n(t)| \le \frac{\pi}{\sqrt{2}}$$

Remark 1.4.3. Because of singularities in equation (1.4.15) at the two points $t = 0$ and $t = \pi$, we discretizing the equations with equidistant points on $[0, 2\pi]$ would lead to a poor accuracy. For this reason, it is more appropriate to use a mesh that is graded towards the intersection points. Such a grading can be achieved most efficiently by using a sigmoidal transformation, i.e., a strictly monotonically increasing function as the follows [19].

Definition 1.4.4 (Sigmoidal Transformation)**.** [43] We define the quadrature weights given by $w_p : [0, 2\pi] \to [0, 2\pi]$

$$w_p(t) = 2\pi \frac{[\upsilon(t)]^p}{[\upsilon(t)]^p + [\upsilon(2\pi - t)]^p}, \qquad 0 \le t \le 2\pi \tag{1.4.19}$$

with υ_p is a cubic polynomial given by

$$\upsilon_p(t) = (\frac{1}{p} - \frac{1}{2})\left(\frac{\pi - t}{\pi}\right)^3 + \frac{1}{p}(\frac{t-\pi}{\pi}) + \frac{1}{2}$$

the function w_p is strictly monotonically increasing, with derivatives vanishing up to a certain order $p-1$, $p \ge 2$, at the two intersection points $t = 0$ and $t = \pi$ with $w_p'(\pi) = 2$, $\upsilon_p(0) = 0$, $\upsilon_p(2\pi) = 1$.

Remark 1.4.4. The parameter p in the substitution functions is the so-called grading parameter. For larger values of p the grid points are more densely accumulated at the end points of the integration interval (see [11]). To this end we take the weights a_j, and the mesh t_j as : $a_j = w_p'(t_j)$, $t_j = w_p(t_j)$, $j = 0, ..., 2n-1$.

1.4.6 Spline interpolation

In our cases approximation, consists of problems where it is required to construct an approximation to an unknown function based on some finite amount of data. We define

$$\mathcal{P}_m = \left\{p(x) : p(x) = \sum_{i=1}^{n} c_i x^{i-1},\ x, c_1, ..., c_n\ real\right\}$$

The space of polynomials of order m, which has played an important role in approximation theory and numerical analysis.

Theorem 1.4.6. *$\mathcal{P}_m$ is a linear subspace of $C^\infty(\mathbb{R})$. Moreover, given any real number a, the functions $1, x-a, ..., (x-a)^{m-1}$ form a basis for $\mathcal{P}_m$.*

Proof. See [47] □

Example 1.4.1. A given function $f \in C[a, b]$ can be approximated by a continuous piece-wise linear function by linear interpolation on each of the sub-intervals, by

$$s_n(x) = \frac{1}{x_j - x_{j-1}}[f(x_{j-1})(x_j - x) + f(x_j)(x - x_{j-1})], \quad x \in [x_{j-1}, xj].$$

Theorem 1.4.7 (Hermite Interpolation)**.** *Let $\tau_1, \tau_2, ..., \tau_n$ and positive integers $\ell_1, \ell_2, ..., \ell_n$ be prescribed with $\sum_{i=1}^n \ell_i = m$. Then for any given set of real numbers $\{z_{i,j}\}_{j=1,i=1}^{\ell_i,\ n}$. there exists a unique $p \in \mathcal{P}_m$ with*

$$D^{j-1}p(\tau_i) = z_{i,j}, \quad j = 1, 2, .., \ell_i, \ \ i = 1, 2, ..., n$$

Proof. See [47] □

Remark 1.4.5. The practical significance of Theorem 3.1.1 is that once having chosen a basis for $\mathcal{P}_m$ each polynomial will have a unique set of coefficients associated with it. This formally establishes the fact that polynomials can be stored on a digital computer.

Definition 1.4.5 (Piecewise Polynomials)**.** Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$, and write $\Delta = \{x_i\}_0^{k+1}$. The set Δ partitions the interval $[a, b]$ into $k + 1$ subintervals, $I_i = [x_i, x_{i+1}),\ i = 0, 1, ..., k-1$, and $I_k = [x_k, x_{k+1}]$. Given a positive integer m, let

$$\mathcal{PP}_m = \left\{ \begin{array}{l} there\ exist\ polynomials: \\ \mathcal{PP}_m(\Delta) = p_0, p_1, ..., p_k \ \in \ \mathcal{P}\ with\ f(x) = p_i(x), \\ \ for\ x \in \ I_i, \ i = 0, 1, ..., k \end{array} \right\}$$

We call $\mathcal{PP}_m$ the space of piecewise polynomials of order m with knots $x_1, \cdots\cdots, x_k$

Definition 1.4.6 (Polynomial Splines With Simple Knots)**.** Let d be a partition of the interval $[a, b]$ as in Definition 1.4.5, and let m be a positive integer. Let

$$\mathcal{S}_m(\Delta) = \mathcal{PP}_m(\Delta) \cap C^{m-2}[a, b],$$

where $\mathcal{PP}_m(\Delta)$ is the space of piecewise polynomials defined in 1.4.5. We call $\mathcal{S}_m(\Delta)$ the space of polynomial splines of order m with simple knots at the points $x_1, \cdots\cdots, x_k$.

In particular, suppose that our problem are located at the points $(x_i, y_i), i = 1, 2, ..., k$ in the Cartesian plane. Then the centerline of the spline is approximately given by the function s with the following properties:

1. s is a piecewise cubic polynomial with knots at $x_1, \cdots\cdots, x_k$.;
2. s is a linear polynomial for $x \leq x_1$ and $x \geq x_k$;
3. s has two continuous derivatives everywhere
4. $s(x_i) = y_i, i = 1, 2, ..., k$;

The function s is a kind of best interpolating function. By S_m^k we denote the set of all splines of degree m for a fixed subdivision

Theorem 1.4.8. *S_m^k is a linear space of dimension $k + m$.*

Theorem 1.4.9. *Let $m = 2\ell - 1$ with $\ell \in \mathbb{N}$ and $\ell \geq 2$. Then, given $k + 1$ values $y_0, ..., y_k$ and $m - 1$ boundary data $a_1, ..., a_{\ell-1}$ and $b_1, ..., b_{\ell-1}$, there exists a unique spline $s \in S_m^k$ satisfying the interpolation conditions*

$$s(x_j) = y_j, \quad j = 0, 1, ..., k$$

and the boundary conditions

$$s^{(j)}(a) = a_j, \quad s^{(j)}(b) = b_j, \quad j = 1, ..., \ell - 1$$

For the sake of simplicity we confine our analysis of B-splines to the case of an equidistant subdivision of step length h. We set

$$B_0 = \begin{cases} 1, & |x| \leq 0.5 \\ 0, & |x| > 0.5 \end{cases}$$

and define recursively

$$B_{m+1}(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} B_m(y), \quad x \in \mathbb{R}, \quad m = 0, 1, ...$$

Corollary 1.4.2. *Let $x_k = a + hk$, $k = 0, ..., n$, be an equidistant subdivision of the interval $[a, b]$ of step size $h = (b - a)/n$ with $n \geq 2$, and let $m = 2\ell - 1$ with $\ell \in \mathbb{N}$. Then the B-splines*

$$B_{m,k}(x) = B_m(\frac{x - a - hk}{h}), \quad x \in [a, b]$$

For $k = -\ell + 1, ..., n + \ell + 1$, from a basis for $B_{m,k}$.

Proof. See [42] □

Therefore, the cubic spline

$$s(x) = \sum_{k=-1}^{n+1} \alpha_k B_3(\frac{x - x_k}{k}), \quad x \in [a, b]$$

with

$$B_3(x) = \frac{1}{6} \begin{cases} (2 - |x|)^3 - 4(1 - |x|)^3, & |x| \leq 1 \\ (2 - |x|)^3, & 1 \leq |x| \leq 2 \\ 0, & |x| \geq 2 \end{cases} \tag{1.4.20}$$

satisfies the interpolation conditions and the boundary conditions in theorem 1.4.9 if and only if the $n + 3$ coefficients $\alpha_{-1}, ..., \alpha_{n+1}$ satisfy the system

$$\begin{aligned} -\frac{1}{2}\alpha_{-1} + \frac{1}{2}\alpha_1 &= ha_1 \\ \frac{1}{6}\alpha_{j-1} + \frac{2}{3}\alpha_j + \frac{1}{6}\alpha_{j+1} &= y_j, \quad j = 0, ..., n, \\ -\frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_{n+1} &= hb_1 \end{aligned} \tag{1.4.21}$$

Chapter 2

Inverse Problem of Identifying an Unknown Boundary

Introduction

This chapter deals with the study of a geometrical inverse problem for the Biharmonic equation to find an unknown boundary from the measured data on the remaining known part of the boundary. Here, we can determine the solution uniquely, everywhere in its domain of definition, by assuming that the available data have a Fourier expansion. The question of the existence and uniqueness of this inverse problem will be answered, and we will conclude with some simple analytical examples.

Theoretical results are standard and well known for the Laplace case. The question of the existence and the uniqueness of the unknown curve is discussed [1], in the case of a doubly connected domain. Further, in the simply connected domain, the results of uniqueness and reconstructions have been demonstrated in [34]. We follow these works and we extend the techniques to our case.

In the first part of this chapter, we present a method that has been suggested in [1], and we show the extension of this method in the remaining part.

2.1 Identifying an unknown boundary for the Harmonic equation

2.1.1 Problem formulation and modeling

Let $\Omega \subset \mathbb{R}^2$, be a bounded domain, and doubly-connected, with regular boundary $\partial\Omega = \Gamma_m \cup \Gamma_c$, where Γ_m and Γ_c are two curves, smooth and closed of class C^2. By n we denote the outward unit normal to $\partial\Omega$, and u is a solution to the following boundary value problem

$$\Delta u = 0 \qquad in\ \Omega \tag{2.1.1}$$

with mixed Cauchy data

$$\begin{cases} u = u_0, & on\ \Gamma_m \\ \dfrac{\partial u}{\partial n} = u_1, & on\ \Gamma_m \end{cases} \tag{2.1.2}$$

Definition 2.1.1. We call a function $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ satisfying Equation (2.1.1) and the boundary conditions (2.1.2) a classical solution to the problem (2.1.1)-(2.1.2).

Theorem 2.1.1 ([86])**.** *The mixed boundary value problem (2.1.1)-(2.1.2) in doubly connected domain with smooth boundary components and boundary data in L^2, have unique solution.*

In what follow we consider $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ as a solution to the problem (2.1.1)-(2.1.2), and assume that Γ_m is a known inner curve and Γ_c an unknown outer curve.

The inverse problem we are considered with is : given Γ_m, u_0 and u_1, determine the curve Γ_c such that:

$$u = 0, \quad sur\, \Gamma_c \tag{2.1.3}$$

Or

$$\tfrac{\partial u}{\partial n} = 0, \quad sur\, \Gamma_c \tag{2.1.4}$$

The problem (2.1.1)-(2.1.2)-(2.1.3)-(2.1.4) can be interpreted as follows: find the curve Γ_c on which the function u or the normal component of its derivatives vanishes. For example, in incompressible fluid, determine the bottom of reservoir, from the knowledge of the velocity u and the pressure $\frac{\partial u}{\partial n}$ on Γ_m. In other, one can measure the pressure and velocity potential on the surface of the water and determine the surface of a submarine [1].

2.1.2 Polar coordinate representation

It is advantageous [92] to apply the variable change $x = rcos\theta$, $y = rsin\theta$ on the problem (2.1.1)-(2.1.2), where the radial coordinate is denoted by r and the angular coordinate is denoted by θ. The following change can be obtained

$$\begin{aligned}
&\frac{\partial r}{\partial x} = \cos(\theta),\ \frac{\partial r}{\partial y} = \sin(\theta),\ \frac{\partial^2 r}{\partial x^2} = \frac{1}{r}\sin^2(\theta),\ \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\cos^2(\theta)\\
&\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r},\ \frac{\partial \theta}{\partial y} = \frac{cos(\theta)}{r},\ \frac{\partial^2 \theta}{\partial x^2} = \frac{\sin(2\theta)}{r^2},\ \frac{\partial^2 \theta}{\partial y^2} = -\frac{\sin(2\theta)}{r^2}
\end{aligned} \tag{2.1.5}$$

By applying (2.1.5) in (2.1.1) therefore, the Laplace equation in polar coordinate system is given as:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2.1.6}$$

Using the variables separation method. Then, the solution is given as:

$$u(r, \theta) = f(r).g(\theta) \tag{2.1.7}$$

with $f \in C^2(]0, \infty[)$, and $g \in C^2([0, 2\pi])$, by replacing (2.1.7) in (2.1.6) this one can be rewritten as two differential equations by

$$r^2\frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = n \tag{2.1.8}$$

We assume that the function $g(\theta)$ can be represented by the Fourier series (see definition 1.2.2). Therefore, the function $g(\theta)$ can be written as

$$g(\theta) = \sum_{n=0}^{\infty} c_n \cos(n\theta) + d_n \sin(n\theta) \tag{2.1.9}$$

By substituting $u(r,\theta) = f(r)\cos(n\theta)$ and $u(r,\theta) = f(r)\sin(n\theta)$ and the variable change $r = e^t$ is used on (2.1.8) to obtain that

$$f''(t) - n^2 f(t) = 0 \tag{2.1.10}$$

The characteristic equation $m^2 - n^2 = 0$ have a roots $m = \pm n$. Clearly, the roots are not repeating if $n \geq 1$ then solution of (2.1.10) is given by

$$f(t) = c_{n,1} r^{-n} + c_{n,2} r^n \tag{2.1.11}$$

The general solution for $n \geq 1$

$$u(r,\theta) = \sum_{n=1}^{\infty} (c_{n,1} r^{-n} + c_{n,2} r^n)\cos(n\theta) + (d_{n,1} r^{-n} + d_{n,2} r^n)\sin(n\theta) \tag{2.1.12}$$

The general solution for the repeating roots $n = 0$ is

$$u(r,\theta) = c_{0,1} + c_{0,2}\ln(r) \tag{2.1.13}$$

Combining the solution shown in (2.1.13), (2.1.12), then, the numerical solution of the Harmonic equation in 2D doubly-connected region [72, 75] is given by

$$\begin{aligned} u(r,\theta) = c_{0,1} + c_{0,2}\ln(r) &+ \sum_{n=1}^{\infty} (c_{n,1} r^{-n} + c_{n,2} r^n)\cos(n\theta) \\ &+ \sum_{n=1}^{\infty} (d_{n,1} r^{-n} + d_{n,2} r^n)\sin(n\theta) \end{aligned} \tag{2.1.14}$$

Without loss of generality, we assume that Γ_m is a circle of radius R_0, and $(u_0, u_1) \in L^2(\Gamma_m) \times L^2(\Gamma_m)$ then both of them have a Fourier expansion as follows (see 1.2.2 in chapter 1).

$$\begin{aligned} u_0 &= A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta) \\ u_1 &= A'_0 + \sum_{n=1}^{\infty} A'_n \cos(n\theta) + B'_n \sin(n\theta) \end{aligned} \tag{2.1.15}$$

Lemma 2.1.1. *The solution of problem (2.1.1)-(2.1.2) in the exterior of a circle with radius R_0 is given by*

$$\begin{aligned} u(r,\theta) &= A_0 + A'_0 r_0 \ln(\frac{r}{r_0}) \\ &+ \sum_{n=1}^{\infty} \left[\frac{A_n}{2} \left((\frac{R_0}{r})^n + (\frac{r}{R_0})^n \right) + \frac{R_0 A'_n}{2n} \left((\frac{r}{R_0})^n - (\frac{R_0}{r})^n \right) \right] \cos(n\theta) \\ &+ \sum_{n=1}^{\infty} \left[\frac{B_n}{2} \left((\frac{R_0}{r})^n + (\frac{r}{R_0})^n \right) + \frac{R_0 B'_n}{2n} \left((\frac{r}{R_0})^n - (\frac{R_0}{r})^n \right) \right] \sin(n\theta) \end{aligned} \tag{2.1.16}$$

where the constants $c_{n,1}, c_{n,2}, d_{n,1}, d_{n,2}$ are uniquely determined by

$$\begin{array}{lll} c_{0,1} = A_0 - A'_0 R_0 \ln R_0, & d_{0,1} = 0 & \\ c_{0,2} = A'_0 R_0, & d_{0,2} = 0 & \\ c_{n,1} = \frac{1}{2} A_n R_0^n - \frac{1}{2n} A'_n R_0^{n+1}, & d_{n,1} = \frac{1}{2} B_n R_0^n - \frac{1}{2n} B'_n R_0^{n+1} & n \geq 1 \\ c_{n,2} = \frac{1}{2} A_n R_0^{-n} + \frac{1}{2n} A'_n R_0^{1-n}, & d_{n,2} = \frac{1}{2} B_n R_0^{-n} + \frac{1}{2n} B'_n R_0^{1-n} & n \geq 1 \end{array} . \tag{2.1.17}$$

Proof. If $r = R_0$ the equation (2.1.14) take the form

$$u(R_0, \theta) = c_{0,1} + c_{0,2}\ln(R_0) + \sum_{n=1}^{\infty}\left(c_{n,1}R_0^{-n} + c_{n,2}R_0^{n}\right)\cos(n\theta) \\ + \sum_{n=1}^{\infty}\left(d_{n,1}R_0^{-n} + d_{n,2}R_0^{n}\right)\sin(n\theta)$$

and

$$\frac{\partial u}{\partial n}(R_0, \theta) = \frac{c_{0,2}}{R_0} + \sum_{n=1}^{\infty} n\left(-c_{n,1}R_0^{-n-1} + c_{n,2}R_0^{n-1}\right)\cos(n\theta) \\ + \sum_{n=1}^{\infty} n\left(-d_{n,1}R_0^{-n-1} + d_{n,2}R_0^{n-1}\right)\sin(n\theta)$$

One can match the boundary conditions (2.1.2) and by considering the Fourier expansion (2.1.15), render to a linear system equations that is uniquely solved to obtain the coefficients (2.1.17). By substituting in (2.1.14), the expression (2.1.16) can be obtained. □

2.1.3 Existence and uniqueness

Starting from a trivial case and based on the lemma 2.1.1 one can verify that the problem (2.1.1)-(2.1.2) accept a solution $u = 0$ if and only if $u_0 = u_1 = 0$, that means both equations (2.1.3)-(2.1.4) admits an infinite number of solution Γ_c. To avoid the trivial case we confine ourselves in the following hypothesis $|u_0| + |u_1| \neq 0$, i.e., at least one of the given data does not vanish identically.

The existence of a solution to the inverse problem (2.1.1)-(2.1.2)-(2.1.3)-(2.1.4) is not assured for arbitrary data u_0, u_1. For example, if $u = 0$ on Γ_c and $u_0 = 0$ on Γ_m, then $u = 0$ on $\partial\Omega$. The maximum-minimum principle (see theorem 1.1.3, and 1.1.1 in Chapter 1) imply that $u \equiv 0$ in its domain of definition, and $u_1 = 0$. Therefore the problem (2.1.1)-(2.1.2)-(2.1.3) has no solution if $u_0 = 0$ and $u_1 \neq 0$. See example 2.1.2.

If $\frac{\partial u}{\partial n} = 0$ on Γ_c and $u_1 = 0$ on Γ_m,i.e., $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Then, lemma (1.1.1) imply that $u = constant$, everywhere in its domain of definition, and $u_0 = constant$. Therefore problem (2.1.1)-(2.1.2)-(2.1.4) has no solution if $u_0 \neq constant$ and $u_1 = 0$, for example.(See example 2.1.2).

Theorem 2.1.2. *Let u be a solution to the problem (2.1.1)-(2.1.2), if $u = 0$ on Γ_c, then the boundary Γ_c is uniquely determined provided that $(u_0, u_1) \neq (0, 0)$.*

Proof. Suppose that Γ_c and Γ_c' are two separate solutions, then there exists a domain Ω', bounded by some parts of Γ_c and Γ_c', in which there exists a harmonic function, u , which vanishes on $\partial\Omega'$. The maximum-minimum principle imply that, $u = 0$ in Ω'. From theorem (1.1.5), then $u = 0$ in it's domain of definition. Therefore $u_0 = u_1 = 0$, which contradicts our assumption. See example 2.1.1. □

Theorem 2.1.3. *Let u be a solution to the problem (2.1.1)-(2.1.2), if $u = 0$ on Γ_c, then the boundary Γ_c is uniquely determined provided that $u_0 \neq constant$ or $u_1 \neq 0$.*

Proof. Suppose there are two different surfaces, Γ_c and Γ_c', on which $\frac{\partial u}{\partial n} = 0$. Then there is a domain Ω' bounded by parts of Γ_c and Γ_c', in which Δu and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega'$. The lemma (1.1.1) imply that, $u = constant$ everywhere in it's domain of definition, and $u_0 = constant$, $u_1 = 0$. Therefore, there is at most one solution Γ_c on which $\frac{\partial u}{\partial n} = 0$ provided that $u_0 \neq constant$ or $u_1 \neq 0$. See example 2.1.3. □

2.1.4 Determination of a non-accessible curve Γ_c

Definition 2.1.2. Let $r = f(\theta)$ be a representation of Γ_c, the following equation

$$u(f(\theta), \theta) = 0 \tag{2.1.18}$$

is a transcendental equation for $f(\theta)$.

Remark 2.1.1. In order to solve (2.1.3) and determine Γ_c, the equation (2.1.18) allows us to determine the unknown function $f(\theta)$ numerically, in some cases analytically, and the following equivalence is satisfy:

$$u(f(\theta), \theta) = 0 \Leftrightarrow r = f(\theta) \tag{2.1.19}$$

Remark 2.1.2. The equation (2.1.4) is equivalent to

$$\frac{\partial u}{\partial n}|_{r=f(\theta)} = \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial f}{\partial \theta}|_{r=f(\theta)} = 0 \tag{2.1.20}$$

with

$$\nabla u = \frac{\partial u}{\partial r}e_r + \frac{1}{r}\frac{\partial u}{\partial \theta}e_\theta, \quad n = e_r - \tfrac{1}{r}f'(\theta)e_\theta$$

where e_r and e_θ are the polar coordinates of unit vectors.

2.1.5 Numerical illustrations

In what follows we assume that Γ_m is the unit circle, in which the coefficients $c_{n,1}, c_{n,2}, d_{n,1}, d_{n,1}$ can be obtained by replacing $R_0 = 1$ in (2.1.17).

Example 2.1.1. Let $u_0 = -1,\ u_1 = 1$. According to (2.1.17) we obtain that $c_{0,1} = -1,\ c_{0,2} = 0$. Therefore : $u = -1 + \ln r$.
Equation (2.1.3) takes the form : $\ln r = 1$. and Γ_c is a circle of radius e.
Equation (2.1.4) takes the form : $\dfrac{1}{r} = 0$. and has no solutions.

Example 2.1.2. Let : $u_0 = 0,\ u_1 = 1$. According to (2.1.17) we obtain that $c_{0,1} = 0,\ c_{0,2} = 1$. Therefore : $u(r, \theta) = \ln r$.
According to the equation (2.1.3) and (2.1.4) there are no external solution.

Example 2.1.3. Let $u_0 = u_1 = \sin(\theta)$. According to (2.1.17) we obtain that : $d_{1,1} = 0,\ d_{1,2} = 1$. Therefore : $u = r\sin(\theta)$.
Equation (2.1.3), has no external solution .
Equation (2.1.4) take the form :

$$\cos(\theta)\frac{\partial f}{\partial \theta} = f(\theta)\sin(\theta)$$

And has solution $f(\theta) = \dfrac{c}{\cos(\theta)}$, with $c = constant$, which is no bounded. Therefore problem (2.1.4) have no solution.

2.2 Identifying an unknown boundary for the Biharmonic equation

2.2.1 Problem formulation and modeling

Let $\Omega \subset \mathbb{R}^2$, be a bounded doubly-connected domain with piece-wise smooth boundary $\partial\Omega = \Gamma_m \cup \Gamma_c$, where Γ_m and Γ_c are two curves, smooth and closed of class C^2, by n we denote the outward unit normal to $\partial\Omega$. Let $u \in C(\bar{\Omega}) \cap C^4(\Omega)$ to be a solution of the following boundary value problem:

$$\Delta^2 u = 0, \qquad in\ \Omega \tag{2.2.1}$$

that is equivalent to system of equations:

$$\begin{cases} \Delta u = w, & in\ \Omega \\ \Delta w = 0, & in\ \Omega \end{cases} \tag{2.2.2}$$

This mathematical model is well known in 2D Stokes flows, and in elasticity, where the functions u and w represent the stream function and vorticity in Stokes flows, whilst they represent deflection and bending moment in elasticity.

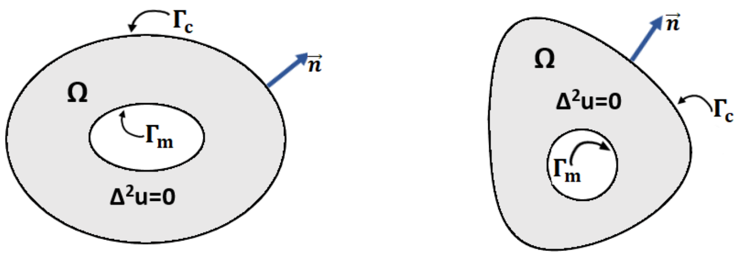


Figure 2.1: Example of doubly connected planar domain.

In [33], it is presented that, if u and its normal derivative $\frac{\partial u}{\partial n}$ or u and w or u and $\frac{\partial w}{\partial n}$ are prescribed at all points of the boundary $\partial\Omega$, this enables us to uniquely identified the solutions u and w everywhere in its definition domain Ω, then it is well-posed direct problem [9, 68]. However, in the practice it is not always possible to specify boundary conditions at all points on the boundary of the considered domain and some other boundary information may be given elsewhere [33, 96], in this case, the problem is called an inverse problem for the Biharmonic equation which is ill-posed. We address the situation where the boundary conditions u, $\frac{\partial u}{\partial n}$, Δu, $\frac{\partial \Delta u}{\partial n}$ are given on a part Γ_m of the boundary, and the unknown part has assumed additional information.

In what follow, we are interested in the Biharmonic equation with mixed boundary conditions given on the accessible part Γ_m of the boundary as:

$$\begin{cases} u = u_0, & on\ \Gamma_m \\ \dfrac{\partial u}{\partial n} = u_1, & on\ \Gamma_m \\ w = u_2, & on\ \Gamma_m \\ \dfrac{\partial w}{\partial n} = u_3, & on\ \Gamma_m \end{cases} \tag{2.2.3}$$

where $\frac{\partial}{\partial n}$ denote the outward unit normal to $\partial\Omega$, we assume that Γ_m is an inside (internal) curve and Γ_c is an outside (external) curve, both have a polar coordinates representations of the form $r = f(\theta)$, where f is a differentiable function and 2π-periodic (see figure 2.1). We assume that Γ_m is known, and Γ_c is unknown.

Remark 2.2.1 (see [7])**.** The mixed boundary value problem (2.2.1)-(2.2.3) admits a unique solution for a compatible data in $L^2(\Gamma_m)$.

Inverse problem The inverse problem we are consider with is: given Γ_m and u_0, u_1, u_2, u_3, to find the shape Γ_c such as:

$$u = \frac{\partial u}{\partial n} = 0, \qquad on\ \Gamma_c \tag{2.2.4}$$

Or

$$u = w = 0, \qquad on\ \Gamma_c \tag{2.2.5}$$

Or

$$\frac{\partial u}{\partial n} = w = 0, \qquad on\ \Gamma_c \tag{2.2.6}$$

This inverse problem is generally encountered in elastic plate [71, 95], in particular, finding the cracks in a medium from the measurements of the elastic field on the surface of the medium.

2.2.2 Polar coordinates Representation

In two-dimensional plane problems, the Biharmonic equation can be solved by a repeated application of variables separation procedures, when, the dependency in one of the coordinate variables is Harmonic.[87]. Each Biharmonic function also called (Airy stress function) has a polar coordinates representation in the plane [37].

It is natural [87] to apply the variable change $x = rcos\theta$, $y = rsin\theta$ on the problem (2.2.1)-(2.2.3), where the radial coordinate is denoted by r and the angular coordinate is denoted by θ. By using the variables change (2.1.5) thus, we have the Biharmonic operator in polar coordinates given as:

$$\Delta^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \tag{2.2.7}$$

In addition, the following variables change can be obtained.

$$\begin{aligned}
\frac{\partial^3 r}{\partial x^3} &= -\frac{3xy^2}{r^5}, & \frac{\partial^3 r}{\partial y^3} &= -\frac{3x^2 y}{r^5},\\
\frac{\partial^4 r}{\partial x^4} &= -3y^2\frac{r^2 - 5x^2}{r^7}, & \frac{\partial^4 r}{\partial y^4} &= -3x^2\frac{r^2 - 5y^2}{r^7}\\
\frac{\partial^3 \theta}{\partial x^3} &= 2y\frac{r^2 - 4x^2}{r^6}, & \frac{\partial^3 \theta}{\partial y^3} &= -2x\frac{r^2 - 4y^2}{r^6}\\
\frac{\partial^4 \theta}{\partial x^4} &= 2y\frac{24x^3 - 12xr^2}{r^8}, & \frac{\partial^4 \theta}{\partial y^4} &= -2x\frac{24y^3 - 12yr^2}{r^8}
\end{aligned} \tag{2.2.8}$$

Therefore, the Biharmonic function (2.2.1) in polar coordinates is given by

$$\frac{\partial^4 u}{\partial r^4} + \frac{2}{r}\frac{\partial^3 u}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^3}\frac{\partial u}{\partial r} - \frac{2}{r^3}\frac{\partial^3 u}{\partial r \theta^2} + \frac{2}{r^2}\frac{\partial^4 u}{\partial r^2 \theta^2} + \frac{4}{r^4}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 u}{\partial \theta^4} = 0 \tag{2.2.9}$$

Let the solution for Biharmonic equation (2.2.1), is separable, i.e., the Airy stress function (see [53])

$$u(r,\theta) = \varphi(t)\psi(\theta) \tag{2.2.10}$$

We assume that the function $\psi(\theta)$ can be represented by the Fourier series. Therefore, the function $\psi(\theta)$ can be written as

$$\psi(\theta) = \sum_{n=0}^{\infty} p_n \cos(n\theta) + \sum_{n=0}^{\infty} q_n \sin(n\theta) \tag{2.2.11}$$

From (2.2.11) and to obtain the function $\varphi(t)$ we depond on the both cases when the solution to be in the form of $\varphi(t)\cos(n\theta)$ or $\varphi(t)\sin(n\theta)$.

i. Solution involving the terms $\cos(n\theta)$

Take the case when the solution $u(r,\theta) = \varphi(r)\cos(n\theta), n \in \mathbb{N}$ and substituting in (2.2.9) then we obtain that

$$\frac{d^4\varphi}{dr^4} + \frac{2}{r}\frac{d^3\varphi}{dr^3} - \frac{(1+2n^2)}{r^2}\frac{d^2\varphi}{dr^2} + \frac{(1+2n^2)}{r^3}\frac{d\varphi}{dr} - \frac{n^2(4-n^2)}{r^4}\varphi = 0 \tag{2.2.12}$$

Use variable change $r = e^t$ then (2.2.12) yieldz

$$\frac{d^4\varphi}{dr^4} - 4\frac{d^3\varphi}{dr^3} + (4-n^2)\frac{d^2\varphi}{dr^2} + 4n^2\frac{d\varphi}{dr} - n^2(4-n^2)\varphi = 0 \tag{2.2.13}$$

The characteristic equation is

$$m^4 - 4m^3 + (4-n^2)m^2 + 4n^2 m - n^2(4-n^2) = 0.$$

and can written as

$$(m^2 - n^2)\left((m-2)^2 - n^2\right) = 0.$$

Thus, roots of characteristic equation can be written as

$$m = \pm n, \quad m = 2 \pm n.$$

Clearly, the roots are not repeating if $n \geq 2$. For repeating roots we need to find some other independent solutions.

i.1. Roots are not reapeting $n \geq 2$

If $n \geq 2$ then, by substituting $r = e^t$, we get

$$\varphi(r) = c_{n,1}r^{-n} + c_{n,2}r^{n} + c_{n,3}r^{2-n} + c_{n,4}r^{2+n}$$

The general solution for $n \geq 2$ take the form

$$u(r,\theta) = \sum_{n=2}^{\infty}\left(c_{n,1}r^{-n} + c_{n,2}r^{n} + c_{n,3}r^{2-n} + c_{n,4}r^{2+n}\right)\cos(n\theta) \tag{2.2.14}$$

i.2. Roots repeating

i.2.1 Case n = 0

By replacing $n = 0$, we obtain the roots $m = 0, 0, 2, 2$. Therefore, a solution for the differential equation (2.2.12) are $c_{0,1}r^0 + c_{0,2}r^2$. We now apply the same technique that was used in above problem to get the general solution. The independent solutions from roots information are $r^n \cos(n\theta)$, $r^{n+2}\cos(n\theta)$. Additional independent solutions can be obtained from [53] as:

$$\left[\frac{d}{dn}(r^n \cos(n\theta))\right]_{n=0} = \ln r$$
$$\left[\frac{d}{dn}(r^{n+2} \cos(n\theta))\right]_{n=0} = r^2 \ln r$$

The general solution for $n = 0$ is:

$$u(r, \theta) = c_{0,1} + c_{0,2}r^2 + c_{0,3}\ln r + c_{0,4}r^2 \ln r$$

i.2.2 Case n = 1

By replacing $n = 1$, we obtain the roots $m = -1, 1, 1, 3$. Therefore, a solution for the differential equation (2.2.12) is $(c_{1,1}r^{-1} + c_{1,2}r + c_{1,3}r^3)\cos\theta$. We now apply the same technique that was used in above problem to get the general solution. The independent solutions from roots information are $r^n \cos(n\theta)$, $r^{n+2}\cos(n\theta)$. The other independent solutions

$$\left[\frac{d}{dn}(r^n \cos(n\theta))\right]_{n=1} = r\ln r\cos\theta - r\theta\sin\theta$$
$$\left[\frac{d}{dn}(r^{2-n} \cos(n\theta))\right]_{n=1} = -r\ln r\cos\theta - r\theta\sin\theta$$

The general solution for $n = 1$ is:

$$u(r, \theta) = c_{1,1}r^{-1} + c_{1,2}r + c_{1,3}r^3 + c_{1,4}(r\ln r)\cos\theta + dr\theta\sin\theta$$

ii. Solution involving terms $\sin(n\theta)$

Following similar steps that of previous solution procedure, for the case when the solution $u(r, \theta) = \varphi(r)\sin(n\theta), n \in \mathbb{N}$, we obtain the general solution for case $n \geq 0$

$$u(r, \theta) = \sum_{n=2}^{\infty}\left(d_{n,1}r^{-n} + d_{n,2}r^n + d_{n,3}r^{2-n} + d_{n,4}r^{2+n}\right)\sin(n\theta) \qquad (2.2.15)$$

We present solution for the cases $n = 0$ and $n = 1$ as there are repeated roots.

ii.1 Case n = 0

By replacing $n = 0$, we obtain the roots $m = 0, 0, 2, 2$. Therefore, a solution for the differential equation (2.2.12) are $c_{0,1}r^0 + c_{0,2}r^2$. We now apply the same technique that was used in above problem to get the general solution. The independent solutions from roots information are $r^n \sin(n\theta)$, $r^{n+2}\sin(n\theta)$. The other independent solutions

$$\left[\frac{d}{dn}(r^n \sin(n\theta))\right]_{n=0} = \theta$$
$$\left[\frac{d}{dn}(r^{n+2} \sin(n\theta))\right]_{n=0} = r^2\theta$$

The general solution for $n = 0$ is

$$u(r,\theta) = (d_{0,1} + d_{0,2}r^2)\sin(0\theta) + d_{0,3}\theta r + d_{0,2}r^2\theta = d_{0,3}\theta r + d_{0,2}r^2\theta$$

ii.2 Case n = 1

By replacing $n = 1$, we obtain the roots $m = -1, 1, 1, 3$. Therefore, a solution for the differential equation (2.2.12) is $(c_{1,1}r^{-1} + c_{1,2}r + c_{1,3}r^3)\sin\theta$. We now apply the same technique that was used in above problem to get the general solution. The independent solutions from roots information are $r^n\sin(n\theta)$, $r^{n+2}\sin(n\theta)$. The other independent solutions

$$\left[\frac{d}{dn}(r^n\sin(n\theta))\right]_{n=1} = r\ln r\sin\theta + r\theta\cos\theta$$
$$\left[\frac{d}{dn}(r^{2-n}\sin(n\theta))\right]_{n=1} = -r\ln r\sin\theta + r\theta\sin\theta$$

The general solution for $n = 1$ is

$$u(r,\theta) = d_{1,1}r^{-1} + d_{1,2}r + d_{1,3}r^3 + d_{1,4}r\ln r)\cos\theta + cr\theta\sin\theta$$

The complete general solution of the Biharmonic equation (2.2.1) in polar coordinates applicable to plane elastic regions with a doubly connected domain was first introduced by Michell (1863 -1940), and thus it is given as follows: (for more details we refer to [37, 87]).

$$\begin{aligned}
u(r,\theta) &= c_{0,1} + c_{0,2}r^2 + c_{0,3}\ln(r) + c_{0,4}r^2\ln(r) + d_{0,3}\theta + d_{0,4}r^2\theta \\
&+ \left(\frac{c_{1,1}}{r} + c_{1,2}r + c_{1,3}r^3 + c_{1,4}(r\ln r) + cr\theta\right)\cos(\theta) \\
&+ \left(\frac{d_{1,1}}{r} + d_{1,2}r + d_{1,3}r^3 + d_{1,4}(r\ln r) + dr\theta\right)\sin(\theta) \\
&+ \sum_{n=2}^{\infty}\left(c_{n,1}r^{-n} + c_{n,2}r^n + c_{n,3}r^{2-n} + c_{n,4}r^{2+n}\right)\cos(n\theta) \\
&+ \sum_{n=2}^{\infty}\left(d_{n,1}r^{-n} + d_{n,2}r^n + d_{n,3}r^{2-n} + d_{n,4}r^{2+n}\right)\sin(n\theta)
\end{aligned} \tag{2.2.16}$$

where $c, d,\ and\ c_{n,1}, c_{n,2}, c_{n,3}, c_{n,4}, d_{n,1}, d_{n,2}, d_{n,3}, d_{n,4},\ for\ n \in \mathbb{N}$, are an unknown coefficients, which, will be retrieved uniquely from to the uniqueness of solution to the Cauchy problem (2.2.1)-(2.2.3), and by satisfying the boundary conditions. [7, 29, 72, 87].

2.2.3 Determination of coefficients

To determine the coefficients in (2.2.16), one must determine the functions $\frac{\partial u}{\partial n}$, $\frac{\Delta u}{\partial n}$ $\frac{\partial \Delta u}{\partial n}$. Here, we consider the variables change (2.1.5)-(2.2.8) then we obtain that

$$\begin{aligned}
\frac{\partial u}{\partial n}(r,\theta) &= 2c_{0,2}r + \frac{c_{0,3}}{r} + c_{0,4}(r + 2r\ln(r)) + 2d_{0,4}r\theta \\
&+ \left(-\frac{c_{1,1}}{r^2} + c_{1,2} + 3c_{1,3}r^2 + c_{1,4}(1 + \ln r) + d\theta\right)\cos(\theta) \\
&+ \left(-\frac{d_{1,1}}{r^2} + d_{1,2} + 3d_{1,3}r^2 + d_{1,4}(1 + \ln r) + c\theta\right)\sin(\theta) \\
&+ \sum_{n=2}^{\infty}\left(-nc_{n,1}r^{-n-1} + nc_{n,2}r^{n-1} + (2-n)c_{n,3}r^{1-n} + (2+n)c_{n,4}r^{1+n}\right)\cos(n\theta) \\
&+ \sum_{n=2}^{\infty}\left(-nd_{n,1}r^{-n-1} + nd_{n,2}r^{n-1} + (2-n)d_{n,3}r^{1-n} + (2+n)d_{n,4}r^{1+n}\right)\sin(n\theta)
\end{aligned} \tag{2.2.17}$$

The representations of the function Δu is given as:

$$\begin{aligned}\Delta u(r,\theta) &= 4c_{0,2} + c_{0,4}(4\ln(r)+4) + 4d_{0,4}\theta \\ &+ \left(8c_{1,3}r + \frac{2}{r}c_{1,4} + \frac{2d}{r}\right)\cos(\theta) + \left(8d_{1,1}r + \frac{2}{r}d_{1,4} - \frac{2c}{r}\right)\sin(\theta) \\ &+ \sum_{n=2}^{\infty}\left((4-4n)c_{n,3}r^{-n} + (4+4n)c_{n,4}r^{n}\right)\cos(n\theta) \\ &+ \sum_{n=2}^{\infty}\left((4-4n)d_{n,3}r^{-n} + (4+4n)d_{n,4}r^{n}\right)\sin(n\theta)\end{aligned} \tag{2.2.18}$$

The representations of the function $\frac{\partial(\Delta u)}{\partial n}$ is given as:

$$\begin{aligned}\frac{\partial(\Delta u)}{\partial n}(r,\theta) &= \frac{4}{r}c_{0,4} + \left(8c_{1,3} - \frac{2}{r^2}c_{1,4} - \frac{2d}{r^2}\right)\cos(\theta) + \left(8d_{1,3} - \frac{2}{r^2}d_{1,4} + \frac{2c}{r^2}\right)\sin(\theta) \\ &+ \sum_{n=2}^{\infty}\left((4n^2-4n)c_{n,3}r^{-n-1} + (4n^2+4n)c_{n,4}r^{n-1}\right)\cos(n\theta) \\ &+ \sum_{n=2}^{\infty}\left((4n^2-4n)d_{n,3}r^{-n-1} + (4n^2+4n)d_{n,4}r^{n-1}\right)\sin(n\theta)\end{aligned} \tag{2.2.19}$$

In order to simplify the expression and without loss of generality, assume that Γ_m is the unit circle, in a conformal manner, one first map the exterior of Γ_m on the exterior of the unit circle. The functions $u_0(\theta),\ u_1(\theta),\ u_2(\theta),\ u_3(\theta)$ are assumed to be L^2 integrable on the interval $[0, 2\pi]$. Hence, all of them admit a development in terms of the Fourier expansion as: [1, 36]

$$\begin{aligned}u_0 &= A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta) \\ u_1 &= A'_0 + \sum_{n=1}^{\infty} A'_n\cos(n\theta) + B'_n\sin(n\theta) \\ u_2 &= A''_0 + \sum_{n=1}^{\infty} A''_n\cos(n\theta) + B''_n\sin(n\theta) \\ u_3 &= A'''_0 + \sum_{n=1}^{\infty} A'''_n\cos(n\theta) + B'''_n\sin(n\theta)\end{aligned} \tag{2.2.20}$$

Therefore, the coefficients of expression (2.2.16) are the solutions of the algebraic systems, which are obtained by matching the boundary condition (2.2.20) as follows:

$$\begin{cases} c_{0,1} + c_{0,2} = A_0 & \\ c_{1,1} + c_{1,2} + c_{1,3} = A_1 & \\ c_{n,1} + c_{n,2} + c_{n,3} + c_{n,4} = A_n & n \geq 2 \\ d_{1,1} + d_{1,2} + d_{1,3} = B_1 & \\ d_{n,1} + d_{n,2} + d_{n,3} + d_{n,4} = B_n & n \geq 2 \end{cases} \tag{2.2.21}$$

$$\begin{cases} 2c_{0,2} + c_{0,3} + c_{0,4} = A'_0 & \\ -c_{1,1} + c_{1,2} + 3c_{1,3} + c_{1,4} = A'_1 & \\ -nc_{n,1} + nc_{n,2} + (2-n)c_{n,3} + (2+n)c_{n,4} = A'_n & n \geq 2 \\ -d_{1,1} + d_{1,2} + 3d_{1,3} + d_{1,4} = B'_1 & \\ -nd_{n,1} + nd_{n,2} + (2-n)d_{n,3} + (2+n)d_{n,4} = B'_n & n \geq 2 \end{cases} \tag{2.2.22}$$

$$
\begin{cases}
4c_{0,2} + 4c_{0,4} = A_0'' \\
8c_{1,3} + 2c_{1,4} = A_1'' \\
(4-4n)c_{n,3} + (4+4n)c_{n,4} = A_n'' \quad n \geq 2 \\
8d_{1,3} + 2d_{1,4} = B_1'' \\
(4-4n)d_{n,3} + (4+4n)d_{n,4} = B_n'' \quad n \geq 2
\end{cases}
\tag{2.2.23}
$$

$$
\begin{cases}
4c_{0,4} = A_0''' \\
8c_{1,3} - 2c_{1,4} = A_1''' \\
(4n^2-4n)c_{n,3} + (4n^2+4n)c_{n,4} = A_n''' \quad n \geq 2 \\
8d_{1,3} - 2d_{1,4} = B_1''' \\
(4n^2-4n)d_{n,3} + (4n^2+4n)d_{n,4} = B_n''' \quad n \geq 2
\end{cases}
\tag{2.2.24}
$$

Thus, the solution of the boundary value problem (2.2.1)-(2.2.3) in the exterior of the unit circle is given as

$$
\begin{aligned}
u(r,\theta) = {} & c_{0,1} + c_{0,2}r^2 + c_{0,3}\ln(r) + c_{0,4}r^2\ln(r) \\
& + \left(\frac{c_{1,1}}{r} + c_{1,2}r + c_{1,3}r^3 + c_{1,4}r\ln r\right)\cos(\theta) \\
& + \left(\frac{d_{1,1}}{r} + d_{1,2}r + d_{1,3}r^3 + d_{1,4}r\ln r\right)\sin(\theta) \\
& + \sum_{n=2}^{\infty}\left(c_{n,1}r^{-n} + c_{n,2}r^n + c_{n,3}r^{2-n} + c_{n,4}r^{2+n}\right)\cos(n\theta) \\
& + \sum_{n=2}^{\infty}\left(d_{n,1}r^{-n} + d_{n,2}r^n + d_{n,3}r^{2-n} + d_{n,4}r^{2+n}\right)\sin(n\theta)
\end{aligned}
\tag{2.2.25}
$$

with their coefficients given as:

$$
\begin{array}{ll}
c_{0,1} = A_0 + \frac{1}{4}(A_0''' - A_0''), & d_{0,1} = d_{0,2} = d_{0,3} = d_{0,4} = 0 \\
c_{0,2} = \frac{1}{4}(A_0'' - A_0'''), & c = d = 0 \\
c_{0,3} = A_0' + \frac{1}{4}(A_0''' - 2A_0'') & \\
c_{0,4} = \frac{1}{4}A_0''' & \\
c_{1,1} = \frac{1}{2}A_1 - \frac{1}{2}A_1' + \frac{1}{16}(3A_1'' - A_1'''), & d_{1,1} = \frac{1}{2}B_1 - \frac{1}{2}B_1' + \frac{1}{16}(3B_1'' - B_1''') \\
c_{1,2} = \frac{1}{2}A_1 + \frac{1}{2}A_1' - \frac{1}{4}A_1'', & d_{1,2} = \frac{1}{2}B_1 + \frac{1}{2}B_1' - \frac{1}{4}B_1'' \\
c_{1,3} = \frac{1}{16}(A_1'' + A_1'''), & d_{1,3} = \frac{1}{16}(B_1'' + B_1''') \\
c_{1,4} = \frac{1}{4}(A_1'' - A_1'''), & d_{1,4} = \frac{1}{4}(B_1'' - B_1''') \\
c_{n,1} = \frac{1}{2}A_n - \frac{1}{2n}A_n' + \frac{1-n}{n}c_{n,3} + \frac{1}{n}c_{n,4}, & d_{n,1} = \frac{1}{2}B_n - \frac{1}{2n}B_n' + \frac{1-n}{n}d_{n,3} + \frac{1}{n}d_{n,4} \\
c_{n,2} = \frac{1}{2}A_n + \frac{1}{2n}A_n' - \frac{1}{n}c_{n,3} - \frac{n+1}{n}c_{n,4}, & d_{n,2} = \frac{1}{2}B_n + \frac{1}{2n}B_n' - \frac{1}{n}d_{n,3} - \frac{n+1}{n}d_{n,4} \\
c_{n,3} = \frac{1}{8n^2-8n}(A_n''' - nA_n''), & d_{n,3} = \frac{1}{8n^2-8n}(B_n''' - nB_n'') \\
c_{n,4} = \frac{1}{8n^2+8n}(nA_n'' + A_n'''), & d_{n,4} = \frac{1}{8n^2+8n}(nB_n'' + B_n''')
\end{array}
\quad . \tag{2.2.26}
$$

One can state the following result:

Lemma 2.2.1. *The numerical solution of the problem (2.2.1)-(2.2.3) in the exterior of unit disc is given by (2.2.25), with the coefficients (2.2.26).*

Remark 2.2.2. It can be seen that, if $u_2 = u_3 = 0$, then by substituting in (2.2.20) and (2.2.26) one can obtains:

$$
u(r,\theta) = c_{0,1} + c_{0,3}\ln(r) + \sum_{n=1}^{\infty}(c_{n,1}r^{-n} + c_{n,2}r^n)\cos(n\theta) + (d_{n,1}r^{-n} + d_{n,2}r^n)\sin(n\theta)
$$

which correspond to the polar coordinates representation of the Harmonic function in the exterior of the unit circle (see section 1 in chapter 2).

2.2.4 Existence and uniqueness

Starting from trivial case, if $u_0 = u_1 = u_2 = u_3 = 0$, then by substituting in (2.2.20) and (2.2.26) one obtain that $u \equiv 0$ in Ω, it means, there are infinite numbers of solutions Γ_c that satisfy the equations (2.2.4)-(2.2.5)-(2.2.6). In what follow, we assume that Γ_m is known, and $|u_0| + |u_1| + |u_2| + |u_3| \neq 0$, i.e., at least one of the given data does not vanish identically.

Lemma 2.2.2. *The existence of a solution to (2.2.1)-(2.2.3)-(2.2.4)-(2.2.5)-(2.2.6) cannot be guaranteed for arbitrary data u_0, u_1, u_2, u_3.*

Proof. If $u = \frac{\partial u}{\partial n} = 0$ on Γ_c and $u_0 = u_1 = 0$, i.e., $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, therefore, $u = 0$ in Ω and $u_0 = u_1 = u_2 = u_3 = 0$, however, this provides a contradiction if $u_2 \neq 0$ and $u_0 = u_1 = 0$, for example.

If $u = w = 0$ on Γ_c and $u_0 = u_2 = 0$, i.e., $u = w = 0$ on $\partial\Omega$. The maximum-minimum principle for Harmonic functions implies that $w = 0$ in Ω, then, $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$, therefore, $u = 0$ in Ω and $u_0 = u_1 = u_2 = u_3 = 0$ however, this provides a contradiction if $u_1 \neq 0$ and $u_0 = u_2 = 0$, for example.

In the case where $\frac{\partial u}{\partial n} = w = 0$ on Γ_c and $u_1 = u_2 = 0$, i.e., $\frac{\partial u}{\partial n} = w = 0$ on $\partial\Omega$. We have already obtained that $\Delta u = 0$ in Ω, and satisfy $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Thus, $u = constant$ in Ω and consequently $u_0 = constant$ and $u_1 = u_2 = u_3 = 0$. However, this contradicts if $u_0 \neq constant$ and $u_1 = u_2 = 0$, for example. □

The uniqueness of solution to (2.2.1)-(2.2.3)-(2.2.4) is guaranteed, let Γ_c, Γ_c' two separate solutions, then, there exist Ω' a domain bounded by certain parts of Γ_c and Γ_c', in which there exist a Biharmonic function, u, verify $\Delta^2 u = 0$ in Ω' and $u = \frac{\partial u}{\partial n} = 0$ on the boundary of Ω', then $u = 0$ in Ω'. This and the unique continuation property for Biharmonic functions [94] imply that $u = 0$ in it's definition domain Ω and $u_0 = u_1 = u_2 = u_3 = 0$. However, this contradicts our assumption. (see example 2.2.1). we can state the following result:

Theorem 2.2.1. *Assume that in (2.2.1)-(2.2.4)-(2.2.5)-(2.2.6) we have $u = \frac{\partial u}{\partial n} = 0$ on Γ_c, then the data u_0, u_1, u_2, u_3 uniquely determine Γ_c provided that, $|u_0| + |u_1| + |u_2| + |u_3| \neq 0$.*

The uniqueness of solution to (2.2.2)-(2.2.3)-(2.2.5) is guaranteed, let Γ_c, Γ_c' two separate solutions then, there exist Ω' a domain bounded by certain parts of Γ_c and Γ_c', in which there exist a Biharmonic function ,u, verify $\Delta^2 u = \Delta w = 0$ in Ω' and satisfy $u = w = 0$ on the boundary of Ω', now, the maximum-minimum principle for Harmonic functions implies that $w = 0$ in Ω' and $\Delta u = 0$ in Ω', then, $u = 0$ in Ω'. Thus, by the unique continuation property for Harmonic functions [94] we obtain that $u = 0$ in it's definition domain Ω, and $u_0 = u_1 = u_2 = u_3 = 0$. However, this contradicts our assumption. (see example 2.2.2). We have the following result:

Theorem 2.2.2. *Suppose that in (2.2.2)-(2.2.4)-(2.2.5)-(2.2.6) we have $u = w = 0$ on Γ_c, then $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $w|_{\Gamma_m} = u_2$ and $\frac{\partial w}{\partial n}|_{\Gamma_m} = u_3$ uniquely determine Γ_c provided that, $|u_0| + |u_1| + |u_2| + |u_3| \neq 0$.*

For the uniqueness of solution to (2.2.2)-(2.2.3)-(2.2.6) , let Γ_c, Γ_c' two separate solutions, then, there exist Ω' a domain bounded by certain parts of Γ_c and Γ_c', in which there exist a Biharmonic function, u, verify $\Delta^2 u = \Delta w = 0$ in Ω' and satisfy $\frac{\partial u}{\partial n} = w = 0$ on the boundary of Ω', therefore, $w = 0$ in Ω', and $\Delta u = 0$ in Ω' and satisfy $\frac{\partial u}{\partial n} = 0$ on the boundary of Ω', therefore, $u = constant$ in Ω' and based on the unique property of continuity of an elliptical function we found $u = constant$ in it's definition domain Ω, thus, $u_0 = constant$ and $u_1 = u_2 = u_3 = 0$, then, there is at most one solution Γ_c provided that $u_0 \neq constant$ or $|u_1| + |u_2| + |u_3| \neq 0$, (see example 2.2.3), and we can state the following result:

Theorem 2.2.3. *Suppose that in (2.2.2)-(2.2.4)-(2.2.5)-(2.2.6) we have* $\frac{\partial u}{\partial n} = w = 0$ *on* Γ_c*, then* $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $w|_{\Gamma_m} = u_2$ *and* $\frac{\partial w}{\partial n}|_{\Gamma_m} = u_3$ *uniquely determine* Γ_c *provided that* $u_0 \neq$ *constant or* $|u_1| + |u_2| + |u_3| \neq 0$.

2.2.5 Determination of a non-accessible curve Γ_c

Suppose that $r = f(\theta)$ is a representation of Γ_c. From (2.2.4), (2.2.5) and (2.2.6), we can find the unknown function $f(\theta)$ numerically, by solving the equations:

$$u(f(\theta), \theta) = \frac{\partial u}{\partial n}(f(\theta), \theta) = 0 \tag{2.2.27}$$

if condition (2.2.4) is considered, and

$$u(f(\theta), \theta) = w(f(\theta), \theta) = 0 \tag{2.2.28}$$

if condition (2.2.5) is considered, and

$$\frac{\partial u}{\partial n}(f(\theta), \theta) = w(f(\theta), \theta) = 0 \tag{2.2.29}$$

if condition (2.2.6) is considered, by applying the formula

$$\frac{\partial u}{\partial n}|_{r=f(\theta)} = \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial f}{\partial \theta}|_{r=f(\theta)}$$

where

$$\nabla u = \frac{\partial u}{\partial r} e_r + \frac{1}{r}\frac{\partial u}{\partial \theta} e_\theta, \text{ and } n = e_r - \frac{1}{r} f'(\theta) e_\theta$$

and the vectors e_r and e_θ are unit vectors in polar coordinates.

This Cauchy's problem is ill-posed and its numerical solution is difficult. For some simple cases, we try to describe (2.2.27), (2.2.28) and (2.2.29) as a transcendental equations, which can be solved analytically using the inverse functions.

2.2.6 Numerical illustrations

In what follows, we consider that Γ_m to be the unit circle and we wish to find Γ_c. Here, $u(r, \theta)$ is given by (2.2.25) and the coefficients can be determined by (2.2.26).

Example 2.2.1. Let $u_0 = -2 - e^2$, $u_1 = -2 + 2e^2$, $u_2 = -4, u_3 = 0$. From (2.2.26) we obtain: $c_{0,1} = -e^2$, $c_{0,2} = -1$, $c_{0,3} = 2e^2$, $c_{0,4} = 0$. Therefore : $u = -e^2 - r^2 + 2e^2 \ln r$.

Equations (2.2.4) take the forms $-e^2 - r^2 + 2e^2 \ln r = 0$ and $-2r + \frac{2e^2}{r} = 0$, then Γ_c is a circle of radius e as shown in Figure 2.2a.

Equations (2.2.5) take the forms $-e^2 - r^2 + 2e^2 \ln r = 0$ and $\Delta u = -4 \neq 0$. They have no common solutions.

Equations (2.2.6) take the forms $-2r + \frac{2e^2}{r} = 0$ and $\Delta u = -4 \neq 0$. They have no common solutions.

Example 2.2.2. Let $u_0 = -2$, $u_1 = -3 - e^2$, $u_2 = 4$, $u_3 = -4$. According to (2.2.26) we obtain $c_{0,1} = 0$, $c_{0,2} = 2$, $c_{0,3} = -e^2$, $c_{0,4} = -1$. Therefore $u = 2r^2 - e^2 \ln r - r^2 \ln r$.

The equations (2.2.4), become $2r^2 - e^2 \ln r - r^2 \ln r = 0$ and $3r - \frac{e^2}{r} - 2r \ln r = 0$. They have no common solutions.

The equations (2.2.5), take the forms $2r^2 - e^2 \ln r - r^2 \ln r = 0$, $4 - 4 \ln r = 0$, then Γ_c is a circle of radius e as shown in Figure 2.2a.

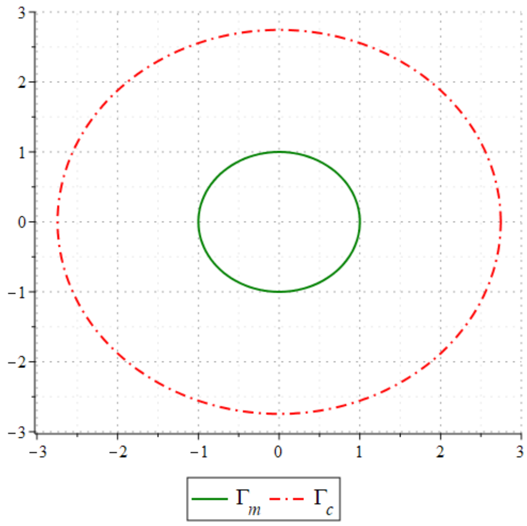
The equations (2.2.6) become $3r - \frac{e^2}{r} - 2r \ln r = 0$ and $4 - 4 \ln r = 0$. They have no common solutions.

Example 2.2.3. Let $u_0 = (\frac{21}{4} + 4\ln 2)\cos(\theta)$, $u_1 = (\frac{7}{4} + 4\ln 2)\cos(\theta)$, $u_2 = -6$, $u_3 = 10$. According to (2.2.26) we obtain: $c_{1,1} = 0$, $c_{1,2} = 1 + 4\ln 2$, $c_{1,3} = \frac{1}{4}$, $c_{1,4} = -4$. Therefore: $u = \left[(1 + 4\ln 2)r + \frac{r^3}{4} - 4r\ln r\right]\cos(\theta)$.

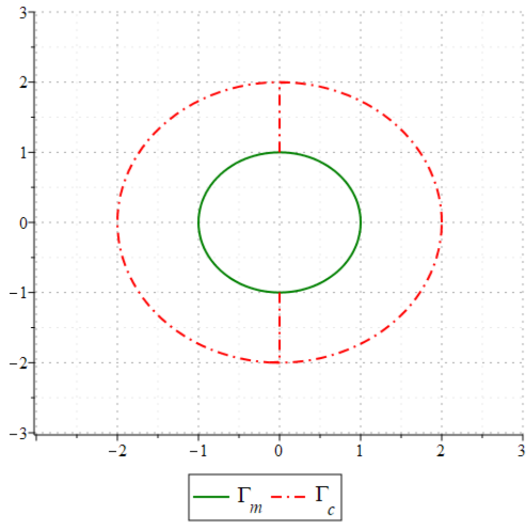
The equations (2.2.4), become $\left[(1 + 4\ln 2)r + \frac{r^3}{4} - 4r\ln r\right]\cos(\theta) = 0$ and $(-3 + 4\ln 2 + \frac{3r^2}{4} - 4\ln r)\cos(\theta) = 0$. They have no common solutions.

The equations (2.2.5), take the forms $\left[(1 + 4\ln 2)r + \frac{r^3}{4} - 4r\ln r\right]\cos(\theta) = 0$ and $(2r - \frac{8}{r})\cos\theta = 0$. They have no common solutions.

The equations (2.2.6) become $(-3 + 4\ln 2 + \frac{3r^2}{4} - 4\ln r)\cos(\theta) = 0$ and $(2r - \frac{8}{r})\cos\theta = 0$ then Γ_c is a circle of radius 2 union the dash-dotted line in the annular space as shown in Figure 2.2b.



(a) For the examples 2.2.1 and 2.2.2.



(b) For the example 2.2.3.

Figure 2.2: For examples (2.2.1)-(2.2.2)-(2.2.3). The geometric shape of the boundary Γ_c.

Chapter 3

Regularization Method of the Mixed Problem for the Harmonic

Introduction

In this chapter, we will be interested in an ill-posed inverse problem for the Laplace equation in an annulus domain, with Dirichlet-Neumann data on an internal boundary. The first section contains the problem formulation and modeling. In section two, the polar coordinates representation of the solution and some tools for our development will be presented. In section three, we examine the convergence of this solution and its stability with and without noise. In section four, the finite term truncation and the collocation method were considered to approximate the solution and allow us to construct our new regularized collocation Trefftz method (NRCTM). In addition, the conjugate gradient method was used to solve the linear system and determine the coefficients. Numerical examples are provided in section five to test our NRCTM and to compare it with the modified collocation Trefftz method (MCTM) to show its feasibility.

3.1 Modeling and problem formulation

The study of flow through an annular region defined by two coaxial pipes has found considerable practical application in many fields such as bio-medical, petroleum, aerospace and processing industries [93]. Here we consider a mathematical modeling of this problem and give an effective numerical algorithm for a method to detect the velocity in the annular space from a measurements of flow field on the accessible boundary.

Let $\Omega \subset \mathbb{R}^2$, be an annulus domain with boundary $\partial\Omega = \Gamma_m \cup \Gamma_c$, where Γ_0 and Γ_1 are a circles with a radius R_0 and R_1, respectively. By n we denote the outward unit normal to $\partial\Omega$, and consider $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ a solution to the following boundary value problem

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{3.1.1}$$

$$u(R_0, \theta) = u_0(\theta), \qquad 0 \le \theta \le 2\pi \tag{3.1.2}$$

$$\frac{\partial u}{\partial r}(R_0, \theta) = u_1(\theta), \qquad 0 \le \theta \le 2\pi \tag{3.1.3}$$

This Cauchy's problem provide a severely ill-posed problem in Hadamard sense, when the experimental measurements are partial available and a small perturbation of these measures in-

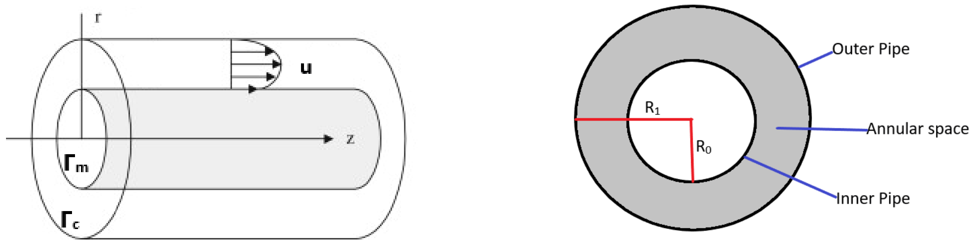


Figure 3.1: Schematic description of annular space bounded by concentric pipes (radius of the inner pipe: R_0 and radius of the outer pipe: R_1).

fluence the comportment of solution, and here the direct methods are very difficult to apply, in addition, it leads to very unstable solutions [85].

We can replace Eqs. (3.1.2) by the following boundary conditions [10, 84]

$$\begin{cases} u(R_0,\theta) = & u_0(\theta)\ , \quad 0 \le \theta \le 2\pi \\ u(R_1,\theta) = & g(\theta)\ , \quad 0 \le \theta \le 2\pi \end{cases} \tag{3.1.4}$$

where $g(\theta)$ is an unknown function assumed to be determined, then the Dirichlet data are completed on the whole boundary, in addition the solution of Laplace equation can be obtained in the whole domain. Therefore, we face the following inverse problem:
Inverse problem. Given the data Γ_0, u_0 and u_1, determine the function $g(\theta)$.

3.2 Preliminaries

As shown in chapter2, the numerical solution of the Harmonic equation in doubly-connected planar domain is given as:

$$u(r,\theta) = \sum_{n=0}^{\infty}(f_n(r)\cos(n\theta) + g_n(r)\sin(n\theta)) \tag{3.2.1}$$

where

$$\begin{aligned} f_0(r) &= c_{0,1} + c_{0,2}\ln(r), \qquad g_0(r) = 0 \\ f_n(r) &= c_{n,1}r^{-n} + c_{n,2}r^n, \qquad g_n(r) = d_{n,1}r^{-n} + d_{n,2}r^n,\ \ pour \quad n \ge 1 \end{aligned} \tag{3.2.2}$$

We assume that both functions $u_0(\theta)$ and $u_1(\theta)$ are L^2 integrable on the interval $[0, 2\pi]$. Hence, both of them admit development in terms of the Fourier expansion as:

$$\begin{aligned} u_0 &= A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta) \\ u_1 &= A'_0 + \sum_{n=1}^{\infty} A'_n\cos(n\theta) + B'_n\sin(n\theta) \end{aligned} \tag{3.2.3}$$

Therefore, the coefficients $c_{n,1}, c_{n,2}, d_{n,1}, d_{n,2}$ are uniquely determined [86] by matching the boundaries condition (3.2.3) as:

$$\begin{array}{llll} c_{0,1} = A_0 - A'_0 r_0 \ln r_0, & d_{0,1} = 0 & & \\ c_{0,2} = A'_0 r_0, & d_{0,2} = 0 & & \\ c_{n,1} = \frac{1}{2}A_n r_0^n - \frac{1}{2n}A'_n r_0^{n+1}, & d_{n,1} = \frac{1}{2}B_n r_0^n - \frac{1}{2n}B'_n r_0^{n+1} & n \ge 1 & . \\ c_{n,2} = \frac{1}{2}A_n r_0^{-n} + \frac{1}{2n}A'_n r_0^{1-n}, & d_{n,2} = \frac{1}{2}B_n r_0^{-n} + \frac{1}{2n}B'_n r_0^{1-n} & n \ge 1 & \end{array} \tag{3.2.4}$$

From the lemma 2.1.1 (in Chapter 2), then the numerical solution of problem (3.1.1)-(3.1.2)-(3.1.3) in region $R_0 \le r \le R_1$ can be obtained as:

$$\begin{aligned} u(r,\theta) &= A_0 + A_0' R_0 \ln(\frac{r}{R_0}) \\ &+ \sum_{n=1}^{\infty}\left[\frac{A_n}{2}\left((\frac{R_0}{r})^n + (\frac{r}{R_0})^n\right) + \frac{R_0 A_n'}{2n}\left((\frac{r}{R_0})^n - (\frac{R_0}{r})^n\right)\right]\cos(n\theta) \\ &+ \sum_{n=1}^{\infty}\left[\frac{B_n}{2}\left((\frac{R_0}{r})^n + (\frac{r}{R_0})^n\right) + \frac{R_0 B_n'}{2n}\left((\frac{r}{R_0})^n - (\frac{R_0}{r})^n\right)\right]\sin(n\theta) \end{aligned} \tag{3.2.5}$$

Definition 3.2.1. For $u(r,.) \in L^2(0,2\pi)$, $r \ge 0$ the following norm can be defined

$$\|u(r,.)\| = \|u(r,.)\|_{L^2(0,2\pi)} = \langle u,u\rangle^{\frac{1}{2}} = \left(\int_0^{2\pi} u(r,\theta)^2 d\theta\right)^{\frac{1}{2}} \tag{3.2.6}$$

Lemma 3.2.1. *For $n \in \mathbb{N}\setminus\{0\}$ the sequence*

$$\frac{1}{\sqrt{2\pi}},\quad \frac{\cos(n\theta)}{\sqrt{\pi}},\quad \frac{\sin(n\theta)}{\sqrt{\pi}},\ldots \tag{3.2.7}$$

is a Hilbert basis for $L^2(0,2\pi)$, and the function u given by (3.2.1) and verify:

$$\|u(r,.)\|^2 = 2\pi f_0(r)^2 + \pi\sum_{n=1}^{\infty}(f_n(r)^2 + g_n(r)^2) \tag{3.2.8}$$

Proof. for $n,m \in \mathbb{N}\setminus\{0\}$, using integration by Party therefore the following properties are satisfies

$$\int_0^{2\pi}\cos(n\theta)\sin(m\theta)d\theta = \int_0^{2\pi}\cos(m\theta)\sin(n\theta)d\theta = 0,$$
$$\int_0^{2\pi}\cos(n\theta)\cos(m\theta)d\theta = \int_0^{2\pi}\sin(m\theta)\sin(n\theta)d\theta = \begin{cases}0, & if \quad m \ne n\\ \pi, & if \quad m = n\end{cases}$$

that mean's if $m \ne n$

$$\langle\cos(n\theta),\sin(m\theta)\rangle = \langle\cos(m\theta),\sin(n\theta)\rangle = \langle\cos(n\theta),\cos(m\theta)\rangle = \langle\sin(m\theta),\sin(n\theta)\rangle = 0$$

if $m = n$

$$\langle\cos(n\theta),\cos(n\theta)\rangle = \langle\sin(n\theta),\sin(n\theta)\rangle = \pi$$

for $n = 0$ we have that $\langle 1,1\rangle = \int_0^{2\pi} d\theta = 2\pi$, then for $n \in \mathbb{N}$ the sequence (3.2.7) is a Hilbert basis for $L^2(0,2\pi)$.

Secondly, we have that

$$\begin{aligned} \|u(r,.)\|^2 &= \langle\sum_{n=0}^{\infty}(f_n(r)\cos(n\theta) + g_n(r)\sin(n\theta)), \sum_{m=0}^{\infty}(f_m(r)\cos(m\theta) + g_m(r)\sin(m\theta))\rangle \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle f_n(r)\cos(n\theta) + g_n(r)\sin(n\theta), f_m(r)\cos(m\theta) + g_m(r)\sin(m\theta)\rangle \\ &= \sum_{n=0}^{\infty} f_n(r)^2\langle\cos(n\theta),\cos(n\theta)\rangle + g_n(r)^2\langle\sin(n\theta),\sin(n\theta)\rangle \\ &= 2\pi f_0(r)^2 + \pi\sum_{n=1}^{\infty}(f_n(r)^2 + g_n(r)^2) \end{aligned}$$

□

Example of instability

Example 3.2.1. From to [23, 40] the classic example of an ill-posed problem given by Hadamard is an initial value problem for the Laplace equation.

In our situation, take $R_0 = 1$, $R_1 = 2$, and the Fourier series associated to u_0 and u_1 are given as:

$$u_0(\theta) = \sum_{n\geq 1} A_n \cos(n\theta), \quad u_1 = 0.$$

From (3.2.4) then the solution u associated to the data u_0 and u_1 is given as:

$$u(r, \theta) = \sum_{n\geq 1} \frac{1}{2} A_n \left(r^{-n} + r^n\right) \cos(n\theta), \quad 1 \leq r \leq 2$$

Let the Fourier coefficients A_k of the function u_0 have small perturbations: $\tilde{A}_n = A_n + \frac{\delta}{n}$, we consider the norm

$$\|\tilde{u}(r, .) - u(r, .)\|^2 = \delta \sum_{n\geq 1} \frac{\pi}{4n^2} \left(r^{-n} + r^n\right)^2, \quad 1 \leq r \leq 2 \tag{3.2.9}$$

Note that for $r = 2$ the series (3.2.9) take the form $\delta \sum_{n\geq 1} \frac{\left(e^{-n\ln 2} + e^{n\ln 2}\right)}{4n^2}$, which is diverge.

3.3 Regularization method and convergence estimates

3.3.1 Regularized problem

We define the regularized problem associated to the ill-posed problem (3.1.1)-(3.1.2)-(3.1.3) with $\alpha > 0$ is the regularization parameter

$$\begin{cases} \Delta u_\alpha = 0, & in\ \Omega \\ u_\alpha = u_0, & on\ \Gamma_m \\ \frac{\partial u_\alpha}{\partial n} = u_1, & on\ \Gamma_m \end{cases} \tag{3.3.1}$$

The main idea of the regularization problem is to approximate the considered ill-posed problem by a family of well-posed problems depending on a (small) regularization parameter [2]. As similar we consider the regularized problem associated to the ill-posed problem (3.1.1)-(3.1.2)-(3.1.3) with $\alpha > 0$ is the regularization parameter and δ is the noisy level resulting from the measurements of the given data u_0, u_1, with condition [4]

$$\|u_0^\delta - u_0\| \leq \delta, \qquad \|u_1^\delta - u_1\| \leq \delta \tag{3.3.2}$$

In what follow we consider that u is the exact solution obtained from the exact data u_0, u_1, and u^δ the approximate solution obtained from the noisy data u_0^δ, u_1^δ. We note by u_α is the regularized solution associated to u_0, u_1 and u_α^δ is the regularized solution associated to u_0^δ, u_1^δ , thus the following regularized problem can be defined as:

$$\Delta u_\alpha^\delta = 0, \tag{3.3.3}$$

$$u_\alpha^\delta = u_0^\delta, \quad 0 \leq \theta \leq 2\pi \tag{3.3.4}$$

$$\frac{\partial u_\alpha^\delta}{\partial n} = u_1^\delta, \quad 0 \leq \theta \leq 2\pi \tag{3.3.5}$$

From (3.2.3) and the property (3.3.2) then u_0^δ, u_1^δ are L^2 integrable on the interval $[0, 2\pi]$, and both have a Fourier expansion as:

$$\begin{aligned} u_0^\delta &= A_0^\delta + \sum_{n=1}^{\infty} A_n^\delta \cos(n\theta) + B_n^\delta \sin(n\theta) \\ u_1^\delta &= A_0'^\delta + \sum_{n=1}^{\infty} A_n'^\delta \cos(n\theta) + B_n'^\delta \sin(n\theta) \end{aligned} \tag{3.3.6}$$

Definition 3.3.1. Define the suite of functions $q_n(\alpha, \mu)$, for $\alpha > 0$ and $0 < \mu < \infty$ defined as:

$$q_n(\alpha, \mu) = \frac{\mu^{-2n}}{\alpha + \mu^{-2n}}, \ \forall n \in \mathbb{N} \tag{3.3.7}$$

which satisfy the following properties [4, 32, 40, 41]

(a) $q_n(\alpha, \mu) \leq 1$ and $q_n(\alpha, \mu) \longrightarrow 1$, *where* $\alpha \longrightarrow 0$.

(b) $q_n(\alpha, \mu) \leq \frac{1}{2\sqrt{\alpha}}\mu^{-n}$

(c) $|q_n(\alpha, \mu) - 1| \leq \frac{\sqrt{\alpha}}{2}\mu^n$.

Regularized scheme

The properties of function (3.3.7) allow to insert it legibly into (3.2.1) for the parameter $\alpha > 0$, here μ is considered as a damping factor for the expression 3.3.8 which can be chosen by trial under the condition $\mu \geq \frac{R_1}{R_0}$. Thus, a new regularized scheme associated to the noisy data u_0^δ, u_1^δ can be constructed as:

$$u_{\alpha,\mu}^\delta(r, \theta) = \sum_{n=0}^{\infty} \frac{\mu^{-2n}}{\alpha + \mu^{-2n}} (f_n^\delta(r) \cos(n\theta) + g_n^\delta(r) \sin(n\theta)) \tag{3.3.8}$$

where

$$\begin{aligned} &f_0^\delta(r) = c_{0,1}^\delta + c_{0,2}^\delta \ln(r), \quad g_0^\delta(r) = 0 \\ &f_n^\delta(r) = c_{n,1}^\delta r^{-n} + c_{n,2}^\delta r^n, \quad g_n^\delta(r) = d_{n,1}^\delta r^{-n} + d_{n,2}^\delta r^n, \quad \textit{pour } n \geq 1 \end{aligned} \tag{3.3.9}$$

Remark 3.3.1. It can be seen that $\Delta(f_n(r)\cos(n\theta)) = \Delta(g_n(r)\sin(n\theta)) = 0$, for $n \in \mathbb{N}$. Therefore $\Delta u_{\alpha,\mu} = 0$, thus $u_{\alpha,\mu}$ provide a solution to the equation (3.3.3) and the following convergence can be obtained

$$\begin{aligned} u_{\alpha,\mu}(r, \theta) &\longrightarrow u(r, \theta)\ , \qquad \textit{where } \alpha \longrightarrow 0, \\ \frac{\partial u_{\alpha,\mu}}{\partial n}(r, \theta) &\longrightarrow \frac{\partial u}{\partial n}(r, \theta)\ , \quad \textit{where } \alpha \longrightarrow 0. \end{aligned}$$

3.3.2 Convergence estimates under exact data

Lemma 3.3.1. *Let* $u_0, u_1 \in L^2(\Gamma_0)$*, for* $\mu \geq \frac{R_1}{R_0}$, $\alpha > 0$ *then* $\|u_{\alpha,\mu}(r, .)\|$ *is bounded.*

Proof. Let $u_{\alpha,\mu}(r,\theta)$ given by (3.3.8), for $R_0 \leq r \leq R_1$, then from (3.2.8) we have that

$$\begin{aligned}
\|u_{\alpha,\mu}(r,.)\|^2 &= 2\pi q_0(\alpha,\mu)^2 f_0(r)^2 + \pi \sum_{n\geq 1} q_n(\alpha,\mu)^2 \left(f_n(r)^2 + g_n(r)^2\right) \\
&\leq \frac{\pi}{2\alpha} f_0(r)^2 + \frac{\pi}{4\alpha} \sum_{n\geq 1} \mu^{-2n} \left(f_n(r)^2 + g_n(r)^2\right) \\
&= \frac{\pi}{2\alpha} \left(c_{0,1} + c_{0,2}\ln(r)\right)^2 \\
&\quad + \frac{\pi}{4\alpha} \sum_{n\geq 1} \mu^{-2n} \left((c_{n,1}r^{-n} + c_{n,2}r^n)^2 + (d_{n,1}r^{-n} + d_{n,2}r^n)^2\right) \\
&\leq \frac{\pi}{\alpha}\left(A_0^2 + A_0'^2 R_0^2 \left(\ln \frac{r}{R_0}\right)^2\right) \\
&\quad + \frac{\pi}{4\alpha} \sum_{n=1}^{\infty} \frac{R_0^{2n}}{R_1^{2n}} \left[A_n^2 \left((\frac{R_0}{r})^{2n} + (\frac{r}{R_0})^{2n}\right) + \frac{R_0^2 A_n'^2}{n^2}\left((\frac{r}{R_0})^{2n} + (\frac{R_0}{r})^{2n}\right)\right] \\
&\quad + \frac{\pi}{4\alpha} \sum_{n=1}^{\infty} \frac{R_0^{2n}}{R_1^{2n}} \left[B_n^2 \left((\frac{R_0}{r})^{2n} + (\frac{r}{R_0})^{2n}\right) + \frac{R_0^2 B_n'^2}{n^2}\left((\frac{r}{R_0})^{2n} + (\frac{R_0}{r})^{2n}\right)\right] \\
&\leq \frac{\pi}{\alpha}\left(A_0^2 + A_0'^2 R_0^2 \left(\ln \frac{R_1}{R_0}\right)^2\right) + \frac{\pi}{2\alpha}\sum_{n\geq 1}\left(A_n^2 + B_n^2 + R_0^2 A_n'^2 + R_0^2 B_n'^2\right) \\
&\leq \frac{2\pi}{\alpha} C_0^2 (A_0^2 + A_0'^2) + \frac{\pi}{\alpha} C_1^2 \sum_{n\geq 1}\left(A_n^2 + B_n^2 + A_n'^2 + B_n'^2\right)
\end{aligned}$$

where $C_0^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\left(\ln \frac{R_1}{R_0}\right)^2\right\}$ and $C_1^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\right\}$ are positive constants. By taking $C_2^2 = \max\left\{C_0^2, C_1^2\right\}$ and from the Bessel's inequality (see 1.2.2 Chapter 1) one obtain that

$$\|u_{\alpha,\mu}(r,.)\| \leq \frac{C_2}{\sqrt{\alpha}}\left(\|u_0\|^2 + \|u_1\|^2\right)^{\frac{1}{2}}$$

□

Lemma 3.3.2. *If $\|u(r,.)\|$ is bounded and $\mu \geq \frac{R_1}{R_0}$, then $u_{\alpha,\mu}$ converge to u as α tends to zero.*

Proof. We have that $\lim_{\alpha\to 0} q_n(.,.) = 1$, therefore $\lim_{\alpha\to 0} u_{\alpha,\mu} = u$. Using the limit definition then:

$\forall\epsilon > 0, \exists\ \alpha_0 > 0,\ for: 0 < \alpha \leq \alpha_0, we\ have\ that: |q_n(\alpha,\mu) - 1|^2 < \epsilon$. We can choose that $\epsilon = \frac{\sqrt{\alpha}}{\mu^n}$, where there exists $\alpha_0(n)$, such as $0 < \alpha \leq \alpha_0(n)$, for $\alpha_0(n) \leq \sup_{n\in\mathbb{N}} \alpha_0(n) = \alpha_0$, then from the lemma 3.3.1 we have the estimation

$$\begin{aligned}
\|u_{\alpha,\mu}(r,.) - u(r,.)\|^2 &= 2\pi\left(q_0(\alpha,\mu) - 1\right)^2 f_0(r)^2 + \pi \sum_{n\geq 0}\left(q_n(\alpha,\mu) - 1\right)^2 \left(f_n(r)^2 + g_n(r)^2\right) \\
&\leq 2\pi\alpha C_0^2 (A_0^2 + A_0'^2) + \pi\alpha C_1^2 \sum_{n\geq 1}\left(A_n^2 + B_n^2 + A_n'^2 + B_n'^2\right)
\end{aligned}$$

where $C_0^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\left(\ln \frac{R_1}{R_0}\right)^2\right\}$ and $C_1^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\right\}$ are positive constants. By taking $C_2^2 = \max\left\{C_0^2, C_1^2\right\}$ we obtain that

$$\|u_{\alpha,\mu}(r,.) - u(r,.)\| \leq C_2\sqrt{\alpha}\left(\|u_0\|^2 + \|u_1\|^2\right)^{\frac{1}{2}} \longrightarrow 0,\ where\ \alpha \longrightarrow 0$$

□

Lemma 3.3.3. *Let $\{u_0, u_1\} \in L^2(\Gamma_0)$, the following normes $\|u_{\alpha,\mu}(R_0,.) - u_0\|_{L^2(\Gamma_0)}$, $\|\frac{\partial u_{\alpha,\mu}}{\partial n}(R_0,.) - u_1\|_{L^2(\Gamma_0)}$ converges to zero when $\alpha \to 0$.*

Proof. The effect that $\mu \geq \frac{R_1}{R_0}$, from the property (3.2.8) we obtain that

$$\begin{aligned}
\|u_{\alpha,\mu}(R_0,.) - u_0\|^2 &= \|\sum_{n\geq 0} q_n(\alpha,\mu)[f_n(R_0)\cos(n\theta) + g_n(R_0)\sin(n\theta)] - u_0\|^2 \\
&= \|\sum_{n\geq 0} [q_n(\alpha,\mu) - 1]\,[f_n(R_0)\cos(n\theta) + g_n(R_0)\sin(n\theta)]\|^2 \\
&= 2\pi\,(q_0(\alpha,\mu) - 1)^2 f_0(R_0)^2 + \pi \sum_{n\geq 0} (q_n(\alpha,\mu) - 1)^2 \left(f_n(R_0)^2 + g_n(R_0)^2\right) \\
&\leq \frac{\alpha}{4}\left[2\pi f_0(R_0)^2 + \pi \sum_{n\geq 0}\left(f_n(R_0)^2 + g_n(R_0)^2\right)\right] \\
&\leq \frac{\alpha}{4}\|u_0\|^2 \to 0, \quad \text{as,} \quad \alpha \to 0.
\end{aligned}$$

By the same way one obtain that:

$$\begin{aligned}
\|\frac{\partial u_{\alpha,\mu}}{\partial n}(R_0,.) - u_1\|^2 =& \|\sum_{n\geq 0} q_n(\alpha,\mu)\left[\frac{\partial f_n}{\partial r}(R_0)\cos(n\theta) + \frac{\partial g_n}{\partial r}(R_0)\sin(n\theta)\right] - u_1\|^2 \\
\leq& \frac{\alpha}{4}\|u_1\|^2 \to 0, \quad \text{as,} \quad \alpha \to 0.
\end{aligned}$$

□

3.3.3 Error estimate under noisy data

Lemma 3.3.4 (Uniqueness of solution)**.** *For all noisy data $\left\{u_0^\delta, u_1^\delta\right\} \in L^2(\Gamma_0)$ the function given by (3.3.8) is the unique solution to the problem (3.3.3)-(3.3.4)-(3.3.5) and it's continuously dependent on u_0^δ, u_1^δ.*

Proof. Let $u^\delta_{\alpha_1}, u^\delta_{\alpha_2}$ be two solutions to the problem (3.3.3)-(3.3.4)-(3.3.5), which is corresponding to the given data (u_0^δ, u_1^δ), (v_0^δ, v_1^δ), respectively, given as

$$\begin{aligned}
u^\delta_{\alpha,\mu}(r,\theta) &= \sum_{n=0}^{\infty} \frac{\mu^{-2n}}{\alpha + \mu^{-2n}}(f_n^\delta(r)\cos(n\theta) + g_n^\delta(r)\sin(n\theta)) \\
v^\delta_{\alpha,\mu}(r,\theta) &= \sum_{n=0}^{\infty} \frac{\mu^{-2n}}{\alpha + \mu^{-2n}}(\varphi_n^\delta(r)\cos(n\theta) + \psi_n^\delta(r)\sin(n\theta))
\end{aligned}$$

where u_0^δ, u_1^δ, v_0^δ, v_1^δ are assumed to have Fourier expansions as:

$$\begin{aligned}
u_0^\delta &= A_0^\delta + \sum_{n=1}^{\infty} A_n^\delta \cos(n\theta) + B_n^\delta \sin(n\theta) \\
u_1^\delta &= A_0'^\delta + \sum_{n=1}^{\infty} A_n'^\delta \cos(n\theta) + B_n'^\delta \sin(n\theta) \\
v_0^\delta &= E_0^\delta + \sum_{n=1}^{\infty} E_n^\delta \cos(n\theta) + F_n^\delta \sin(n\theta) \\
v_1^\delta &= E_0'^\delta + \sum_{n=1}^{\infty} E_n'^\delta \cos(n\theta) + F_n'^\delta \sin(n\theta)
\end{aligned}$$

and

$$\begin{aligned}
f_0^\delta(r) &= c_{0,1}^\delta + c_{0,2}^\delta \ln(r), \quad g_0^\delta(r) = 0 \\
f_n^\delta(r) &= c_{n,1}^\delta r^{-n} + c_{n,2}^\delta r^n, \quad g_n^\delta(r) = d_{n,1}^\delta r^{-n} + d_{n,2}^\delta r^n, \qquad \textit{pour } n \geq 1 \\
\varphi_0^\delta(r) &= y_{0,1}^\delta + y_{0,2}^\delta \ln(r), \quad \psi_0^\delta(r) = 0 \\
\varphi_n^\delta(r) &= y_{n,1}^\delta r^{-n} + y_{n,2}^\delta r^n, \quad \psi_n^\delta(r) = z_{n,1} r^{-n} + z_{n,2} r^n, \qquad \textit{pour } n \geq 1
\end{aligned}$$

By substituting the coefficients as in the expression (3.2.5). For $\mu \geq \frac{R_1}{R_0}$ and from the lemma 3.3.1 one obtain that:

$$\begin{aligned}
\|u_{\alpha_1}^\delta(r,\theta) - u_{\alpha_2}^\delta(r,\theta)\|^2 &= \|\sum_{n\geq 0} q_n(\alpha,\mu)[(f_n^\delta(r) - \varphi_n^\delta(r))\cos(n\theta) + (g_n^\delta(r) - \psi_n^\delta(r))\sin(n\theta)]\|^2 \\
&\leq \frac{\pi}{2\alpha}(f_0^\delta(r) - \varphi_0^\delta(r))^2 + \frac{\pi}{4\alpha}\sum_{n\geq 1}\mu^{-2n}\left((f_n^\delta(r) - \varphi_n^\delta(r))^2 + (g_n^\delta(r) - \psi_n^\delta(r))^2\right) \\
&\leq \frac{C_2^2}{\alpha}(\|v_0^\delta - u_0^\delta\|^2 + \|v_1^\delta - u_1^\delta\|^2) \\
&\leq \frac{2^3 C_2^2 \delta^2}{\alpha} \to 0, \quad \text{when}, \quad \delta \to 0
\end{aligned}$$

where $C_0^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\left(\ln\frac{R_1}{R_0}\right)^2\right\}$, $C_1^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\right\}$ and $C_2^2 = \max\left\{C_0^2, C_1^2\right\}$ are a positive constants. □

Theorem 3.3.1. *If $\|u(r,.)\|$ is bounded and $\mu \geq \frac{R_1}{R_0}$, then $u_{\alpha,\mu}^\delta$ converge to u as δ tends to zero.*

Proof. We have that

$$\begin{aligned}
\|u_{\alpha,\mu}^\delta(r,\theta) - u(r,\theta)\| &\leq \|u_{\alpha,\mu}^\delta(r,\theta) - u_{\alpha,\mu}(r,\theta)\| + \|u_{\alpha,\mu}(r,\theta) - u(r,\theta)\| \\
&\leq \|u_{\alpha,\mu}^\delta(r,\theta) - u_{\alpha,\mu}(r,\theta)\| + C_2\sqrt{\alpha}\left(\|u_0\|^2 + \|u_1\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

where $C_0^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\left(\ln\frac{R_1}{R_0}\right)^2\right\}$, $C_1^2 = \max\left\{\frac{1}{2}, \frac{R_0^2}{2}\right\}$ and $C_2^2 = \max\left\{C_0^2, C_1^2\right\}$ are a positive constants given in lemma 3.3.1. Frome the property (4.5.15) and the lemma 3.3.4, one can obtain that:

$$\begin{aligned}
\|u_{\alpha,\mu}^\delta(r,\theta) - u_{\alpha,\mu}(r,\theta)\|^2 &= \|\sum_{n\geq 0} q_n(\alpha,\mu)\left[(f_n^\delta(r) - f_n(r))\cos(n\theta) + (g_n^\delta(r) - g_n(r))\sin(n\theta)\right]\|^2 \\
&\leq \frac{\pi}{2\alpha}(f_0^\delta(r) - f_0(r))^2 + \frac{\pi}{4\alpha}\sum_{n\geq 1}\mu^{-2n}\left((f_n^\delta(r) - f_n(r))^2 + (g_n^\delta(r) - g_n(r))^2\right) \\
&\leq \frac{C_2^2}{\alpha}(\|u_0^\delta - u_0\|^2 + \|u_1^\delta - u_1\|^2) \\
&\leq \frac{2C_2^2\delta^2}{\alpha}
\end{aligned}$$

Then

$$\|u_{\alpha,\mu}^\delta(r,\theta) - u_{\alpha,\mu}(r,\theta)\| \leq \frac{\sqrt{2}C_2}{\sqrt{\alpha}}\delta$$

Therefore

$$\|u_{\alpha,\mu}^\delta(r,\theta) - u(r,\theta)\| \leq \frac{\sqrt{2}C_2}{\sqrt{\alpha}}\delta + C_2\sqrt{\alpha}\left(\|u_0\|^2 + \|u_1\|^2\right)^{\frac{1}{2}}$$

take $\alpha(\delta) = C_3\delta$, where C_3 is a constant to be determined, then we obtain that

$$\|u_{\alpha,\mu}^\delta(r,\theta) - u(r,\theta)\| \leq C_2\left(\sqrt{C_3}\left(\|u_0\|^2 + \|u_1\|^2\right)^{\frac{1}{2}} + \frac{\sqrt{2}}{\sqrt{C_3}}\right)\sqrt{\delta} \to 0, \ as\ \delta \to 0.$$

□

3.4 Collocation method

We are already acquainted with the Trefftz method and the modified Trefftz method presented in 1.4.1 of Chapter 1. Here, we are interested in the numerical solution for the two-dimensional Laplace equation in doubly-connected planar domain which is expressed by linear summation of the following bases [79, 81, 82].

$$\{1, \ln r, r^{\pm n}\cos(n\theta), r^{\pm n}\sin(n\theta), n = 1, 2...\} \tag{3.4.1}$$

In, [9, 10, 83], the modified T-complete functions in (3.4.1) is defined by considering the characteristic length of the computational domain to stabilize the numerical scheme as:

$$\left\{1, \ln r, \left(\frac{r}{R_1}\right)^n \cos(n\theta), \left(\frac{R_0}{r}\right)^n \cos(n\theta), \left(\frac{r}{R_1}\right)^n \sin(n\theta), \left(\frac{R_0}{r}\right)^n \sin(n\theta), n = 1, 2...\right\} \tag{3.4.2}$$

Our starting point in Eq. (3.2.2) by inserting the damping function noted by $q_n(\alpha, \mu)$ for the regularization parameter α and the damping factor $\mu \geq \frac{R_1}{R_0}$. Therefore the new set of T-complete bases is taken as

$$\{q_0(\alpha, \mu), q_0(\alpha, \mu)\ln r, q_n(\alpha, \mu)r^{\pm n}\cos(n\theta), q_n(\alpha, \mu)r^{\pm n}\sin(n\theta), n = 1, 2...\} \tag{3.4.3}$$

In the NRCTM, we approximate the regularized scheme (3.3.8) by a linear combination of T-complete functions (3.4.3) given on the form of admissible functions in finite term with regularized parameter α and a damping factor μ as:

$$u_{\alpha,\mu}(r, \theta) = \sum_{n=0}^{m} q_n(\alpha, \mu)(f_n(r)\cos(n\theta) + g_n(r)\sin(n\theta)) \tag{3.4.4}$$

It is known that the collocation method has a great advantage to apply on different geometric shapes, and the simplicity for computer programming. In order to apply the collocation method, we define θ_i as the collocated points on Γ_0 given by

$$\theta_i = ih, \qquad for\ i = 0, ..., m, \qquad and, \qquad h = \frac{2\pi}{m+1}. \tag{3.4.5}$$

In Eq. (3.4.4) there are $4m+2$ unknown coefficients, which can be obtained by imposing the different collocated points (3.4.5) in (3.4.4) for $i = 1, ..., 2m+1$, and by matching the boundary conditions (3.1.2)-(3.1.3) one can obtain that:

$$\sum_{n=0}^{m} q_n(\alpha, \mu)(f_n(R_0)\cos(n\theta_i) + g_n(R_0)\sin(n\theta_i)) = u_0(\theta_i) \tag{3.4.6}$$

$$\sum_{n=0}^{m} q_n(\alpha, \mu)(f_n'(R_0)\cos(n\theta_i) + g_n'(R_0)\sin(n\theta_i)) = u_1(\theta_i) \tag{3.4.7}$$

we obtain a linear equations system with dimensions $n = 4m + 2$ denoted by:

$$Ax = b \tag{3.4.8}$$

where the matrix $A \in \mathbb{R}^{4m+2} \times \mathbb{R}^{4m+2}$ is given by

$$\begin{aligned}
&\left[\begin{matrix}
q_0 & q_0 \ln(R_0) & q_1 R_0 \cos\theta_0 & q_1 R_0^{-1} \cos\theta_0 & q_1 R_0 \sin\theta_0 & q_1 R_0^{-1} \sin\theta_0 & \cdots \\
0 & \frac{q_0}{R_0} & q_1 \cos\theta_0 & -q_1 R_0^{-2} \cos\theta_0 & q_1 \sin\theta_0 & -q_1 R_0^{-2} \sin\theta_0 & \cdots \\
q_0 & q_0 \ln(R_0) & q_1 R_0 \cos\theta_1 & q_1 R_0^{-1} \cos\theta_1 & q_1 R_0 \sin\theta_1 & q_1 R_0^{-1} \sin\theta_1 & \cdots \\
0 & \frac{q_0}{R_0} & q_1 \cos\theta_1 & -q_1 R_0^{-2} \cos\theta_1 & q_1 \sin\theta_1 & -q_1 R_0^{-2} \sin\theta_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
q_0 & q_0 \ln(R_0) & q_1 R_0 \cos\theta_m & q_1 R_0^{-1} \cos\theta_m & q_1 R_0 \sin\theta_m & q_1 R_0^{-1} \sin\theta_m & \cdots \\
0 & \frac{q_0}{R_0} & q_1 \cos\theta_m & -q_1 R_0^{-2} \cos\theta_m & q_1 \sin\theta_m & -q_1 R_0^{-2} \sin\theta_m & \cdots
\end{matrix}\right. \\
&\left.\begin{matrix}
\cdots & q_m R_0^{m} \cos\theta_0 & q_m R_0^{-m} \cos\theta_0 & q_m R_0^{m} \sin\theta_0 & q_m R_0^{-m} \sin\theta_0 \\
\cdots & m q_m R_0^{m-1} \cos\theta_0 & -m q_m R_0^{-m-1} \cos\theta_0 & m q_m R_0^{m-1} \sin\theta_0 & -m q_m R_0^{-m-1} \sin\theta_0 \\
\cdots & q_m R_0^{m} \cos\theta_1 & q_1 R_0^{-m} \cos\theta_1 & q_1 R_0^{m} \sin\theta_1 & q_1 R_0^{-m} \sin\theta_1 \\
\cdots & m q_m R_0^{m-1} \cos\theta_1 & -m q_m R_0^{-m-1} \cos\theta_1 & m q_m R_0^{m-1} \sin\theta_1 & -m q_m R_0^{-m-1} \sin\theta_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & q_m R_0^{m} \cos\theta_m & q_m R_0^{-m} \cos\theta_m & q_m R_0^{m} \sin\theta_m & q_m R_0^{-m} \sin\theta_m \\
\cdots & m q_m R_0^{m-1} \cos\theta_m & -m q_m R_0^{-m-1} \cos\theta_m & m q_m R_0^{m-1} \sin\theta_m & -m q_m R_0^{-m-1} \sin\theta_m
\end{matrix}\right]
\end{aligned} \tag{3.4.9}$$

and

$$x = [c_{0,1}, c_{0,2}, c_{1,1}, c_{1,2}, d_{1,1}, d_{1,2}, ..., c_{m,1}, c_{m,2}, d_{m,1}, d_{m,2}] \in \mathbb{R}^{4m+2} \tag{3.4.10}$$

$$b = [u_0(\theta_0), u_1(\theta_0), u_0(\theta_1), u_1(\theta_1), ..., u_0(\theta_{2m}), u_0(\theta_{2m})] \in \mathbb{R}^{4m+2} \tag{3.4.11}$$

The conjugate gradient method can be used to solve the following normal equation:

$$A^\top A x = A^\top b \tag{3.4.12}$$

By inserting the calculated x into Eq. (3.4.8), thus we find a semi-analytical solution for $u_\alpha^\delta(r, \theta)$ as:

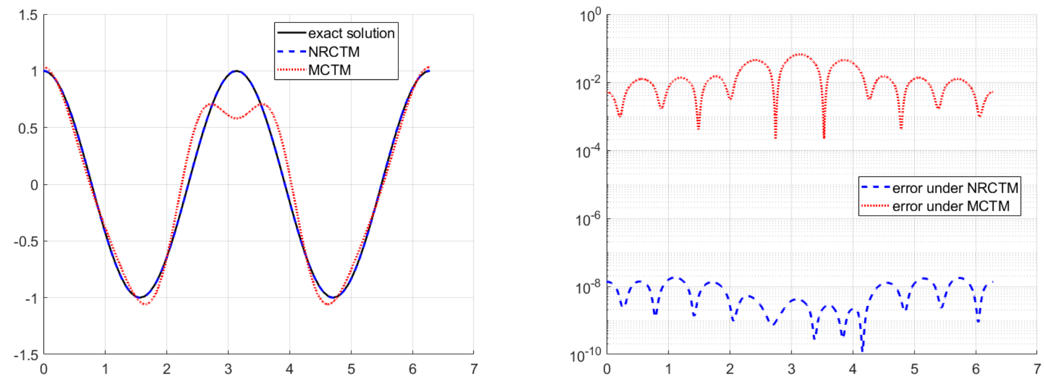
$$\begin{aligned}
u_{\alpha,\mu}^\delta(r, \theta) = q_0(\alpha, \mu)(x_0 + x_1 \ln(r)) &+ \sum_{n=1}^{m} q_n(\alpha, \mu)\left(x_{4n-2} r^{-n} + x_{4n-1} r^n\right) \cos(n\theta) \\
&+ \sum_{n=1}^{m} q_n(\alpha, \mu)\left(x_{4n} r^{-n} + x_{4n+1} r^n\right) \sin(n\theta)
\end{aligned} \tag{3.4.13}$$

where $x = (x_0, x_1, x_2, x_3, x_4, ..., x_{4m+1}) \in \mathbb{R}^{4m+2}$ are the components of x.

3.5 Numerical tests

In order to test numerical stability of the NRCTM, the parameter α and the damping factor $\mu \geq \frac{R_1}{R_0}$ are chosen by trial and error. In the following examples we give a simple exact solution u for the equation (3.1.1) and we verify the boundary condition (3.1.2)-(3.1.3), we consider the approximate solution which corresponds to the data with and without noise (see section 3.4), we show the comparison between this one with the MCTM. Here we consider an annulus domain with radius R_0 and R_1. The solution of Eq. (3.4.12) is obtained under a stopping criterion 10^{-15}. The noisy data (4.5.15) has been generated with noise added to the Dirichlet-Neumann data in the form.

$$u_0^\delta = u_0 + \epsilon \frac{||u_0||_{L^2}}{||\xi||_{L^2}} \xi, \qquad u_1^\delta = u_1 + \epsilon \frac{||u_1||_{L^2}}{||\xi||_{L^2}} \xi \tag{3.5.1}$$



(a) (b)

Figure 3.2: For Example 3.5.1. Comparing the exact solution and numerical solutions without noise, in (3.2a), and the numerical errors are plotted in (3.2b). For $m = 40$, $\alpha = 0.1$.

where ξ is a normally distributed random variable and ϵ is the relative noisy level.

In our algorithm, we note with $\mathbf{Error_u}$ the evaluate relative error between the exact solution u, and its computed approximations $u^{\delta}_{\alpha,\mu}$, and can be given in the sense of Root Mean Square Error (RMSE) by

$$\mathbf{Error}_u = \frac{\|u^{\delta}_{\alpha,\mu} - u\|_{L^2}}{\|u\|_{L^2}} \tag{3.5.2}$$

with $R_0 \leq r \leq R_1$, and using the collocate points θ_i given in (3.4.5).

Example 3.5.1. We start with a simple example of an annulus defined by the radius $R_0 = \frac{1}{2}$, $R_1 = 1$, and we consider the exact solution given by

$$u(r, \theta) = x^2 - y^2 = r^2 \cos 2\theta \tag{3.5.3}$$

Therefore, the data on the circle with a radius $R_0 = \frac{1}{2}$ are given by

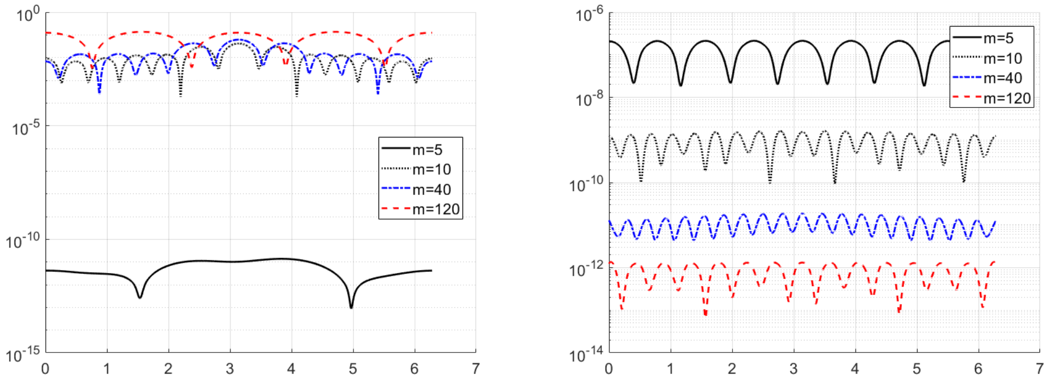
$$u_0(\theta) = R_0^2 \cos 2\theta, \qquad 0 \leq \theta \leq 2\pi \tag{3.5.4}$$

$$u_1(\theta) = 2R_0 \cos 2\theta, \qquad 0 \leq \theta \leq 2\pi \tag{3.5.5}$$

We solve this problem by the new regularized collocation Trefftz method NRCTM presented in Section 3.4, whose the results along a unit circle $r = 1$ are shown by chosen the damping factor as $\mu = 2$.

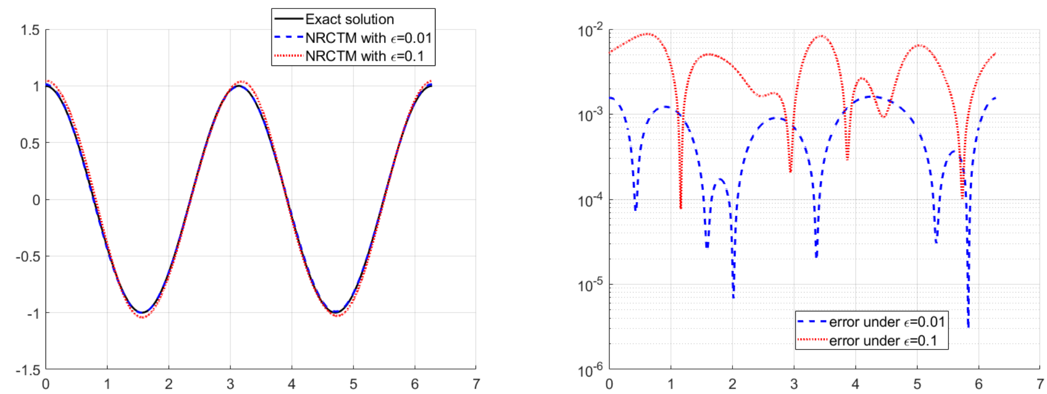
In Fig. 3.2a we show the comparisons between the exact solution u, and its computed approximations by using the MCTM and the NRCTM, without noise , the errors was ploted in 3.2b, respectively with regularization parameter chosen by the trial as $\alpha = 0.1$ for $m = 40$.

We can compare the NRCTM and MCTM on this example with very high accuracy, as shown in fig 3.3b and 3.3a, respectively, for the cases $m = 5; 10; 40; 120$ with a regularized parameter chosen by trial as $\alpha = 10^{-2}$, 10^{-1}, 10^{-1}, 10^{-1}, respectively. It can be seen that by the NRCTM the error decreases when m increases, but this is contrary by the MCTM.



(a) Errors under MCTM (b) Errors under NRCTM

Figure 3.3: Plotting the numerical errors for example 3.5.1.



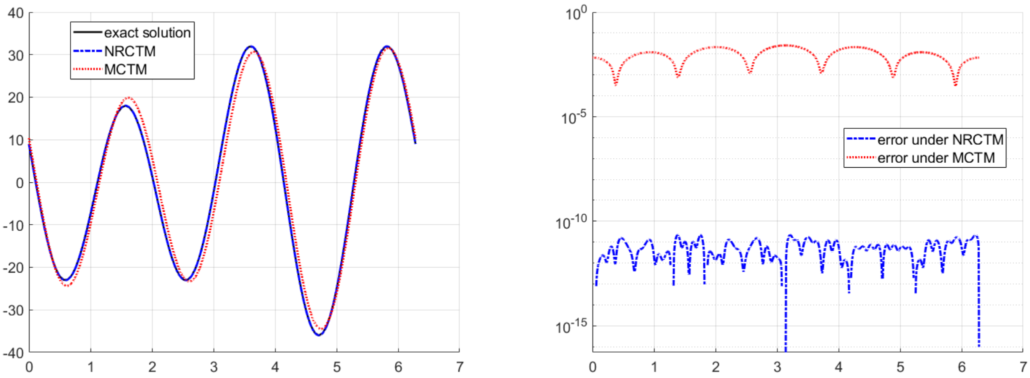
(a) (b)

Figure 3.4: For example 3.5.1. Exact solution and numerical solution by NRCTM with noises in (3.4a), the numerical errors are plotted in (3.4b). For $m = 70$, $\alpha(0.01) = 10^{-6}$ and $\alpha(0.1) = 10^{-5}$

In Fig. 3.4a we compare the exact solution with the numerical solutions by using the NRCTM under the noises $\epsilon = 0.01$ and 0.1, the corresponding errors was plotted in 3.4b, the regularization parameter was chosen by trial and error by $\alpha = 10^{-6}$ and 10^{-5} respectively, with truncation number $m = 70$. It can be seen that the numerical solutions are close to the exact solution, which indicates that the present method is robust against the noise, and even whose level was taken up to 1% (0.01) and 10% (0.1), the numerical error was still with an L^2 error smaller than 0.8% (0.008) and 5% (0.05) respectively.

Example 3.5.2. Next, we consider $R_0 = 1$, $R_1 = 3$, and the damping factor is choosen by $\mu = 5$. Let the following Harmonic function be an exact solution to our problem, given by

$$u(r, \theta) = -r^3 \sin 3\theta + r^2 \cos 2\theta \tag{3.5.6}$$



(a) (b)

Figure 3.5: For Example 3.5.2. Comparing the exact solution and numerical solutions without noise, in (3.5a), and the numerical errors are plotted in (3.5b). For $m = 80$, $\alpha = 0.1$.

Therefore, the data on the circle with a radius $R_0 = 1$ are given by

$$u_0(\theta) = -R_0^3 \sin 3\theta + R_0^2 \cos 2\theta, \qquad 0 \le \theta \le 2\pi \tag{3.5.7}$$

$$u_1(\theta) = -3R_0^2 \sin 3\theta + 2R_0 \cos 2\theta, \qquad 0 \le \theta \le 2\pi \tag{3.5.8}$$

We solve this problem by the NRCTM, whose result along a circle with the radius $R_1 = 3$ and the damping factor is chosen by $\mu = 5$.

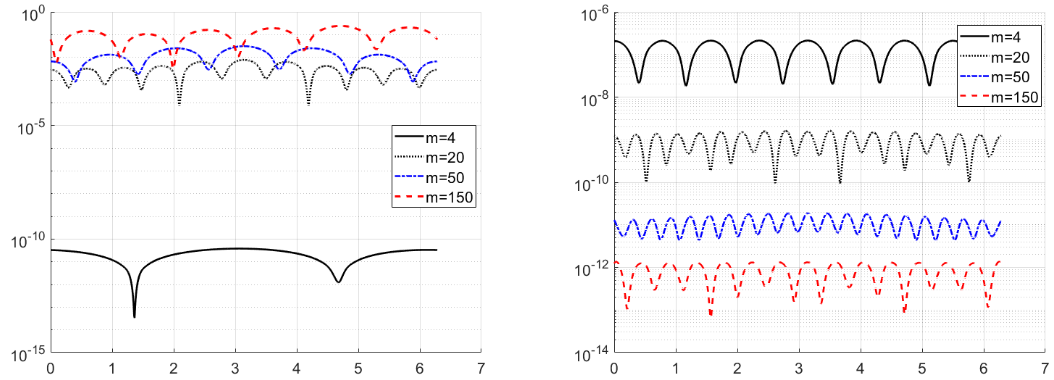
In Fig. 3.5a we show the comparisons between the exact solution u, and its computed approximations by using the MCTM and the NRCTM, without noise, the errors was ploted in 3.5b, respectively with regularization parameter chosen by the trial as 0.1 for $m = 80$.

We can compare the NRCTM and MCTM on this example with very high accuracy, as shown in fig 3.6b and 3.6a, respectively , for the cases $m = 4; 20; 50; 150$ with a regularized parameter chosen by trial as $\alpha = 10^{-2}$, 10^{-6}, 10^{-6}, 10^{-2}, respectively. It can be seen that by the NRCTM the error decreases when m increases, on the contrary in the MCTM.

In Fig. 3.7a we compare the exact solution with the numerical solutions by using the NRCTM under the noises $\epsilon = 0.01$ and 0.07, the corresponding errors was plotedd in 3.7b, the regularization parameter was chosen by trial and error by $\alpha = 0.5$ and 0.1 respectively, with truncation number $m = 50$. It can be seen that the numerical solutions are close to the exact solution, which indicates that the present method is robust against the noise, and even whose level was taken up to 1% (0.01) and 7% (0.07), the numerical error was still with an L^2 error smaller than 2% (0.02) and 8% (0.08) respectively.

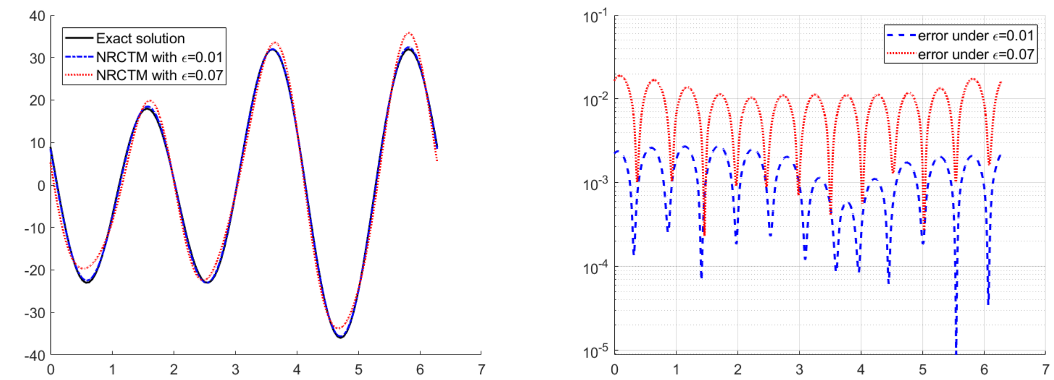
Example 3.5.3. For this example the domain is considered between the radius $R_0 = 2$ and $R_1 = 5$, the damping factor is chosen by $\mu = 4$. To illustrate the accuracy and stability of the new method we consider the following analytical solution

$$u(r, \theta) = \cos x \cosh y + \sin x \cosh y.$$



(a) Errors under MCTM (b) Errors under NRCTM

Figure 3.6: Plotting the numerical errors for example 3.5.2.



(a) (b)

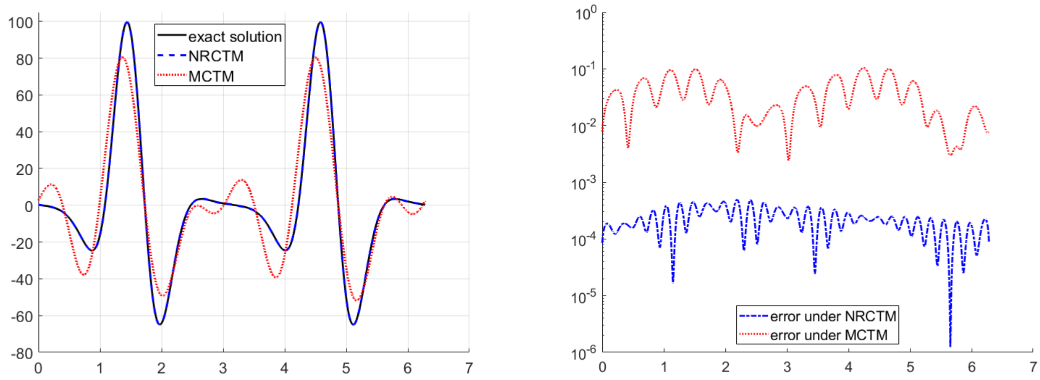
Figure 3.7: For example 3.5.2. Exact solution and numerical solution by NRCTM with noises in (3.7a), the numerical errors are plotted in (3.7b). For $m = 50$, $\alpha(0.01) = 0.5$ and $\alpha(0.07) = 0.1$

The exact boundary data can be derived as:

$$\begin{aligned} u_0(\theta) &= \cos(R_0 \cos\theta)\cosh(R_0 \sin\theta) + \sin(R_0\cos\theta)\cosh(R_0\sin\theta) \\ u_1(\theta) &= -\cos\theta\sin(R_0\cos\theta)\cosh(R_0\sin\theta) + \sin\theta\cos(R_0\cos\theta)\sinh(R_0\sin\theta) \\ &\quad + \cos\theta\cos(R_0\cos\theta)\sinh(R_0\sin\theta) + \sin\theta\sin(R_0\cos\theta)\cosh(R_0\sin\theta) \end{aligned}$$

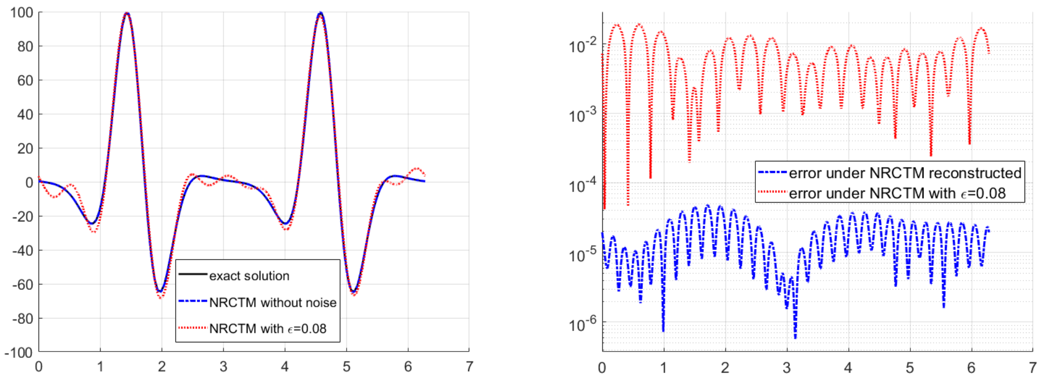
We apply the NRCTM on this example as was done in examples 1 and 2. In Fig. 3.8a we show the comparisons between the exact solution u, and its computed approximations by using the MCTM and the NRCTM, without noise, the errors was ploted in 3.8b, respectively with regularization parameter chosen by the trial as 10^{-9} for $m = 30$.

The result is accurate by using the NRCTM as shown in Fig. 3.9, where $m = 60$, the regularization parameter for the numerical reconstruction is $\alpha = 10^{-9}$ and $\alpha = 10^{-1}$ for noisy data with $\epsilon = 0.08$. The boundary data $g(\theta)$ as shown in 3.9a were plotted for the circle with radius $R = 5$, and the errors were plotted in Fig. 3.9b. Also, it can be seen that in Fig.3.8b the errors are larger than in Fig.3.9b when the truncation number changes



(a) (b)

Figure 3.8: For Example 3.5.3. Comparing the exact solution and numerical solutions without noise, in (3.8a), and the numerical errors are plotted in (3.8b). For $m = 30$, $\alpha = 10^{-9}$.



(a) (b)

Figure 3.9: For example 3.5.3. Exact solution and numerical solution by NRCTM with and without noises in (3.9a), the numerical errors are plotted in (3.9b). For $m = 60$, $\alpha = 10^{-9}$ and $\alpha(0.08) = 0.1$.

from $m = 30$ to $m = 60$, respectively.

Chapter 4

Inverse Problem for the Biharmonic Equation in Detection of Robin Coefficients.

Introduction

In this chapter, we are interested in an inverse problem for a Biharmonic function u to recover Robin coefficients on a non-accessible boundary Γ_c of a simply connected planar domain Ω. From a measured Riquier-Neumann data on the remaining part $\partial\Omega/\Gamma_c$, we search to determine Robins coefficients on Γ_c, when u satisfies homogeneous Robin boundary conditions. Our approach extends a method for the Harmonic equation that has been suggested in [17].

This chapter is organized as follows: The section 4.1, contains a modeling and a general formulation of the problem. In section 4.2, we briefly discuss the open issue of existence and uniqueness to determine the non-accessible portion and the Robin coefficients. The section 4.3 is devoted to recover the Robin coefficients by assuming that the non-accessible part is known. Here, the ill-posed non linear integral equations system equivalent to our inverse problem will be derived. In section 4.4, we deal with completing the missing Cauchy data. The main condition to apply the Tikhonov regularization method will be treated in an L^2 space that is appropriate for quantifying errors on the measured data u_1, u_3, in the image space ℓ^2. In section 4.5, we describ the solution of the inverse problem by the least square sense method, and we conclude with some numerical examples to show the feasibility of the algorithm and the smoothness of boundary.

4.1 Modeling and problem formulation

The problem arises from the static deflection of an elastic bending beam in the plate are subject to a linear boundary conditions that include all types of the conventional boundary conditions, which modeling by Robin coefficients types [48, 49]. In this study we deal with determining this Robin coefficients on a known boundary non-accessible to measurements from an available data on an accessible boundary, via an important step which is the completion of missing Cauchy data on the whole boundary.

In what follow, we assume that Ω is a simply connected bounded domain in the plane, with piece-wise smooth boundary $\partial\Omega = \overline{\Gamma}_m \cup \overline{\Gamma}_c$, where Γ_m and Γ_c are two open disjoint portions of the boundary, and u is a solution of the boundary value problem :

$$\Delta^2 u = 0 \qquad in\, \Omega \tag{4.1.1}$$

with Navier-boundary condition [46] on the Party Γ_m

$$\begin{cases} u = u_0, & on\, \Gamma_m \\ \Delta u = u_2, & on\, \Gamma_m \end{cases} \tag{4.1.2}$$

and homogeneous robin-conditions [3, 48] on the Party Γ_c,

$$\begin{cases} \dfrac{\partial u}{\partial n} + \mu u = 0, & on\, \Gamma_c \\ \dfrac{\partial(\Delta u)}{\partial n} + \lambda \Delta u = 0, & on\, \Gamma_c \end{cases} \tag{4.1.3}$$

where $(\mu, \lambda) \in (L^\infty(\Gamma_c) \times L^\infty(\Gamma_c))$ and $\mu \geq 0$, $\lambda \geq 0$ are two specified functions [14, 97].

Inverse problem. The inverse problem we are concerned with is to determine the functions λ and μ from a given Navier data (u_0, u_2) and the measured Riquier-Neumann data [49]

$$\begin{cases} \dfrac{\partial u}{\partial n} = u_1, & on\, \Gamma_m \\ \dfrac{\partial(\Delta u)}{\partial n} = u_3, & on\, \Gamma_m \end{cases} \tag{4.1.4}$$

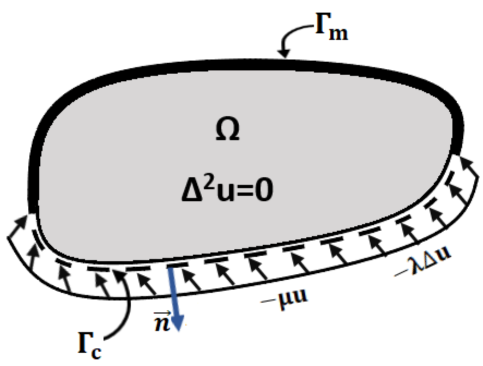


Figure 4.1: Accessible and non-accessible part of the boundary of a simply connected domain

The problem (4.1.1)-(4.1.3) resulting from the study of static deflection of an elastic bending beam, where u denotes the transverse deflection of the beam. This case is known by a linear quasi-static plate problem with unit stiffness, and here, the fictitious force distribution (also called, the transverse loading force) may depend on the deflection and the curvature [14, 97] is assumed to be zero. The coefficients λ and μ are a specified at the boundary of the beam. Therefore, the above inverse problem can be interpreted to determine the specified coefficients λ and μ from the knowledge of deflection $u|_{\Gamma_m}$, the curvature $\Delta u|_{\Gamma_m}$, the measured resulting for each of bending moment $\frac{\partial u}{\partial n}|_{\Gamma_m}$, and the effective shear force $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m}$ on the accessible part Γ_m (for more detail see [14, 54, 97, 98]).

Remark 4.1.1. It should be pointed out that, V.V. Karachik [48, 49] gives certain sufficient conditions of resolvability to the problem of Robin types for Biharmonic Equation. In particular, the Robin's problem (4.1.1)-(4.1.3) is unconditionally resolvable in the unit ball, and its solution is unique. This can be shown by verifying that hypothesis $\lambda \geq 0$ and $\mu \geq 0$ satisfy the conditions of (Theorem1 in [49]).

Remark 4.1.2. From both equations in (4.1.3), one can distinguish particular cases on the portion Γ_c as follows:

(1) $\mu = \lambda = \infty$ and $\mu = \lambda = 0$, which correspond to the homogeneous Navier boundary condition, and homogeneous Riquier–Neumann boundary condition.

(2) The remainder cases $\mu = \infty$, $\lambda = 0$ and $\mu = 0$, $\lambda = \infty$ are correspond to the homogeneous mixed boundary condition.

Resolution Methods

Most of the methods which developed for solving data completion problem are based on a control approach, ie., minimization of a functional by taking functions of the non-accessible part of the boundary as minimization parameters. The regularization methods are the most known and interesting, another class of methods includes the iterative methods. In [85], the advantages and disadvantages of each of them had presented. Having distinguished this different methods, we opted for the group of regularization methods.

Recently, in [7], an iterative method based on the boundary element method (BEM) for the Biharmonic data completion has been proposed. Further, in [44], a single-layer approach is proposed for the Biharmonic data completion using the Tikhonov regularization method that provide higher accuracy than the iterative procedure with respect to the exact data, however, the hyper singular integral arising from some derivatives [21] leads to instability [60].

We undertake the task of deriving an integral equation system that equivalent to our inverse problem by using an indirect integral equation approch based on the fundamental solution method to avoid the hyper singular integral. A numerical solution strategy is based on the quadrature to compute the integral equations and the Tikhonov regularization method to complete missing data.

The completion of missing Cauchy data in the Tikhonov scheme is represented by the indirect integral equations method with densities to be determined. Matching the given data on the accessible part of the boundary, leads to a system of boundary integral equations that can be solved to obtain the densities. This system is discretised using the Nyström method. After completing data, the least squares sense method is used to obtain the functions λ and μ on the non-accessible portion of the boundary.

As a byproduct, we will discuss the question of existence and uniqueness of the non-accessible portion Γ_c, in practice, the reconstruction of the shape Γ_c this is not our case.

Inverse Problem for Harmonic Equation in Detecting Robin's Coefficient

The inverse problem of detecting a Robin's coefficient for the Harmonic equation is widely addressed, so that many scientific researches are devoted to this problem. The uniqueness was obtained using Holmgren's [17, 56] theorem (see also theorem 1.1.6 and theorem 1.1.7). Various stability estimates have been studied in the literature [6, 8]. Concerning the numerical computations, we can find some research works. For example, in [50] the Maz'ya iterative algorithm, a regularized BEM method is considered to obtain the corrosion occurring in an inaccessible interior part of a pipe from the measurements on the outer boundary. In [17, 18, 19, 20], the authors use the direct and indirect boundary integral equations method to recover the impedance for the Harmonic equation. Moreover, in [27], the author transforms the inverse problem into an optimization problem based on the MFS and the Tikhonov's regularization method to recover the impedance for the Harmonic equation. We follow these works and we extend the techniques to (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4).

Problem Formulation using Sobolev Spaces

Obviously, to conduct our study, we need to formulate the boundary value problem (4.1.1)-(4.1.3) and the inverse problems (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4) more precisely [17, 18, 19, 20]. We recall the definitions of some usual Sobolev spaces (see Chapter 1 subsection 1.2.4).

Remark 4.1.3 (Direct problem)**.** It is known [7, 99, 45], that for $(u_0, u_2) \in H^{\frac{3}{2}}(\Gamma_m) \times H^{-\frac{1}{2}}(\Gamma_m)$, there exists a unique solution $u \in H^2(\Omega)$ to the problem (4.1.1)-(4.1.3).

In what follows, we consider that both λ and μ are functions in space on the non-accessible portion Γ_c of the boundary. We understand the inverse problem of determine λ and μ from a given quad of the Cauchy data u_0, u_1, u_2, u_3 on Γ_m by assuming that the whole boundary $\partial\Omega$ is known. That means, given Γ_c and $u_0 \in H^{\frac{3}{2}}(\Gamma_m)$, $u_1 \in H^{\frac{1}{2}}(\Gamma_m)$, $u_2 \in H^{-\frac{1}{2}}(\Gamma_m)$, $u_3 \in H^{-\frac{3}{2}}(\Gamma_m)$, we determine λ and μ such that the unique solution $u \in H^2(\Omega)$ of (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4) again satisfies $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$ and $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$.

4.2 Preliminary results

In this entry section, we show that it is impossible to recover the non-accessible portion Γ_c and the Robin coefficients simultaneously for a single quad of Cauchy data u_0, u_1, u_2, u_3 is given on the accessible part Γ_m, in addition the case when the portion Γ_c is known will be considered. As byproduct, if λ and μ are assumed to be knowns, we discuss the question of whether a single quad of Cauchy data is given on Γ_m uniquely determines the portion Γ_c , also we show that for a fixed constants λ and μ a single quad of Cauchy data on Γ_m can gives rise to infinitely many different domains Ω and we can not assure the existence of portion Γ_c for an arbitrary data on Γ_m.

Begin with the following example to show that we can't determine the portion Γ_c and the functions λ , μ simultaneously.

Example 4.2.1. Let $\Omega = \Big\{(x, y) \in \mathbb{R}^2 : -1 < x < 0,\ 0 < y < t\Big\}$ for $t > 0$, and

$$\Gamma_m = \{(x, 0) : -1 < x < 0\} \cup \{(0, y) : 0 < y < t\} \cup \{(-1, y) : 0 < y < t\},$$
$$\Gamma_c = \{(x, t) : -1 < x < 0\}.$$

Let the Biharmonic function $u(x, y) = \frac{1}{3}(x^3 + y^3)$. According to the equations (4.1.3) we obtain the following system:

$$\begin{cases} 3t^2 + \mu(x^3 + t^3) = 0, \\ 1 + \lambda(x + t) = 0 \end{cases}$$

with $\vec{n} = (0, 1)^\top$. This homogeneous non-linear system did not have a fixed solution (μ, λ, t) for all $x \in \Gamma_c$. We observe that each equation provides only one equation for two unknowns, that mean's, we cannot recover simultaneously both μ and t or λ and t. Also, it can be shown that the above system is equivalent to solve the equation

$$3t^2 + (t + x)\Big(\mu(t^2 - tx + x^2) - \lambda\Big) - 1 = 0$$

which provides only one equation for three unknowns, therefore, we cannot recover simultaneously all λ , μ and t.

We consider an example used in [19] to indicates that if we have fixed only μ or (both μ and λ) by a constants, we can't ensure the uniqueness of Γ_c.

Example 4.2.2. Let Ω be a rectangular domain, given For $t > 0$ by
$\Omega = \Big\{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, -t < y < 1\Big\}$, with the portions Γ_m, Γ_c given as

$$\Gamma_m = \{(0, y) : -t < y < 1\} \cup \{(x, 1) : 0 < x < \pi\} \cup \{(\pi, y) : -t < y < 1\},$$
$$\Gamma_c = \{(x, -t) : 0 < x < \pi\}.$$

Let the Harmonic function $u(x, y) = e^y(\cos(x) + \sin(x))$, therefore, u is Biharmonic. We choose $\mu = 1$, then, some simple calculations in (4.1.3) show that $\frac{\partial u}{\partial n} + u = 0$, and $\frac{\partial \Delta u}{\partial n} + \lambda\Delta u = 0$ for $\lambda \geq 0$, with $\vec{n} = (0, -1)^\top$ is the normal orthogonal vector on Γ_c , we can see that both equations are satisfies for all $(x, y) \in \Gamma_c$ and $t > 0$, it mean's, there are infinite numbers of portion Γ_c that satisfy the above equations.

Remark 4.2.1. By the coupled equation technique [7], we make that the Biharmonic equation (4.1.1) is equivalent to system of equations:

$$\begin{cases} \Delta u = w, & in\ \Omega \\ \Delta w = 0, & in\ \Omega \end{cases}$$

where w is uniquely determined. Thus we obtain two inverse Cauchy problems associated with the Laplace and Poisson equations that equivalent to our inverse problem (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4) as

$$\begin{cases} \Delta u = w, & in\ \Omega \\ u = u_0, & on\ \Gamma_m \\ \dfrac{\partial u}{\partial n} = u_1, & on\ \Gamma_m \\ \dfrac{\partial u}{\partial n} + \mu u = 0, & on\ \Gamma_c \end{cases} (P1)\,, \quad and \quad \begin{cases} \Delta w = 0, & in\ \Omega \\ w = u_2, & on\ \Gamma_m \\ \dfrac{\partial w}{\partial n} = u_3, & on\ \Gamma_m \\ \dfrac{\partial w}{\partial n} + \lambda w = 0, & on\ \Gamma_c \end{cases} (P2).$$

Existence of the non-accessible part Γ_c

The existence of a portion Γ_c cannot be guaranteed for arbitrary data u_0,u_1,u_2,u_3. Indeed, if we fixed the domain Ω in which Γ_c is exist and known, for example, we take $u_0 = u_2 = 0$ and $\mu = \lambda = \infty$ which correspond to $u = w = 0$ on Γ_c, then the Harmonic function w given in (P2) satisfy $w = 0$ on $\partial\Omega = \overline{\Gamma}_m \cup \overline{\Gamma}_c$. Let v be another solution to the problem (P2) verify $\Delta v = 0$ in Ω and satisfy $v = w = 0$ on $\partial\Omega$ and $\frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = u_3$ on Γ_m. Then Holmgren's theorem implies that $v = w$ in Ω, and necessarily forces $w = 0$ in Ω (see [56]). By replacing $w = 0$ in (P1) on the same way, then we obtain that $u = 0$ in Ω and $u_0 = u_1 = u_2 = u_3 = 0$. However, this provides a contradiction if $u_0 = u_2 = 0$ and $u_1 \neq 0$ or $u_3 \neq 0$ for example.

Uniqueness of the non-accessible part Γ_c

We investigate the uniqueness of the portion Γ_c from a single quad of Cauchy data in the particular cases when $\lambda = \mu = \infty$, and $\lambda = \infty, \mu = 0$. The following results can be shown.

Theorem 4.2.1. *Let u be a solution to the problem (4.1.1)-(4.1.3), if $u = \Delta u = 0$ on Γ_c, then $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $\Delta u|_{\Gamma_m} = u_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$ uniquely determine the portion Γ_c provided that $|u_0| + |u_2| \neq 0$.*

Proof. Let Ω_1 and Ω_2 are two bounded domains having Γ_m as part of their boundary in which their corresponding solutions f and g, respectively, verify $\Delta^2 f = 0$ in Ω_1 and $\Delta^2 g = 0$ in Ω_2 and satisfy $f = \Delta f = 0$ on $\partial\Omega_1 \setminus \overline{\Gamma}_m$, $g = \Delta g = 0$ on $\partial\Omega_2 \setminus \overline{\Gamma}_m$ and $f = g = u_0$ on Γ_m, $\frac{\partial f}{\partial n} = \frac{\partial g}{\partial n} = u_1$ on Γ_m, $\Delta f = \Delta g = u_2$ on Γ_m, $\frac{\partial \Delta f}{\partial n} = \frac{\partial \Delta g}{\partial n} = u_3$ on Γ_m. Then Holmgren's theorem [56] implies that $f = g$ in $\Omega_1 \cap \Omega_2$.

By applying the coupled equation technique, we take $\Delta f = w_1$ in Ω_1 where $\Delta w_1 = 0$ in Ω_1, and we take $\Delta g = w_2$ in Ω_2 where $\Delta w_2 = 0$ in Ω_2, and the Harmonic functions w_1 and w_2 satisfy $w_1 = 0$ on $\partial\Omega_1 \setminus \overline{\Gamma}_m$, $w_2 = 0$ on $\partial\Omega_2 \setminus \overline{\Gamma}_m$ and $w_1 = w_2 = u_2$ on Γ_m, $\frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n} = u_3$ on Γ_m. In particulary, and without loss of generality, we suppose that there exists a nonempty connected component D of $\Omega_1 \setminus \overline{\Omega}_2$. From $f = g$ in $\Omega_1 \cap \Omega_2$, and by considering the boundary conditions of f and g then we can conclude that $f = w_1 = 0$ on the boundary of D (see [17]). Now the maximum-minimum principle for Harmonic functions (see corollary 1.9 in [45]) implies that $w_1 = 0$ in D, by substituting in the above equation, we obtain $\Delta f = 0$ in D, and satisfay $f = 0$ on the boundary of D, thus, $f = 0$ in D, and consequently, by analyticity (see theorem 1.27 in [45]), $f = w_1 = 0$ in Ω_1 and $u_0 = u_2 = 0$. However, this contradicts our assumption $|u_0| + |u_2| \neq 0$, i.e., at least one of the functions u_0 or u_2 does not vanish identically. □

Theorem 4.2.2. *Let u be a solution to the problem (4.1.1)-(4.1.3), if $\frac{\partial u}{\partial n} = \Delta u = 0$ on Γ_c, then $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $\Delta u|_{\Gamma_m} = u_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$ uniquely determine the part Γ_c, provided that $u_0 \neq$ constant or $u_2 \neq 0$.*

Proof. We assume that there are two bounded domains Ω_1 and Ω_2 having Γ_m as part of their boundary such that the corresponding solutions f and g, respectively, verify $\Delta^2 f = 0$ in Ω_1 and $\Delta^2 g = 0$ in Ω_2 and satisfy $\frac{\partial f}{\partial n} = \Delta f = 0$ on $\partial\Omega_1 \setminus \overline{\Gamma}_m$, $\frac{\partial g}{\partial n} = \Delta g = 0$ on $\partial\Omega_2 \setminus \overline{\Gamma}_m$ and $f = g = u_0$ on Γ_m , $\frac{\partial f}{\partial n} = \frac{\partial g}{\partial n} = u_1$ on Γ_m , $\Delta f = \Delta g = u_2$ on Γ_m , $\frac{\partial \Delta f}{\partial n} = \frac{\partial \Delta g}{\partial n} = u_3$ on Γ_m, respectively. Then Holmgren's theorem [56] implies that $f = g$ in $\Omega_1 \cap \Omega_2$.

We take $\Delta f = w_1$ in Ω_1 where $\Delta w_1 = 0$ in Ω_1, and we take $\Delta g = w_2$ in Ω_2 where $\Delta w_2 = 0$ in Ω_2. In particulary, and without loss of generality, we suppose that there exists a nonempty connected component D of $\Omega_1 \setminus \overline{\Omega_2}$. From $f = g$ in $\Omega_1 \cap \Omega_2$, and by considering the boundary conditions of f and g then we can conclude that $\frac{\partial f}{\partial n} = w_1 = 0$ on the boundary of D. Then, $w_1 = 0$ in D, by substituting in the above equation, we obtain $\Delta f = 0$ in D, and satisfay $\frac{\partial f}{\partial n} = 0$ on the boundary of D. Therefore (see proof of theorem 4.2.1), $w_1 = 0$ in Ω_1 and $f = constant$ in D, and by the unique continuation property for solutions to elliptic equations, $f = constant$ in Ω_1 (see [1]). Consequently we obtain $u_0 = constant$ and $u_2 = 0$, which contradicts our assumption. Then we conclude that, there exist at most one portion Γ_c provided that $u_0 \neq constant$ or $u_2 \neq 0$. □

Lemma 4.2.1. *A single quad of data $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $\Delta u|_{\Gamma_m} = u_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$ is uniquely determine the Robin coefficients λ and μ as functions of space on the known non-accessible portion Γ_c of the boundary $\partial\Omega$.*

Proof. Let Ω is a bounded domain having Γ_m and Γ_c as parts of their boundary where $\partial\Omega = \Gamma_m \cup \Gamma_m$. We assume that f and g are two different solutions, verify $\Delta^2 f = \Delta^2 g = 0$ in Ω and satisfy $f = g = u_0$, $\frac{\partial f}{\partial n} = \frac{\partial g}{\partial n} = u_1$, $\Delta f = \Delta g = u_2$, and $\frac{\partial \Delta f}{\partial n} = \frac{\partial \Delta g}{\partial n} = u_3$ on Γ_m. Then Holmgren's theorem 1.1.7 implies that $f = g$ in Ω. Thus, the uniqueness of solution u to the problem (4.1.1)-(4.1.2)-(4.1.4) is guarantied

By pointed out that, f and Δf required a quantitative control of possible vanishing. Hence, the functions μ and λ are uniquely determined as: $\lambda = -\frac{1}{\Delta f}\frac{\partial(\Delta f)}{\partial n}$ and $\mu = -\frac{1}{f}\frac{\partial f}{\partial n}$ on Γ_c. □

4.3 Non linear integral equation

The method of fundamental solutions (MFS) is a form of IBIEMs, which belongs to the class of BIEMs [35, 89]. According to [90], any Biharmonic function can be expressed by the MFS based on Chakrabarty or Almansi formulas. The unique solvability results of these formulas were given in [13]. The MFS has another advantage based on a boundary integral representation, so that, a quadrature is required in order to compute its value at any point in the region of interest and similarly for its derivatives [13]. On the other hand, the form of approximation used by the MFS can be evaluated in a straightforward manner, derivative values can also be obtained by a direct evaluation process (see [2, 16, 89]).

In this section, we derive the equivalent systems of integral equations that we employ for the solution of the inverse problem to determine the functions λ and μ, here the formulations given in Chapter 1 subsection 1.2.4 will be recalled. We have seen in section 4.2, it is impossible to determine the portion Γ_c and the Robin coefficients simultaneously, in addition, if Γ_c is known then the functions λ and μ are uniquely determined from a single quad of Cauchy data (see lemma 4.2.1), that, allows us to continue by assuming that the portion Γ_c is known.

Remark 4.3.1. One can see that our inverse problem is linked with Cauchy's problem, which is defined as follows: given the single quad $u_0 \in H^{\frac{3}{2}}(\Gamma_m)$, $u_1 \in H^{\frac{1}{2}}(\Gamma_m)$, $u_2 \in H^{\frac{-1}{2}}(\Gamma_m)$, $u_3 \in H^{\frac{-3}{2}}(\Gamma_m)$ to find $\alpha_0 \in H^{\frac{3}{2}}(\Gamma_c)$, $\alpha_1 \in H^{\frac{1}{2}}(\Gamma_c)$, $\alpha_2 \in H^{\frac{-1}{2}}(\Gamma_c)$, $\alpha_3 \in H^{\frac{-3}{2}}(\Gamma_c)$, such that there exists a Biharmonic function $u \in H^2(\Omega)$ which provide a solution to the equation (4.1.1) and satisfying $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $\Delta u|_{\Gamma_m} = u_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$ and $u|_{\Gamma_c} = \alpha_0$,

$\frac{\partial u}{\partial n}|_{\Gamma_c} = \alpha_1$, $\Delta u|_{\Gamma_c} = \alpha_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_c} = \alpha_3$. This Cauchy problem admits at most one solution (see theorem 1.1.6 and theorem 1.1.7 in chapter 1), and is known to be ill-posed [7]. From the Green's representation theorem, many nonlinear integral equations can be obtained to represent our inverse problem [21].

Definition 4.3.1. Our solution method for the Cauchy problem by the indirect boundary integral equation is based on Chakrabarty representation [60]. Here we represent the solution u of (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4) as surface superposition of the point sources [19] given by the standard fundamental solutions E_1 and E_2 of the Laplacian and Bi-Laplacian, respectively as:

$$u(x) = \int_{\partial\Omega} E_1(x, y)\psi(y)ds(y) + \int_{\partial\Omega} E_2(x, y)\varphi(y)ds(y), \qquad x \in \Omega \tag{4.3.1}$$

where

$$E_1(x, y) = \frac{1}{2\pi} \ln r, \quad E_2(x, y) = \frac{1}{8\pi} r^2 \ln r, \quad r = |x - y|, \quad x \neq y$$

and $(\psi, \varphi) \in H^{-\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, are an unknown densities which assumed to be continuous functions verify $\int_{\partial\Omega} \varphi(y)ds(y) = \int_{\partial\Omega} \psi(y)ds(y) = 0$ (see also theorem 1.2.10 in chapter 1).

4.3.1 Integral equations representation

To set up a system of integral equations that represents our inverse problem we restrict (4.3.1) to the boundary $\partial\Omega$ from inside Ω requiring that $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $\Delta u|_{\Gamma_m} = u_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$ and by using the properties of the single and double layer potential (see theorem 1.2.9 and theorem 1.2.10 in Chapter 1). Thus we obtain:

$$\begin{aligned} A\psi + S\varphi &= u_0, && on\ \Gamma_m \\ B'\psi + K'\varphi - \frac{1}{2}\varphi &= u_1, && on\ \Gamma_m \\ S\psi &= u_2, && on\ \Gamma_m \\ K'\psi - \frac{1}{2}\psi &= u_3, && on\ \Gamma_m \end{aligned} \tag{4.3.2}$$

where

$$\begin{aligned} A &: H^{s-1}(\partial\Omega) \longmapsto H^{s+2}(\partial\Omega), & B' &: H^{s-1}(\partial\Omega) \longmapsto H^{s+1}(\partial\Omega), \\ S &: H^{s-1}(\partial\Omega) \longmapsto H^{s}(\partial\Omega), & K' &: H^{s-1}(\partial\Omega) \longmapsto H^{s-1}(\partial\Omega) \end{aligned}$$

for $-\frac{1}{2} \leq s \leq \frac{3}{2}$, are continuous boundary integral operators [55] defined for $x \in \partial\Omega$ by:

$$\begin{aligned} (A\psi)(x) &= \int_{\partial\Omega} E_2(x, y)\psi(y)ds(y) \quad , \quad (S\varphi)(x) = \int_{\partial\Omega} E_1(x, y)\varphi(y)ds(y), \\ (B'\psi)(x) &= \int_{\partial\Omega} \frac{\partial E_2(x, y)}{\partial n_x}\psi(y)ds(y) \quad , \quad (K'\psi)(x) = \int_{\partial\Omega} \frac{\partial E_1(x, y)}{\partial n_x}\psi(y)ds(y). \end{aligned} \tag{4.3.3}$$

with their following kernels: (see G.C.Hsiao[21] Chapt. 10.4.4)

$$\frac{\partial E_2(x,y)}{\partial n_x} = \frac{1}{8\pi} n_x \cdot (x-y)\,(2\ln r + 1)\,, \qquad \frac{\partial E_1(x,y)}{\partial n_x} = \frac{1}{2\pi}\frac{n_x \cdot (x-y)}{r^2}$$

$$\Delta_x E_2(x,y) = \frac{1}{8\pi}\,(4\ln r + 4)\,, \qquad \frac{\partial \Delta_x E_2(x,y)}{\partial n_x} = \frac{\partial E_1(x,y)}{\partial n_x}$$

Note that if $|x - x_0| \neq 1$ for a some $x_0 \in \Omega$, and for all $x \in \partial\Omega$ the operator S is injective (see lemma 1.2.3 in Chapter 1).

In addition, the equations on the non-accessible part $\Gamma_c = \partial\Omega \setminus \overline{\Gamma}_m$ is given by :

$$B'\psi + K'\varphi - \frac{\varphi}{2} + (A\psi + S\varphi)\mu = 0, \qquad on\ \Gamma_c \tag{4.3.4}$$

$$K'\psi - \frac{\psi}{2} + (S\psi)\lambda = 0, \qquad on\ \Gamma_c \tag{4.3.5}$$

Inversely, we suppose that $(\psi, \varphi) \in H^{-\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ are satisfy the system (4.3.2) and (4.3.4)-(4.3.5), then λ and μ solve the inverse problem. Indeed, if we define $u \in H^2(\Omega)$ as in (4.3.1), then u is Biharmonic function [60]. By approaching the boundary $\partial\Omega$ from inside Ω and according to (4.3.2) and (4.3.4)-(4.3.5) we also have that u verify the mixed boundary value problem (4.1.1)-(4.1.3)-(4.1.4). Therefore, λ and μ are a solutions of the inverse problem. Thus, we can state the following lemma.

Lemma 4.3.1. *The inverse problem (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4) and the system of integral equations (4.3.2)-(4.3.4)-(4.3.5) are equivalent.*

Remark 4.3.2. The system (4.3.2)-(4.3.4)-(4.3.5) equivalent to our inverse problem is not unique. For example, representing the solution u as a combination of a single-layer and double-layer potential (see 1.1.14 in chapter 1) one can derive a different system of integral equations equivalent to our inverse problem. The advantage of this aproch is to avoids the hyper singularity of integral operator kernels (for more detail see [21, 60]).

4.3.2 Paremeterization of integral equations

Investigations have been carried out into the integral equations and their numerical solution using the parameterization (see [17, 18, 19, 20] and 1.2.10 in chapter 1). In this study, for the sake of simplicity we assume that the boundary $\partial\Omega$ is smooth of class C^2 that is, we represent:

$$\partial\Omega = \{z(t) : t \in [0, 2\pi]\} \tag{4.3.6}$$

with $z : \mathbb{R} \longmapsto \mathbb{R}^2$ is an injective of class C^2, and 2π periodic such as $z'(t) \neq 0, \forall\ t \in \mathbb{R}$. Without loss of generality, suppose that:

$$\Gamma_m = \{z(t) : t \in [0, \pi]\}\,, \qquad \Gamma_c = \{z(t) : t \in [\pi, 2\pi]\} \tag{4.3.7}$$

According to (4.3.6), we introduce the setting:

$$\tilde{\psi}(t) = |z'(t)|\psi(z(t)) \quad , \quad \tilde{\varphi}(t) = |z'(t)|\varphi(z(t))$$

then, we obtain from (4.3.3) the following parameterized integral operators:

$$
\begin{aligned}
(\tilde{A}\tilde{\psi})(t) &= \frac{1}{8\pi}\int_0^{2\pi} |z(t)-z(\tau)|^2 \ln|z(t)-z(\tau)|\tilde{\psi}(\tau)d\tau \\
(\tilde{B}'\tilde{\psi})(t) &= \frac{1}{8\pi|z'(t)|}\int_0^{2\pi} [z'(t)]^{\perp}\cdot[z(t)-z(\tau)]\,(2\ln|z(t)-z(\tau)|+1)\,\tilde{\psi}(\tau)d\tau \\
(\tilde{S}\tilde{\varphi})(t) &= \frac{1}{2\pi}\int_0^{2\pi} \ln|z(t)-z(\tau)|\tilde{\varphi}(\tau)d\tau \\
(\tilde{K}'\tilde{\psi})(t) &= \frac{1}{2\pi|z'(t)|}\int_0^{2\pi} \frac{[z'(t)]^{\perp}\cdot[z(t)-z(\tau)]}{|z(\tau)-z(t)|^2}\tilde{\psi}(\tau)d\tau
\end{aligned}
\tag{4.3.8}
$$

for $t \in [0.2\pi]$ and $[z'(t)] = (z_1'(t), z_2'(t))^{\top}$, with the notation $a^{\perp} = (a_2, -a_1)^{\top}$ is the vector orthogonal of $a = (a_1, a_2)^{\top}$. Therefore, we have obtained the parameterized form of (4.3.2)-(4.3.4)-(4.3.5) as:

$$
\begin{aligned}
\tilde{A}\tilde{\psi} + \tilde{S}\tilde{\varphi} &= u_0 \circ z, && on\ [0,\pi] \\
\tilde{B}'\tilde{\psi} + \tilde{K}'\tilde{\varphi} - \frac{1}{2}\tilde{\varphi} &= u_1 \circ z, && on\ [0,\pi] \\
\tilde{S}\tilde{\psi} &= u_2 \circ z, && on\ [0,\pi] \\
\tilde{K}'\tilde{\psi} - \frac{1}{2}\tilde{\psi} &= u_3 \circ z, && on\ [0,\pi]
\end{aligned}
\tag{4.3.9}
$$

and

$$
\tilde{B}'\tilde{\psi} + \tilde{K}'\tilde{\varphi} - \frac{1}{2}\tilde{\varphi} + (\tilde{A}\tilde{\psi} + \tilde{S}\tilde{\varphi})\mu = 0, \qquad on\ [\pi, 2\pi] \tag{4.3.10}
$$

$$
\tilde{K}'\tilde{\psi} - \frac{1}{2}\tilde{\psi} + (\tilde{S}\tilde{\psi})\lambda = 0, \qquad on\ [\pi, 2\pi] \tag{4.3.11}
$$

The decomposition of kernels

The discretizations of the integral operators defined in (4.3.8) are given by their 2π periodic kernels that decomposed as [19, 41]:

$$
\ln|z(t)-z(\tau)| = \ln|\sin\frac{t-\tau}{2}| - \ln\frac{|\sin\frac{t-\tau}{2}|}{|z(t)-z(\tau)|}
$$

where the second term is smooth with diagonal value

$$
-\lim_{\tau\longrightarrow t} \ln\frac{|\sin\frac{t-\tau}{2}|}{|z(t)-z(\tau)|} = \ln 2|z'(t)|.
$$

The 2π periodic kernel $k(t,\tau)$ of the integral operator $\tilde{K}'$ is smooth, given by their diagonal value as [28]:

$$
k(t,\tau) = \begin{cases} \dfrac{[z'(t)]^{\perp}\cdot[z(t)-z(\tau)]}{|z(\tau)-z(t)|^2}, & t \neq \tau \\[2ex] \dfrac{[z'(t)]^{\perp}\cdot z''(t)}{|z'(t)|^2}, & t = \tau \end{cases}
$$

The kernels $a(t,\tau)$ and $b(t,\tau)$ associated to the integrals operator $\tilde{A}$ and $\tilde{B}'$ respectively, are smooth with vanish diagonals values given as:

$$a(t,\tau) = \begin{cases} |z(t) - z(\tau)|^2 \ln|z(t) - z(\tau)|, & t \neq \tau \\ 0, & t = \tau \end{cases}$$

$$b(t,\tau) = \begin{cases} \dfrac{[z'(t)]^{\perp} \cdot [z(t) - z(\tau)]}{|z'(t)|} \left(2\ln|z(t) - z(\tau)| + 1\right), & t \neq \tau \\ 0, & t = \tau \end{cases}.$$

Algorithm

In order to solve the inverse problem (4.1.1)-(4.1.2)-(4.1.3)-(4.1.4) via equations (4.3.9)-(4.3.10)-(4.3.11), this last can be summarized by the following algorithm which based on the above discussion,

1 Let Ω be a simply connected domain in which the portions Γ_m and Γ_c are knowns parameterized by (4.3.7). Given a single quad of Cauchy data u_0,u_1,u_2,u_3 on Γ_m.

2 Then, we find the densities $\tilde{\psi}$ and $\tilde{\varphi}$ on $\partial\Omega$ by solving the ill-posed system (4.3.9). For this, we propose the Tikhonov regularization method in order to achieve stability.

3 From the knowledge of the densities $\tilde{\psi}$ and $\tilde{\varphi}$ on $\partial\Omega$. Therefore, the Cauchy data α_0,α_1,α_2,α_3 can be obtained on Γ_c by $(\tilde{A}\tilde{\psi} + \tilde{S}\tilde{\varphi})|_{\Gamma_c} = \alpha_0$, $(\tilde{B}'\tilde{\psi} + \tilde{K}'\tilde{\varphi} - \frac{\tilde{\varphi}}{2})|_{\Gamma_c} = \alpha_1$, $(\tilde{S}\tilde{\psi})|_{\Gamma_c} = \alpha_2$ and $(\tilde{K}'\tilde{\psi} - \frac{\tilde{\psi}}{2})|_{\Gamma_c} = \alpha_3$.

4 Thus, the unknowns functions λ and μ can be uniquely determined from the equations (4.3.10) and (4.3.11) by $\alpha_1 + \mu\alpha_0 = 0$ and $\alpha_3 + \lambda\alpha_2 = 0$, respectively.

4.4 Data completion

In order to determine the unknown functions λ and μ, we will be interested in an important step which is the completion of the missing Cauchy data. For this, we recall the inverse problem which is : given $(u_0, u_1, u_2, u_3) \in H^{\frac{3}{2}}(\Gamma_m) \times H^{\frac{1}{2}}(\Gamma_m) \times H^{-\frac{1}{2}}(\Gamma_m) \times H^{-\frac{3}{2}}(\Gamma_m)$, and determine $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in H^{\frac{3}{2}}(\Gamma_c) \times H^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c) \times H^{-\frac{3}{2}}(\Gamma_c)$ such that, there exists a Biharmonic function $u \in H^2(\Omega)$ satisfaying

$$u|_{\Gamma_m} = u_0 \quad , \quad \frac{\partial u}{\partial n}|_{\Gamma_m} = u_1 \quad , \quad \Delta u|_{\Gamma_m} = u_2 \quad , \quad \frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$$

and

$$u|_{\Gamma_c} = \alpha_0 \quad , \quad \frac{\partial u}{\partial n}|_{\Gamma_c} = \alpha_1 \quad , \quad \Delta u|_{\Gamma_c} = \alpha_2 \quad , \quad \frac{\partial \Delta u}{\partial n}|_{\Gamma_c} = \alpha_3.$$

It's known that, this Cauchy problem is ill-posed, and admits at most one solution. In particular, if $(\psi, \varphi) \in H^{-\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve (4.3.2) then the solution $u \in H^2(\Omega)$ given by (4.3.1) verify : $u|_{\Gamma_c} = \alpha_0$, $\frac{\partial u}{\partial n}|_{\Gamma_c} = \alpha_1$, $\Delta u|_{\Gamma_c} = \alpha_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_c} = \alpha_3$. [19].

Practically, we give $(u_0, u_1, u_2, u_3) \in H^{\frac{3}{2}}(\Gamma_m) \times H^{\frac{1}{2}}(\Gamma_m) \times H^{-\frac{1}{2}}(\Gamma_m) \times H^{-\frac{3}{2}}(\Gamma_m)$ and solve (4.3.2) by a Tikhonov regularization method applying in the space of square integrable functions, with L^2-norm is the appropriate norm to measure the data error, For this we recall the ill-posed linear system (4.3.2), and consider the corresponding operator (see [19, 20, 21]) $T : X \to Y$ defined by:

$$T(\psi, \varphi) = \begin{pmatrix} A\psi + S\varphi \\ B'\psi + K'\varphi - \frac{\varphi}{2} \\ S\psi \\ K'\psi - \frac{\psi}{2} \end{pmatrix}.$$

with : $X = L^2(\partial\Omega) \times L^2(\partial\Omega)$, et $Y = L^2(\Gamma_m) \times L^2(\Gamma_m) \times L^2(\Gamma_m) \times L^2(\Gamma_m)$.

In order to apply the Tikhonov regularization scheme to (4.3.2), the following result must be verified (see theorem 1.3.6 in Chapter 1) :

Theorem 4.4.1. *The operator T is injective with a dense range.*

Proof. If $T(\psi, \varphi) = 0$ for some $(\psi, \varphi) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ then u is given by (4.3.1) and verify $u|_{\Gamma_m} = \frac{\partial u}{\partial n}|_{\Gamma_m} = \Delta u|_{\Gamma_m} = \frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = 0$ from inside Ω. The trace theorem implies that $A\psi + S\varphi = B'\psi + K'\varphi - \frac{\varphi}{2} = S\psi = K'\psi - \frac{\psi}{2} = 0$. The geometry assumed on Ω, ensured the injectivity of the operator S therefore $\psi = 0$ and we deduce that $\varphi = 0$, this proves that T is injective.

Next, to prove that T has a dense image, we need to show that T^* is injective. For that, we consider the adjoint operator $T^* : Y \longmapsto X$ which is given by:

$$(T(\psi, \varphi), [\alpha_0, \alpha_1, \alpha_2, \alpha_3])_{Y,Y} = ((\varphi, \psi), T^*[\alpha_0, \alpha_1, \alpha_2, \alpha_3])_{X,X}$$

where, $(.,.)$ denotes the respective inner product in L^2. Let $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ be the extensions of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ by zero to boundary $\partial\Omega/\overline{\Gamma}_m$ (see subsection 4.1). Then, for all $(\psi, \varphi) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ we have that :

$$\begin{aligned}(T(\psi, \varphi), [\alpha_0, \alpha_1, \alpha_2, \alpha_3])_{Y,Y} &= (\psi, A\tilde{\alpha}_0 + B'\tilde{\alpha}_1 + S\tilde{\alpha}_2 + K'\tilde{\alpha}_3 - \frac{\tilde{\alpha}_3}{2})_{X,X} \\ &+ (\varphi, S\tilde{\alpha}_0 + K'\tilde{\alpha}_1 - \frac{1}{2}\tilde{\alpha}_1)_{X,X}\end{aligned}$$

therefore, we have obtained that

$$T^*[\alpha_0, \alpha_1, \alpha_2, \alpha_3] = \begin{pmatrix} A_m\alpha_0 + B'_m\alpha_1 + S_m\alpha_2 + K'_m\alpha_3 - \dfrac{\alpha_3}{2} \\ S_m\alpha_0 + K'_m\alpha_1 - \dfrac{1}{2}\alpha_1 \end{pmatrix}$$

where A_m, B'_m, S_m, K'_m are defined for $x \in \partial\Omega$ by

$$(A_m\alpha_0)(x) = \int_{\Gamma_m} E_2(x,y)\alpha_0(y)ds(y), \quad (B'_m\alpha_1)(x) = \int_{\Gamma_m} \frac{\partial E_2(x,y)}{\partial n_x}\alpha_1(y)ds(y),$$

$$(S_m\alpha_2)(x) = \int_{\Gamma_m} E_1(x,y)\alpha_1(y)ds(y), \quad (K'_m\alpha_3)(x) = \int_{\Gamma_m} \frac{\partial E_1(x,y)}{\partial n_x}\alpha_1(y)ds(y)$$

Let $T^*(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = 0$, for some $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in Y$. We define the Harmonic function given by

$$v = S_m\alpha_0 + K'_m\alpha_1,$$

which is a solution of the Laplace equation in $\mathbb{R}^2/\overline{\Gamma}_m$. Letting $x \to \partial\Omega$ from outside Ω and using the jump relations for single and double-layer potentials (see theorem 1.2.9 and theorem 1.2.10) with L^2 densities we obtain that

$$v|_{\partial\Omega} = S_m\alpha_0 + K'_m\alpha_1 - \frac{1}{2}\alpha_1 = 0.$$

From the logarithmic behavior of the single-layer potential at infinity (see theorem 1.2.11 in Chapter 1) we deduce that $v = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, and consequently, by analyticity [17] (see also theorem 1.1.5) we obtain that $v = 0$ in the definition domain Ω. As consequence, the jump relations across $\partial\Omega$ imply that $\alpha_0 = \alpha_1 = 0$.

By substituting this last result into the first component of T^* we obtain that

$$S_m\alpha_2 + K'_m\alpha_3 - \frac{1}{2}\alpha_3 = 0.$$

Using the same steps we saw above leads to $\alpha_2 = \alpha_3 = 0$, finally we conclude that $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ which proves that T^* is injective. $\square$

4.5 Numerical methods and examples

For the numerical methods, the Nyström method provides a highly efficient method for the approximate solution of the boundary integral equations of the second kind for two-dimensional boundary value problems [44]. In our situation we take into account the parameterized decomposition of the kernels given in section 4.3 and applying the Nyströom method to (4.3.9) based on the following trigonometric quadrature rules (see [41, 43] and 1.4.5 in chapter 1).

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)d\sigma &\approx \frac{1}{2n}\sum_{j=0}^{2n-1} a_j^{(n)} f(s_j^n) \\
\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\ln\left(\frac{4}{e}\sin^2(\frac{\sigma-t}{2})\right)d\sigma &\approx \frac{1}{2n}\sum_{j=0}^{2n-1} a_j^{(n)} R_j(t) f(s_j^{(n)})
\end{aligned} \tag{4.5.1}$$

on an equidistant mesh points

$$t_j = jh, \quad for: \ j = 0, ..., 2n-1, \quad and \quad h = \frac{\pi}{n}. \tag{4.5.2}$$

The weights functions are given by

$$\begin{aligned}
R_j(t) &= -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\left(m(t - s_j^{(n)})\right) - \frac{\pi}{n^n}\cos\left(n(t - s_j^{(n)})\right), \\
a_j^{(n)} &= w'(t_j), \\
s_j^{(n)} &= w(t_j)
\end{aligned}$$

with a sigmoidal transformation defined as (see [43] and 1.4.4 in chapter 1). $w : [0, 2\pi] \longmapsto [0, 2\pi]$

$$w_p(t) = 2\pi\frac{[\upsilon(t)]^p}{[\upsilon(t)]^p + [\upsilon(2\pi - t)]^p}, \qquad 0 \le t \le 2\pi, \tag{4.5.3}$$

and υ is a cubic polynomial given by

$$\upsilon(t) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - t}{\pi}\right)^3 + \frac{1}{p}\left(\frac{t - \pi}{\pi}\right) + \frac{1}{2},$$

the parameter p in the substitution functions is the so-called grading parameter. For larger values of p the grid points are more densely accumulated at the end points of the integration interval (see [19] and 1.4.4 in chapter 1).

Method description

To show the feasibility of this method to complete Cauchy data we want to use it for the inverse problem of determining the functions λ, μ for a fixed domain Ω, i.e. we want to recover the specified functions λ, μ on Γ_c, from a single quad of Cauchy data (u_0, u_1, u_2, u_3) on Γ_m (see [17, 18, 19]). To this end, using the notations that are introduced in section 3, we observed that after the determination of ψ, φ, and having completed the Cauchy data $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ on Γ_c, we obtained the functions λ, μ from the following equations :

$$\begin{aligned} \alpha_1 + \mu\alpha_0 &= 0, \qquad on\, \Gamma_c \\ \alpha_3 + \lambda\alpha_2 &= 0, \qquad on\, \Gamma_c \end{aligned}$$

We follow these three steps: first, we need to solve the ill-posed equation (4.3.9), for example, by Tikhonov regularization (see 1.3.1 in chapter 1) for the densities ψ, φ on $\partial\Omega$. For this, of course, we use the parameterized version (4.3.9)-(4.3.10)-(4.3.11) and trigonometric quadrature (4.5.1) with sigmoidal transformation (4.5.3). Then we obtain the densities ψ and φ on $\partial\Omega$, and we can deduce the traces of u on Γ_c, i.e. $(A\psi + S\varphi)|_{\Gamma_c} = \alpha_0$, $(B'\psi + K'\varphi - \frac{\varphi}{2})|_{\Gamma_c} = \alpha_1$, $(S\psi)|_{\Gamma_c} = \alpha_2$ and $(K'\psi - \frac{\psi}{2})|_{\Gamma_c} = \alpha_3$. Finally, we calculate the functions λ and μ at collocation points $x_i = z(t_{n+i})$, $i = 1, ..., n$, on Γ_c by resolving

$$\alpha_1(x_i) + \mu(x_i)\alpha_0(x_i) = 0, \qquad i = 1, ..., n \tag{4.5.4}$$

$$\alpha_3(x_i) + \lambda(x_i)\alpha_2(x_i) = 0, \qquad i = 1, ..., n \tag{4.5.5}$$

To avoid instabilities arising from dividing by small values of $\alpha_0(x_i)$ and $\alpha_2(x_i)$, as mentioned in [17, 18, 19], we represent the unknowns λ and μ as a linear combinations of the Gaussian basis functions e_j (see [31, 74] and 1.3.6 in chapter 1).

$$\mu(x) \approx \sum_{j=1}^{K} c_j e_j(x) \tag{4.5.6}$$

$$\lambda(x) \approx \sum_{j=1}^{M} d_j e_j(x) \tag{4.5.7}$$

In matrix-vector notation, the models (4.5.6) and (4.5.7) are given

$$\mathcal{A}c \approx \mu \tag{4.5.8}$$

$$\mathcal{B}d \approx \lambda \tag{4.5.9}$$

The residuals associated to (4.5.8) and (4.5.9) are given by

$$r_1 = \mu - \mathcal{A}c \tag{4.5.10}$$
$$r_2 = \lambda - \mathcal{B}d \tag{4.5.11}$$

Least squares sens method (see 1.3.2 in chapter 1) consist to research on the best approximation of λ and μ, i.e., we solve the equations that is obtained by inserting (4.5.6), (4.5.7) into (4.5.4), (4.5.5), in the least squares sense for the coefficients c_j and d_j that makes the residuals r_1 and r_2 , respectively, smallest possible, as follow

$$\alpha_1(x_i) + \sum_{j=1}^{K} c_j e_j(x_i)\alpha_0(x_i) = r_{1,i}, \qquad i = 1, ..., n \tag{4.5.12}$$

$$\alpha_3(x_i) + \sum_{j=1}^{M} d_j e_j(x_i)\alpha_2(x_i) = r_{2,i}, \qquad i = 1, ..., n \tag{4.5.13}$$

In the numerical examples, we used cubic B-splines (see 1.4.6 in chapter 1) on an equidistant subdivision (4.5.2) with respecting t parameter in (4.3.6)-(4.3.7) and the following parameterization is considered

$$\Gamma_m = \left\{ z(w_{p_1}(t)) \,, \quad t \in [0, \pi] \right\}, \qquad \Gamma_c = \left\{ z(w_{p_2}(t)) \,, \quad t \in [\pi, 2\pi] \right\}.$$

In order to test the numerical stability of our method, the noisy data u_1^δ, u_3^δ has been generated with arbitrary small noise level δ added to the Riquier-Neumann data in the form [4, 19]

$$u_1^\delta = u_1 + \epsilon \frac{\|u_1\|_{L^2(\Gamma_m)}}{\|\xi\|_{L^2(\Gamma_m)}} \xi, \qquad u_3^\delta = u_3 + \epsilon \frac{\|u_3\|_{L^2(\Gamma_m)}}{\|\xi\|_{L^2(\Gamma_m)}} \xi \tag{4.5.14}$$

with

$$\|u_1^\delta - u_1\|_{L^2(\Gamma_m)} \leq \delta, \qquad \|u_3^\delta - u_3\|_{L^2(\Gamma_m)} \leq \delta \tag{4.5.15}$$

where ξ is a normally distributed random variable and ϵ is the relative noise level. The discrete norm $\|.\|_{\ell^2}$ associated to the norm $\|.\|_{L^2(\Gamma_m)}$ with respect to the mesh points (4.5.2) on Γ_c is defined for $v \in \ell^2$ by

$$\|v\|_{\ell^2} = \left(\frac{1}{n+1} \sum_{i=1}^{n+1} v_i^2 \right)^{\frac{1}{2}} \tag{4.5.16}$$

4.5.1 Numerical examples

Example 4.5.1. We start with smooth boundary by consider an ellipse with the parameterization

$$\Gamma_m = (0.1 \cos t \,,\ 0.4 \sin t) \,, \quad t \in [0, \pi],$$
$$\Gamma_c = (0.1 \cos t \,,\ 0.4 \sin t) \,, \quad t \in [\pi, 2\pi].$$

Secondly, we make a slightly change in the smoothness of boundary by consider corners at the connection of the two parts of the boundary [19], that parameterized by half of a bowl shaped contour

$$\Gamma_m = (1 + \sin t)(0.1\cos t\ ,\ 0.4\sin t)\ , \quad t \in [0, \pi],$$

and by half of an ellipse

$$\Gamma_c = \ (0.1\cos t\ ,\ 0.4\sin t), \quad t \in [\pi, 2\pi].$$

In both examples, the profiles of specified functions are given as:

$$\mu(t) = \begin{cases} 0, & t \in [0, \pi] \\ 1\ +\ \sin^4 t, & t \in [\pi, 2\pi] \end{cases},$$

$$\lambda(t) = \begin{cases} 0, & t \in [0, \pi] \\ 1\ +\ \cos^4 t, & t \in [\pi, 2\pi] \end{cases},$$

and the synthetic Cauchy data (u_0, u_1, u_2, u_3) on Γ_m were obtained by solving the Robin problem in Ω, with the boundary conditions

$$\frac{\partial u}{\partial n} + \mu u = h \quad , \quad \frac{\partial \Delta u}{\partial n} + \lambda \Delta u = g\ ,$$

with

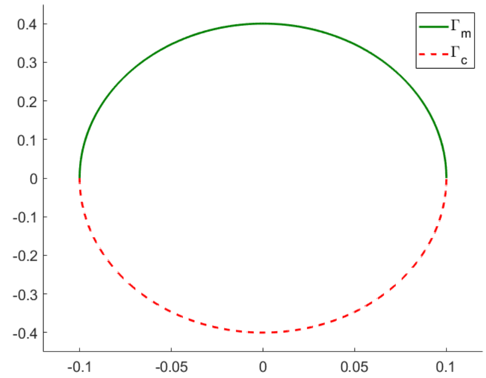
$$h(t) = \begin{cases} 1\ +\ \sin^4 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases},$$

$$g(t) = \begin{cases} 1\ +\ \cos^4 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases}$$

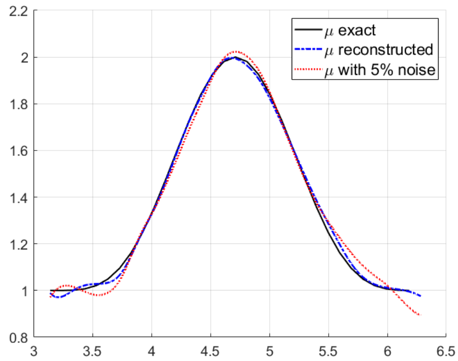
Based on the double-layer boundary integral equation approach, with double number of discretization points, the characteristics of the sigmoidal transformation (4.5.3) allow us to avoiding singularities arising in two intersection points [19]. In addition, (to avoid an inverse crime) we choose the grading parameter $p_1 = 10$ for the forward problem and $p_2 = 8$ for the inverse algorithms.

The reconstructions were performed using $2n = 64$ grid points for discretizing the integral operators on the boundary. Figures 4.2b, 4.2c and 4.3b, 4.3c show the reconstructed profile for both exact data and for relative noise level $\epsilon = 0.01$ (with respect to the L^2 norm). The exact functions profiles are represented by the full lines (black) and the reconstructions are represented by the dash-dotted lines (blue), and the dotted lines (red) for the noise. To obtain the densities ψ, φ we solve the two equations correspond to the Riquier-Neumann data u_1, u_3 in (4.3.9), by the Tikhonov's regularization method with parameters 10^{-13} for the exact data and 10^{-8} for the noisy data, respectively. For the B-spline approximation of the functions profile, we take the dimensions $K = M = 21$. In 4.2d, 4.3d and 4.2e, 4.3e we plot the errors between exact functions μ and λ with their computed approximations, respectively, with and without noise.

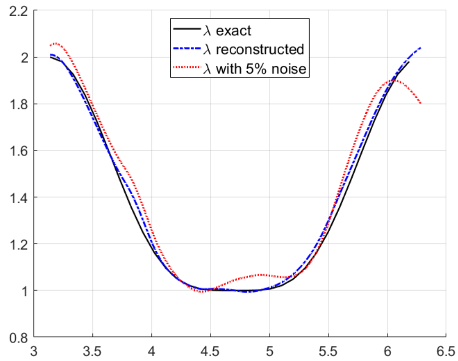
We can summarize that the numerical experiments show satisfactory reconstructions for the functions μ and λ also with reasonable stability against noisy data. By comparing errors plotting in Fig. 4.2d and Fig. 4.2e with Fig. 4.3d and Fig. 4.3e, we can note, the quality of the reconstructions in the case of the bowl-ellipse shaped contour (4.3a) is not as accurate as in the smooth boundary (4.2a).



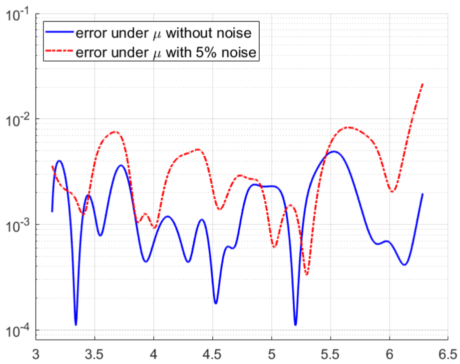
(a) Geometry of the boundary



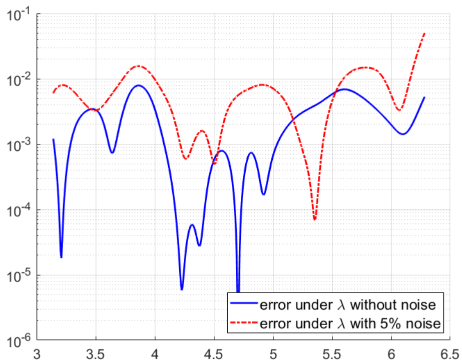
(b) Reconstruction of μ.



(c) Reconstruction of λ.



(d) Plotting the numerical errors for μ



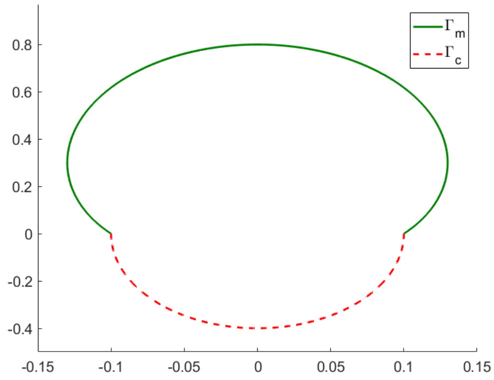
(e) Plotting the numerical errors for λ

Figure 4.2: For example 4.5.1. Reconstruction of a function specified profile for an ellipse with the semi-axis $a = 0.1, 0.4$.

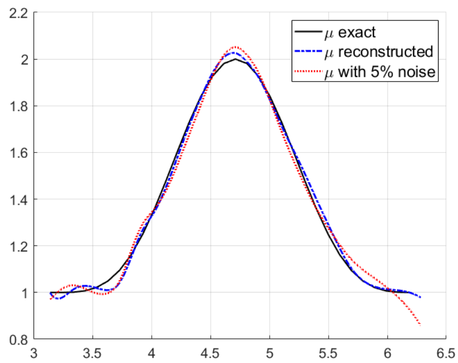
Example 4.5.2. For this example we consider an ellipse with the parameterization

$$\Gamma_m = (0.2\cos t\,,\; 0.3\sin t)\,, \qquad t \in [0, \pi],$$
$$\Gamma_c = (0.2\cos t\,,\; 0.3\sin t)\,, \qquad t \in [\pi, 2\pi].$$

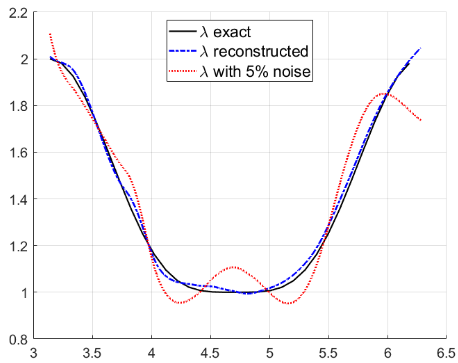
and we make corners at the connection of the two parts of the boundary, that parameter-



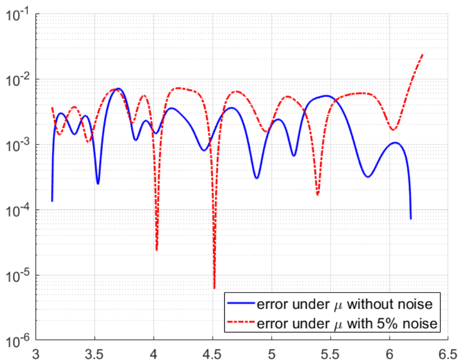
(a) Geometry of the boundary



(b) Reconstruction of μ.



(c) Reconstruction of λ.



(d) Plotting the numerical errors for μ

(e) Plotting the numerical errors for λ

Figure 4.3: For example 4.5.1. Reconstruction of a function specified profile for a bowl-ellipse shaped contour.

ized by half of a bowl shaped contour as:

$$\Gamma_m = (1 + \sin t)(0.2 \cos t\ ,\ 0.3 \sin t)\ , \quad t \in [0, \pi],$$

and by half of an ellipse

$$\Gamma_c = \ (0.2 \cos t\ ,\ 0.3 \sin t), \quad t \in [\pi, 2\pi].$$

The profiles of specified functions are given as:

$$\mu(t) = \begin{cases} 0, & t \in [0, \pi] \\ \sin^2 t, & t \in [\pi, 2\pi] \end{cases},$$

$$\lambda(t) = \begin{cases} 0, & t \in [0, \pi] \\ \cos^2 t, & t \in [\pi, 2\pi] \end{cases},$$

and the synthetic Cauchy data (u_0, u_1, u_2, u_3) on Γ_m were obtained by solving the Robin problem in Ω, with the boundary conditions

$$\frac{\partial u}{\partial n} + \mu u = h \quad , \quad \frac{\partial \Delta u}{\partial n} + \lambda \Delta u = g \, ,$$

with

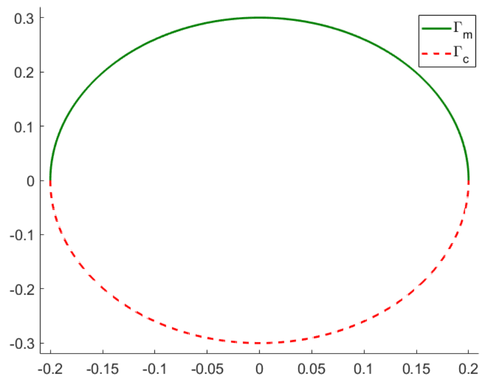
$$h(t) = \begin{cases} \sin^2 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases},$$

$$g(t) = \begin{cases} \cos^2 t, & t \in [0, \pi] \\ 0, & t \in [\pi, 2\pi] \end{cases}$$

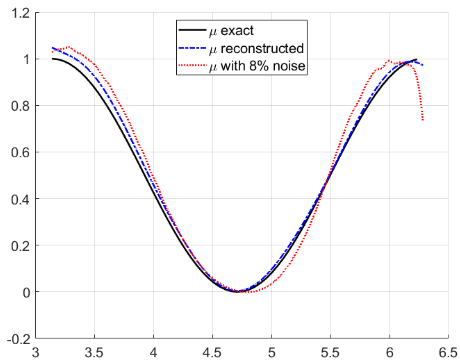
We choose the grading parameter $p_1 = 8$ for the forward problem and $p_2 = 6$ for the inverse algorithms.

The reconstructions were performed using $2n = 128$ grid points for discretizing the integral operators on the boundary. Figures 4.4b, 4.4c and 4.5b, 4.5c show the reconstructed profile for both exact data and for relative noise level $\epsilon = 0.05$ (with respect to the L^2 norm). The exact functions profiles are represented by the full lines (black) and the reconstructions are represented by the dash-dotted lines (blue), and the dotted lines (red) for the noise. To obtain the densities ψ, φ we solve the two equations correspond to the Riquier-Neumann data u_1, u_3 in (4.3.9), by the Tikhonov's regularization method with parameters 10^{-13} for the exact data and 10^{-9} for the noisy data, respectively. For the B-spline approximation of the functions profile, we take the dimensions $K = M = 35$. In 4.4d, 4.5d and 4.4e, 4.5e we plot the errors between exact functions μ and λ with their computed approximations, respectively, with and without noise.

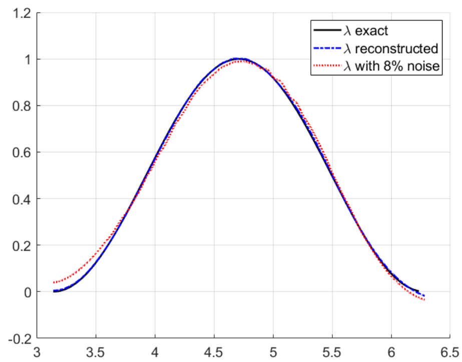
As shown in the example (4.5.1). The numerical experiments show satisfactory reconstructions for the functions μ and λ also with reasonable stability against noisy data. By comparing errors plotting in Fig. 4.4d and Fig. 4.4e with Fig. 4.5d and Fig. 4.5e, we can note, the quality of the reconstructions in the case of the bowl-ellipse shaped contour (4.5a) is not as accurate as in the smooth boundary (4.4a).



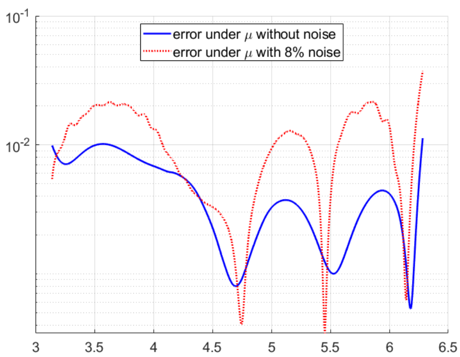
(a) Geometry of the boundary



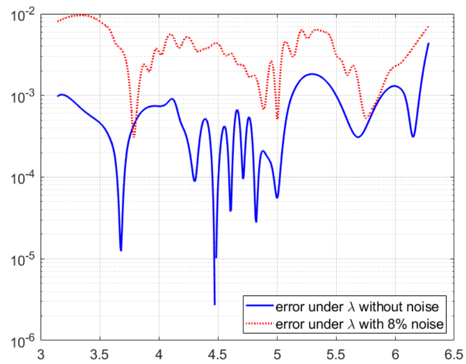
(b) Reconstruction of μ.



(c) Reconstruction of λ.

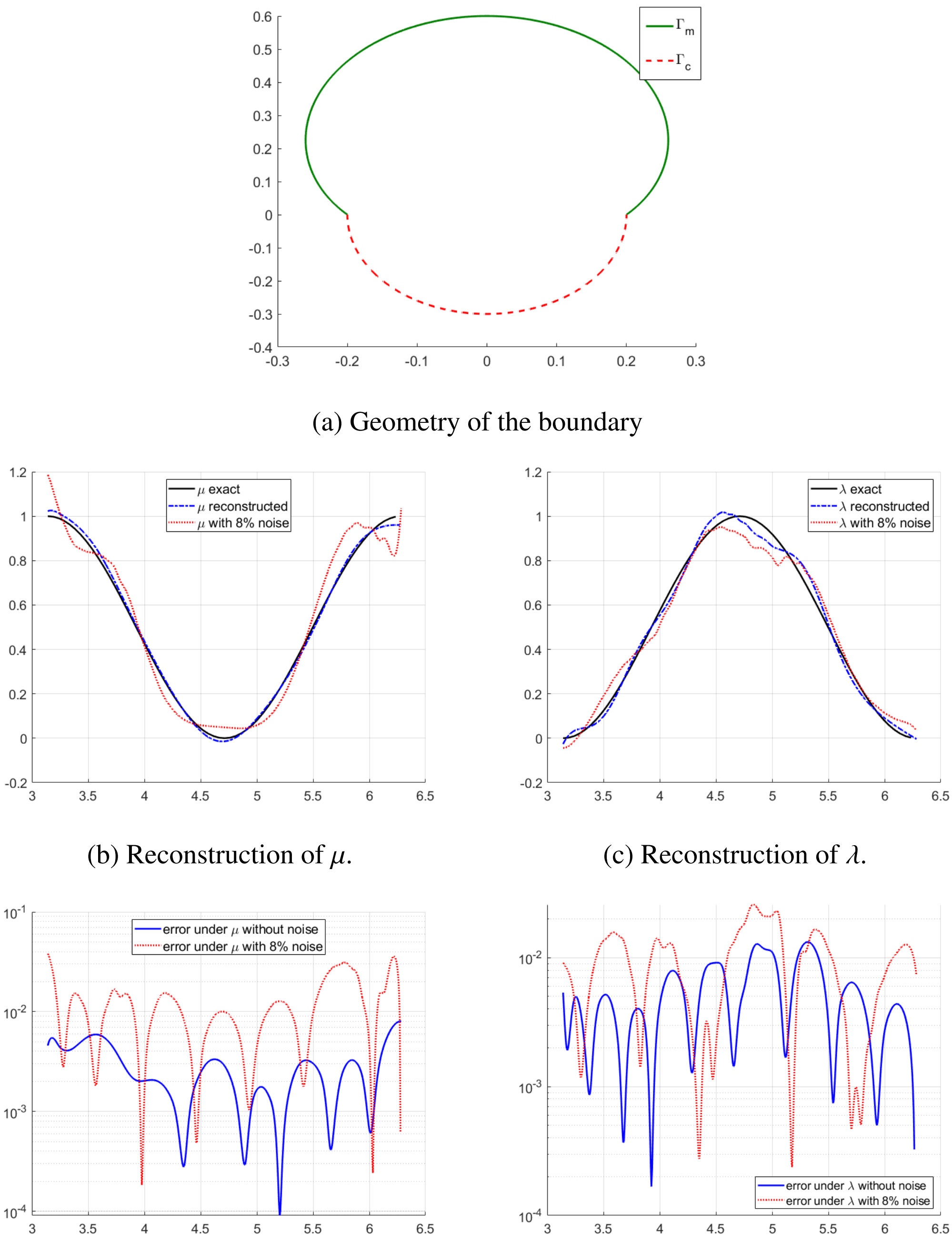


(d) Plotting the numerical errors for μ



(e) Plotting the numerical errors for λ

Figure 4.4: For example 4.5.2. Reconstruction of a function specified profile for an ellipse with the semi-axis $a = 0.2, 0.3$.



(a) Geometry of the boundary

(b) Reconstruction of μ.

(c) Reconstruction of λ.

(d) Plotting the numerical errors for μ

(e) Plotting the numerical errors for λ

Figure 4.5: For example 4.5.2. Reconstruction of a function specified profile for a bowl-ellipse shaped contour.

Conclusion and prospects

In this thesis, two inverses problems caused by the Biharmonic were addressed the first one is: (1)to find a non-accessible boundary of doubly-connected planar domain, and the second is: (2)to recover the Robin coefficients on a non-accessible known part in a simply-connected planar domain. As a result, (3)a method of regularization has been proposed for an inverse Cauchy problem governed by the Harmonic, here we rely on Tikhonov's regularization method where the errors estimation have been established in appropriate correction classes.

As a prospect, we are planning to (1)Explore the regularization approaches applied to the Biharmonic problem. (2)Reconstruct the non-accessible portion of the boundary. (3)Although we have only considered our study in a disc or annulus, we are looking to extend our study to other problems in a more complex region.

Bibliography

[1] A.G.Ramm, *A geometrical inverse problem*. 2-2 Inv Prob (1986) 2 L19.

[2] A. Benrabah and N. Boussetila, *Modified nonlocal boundary value problem method for an ill-posed problem for the biharmonic equation.* Inv Probs in Sci and Engi (2018) pp. 340-368.

[3] A. Gomez-Polanco, J.M. Guevara-Jordan. B. Molina, *A mimetic iterative scheme for solving biharmonic equations.* 57 Math. Comput. Model (2011) 2132-2139.

[4] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems.* Springer-Verlag 2011.

[5] A. Zeb, D.B. Ingham and D. Lesnic, *The method of fundamental solutions for a biharmonic inverse boundary determination problem.* 42 Comput Mech (2008) 371–379.

[6] Chaabane S and Jaoua M.: *Identification of Robin coefficients by the means of boundary measurements*. Inverse Problems (1999) 15 1425–38.

[7] Ch. Tajani, H. Kajtih and A. Daanoun, *Iterative Method to Solve a Data Completion Problem for biharmonic Equation for Rectangular Domain.* 55-1 annals of West Univ of Timi-Math and Comp Sci (2017) 129-147.

[8] Alessandrini G, Del Piero L, Rondi L.: *Stable determination of corrosion by a single electrostatic boundary measurement*. Inverse Problem (2003);19(4): 973–84.

[9] Ch. Sh. Liu. *A Highly Accurate MCTM for Direct and Inverse Problems of Biharmonic Equation in Arbitrary Plane Domains.* 30-2 CMES, (2008) pp 65-75.

[10] Chein-Shan Liu. *A modified collocation Trefftz method for the inverse Cauchy problem of Laplace equation.* 32-9 Engine anal with Bound elem (2008) 778-785.

[11] Ch. Sh. Liu. *Solving the Inverse Problems of Laplace Equation to Determine the Robin Coefficient Cracks Position Inside a Disk.* 40-1 CMES, (2009) pp 1-28.

[12] Yu.P. Petrov, V.S. Sizikov, *Well-posed, ill-posed, and intermediate problems with applications.* VSP (2005)

[13] Boris N. Zakhariev, Allina A. Suzko. : *Direct and Inverse Problems- Potentials in Quantum Scattering*-Springer-Verlag Berlin Heidelberg (1990).

[14] C.V. Pao, *On fourth order elliptic boundary value problem.* 4-128 Proc of the Am Math Soc (2000) pp. 1023-1030.

[15] Dang Q. A and M. Xuan Thao.: *Iterative method for solving a problem with mixed boundary conditions for biharmonic equation arising in fracture mechanics.* 31-1 Bol. Soc. Paran. Mat. (2013).

[16] E. Klann and R. Ramlau.: *Regularization by Fractional Filter Methods and Data Smoothing.* 24-2 Inverse problems , (2008)

[17] F.Cakoni and R. Kress, *Integral equations for inverse problems in corrosion detection from partial Cauchy data*, 1(2) Inv prob and Imag (2007) 229-245.

[18] F. Cakoni, G.C. Hsiao, *On the Boundary Integral Equation Method for a Mixed Boundary Value Problem of the biharmonic Equation.* 7-11 Complex Var and Ellip Eq (2005) pp 681-696.

[19] F. Cakoni, R. Kress, and Ch. Schuft, *Simultaneous reconstruction of shape and impedance in corrosion detection*. 17 Methods Appl. Anal (2010) 357-378.

[20] F. Cakoni, R. Kress and Ch. Schuft, *Integral equations for shape and impedance reconstruction in corrosion detection*, 9-26 Inverse Problems (2010) 095012.

[21] G.C. Hsiao and W.L. Wendland, *Boundary Integral Equations*, Springer-Verlag 2008.

[22] G. Filippo, G.H. Christoph, S. Guido, *Polyharmonic boundary value problems. A monograph on positivity preserving and nonlinear higher order elliptic equations in bounded domains.* Springer-Verlag 2010.

[23] J. Hadamard *Lecture note on Cauchys problem in linear partial differential equations.* Yale University Press, (1923).

[24] Haim Brezis.: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag (2010).

[25] T. Sh. Kal'menov and U. A. Iskakova , *On a boundary value problem for the biharmonic equation.* 1676-1 AIP Conference Proceedings (2015) 020031.

[26] V.V. Karachik, B.T. Torebek, *On One Mathematical Model Described by Boundary Value Problem for the Biharmonic Equation.* Bulletin SUSU MMCS (2016) pp 40-52.

[27] T. Wei n, Y.G.Chen.: *Numerical identification for impedance coefficient by a MFS-based optimization method.* Engineering Analysis with Boundary Elements, (2012) 1445–1452.

[28] Kendall E. Atkinson.: *The numerical solution of integral equations of the second kind.* cambridge monographs on applied and computational mathematics (2009)

[29] Lars-Erik Andersson and Tommy Elfving and Gene H. Golub.: *Solution of biharmonic equations with application to radar imaging.* Jour of Comp and App Math (1998) pp 153-180.

[30] Lesnic D, Elliott L, Ingham DB.: *The boundary element solution of the Laplace and biharmonic equations subject to noisy data.* 43-3 Int J Numer Meth Eng (1998) pp 479-492.

[31] J. Li, *Application of radial basis meshless methods to direct and inverse biharmonic boundary value problems.* 21-4 Commun Numer Methods Eng 2005 pp 169-182.

[32] Liquan Mei and Weigeng Cui. *A new regularization method based on the magnitude response function of Chebyshev filter for inverse Problems.* 8(1) World Jour of Model and Sim, (2012). pp 19-26

[33] L. Marin, D.Lesnic, *The method of fundamental solutions for inverse boundary value problems associated with the two dimentional biharmonic equation.* Math and Comp Modell (2005) pp 261-278.

[34] W. Rundell , *Inverse Problems Recovering an obstacle and a nonlinear conductivity from Cauchy data*, Inverse Problems 24, (2008), 055015.

[35] V.V. Meleshko, *Selected topics in the history of the two-dimensional biharmonic problem.* 56(1) Appl. Mech. Rev. (2003) pp 33-85.

[36] Ming-chlai and Hsi-chi.Lui *Fast direct solver of the biharmonic equation on a disk and it's application to incompressible flows.* 164-3 App Math and Comp (2005) pp 679-695.

[37] M. Doschoris, *Towards a generalization of the separation of variables technique.* Methods and applications of analysis. v19 (2012) pp 381–402.

[38] M. Thamban Nair, *Linear Operator Equations-Approximation and Regularization*, World Scientific Publishing Company (2009).

[39] R. Glowinski and O. Pironneau, *Numerical methods for the first biharmonic equation and for the two-dimensional stokes problem.* 21-2 (1977) pp 167-212.

[40] R. Kress, *Linear Integral Equations.* Springer-Verlag (1999).

[41] R. Kress, *Linear Integral Equations*. Springer (2014).

[42] R. Kress and F.W. Gehring, *Numerical Analysis*, Springer (1998).

[43] R. Kress, *A Nystrom method for boundary integral equations in domains with corners*, 58 Numerische Mathematik volume (1990) 145-161.

[44] R. Chapko, B.T. Johansson, *Integral equations for biharmonic data completion*, 13(5) Inv prob and Imag (2019), 1095-1111.

[45] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, Springer (2001).

[46] G. Sweers, *A survey on boundary conditions for the biharmonic.* Comp Var and Ell Eq, (2008) pp 79-93.

[47] Schumaker Larry, *Spline Functions - Basic Theory*, Cambridge University Press (2007).

[48] V.V. Karachik, *Solving a Problem of Robin Type for biharmonic Equation,* 62(2) Russian Mathematics, (2018) pp 34-48.

[49] V.V. Karachik, *Riquier-Neumann Problem for the Polyharmonic Equation in a Ball,* 54 Diff Eq (2018), 648–657.

[50] J. Cheng, M. Choulliz and X. Yangy, *An Iterative BEM for the Inverse Problem of Detecting Corrosion in a Pipe*, Series in Contemporary Applied Mathematics, Frontiers and Prospects of Contemporary Applied Mathematics, pp. 1-17 (2006)

[51] Ch. Constanda, D. Doty and W. Hamill, *Boundary Integral Equation Methods and Numerical Solutions-Thin Plates on an Elastic Foundation*, Springer, (2016).

[52] K. B. Howell, *Principles of Fourier Analysis*, Chapman and Hall-CRC, (2001).

[53] W Michael Lai, Er. Krempl and D. Rubin, *Introduction to Continuum Mechanics*, Butterworth-Heinemann (1996).

[54] A. Constantinescu and A. Korsunsky, *Elasticity with Mathematica- An introduction to continuum mechanics and linear elasticity*, Cambridge University Press (2007).

[55] H. Saker and N. Bouselsal, *On the bilaplacian problem with nonlinear boundary conditions*, Ind jour of pure and app math (2016) 425-435.

[56] H. Hedenmalm, *On the uniqueness theorem of Holmgren.* 281 Mathematische Zeitschrift (2015) 357–378.

[57] CHAZARIN and J and PIRIOU, A, *Introduction à la théorie des équatons aux dérivées partielles linéaires*, Gauthier-Villars, Paris (1981).

[58] Roman Chapko, Leonidas Mindrinos.: *On the non-linear integral equation approach for an inverse boundary value problem for the heat equation* 119 Journal of Eng Math (2019) pp 255–268.

[59] C.A. Brebbia, J. C. F. Telles and L.C. Wrobel, *Boundary Element Techniques*. Springer. Verlag,Berlin 1984.

[60] Y. Jeon, *New indirect scalar boundary integral equation formulas for the biharmonic equation*, 135 Jour of Comput and App Math (2001) 313–324.

[61] Y. Jeon, *An indirect boundary integral equation method for the Biharmonic equation*. SIAM J. Numer. Anal. 31 (2) 461–476, (1994) .

[62] Yu.P. Petrov, V.S. Sizikov.: *Well-posed, ill-posed, and intermediate problems with applications*. VSP (2005).

[63] A. A. Samarskii and Petr N. Vabishchevich *Numerical methods for solving inverse problems of mathematical physics* 52 Walter de Gruyter (2007).

[64] Lu S., Pereverzev S.V, *Regularization Theory for Ill-posed Problems*. Selected Topics-de Gruyter (2013).

[65] S I Kabanikhin.: *Inverse and ill-posed problems - theory and applications* 55 De Gruyter (2012).

[66] Gazzola, Filippo, Grunau, Hans-Christoph, Sweers, Guido, *Polyharmonic Boundary Value Problems. Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains*. Springer-Verlag, (2010).

[67] V.V Meleshko.: *Selected topics in the history of the two-dimensional Biharmonic problem*. Appl. Mech. Rev. Jan , 56(1) (2003) pp 33-85.

[68] A. Zeb, D. B. Ingham, D. Lesnic.: *The method of fundamental solutions for a Biharmonic inverse boundary determination problem* 42 Comput Mech (2008) pp 371–379.

[69] Å. Björck.: *Algorithms for Linear Least Squares Problems*. 77 Computer Algorithms for Solving Linear Algebraic Equations (1991) pp 57-92.

[70] Ch. L. Lawson, R. J. Hanson.: *Solving Least Squares Problems*, Society for Industrial Mathematics, (1987).

[71] A.G. Ramm, *An inverse problem for biharmonic equation.* 11 Internat J Math and Math Sci (1988) 152649.

[72] J. Adam Kolodziej, A. Piotr Zielinski, *Boundary Collocation Techniques and Their Application in Engineering*, WIT Press (2009).

[73] H. P. Langtangen.: *Python Scripting for Computational Science*, Springer-Verlag Berlin Heidelberg (2008).

[74] Hongwei Guo.: *A Simple Algorithm for Fitting a Gaussian Function*, 28(5) IEEE Signal Processing Magazine (2011) pp 134-137.

[75] Chia-Ming Fan and Hsin-Fang Chan.: *Modified collocation Trefftz method for the geometry boundary identification problem of heat conduction*. 59-1 Inter Jour of Comp and Method (2011) pp 58–75.

[76] S. I. Kabanikhin, *Definitions and examples of inverse and ill-posed problems*. 16, Jour Inv Ill-Posed Problems (2008), 317–357.

[77] D. Medkova, *The Laplace Equation-Boundary Value Problems on Bounded and Unbounded Lipschitz Domains*. Springer (2018).

[78] C.T.Chen, K.H.Chen, J.F.Lee, J.T.Chen. *Adaptive error estimation of the Trefftz method for solving the Cauchy problem*, 44 WIT Transactions on Modelling and Simulation (2007).

[79] Z.C.Li, T.T.Lu, H-Y.Hu, A.H.D.Cheng. *Trefftz and Collocation Methods*, WIT Press (2008).

[80] Z.C.Li, T.T.Lu, H.T. Huang, A.H.D.Cheng, *Trefftz, Collocation, and Other Boundary Methods - A Comparison*, 23-1 Numerical meth for part diff equation (2008) pp 93-144.

[81] Kita E, Kamiya N. *Trefftz method: an overview.* 24-1-3 Adv Eng Software, (1995) 3–12.

[82] A. Karageorghis, D. Lesnic and L. Marin. *Regularized collocation Trefftz method for void detection in two-dimensional steady-state heat conduction problems*, Vol. 22, No. 3 Inverse Problems in Science and Engineering, (2014) pp 395–418.

[83] Ch. Sh. Liu. *A Highly Accurate MCTM for Direct and Inverse Problems of Biharmonic Equation in Arbitrary Plane Domains.* 30-2 CMES, (2008) pp 65-75.

[84] Ch. Sh. Liu. *A Highly Accurate Collocation Trefftz Method for Solving the Laplace Equation in the Doubly Connected Domains*, 24-1 Numeric Meth Part diff Eq (2008) pp 179-192.

[85] C. Tajani and J. Abouchabaka, *On the data completion problem for Laplace's equation.* 45(1) Annals of the Univ of Cra, Math and Comp Sci Ser (2018) Pages 11-36.

[86] L. Lanzani, L. Capogna and R.M. Brown. *The solution of mixed boundary value problem for the Laplace equation in a multiply connected domain*, 8(26)-2 Probl. Anal. Issues Anal. (2019), pp. 51–66.

[87] A. P. S. Selvadurai, *Partial Differential Equations in Mechanics 2 - The Biharmonic Equation, Poisson's Equation*, Springer - Verlag Berlin Heidelberg (2000).

[88] W. McLean.: *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press (2000).

[89] G. Fairweather and R.L. Johnston, *The method of fundamental solutions for problems in potential theory.* 8-4 App Math Modell (1984) pp 265-270

[90] M.A. Jaswon, *Integral Equation Methods in Potential Theory I.* 275-1360 Proc of roy soc A, math phys and eng sci 1963 pp 23-32.

[91] C.A. Brebia, J.C.F. Telles and L.C. Wrobel, *Boundary Element Techniques: Theory and Application in Engineering.* Springer 1984.

[92] Ch-Sh. Liu. *A Modified Trefftz Method for Two-Dimensional Laplace Equation Considering the Domain's Characteristic Length.* 21-1 CMES, (2007), pp 53-65.

[93] V. Thirumaran, J. A. Weliwita and M. I. M. Ishak, *An Analysis of Axial Couette Flow in Annular Region of Abruptly Stopped Pipes.* Phy Sci Inter Jour, 17-4 (2018) pp 1-12.

[94] S. Granlund and N. Marola, *On the problem of unique continuation for the p-Laplace equation.* 101 Nonlinear Anal Theo, Meths and Apps (2014) pp 89-97.

[95] H. Blum, R. Rannacher and R. Leis, *On the boundary value problem of the biharmonic operator on domains with angular corners.* 2-4 Math Meth in the Appl Sci (1980) pp 556-581.

[96] A. Zeb, L. Elliott, D. B. Ingham and D. Lesnic, *A comparison of different methods to solve inverse biharmonic boundary value problems.* 45-12 Numer Meth Engng (1999) pp 1791-1806.

[97] E. J. Sapountzakis, J. T. Katsikadelis, *Elastic deformation of ribbed plates under static, transverse and in-plane loading.* 74 Comp and St (2000) 571-581.

[98] M. S. Nerantzaki, *Solving plate bending problems by the analog equation method* 7 Trans on Mod and Sim (1994) 11153.

[99] D. Lesnic, L. Elliot, D. B. Iigham and A. Zeb, *A numerical method for an inverse biharmonic problem.* 7 Inv Prob in Eng (1999) pp 409-431.