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Periodicity and stability of solutions for nonlinear delay dynamic equations on time scales by the fixed point technique

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A Doctoral Thesis,

By Manel Gouasmia
Advisors: Pr. A. Ardjouni and Pr. A. Djoudi

Dedication

I dedicate this work to:
My very dear parents,
My dear brothers and sisters,
All my family,
And all my friends and all my teachers.

Acknowledgement

In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohammed His servant and messenger.

First and foremost I wish to thank Allah for his help and making all kind of task easy, he is indeed the merciful and compassionate.

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ملخـــــــص

تظهر المعادلات الديناميكية ذات تأخر على مقياس زمني بشكل تلقائي في مختلفة المجالات العلمية مثل الفيزياء والهندسة والطب والكيمياء الكهربائية ونظرية التحكم وما إلى ذلك.

الفعالية الكبيرة لهذا النوع من المعادلات في نمذجة العديد من الظواهر الطبيعية والفيزيائية شجعت الكثير من الباحثين لدراسة جوانبها الكمية والنوعية.

الهدف من هذه الأطروحة هو دراسة وجود و دورية وايجابية واستقرار حلول المعادلات الديناميكية ذات تأخير على مقياس زمني. تستند النتائج في هذا العمل إلى تقنيات النقطة الثابتة.

سنستعمل نظرية النقطة الثابتة لكراسنوسلسكي-بيرتون لإظهار نتائج وجود حلول دورية وغير سلبية واستقرارها للمعادلات الديناميكية غير الخطية من النوع المحايد على مقياس زمني. وكما سندرس أيضًا السلوك المقارب لحلول المعادلات الديناميكية المحايدة المختلطة على مقياس زمني باستعمال مبدأ التطبيق المقلص.

الكلمـات المفتاحيـة: معـادلات ديناميكية حيادية، نظريـات النقطة الثابتـة، الدوريـة، الايجابيـة، الاستقرار، أزمنة سلمية.

Abstract

Delay dynamic equations on time scales appear as a natural description of observed evolution phenomena in various scientific areas such as physics, engineering, medicine, electrochemistry, control theory, etc. The effectiveness of these equations in the modeling of several real-world phenomena has motivated many researchers to study their quantitative and qualitative aspects. The objective of this thesis is the study the existence, periodicity, positivity and stability of solutions of neutral dynamic equations with delay on time scale. Results in this work are based on fixed point techniques. Using Krasnoselskii-Burton's fixed point theorem for show the existence results of periodic and nonnegative solutions and their stability for nonlinear dynamic equations of neutral type on time scale, and also study of asymptotic behavior of solutions of mixed type neutral dynamic equations on time scales using the contraction principle.

Keywords: Fixed point theory, periodicity, positivity, stability, neutral dynamic equations, asymptotic behavior, time scales.

Mathematics Subject Classification: 34K20, 34K30, 34K40.

Résumé

Les équations dynamiques à retard sur des échelles de temps apparaissent naturellement dans différents domaines scientiques comme la physique, l'ingénierie, la médicine, l'électrochimie, la théorie du contrôle, etc. L'efficacité de ces équations dans la modélisation de plusieurs phénomènes du monde réel a motivé beaucoup de chercheurs à étudier leurs aspects quantitatifs et qualitatifs. L'objectif de cette thèse est l'étude de l'existence, la périodicité, la positivité et la stabilité des solutions d'équations dynamiques neutrales à retard sur des échelle de temps. Les résultats dans ce travail sont basée sur les techniques du point fixe. En utilisant le théorème du point fixe de Krasnoselskii-Burton pour montrer les résultats d'existence de solutions périodiques et non négatives et leur stabilité pour des équations dynamiques non linéaires de type neutre sur des échelles de temps, et étude aussi le comportement asymptotique de solutions d'équations dynamiques de type neutre mixte sur une échelle de temps à l'aide le principe d'application contractante.

Mots-clés: Théorie de point fixe, periodicité, positivité, stabilité, equations dynamique neutral, le comportement asymptotique, échelles de temps.

Mathematics Subject Classification: 34K20, 34K30, 34K40.

Contents

Introduction

Delay differential equations play significant roles in every facet of real life applications. For example, in the birth process biological populations. Many phenomena encountered in physics, biology, chemistry, engineering, medicine, electrochemistry, control theory, etc, have found in the theory of delay differential equations, late, a good modeling means, (a more realistic means than in the case of ordinary differential equations). DDEs are differential equations where the derivatives of some unknown functions at two different time instants (past and present) are correlated.

From years (40), the theory of delay equations has known a great development, notably we find Belman and Cooke (1963), Hale (1977), etc. But recently, many phenomena have been proposed for the modeling of some complicated situations, the delay equations were introduced to model phenomena in which there is a time lag between the action on the system and the response of the system to this action. The nature of the delay (discrete, continuous, infin, dependent on the state, ...) potentially complicates the study of the equations.

Many researchers are studying the existence, uniqueness, periodicity and positivity of solutions of delay differential equations [1], [4]-[6], [25], [26], [27], [41], [49], [52].

Fixed point theory is at the heart of nonlinear analysis and then provides the tools necessary to have existence theorems in many different non-linear problems. In the 19th century The study of fixed point theory was initiated by Poincare and in 20th century developed by many mathematicians like Brouwer, Schauder, Kakutani, Banach, Kannan, Tarski, and others. The precursors of the approximate fixed-point theory are explicit in Picard's work, and in 1922 the Polish mathematician Stefan Banach, who is credited on the placement of an abstract idea, Banach established the existence and the uniqueness of the fixed point of a contraction mapping in a complete metric space. This theorem gives a regular behavior of the fixed point with respect to the parameters. In addition, it provides a fixed point approximation algorithm as the limit of an iterated sequence.

Among the hundreds of fixed point theorems, Brouwer's is particularly famous, in partly because it is used in many mathematical branches. The theorem is supposed to have originated from Brouwer's observation of a cup of coffee. If one stirs to dissolve a lump of sugar, it appears there is always a point without motion. He drew the conclusion that at any moment, there is a point on the surface that is not moving. The fixed point is not necessarily the point that seems to be motionless, since the center of the turbulence moves a little bit. The result is not intuitive, since the original fixed point may become mobile when another fixed point appears.

Schauder's Fixed Point Theorem established in 1930, and states that a continuous application on a compact convex admits a fixed point, which is not necessarily unique. In 1955, Krasnoselskii combined the geometrical fixed point theory of Banach and the topological fixed point theory of Schauder and established a theorem of point fixes hybrid known under its name. In 1996, Burton introduced the concept of large contraction and established that a large contraction possesses a fixed point in a complete metric space. Subsequently, Burton has investigated Krasnoselskii's fixed point theorem and has established, what we have called, Krasnoselskii-Burton theorem,

In 1892 Liapunov published a large work on the stability of ordinary differential equations based on definite positive functions. His work was the foundation of the theory of stability as we know it today for EDP, EDO, differential equations and integral equations as well as in control theory.

There is still a multitude of difficulties that persist in this theory and it seems that there are no other avenues to study. During this last decade several investigators have undertaken a study, with the aim of overcoming these difficulties, based on the fixed point theory.

The fixed point method used for stability purposes has been used in a number of recent works such as [1], [4], [5], [6], [7], [12], [20], [21], [22], [23], [25], [26], [36], [37], [40], [49], [50], [56]. This method is based on three fundamental elements. Namely,

- A fixed point application,
- A functional space suitable for containing the desirable solutions,
- A fixed point theorem.

This method has shown significant advantages over that of Liapunov. In particular when the coefficients are unbounded and / or if the delay is unbounded the direct method of Liapunov showed its limits in contrast to that of the fixed point. In addition, the conditions of the fixed point method are average, on the other hand, those of the second are always punctual.

The concept of time scales analysis is a fairly new idea, was introduced in 1988 by the German mathematician Stefan Hilger [35] in its thesis of doctorate under the direction of professor Bern Aulbach. However, similar ideas have been used before and go back at least to the introduction of the Riemann-Stieltjes integral which unifies sums and integrals. Its principal objective is to unify the discrete analysis and the continuous analysis, Bohner and Peterson [8] and [9] (2001, 2003) further develop TSC using many of the usual notions of calculus over time scales, including a generalized derivative, a unified set of differentiation rules for finding derivatives (power, product, quotient, and chain rules), and solutions to first-order equations.

A time scale is simply any closed subset of real numbers with the purpose of developing an equation that evolves over values in this scale.It is clear that if $\mathbb{T} = \mathbb{R}$, the dynamic equations on a the time scales become differential equations. If $\mathbb{T} = \mathbb{Z}$, the dynamic equations on a the time scales become difference equations.

For example, for a differential equation that models population density, the time scale would begin at time equals zero and run over all positive real numbers. In the case of a difference equation model that describes a population of dividing cells, time is discrete. Each time step is the amount of time it takes for a single cell to divide. In this case, the time scale would be positive integers. Time-scale calculus provides a unified theoretical tool for any combination of differential and difference equations.

For example, they can model insect populations that evolve continuously while in season, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a non-overlapping population. The time scale for a mosquito population would be $\mathbb{T} =$ mosquito season 1 $\cup$ winter moths $\cup$ mosquito season 2 $\cup$ winter moths ...

By working under a general time scale, it is possible to simultaneously advance these two fields of mathematics. In a second step, the theory developed around the time scales allows the study of phenomena modeling in a way that calls simultaneously discrete and continuous.

Presentation of the thesis

This thesis consists of five chapters. In the following, a summary of the content of each part is presented:

Chapter 1: This chapter is devoted to the presentation of some useful preliminaries as well as reminders of some essential definitions and necessary to better understand the manuscript.

Chapter 2: This Chapter presents a theoretical basis for calculating at time scales.

The essential of this thesis is presented in the Third, Fourth and Fifth chapters which correspond to published articles.

Chapter 3: The subject of the third chapter is the study of the existence of periodic and nonnegative periodic solutions of nonlinear neutral dynamic equation with variable delay of the form

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t-\tau(t)))^{\Delta} + G(t, x(t), x(t-\tau(t))).$$

This existence is obtained by the Krasnoselskii-Burton theorem. We invert the given equation to obtain an equivalent integral equation from which we define a fixed point mapping written as a sum of a large contraction and a completely continuous map. The Caratheodory condition is used for the functions Q and G. The results presented in this chapter are published in International Journal of Analysis and Applications (2018) (see [28]).

Chapter 4: We use the Krasnoselskii-Burton fixed point theorem to obtain stability results about the zero solution for the following neutral nonlinear dynamic equations with variable delay given by

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t-\tau(t)))^{\Delta} + G(t, x(t), x(t-\tau(t))),$$

with an assumed initial function

$$x(t) = \psi(t),\ t \in [m_0, 0] \cap \mathbb{T}.$$

The results presented in this chapter are accepted publication in Memoirs on Differential Equations and Mathematical Physics (2020) (see [30]).

Chapter 5: We concentrate on the asymptotic behavior of solutions for the neutral mixed type dynamic equation

$$x^{\Delta}(t) + a(t)\, x^{\widetilde{\Delta}}(\tau(t)) + \sum_{i=1}^{k} b_i(t)\, x(\tau_i(t)) + \sum_{j=1}^{l} c_j(t)\, x(r_j(t)) = 0,$$

with the initial condition where $\theta \in C_{rd}([\tau_0, t_0] \cap \mathbb{T}, \mathbb{R})$. The results presented in this chapter are published in Mathematics in Engineering, Science and Aerospace MESA (2019) (see [29]).

We conclude this thesis with a general conclusion, as well as the perspectives of our future research.

Chapter 1

Fixed point theory, functional differential equations and stability

Keywords. Fixed point theory, functional differential equations, stability.

In this chapter, We introduce the concepts necessary for the proper understanding of manuscript, It shares in four sections, the first section includes a brief reminder on the basic elements of functional analysis. the second section reserved for the various fixed point theorems used in this work. the third section is devoted to the presentation of the basic notion of the theory of delay functional differential equations. we conclude the chapter by the theory of stability. The illustrated in this chapter for specific references and many more examples and applications, (see [2], [10], [11], [14], [32], [33] [41], [42], [43], [45], [53], [54] and [55]).

1.1 Notation and preliminaries

Definition 1.1 (Metric space) A pair (X, d) is a metric space if X is a set and $d : X \times X \to [0, \infty)$ such that when y, z and u are in X then

i) $d(y, z) \geq 0$, $d(y, y) = 0$ and $d(y, z) = 0$ implies $y = z$,

ii) $d(y, z) = d(z, y)$,

iii) $d(y, z) \leq d(y, u) + d(u, z)$.

The metric space is complete if every Cauchy sequence in (X, d) has a limit in that space. A sequence $\{x_n\} \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $d(x_n, x_m) < \varepsilon$.

Theorem 1.1 (Convergent sequence) *Every convergent sequence in a metric space is a Cauchy sequence.*

Definition 1.2 (Normed space) A vector space $(X,+,.)$ is a normed space if for each $x,y \in X$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

i) $\|x\| = 0$ if and only if $x = 0$,

ii) $\|\alpha x\| = |\alpha| \, \|x\|$for each $\alpha \in \mathbb{R}$,

iii) $\|x+y\| \leq \|x\| + \|y\|$.

A normed space is a vector space and it a metric space with $d(x,y) = \|x-y\|$. But a vector space with a metric is not always a normed space.

Definition 1.3 A normed space $(X, \|.\|)$ is said to be **complete** if it is complete as a metric space (X,d), i.e., every Cauchy sequence is convergent in X.

Definition 1.4 (Banach space) A Banach space is a complete normed space.

Example 1.1 (a) Let $X = \mathbb{R}^n$, $n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms

$\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$;

$\|x\|_p = \left(\sum\limits_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $p \in (1, \infty)$;

$\|x\|_\infty = \max\limits_{1 \leq i \leq n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

(b) With any of these norms, $(\mathbb{R}^n, \|.\|)$ is a Banach space. It is complete because the real numbers are complete.

Theorem 1.2 *A closed subspace of a Banach space is a Banach space.*

Definition 1.5 (Compactness) A subset M of a metric space $(X;d)$ is compact if any sequence $\{x_n\}$ of M admits a subsequence with limit in M. M is relatively compact if every sequence of M admits a subsequence converging towards a limit belonging to X (i.e. $\overline{M}$ is compact).

Lemma 1.1 *A compact subset M of a metric space is closed and bounded.*

Proposition 1.1 *The set M is compact if f it is relatively compact and closed.*

Proposition 1.2 *Each relatively compact set is bounded.*

Definition 1.6 Let $\{f_n\}$ be a sequence of real valued functions with $f_n : [a,b] \to \mathbb{R}$.

a) $\{f_n\}$ is **uniformly bounded** on $[a,b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a,b]$.

b) $\{f_n\}$ is **equicontinuous** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \varepsilon$ for all n.

Theorem 1.3 (Ascoli-Arzela [14]) *If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

But here we manipulate function spaces defined on infinite t-intervals. So, for compactness we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([19], Theorem 1.2.2 p. 20) and is as follows.

Theorem 1.4 *Let $q : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{f_n\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}^+$ with $|f_n(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $f(t)$ with $|f(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

Definition 1.7 Let M be a subset of a Banach space X and $T : M \to X$. If T is continuous and $T(M)$ is contained in a compact subset of X, then T is a **compact mapping**.

Definition 1.8 Let M and Y be *Banach* space, and $T : D(T) \subseteq M \to Y$ an operator.T is called compact if

(i) T is continuous,

(ii) T maps bounded sets into relatively compact sets.

Example 1.2 a) The space $C([a, b], \mathbb{R}^n)$ consisting of all continuous functions $f : [a, b] \to \mathbb{R}^n$ is a vector space over the reels.

b) If $\|f\| = \max_{t\in[a,b]} |f(t)|$, where $|\cdot|$ is a norm in $\mathbb{R}^n$, then it is a Banach space.

c) For a given pair of positive constants M and K, the set

$$L = \{f \in C([a, b], \mathbb{R}^n) \mid \|f\| \leq M, |f(u) - f(v)| \leq K|u - v|\},$$

is compact. To see this, note first that Ascoli's theorem is also true for vector sequences; apply it to each component successively. If $\{f_n\}$ is any sequence in L, then it is uniformly bounded and equicontinuous. By Ascoli's theorem it has a subsequence converging uniformly to a continuous function $f : [a, b] \to \mathbb{R}^n$. But

$|f_n(t)| \leq M$ for any fixed t, so $\|f\| \leq M$. Moreover, if we denote the subsequence by $\{f_n\}$ again, then for fixed u and v there exist $\varepsilon_n > 0$ and $\delta_n > 0$ with

$$\begin{aligned} |f(u) - f(v)| &\leq |f(u) - f_n(u)| + |f_n(u) - f_n(v)| + |f_n(v) - f(v)| \\ &= \varepsilon_n + |f_n(u) - f_n(v)| + \delta_n \\ &\leq v + \delta_n + k|u - v| \to k|u - v|, \end{aligned}$$

as $n \to \infty$. Hence, $f \in L$ and L is compact.

Definition 1.9 Let P be a mapping from a metric space (X, d) into another metric space (Y, ρ). Then P is said to satisfy **Lipschitz condition** on X if there exists a constant $L > 0$ such that

$$\rho(Px, Py) \leq Ld(x, y) \text{ for all } x, y \in X.$$

If L is the least number for which Lipschitz condition holds, then L is called Lipschitz constant. In this case, we say that P is an L-**Lipschitz mapping** or simply a Lipschitzia mapping with Lipschitz constant L. Otherwise, it is called **non-Lipschitzian mapping**. An L-Lipschitz mapping P is said to be **contraction** if $L < 1$ and nonexpansive if $L = 1$. The mapping P is said to be contractive if

$$\rho(Px, Py) < d(x, y) \text{ for all } x, y \in X, \ x \neq y.$$

Definition 1.10 Let (X, d) be a complete metric space and $P : X \to X$. The mapping P is a **contraction** if there is an $\alpha \in (0, 1)$ such that $x, y \in X$ imply

$$d(Px, Py) \leq \alpha d(x, y).$$

Definition 1.11 Let (X, d) be a metric space and assume that $P : X \to X$. P is said to be a **large contraction**, if for $x, y \in X$, with $x \neq y$, we have
$d(Px, Py) < d(x, y)$, and $\forall \varepsilon > 0$, $\exists \delta < 1$ such that

$$[x, y \in X, d(x, y) \geq \varepsilon] \Rightarrow d(Px, Py) < \delta d(x, y).$$

Definition 1.12 A set X in a vector space is **convex** if $x, y \in X$ and $0 \leq k \leq 1$ imply $kx + (1 - k)y \in X$.

1.2 Fixed point theorems

Definition 1.13 Let f be a mapping in the set M. we call **fixed point** of f any point x satisfying $f(x) = x$. If there exists such x, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.

Theorem 1.5 (Brouwer Fixed Point Theorem (1912)) *Suppose that M is a nonempty, convex, compact subset of $\mathbb{R}^N$ where $N > 1$, and that $f : M \to M$ is a continuous mapping. Then f has a fixed point.*

Theorem 1.6 (Contraction Mapping Principle (1922) [14]) *Let (X, d) a complete metric space and let $P : X \to X$ a contraction mapping. Then there is one and only one point $z \in X$ with $Pz = z$. Moreover $z = \lim z_n$ where $z_{n+1} = Pz_n$ and z_1 chosen arbitrarily in X.*

Theorem 1.7 ([14]) *Let (X, d) a compact nonempty metric space and let $P : X \to X$. If*

$$d(Px, Py) < d(x, y), \text{ for } x \neq y$$

Then P has a fixed point.

Theorem 1.8 ([14]) *If (X, d) is a complete metric space and $P : X \to X$ is a α-contraction operator with fixed point x, then for any $y \in X$ we have*
(a) $d(x, y) \leq d(y, Py) \diagup (1 - \alpha)$.
(b) $d(P^n y, x) \leq \alpha^n d(y, Py) \diagup (1 - \alpha)$.

1.2.1 Krasnoselskii fixed point theorem

The fixed point theorem of Krasnoselskii is an hybrid result and is based on Banach and Schauder theorems. Firstly, we recall the theorem of Schauder

Definition 1.14 ([53]) A topological space X has the fixed-point property if whenever $P : X \to X$ is continuous, then P has a **fixed point.**

Theorem 1.9 (Schauder's first fixed-point theorem (1930) [53]) *Any compact convex nonempty subset M of a Banach space has the X fixed-point property.*

Theorem 1.10 (Schauder's second fixed point theorem [53]) *Let M be a nonempty closed convex bounded subset of a Banach space $(X, \|.\|)$. Then every continuous compact mapping $P : M \to M$ has a fixed point.*

The fixed point theorem of Krasnoselskii is a combination of Banach theorem and that of Schauder. It was the object of several studies these last years and one meets it in several forms. In particular, the theorem of Krasnoselskii gives the existence and the stability of the solutions of the functional differential equations and the nonlinear integral equations with delay of mixed type.

In 1955 Krasnoselskii (see [52], [53]) observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,

Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

Consider the differential equation

$$x'(t) = -a(t)\,x(t) - g(t,x). \tag{1.1}$$

We can transform this equation in another form while writing, formally

$$x'(t)\,e^{\int_0^t a(s)ds} = -a(t)\,x(t)\,e^{\int_0^t a(s)ds} - g(t,x)\,e^{\int_0^t a(s)ds},$$

thus

$$x'(t)\,e^{\int_0^t a(s)ds} + a(t)\,x(t)\,e^{\int_0^t a(s)ds} = -g(t,x)\,e^{\int_0^t a(s)ds},$$

or

$$\left(x(t)\,e^{\int_0^t a(s)ds}\right)' = -g(t,x)\,e^{\int_0^t a(s)ds},$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^{t}\left(x(u)\,e^{\int_0^u a(s)ds}\right)' du = -\int_{t-T}^{t} g(u,x)\,e^{\int_0^u a(s)ds}du,$$

what gives

$$x(t) = x(t-T)\,e^{-\int_{t-T}^{t} a(s)ds} - \int_{t-T}^{t} g(u,x)\,e^{-\int_u^t a(s)ds}du. \tag{1.2}$$

If we suppose that $e^{-\int_{t-T}^{t} a(s)ds} = \alpha < 1$ and if $(M, \|.\|)$ is the Banach space of functions continuous and T-periodic $\varphi : \mathbb{R} \to \mathbb{R}$, then the equation (1.2) can be written as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (P\varphi)(t),$$

where B is contraction provides that the constant $\alpha < 1$ and A is compact mapping.

This example shows the birth of the mapping $(P\varphi) := (B\varphi) + (A\varphi)$ who is identified with a sum of a contraction and a compact mapping.

Thus, the search of the solution for (1.2) requires an adequate theorem which applies to this hybrid operator P and who can conclude the existence for a fixed point which will be, in his turn, solution of the initial equation (1.1). Krasnoselskii found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result ([15], [53]).

Theorem 1.11 (Krasnoselskii (1955)) *Let M be a closed bounded convex nonempty subset of a Banach space $(X, \|.\|)$. Suppose that A and B map M into X such that*

(i) *A is compact and continuous,*

(ii) *B is a contraction mapping,*

(iii) *$x, y \in M$, implies $Ax + By \in M$,*

Then there exists $z \in M$ with $z = Az + Bz$.

Note that if $A = 0$, the theorem become the theorem of Banach. If $B = 0$, then the theorem is not other than the theorem of Schauder.
Proof. According to the condition (iii) we have

$$\begin{aligned} \|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\ &\leq \|x-y\|+\|Bx-By\| \\ &\leq \|x-y\|+\alpha\,\|x-y\| \\ &= (1+\alpha)\,\|x-y\|\,, \end{aligned}$$

and

$$\begin{aligned} \|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\ &\geq \|x-y\|-\|Bx-By\| \\ &\geq \|x-y\|-\alpha\,\|x-y\| \\ &= (1-\alpha)\,\|x-y\|\,. \end{aligned}$$

In short

$$(1-\alpha)\,\|x-y\| \leq \|(I-B)\,x-(I-B)\,y\| \leq (1+\alpha)\,\|x-y\|\,.$$

This inequality shows that $(I-B): M \to (I-B)M$ is continuous and bijective. Thus, $(I-B)^{-1}$ exist and is continuous. Let us pose $U := (I-B)^{-1}A$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in M \text{ such that } (I-B)^{-1}Az = z.$$

This is equivalent to $z = Az + Bz$. ∎

1.2.2 Krasnoselskii-Burton fixed point theorem

In this many work on stability with the help of the technique of fixed point T.A. Burton ([11]) observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated the Theorem 1.13 below (see [11] for the proof).

Burton ([11]) remarked that in certain problems the situation does not arise in contraction form. For example, if we consider the equation $x' = -x^3 = -x + (x - x^3)$.

It is proved in [11] that a large contraction defined on a bounded and complete metric space has a unique fixed point.

Theorem 1.12 (Burton [11]) *Let (X, d) be a complete metric space and P be a large contraction. Suppose there is an $x \in X$ and an $L > 0$, such that $(x, P^n x) \leq L$ for all $n \geq 1$. Then P has a unique fixed point in X.*

Proof. Suppose there exist $x \in X$, consider $\{P^n x\}$. If this is a Cauchy sequence then by the triangle inequality we have for $m \geq n$

$$\begin{aligned} d(P^n x, P^m x) &\leq d\left(P^n x, P^{n+1} x\right) + d\left(P^{n+1} x, P^{n+2} x\right) + ... + d\left(P^{m-1} x, P^m x\right) \\ &\leq \left(\delta^n + \delta^{n+1} + ... + \delta^{m-1}\right) d(x, Px) \\ &\leq \frac{\delta^n}{1-\delta} d(x, Px). \end{aligned}$$

Thus $d(P^n x, P^m x) \to 0$ if $n, m \to \infty$, since (X, d) is a complete metric space the sequence $\{P^n x\}$ has a limit y in X. This fixed point is unique since $Pz = z$ and $Pw = w$ we have

$$d(z, w) = d(Pz, Pw) \leq \delta d(z, w),$$

so that $d(z, w) = 0$, that is $z = w$.

Suppose now the contradiction, if $\{P^n x\}$ is not a Cauchy sequence, then there

$$\exists \varepsilon > 0, \{N_k\} \uparrow \infty, n_k > N_k, m_k > n_k,$$

with $d(P^{m_k} x, P^{n_k} x) \geq \varepsilon$. Thus

$$\begin{aligned} \varepsilon &\leq d(P^{m_k} x, P^{n_k} x) \leq d\left(P^{m_k-1} x, P^{n_k-1} x\right) \leq d\left(P^{m_k-2} x, P^{n_k-2} x\right) \\ &\leq ... \leq d\left(P^{m_k-n_k+1} x, Px\right) \leq d\left(P^{m_k-n_k} x, x\right). \end{aligned}$$

Since P is large contraction, for this $\varepsilon > 0$ there is a $\delta < 1$ such that

$$\begin{aligned} \varepsilon &\leq d(P^{m_k} x, P^{n_k} x) \leq d\left(P^{m_k-1} x, P^{n_k-1} x\right) \\ &\leq ... \leq \delta^{n_k} d\left(P^{m_k-n_k} x, x\right), \end{aligned}$$

which contradict the fact that $\varepsilon > 0$ and $\delta < 1$ for $n_k \to \infty$. Then P has a unique fixed point in X. ■

Lemma 1.2 ([11]) *If $(X, \|.\|)$ is a normed space, if $M \subset X$, if $B : M \to X$ is a large contraction, then $(I - B)$ is a homeomorphism of M onto $(I - B) M$.*

Proof. Clearly, $I - B$ is continuous. To see that if $x \neq y$, then

$$\begin{aligned} \|(I-B) x - (I-B) y\| &= \|(x-y) - (Bx - By)\| \\ &\geq \|x-y\| - \|Bx - By\| \\ &\geq \|x-y\| - \|x-y\| \\ &= 0. \end{aligned}$$

Hence, $I-B$ is $1-1$ and $(I-B)^{-1}$ exists.

Suppose that $(I-B)^{-1}$ is not continuous. Then $\exists\,(I-B)\,y$ and $(I-B)\,x_n \to (I-B)\,y$ but $x_n \nrightarrow y$.

Now for

$$\begin{aligned} \forall\varepsilon &> 0\exists N \text{ such that } n > N \Rightarrow \\ \varepsilon &\geq \|(I-B)\,x_n - (I-B)\,y\| \geq \|x_n - y\| - \|Bx_n - By\|\,. \end{aligned} \tag{1.3}$$

Since $x_n \nrightarrow y$ $\exists\varepsilon_0 > 0$ and $\{x_{n_k}\}$ with $\|y - x_{n_k}\| \geq \varepsilon_0$; as B is a large contraction there is a $\delta < 1$ with $\|By - Bx_{n_k}\| \leq \delta\,\|y - x_{n_k}\|$. Thus, from (1.3) we have

$$\begin{aligned} \varepsilon &\geq \|(I-B)\,x_{n_k} - (I-B)\,y\| \\ &\geq \|x_{n_k} - y\| - \delta\,\|x_{n_k} - y\| \\ &= (1-\delta)\,\|x_{n_k} - y\| \\ &\geq (1-\delta)\,\varepsilon_0. \end{aligned}$$

But ε_0 is fixed, $\delta < 1$, and a contradiction occurs as $\varepsilon \to 0$; that is, as $\varepsilon \to 0$, $n_k \to \infty$, but ε_0 remains fixed. This completes the proof. ■

Theorem 1.13 (Krasnoselskii-Burton's (1996) [11]) *Let M be a closed bounded convex non-empty subset of a Banach space $(X, \|.\|)$. Suppose that A, B map M into M and that*

(i) *A is continuous and AM is contained in a compact subset of M,*

(ii) *B is a large contraction,*

(iii) *for all $x, y \in M \Longrightarrow Ax + By \in M$,*

Then there is $y \in M$ with $y = Ay + By$.

Proof. For each fixed $y \in M$ the mapping $Pz = Bz + Ay$ is a large contraction on M with unique fixed point z (since M is bounded the L is assured in Theorem 1.12) so that $z = Bz + Ay$ has a unique solution z. Thus, $(I-B)\,z = Ay$ and by the lemma 1.2 $Hy := (I-B)^{-1}\,Ay$ is a continuous mapping of M into M. Now AM is contained in a compact subset of M and $(I-B)^{-1}$ is a continuous mapping of A into M; it is then well known that $(I-B)^{-1}\,AM$ is contained in a compact subset of M. By Schauder's second theorem there is a fixed point $y = (I-B)^{-1}\,Ay$ or $y = Ay + By$, as required. ■

1.3 Functional differential equations with delay

Delay-differential equations (DDEs) are a large and important class of dynamical systems. They often arise in either natural or technological control problems The delay is a natural component of different processes in biology, chemistry and communications to mention a few.

A delay differential equation (DDE) is an equation where the evolution of the system at a certain time, solution t, depends on the state of the system at an earlier time;

Depending on the kind of delay, we distinguish

Constant delay problems ($\tau = const$), time dependent delay problems ($\tau = \tau(t)$), state dependent delay problems ($\tau = \tau(t, y(t))$), neutral delay problems ($\tau = \tau(t, y(t), \dot{y}(t))$).

1.3.1 Basic statements on functional differential equations with delays

suppose $\tau \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, denote $C([a, b], \mathbb{R}^n)$, is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a, b] = [-\tau, 0]$, we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element φ in C by $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. Even though single bars are used for norms is different spaces, no confusion should arise.

Let $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$, be defined by

$$x_t(\theta) = x(t + \theta), \text{ for } -\tau \leq \theta \leq 0.$$

Definition 1.15 ([32]) If D is a subset of $\mathbb{R} \times C$, and $f : D \to \mathbb{R}^n$ is a given function and represents the right - hand derivative, we say that the relation

$$\begin{aligned} x' &= f(t, x_t), \\ x_t(\theta) &= x(t + \theta), \text{ for } \theta \in [-\tau, 0]. \end{aligned} \tag{1.4}$$

is a retarded functional differential equation on D and will denote this equation by **RFDE**.

Definition 1.16 ([32]) A function x is said to be a solution of equation (1.4) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n)$, $(t, x_t) \in D$ and $x(t)$ satisfies equation (1.4) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}$, $\varphi \in C$, we say $x(t, t_0, \varphi)$ is a solution of equation (1.4) with initial value φ at t_0 or simply a solution through (t_0, φ) if there is an $A > 0$ such that $x(t, t_0, \varphi)$ is a solution of equation (1.4) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t, t_0, \varphi) = \varphi$.

Equation (1.4) is a very general type of equation and includes ordinary differential equations ($\tau = 0$)

$$x'(t) = f(x(t)),$$

differential-difference equations of the type

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_p(t))),$$

with $0 \leq \tau_j(t) \leq r$, $j = 1, 2, ..., p$, as well as the integro-differential equation

$$x'(t) = \int_{-\tau}^{0} g(t, \theta, x(t + \theta))\, d\theta.$$

Much more general equation are also included in equation (1.4).

We say equation (1.4) is linear delay differential equation if $f(t, \varphi) = L(t, \varphi) + h(t)$ where $L(t, \varphi)$ is linear in φ and; Linear homogeneous if $h \equiv 0$. The equation (1.4) is autonomous if $f(t, \varphi) = g(\varphi)$ where g does not depend on t.

Example 1.3 For example the following equation **EDFR**.

a) $x'(t) = x(t - 1)$,

b) $x'(t) = -x(t) + x^2(\frac{t}{2})$,

c) $x'(t) = -x^3(t) + x(t - \sin^2 t)$.

Equation a) has a constant delay $\tau(t) = 1$. Equation b) has a variable delay $\tau(t) = \frac{t}{2}$ and equation c) has a variable and bounded funtional delay $\tau(t) = \sin^2(t)$.

Remark 1.1 A natural classification of functional differential equation is **retarded FDEs, neutral FDEs, advanced FDEs, and mixed FDEs**. This classification depends on whether the rate of change of the current state of the system depends on **past values** or **future values** or **both**.

Lemma 1.3 ([32]) *If $t_0 \in \mathbb{R}$, $\varphi \in C$ are given and $f(t, \varphi)$ is continuous, then finding a solution of Equation (1.4) through (t_0, φ) is equivalent to solving the integral equation*

$$\begin{aligned} x_{t_0} &= \varphi, \\ x(t) &= \varphi(0) + \int_{t_0}^{t} f(t, x_s)\, ds, \quad t \geq t_0. \end{aligned}$$

Lemma 1.4 ([32]) *If $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then x_t is a continuous function of t for t in $[t_0, t_0 + A]$.*

Proof. Since x is continuous on $[t_0 - \tau, t_0 + A]$, it is uniformly continuous and thus for any $\varepsilon > 0$ there is $\delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently for t in $[t_0, t_0 + A]$, $|t - s| < \delta$ we have

$$|x(t+\theta) - x(s+\theta)| < \varepsilon, \quad \forall \theta \in [-\tau, 0].$$

■

Theorem 1.14 (Existence, [32]) *Let M be an open subset of $\mathbb{R} \times C$ and $f : M \to \mathbb{R}^n$ be continuous. For any $(t_0, \varphi) \in M$, there exists a solution of equation (1.4) through (t_0, φ).*

Theorem 1.15 (uniqueness [32]) *Let M be an open subset of $\mathbb{R} \times C$ and suppose that $f : M \to \mathbb{R}^n$ be continuous and $f(t, \varphi)$ be lipschitzian with respect to φ in every compact subset of M. If $(t_0, \varphi) \in M$, then equation (1.4) has a unique solution passing through (t_0, φ).*

1.3.2 The Method of steps

One way of solving the delay differential equations (DDEs) is using the so-called Method of Steps. The idea is to start with the initial history on the interval $[-\tau, 0]$ and then use the differential equation to obtain a piece of solution on the next interval $[0, \tau]$. This process can then be repeated to generate the solution on succeeding intervals.n for understanding these method we give illustrative example.

Example 1.4 A delay differential equation is given by the following relation

$$\begin{aligned} x'(t) &= 2x\left(t - \frac{1}{2}\right), \\ x(t) &= 1 \quad \text{for } t \in \left[-\frac{1}{2}, 0\right], \end{aligned}$$

to find a solution for $t \in \left[0, \frac{1}{2}\right]$

$$0 \leq t \leq \frac{1}{2} \Rightarrow -\frac{1}{2} \leq t - \frac{1}{2} \leq 0 \Rightarrow x\left(t - \frac{1}{2}\right) = 1,$$

the equation becomes $x'(t) = 2$ and the general solution will be

$$x(t) = 2t + c.$$

Now we replace t by zero to obtain

$$x(t) = 1 + 2t \text{ on } \left[0, \frac{1}{2}\right].$$

We repeat this process now for $t \in \left[\frac{1}{2}, 1\right]$,

$$\frac{1}{2} \leq t \leq 1 \Rightarrow 0 \leq t - \frac{1}{2} \leq \frac{1}{2} \Rightarrow x\left(t - \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right),$$

and

$$x'(t) = 2\left(1 + 2\left(t - \frac{1}{2}\right)\right) = 4t.$$

It has the general solution

$$x(t) = 2t^2 + d, \quad d = \text{constant}.$$

We replace $t = \frac{1}{2}$ in the solution which we found for $t \in \left[0, \frac{1}{2}\right]$ to obtain $d = \frac{3}{2}$, we have

$$x(t) = 2t^2 + \frac{3}{2}, \quad t \in \left[\frac{1}{2}, 1\right].$$

We can progress in this process and solve the **DDE** (1.4) like an infinite series of **EDO**.

1.3.3 Neutral delay differential equations

Definition 1.17 ([32]) Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, φ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω. if D is continuous together with its first and second Fréchet derivatives with respect to 'φ and D_φ', the derivative with respect to 'φ, is atomic at β on Ω.

Definition 1.18 ([32]) Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. The equation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t), \tag{1.5}$$

is called the **neutral delay differential equation NDDE**.

Definition 1.19 ([32]) A function x is said to be a solution of Equation (1.5) if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that

$$x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n), \ (t, x_t) \in \Omega, \ t \in [t_0, t_0 + A),$$

$D(t, x_t)$ is continuously differentiable and satisfies Eq. (1.5) on $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}$, $\varphi \in C$, and $(t_0, \varphi) \in \Omega$, we say $x(t_0, \varphi)$ is a solution of Eq. (1.5) with initial value φ at t_0 or simply a solution through (t_0, φ) if there is an $A > 0$ such that $x(t_0, \varphi)$ is a solution of Eq. (1.5) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \varphi) = \varphi$.

Example 1.5 ([32]) If $\tau > 0$, B is an $n \times n$ constant matrix, $D(\varphi) = \varphi(0) - B(-r)$ and $f : \Omega \to \mathbb{R}^n$ is continuous, then the pair (D, f) defines an NDDE,

$$\frac{d}{dt}\left[x(t) - Bx(t-r)\right] = f(t, x_t).$$

Theorem 1.16 ([32]) *Let Ω be an open subset of $\mathbb{R} \times C$ and $f : \Omega \to \mathbb{R}^n$ be continuous. For any $(t_0, \varphi) \in \Omega$, there exists a solution of equation (1.5) through (t_0, φ).*

Theorem 1.17 (Existence and uniqueness [32]) *Let Ω be an open subset of $\mathbb{R} \times C$ and suppose that $f : \Omega \to \mathbb{R}^n$ be continuous and $f(t, \varphi)$ be lipschitzian with respect to φ in every compact subset of Ω. If $(t_0, \varphi) \in \Omega$, then equation (1.5) has a unique solution passing through (t_0, φ).*

1.3.4 Real examples of delay differential equations

Modeling of Cancer with delays

Time delays in the immune response (Delay equations for Tumor modelling)

In [44], a further class of models deals with interactions of tumor cells with the cells of the immune system.

These models are simplifications of the physiological phenomenon and consider mainly the following two-populations dynamics

tumor cells ≡ target cells ≡ prey,

immune system cells ≡ effectors ≡ predators.

Hypothesis of antitumoral immune surveillance (Burnet, 1970): the immune system patrols the cells of the body, and, upon recognition of a (group of) cell(s) that has become cancerous, it will attempt to destroy it (them), thus preventing the growth of some tumors.

Observed facts

- The immune system can, in some cases, eradicate the tumor.
- The tumor may escape from the immune system control and grow.
- An equilibrium is established: the tumor survives in a microscopic (steady) state.

Idea: do not treat the tumor but the immune system!

Immunotherapy focuses on the stimulation of the immune system

- Cancer vaccine trains the immune system to recognize tumor cells as targets manipulation of therapeutic antibodies stimulates the immune system to attack the tumor.

Aim: if it is not possible to eradicate the tumor, reduce its size to a life-compatible dimension.

One of the first models. Starting from an existing ODE Model (Mayer, 1995), Búric et al. investigated the effects of time delays in the immune system response. Their motivation was essentially mathematical: since the ODE model could not describe the frequently observed, irregular or chaotic dynamics, they introduced chaotic behavior as an effect of the time delay

$$T' = rT - kTI, \text{ target cells},$$

$$I' = pf(aT + (1-a)T_{\tau_T}) + sg(bI + (1-b)I_{\tau_I}) - I, \text{ immuneagent (IS)},$$

where

$$X = X(t), \ X_\tau = X(t-\tau),$$

and with

$$f(T) = \frac{T^4}{1+T^4}, \text{ activation of IS duetothetumor},$$

$$g(I) = p\frac{I^3}{1+I^3}, \text{ self-regulation of IS}.$$

This model includes two constant delays for the activation of the immune system:

1) τ_T is the delay due to the size of the tumor

2) τ_I is the delay due to the self-regulation processes in the IS effectors.

Predator-prey models by A. d'Onofrio A. d'Onofrio worked from 2005 to a general class of predator-prey models for tumor-immune system interplay. The general model is an ODE system

$$\begin{aligned} T' &= T(f(T) - \phi(T, I)), \text{ tumor}, \\ I' &= \beta(T)I - \mu(T)I + \sigma q(T) + \theta(t), \text{ immune agent (IS)}. \end{aligned}$$

- $f(T)$ is a bounded, positive, non-growing function which describes the tumor

growth.

- $\phi(T, I)$ is the loss of tumor cells due to the attack by the immune system.
- $\sigma q(T)$, with $q(0) = 1$, represents the influx of immune cells in tumor in situ (may depend on the tumor size).
- $\beta(T)$ is a growing function of T and models the stimulating effect of the tumor on immune cells proliferation.

- $\mu(T)$ is the loss rate of effectors(IS) due to their interactions with the tumor.
- $\theta(t)$ models the immunotherapy (constant, periodic or absent).

In 2010 the basic ODE model was modified by including a delay for the immune system response

$$\begin{aligned} T' &= T(f(T) - \phi(T,I)), \text{ tumor}, \\ I' &= \beta(T_\tau)I - \mu(T)I + \sigma q(T) + \theta(t), \text{ immune agent (IS)}. \end{aligned}$$

Interactions between delayed immune response and immunotherapy were investigated.

The therapy was assumed to be ω-periodic

$$\theta(t) = \theta_A \exp\left(-\frac{1}{\gamma}\left(t - \omega\left\lfloor\frac{t}{\omega}\right\rfloor\right)\right),$$

where θ_A is the maximal 'boost' for the influx of effectors, ω is the time between two consecutive deliveries and γ is a measure of decay time.

Blowfly equation

The blowfly equation describes the (adult) population of flies P at time t

$$\begin{aligned} P'(t) &= b(P(t-\tau))\,P(t-\tau) - \mu(P(t))\,P(t), \\ P(t) &= \phi_0(t), \ t \leq 0. \end{aligned}$$

Here the delay occurs in the birth-term b: individuals have to grow up before they can reproduce.

1.4 Stability theory for the functional differential equations with delay

This section is devoted to the study of stability properties of solutions of the differential system

$$x'(t) = f(t, x_t), \ f(t, 0) = 0, \tag{1.6}$$

where $f : (-\infty, +\infty) \times C \to \mathbb{R}^n$, with $C = C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$, $\tau > 0$ equipped with the supremum norm $\|\varphi\| = \sup_{-\tau\leq t\leq 0} |\varphi(t)|$, $\varphi \in C$. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution. We define

$$\beta(t) = \{\varphi : [t-\tau, t] \to \mathbb{R}^n, \ \varphi \text{ is continuous}\}.$$

Definition 1.20 ([14]) The zero solution of (1.6) is

a) Stable if for every $\varepsilon > 0$ and to $t_0 \geq 0$ there exists $\delta > 0$ such that $[\varphi \in \beta(t_0), \ \|\varphi\| < \delta$ and $t \geq t_0]$ imply that $|x(t, t_0, \varphi)| < \varepsilon$.

b) Uniformly stable if it is stable and if δ is independent of t_0.

c) Asymptotically stable if it is stable and for each $t_1 \geq t_0$ there is an $\eta > 0$ such that

$$[\varphi \in \beta(t_1), \ \|\varphi\| < \eta \text{ and } t \geq t_1] \Rightarrow |x(t, t_0, \varphi)| \to 0 \text{ as } t \to +\infty.$$

d) Uniformly asymptotically stable if it is uniformly stable and if there is an $\eta > 0$ and for $\gamma > 0$, $\exists T > 0$ such that

$$[t_1 \geq t_0, \ \varphi \in \beta(t_1), \ \|\varphi\| < \eta \text{ and } t \geq t_0 + T] \Rightarrow |x(t, t_0, \varphi)| < \gamma.$$

The method of fixed point theory

When one wants to study the stability of the trivial solution of a differential equation with delay by the method of fixed point one will have to proceed as follows

1) A delay differential equation requires primarily a an initial function defined on an appropriate initial interval I_{t_0} i.e. $\psi : I_{t_0} \to R^n$. We must fall immediately after a suitable space C of functions $\varphi : I_{t_0} \cup [t_0, \infty) \to R^n$ which coincide on I_{t_0} with ψ. According to the case of needs we can always add other restrictions to the functions φ of C such as the boundary or the condition $\varphi(t) \to 0$ when $t \to \infty$. This last condition is necessary if we wish to study asymptotic stability.

2) Then we have to invert the differential equation to define what we call a fixed point application i.e., a mapping $S : C \to C$ whose fixed point is the solution of the given delay equation (the original equation). Nevertheless, this inversion can be a delicate task in many cases. For example if the equation does not have a linear term in its structure we will not be able to use the variation of the parameters. It is therefore essential to act differently and to try if a transformation of this equation is possible.

3) A fixed point theorem must be chosen allowing the equation $S(x) = x$ to have a solution. Especially if S is a contraction we can apply the Banach fixed point theorem, if S is compact then we will apply the theorem of Schauder or Schaeffer and if S is puts in the form of a sum of a contraction and a compact application then the Krasnoselskii hybrid theorem can give satisfaction. It thus becomes clear that the stability method by the fixed point method relies on three essential things, the variation of the parameters, a complete space and a fixed point theorem. In one stage we can conclude the existence(or even uniqueness) and stability. In addition, it will be seen that this method always requires conditions on average however the conditions of the Lyapunov method are always punctual.

1.4. Stability theory for the functional differential equations with delay

Chapter 2

Time scale calculus

Keywords. Time scales, differentiation, integration, exponential function.

In this chapter, We introduce the time scales. We also present usual derivation and integration results for denoted functions on arbitrary time scales. In addition, We recall the exponential function on time scales. which will allow us to study the nonlinear delay differential equations of neutral type on time scales. For specific references and many more examples and applications, see [8], [9] and [51].

2.1 Terminology of time scales

Definition 2.1 A **time scale** is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$ and denoted by $\mathbb{T}$. **Examples** of time scales are the reals $\mathbb{R}$, the integers $\mathbb{Z}$, the positive integers $\mathbb{N}$, and the nonnegative integers $\mathbb{N}_0$, along with any finite union of closed intervals, such as $[0,1] \cup [2,3]$. The most common time scales are $\mathbb{T} = \mathbb{R}$ for continuous calculus and $\mathbb{T} = \mathbb{Z}$ for discrete calculus.

Definition 2.2 Let $\mathbb{T}$ be a time scale.

- The **forward** and **backward jump** operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ for all } t \in \mathbb{T},$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ for all } t \in \mathbb{T},$$

respectively.

Let $\varnothing$ denotes the empty set, we put $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t).

Definition 2.3 ([51]) For $t \in \mathbb{T}$ we have the following cases

(a) If $\sigma(t) > t$, then we say that t is right-scattered.

(b) If $t < \sup \mathbb{T}$and $\sigma(t) = t$, then we say that t is right-dense.
(c) If $\rho(t) < t$, then we say that t is left-scattered.
(d) If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.
(e) If $\rho(t) < t < \sigma(t)$, then we say that t is isolated.
(f) If $\rho(t) = t = \sigma(t)$, then we say that t is dense.

In addition to the set $\mathbb{T}$, the set $\mathbb{T}^k$ is defined as follows. If $\mathbb{T}$ contains the left scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$. Else $\mathbb{T}^k = \mathbb{T}$. Therefore,

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.4 Let $\mathbb{T}$ be a time scale. The **graininess function** $\mu : \mathbb{T} \to [0, \infty)$ is given by the formula

$$\mu(t) = \sigma(t) - t \text{ for all } t \in \mathbb{T}.$$

Example 2.1 We consider the two examples of time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

i) If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf\,]t, \infty[= t,$$

and

$$\mu(t) = \sigma(t) - t = 0 \text{ for all } t \in \mathbb{T}.$$

ii) If $\mathbb{T} = \mathbb{Z}$, then for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf\{t+1, t+2, ...\} = t + 1,$$

and

$$\mu(t) = (t + 1) - t = 1 \text{ for all } t \in \mathbb{T}.$$

The next definitions are about periodic time scale.

Definition 2.5 We say that a time scale $\mathbb{T}$ is periodic if there exist a $w > 0$ such that if $t \in \mathbb{T}$ then $t \pm w \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive w is called the **period** of the time scale.

Example 2.2 The following time scales are periodic.

1) $\mathbb{T} = \bigcup\limits_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ has period $w = 2h$.

2) $\mathbb{T} = h\mathbb{Z}$ has period $w = h$.

3) $\mathbb{T} = \mathbb{R}$.

Remark 2.1 ([39]) All periodic time scales are unbounded above and below.

Definition 2.6 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period w. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = nw$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.2 If $\mathbb{T}$ is a periodic time scale with period w, then $\sigma(t \pm nw) = \sigma(t) \pm nw$. Consequently, the graininess function μ satisfies $\mu(t \pm nw) = \sigma(t \pm nw) - (t \pm nw) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period w.

2.2 Differentiation on time scales

Further we explore new properties of delta derivative (Hilger derivative) of function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$.

Definition 2.7 ([8]) The function $f : \mathbb{T} \to \mathbb{R}$ is called **Δ-differentiable** at a point $t \in \mathbb{T}^k$ if there exists $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a U-neighborhood of $t \in \mathbb{T}^k$ satisfying

$$\left|\left[f(\sigma(t)) - f(s)\right] - \gamma\left[\sigma(t) - s\right]\right| \leq \varepsilon \left|\sigma(t) - s\right|, \text{ for all } s \in U.$$

In this case we shall write $f^{\Delta}(t) = \gamma$.

The function f is Δ-differentiable for any $t \in \mathbb{T}^k$, then $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable on $\mathbb{T}^k$.

Theorem 2.1 ([8]) *Assume that $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^k$. Then we have the following*

1) *If f is differentiable at t, then f is continuous at t.*

2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

3) *If t is right-dense, then f is differentiable at t if f there exists the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

as a finite number, and then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + f^{\Delta}(t)\,\mu(t).$$

Note that, if $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$, which is the usual derivative of f, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, which is the forward difference of f.

Theorem 2.2 ([8]) *Assume that the functions $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the following assertions are valid:*

1) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t and

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2) For any constant α, the function $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t and

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3) The product $f, g : \mathbb{T} \to \mathbb{R}$ is differentiable at t and

$$(fg)^{\Delta}(t) = f^{\Delta}(t)\,g(t) + f^{\sigma}(t)\,g^{\Delta}(t),$$
$$(fg)^{\Delta}(t) = f(t)\,g^{\Delta}(t) + f^{\Delta}(t)\,g^{\sigma}(t).$$

3) If $f(t)\,f^{\sigma}(t) \neq 0$ then the function $\frac{1}{f}$is differentiable at t and

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)\,f^{\sigma}(t)};$$

5) If $g(t)\,g^{\sigma}(t) \neq 0$ then the function $\frac{f}{g}$ is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)\,g(t) - f(t)\,g^{\Delta}(t)}{g(t)\,g^{\sigma}(t)}.$$

Theorem 2.3 (Chain Rule [8]) *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $w^{\widetilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(w \circ v)^{\Delta} = (w^{\widetilde{\Delta}} \circ v)v^{\Delta}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(v(t))$ where, $v(t) := t - \tau(t)$. Our next theorem is the substitution rule

2.3 Integration on time scales

Definition 2.8 (**[8]**) A function $f : \mathbb{T} \to \mathbb{R}$ is called **regulated** provided its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.9 (**[8]**) • A function $f : \mathbb{T} \to \mathbb{R}$ is called **rd-continuous** provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

• The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable and whose derivatives are rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Proposition 2.1 *If $\mathbb{T}$ is compact, $C_{rd}(\mathbb{T}, \mathbb{R})$ is a Banach space.*

Theorem 2.4 (**[8]**) *Assume that $f : \mathbb{T} \to \mathbb{R}$. Then the following assertions are true.*

1) If f is continuous on $\mathbb{T}$, then it is rd-continuous on $\mathbb{T}$;

2) if f is rd-continuous on $\mathbb{T}$, then it is regulated on $\mathbb{T}$;

3) the jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is rd-continuous;

4) if f is regulated or rd-continuous on $\mathbb{T}$, then the function $f \circ \sigma$ possesses the same property;

5) if $f : \mathbb{T} \to \mathbb{R}$ is continuous and $g : \mathbb{T} \to \mathbb{R}$ is regulated and rd-continuous, then the function $f \circ g$ possesses the same property.

Definition 2.10 (**[8]**) A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called **pre-differentiable** with (region of differentiation) D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Definition 2.11 (**[8]**) Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F is called a **pre-antiderivative** of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all} \quad s, r \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an **antiderivative** of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^k.$$

Theorem 2.5 (Existence of Antiderivatives [8]) *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) = \int_{t_0}^t f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T},$$

is an antiderivative of f.

Theorem 2.6 ([8]) *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t) f(t).$$

Some properties of integration on $\mathbb{T}$ are presented next.

Theorem 2.7 ([8]) *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}(\mathbb{T})$. Then*

1) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

2) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

3) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

4) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

5) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) + \int_a^b f^{\Delta}(t)g(t)\Delta t$;

5) $\int_a^b f^{\Delta}(t)g(t)\Delta t = (fg)(b) - (fg)(a) + \int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t$;

7) $\int_a^a f(t)\Delta t = 0$;

8) $\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$;

9) If $|f| \le g$ on $[a, b)$, then $\left|\int_a^b f(t)\Delta t\right| \le \int_a^b g(t)\Delta t$;

10) If $f \ge 0$ on $[a, b)$, then $\int_a^b f(t)\Delta t \ge 0$.

Theorem 2.8 ([8]) *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

1) If $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right is the usual Riemann integral from calculus.

2) If $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$ *consists of only isolated points, then*

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t) f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t\in[b,a)} \mu(t) f(t) & \text{if } a > b. \end{cases}$$

3) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, *where* $h > 0$, *then*

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)\,h & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)\,h & \text{if } a > b. \end{cases}$$

4) If $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases}$$

The notion of the improper integral is defined as follows.

Definition 2.12 ([8]) If $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$; then the **improper integral** is defined by

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b F(t)\Delta t \quad \text{for} \quad a \in \mathbb{T}$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Theorem 2.9 ([8]) *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable and* $g : \mathbb{T} \to \mathbb{R}$ *is* Δ*-differentiable on* $\mathbb{T}$. *Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is* Δ*-differentiable, and the formula*

$$(f \circ g)^\Delta (t) = \left\{ \int_0^1 f'\left((g(t)) + h\mu(t)\, g^\Delta(t)\right) dh \right\} g^\Delta(t),$$

holds.

Theorem 2.10 (Substitution [8]) *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in T$,*

$$\int_a^b f(t)v^{\Delta}(t)\Delta t = \int_{v(a)}^{v(b)} \left(f \circ v^{-1}\right)(s)\,\widetilde{\Delta} s.$$

Theorem 2.11 ([8]) *Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}^k$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^{\Delta}(t, .)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $r \in [a, \sigma(t)]$, such that*

$$\left|f(\sigma(t), r) - f(s, \tau) - f^{\Delta}(t, r)(\sigma(t) - s)\right| \leq \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U,$$

where f^{Δ} denotes the derivative of f with respect to the first variable. Then

$$\begin{aligned} (i)\ g(t) \ &: = \int_a^t f(t, r)\Delta r \text{ implies } g^{\Delta}(t) := \int_a^t f^{\Delta}(t, r)\Delta r + f(\sigma(t), t), \\ (ii)\ h(t) \ &: = \int_t^b f(t, r)\Delta r \text{ implies } h^{\Delta}(t) := \int_t^b f^{\Delta}(t, r)\Delta r - f(\sigma(t), t), \end{aligned}$$

Lemma 2.1 ([8]) *Let $a \in \mathbb{T}^k$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^{\Delta}(t, .)$ is rd-continuous on $[a, \sigma(t)]$. Where f^{Δ} denotes the derivative of f with respect to the first variable. then*

$$\left(\int_{\alpha(t)}^{\beta(t)} f(t, r)\Delta r\right)^{\Delta} = \beta^{\Delta}(t)f(\sigma(t), \beta(t)) + \alpha^{\Delta}(t)f(\sigma(t), \alpha(t)) + \int_{\alpha(t)}^{\beta(t)} f^{\Delta}(t, r)\Delta r.$$

2.4 The exponential function on time scales

Definition 2.13 ([8]) We say that a function $p : \mathbb{T} \to \mathbb{R}$ is **regressive** provided

$$1 + \mu(t)\,p(t) \neq 0 \text{ for all } t \in \mathbb{T}^k,$$

holds. The set of all regressive and rd-continuous function $p : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

And an rd-continuous function $p : \mathbb{T} \to \mathbb{R}$ is **positively regressive** if

$$1 + \mu(t)\,p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

Definition 2.14 ([8]) Let $p \in \mathcal{R}$, the **exponential function** e_p is defined by the expression

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log\left(1 + \mu(z)\,p(z)\right)\Delta z\right) \text{ for all } (t, s) \in \mathbb{T} \times \mathbb{T} \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$.

Other properties of the exponential function are given in the following lemma

Theorem 2.12 ([8]) *Let $p, q \in \mathcal{R}$ and $t, r, s \in \mathbb{T}$ Then*

1) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

2) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t,s)$;

3) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

4) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

5) $e_p(t,s) e_p(s,r) = e_p(t,r)$;

6) $e_p(t,s) e_q(t,s) = e_{p\oplus q}(t,s)$;

7) $e_p^\Delta(.,s) = p e_p(.,s)$ *and* $\left(\frac{1}{e_p(.,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(.,s)}$;

8) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

9) *if* $\mathbb{T} = \mathbb{R}$, *then* $e_p(t,s) = e^{\int_s^t p(\tau)d\tau}$;

10) *if* $\mathbb{T} = \mathbb{R}$ *and* $p(t) \equiv \alpha$, *then* $e_p(t,s) = e^{\alpha(t-s)}$;

11) *if* $\mathbb{T} = \mathbb{Z}$, *then* $e_p(t,s) = \prod_{\tau=s}^{t-1} (1 + p(\tau))$;

12) *if* $\mathbb{T} = h\mathbb{Z}$ *with* $h > 0$ *and* $p(t) \equiv \alpha$, *then* $e_p(t,s) = (1 + h\alpha)^{\frac{t-s}{h}}$.

Lemma 2.2 ([1]) *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\,\Delta u\right), \quad \forall t \in \mathbb{T}.$$

Corollary 2.1 *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\,\Delta u\right) < 1.$$

Theorem 2.13 ([8]) *Let $\mathbb{T}$ be a periodic time scale with period $w > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = nw$, then*

$$\int_{a+T}^{b+T} p(u)\Delta u = \int_a^b p(u)\Delta u, \; e_p(b+T, a+T) = e_p(b,a) \text{ if } p \in \mathcal{R},$$

and $e_p(t+T, t)$ is independent of $t \in \mathbb{T}$ whenever $p \in \mathcal{R}$.

Definition 2.15 A function $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is an $L^1_\Delta-$**Caratheodory function** if it satisfies the following conditions

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is $\Delta-$measurable.

(ii) For almost all $t \in [0,T]$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1_\Delta\left([0,T], \mathbb{R}\right)$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|f(t,z)| \leq \alpha_r(t)$.

Chapter 3

Periodic and nonnegative periodic solutions of nonlinear neutral dynamic equations on a time scale

Keywords. Fixed points, Neutral dynamic equations, Time scales.

In this chapter, we expose the work cited in [28] as follow

M. Gouasmia, A. Ardjouni and A. Djoudi, Periodic and nonnegative periodic solutions of nonlinear neutral dynamic equations on a time scale, International Journal of Analysis and Applications 16(2) (2018), 162–177.

We use in this chapter the Krasnoselskii-Burton's fixed point theorem to obtain periodic and nonnegative periodic solutions of nonlinear neutral dynamic equations on a time scale. The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [46].

3.1 Introduction

In 1988, Stephan Hilger [35] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [8], [9], and references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this research, we are interested in the analysis of qualitative theory of periodic and positive periodic solutions of neutral dynamic equations. Motivated by the papers [[1], [18], [[38], [37], [48], [46]] and the references therein, we consider the following nonlinear neutral dynamic equation

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t-\tau(t)))^{\Delta} + G(t, x(t), x(t-\tau(t))),\ t \in \mathbb{T}. \tag{3.1}$$

Throughout this chapter we assume that a and τ are positive rd-continuous functions,

$id - \tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $x(t - \tau(t))$ is well defined over $\mathbb{T}$. The function h is continuous, Q and G satisfying the Caratheodory condition. To reach our desired end we have to transform (3.1) into an integral equation written as a sum of two mapping, one is a contraction and the other is continuous and compact. After that, we use Krasnoselskii-Burton's fixed point theorem, to show the existence of periodic and nonnegative periodic solutions.

3.2 Existence of periodic solutions

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = n\omega$ for some $n \in \mathbb{N}$. By the notation $[a, b] \cap \mathbb{T}$ we mean

$$[a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a, b) \cap \mathbb{T}$, $(a, b] \cap \mathbb{T}$ and $(a, b) \cap \mathbb{T}$ are defined similarly.

For $T > 0$ define

$$P_T = \{\phi \in C(\mathbb{T}, \mathbb{R}),\ \phi(t + T) = \phi(t)\},$$

where $C(\mathbb{T}, \mathbb{R})$ is the space of all real valued rd-continuous functions. Then $(P_T, \|.\|)$ is a Banach space when it is endowed with the supremum norm

$$\|\phi\| = \sup_{t \in [0,T]} |\phi(t)|.$$

Lemma 3.1 *Let $x \in P_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this chapter we assume that h is continuous, $a \in \mathcal{R}^+$ is rd-continuous and

$$a(t - T) = a(t),\ \tau(t - T) = \tau(t), \quad \tau(t) \geq \tau^* > 0, \tag{3.2}$$

with τ continuously and τ^* is positive constant, a is positive function and

$$1 - e_{\ominus a}(t, t - T) \equiv \frac{1}{\eta} \neq 0. \tag{3.3}$$

The functions $Q(t, x)$ and $G(t, x, y)$ are periodic in t of period T. That is

$$Q(t - T, x) = Q(t, x),\ G(t - T, x, y) = G(t, x, y). \tag{3.4}$$

The following lemma is fundamental to our results.

Lemma 3.2 *Suppose (3.2)–(3.4) hold. If $x \in P_T$, then x is a solution of equation (3.1) if and only if*

$$\begin{aligned} &x(t)\\ &= \eta \int_{t-T}^{t} k(t,u)a(u)[x^\sigma(u) - h(x^\sigma(u))]\Delta u + Q(t, x(t-\tau(t)))\\ &+ \eta \int_{t-T}^{t} k(t,u)\left[-a(u)Q^\sigma(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))\right]\Delta u, \end{aligned} \tag{3.5}$$

where

$$k(t,u) = e_{\ominus a}(t,u). \tag{3.6}$$

Proof. Let $x \in P_T$ be a solution of (3.1). Rewrite the equation (3.1) as

$$\begin{aligned} &(x(t) - Q(t, x(t-\tau(t))))^\Delta + a(t)[x^\sigma(t) - Q^\sigma(t, x(t-\tau(t)))]\\ &= a(t)\left[x^\sigma(t) - h(x^\sigma(t))\right] - a(t)Q^\sigma(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e_a(t,0)$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^{t} ((x(u) - Q(u, x(u-\tau(u))))e_a(u,0))^\Delta \Delta u\\ &= \int_{t-T}^{t} a(u)[x^\sigma(u) - h(x^\sigma(u))]e_a(u,0)\Delta u\\ &+ \int_{t-T}^{t} [-a(u)Q^\sigma(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))]e_a(u,0)\Delta u. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &(x(t) - Q(t, x(t-\tau(t))))e_a(t,0)\\ &- (x(t-T) - Q(t-T, x(t-T-\tau(t-T))))e_a(t-T,0)\\ &= \int_{t-T}^{t} a(u)[x^\sigma(u) - h(x^\sigma(u))]e_a(u,0)\Delta u\\ &+ \int_{t-T}^{t} [-a(u)Q^\sigma(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))]e_a(u,0)\Delta u. \end{aligned}$$

By dividing both sides of the above equation by $e_a(t,0)$ and using the fact that $x(t) = x(t-T)$, we obtain

$$\begin{aligned} &x(t) - Q(t, x(t-\tau(t)))\\ &= \eta \int_{t-T}^{t} a(u)[x^\sigma(u) - h(x^\sigma(u))]e_{\ominus a}(t,u)\Delta u\\ &+ \eta \int_{t-T}^{t} [-a(u)Q^\sigma(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))]e_{\ominus a}(t,u)\Delta u. \end{aligned} \tag{3.7}$$

The converse implication is easily obtained and the proof is complete. ■

To apply Theorem 1.13, we need to define a Banach space $\mathcal{B}$, a closed bounded convex subset $\mathcal{M}$ of $\mathcal{B}$ and construct two mappings; one is a completely continuous and the other is large contraction. So, we let $(\mathcal{B}, \|.\|) = (P_T, \|.\|)$ and

$$\mathcal{M} = \{\varphi \in P_T, \ \ \|\varphi\| \leq L\}, \tag{3.8}$$

with $L \in (0, 1]$. For $x \in \mathcal{M}$, let the mapping H be defined by

$$H(x) = x^\sigma - h(x^\sigma), \tag{3.9}$$

and by (3.5), define the mapping $S : P_T \to P_T$ by

$$\begin{aligned}
&(S\varphi)(t) \\
&= \eta \int_{t-T}^{t} k(t,u)a(u)H(\varphi(u))\Delta u + Q(t, \varphi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} k(t,u)[-a(u)Q^\sigma(u, \varphi(u-\tau(u))) + G(u, \varphi(u), \varphi(u-\tau(u)))]\Delta u.
\end{aligned} \tag{3.10}$$

Therefore, we express the above equation as

$$(S\varphi)(t) = (A\varphi)(t) + (B\varphi)(t),$$

where $A, B : P_T \to P_T$ are given by

$$\begin{aligned}
&(A\varphi)(t) \\
&= Q(t, \varphi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} k(t,u)\left[-a(u)Q^\sigma(u, \varphi(u-\tau(u))) + G(u, \varphi(u), \varphi(u-\tau(u)))\right]\Delta u.
\end{aligned} \tag{3.11}$$

and

$$(B\varphi)(t) = \eta \int_{t-T}^{t} k(t,u)a(u)H(\varphi(u))\Delta u. \tag{3.12}$$

We will assume that the following conditions hold.

(H1) $a \in L^1_\Delta[0, T]$ is bounded.

(H2) Q, G satisfies Caratheodory conditions with respect to $L^1_\Delta[0, T]$.

(H3) There exists periodic functions $q_1, q_2 \in L^1_\Delta[0, T]$, with period T, such that

$$|Q(t, x)| \leq q_1(t)|x| + q_2(t).$$

(H4) There exists periodic functions $g_1, g_2, g_3 \in L^1_\Delta [0, T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)\,|x| + g_2(t)\,|y| + g_3(t).$$

Now, we need the following assumptions

$$q_1(t)L + q_2(t) \leq \frac{\gamma_1}{2} L, \tag{3.13}$$

$$g_1(t)L + g_2(t)L + g_3(t) \leq \gamma_2 L a(t), \tag{3.14}$$

$$J\left(\gamma_1 + \gamma_2\right) \leq 1, \tag{3.15}$$

where γ_1, γ_2 and J are positive constants with $J \geq 3$.

Lemma 3.3 *For A defined in (3.11), suppose that (3.2)–(3.4), (3.13)–(3.15) and (H1)–(H4) hold. Then $A : \mathcal{M} \to \mathcal{M}$.*

Proof. Let A be defined by (3.11). Obviously, $A\varphi$ is rd-continuous. First by (3.2) and (3.4), a change of variable in (3.11) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T. Next, let $\varphi \in \mathcal{M}$, by (3.13)–(3.15) and (H1)–(H4) we have

$$\begin{aligned}
&\left|\left(A\varphi\right)(t)\right| \\
&\leq |Q(t, \varphi(t-\tau(t)))| \\
&+ \eta \int_{t-T}^{t} k(t,u)\left[a(u)\left|Q^{\sigma}(u, \varphi(u-\tau(u)))\right| + \left|G(u, \varphi(u), \varphi(u-\tau(u)))\right|\right] \Delta u \\
&\leq q_1(t)\left|\varphi(t-\tau(t))\right| + q_2(t) \\
&+ \eta \int_{t-T}^{t} k(t,u)a(u)[q_1(u)\left|\varphi(u-\tau(u))\right| + q_2(u)]\Delta u \\
&+ \eta \int_{t-T}^{t} k(t,u)[g_1(u)\left|\varphi(u)\right| + g_2(u)\left|\varphi(u-\tau(u))\right| + g_3(u)]\Delta u \\
&\leq \gamma_1 L + \gamma_2 L \leq \frac{L}{J} \leq L.
\end{aligned}$$

That is $A\varphi \in \mathcal{M}$. ■

Lemma 3.4 *For $A : \mathcal{M} \to \mathcal{M}$ defined in (3.11), suppose that (3.2)–(3.4), (3.13)–(3.15) and (H1)–(H4) hold. Then A is completely continuous.*

Proof. We show that A is continuous in the supremum norm, Let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$.

$$\begin{aligned}
&\left|(A\varphi_n)(t) - (A\varphi)(t)\right| \\
&\leq \left|Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))\right| \\
&+ \eta \int_{t-T}^{t} k(t,u)a(u)\left|Q^{\sigma}(u, \varphi_n(u-\tau(u))) - Q^{\sigma}(u, \varphi(u-\tau(u)))\right| \Delta u \\
&+ \eta \int_{t-T}^{t} k(t,u)\left|G(u, \varphi_n(u), \varphi_n(u-\tau(u))) - G(u, \varphi(u), \varphi(u-\tau(u)))\right| \Delta u.
\end{aligned}$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty} \left|(A\varphi_n)(t) - (A\varphi)(t)\right| = 0$. Then A is continuous. We next show that A is completely continuous. Let $\varphi \in \mathcal{M}$, then, by Lemma 3.3, we see that

$$\|A\varphi\| \leq L.$$

And so the family of functions $A\varphi$ is uniformly bounded. Again, let $\varphi \in \mathcal{M}$. Without loss of generality, we can pick $\omega < t$ such that $t - \omega < T$. Then

$$\begin{aligned}
&\left|(A\varphi)(t) - (A\varphi)(\omega)\right| \\
&\leq \left|Q(t, \varphi(t-\tau(t))) - Q(\omega, \varphi(\omega-\tau(\omega)))\right| \\
&+ \eta \left| \int_{t-T}^{t} k(t,u)a(u)Q^{\sigma}(u, \varphi(u-\tau(u)))\Delta u \right. \\
&- \left. \int_{\omega-T}^{\omega} k(\omega,u)a(u)Q^{\sigma}(u, \varphi(u-\tau(u)))\Delta u \right| \\
&+ \eta \left| \int_{t-T}^{t} k(t,u)G(u, \varphi(u), \varphi(u-\tau(u)))\Delta u \right. \\
&- \left. \int_{\omega-T}^{\omega} k(\omega,u)G(u, \varphi(u), \varphi(u-\tau(u)))\Delta u \right| \\
&\leq \left|Q(t, \varphi(t-\tau(t))) - Q(\omega, \varphi(\omega-\tau(\omega)))\right| \\
&+ 2\eta k_0 \int_{\omega-T}^{t-T} \left[a(u)q_L(u) + g_{\sqrt{2}L}(u)\right] \Delta u \\
&+ \eta \int_{\omega-T}^{\omega} \left|k(t,u) - k(\omega,u)\right| \left[a(u)q_L(u) + g_{\sqrt{2}L}(u)\right] \Delta u \\
&\leq \left|Q(t, \varphi(t-\tau(t))) - Q(\omega, \varphi(\omega-\tau(\omega)))\right| \\
&+ 2\eta k_0 \int_{\omega}^{t} \left[a(u)q_L(u) + g_{\sqrt{2}L}(u)\right] \Delta u \\
&+ \eta \int_{0}^{T} \left|k(t,u) - k(\omega,u)\right| \left[a(u)q_L(u) + g_{\sqrt{2}L}(u)\right] \Delta u,
\end{aligned}$$

where $k_0 = \max_{u\in[t-T,t]} \{k(t,u)\}$, then by the Dominated Convergence Theorem $\left|(A\varphi)(t) - (A\varphi)(\omega)\right| \to 0$ as $t - \omega \to 0$ independently of $\varphi \in \mathcal{M}$. Thus $(A\varphi)$ is equicontinuous. Hence by Ascoli-Arzela's theorem A is completely continuous. ■

3.2. Existence of periodic solutions

Now, we state an important result see [1, Theorem 3.4] and for convenience we present below its proof, we deduce by this theorem that the following are sufficient conditions implying that the mapping H given by (3.9) is a large contraction on the set $\mathcal{M}$.

(H5) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

(H6) the function h is strictly increasing on $[-L, L]$,

(H7) $\sup\limits_{t\in(-L,L)} h'(t) \leq 1$.

Theorem 3.1 *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H5)–(H7). Then the mapping H in (3.9) is a large contraction on the set $\mathcal{M}$.*

Proof. Let $\varphi^\sigma, \psi^\sigma \in \mathcal{M}$ with $\varphi^\sigma \neq \psi^\sigma$. Then $\varphi^\sigma(t) \neq \psi^\sigma(t)$ for some $t \in \mathbb{T}$. Let us denote the set of all such t by $D(\varphi, \psi)$, i.e.,

$$D(\varphi, \psi) = \left\{t \in \mathbb{T} : \ \varphi^\sigma(t) \neq \psi^\sigma(t)\right\}.$$

For all $t \in D(\varphi, \psi)$, we have

$$\begin{aligned} &|(H\varphi)(t) - (H\psi)(t)| \\ &\leq |\varphi^\sigma(t) - \psi^\sigma(t) - h(\varphi^\sigma(t)) + h(\psi^\sigma(t))| \\ &\leq |\varphi^\sigma(t) - \psi^\sigma(t)| \left|1 - \frac{h(\varphi^\sigma(t)) - h(\psi^\sigma(t))}{\varphi^\sigma(t) - \psi^\sigma(t)}\right|. \end{aligned} \tag{3.16}$$

Since h is a strictly increasing function we have

$$\frac{h(\varphi^\sigma(t)) - h(\psi^\sigma(t))}{\varphi^\sigma(t) - \psi^\sigma(t)} > 0 \text{ for all } t \in D(\varphi, \psi). \tag{3.17}$$

For each fixed $t \in D(\varphi, \psi)$ define the interval $I_t \subset [-L, L]$ by

$$I_t = \begin{cases} (\varphi^\sigma(t), \psi^\sigma(t)) \text{ if } \varphi^\sigma(t) < \psi^\sigma(t), \\ (\psi^\sigma(t), \varphi^\sigma(t)) \text{ if } \psi^\sigma(t) < \varphi^\sigma(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \psi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi^\sigma(t)) - h(\psi^\sigma(t))}{\varphi^\sigma(t) - \psi^\sigma(t)} = h'(c_t).$$

By (H6) and (H7) we have

$$0 \leq \inf_{u\in(-L,L)} h'(u) \leq \inf_{u\in I_t} h'(u) \leq h'(c_t) \leq \sup_{u\in I_t} h'(u) \leq \sup_{u\in(-L,L)} h'(u) \leq 1. \tag{3.18}$$

Hence, by (3.16)–(3.18) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \leq |\varphi^\sigma(t) - \psi^\sigma(t)| \left| 1 - \inf_{u\in(-L,L)} h'(u) \right|, \tag{3.19}$$

for all $t \in D(\varphi, \psi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\varepsilon \in (0,1)$ and assume that φ and ψ are two functions in $\mathcal{M}$ satisfying

$$\varepsilon \leq \sup_{t\in(-L,L)} |\varphi(t) - \psi(t)| = \|\varphi - \psi\|.$$

If $|\varphi^\sigma(t) - \psi^\sigma(t)| \leq \frac{\varepsilon}{2}$ for some $t \in D(\varphi,\psi)$, then we get by (3.18) and (3.19) that

$$|(H\varphi)(t) - (H\psi)(t)| \leq |\varphi^\sigma(t) - \psi^\sigma(t)| \leq \frac{1}{2}\|\varphi - \psi\|. \tag{3.20}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\varepsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\varepsilon}{2} \leq |\varphi^\sigma(t) - \psi^\sigma(t)|$ for some $t \in D(\varphi,\psi)$, then by (H6) and (H7) we conclude that

$$1 \geq \frac{h(\varphi^\sigma(t)) - h(\psi^\sigma(t))}{\varphi^\sigma(t) - \psi^\sigma(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min\left\{ h(u + \frac{\varepsilon}{2}) - h(u) : u \in [-L, L] \right\} > 0.$$

Hence, (3.16) implies

$$|(H\varphi)(t) - (H\psi)(t)| \leq (1-\lambda)\|\varphi - \psi\|. \tag{3.21}$$

Consequently, combining (3.20) and (3.21) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \leq \delta\|\varphi - \psi\|, \tag{3.22}$$

where

$$\delta = \max\left\{\frac{1}{2}, 1-\lambda\right\}.$$

The proof is complete. ■

The next result shows the relationship between the mappings H and B in the sense of large contractions. Assume that

$$\max\{|H(-L)|, |H(L)|\} \leq \frac{2L}{J}. \tag{3.23}$$

Lemma 3.5 *Let B be defined by (3.12), suppose (H5)–(H6) hold. Then $B : \mathcal{M} \to \mathcal{M}$ is a large contraction.*

3.2. Existence of periodic solutions

Proof. Let B be defined by (3.12). Obviously, $B\varphi$ is continuous and it is easy to show that $(B\varphi)(t+T) = (B\varphi)(t)$. Let $\varphi \in \mathcal{M}$

$$\begin{aligned}|(B\varphi)(t)| &\leq \eta \int_{t-T}^{t} k(t,u)a(u) \max\{|H(-L)|, |H(L)|\} \Delta u \\ &\leq \frac{2L}{J} < L,\end{aligned}$$

which implies $B : \mathcal{M} \to \mathcal{M}$.

By Theorem 3.1, H is large contraction on $\mathcal{M}$, then for any φ, $\psi \in \mathcal{M}$, with $\varphi \neq \psi$ and for any $\varepsilon > 0$, from the proof of that Theorem, we have found a $\delta < 1$, such that

$$\begin{aligned}|(B\varphi)(t) - (B\psi)(t)| &= \left|\eta \int_{t-T}^{t} k(t,u)a(u)[H(\varphi(u)) - H(\psi(u))]\Delta u\right| \\ &\leq \delta \|\varphi - \psi\| \eta \int_{t-T}^{t} k(t,u)a(u)\Delta u \leq \delta \|\varphi - \psi\|.\end{aligned}$$

The proof is complete. ■

Theorem 3.2 *Suppose the hypothesis of Lemmas 3.3, 3.4 and 3.5 hold. Let $\mathcal{M}$ defined by (3.8). Then the equation (3.1) has a T-periodic solution in $\mathcal{M}$.*

Proof. By Lemma 3.3, 3.4, A is continuous and $A(\mathcal{M})$ is contained in a compact set. Also, from Lemma 3.5, the mapping B is a large contraction. Next, we show that if $\varphi, \psi \in \mathcal{M}$, we have $\|A\psi + B\varphi\| \leq L$. Let $\varphi, \psi \in \mathcal{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (3.13)–(3.15)

$$\begin{aligned}\|A\psi + B\varphi\| &\leq (\gamma_1 + \gamma_2)L + \frac{2L}{J} \\ &\leq \frac{L}{J} + \frac{2L}{J} \leq L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 3.2 this fixed point is a solution of (3.1). Hence (3.1) has a T-periodic solution. ■

3.3 Existence of nonnegative periodic solutions

In this section we obtain the existence of a nonnegative periodic solution of (3.1). By applying Theorem 1.13, we need to define a closed, convex, and bounded subset $\mathbb{M}$ of P_T. So, let

$$\mathbb{M} = \{\phi \in P_T : 0 \leq \phi \leq K\}. \tag{3.24}$$

where K is positive constant. To simplify notation, we let

$$m = \min_{u\in[t-T,t]} e_{\ominus a}(t,u), \ M = \max_{u\in[t-T,t]} e_{\ominus a}(t,u). \tag{3.25}$$

It is easy to see that for all $(t,u) \in [0,2T]^2$,

$$m \le k(t,u) \le M. \tag{3.26}$$

Then we obtain the existence of a nonnegative periodic solution of (3.1) by considering the two cases;

1) $Q(t,y) \ge 0$, $\forall t \in [0,T]$, $y \in \mathbb{M}$.

2) $Q(t,y) \le 0$, $\forall t \in [0,T]$, $y \in \mathbb{M}$.

In the case one, we assume for all $t \in [0,T]$, $x,y \in \mathbb{M}$, that there exist a positive constant c_1 such that

$$0 \le Q(t,y) \le c_1 y, \tag{3.27}$$

$$c_1 < 1, \tag{3.28}$$

$$0 \le -a(t)Q^\sigma(t,y) + G(t,x,y), \tag{3.29}$$

$$a(t)H(\varphi(t)) - a(t)\,Q^\sigma(t,y) + G(t,x,y) \le \frac{K(1-c_1)}{M\eta T}. \tag{3.30}$$

Lemma 3.6 *Let A, B given by (3.11), (3.12) respectively, assume (3.27)–(3.30) hold. Then $A, B : \mathbb{M} \to \mathbb{M}$.*

Proof. Let A defined by (3.12). So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}
0 &\le (A\varphi)(t) \le Q(t,\varphi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} k(t,u)\left[-a(u)Q^\sigma(u,\varphi(u-\tau(u))) + G(u,\varphi(u),\varphi(u-\tau(u)))\right]\Delta u \\
&\le \eta \int_{t-T}^{t} M \frac{K(1-c_1)}{M\eta T}\Delta u + c_1 K = K,
\end{aligned}$$

That is $A\varphi \in \mathbb{M}$.

Now, let B defined by (3.12). So, for any $\varphi \in \mathbb{M}$, we have

$$0 \le (B\varphi)(t) \le \eta \int_{t-T}^{t} M \frac{K(1-c_1)}{M\eta T}\Delta u \le M\eta T \frac{K}{M\eta T} = K.$$

That is $B\varphi \in \mathbb{M}$. ■

Theorem 3.3 *Suppose the hypothesis of Lemmas 3.4, 3.5 and 3.6 hold. Then equation (3.1) has a nonnegative T-periodic solution x in the subset $\mathbb{M}$.*

Proof. By Lemma 3.4, A is completely continuous. Also, from Lemma 3.5, the mapping B is a large contraction. By Lemma 3.6, $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \le A\psi + B\varphi \le K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \le \varphi, \psi \le K$. By (3.27)–(3.30)

$$
\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= \eta \int_{t-T}^{t} k(t,u)a(u)H(\varphi(u))\Delta u + Q(t, \psi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} k(t,u)\left[-a(u)Q^{\sigma}(u, \psi(u-\tau(u))) + G(u, \psi(u), \psi(u-\tau(u)))\right]\Delta u \\
&\le \eta \int_{t-T}^{t} k(t,u)\frac{K(1-c_1)}{M\eta T}\Delta u + c_1 K \\
&\le \eta \int_{t-T}^{t} M\frac{K(1-c_1)}{M\eta T}\Delta u + c_1 K = K.
\end{aligned}
$$

On the other hand,

$$(A\psi)(t) + (B\varphi)(t) \ge 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 3.2 this fixed point is a solution of (3.1) and the proof is complete. ∎

In the case two, we substitute conditions (3.27)–(3.30) with the following conditions respectively. We assume that there exist a negative constant c_2 such that

$$c_2 y \le Q(t, y) \le 0, \tag{3.31}$$

$$-c_2 < 1, \tag{3.32}$$

$$\frac{-c_2 K}{M\eta T} \le a(t)H(\varphi(t)) - a(t)Q(t,y) + G(t,x,y), \tag{3.33}$$

$$a(t)H(\varphi(t)) - a(t)\,Q(t,y) + G(t,x,y) \le \frac{K}{M\eta T}. \tag{3.34}$$

Theorem 3.4 *Suppose (3.31)–(3.34) and the hypothesis of Lemmas 3.3, 3.4 and 3.5 hold. Then equation (3.1) has a nonnegative T-periodic solution x in the subset $\mathbb{M}$.*

Proof. By Lemma 3.3, 3.4, A is completely continuous. Also, from Lemma 3.5, the mapping B is a large contraction. To see that, it is easy to show as in Lemma 3.6, $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \le A\psi + B\varphi \le K$. Let

$\varphi, \psi \in \mathbb{M}$, with $0 \leq \varphi, \psi \leq K$. By (3.31)– (3.34)

$$\begin{aligned}
&\left(A\psi\right)(t) + \left(B\varphi\right)(t) \\
&= \eta \int_{t-T}^{t} k(t,u)a(u)H(\varphi(u))\Delta u + Q(t, \psi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} k(t,u)\left[-a(u)Q^{\sigma}(u, \psi(u-\tau(u))) + G(u, \psi(u), \psi(u-\tau(u)))\right]\Delta u \\
&\leq \eta \int_{t-T}^{t} k(t,u)\frac{K}{M\eta T}\Delta u = \eta \int_{t-T}^{t} M\frac{K}{M\eta T}\Delta u = K.
\end{aligned}$$

On the other hand,

$$(A\psi)(t) + \left(B\varphi\right)(t) \geq \eta \int_{t-T}^{t} M\frac{-c_2 K}{M\eta T}\Delta u + c_2 K = 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 3.2 this fixed point is a solution of (3.1) and the proof is complete. ■

3.3. Existence of nonnegative periodic solutions

Chapter 4

Study of stability in nonlinear neutral dynamic equations on time scales using Krasnoselskii-Burton's fixed point

Keywords. Fixed points, neutral dynamic equations, time scales.

In this chapter, we expose the work cited in [30] as follow

M. Gouasmia, A. Ardjouni, A. Djoudi, Study of stability in nonlinear neutral dynamic equations on time scales using Krasnoselskii–Burton's fixed point, Memoirs on Differential Equations and Mathematical Physics, Accepted 2020.

In this chapter, we use the Krasnoselskii-Burton's fixed point theorem to obtain stability results about the zero solution for a nonlinear neutral dynamic equation with variable delay.

4.1 Introduction

In this chapter, we consider the following nonlinear neutral dynamic equations with variable delay given by

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + (Q(t, x(t-\tau(t))))^{\Delta} + G(t, x(t), x(t-\tau(t))), \tag{4.1}$$

with an assumed initial function

$$x(t) = \psi(t),\ t \in [m_0, 0] \cap \mathbb{T},$$

where $\mathbb{T}$ is an unbounded above and below time scale and such that $0 \in \mathbb{T}$.

Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [16, Theorem 3]) to show the asymptotic stability and the stability of the zero solution for equation (4.1). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [16], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (4.1) into an integral equation written as a sum of two mapping; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, to show the asymptotic stability and the stability of the zero for equation (4.1). In the special case $\mathbb{T} = \mathbb{R}$, Mesmouli, Ardjouni and Djoudi [47] show the zero solution of (4.1) is asymptotically stable by using Krasnoselskii-Burton's fixed point theorem. Then, the results presented in this chapter extend the main results in [47].

In this chapter, we give the assumptions as follows that will be used in the main results.

(H1) $\tau : [0, \infty) \cap \mathbb{T} \to \mathbb{T}$ is a positive rd-continuous function, $id - \tau : [0, \infty) \cap \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id - \tau)([0, \infty) \cap \mathbb{T})$ is closed where id is the identity function. Moreover, there exists a constants $l_2 > 0$ such that for $0 \leq t_1 < t_2$

$$|\tau(t_2) - \tau(t_1)| \leq l_2 |t_2 - t_1|.$$

(H2) $\psi : [m_0, 0] \cap \mathbb{T} \to \mathbb{R}$ is a rd-continuous function with $m_0 = -\tau(0)$.

(H3) $a : [0, \infty) \cap \mathbb{T} \to (0, \infty)$ is a bounded rd-continuous function and there exists a constant $l_3 > 0$ such that for $0 \leq t_1 < t_2$

$$\left| \int_{t_1}^{t_2} a(u) \Delta u \right| \leq l_3 |t_2 - t_1|.$$

(H4) $Q : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $Q(t, 0) = 0$, that is, for $t_1, t_2 \geq 0$ and $x, y \in [-R, R]$ where $R \in (0, 1]$, there exist constants $l_0, E_Q > 0$, such that

$$|Q(t_1, x) - Q(t_2, y)| \leq l_0 |t_1 - t_2| + E_Q |x - y|.$$

Also, Q is a bounded function and satisfies the Caratheodory condition with respect to $L^1_\Delta([0, \infty) \cap \mathbb{T})$, such that

$$|Q(t, \varphi(t - \tau(t)))| \leq q_R(t) \leq \frac{\alpha_1}{2} R,$$

where α_1 is a positive constant.

(H5) The function $G : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition with respect to $L^1_\Delta([0, \infty) \cap \mathbb{T})$, G/a is a bounded function and $G(t, 0, 0) = 0$, such that for $t \geq 0$,

$$|G(t, \varphi(t), \varphi(t - \tau(t)))| \leq g_{\sqrt{2}R}(t) \leq \alpha_2 a(t) R,$$

where α_2 is a positive constant.

(H6) There exists a constant $J > 3$, such that

$$J(\alpha_1 + \alpha_2) \leq 1,$$

and

$$(E_Q + E_Q l_2)\, l_1 + l_0 + 3R\left(\frac{\alpha_1}{2} + \alpha_2 + \frac{2}{J}\right) l_3 < l_1,$$

where l_1 is a positive constant.

(H7) $h : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing on $[-R, R]$, $h(0) = 0$, h is differentiable on $(-R, R)$ with $h'(x) \leq 1$ for $x \in (-R, R)$.

(H8) For $\gamma > 0$ small enough,

$$[1 + E_Q]\,\gamma + (E_Q + E_Q l_2)\, l_1 + l_0 + 3R\left(\frac{\alpha_1}{2} + \alpha_2 + \frac{2}{J}\right) l_3 \leq l_1,$$

and

$$[1 + E_Q]\,\gamma e_{\ominus a}(t, 0) + \frac{3R}{J} \leq R.$$

Also,

$$\max\{|H(-R)|\,, |H(R)|\} \leq \frac{2R}{J},$$

where $H(x) = x^\sigma - h(x^\sigma)$.

(H9) $t - \tau(t) \to \infty$, $e_{\ominus a}(t, 0) \to 0$, $q_R(t) \to 0$ and $\dfrac{g_{\sqrt{2}R}(t)}{a(t)} \to 0$ as $t \to \infty$.

4.2 Stability

We begin this section by the following lemma.

Lemma 4.1 *x is a solution of equation (4.1) if and only if*

$$\begin{aligned} x(t) &= [\psi(0) - Q(0, \psi(-\tau(0)))]\, e_{\ominus a}(t, 0) \\ &\quad + \int_0^t a(s) e_{\ominus a}(t, s) H(x(s))\Delta s + Q(t, x(t - \tau(t))) \\ &\quad + \int_0^t e_{\ominus a}(t, s)\left[-a(s) Q^\sigma(s, x(s - \tau(s))) + G(s, x(s), x(s - \tau(s)))\right]\Delta s, \end{aligned} \tag{4.2}$$

where

$$H(x) = x^\sigma - h(x^\sigma). \tag{4.3}$$

Proof. Let x be a solution of (4.1). Rewrite the equation (4.1) as

$$\begin{aligned} &(x(t) - Q(t, x(t - \tau(t))))^\Delta + a(t)[x^\sigma(t) - Q^\sigma(t, x(t - \tau(t)))] \\ &= a(t)\left[x^\sigma(t) - h(x^\sigma(t))\right] - a(t) Q^\sigma(t, x(t - \tau(t))) + G(t, x(t), x(t - \tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e_a(t,0)$ and then integrate from 0 to t , we obtain

$$\begin{aligned}
&\int_0^t ((x(s) - Q(s, x(s-\tau(s))))e_a(s,0))^{\Delta} \Delta s \\
&= \int_0^t a(s)[x^{\sigma}(s) - h(x^{\sigma}(s))]e_a(s,0)\Delta s \\
&+ \int_0^t [-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))]e_a(s,0)\Delta s.
\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&[x(t) - Q(t, x(t-\tau(t)))]\, e_a(t,0) - \psi(0) + Q(0, \psi(-\tau(0))) \\
&= \int_0^t a(s)[x^{\sigma}(s) - h(x^{\sigma}(s))]e_a(s,0)\Delta s \\
&+ \int_0^t [-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))]e_a(s,0)\Delta s.
\end{aligned}$$

By dividing both sides of the above equation by $e_a(t,0)$ we obtain

$$\begin{aligned}
&x(t) - Q(t, x(t-\tau(t))) - [\psi(0) - Q(0, \psi(-\tau(0)))]\, e_{\ominus a}(t,0) \\
&= \int_0^t a(s)[x^{\sigma}(s) - h(x^{\sigma}(s))]e_{\ominus a}(t,s)\Delta s \\
&+ \int_0^t [-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))]e_{\ominus a}(t,s)\Delta s. \qquad (4.4)
\end{aligned}$$

The converse implication is easily obtained and the proof is complete. ■

From the existence theory, which can be found in [18] or [32], we conclude that for each rd-continuous initial function $\psi \in C_{rd}([m_0, 0] \cap \mathbb{T}, \mathbb{R})$, there exists a rd-continuous solution $x(t, 0, \psi)$ which satisfies (4.1) on an interval $[0, \sigma) \cap \mathbb{T}$ for some $\sigma > 0$ and $x(t, 0, \psi) = \psi(t)$, $t \in [m_0, 0] \cap \mathbb{T}$.

To apply Theorem 1.13, we need to define a Banach space χ, a closed bounded convex subset $\mathcal{M}$ of χ and construct two mappings; one large contraction and the other is compact operator. So, let $\omega : [m_0, \infty) \cap \mathbb{T} \to [1, \infty)$ be any strictly increasing and rd-continuous function with $\omega(m_0) = 1$, $\omega(t) \to \infty$ as $t \to \infty$. Let $(S, |.|_{\omega})$ be the Banach space of rd-continuous $\varphi : [m_0, \infty) \cap \mathbb{T} \to \mathbb{R}$ for which

$$|\varphi|_{\omega} = \sup_{t \geq m_0} \left| \frac{\varphi(t)}{\omega(t)} \right| < \infty.$$

Let $R \in (0, 1]$ and define the set

$$\begin{aligned}
\mathcal{M} \ :\ &= \{\varphi \in S : \varphi \text{ is } l_1\text{-Lipschitzian},\ \ |\varphi(t)| \leq R,\ \ t \in [m_0, \infty) \cap \mathbb{T} \\
&\text{and } \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0] \cap \mathbb{T}\}. \qquad (4.5)
\end{aligned}$$

4.2. Stability

Clearly, if $\{\varphi_n\}$ is a sequence of l_1-Lipschitzian functions converging to some function φ, then

$$\begin{aligned}|\varphi(t)-\varphi(s)| &= |\varphi(t)-\varphi_n(t)+\varphi_n(t)-\varphi_n(s)+\varphi_n(s)-\varphi(s)|\\ &\leq |\varphi(t)-\varphi_n(t)|+|\varphi_n(t)-\varphi_n(s)|+|\varphi_n(s)-\varphi(s)|\\ &\leq l_1|t-s|,\end{aligned}$$

as $n\to\infty$, which implies φ is l_1-Lipschitzian. It is clear that $\mathcal{M}$ is closed convex and bounded. For $\varphi\in\mathcal{M}$ and $t\geq 0$, we define by (4.2) the mapping $P:\mathcal{M}\to S$ as follows

$$\begin{aligned}(P\varphi)(t) &= [\psi(0)-Q(0,\psi(-\tau(0)))]\, e_{\ominus a}(t,0)\\ &\quad+\int_0^t a(s)e_{\ominus a}(t,s)H(\varphi(s))\Delta s+Q(t,\varphi(t-\tau(t)))\\ &\quad+\int_0^t e_{\ominus a}(t,s)\left[-a(s)Q^{\sigma}(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]\Delta s. \qquad (4.6)\end{aligned}$$

Therefore, we express mapping (4.6) as

$$P\varphi=A\varphi+B\varphi,$$

where $A,B:\mathcal{M}\to S$ are given by

$$\begin{aligned}(A\varphi)(t) &= Q(t,\varphi(t-\tau(t)))\\ &\quad+\int_0^t e_{\ominus a}(t,s)\left[-a(s)Q^{\sigma}(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]\Delta s, \qquad (4.7)\end{aligned}$$

and

$$(B\varphi)(t)=[\psi(0)-Q(0,\psi(-\tau(0)))]\, e_{\ominus a}(t,0)+\int_0^t a(s)e_{\ominus a}(t,s)H(\varphi(s))\Delta s. \qquad (4.8)$$

By applying Theorem 1.13, we need to prove that P has a fixed point φ on the set $\mathcal{M}$, where $x(t,0,\psi)=\varphi(t)$ for $t\geq 0$ and $x(t,0,\psi)=\psi(t)$ on $[m_0,0]\cap\mathbb{T}$, $x(t,0,\psi)$ satisfies (4.1) and $|x(t,0,\psi)|\leq R$ with $R\in(0,1]$.

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 1.13.

Lemma 4.2 *For A defined in (4.7), suppose that (H1)–(H6) hold. Then, $A:\mathcal{M}\to\mathcal{M}$ and A is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$.*

Proof. Let A be defined by (4.7). Then, for any $\varphi\in\mathcal{M}$, we have

$$\begin{aligned}|(A\varphi)(t)| &\leq |Q(t,\varphi(t-\tau(t)))|+\int_0^t e_{\ominus a}(t,s)\left[a(s)\left|Q^{\sigma}(s,\varphi(s-\tau(s)))\right|\right.\\ &\quad\left.+\left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right]\Delta s\\ &\leq q_R(t)+R\int_0^t e_{\ominus a}(t,s)\left(a(s)\frac{q_R(s)}{R}+\frac{g_{\sqrt{2}R}(s)}{R}\right)\Delta s\\ &\leq \frac{\alpha_1}{2}R+\frac{\alpha_1}{2}R+\alpha_2 R\leq\frac{R}{J}<R.\end{aligned}$$

That is $|(A\varphi)(t)| < R$. Second we show that, for any $\varphi \in \mathcal{M}$ the function $A\varphi$ is l_1-Lipschitzian. Let $\varphi \in \mathcal{M}$, and let $0 \leq t_1 < t_2$, then

$$\begin{aligned}
&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\
&\leq |Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))| \\
&+ \left| \int_0^{t_2} e_{\ominus a}(t_2, s)\left[-a(s)Q^{\sigma}(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right]\Delta s \right. \\
&\left. - \int_0^{t_1} e_{\ominus a}(t_1, s)\left[-a(s)Q^{\sigma}(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right]\Delta s \right|.
\end{aligned} \tag{4.9}$$

By hypotheses (H1), (H3) and (H4), we have

$$\begin{aligned}
&|Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))| \\
&\leq l_0|t_2 - t_1| + E_Q l_1 |(t_2 - t_1) - (\tau(t_2) - \tau(t_1))| \\
&\leq (l_0 + E_Q l_1 + E_Q l_1 l_2)|t_2 - t_1|,
\end{aligned} \tag{4.10}$$

where l_1 is the Lipschitz constant of φ. In the same way, by (H3)–(H5), we have

$$\begin{aligned}
&\left| \int_0^{t_2} e_{\ominus a}(t_2, s)\left[-a(s)Q^{\sigma}(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right]\Delta s \right. \\
&\left. - \int_0^{t_1} e_{\ominus a}(t_1, s)\left[-a(s)Q^{\sigma}(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right]\Delta s \right| \\
&\leq \left| \int_0^{t_1} \left[-a(s)Q^{\sigma}(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right]\Delta s \right. \\
&\times e_{\ominus a}(t_1, s)\left(e_{\ominus a}(t_2, t_1) - 1\right)\Delta s| \\
&+ \left| \int_{t_1}^{t_2} e_{\ominus a}(t_2, s)\left[-a(s)Q^{\sigma}(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right]\Delta s \right| \\
&\leq \left(\frac{\alpha_1}{2} + \alpha_2\right) R \left|e_{\ominus a}(t_2, t_1) - 1\right| \int_0^{t_1} a(s) e_{\ominus a}(t_1, s)\Delta s \\
&+ \int_{t_1}^{t_2} e_{\ominus a}(t_2, s)\left(a(s)q_R(s) + g_{\sqrt{2}R}(s)\right)\Delta s \\
&\leq \left(\frac{\alpha_1}{2} + \alpha_2\right) R \int_{t_1}^{t_2} a(s)\Delta s \\
&+ \int_{t_1}^{t_2} a(s) e_{\ominus a}(t_2, s)\left(\int_{t_1}^{s}\left(a(r)q_R(r) + g_{\sqrt{2}R}(r)\right)\Delta r\right)^{\Delta}\Delta s,
\end{aligned}$$

so,

$$\begin{aligned}
&\left| \int_0^{t_2} e_{\ominus a}(t_2,s)\left[-a(s)Q^\sigma(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]\Delta s \right. \\
&\left. - \int_0^{t_1} e_{\ominus a}(t_1,s)\left[-a(s)Q^\sigma(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]\Delta s \right| \\
&\le \left(\frac{\alpha_1}{2}+\alpha_2\right) R \int_{t_1}^{t_2} a(s)\Delta s + \left[e_{\ominus a}(t_2,s)\int_{t_1}^{s}\left(a(r)q_R(r)+g_{\sqrt{2}R}(r)\right)\Delta r\right]_{t_1}^{t_2} \\
&+ \int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\int_{t_1}^{s}\left(a(r)q_R(r)+g_{\sqrt{2}R}(r)\right)\Delta r\Delta s \\
&\le \left(\frac{\alpha_1}{2}+\alpha_2\right) R \int_{t_1}^{t_2} a(s)\Delta s \\
&+ \int_{t_1}^{t_2}\left(a(s)q_R(s)+g_{\sqrt{2}R}(s)\right)\Delta s\left(1+\int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\Delta s\right) \\
&\le \left(\frac{\alpha_1}{2}+\alpha_2\right) R \int_{t_1}^{t_2} a(s)\Delta s + 2\int_{t_1}^{t_2}\left(a(s)q_R(s)+g_{\sqrt{2}R}(s)\right)\Delta s \\
&\le \left(\frac{\alpha_1}{2}+\alpha_2\right) R \int_{t_1}^{t_2} a(s)\Delta s + 2\left(\frac{\alpha_1}{2}+\alpha_2\right) R\int_{t_1}^{t_2} a(s)\Delta s \\
&\le 3R\left(\frac{\alpha_1}{2}+\alpha_2\right) l_3\left|t_2-t_1\right|. \qquad (4.11)
\end{aligned}$$

Thus, by substituting (4.10) and (4.11) in (4.9), we obtain

$$\begin{aligned}
&\left|\left(A\varphi\right)(t_2)-\left(A\varphi\right)(t_1)\right| \\
&\le (l_0+E_Q l_1+E_Q l_1 l_2)|t_2-t_1| + 3R\left(\frac{\alpha_1}{2}+\alpha_2\right) l_3\left|t_2-t_1\right| \\
&\le l_1\left|t_2-t_1\right|.
\end{aligned}$$

This shows that $A\varphi$ is l_1-Lipschitzian if φ is. This completes the proof of $A : \mathcal{M} \to \mathcal{M}$.

Since $A\varphi$ is l_1-Lipschitzian, then $A\mathcal{M}$ is equicontinuous, which implies that the set $A\mathcal{M}$ resides in a compact set in the space $(S, |.|_\omega)$.

Now, we show that A is continuous in the weighted norm, let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\left|\frac{(A\varphi_n)(t) - (A\varphi)(t)}{\omega(t)}\right|$$
$$\leq |Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))|_\omega$$
$$+ \int_0^t a(s)e_{\ominus a}(t,s) |Q^\sigma(s, \varphi_n(s-\tau(s))) - Q^\sigma(s, \varphi(s-\tau(s)))|_\omega \Delta s$$
$$+ \int_0^t e_{\ominus a}(t,s) |G(s, \varphi_n(s), \varphi_n(s-\tau(s))) - G(s, \varphi(s), \varphi(s-\tau(s)))|_\omega \Delta s.$$

By the dominated convergence theorem, $\lim_{n\to\infty} |(A\varphi_n)(t) - (A\varphi)(t)|_\omega = 0$. Then A is continuous. This completes the proof of $A : \mathcal{M} \to \mathcal{M}$ is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$. ■

Now, we state an important result implying that the mapping H given by (4.3) is a large contraction on the set $\mathcal{M}$. This result was already obtained in [1, Theorem 3.4].

Theorem 4.1 *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H7). Then the mapping H in (4.3) is a large contraction on the set $\mathcal{M}$.*

The relations of (H8) will be used below in Lemma 4.3 and Theorem 4.2 to show that if $\varepsilon = R$ and if $\|\psi\| < \gamma$, then the solutions satisfies $|x(t, 0, \psi)| < \varepsilon$.

Lemma 4.3 *Let B be defined by (4.8). Suppose that (H1)–(H3), (H7) and (H8) hold. Then $B : \mathcal{M} \to \mathcal{M}$ and B is a large contraction.*

Proof. Let B be defined by (4.8). Obviously, B is continuous with the weighted norm. Let $\varphi \in \mathcal{M}$,

$$\begin{aligned} |(B\varphi)(t)| &\leq |\psi(0) - Q(0, \psi(-\tau(0)))| e_{\ominus a}(t,0) + \int_0^t a(s)e_{\ominus a}(t,s) |H(\varphi(s))| \Delta s \\ &\leq [1 + E_Q] \gamma e_{\ominus a}(t,0) + \int_0^t a(s)e_{\ominus a}(t,s) \max\{|H(-R)|, |H(R)|\} \Delta s \leq R, \end{aligned}$$

and we use a method like in Lemma 4.2, we deduce that, for any $\varphi \in \mathcal{M}$ the function $B\varphi$ is l_1-Lipschitzian, which implies $B : \mathcal{M} \to \mathcal{M}$.

By Theorem 4.1, H is large contraction on $\mathcal{M}$, then for any $\varphi, \phi \in \mathcal{M}$, with $\varphi \neq \phi$ and for any $\varepsilon > 0$, from the proof of that Theorem, we have found a $\delta < 1$, such that

$$\begin{aligned} \left|\frac{B\varphi(t) - B\phi(t)}{\omega(t)}\right| &\leq \int_0^t a(s)e_{\ominus a}(t,s) |H(\varphi(s)) - H(\phi(s))|_\omega \Delta s \\ &\leq \delta |\varphi - \phi|_\omega . \end{aligned}$$

The proof is complete. ■

4.2. Stability

Theorem 4.2 *Assume that (H1)–(H8) hold. Then the zero solution of (4.1) is stable.*

Proof. By Lemmas 4.2 and 4.4, $A : \mathcal{M} \to \mathcal{M}$ is continuous and $A\mathcal{M}$ is contained in a compact set. Also, from Lemma 4.3, the mapping $B : \mathcal{M} \to \mathcal{M}$ is a large contraction. Firstly, we show that if $\varphi, \phi \in \mathcal{M}$, we have $\|A\varphi + B\phi\| \leq R$. Let $\varphi, \phi \in \mathcal{M}$ with $\|\varphi\|, \|\phi\| \leq R$, then

$$\begin{aligned} \|A\varphi + B\phi\| &\leq (1 + E_Q)\, \gamma e_{\ominus a}(t,0) + (\alpha_1 + \alpha_2)R + \frac{2R}{J} \\ &\leq (1 + E_Q)\gamma e_{\ominus a}(t,0) + \frac{R}{J} + \frac{2R}{J} \leq R. \end{aligned}$$

Secondly, we prove that, for any $\varphi, \phi \in \mathcal{M}$ the function $A\varphi + B\phi$ is l_1-Lipschitzian. Let $\varphi, \phi \in \mathcal{M}$, and let $0 \leq t_1 < t_2$, then

$$\begin{aligned} &\left|\left(A\varphi + B\phi\right)(t_2) - \left(A\varphi + B\phi\right)(t_1)\right| \\ &\leq \left(\left[1 + E_Q\right]\gamma + \left(E_Q + E_Q l_2\right) l_1 + l_0 + 3R\left(\frac{\alpha_1}{2} + \alpha_2 + \frac{2}{J}\right) l_3\right) |t_2 - t_1| \\ &\leq l_1 |t_2 - t_1|. \end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 4.1 this fixed point is a solution of (4.1). Hence the zero solution of (4.1) is stable. ■

Remark 4.1 When $\mathbb{T} = \mathbb{R}$, Theorem 4.2 reduces to Theorem 4 of [47]. Therefore, Theorem 4.2 is a generalization of Theorem 4 of [47].

4.3 Asymptotic stability

Now, for the asymptotic stability, define $\mathcal{M}_0$ by

$$\begin{aligned} \mathcal{M}_0 := \{\varphi \in S : \varphi \text{ is } l_1\text{-Lipschitzian, } |\varphi(t)| \leq R,\ t \in [m_0, \infty) \cap \mathbb{T}, \\ \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0] \cap \mathbb{T} \text{ and } |\varphi(t)| \to 0 \text{ as } t \to \infty\}. \end{aligned} \tag{4.12}$$

All of the calculations in the proof of Theorem 4.2 hold with $\omega(t) = 1$ when $|.|_\omega$ is replaced by the supremum norm $\|.\|$.

Lemma 4.4 *Let (H1)–(H6) and (H9) hold. Then, the operator A maps $\mathcal{M}$ into a compact subset of $\mathcal{M}$.*

Proof. First, we deduce by the Lemma 4.2 that $A\mathcal{M}$ is equicontinuous. Next, we notice that for arbitrary $\varphi \in \mathcal{M}$ we have

$$\left|\left(A\varphi\right)(t)\right| \leq q_R(t) + \int_0^t e_{\ominus a}(t,s)a(s)\left(q_R(s) + \frac{g_{\sqrt{2}R}(s)}{a(s)}\right)\Delta s := q(t).$$

We see that $q(t) \to 0$ as $t \to \infty$ which implies that the set $A\mathcal{M}$ resides in a compact set in the space $(S, \|.\|)$ by Theorem 1.4. ■

Theorem 4.3 *Assume that (H1)–(H9) hold. Then the zero solution of (4.1) is asymptotically stable.*

Proof. Note that, all of the steps in the proof of Theorem 4.2 hold with $\omega(t) = 1$ when $|.|_\omega$ is replaced by the supremum norm $\|.\|$. It is sufficient to show, for $\varphi \in \mathcal{M}_0$ then $A\varphi \to 0$ and $B\varphi \to 0$. Let $\varphi \in \mathcal{M}_0$ be fixed, we will prove that $|(A\varphi)(t)| \to 0$ as $t \to \infty$, as above we have

$$\begin{aligned}|(A\varphi)(t)| &\leq |Q(t, \varphi(t-\tau(t)))| \\ &\quad + \int_0^t e_{\ominus a}(t,s)\left[a(s)\left|Q^\sigma(s,\varphi(s-\tau(s)))\right| + |G(s,\varphi(s),\varphi(s-\tau(s)))|\right]\Delta s.\end{aligned}$$

First, we have

$$|Q(t, \varphi(t-\tau(t)))| \leq q_R(t) \to 0 \text{ as } t \to \infty,$$

Second, let $\varepsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))|, |\varphi(t)| < \varepsilon$, for $t \geq T$. Then we have

$$\begin{aligned}&\int_0^t e_{\ominus a}(t,s)\left[a(s)\left|Q^\sigma(s,\varphi(s-\tau(s)))\right| + |G(s,\varphi(s),\varphi(s-\tau(s)))|\right]\Delta s \\ &= e_{\ominus a}(t,T)\int_0^T e_{\ominus a}(T,s)\left[a(s)\left|Q^\sigma(s,\varphi(s-\tau(s)))\right| + |G(s,\varphi(s),\varphi(s-\tau(s)))|\right]\Delta s \\ &+ \int_T^t e_{\ominus a}(t,s)\left[a(s)\left|Q^\sigma(s,\varphi(s-\tau(s)))\right| + |G(s,\varphi(s),\varphi(s-\tau(s)))|\right]\Delta s \\ &\leq e_{\ominus a}(t,T)\left(\frac{\alpha_1}{2}+\alpha_2\right)R + \left(\frac{\alpha_1}{2}+\alpha_2\right)\varepsilon.\end{aligned}$$

By (H9) the term $e_{\ominus a}(t,T)\left(\frac{\alpha_1}{2}+\alpha_2\right)R$ is arbitrarily small, as $t \to \infty$, In the same way, we obtain $B\varphi \to 0$. Then, by the Krasnoselskii-Burton theorem, there exists a fixed point $z \in \mathcal{M}_0$ such that $z = Az + Bz$. By Lemma 4.1 this fixed point is a solution of (4.1). Hence the zero solution of (4.1) is asymptotically stable.This completes the proof. ■

Remark 4.2 1) When $\mathbb{T} = \mathbb{R}$, Theorem 4.3 reduces to Theorem 5 of [47]. Therefore, Theorem 4.3 is a generalization of Theorem 5 of [47].

2) The sufficient conditions (H1)–(H9) of Theorem 4.3 are essentially for applying Theorems 1.13 and 1.4.

Chapter 5

Study of asymptotic behavior of solutions of neutral mixed type dynamic equations on a time scale

Keywords. Contraction mapping, neutral dynamic equations, mixed type, asymptotic behavior, time scales.

In this chapter, we expose the work cited in [29] as follow

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The objective of this work is to study the asymptotic behavior of solutions of a neutral mixed type dynamic equation on a time scale.

5.1 Introduction

Let $\mathbb{T}$ be a time scale which is unbounded above and below and such that $t_0 \in \mathbb{T}$. In this work, we consider the mixed type neutral dynamic equation on time scale of the from:

$$x^{\Delta}(t) + a(t) x^{\widetilde{\Delta}}(\tau(t)) + \sum_{i=1}^{k} b_i(t) x(\tau_i(t)) + \sum_{j=1}^{l} c_j(t) x(r_j(t)) = 0, \tag{5.1}$$

with the initial condition

$$x(t) = \theta(t) \text{ for } t \in [\tau_0, t_0] \cap \mathbb{T}, \tag{5.2}$$

where $\theta \in C_{rd}([\tau_0, t_0] \cap \mathbb{T}, \mathbb{R})$ and

$$\tau_0 = \inf\{\tau_i(s) : s \geq t_0,\ i = 1, ...k\}.$$

Also, a, b_i and c_j are rd-continuous functions and τ, τ_i and r_j are non-negative rd-continuous functions such that

$$\begin{aligned} \tau(t) &\to \infty \text{ as } t\to\infty,\ \tau(t)\geq t,\ t\geq t_0 \\ \tau_i(t) &\to \infty \text{ as } t\to\infty,\ i=1,...,k,\ \tau_i(t)\leq t,\ t\geq t_0. \\ r_j(t) &\to \infty \text{ as } t\to\infty,\ j=1,...,l,\ r_j(t)\geq t,\ t\geq t_0. \end{aligned}$$

In order for the functions $x(\tau(t))$, $x(\tau_i(t))$ and $x(r_j(t))$ to be well-defined and rd-continuous over $[t_0,\infty)\cap\mathbb{T}$, we assume that $\tau,\tau_i,r_j:[t_0,\infty)\cap\mathbb{T}\to\mathbb{T}$ are increasing mappings such that $\tau([t_0,\infty)\cap\mathbb{T})$, $\tau_i([t_0,\infty)\cap\mathbb{T})$ and $r_j([t_0,\infty)\cap\mathbb{T})$ are closed. To show the asymptotic behavior of solutions of (5.1), we transform (5.1) into an integral equation and then use the contraction mapping principle. Further, we will establish necessary and sufficient conditions for all solutions of (5.1) to converge to zero. In the special case $\mathbb{T}=\mathbb{R}$, Bicer [24] show the asymptotic behavior of solutions of (5.1) by using the fixed point theorem. Then, the results obtained here extend the work of Bicer [24].

5.2 Asymptotic behavior of solutions

Theorem 5.1 *Let a, b_i and c_j non-positive functions. Assume that the following inequality has a nonnegative solution $\lambda\in\mathcal{R}^+$*

$$-a(t)\lambda(\tau(t))e_\lambda(\tau(t),t)-\sum_{i=1}^{k}b_i(t)e_{\ominus\lambda}(t,\tau_i(t))-\sum_{j=1}^{l}c_j(t)e_\lambda(r_j(t),t)\leq\lambda(t), \quad (5.3)$$

for $t\geq t_0$. Then (5.1) has a positive solution.

Proof. Let $\lambda_0\in\mathcal{R}^+$ be a nonnegative solution of (5.3). Set

$$\lambda_n(t)=\begin{cases}\lambda_{n-1}(t), & \text{if } \tau_0\leq t\leq t_0, \\ -a(t)\lambda_{n-1}(\tau(t))e_{\lambda_{n-1}}(\tau(t),t)-\sum_{i=1}^{k}b_i(t)e_{\ominus\lambda_{n-1}}(t,\tau_i(t)) \\ -\sum_{j=1}^{l}c_j(t)e_{\lambda_{n-1}}(r_j(t),t), & t\geq t_0,\end{cases}$$

for $n=1,2,....$ Then, by (5.3), we get

$$\begin{aligned}\lambda_0(t)\geq{}&-a(t)\lambda_0(\tau(t))e_{\lambda_0}(\tau(t),t)-\sum_{i=1}^{k}b_i(t)e_{\ominus\lambda_0}(t,\tau_i(t)) \\ &-\sum_{j=1}^{l}c_j(t)e_{\lambda_0}(r_j(t),t) = \lambda_1(t).\end{aligned}$$

Then, we obtain $\lambda_0(t)\geq\lambda_1(t)\geq...\geq\lambda_n(t)\geq0$. So, there exists a pointwise limit $\lambda(t)=\lim\limits_{n\to\infty}\lambda_n(t)$. So, from the Lebesgue convergence theorem, we obtain

$$\lambda(t)=-a(t)\lambda(\tau(t))e_\lambda(\tau(t),t)-\sum_{i=1}^{k}b_i(t)e_{\ominus\lambda}(t,\tau_i(t))-\sum_{j=1}^{l}c_j(t)e_\lambda(r_j(t),t).$$

Hence,

$$x(t)=\begin{cases}\lambda(t), & \text{if } \varphi_0\le t\le t_0,\\ \lambda(t_0)\,e_\lambda(t,t_0), & t\ge t_0\end{cases}$$

is a positive solution of (5.1). ■

Theorem 5.2 *Let $\tau^\Delta(t)\neq 0$, $\frac{a}{\tau^\Delta}$, b_i and c_j be non-positive functions and let $\left[\frac{a(t)}{\tau^\Delta(t)}\right]^\Delta>0$, $\frac{a(t_0)}{\tau^\Delta(t_0)}\neq-\infty$. If*

$$\int_{t_0}^{\infty}\sum_{j=1}^{l}c_j(u)\,\Delta u=-\infty,$$

and x is a eventually positive solution of (5.1), then $x(t)\to\infty$ as $t\to\infty$.

Proof. Assume that $x(t)>0$ for $t\ge T_1$. Choose $T\ge T_1$ such that $T_1\le\inf\{\tau_i(s):s\ge T,\ i=1,...,k\}$. Let $m(t)=\frac{a(t)}{\tau^\Delta(t)}$. Then $x^\Delta(t)+a(t)x^{\widetilde{\Delta}}(\tau(t))\ge 0$, for $t\ge T$, and

$$x^\Delta(t)+a(t)\,x^{\widetilde{\Delta}}(\tau(t))=-\sum_{i=1}^{k}b_i(t)\,x(\tau_i(t))-\sum_{j=1}^{l}c_j(t)\,x(r_j(t)),$$

that is

$$[x(t)+m(t)x(\tau(t))]^\Delta-m^\Delta(t)\,x^\sigma(\tau(t))\ge-\sum_{j=1}^{l}c_j(t)\,x(r_j(t)).$$

From this, we can write

$$[x(t)+m(t)x(\tau(t))]^\Delta\ge-\sum_{j=1}^{l}c_j(t)\,x(r_j(t)),$$

$$[x(t)+m(t)x(\tau(t))]^\Delta\ge-x(T)\sum_{j=1}^{l}c_j(t),$$

which implies

$$x(t)+m(t)x(\tau(t))\ge m(t_0)x(\tau(t_0))-x(T)\int_{t_0}^{t}\sum_{j=1}^{l}c_j(u)\,\Delta u.$$

So, we get

$$x(t)\ge m(t_0)x(\tau(t_0))-x(T)\int_{t_0}^{t}\sum_{j=1}^{l}c_j(u)\,\Delta u.$$

Then $x(t)\to\infty$ as $t\to\infty$. ■

Theorem 5.3 *Let* $\frac{a(t)}{\tau^{\Delta}(t)} > 0$, b_i *and* c_j *be non-negative functions and let* $\left[\frac{a(t)}{\tau^{\Delta}(t)}\right]^{\Delta} < 0$, $\frac{a(t_0)}{\tau^{\Delta}(t_0)} \neq \infty$. *If*

$$\int_{t_0}^{\infty} \sum_{j=1}^{l} c_j(u)\, \Delta u = \infty,$$

and x *is a eventually positive solution of (5.1), then* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let $m(t) = \frac{a(t)}{\tau^{\Delta}(t)}$. For $t \geq T_1$, since $x(t) > 0$ we choose $T \geq T_1$ such that $T_1 \leq \inf\{\tau_i(s) : s \geq T,\ i = 1, ..., k\}$. Then $x^{\Delta}(t) + a(t)x^{\tilde{\Delta}}(\tau(t)) \leq 0$, for $t \geq T$, and

$$x^{\Delta}(t) + a(t)\, x^{\tilde{\Delta}}(\tau(t)) \leq -\sum_{j=1}^{l} c_j(t)\, x(r_j(t)),$$

that is

$$[x(t) + m(t)x(\tau(t))]^{\Delta} - m^{\Delta}(t)\, x^{\sigma}(\tau(t)) \leq -\sum_{j=1}^{l} c_j(t)\, x(r_j(t)).$$

From this, we can write

$$[x(t) + m(t)x(\tau(t))]^{\Delta} \leq -\sum_{j=1}^{l} c_j(t)\, x(r_j(t)) \leq -x(T)\sum_{j=1}^{l} c_j(t),$$

which implies

$$x(t) + m(t)x(\tau(t)) \leq m(t_0)x(\tau(t_0)) - x(T)\int_{t_0}^{t} \sum_{j=1}^{l} c_j(u)\, \Delta u.$$

So, we get

$$x(t) \leq m(t_0)x(\tau(t_0)) - x(T)\int_{t_0}^{t} \sum_{j=1}^{l} c_j(u)\, \Delta u.$$

Since $x(t) > 0$, we get a contradiction.Then $x(t) \to 0$ as $t \to \infty$. ∎

Now, we investigate the asymptotic behavior of (5.1), free of the sign of the coefficients. During the process of inverting (5.1), an integration by parts will have to performed on the term involving $x^{\tilde{\Delta}}(\tau(t))$. Thus, we require that

$$\tau^{\Delta}(t) \neq 0, \quad \forall t \in \mathbb{T}. \tag{5.4}$$

Lemma 5.1 *Suppose (5.4) holds. A function* x *is a solution of equation (5.1)–(5.2) if and only if*

$$\begin{aligned} x(t) = {} & \left(x(t_0) + \frac{a(t_0)}{\tau^{\Delta}(t_0)} x(\tau(t_0))\right) e_{\ominus B}(t, t_0) - \frac{a(t)}{\tau^{\Delta}(t)} x(\tau(t)) \\ & + \int_{t_0}^{t} h(u)\, x(\tau(u))\, e_{\ominus B}(t, u)\, \Delta u + \int_{t_0}^{t} e_{\ominus B}(t, u)\, B(u)\, x^{\sigma}(u)\, \Delta u \\ & - \sum_{i=1}^{k} \int_{t_0}^{t} e_{\ominus B}(t, u)\, b_i(u)\, x(\tau_i(u))\, \Delta u - \sum_{j=1}^{l} \int_{t_0}^{t} e_{\ominus B}(t, u)\, c_j(u)\, x(r_j(u))\, \Delta u, \end{aligned} \tag{5.5}$$

for $t \geq t_0$, *where*

$$B(u) = \sum_{i=1}^{k} b_i(u) + \sum_{j=1}^{l} c_j(u),$$

and

$$h(u) = \frac{\tau^{\Delta}(u)\left(a^{\sigma}(u) B(u) + a^{\Delta}(u)\right) - \tau^{\Delta\Delta}(u) a(u)}{\tau^{\Delta}(u) \tau^{\Delta}(\sigma(u))}.$$

Proof. Since

$$\begin{aligned} x(r_j(t)) &= x^{\sigma}(t) + \int_{\sigma(t)}^{r_j(t)} x^{\Delta}(u) \Delta u, \\ x(\tau_i(t)) &= x^{\sigma}(t) - \int_{\tau_i(t)}^{\sigma(t)} x^{\Delta}(u) \Delta u, \end{aligned}$$

we can rewrite (5.1) as

$$\begin{aligned} x^{\Delta}(t) + B(t) x^{\sigma}(t) = -a(t) x^{\widetilde{\Delta}}(\tau(t)) &+ \sum_{i=1}^{k} b_i(t) \int_{\tau_i(t)}^{\sigma(t)} x^{\Delta}(u) \Delta u \\ &- \sum_{j=1}^{l} c_j(t) \int_{\sigma(t)}^{r_j(t)} x^{\Delta}(u) \Delta u. \end{aligned} \tag{5.6}$$

Multiplying both sides of (5.6) with $e_B(t, t_0)$, and integrating from t_0 to t, we obtain

$$\begin{aligned} &\int_{t_0}^{t} [x(u) e_B(u, t_0)]^{\Delta} \Delta u \\ &= \int_{t_0}^{t} -a(u) x^{\widetilde{\Delta}}(\tau(u)) e_B(u, t_0) \Delta u \\ &+ \int_{t_0}^{t} e_B(u, t_0) \left(\sum_{i=1}^{k} b_i(u) \int_{\tau_i(u)}^{\sigma(u)} x^{\Delta}(s) \Delta s - \sum_{j=1}^{l} c_j(u) \int_{\sigma(u)}^{r_j(u)} x^{\Delta}(s) \Delta s \right) \Delta u. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} x(t) = x(t_0) e_{\ominus B}(t, t_0) &- \int_{t_0}^{t} a(u) x^{\widetilde{\Delta}}(\tau(u)) e_{\ominus B}(t, u) \Delta u \\ + \int_{t_0}^{t} e_{\ominus B}(t, u) &\left(\sum_{i=1}^{k} b_i(u) \int_{\tau_i(u)}^{\sigma(u)} x^{\Delta}(s) \Delta s - \sum_{j=1}^{l} c_j(u) \int_{\sigma(u)}^{r_j(u)} x^{\Delta}(s) \Delta s \right) \Delta u, \end{aligned} \tag{5.7}$$

Rewrite

$$\int_{t_0}^{t} a(u) x^{\widetilde{\Delta}}(\tau(u)) e_{\ominus B}(t, u) \Delta u = \int_{t_0}^{t} a(u) x^{\widetilde{\Delta}}(\tau(u)) \frac{\tau^{\Delta}(u)}{\tau^{\Delta}(u)} e_{\ominus B}(t, u) \Delta u.$$

By performing an integration by parts on the above integral we get

$$\int_{t_0}^{t} a(u)\, x^{\widetilde{\Delta}}(\tau(u))\, e_{\ominus B}(t,u)\, \Delta u = \frac{a(t)}{\tau^{\Delta}(t)} x(\tau(t)) - \frac{a(t_0)}{\tau^{\Delta}(t_0)} x(\tau(t_0)) e_{\ominus B}(t,t_0) - \int_{t_0}^{t} h(u)\, x(\tau(u))\, e_{\ominus B}(t,u)\, \Delta u. \tag{5.8}$$

Therefore, we obtain (5.5) by substituting (5.8) in (5.7). Since each step is reversible, the converse follows easily. This completes the proof. ■

Theorem 5.4 *Assume that $B \in \mathcal{R}^{+}$, (5.4) and the following conditions hold*

$$\int_{t_0}^{t} \frac{1}{\mu(s)} \log(1 + \mu(s) B(s))\, \Delta s \to \infty \text{ as } t \to \infty, \tag{5.9}$$

and

$$\left| \frac{a(t)}{\tau^{\Delta}(t)} \right| + \int_{t_0}^{t} |h(u)|\, e_{\ominus B}(t,u)\, \Delta u + \int_{t_0}^{t} e_{\ominus B}(t,u)\, |B(u)|\, \Delta u + \sum_{i=1}^{k} \int_{t_0}^{t} e_{\ominus B}(t,u)\, |b_i(u)|\, \Delta u + \sum_{j=1}^{l} \int_{t_0}^{t} e_{\ominus B}(t,u)\, |c_j(u)|\, \Delta u \le \beta < 1. \tag{5.10}$$

Then for each initial condition (5.2), every solution of (5.1) converges to zero.

Proof. Let $C_{rd}([\tau_0,\infty) \cap \mathbb{T})$ is the space of rd-continuous functions on $[\tau_0,\infty) \cap \mathbb{T}$ and

$$M = \{x \in C_{rd}([\tau_0,\infty) \cap \mathbb{T}) : x(t) \to 0, \text{ as } t \to \infty\},$$

be a closed subspace of $C_{rd}([\tau_0,\infty) \cap \mathbb{T})$. Then $(M, \|.\|)$ is a Banach space with the norm

$$\|x\| = \sup_{t \ge \tau_0} |x(t)|.$$

Define the operator $\phi : M \to M$ by

$$(\phi x)(t) = \begin{cases} \theta(t), \quad \text{if } \tau_0 \le t \le t_0, \\ \left(x(t_0) + \frac{a(t_0)}{\tau^{\Delta}(t_0)} x(\tau(t_0)) \right) e_{\ominus B}(t,t_0) - \frac{a(t)}{\tau^{\Delta}(t)} x(\tau(t)) \\ + \int_{t_0}^{t} h(u)\, x(\tau(u))\, e_{\ominus B}(t,u)\, \Delta u + \int_{t_0}^{t} e_{\ominus B}(t,u)\, B(u)\, x^{\sigma}(u)\, \Delta u \\ - \sum\limits_{i=1}^{k} \int_{t_0}^{t} e_{\ominus B}(t,u)\, b_i(u)\, x(\tau_i(u))\, \Delta u \\ - \sum\limits_{j=1}^{l} \int_{t_0}^{t} e_{\ominus B}(t,u)\, c_j(u)\, x(r_j(u))\, \Delta u, \qquad t \ge t_0. \end{cases} \tag{5.11}$$

It is clear that for $x \in M$, ϕx is rd-continuous. Now, we will show that, $(\phi x) \to 0$ as $t \to \infty$. Actually, for $x \in M$, we have

$$\begin{aligned}
|(\phi x)(t)| \leq & \left| x(t_0) + \frac{a(t_0)}{\tau^\Delta(t_0)} x(\tau(t_0)) \right| e_{\ominus B}(t, t_0) + \left| \frac{a(t)}{\tau^\Delta(t)} \right| |x(\tau(t))| \\
& + \int_{t_0}^{t} |h(u)| |x(\tau(u))| e_{\ominus B}(t, u) \Delta u \\
& + \int_{t_0}^{t} e_{\ominus B}(t, u) |B(u)| |x^\sigma(u)| \Delta u \\
& + \sum_{i=1}^{k} \int_{t_0}^{t} |b_i(u)| |x(\tau_i(u))| e_{\ominus B}(t, u) \Delta u \\
& + \sum_{j=1}^{l} \int_{t_0}^{t} |c_j(u)| |x(r_j(u))| e_{\ominus B}(t, u) \Delta u. \qquad (5.12)
\end{aligned}$$

Note that by (5.9),

$$\left| x(t_0) + \frac{a(t_0)}{\tau^\Delta(t_0)} x(\tau(t_0)) \right| e_{\ominus B}(t, t_0) \to 0 \text{ as } t \to \infty.$$

Moreover, since $x(t) \to 0$ as $t \to \infty$, for each $\varepsilon > 0$, there exists $T_1 > t_0$ such that $u \geq T_1$ implies that $|x(\tau(u))| < \frac{\varepsilon}{2}$. Thus, for $t \geq T_1$, the third term I_3 in (5.11) satisfies

$$\begin{aligned}
I_3 &= \int_{t_0}^{t} h(u) x(\tau(u)) e_{\ominus B}(t, u) \Delta u \\
&\leq \int_{t_0}^{T_1} e_{\ominus B}(t, u) |h(u)| |x(\tau(u))| \Delta u + \int_{T_1}^{t} e_{\ominus B}(t, u) |h(u)| |x(\tau(u))| \Delta u \\
&\leq \int_{t_0}^{T_1} e_{\ominus B}(t, u) |h(u)| |x(\tau(u))| \Delta u + \frac{\varepsilon}{2} \int_{T_1}^{t} e_{\ominus B}(t, u) |h(u)| \Delta u \\
&\leq \frac{\varepsilon}{2} + \beta \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

Thus $I_3 \to 0$ as $t \to \infty$. By a similar technique, we can prove that the rest of terms in (5.11) tend zero as $t \to \infty$. Therefore $(\phi x)(t) \to 0$ as $t \to \infty$. Now, we will show that ϕ is a contraction. Let x, y be two continuous function on $[t_0, \infty) \cap \mathbb{T}$ and satisfying same initial condition (5.2). Then for $t \geq t_0$, we get

$$
\begin{aligned}
&\left|(\phi x)(t) - (\phi y)(t)\right| \\
&\leq \left|\frac{a(t)}{\tau^{\Delta}(t)}\right| \left|x(\tau(t)) - y(\tau(t))\right| + \int_{t_0}^{t} |h(u)| \left|x(\tau(u)) - y(\tau(u))\right| e_{\ominus B}(t,u)\,\Delta u \\
&+ \int_{t_0}^{t} e_{\ominus B}(t,u) \left|x^{\sigma}(u) - y^{\sigma}(u)\right| |B(u)|\,\Delta u \\
&+ \sum_{i=1}^{k} \int_{t_0}^{t} |b_i(u)| \left|x(\tau_i(u)) - y(\tau_i(u))\right| e_{\ominus B}(t,u)\,\Delta u \\
&+ \sum_{j=1}^{l} \int_{t_0}^{t} |c_j(u)| \left|x(r_j(u)) - y(r_j(u))\right| e_{\ominus B}(t,u)\,\Delta u \\
&\leq \beta \left\|x - y\right\|.
\end{aligned}
$$

Thus, by the contraction mapping principle (see [24]), the operator ϕ has a unique fixed point in M, which solves (5.1) and tends to zero as $t \to \infty$. This completes the proof. ■

Theorem 5.5 *Suppose that $B \in \mathcal{R}^{+}$ (5.4) and(5.10) holds, and*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \frac{1}{\mu(s)} \log\left(1 + \mu(s)B(s)\right)\Delta s > -\infty. \tag{5.13}$$

If all solutions of (5.1) converge to zero, then (5.9) holds.

Proof. Suppose that (5.9) does not hold. That is,

$$\liminf_{t\to\infty} \int_{t_0}^{t} \frac{1}{\mu(s)} \log\left(1 + \mu(s)B(s)\right)\Delta s = \delta < \infty. \tag{5.14}$$

So, from (5.13), we can write $\delta > -\infty$. Then, there exists a sequence $\{t_n\}$ approaching ∞, such that

$$\int_{t_0}^{t_n} \frac{1}{\mu(s)} \log\left(1 + \mu(s)B(s)\right)\Delta s \to \delta, \text{ as } n \to \infty.$$

For $x(t_0) \neq 0$, let x be a solution. Then,

$$\lim_{n\to\infty} \left(x(t_0) + \frac{a(t_0)}{\tau^{\Delta}(t_0)} x(\tau(t_0))\right) e_{\ominus B}(t_n, t_0) = \left(x(t_0) + \frac{a(t_0)}{\tau^{\Delta}(t_0)} x(\tau(t_0))\right) e^{-\delta} \neq 0. \tag{5.15}$$

From Lemma 5.1, $x(t_n)$ satisfies (5.5). On the other hand, we know that

$$\begin{aligned}&\lim_{n\to\infty}\left[\int_{t_0}^{t_n} h(u)\,x(\tau(u))\,e_{\ominus B}(t_n,u)\,\Delta u-\frac{a(t_n)}{\tau^{\Delta}(t_n)}x(\tau(t_n))\right.\\&+\int_{t_0}^{t_n} e_{\ominus B}(t_n,u)\,B(u)\,x^{\sigma}(u)\,\Delta u\\&-\sum_{i=1}^{k}\int_{t_0}^{t_n} e_{\ominus B}(t_n,u)\,b_i(u)\,x(\tau_i(u))\,\Delta u\\&\left.-\sum_{j=1}^{l}\int_{t_0}^{t_n} e_{\ominus B}(t_n,u)\,c_j(u)\,x(r_j(u))\,\Delta u\right]=0. \qquad (5.16)\end{aligned}$$

Since all solutions tend zero, from (5.5), (5.15) and (5.16), we get

$$\lim_{n\to\infty} x(t_n)=\left(x(t_0)+\frac{a(t_0)}{\tau^{\Delta}(t_0)}x(\tau(t_0))\right)e^{-\delta}\neq 0,$$

which contradicts all solutions of (5.1) converge to zero. The proof is completed. ■

Conclusion and perspectives

The works presented in this thesis study the existence, periodicity, positivity and stability of solutions of neutral dynamic equations on the time scale. The fixed point technique used for this type of equations which shows in this thesis are efficiency.

At the beginning, we gave the preliminary notions useful for a good understanding of this work. Then we have established the conditions of the existence results of periodic and non-negative solutions and their stability for nonlinear dynamic equations of neutral type on time scale using Krasnoselskii-Burton's theorem. Finally, we have found the conditions the asymptotic behavior of solutions of mixed-type dynamic equations on a time scale using the contraction principle.

As a future work on the presented results in this thesis, we can mention a few points that can be developed and improved

- It can make change in space from either in the type of delay or add stochastic term.
- It can study dynamic equations with a delay superior than two orders, extend to dynamical equations with distributed delay or dynamic equations with fractional delay.
- It can also do numerical study.

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