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THÈSE

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Sur la stabilité de quelques problèmes d'évolutions avec des conditions aux limites

Option
Mathématiques Appliquées

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Abstact

In this dissertation, we study the well-posedness and the stability of the solutions of some evolution problems in the presence of delay, and under assumptions on initial and boundary conditions. In this regard, we study four problems and establish an exponential decay result under some suitable assumptions. The first and second problems relate to a laminated beam system, while the third and fourth problems relate to a flexible structure system.
We demonstrate the existence of the solution by the semi-group method, then we demonstrate the exponential stability by the disturbed energy method by an adequate construction of the Lyapunov functional.

Key words: Evolution problems, laminated beam system, flexible structure system, delay term, distributed delay term, well-posedness, exponential decay.

Résumé

Dans cette thèse, nous étudions l'existence, l'unicité et la stabilité des solutions de quelques problèmes d'évolution en présence de retard, et sous des hypothèses sur les données initiales et les conditions aux limites. À cet égard, nous étudions quatre problèmes et nous établissons un résultat de décroissance exponentielle sous certaines hypothèses appropriées. Les premier et second problèmes concernent un système de poutre feuilletée, tandis que les troisième et quatrième problèmes concernent un système de structure flexible.
Nous démontrons l'existence de la solution par la méthode de semi-groupe, ensuite nous démontrons la stabilité exponentielle par la méthode de l'énergie perturbée par une construction adéquate de la fonctionnelle de Lyapunov.

Mots clés: Problèmes d'évolution, système de poutres laminées, système de structure flexible, terme de retard, terme de retard distribué, l'existence, l'unicité, décroissance exponentielle.

ملخص

ندرس في هذه الأطروحة وجود ووحدانية واستقرار حلول بعض مسائل التطور في وجود حد تأخير، وفي ظل افتراضات بشأن الشروط الأولية والحدية. في هذا الصدد، ندرس أربع مسائل ونحدد نتيجة الاضمحلال الأسي تحت بعض الافتراضات المناسبة. تتعلق المسألتان الأولى والثانية بنظام حزمة صفائح، بينما تتعلق المسألتان الثالثة والرابعة بنظام هيكل مرن. نثبت وجود الحل بطريقة شبه المجموعة، ثم نبين الاستقرار الأسي بطريقة الطاقة من خلال البناء المناسب لدالة ليابونوف.

الكلمات المفتاحية:

مسائل التطور، نظام حزمة صفائح، نظام هيكل مرن، حد التأخير، حد التأخير الموزع، الوجود والوحدانية، الاضمحلال الأسي.

Contents

Introduction

Many problems in physics can be modeled by partial differential equations (PDEs) which generally depend on time (evolution problems). The study of the existence and stability of development problems has been the subject of many recent works. In this thesis we were interested in the study of the global existence and the stabilization of some evolution equations. The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to zero in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero. There are several types of stabilization, of which:

- Strong stabilization: $E(t) \longrightarrow 0$, as $t \longrightarrow \infty$,
- polynomial stabilization: $E(t) \leq \alpha t^{-\beta}$, $\forall t > 0$, $(\alpha, \beta > 0)$,
- exponential (uniform) stabilization: $E(t) \leq \alpha e^{-\beta t}$, $\forall t > 0$, $(\alpha, \beta > 0)$.

We use the multiplier method to establish the desired stability results of the systems. Multiplier method relies mostly on the construction of an appropriate Lyapunov functional L equivalent to the energy of the solution E. By equivalence $L \sim E$, we mean

$$\alpha E(t) \leq L(t) \leq \beta E(t), \quad \forall t > 0, \tag{1}$$

for two positive constants α and β. To prove the exponential stability, we show that L satisfies

$$L'(t) \leq \gamma L(t), \ \ \forall t > 0, \tag{2}$$

for some $\gamma > 0$. A simple integration of (2) over $(0, t)$ together with (1) gives the desired exponential stability result.

Time delays arise in many applications of most phenomena naturally modulate by partial differential equations problems, where the rate of change in a state is not only determined by the present states but also by the past states. The delay differential equations (DDEs) are differential equations in which the derivatives of some unknown functions at present time depend on the values of the functions at previous times. Mathematically, a simple delay differential equation for $x(t) \in \mathbb{R}^n$ takes the form

$$\frac{d}{dt}x(t) = f(t, x_t),$$

where $x_t = \{x(\tau),\ \tau \leq t\}$ represents the trajectory of the solution in the past. The functional operator f takes a time input and a continuous function x_t and generates a real number $\frac{d}{dt}x(t)$ as its output. Examples of such equation include:

- discrete/constant delay $\frac{d}{dt}x(t) = f(t, x(t-\tau))$,
- time-varying delay $\frac{d}{dt}x(t) = f(t, x(t-\tau(t)))$,
- distributed delay $\frac{d}{dt}x(t) = f\left(t, \int_0^\tau \mu(s) x(t-s)\, ds\right)$,

  where τ is the delay in time.

In recent years, the PDEs with time delay effects have become an active area of research. Many authors have focused on this problem (see [9, 10, 23, 44, 45, 52, 53, 54]). The presence of delay may be a source of instability. It may turn a well-behaved system into a wild one. In [9] for example, R. Datko, J. Lagnese and M. P. Polis proved that a small delay may destabilize a system. In [45], Nicaise and Pignotti considered wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \triangle u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)\, ds = 0, \text{ in } \Omega \times (0, \infty),$$

with initial and Dirichlet-Neumann boundary conditions and a is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)\,ds < \mu_1.$$

Regarding the similar result concerning boundary distributed delay see [4, 42, 43]. Moreover, Nicaise, Pignotti and Valein [46] replaced the constant delay term in the boundary condition of [44] by a time-varying delay term and obtained an exponential decay result under an appropriate assumption on the weights of the damping and delay. Moreover, Kafini et al. [27] examined a coupling Timoshenko-thermoelasticity of type III system with time delay and established exponential and polynomial stability results depending on the wave propagation speeds. For other related results, we refer the reader to [8, 11, 15, 26, 30].

The aim of this thesis is to investigate the well-posedness and asymptotic behavior of solutions of some evolution problems in the presence of delay, and under assumptions on initial data and boundary conditions. In this regard, we study four problems and establish an exponential decay result for the one-dimensional case under some suitable assumptions. The first and second problems relate to a laminated beam system, while the third and fourth problems relate to a flexible structure system.

A laminated beam system

Laminated beam, which is a relevant research subject due to the high applicability of such materials in the industry, was firstly introduced by Hansen and Spies [24, 25]. Hansen [24] proposed a model of laminated beam based on the Timoshenko system which is one of particular interest. In [25], Hansen and Spies derived three mathematical models for two-layered beams with structural damping due to the interfacial slip. Assume that each individual beam (of length 1) satisfies the Timoshenko system [50],

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho\psi_{tt} + G(\psi - \varphi_x) - D\psi_{xx} = 0, \end{cases}$$

where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, φ represents the transverse displacement, ψ is the rotation angle produced by the beam deflection, and the parameters ρ, G, I_ρ, D mean respectively, mass density, shear stiffness, moment of mass inertia and flexural rigidity. Then, as shown in [24, 25], the laminated beam featuring longitudinal slip can be modeled through the system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3\omega - \psi)_{tt} - G(\psi - \varphi_x) - D(3\omega - \psi)_{xx} = 0, \\ I_\rho\omega_{tt} + G(\psi - \varphi_x) + \dfrac{4}{3}\gamma\omega + \dfrac{4}{3}\beta\omega_t - D\omega_{xx} = 0, \end{cases} \tag{3}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, with ω accounts for the interfacial slip, $\gamma \geq 0$ represents the adhesive stiffness and $\beta \geq 0$ is a damping parameter.

In recent years, an increasing interest has been developed to determine the asymptotic behavior of the solution of several laminated beam problems, we refer the reader to [5, 29, 31, 32, 33, 48, 49, 51] and the references therein. In [48], Raposo considered system (3) with two frictional dampings of the form

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi - \varphi_x)_x + k_1\varphi_t = 0, \\ \rho_2(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + k_2(3w - \psi)_t = 0, \\ \rho_2 w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and obtained the exponential decay result under appropriate initial and boundary conditions. In [51], Wang, Xu and Yung considered system (3) with the cantilever boundary conditions and two different wave speeds ($\sqrt{\frac{G}{\rho_1}}$ and $\sqrt{\frac{D}{\rho_2}}$). W. Liu and W. Zhao [32] considered a coupled system of a laminated beam with Fourier's type heat conduction, which has the form

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + \sigma\theta_x = 0, \\ I_\rho w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w - \psi)_{tx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, they used the energy method to prove an exponential decay result for the case of equal wave speeds. (See also [2, 8, 28, 36, 40]).

For the Timoshenko system of thermo-viscoelasticity of type III, Messaoudi and Said-Houari [37] considered the following one-dimensional linear Timoshenko system of thermoelastic type

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, they used the energy method to prove an exponential decay under the condition $\frac{\rho_1}{K} = \frac{\rho_2}{b}$. A similar result was also obtained by Rivera and Racke [40]. Since this theory predicts an infinite speed of heat propagation, many theories have emerged, to overcome this physical paradox. Green and Naghdi [20, 21, 22], suggest a replacing Fourier's law by the so-called thermoelasticity of type III. This is for heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. For more details, see [7]. A large number of interesting decay results depending on the stability number have been established, (see [16, 35, 36, 41] and references therein). In [34],Y. Luan, W. Liu and G. Li considered a coupled system of a laminated beam with thermoelasticity of type III, which has the form:

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_{\rho_1}(3\omega - \psi)_{tt} - D(3\omega - \psi)_{xx} - G(\psi - \varphi_x) + \alpha\theta_x = 0, \\ I_{\rho_1}\omega_{tt} - D\omega_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\beta_1\omega + \frac{4}{3}\beta_2\omega_t = 0, \\ \rho_2\theta_{tt} - \delta\theta_{xx} + \gamma(3\omega - \psi)_{ttx} - k\theta_{txx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, they used the energy method to prove an exponential decay result for the case of equal wave speeds.

A flexible structure system

One of the main issues concerning the vibrations in models of flexible structural systems is the question of the stabilization of the structure, the linear differential equation describing the vibrations of an inhomogeneous flexible structure with an exterior disturbing force can be described by the following equation

$$m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x = f(x), \text{ on } (0,L) \times \mathbb{R}^+, \tag{4}$$

where $u = u(x,t)$ is the displacement of a particle at position $x \in (0,L)$ and time $t > 0$. The parameters $m(x)$, $\delta(x)$ and $p(x)$ is responsible for the non-uniform structure of the body, where $m(x)$ denote mass per unit length of structure, $\delta(x)$ coefficient of internal material damping and $p(x)$ a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m(x), \delta(x)$ and $p(x)$ in [3] such that

$$m, \delta, p \in W^{1,\infty}(0, L), \quad m(x), \delta(x), p(x) > 0, \quad \forall x \in [0, L].$$

The distributed force $f : (0, L) \times \mathbb{R}^{+} \longrightarrow \mathbb{R}$ is the uncertain disturbance appearing in the model which is assumed to be continuously differentiable for all $t \geq 0$. In [19], Gorain has established uniform exponential stability of the problem (4). It is physically relevant to take into account thermal effects in flexible structures, in 2014, M. Siddhartha et al. [38] showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{cases} m(x) u_{tt} - (p(x) u_x + 2\delta(x) u_{xt})_x + \kappa \theta_x = f, \\ \theta_t - \theta_{xx} + \kappa u_{tx} = 0. \end{cases} \tag{5}$$

In the above model, the temperature has an infinite velocity of propagation (heat equation). this property of the model is not consistent with the reality, where the heating or cooling of a flexible structure will usually take some time. Many researches have thus been conducted in order to modify the model of thermal effect.

In [17], the Authors consider a non-uniform flexible structure system with time delay under Cattaneo's law of heat condition:

$$\begin{cases} m(x) u_{tt} - (p(x) u_x + 2\delta(x) u_{xt})_x + \eta \theta_x + \mu u_t(x, t - \tau_0) = 0, x \in (0, L), t > 0, \\ \theta_t + \kappa q_x + \eta u_{tx} = 0, \ \ x \in (0, L), \ t > 0, \\ \tau q_t + \beta q + \kappa \theta_x = 0, \ \ x \in (0, L), \ t > 0, \end{cases} \tag{6}$$

with boundary condition

$$u(0, t) = u(L, t) = 0, \ \theta(0, t) = \theta(L, t) = 0, \ t \geq 0, \tag{7}$$

and initial condition

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x), x \in [0, L]. \tag{8}$$

They proved that system (6)-(8) is well-posed, and the system is exponential decay under a small condition on time delay. M. S. Alves et al. [3] consider the system (6)-(8) without delay term, and obtained an exponential stability result for one set of boundary conditions, and at least polynomial for another set of boundary conditions.

The plan of the thesis is as follows.

Chapter 1. In this chapter, we recall some basic knowledge in fonctional analysis.

Part I. We study two problems relate to a laminated beam system.

Chapter 2. In this chapter, we study the well-posedness and the asymptotic behavior of a one-dimensional laminated beam system with a distributed delay term in the first equation, where the heat conduction is given by Fourier's law effective in the rotation angle displacements. We first give the well-posedness of the system by using the semigroup method. Then, we show that the system is exponentially stable under the assumption of equal wave speeds.

Chapter 3. In this chapter, we study the well-posedness and asymptotic behaviour of solutions to a laminated beam in thermoelasticity of type III with delay term in the first equation. We show that the system is well-posed by using Lumer-Phillips theorem and prove that the system is exponentially stable if and only if the wave speeds are equal.

Part II. We study two problems relate to a flexible structure system.

Chapter 4. In this chapter, we study the well-posedness and asymptotic behaviour of solutions to a flexible structure with Fourier's type heat conduction and distributed delay. we prove well-posedness by using the semigroup theory. Also we establish a decay result by introducing a suitable Lyaponov functional.

Chapter 5. In this chapter, we study well-posedness and exponential stability for coupled flexible structure system with distributed delay in the two equations. We first give the well-posedness of the system by using semigroup method. Then, by using the perturbed energy method and construct some Lyapunov functionals, we then obtain the exponential decay result.

Chapter 1

Preliminary

Contents

In this chapter, we recall some concepts and properties in functional analysis related to the subsequent chapters, reader should consult [1], [6], [18] and [47] for proofs and more details.

1.1 Some functional spaces

1.1.1 Banach spaces

Definition 1.1. ***(Normed linear space)*** *A normed vector space X is a vector space equipped with a norm $\|.\| : X \longrightarrow \mathbb{R}$ that satisfies the following properties:*

1. $\|x\| \geq 0$, *and* $\|x\| = 0 \Longleftrightarrow x = 0$,
2. $\|ax\| = |a| \|x\|$ *for any scalar* a,
3. $\|x+y\| \leq \|x\| + \|y\|$.

Recall that completeness of a normed vector space X means that all Cauchy sequences in X converge in X.

Definition 1.2. ***(Banach spaces)*** *A Banach space is a complete normed vector space.*

1.1.2 Hilbert spaces

Definition 1.3. ***(Inner product space)*** *An inner product space X is a vector space equipped with an inner product $\langle .,. \rangle : X \times X \longrightarrow \mathbb{R}$ that satisfies the following properties:*

1. $\langle u, v \rangle = \langle v, u \rangle$,
2. $\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$,
3. $\langle u, u \rangle \geq 0$ *with* $\langle u, u \rangle = 0 \Longleftrightarrow u = 0$.

An inner product induces a norm, $\|u\| = \sqrt{\langle u, u \rangle}$, and hence a metric.

Definition 1.4. *A Hilbert space $\mathcal{H}$ is a vectorial space supplied with inner product $\langle u, v \rangle$ such that $\|u\| = \sqrt{\langle u, u \rangle}$ is the norm which let $\mathcal{H}$ complete.*

Theorem 1.1. *Let $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in the Hilbert space $\mathcal{H}$, then it possess a subsequence which converges in the weak topology of $\mathcal{H}$.*

Theorem 1.2. ***(Lax-Milgram)*** *Let $\mathcal{H}$ be a Hilbert space, B be a bilinear form and L be a linear form. Assume that* *(i) The bilinear form B is continuous, i.e., there exists a constant M such that*

$$|B(u,v)| \leq M \|u\| \|v\| \quad \textit{for all} \quad u,v \in \mathcal{H},$$

(ii) The bilinear form B is V-elliptic, i.e., there exists a constant $m > 0$ such that

$$|B(u,u)| \geq m \|u\|^2 \quad \textit{for all} \quad u \in \mathcal{H},$$

(iii) The linear form L is continuous, i.e., there exists a constant C such that

$$|L(u)| \leq C \|u\| \quad \textit{for all} \quad u \in \mathcal{H}.$$

There exists a unique $u \in \mathcal{H}$ that solves the abstract variational problem: Find $u \in \mathcal{H}$ such that

$$\forall v \in \mathcal{H}, \ \ B(u,v) = L(u).$$

Moreover, if $B(.,.)$ is symmetric, then $u \in \mathcal{H}$ is characterized by:

$$\frac{1}{2}B(u,u) - L(u) = \min_{v \in \mathcal{H}} \left(\frac{1}{2}B(v,v) - L(v) \right).$$

1.1.3 The $L^p(\Omega)$ spaces

Definition 1.5. *Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, and let Ω be on open domain in $\mathbb{R}^n$, define the standard Lebesgue space $L^p(\Omega)$, by*

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R}; \ f \textit{ is measurable and } \int_\Omega |u(x)|^p \, dx < +\infty \right\}.$$

If $p = +\infty$, we have

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R}; \begin{array}{c} u \textit{ is measurable and there is a constant } C \\ \textit{such that } |u(x)| \leq C \ \textit{ a.e. on } \Omega \end{array} \right\}.$$

For $u \in L^p(\Omega)$, we define the norms

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad 1 \leq p < +\infty,$$

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = ess \sup_{x \in \Omega} |u(x)| = inf\{C; \ |u(x)| \leq C \ \textit{ a.e. on } \Omega\}, \ \ p = +\infty.$$

1.1.4 Sobolev Spaces

Definition 1.6. *Let Ω be a domain in $\mathbb{R}^n$ and let m be a non-negative integer. We define by $C^m(\Omega)$ the linear space of continuous functions on Ω whose partial derivatives $D^\alpha u$, $|\alpha| \leq m$, exist and continuous, where*

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, \ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \ |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n.$$

Definition 1.7. ***(weak derivative)*** *If $u, v \in L^p(\Omega)$, v is called a weak derivative of order α of u if*

$$\int_\Omega u(x) D^\alpha \Phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x) \Phi(x) dx, \ \forall \Phi \in C_0^\infty(\Omega).$$

Definition 1.8. ***(Sobolev spaces)*** *Let Ω be an open set of $\mathbb{R}^n$,the Sobolev space $W^{m,p}(\Omega)$, $m \in \mathbb{N}^*$, $1 \leq p \leq +\infty$, is defined as*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega); \ D^\alpha u \in L^p(\Omega), \ |\alpha| \leq m\}.$$

$W^{m,p}(\Omega)$ is equipped with the following norm:

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \max\limits_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

Specially, when $m = 0$, $W^{0,p} = L^p$; when $p = 2$, $W^{m,2}(\Omega)$ is denoted as $H^m(\Omega)$ and it is a Hilbert space, the norm of $u \in H^m(\Omega)$ is defined as

$$\|u\|_{H^m(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^2 dx \right)^{\frac{1}{2}},$$

and the inner product is expressed as

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} Re \int_\Omega D^\alpha u(x) \overline{D^\alpha v(x)} dx.$$

Definition 1.9. ***(The Sobolev space $\mathbf{W}^{1,p}(\Omega)$)*** *Let Ω be an open domain of $\mathbb{R}^n$ and $1 \leq p \leq +\infty$, The Sobolev space $\mathbf{W}^{1,p}(\Omega)$ is defined to be*

$$W^{1,p}(\Omega) = \left\{ \begin{array}{c} u \in L^p(\Omega); \exists v_i \in L^p(\Omega) \text{ such that } \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega v_i \varphi dx, \\ i = 1, 2, ..., n, \ \forall \varphi \in C_0^\infty(\Omega) \end{array} \right\}.$$

is called the Sobolev space of order one and it is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{1\leq i\leq n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p},$$

or sometimes, if $1 < p < \infty$, *with the equivalent norm*

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \sum_{1\leq i\leq n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^p \right)^{\frac{1}{p}}.$$

Remark 1.3. $W^{1,p}(\Omega) = H^1(\Omega)$ *is a Hilbert space with respect to the inner product*

$$\langle u, v\rangle_{H^1} = \int_\Omega uv dx + \sum_{1\leq i\leq n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx.$$

Definition 1.10. *(**The Sobolev space** $\mathbf{W}_0^{1,p}(\Omega)$) Let* Ω *be an open domain of* $\mathbb{R}^n$ *and* $1 \leq p < +\infty$, *we define the space* $\mathbf{W}_0^{1,p}(\Omega)$ *to be the closure of* $C_0^1(\Omega)$ *with respect to the norm of* $W^{1,p}(\Omega)$.

1.2 Some useful inequalities

In this section, we shall recall some inequalities which will be used in the subsequent chapters.

Theorem 1.4. *(**Young's Inequality**) The following inequalities hold*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},\ q = \frac{p}{p-1},\ 1 < p < +\infty,\ \text{for all}\ \ a, b > 0,$$

$$ab \leq \frac{\varepsilon}{p} a^p + \frac{1}{\varepsilon^{\frac{1}{p-1}} q} b^q,\ \ q = \frac{p}{p-1},\ 1 < p < +\infty,\ \text{for all}\ \ a, b, \varepsilon > 0.$$

Theorem 1.5. ***(The Cauchy–Schwarz Inequality)*** *There holds that*

$$\left|x.y\right| \leq |x|\left|y\right|,\ \text{for all}\ \ x, y \in \mathbb{R}^n,$$

here $|x| = (x, x)^{1/2} = (\sum_{i=1}^n x_i^2)^{1/2}$ *for all* $x \in \mathbb{R}^n$.

Theorem 1.6. ***(Hölder's Inequality)*** *Let* $\Omega \subseteq \mathbb{R}^n$ *be a domain, assume that* $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ *with* $1 < p, q < +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Theorem 1.7. ***(Minkowski's Inequality)*** *Assume that* $1 \leq p \leq +\infty$*. Then for any* $u, v \in L^p(\Omega)$*,*

$$\|u+v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Theorem 1.8. ***(Poincaré's Inequality)*** *Assume that* Ω *is bounded in one direction and* $1 \leq p < +\infty$*. Then there is a positive constant* $C = C(\Omega, p)$ *such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \ \forall u \in \mathbf{W}_0^{1,p}(\Omega).$$

1.3 The theory of semi-groups

1.3.1 Semigroups

Numerous physical models can be written in the form of an abstract Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} = \mathcal{A}U, \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space $\mathcal{H}$ and $U_0 \in \mathcal{H}$ is given. We are looking for a solution $U : \mathbb{R}_+ \longrightarrow \mathcal{H}$. Therefore, we start by introducing some basic concepts concerning the semigroups.

Definition 1.11. ***(Semigroups)*** *Let* X *be a Banach space. A one parameter family* $S(t)$*,* $0 \leq t < \infty$*, of bounded linear operators from* X *into* X *is a semigroup of bounded linear operators on* X *if*

(i) $S(0) = Id$*, (*Id *is the identity operator on* X*).*

(ii) $S(t+s) = S(t)S(s)$ *for every* $s, t \geq 0$*.*

Definition 1.12. ***(C_0-Semigroups)*** *A semigroup* $S(t)$*,* $0 \leq t < \infty$*, from* X *to* X *is called a strong continuous semigroup of bounded linear operators if*

$$\lim_{t \longrightarrow 0^+} S(t)x = x \quad \text{or} \quad \lim_{t \longrightarrow 0^+} \|S(t)x - x\| = 0, \ \text{ for all } \ x \in X,$$

i.e., $S(t)$ *is a* C_0*-semigroup.*

Definition 1.13. ***(Contraction Semigroups)*** *The semigroup* $S(t)$ *is a contraction semigroup if there exists a constant* $\alpha > 0$ $(0 < \alpha < 1)$ *such that for all* $t > 0$*,*

$$\left\|S(t)x - S(t)y\right\| \leq \alpha \left\|x - y\right\|, \ \text{ for all } \ x, y \in X.$$

Definition 1.14. *The linear operator $\mathcal{A}$ defined by*

$$D(\mathcal{A}) = \left\{ x \in X;\ \lim_{t \longrightarrow 0^+} \frac{S(t)x - x}{t} \text{ exists} \right\},$$

and

$$\mathcal{A}x = \lim_{t \longrightarrow 0^+} \frac{S(t)x - x}{t} = \frac{d(S(t)x)}{dt}\Big|_{t=0} \text{ for all } x \in D(\mathcal{A})$$

is called the infinitesimal generator of the semigroup $S(t)$, $D(\mathcal{A})$ is called the domain of $\mathcal{A}$.

Theorem 1.9. *Let $S(t)$, $0 \leq t < \infty$ be a C_0-semigroup. Then there exist constants $M > 0$ and $\omega \geq 0$ such that*

$$\|S(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

In the above theorem, if $M = 1$ and $\omega = 0$, then we obtain a C_0-semigroup of contractions.

For the existence of solutions, we normally use the Lumer-Phillips Theorem or Hille-Yosida Theorem.

Definition 1.15. *Let $\mathcal{H}$ denotes a Hilbert space, an unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is said to be monotone (or that $-\mathcal{A}$ is dissipative) if it satisfies*

$$\langle \mathcal{A}U, U \rangle \geq 0, \quad \forall U \in D(\mathcal{A}).$$

It is called maximal monotone if, in addition, $R(Id + \mathcal{A}) = \mathcal{H}$, i.e.,

$$\forall F \in \mathcal{H},\ \exists U \in D(\mathcal{A}) \text{ such that } U + \mathcal{A}U = F.$$

Proposition 1.10. *Let $\mathcal{A}$ be a maximal monotone operator. Then*

1. $D(\mathcal{A})$ is dense in $\mathcal{H}$,

2. $\mathcal{A}$ is a closed operator,

3. For every $\lambda > 0$, $(I + \lambda\mathcal{A})$ is bijective from $D(\mathcal{A})$ onto $\mathcal{H}$, $(Id + \lambda\mathcal{A})^{-1}$ is abounded operator, and $\left\|(Id + \lambda\mathcal{A})^{-1}\right\| \leq 1$.

Theorem 1.11. ***(Lumer-Phillips Theorem)*** *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Banach space X.*

(i) If $\mathcal{A}$ is dissipative and there exists a $\lambda_0 > 0$ such that the range $R(\lambda_0 Id - \mathcal{A}) = X$, then A generates a C_0-semigroup of contractions on X.

(ii) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda Id - \mathcal{A}) = X$ for all $\lambda > 0$ and $\mathcal{A}$ is dissipative.

Consequently, $\mathcal{A}$ is maximal dissipative on a Hilbert space $\mathcal{H}$ if and only if it generates a C_0-semigroup of contractions on $\mathcal{H}$ and thus the existence of the solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollaire 1.12. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset \mathcal{H}$ into $\mathcal{H}$. If $\mathcal{A}$ is maximal dissipative then the initial value problem (1.1) has a unique weak solution $U \in C([0;+\infty);\mathcal{H})$, for each initial data $U_0 \in \mathcal{H}$. Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty);D(\mathcal{A}))$.*

Theorem 1.13. ***(Hille-Yosida Theorem)*** *Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $U_0 \in D(\mathcal{A})$ there exists a unique function*

$$U \in C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty);D(\mathcal{A})),$$

satisfying

$$\begin{cases} \dfrac{dU}{dt} + \mathcal{A}U = 0, \;\; on \;\; [0,+\infty), \\ U(0) = U_0. \end{cases}$$

Moreover,

$$|U(t)| \leq |U_0| \;\text{ and }\; \left|\frac{dU}{dt}(t)\right| = |\mathcal{A}U(t)| \leq |\mathcal{A}U_0| \quad \forall t \geq 0.$$

Part I

A laminated beam system

Chapter 2

A laminated beam with Fourier's type heat conduction and distributed delay term

Contents

2.1 Presentation of the problem

In the present chapter, we consider the laminated beam system where the heat flux is given by Fourier's law with distributed delay term (See [14]). The system is written as

$$\begin{cases} \rho_1 \varphi_{tt} + G(\psi - \varphi_x)_x + \mu_0 \varphi_t + \int_{\tau_1}^{\tau_2} \mu(s) \varphi_t(x, t-s)\, ds = 0, \\ \rho_2 (3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + \sigma \theta_x = 0, \\ \rho_2 w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - D w_{xx} = 0, \\ k\theta_t - \tau \theta_{xx} + \sigma (3w - \psi)_{tx} = 0, \end{cases} \tag{2.1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and ρ_1, G, ρ_2, D, σ, γ, β, k, τ are positive constant coefficients, with Dirichlet-Neumann boundary conditions

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty). \end{cases} \tag{2.2}$$

and the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x), & x \in [0,1], \\ \varphi_t(x,-t) = f_0(x,t), & (x;t) \in (0,1) \times (0,\tau_2), \end{cases} \tag{2.3}$$

where τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$, μ_0 is a positive constant, and $\mu : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is an L^∞ function, $\mu \geq 0$ almost everywhere, and the initial data $(\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, f_0)$ belong to a suitable Sobolev space.

Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption

$$\mu_0 \geq \int_{\tau_1}^{\tau_2} \mu(s)\, ds. \tag{2.4}$$

The rest of the chapter is organized as follows. In Section 2.2, by using Hille-Yosida theorem, we state and prove the well posedness of problem (2.1)-(2.3). In Section 2.3, by using the perturbed energy method, we then establish the exponential result if and only if $\frac{\rho_1}{G} = \frac{\rho_2}{D}$.

2.2 Well-posedness of the problem

In this section, we will prove that system (2.1)-(2.3) are well posed using semigroup theory by introducing the following new variable as in [45].

$$z(x,\rho,t,s) = \varphi_t(x,t-\rho s), \ \ x \in (0,1), \ \rho \in (0,1), \ t > 0, \ s \in (\tau_1,\tau_2).$$

Then, we have

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \ \ x \in (0,1), \ \rho \in (0,1), \ t > 0, \ s \in (\tau_1,\tau_2). \tag{2.5}$$

Therefore, problem (2.1) takes the form

$$\begin{cases} \rho_1 \varphi_{tt} + G(\psi - \varphi_x)_x + \mu_0 \varphi_t + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s) ds = 0, \\ \rho_2 (3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + \sigma\theta_x = 0, \\ \rho_2 w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - D w_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w - \psi)_{tx} = 0, \end{cases} \tag{2.6}$$

with Dirichlet-Neumann boundary conditions

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty), \end{cases} \tag{2.7}$$

and the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ x \in [0,1], \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ x \in [0,1], \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ x \in [0,1], \\ \theta(x,0) = \theta_0(x), \ x \in [0,1], \\ \varphi_t(x,-t) = f_0(x,t), \ (x,t) \in (0,1) \times (0,\tau_2) \\ z(x,0,t,s) = \varphi_t(x,t) \ \text{on} \ (0,1) \times (0,\infty) \times (\tau_1,\tau_2), \\ z(x,\rho,0,s) = f_0(x,\rho,s) \ \text{on} \ (0,1) \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{2.8}$$

Introducing the vector function

$$U = \left(\varphi, \varphi_t, 3w - \psi, (3w - \psi)_t, w, w_t, \theta, z\right)^T,$$

problem (2.6)-(2.8) can be written as

$$\begin{cases} \dfrac{dU(t)}{dt} = AU, \\ U(x,0) = U_0(x) = (\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, f_0)^T, \end{cases} \tag{2.9}$$

where the operator A is defined by

$$AU=\begin{pmatrix} \varphi_t \\ -\frac{G}{\rho_1}(\psi-\varphi_x)_x-\frac{\mu_0}{\rho_1}\varphi_t-\frac{1}{\rho_1}\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,ds \\ (3w-\psi)_t \\ \frac{G}{\rho_2}(\psi-\varphi_x)+\frac{D}{\rho_2}(3w-\psi)_{xx}-\frac{\sigma}{\rho_2}\theta_x \\ w_t \\ -\frac{G}{\rho_2}(\psi-\varphi_x)-\frac{4\gamma}{3\rho_2}w-\frac{4\beta}{3\rho_2}w_t+\frac{D}{\rho_2}w_{xx} \\ \frac{\tau}{\kappa}\theta_{xx}-\frac{\sigma}{\kappa}(3w-\psi)_{tx} \\ -s^{-1}z_\rho \end{pmatrix}$$

We consider the following spaces

$$\begin{aligned} H_*^1(0,1) &= \left\{\chi/\chi\in H^1(0,1):\chi(0)=0\right\}, \\ \widetilde{H}_*^1(0,1) &= \left\{\chi/\chi\in H^1(0,1):\chi(1)=0\right\}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{H} &= H_*^1(0,1)\times L^2(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)\times\widetilde{H}_*^1(0,1) \\ &\quad \times L^2(0,1)\times L^2(0,1)\times L^2\Big((0,1)\times(\tau_1,\tau_2),H^1(0,1)\Big), \end{aligned}$$

be the Hilbert space equipped with the inner product

$$\begin{aligned} \left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^1[\rho_1\varphi_t\widetilde{\varphi}_t+\rho_2(3w-\psi)_t\left(3\widetilde{w}-\widetilde{\psi}\right)_t+3\rho_2w_t\widetilde{w}_tdx+k\theta\widetilde{\theta} \\ &\quad +4\gamma w\widetilde{w}+G(\psi-\varphi_x)\left(\widetilde{\psi}-\widetilde{\varphi}_x\right)+D(3w-\psi)_x\left(3\widetilde{w}-\widetilde{\psi}\right)_x \\ &\quad +3Dw_x\widetilde{w}_x]dx+\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\int_0^1 z(x,\rho,s)\widetilde{z}(x,\rho,s)\,d\rho\, ds\, dx. \end{aligned}$$

The domain of A is

$$D(A)=\left\{\begin{array}{c} U\in\mathcal{H}\mid\varphi\in H^2(0,1)\cap H_*^1(0,1),\ \theta\in H_*^1(0,1), \\ 3w-\psi,w\in H^2(0,1)\cap\widetilde{H}_*^1(0,1), \\ \varphi_t\in H_*^1(0,1),(3w-\psi)_t,w_t\in\widetilde{H}_*^1(0,1), \\ \varphi_x(1,t)=\psi_x(0,t)=w_x(0,t)=0,\varphi_t(x)=z(x,0,s)\,in(0,1) \end{array}\right\},$$

and it is dense in $\mathcal{H}$. The well-posedness of problem (2.9) is ensured by

Theorem 2.1. *Assume that $U^0 \in \mathcal{H}$ and (2.4) holds, then problem (2.9) exists a unique weak solution $U \in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U^0 \in D(A)$, then*

$$U \in C\left(\mathbb{R}^+; D(A) \cap C^1(\mathbb{R}^+;\mathcal{H})\right).$$

Proof. To prove the well-posedness result, it suffices to show that $A : D(A) \to \mathcal{H}$ is a maximal monotone operator, which means A is dissipative and $Id - A$ is surjective.

First, we prove that A is dissipative.
For any $U = \left(\varphi, \varphi_t, 3w-\psi, (3w-\psi)_t, w, w_t, \theta, z\right)^T \in D(A)$, by using the inner product and integrating by parts, we have

$$\begin{aligned}\langle AU, U\rangle_{\mathcal{H}} &= -\mu_0 \int_0^1 \varphi_t^2(x)\,dx - \int_0^1 \varphi_t(x)\left(\int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s)\,ds\right)dx \\ &\quad -4\beta \int_0^1 w_t^2\,dx - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s)\,ds dx \\ &\quad -\tau \int_0^1 \theta_x^2\,dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} \mu(s)\,ds \int_0^1 \varphi_t^2(x)\,dx.\end{aligned}$$

Now, using Young's and Cauchy–Schwarz inequalities, we can estimate

$$\begin{aligned}&-\int_0^1 \varphi_t(x)\left(\int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s)\,ds\right)dx \\ \leq\; & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} \mu(s)\,ds\right)\int_0^1 \varphi_t^2(x)\,dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s)\,ds dx.\end{aligned}$$

Therefore, from the assumption (2.4) we have

$$\begin{aligned}&\langle AU, U\rangle_{\mathcal{H}} \\ \leq\; & -\tau \int_0^1 \theta_x^2\,dx - 4\beta \int_0^1 w_t^2\,dx + \left(-\mu_0 + \int_{\tau_1}^{\tau_2} \mu(s)\,ds\right)\int_0^1 \varphi_t^2(x)\,dx \leq 0.\end{aligned}$$

Consequently, A is a dissipative operator.

Next, we prove that the operator $Id - A$ is surjective.

Given $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, we prove that there exists a unique $U = \left(\varphi, \varphi_t, 3w-\psi, (3w-\psi)_t, w, w_t, \theta, z\right)^T \in D(A)$ such that

$$(Id - A)U = F, \tag{2.10}$$

that is,

$$\begin{cases} \varphi - \varphi_t = f_1, \\ (\rho_1 + \mu_0)\varphi_t - G\varphi_{xx} - G(3w-\psi)_x + 3Gw_x + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\,ds \\ \qquad = \rho_1 f_2, \\ (3w-\psi) - (3w-\psi)_t = f_3, \\ \rho_2 (3w-\psi)_t + G\varphi_x + G(3w-\psi) - D(3w-\psi)_{xx} - 3Gw + \sigma\theta_x \\ \qquad = \rho_2 f_4, \\ w - w_t = f_5, \\ \left(\rho_2 + \frac{4\beta}{3}\right) w_t - G\varphi_x - G(3w-\psi) + \left(3G + \frac{4\gamma}{3}\right) w - Dw_{xx} = \rho_2 f_6, \\ k\theta - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = kf_7, \\ z + s^{-1} z_\rho = f_8. \end{cases} \tag{2.11}$$

From $(2.11)_1$,$(2.11)_3$ and $(2.11)_5$ we have

$$\begin{cases} \varphi_t = \varphi - f_1, \\ (3w-\psi)_t = (3w-\psi) - f_3, \\ w_t = w - f_5. \end{cases} \tag{2.12}$$

Inserting (2.12) into $(2.11)_2$, $(2.11)_4$, $(2.11)_6$ and $(2.11)_7$, we get

$$\begin{cases} (\mu_0 + \rho_1)\varphi - G\varphi_{xx} - G(3w-\psi)_x + 3Gw_x + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\,ds \\ \qquad = \rho_1 (f_1 + f_2) + \mu_0 f_1, \\ \\ \rho_2 (3w-\psi) + G\varphi_x + G(3w-\psi) - D(3w-\psi)_{xx} - 3Gw + \sigma\theta_x \\ \qquad = \rho_2 (f_3 + f_4), \\ \\ \left(\rho_2 + \frac{4\beta}{3}\right) w - G\varphi_x - G(3w-\psi) + \left(3G + \frac{4\gamma}{3}\right) w - Dw_{xx} \\ \qquad = \rho_2 (f_5 + f_6) + \frac{4\beta}{3} f_5, \\ \\ k\theta - \tau\theta_{xx} + \sigma(3w-\psi)_x = \sigma(f_3)_x + kf_7, \\ z + s^{-1} z_\rho = f_8. \end{cases} \tag{2.13}$$

Using (2.12) and the fact that $z(x,0,s) = \varphi_t(x)$, we get

$$z(x,\rho,s) = \varphi(x)e^{-\rho s} - f_1 e^{-\rho s} + s e^{-\rho s} \int_0^\rho f_8(x,\delta,s) e^{\delta s}\, d\delta. \tag{2.14}$$

In order to solve (2.11), we consider the following variational formulation

$$B\left((\varphi, 3w-\psi, w, \theta)^T, \left(\widetilde{\varphi}, 3\widetilde{w} - \widetilde{\psi}, \widetilde{w}, \widetilde{\theta}\right)^T\right) = L\left(\widetilde{\varphi}, 3\widetilde{w} - \widetilde{\psi}, \widetilde{w}, \widetilde{\theta}\right)^T, \tag{2.15}$$

where $B:\left[H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)\right]^2\longrightarrow\mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}
& B\left((\varphi,3w-\psi,w,\theta)^T,\left(\widetilde{\varphi},3\widetilde{w}-\widetilde{\psi},\widetilde{w},\widetilde{\theta}\right)^T\right)\\
= & \int_0^1 G(-\varphi_x+\psi)\left(-\widetilde{\varphi}_x+\widetilde{\psi}\right)dx+\int_0^1(\mu_0+\rho_1)\varphi\widetilde{\varphi}dx+\int_0^1 k\theta\widetilde{\theta}dx\\
& +\int_0^1\rho_2(3w-\psi)\left(3\widetilde{w}-\widetilde{\psi}\right)dx+\int_0^1(3\rho_2+4\beta+4\gamma)w\widetilde{w}dx\\
& +\int_0^1 D(3w-\psi)_x\left(3\widetilde{w}-\widetilde{\psi}\right)_x dx+\int_0^1 3Dw_x\widetilde{w}_x dx+\tau\int_0^1\theta_x\widetilde{\theta}_x dx\\
& +\sigma\int_0^1\theta_x\left(3\widetilde{w}-\widetilde{\psi}\right)dx+\sigma\int_0^1(3w-\psi)_x\widetilde{\theta}dx\\
& +\int_0^1\varphi\widetilde{\varphi}\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}dsdx,
\end{aligned}$$

and $L:\left[H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)\right]\longrightarrow\mathbb{R}$ is the linear form defined by

$$\begin{aligned}
& L\left(\widetilde{\varphi},3\widetilde{w}-\widetilde{\psi},\widetilde{w},\widetilde{\theta}\right)^T\\
= & \int_0^1\rho_1(f_1+f_2)\tilde{\varphi}dx+\int_0^1\mu_0 f_1\tilde{\varphi}dx+\int_0^1\rho_2(f_3+f_4)\left(3\widetilde{w}-\widetilde{\psi}\right)dx\\
& +\int_0^1 3\rho_2(f_5+f_6)\widetilde{w}dx+\int_0^1 4\beta f_5\widetilde{w}dx+\int_0^1\sigma(f_3)_x\widetilde{\theta}dx+\int_0^1 kf_7\widetilde{\theta}dx\\
& -\int_0^1\widetilde{\varphi}\int_{\tau_1}^{\tau_2}\mu(s)z_0(x,s)dsdx.
\end{aligned}$$

Now, for $V=H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)$ equipped with the norm

$$\left\|(\varphi,3w-\psi,w,\theta)\right\|_V^2=\left\|-\varphi_x+\psi\right\|_2^2+\|\varphi\|_2^2+\left\|3w_x-\psi_x\right\|_2^2+\|w_x\|_2^2+\|\theta_x\|_2^2.$$

It is easy to verify that $B(.,.)$ is continuous and coercive, and $L(.)$ is continuous. So applying the Lax-Milgram theorem, problem (2.15) admits a unique solution

$$\varphi\in H_*^1(0,1),\ (3w-\psi)\in\widetilde{H}_*^1(0,1),\ w\in\widetilde{H}_*^1(0,1),\ \theta\in L^2(0,1).$$

The substitution of $\varphi,3w-\psi$ and w into (2.12), we obtain

$$\varphi_t\in H_*^1(0,1),\ \ (3w-\psi)_t\in\widetilde{H}_*^1(0,1),\ w_t\in\widetilde{H}_*^1(0,1).$$

Applying the classical elliptic regularity, it follows from (2.13) that

$$\begin{aligned}
\varphi &\in H^2(0,1)\cap H_*^1(0,1),(3w-\psi)\in H^2(0,1)\cap\widetilde{H}_*^1(0,1),\theta\in H_*^1(0,1),\\
w &\in H^2(0,1)\cap\widetilde{H}_*^1(0,1),\varphi_x(1)=(3w-\psi)_x(0)=w_x(0)=0.
\end{aligned}$$

Therefore, the operator $Id - A$ is surjective. Consequently, the well-posedness result stated in Theorem 2.1 follows from the Hille–Yosida theorem (see [6]). □

2.3 Exponential stability of solution

In this section, we show that, under the assumption $\mu_0 \geq \int_{\tau_1}^{\tau_2} \mu(s)ds$ and for $\frac{\rho_1}{G} = \frac{\rho_2}{D}$, the solution of problem (2.6)-(2.8) decays exponentially to the study state. To achieve our goal we use the energy method to produce a suitable Lyapunov functional. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) &:= \frac{1}{2}\int_0^1 \Big[\rho_1 \varphi_t^2 + \rho_2 (3w_t - \psi_t)^2 + 3\rho_2 w_t^2 + G(\psi - \varphi_x)^2 + 4\gamma w^2 + k\theta^2 \\ &+ D(3w_x - \psi_x)^2 + 3Dw_x^2\Big] dx + \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) z^2(x,\rho,s,t)\, ds d\rho dx. \end{aligned} \tag{2.16}$$

Theorem 2.2. *Assume that* $\frac{\rho_1}{G} = \frac{\rho_2}{D}$ *and (2.4) holds. Let* $U^0 \in \mathcal{H}$ *, then there exists positive constants* c_0 *and* c_1 *such that the energy* $E(t)$ *associated with problem (2.6)-(2.8) satisfies,*

$$E(t) \leq c_0 e^{-c_1 t}, \qquad \forall t \geq 0.$$

In order to prove this result, we need the following lemmas.

Lemma 2.3. *Let* $(\varphi, \psi, w, \theta, z)$ *be the solution of (2.6)-(2.8) and assume (2.4) holds. Then the energy functional, defined by (2.16) satisfies*

$$\frac{d}{dt}E(t) \leq -4\beta \int_0^1 w_t^2 dx - \tau \int_0^1 \theta_x^2 dx - \left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)ds\right)\int_0^1 \varphi_t^2 dx \leq 0. \tag{2.17}$$

Proof. Multiplying $(2.6)_1$, $(2.6)_2$, $(2.6)_3$ and $(2.6)_4$ by $\varphi_t, 3(w-\psi)_t$, $3w_t$ and θ, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions in (2.7), we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\left(\rho_1 \int_0^1 \varphi_t^2 dx + G\int_0^1 (\psi - \varphi_x)^2 dx\right)\right] \\ = \;& G\int_0^1 (\psi - \varphi_x)\psi_t dx - \mu_0 \int_0^1 \varphi_t^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu(s)\varphi_t(x,t-s)\,ds dx, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\left(\rho_2 \int_0^1 (3w_t - \psi_t)^2 dx + D\int_0^1 (3w_x - \psi_x)^2 dx\right)\right] \\ = \;& G\int_0^1 (\psi - \varphi_x)(3w-\psi)_t dx - \sigma \int_0^1 \theta_x (3w-\psi)_t dx, \end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\left(3\rho_2\int_0^1 w_t^2dx+4\gamma\int_0^1 w^2dx+3D\int_0^1 w_x^2dx\right)\right]\\
=\ &-3G\int_0^1(\psi-\varphi_x)w_tdx-4\beta\int_0^1 w_t^2dx,
\end{aligned}\tag{2.20}$$

and

$$\frac{d}{dt}\left[\frac{1}{2}k\int_0^1\theta^2dx\right]=\sigma\int_0^1(3w-\psi)_t\theta_xdx-\tau\int_0^1\theta_x^2dx.\tag{2.21}$$

On the other hand, multiplying (2.5) by $\mu(s)z(x,\rho,s,t)$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)z(x,\rho,s,t)z_t(x,\rho,s,t)dsd\rho dx\\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z(x,\rho,s,t)z_\rho(x,\rho,s,t)dsd\rho dx=0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)z^2(x,\rho,s,t)dsd\rho dx\\
=\ &-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,s,t)dsdx+\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s)ds\int_0^1\varphi_t^2dx.
\end{aligned}\tag{2.22}$$

Summing up (2.18)-(2.22), we arrive at

$$\begin{aligned}
\frac{d}{dt}E(t)\ =\ &-4\beta\int_0^1 w_t^2dx-\left(\mu_0-\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\right)\int_0^1\varphi_t^2dx\\
&-\tau\int_0^1\theta_x^2dx-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)dsdx\\
&-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,s,t)dsdx.
\end{aligned}\tag{2.23}$$

Young's and Cauchy–Schwarz inequalities applied to the fourth term on the right-hand side yield

$$\begin{aligned}
-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)dsdx\ \le\ &\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1\varphi_t^2dx\\
&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,s,t)dsdx.
\end{aligned}\tag{2.24}$$

Simple substitution of (2.24) into (2.23) and using (2.4) give (2.17), which concludes the proof. □

Now, we are going to construct a Lyapunov functional equivalent to the energy. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 2.4. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.6)-(2.8). Then the functional*

$$F_1(t) := \rho_1 \int_0^1 \varphi\varphi_t dx$$

satisfies the estimate

$$\begin{aligned} F_1'(t) \leq\ & -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + C_1\int_0^1 (\psi-\varphi_x)^2 dx + C_2\int_0^1 (3w_x-\psi_x)^2 dx \\ & + C_3\int_0^1 w_x^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s,t) ds dx, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} C_1 &= \frac{3G}{2} + \frac{\mu_0^2}{\rho_1} + \int_{\tau_1}^{\tau_2}\mu(s)ds, \quad C_2 = G + \frac{2\mu_0^2}{\rho_1} + 2\int_{\tau_1}^{\tau_2}\mu(s)ds, \\ C_3 &= 9G + \frac{18\mu_0^2}{\rho_1} + 18\int_{\tau_1}^{\tau_2}\mu(s)ds. \end{aligned}$$

Proof. Taking the derivative of $F_1(t)$ with respect to t, using the first equation in (2.6), and integrating by parts, gives

$$\begin{aligned} F_1'(t) =\ & -\rho_1\int_0^1 \varphi_t^2 dx - G\int_0^1 (\psi-\varphi_x)\varphi_x dx + \mu_0\int_0^1 \varphi_t\varphi dx \\ & + \int_0^1 \varphi\int_{\tau_1}^{\tau_2}\mu(s) z(x,1,t,s) ds dx. \end{aligned}$$

Note that

$$-G\int_0^1 (\psi-\varphi_x)\varphi_x dx = G\int_0^1 (\psi-\varphi_x)^2 dx - G\int_0^1 \psi(\psi-\varphi_x) dx.$$

Then, we deduce that

$$\begin{aligned} F_1'(t) =\ & -\rho_1\int_0^1 \varphi_t^2 dx + G\int_0^1 (\psi-\varphi_x)^2 dx - G\int_0^1 \psi(\psi-\varphi_x) dx \\ & + \mu_0\int_0^1 \varphi_t\varphi dx + \int_0^1 \varphi\int_{\tau_1}^{\tau_2}\mu(s) z(x,1,t,s) ds dx. \end{aligned}$$

We then use Young's inequality, we obtain

$$\begin{aligned} F_1'(t) \leq\ & -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + \frac{3G}{2}\int_0^1 (\psi-\varphi_x)^2 dx + \frac{G}{2}\int_0^1 \psi_x^2 dx \\ & + \left(\frac{\mu_0^2}{2\rho_1} + \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1 \varphi^2 dx \\ & + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s) z^2(x,1,s,t) ds dx. \end{aligned}$$

By using (2.4) and the trivial relation

$$\int_0^1 \varphi^2 dx \le 2\int_0^1 (\psi-\varphi_x)^2\,dx + 2\int_0^1 \psi_x^2 dx,$$

we obtain

$$\begin{aligned} F_1'(t) \le\; & -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + \left(\frac{3G}{2}+\frac{\mu_0^2}{\rho_1}+\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^1 (\psi-\varphi_x)^2\,dx \\ & +\left(\frac{G}{2}+\frac{\mu_0^2}{\rho_1}+\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^1 \psi_x^2 dx \\ & +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z^2(x,1,s,t)\,ds dx. \end{aligned}$$

Note that

$$\int_0^1 \psi_x^2 dx = \int_0^1 (\psi_x - 3w_x + 3w_x)^2\,dx \le 2\int_0^1 (3w_x-\psi_x)^2\,dx + 18\int_0^1 w_x^2 dx.$$

Then the estimate (2.25) is established. □

Lemma 2.5. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.6)-(2.8). Then the functional*

$$F_2(t) := \rho_2\int_0^1 (3w-\psi)(3w-\psi)_t\,dx$$

satisfies the estimate

$$\begin{aligned} F_2'(t) \le\; & -\frac{D}{2}\int_0^1 (3w_x-\psi_x)^2\,dx + \rho_2\int_0^1 (3w_t-\psi_t)^2\,dx \\ & +\frac{G^2}{D}\int_0^1 (\psi-\varphi_x)^2\,dx + \frac{\sigma^2}{D}\int_0^1 \theta^2 dx. \end{aligned} \tag{2.26}$$

Proof. By differentiating $F_2(t)$ with respect to t, then exploiting the second equation in (2.6), and integrating by parts, we obtain

$$\begin{aligned} F_2'(t) =\; & -D\int_0^1 (3w_x-\psi_x)^2\,dx + \rho_2\int_0^1 (3w_t-\psi_t)^2\,dx \\ & +G\int_0^1 (\psi-\varphi_x)(3w-\psi)\,dx + \sigma\int_0^1 (3w-\psi)_x\,\theta dx. \end{aligned} \tag{2.27}$$

Using Young's inequality, we obtain estimate (2.26). □

Lemma 2.6. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.6)-(2.8). Then the functional*

$$F_3(t) := \rho_2\int_0^1 w w_t dx$$

satisfies, for any $\varepsilon_1 > 0$, *the estimate*

$$\begin{aligned} F_3'(t) &\leq -\left(\frac{4\gamma}{3} - \varepsilon_1\right)\int_0^1 w^2 dx - D\int_0^1 w_x^2 dx + C_4(\varepsilon_1)\int_0^1 w_t^2 dx \\ &+\frac{G^2}{2\varepsilon_1}\int_0^1 (\psi - \varphi_x)^2 dx, \end{aligned} \tag{2.28}$$

where

$$C_4(\varepsilon_1) = \rho_2 + \frac{8\beta^2}{9\varepsilon_1}.$$

Proof. By differentiating $F_3(t)$ with respect to t, then exploiting the third equation in (2.6), and integrating by parts, we obtain

$$\begin{aligned} F_3'(t) &= \rho_2\int_0^1 w_t^2 dx - G\int_0^1 w(\psi - \varphi_x) dx - \frac{4}{3}\gamma\int_0^1 w^2 dx \\ &-\frac{4}{3}\beta\int_0^1 w w_t dx - D\int_0^1 w_x^2 dx. \end{aligned}$$

Using Young's inequality with $\varepsilon_1 > 0$, we obtain estimate (2.28). □

Lemma 2.7. *Let* $(\varphi, \psi, w, \theta, z)$ *be the solution of (2.6)-(2.8). Then the functional*

$$F_4(t) := \frac{k\rho_2}{\sigma}\int_0^1 (3w - \psi)_t \int_0^x \theta dy dx$$

satisfies, for any $\varepsilon_2 > 0$, *the estimate*

$$\begin{aligned} F_4'(t) &\leq -\frac{\rho_2}{2}\int_0^1 (3w_t - \psi_t)^2 dx + C_5(\varepsilon_2)\int_0^1 \theta^2 dx + \varepsilon_2\int_0^1 (\psi - \varphi_x)^2 dx \\ &+\varepsilon_2\int_0^1 (3w_x - \psi_x)^2 dx + \frac{\tau\rho_2}{2\sigma^2}\int_0^1 \theta_x^2 dx, \end{aligned} \tag{2.29}$$

where

$$C_5(\varepsilon_2) = k + \frac{k^2 D^2}{4\varepsilon_2\sigma^2} + \frac{k^2 G^2}{4\varepsilon_2\sigma^2}.$$

Proof. By differentiating $F_4(t)$ with respect to t, using the second and the fourth equations in (2.6), and integrating by parts, we obtain

$$\begin{aligned} F_4'(t) &= -\rho_2\int_0^1 (3w_t - \psi_t)^2 dx + \frac{kG}{\sigma}\int_0^1 (\psi - \varphi_x)\int_0^x \theta dy dx \\ &-\frac{kD}{\sigma}\int_0^1 (3w - \psi)_x \theta dx + k\int_0^1 \theta^2 dx + \frac{\tau\rho_2}{\sigma}\int_0^1 (3w - \psi)_t \theta_x dx. \end{aligned} \tag{2.30}$$

Then, using Young's and Poincaré inequalities with $\varepsilon_2 > 0$, we arrive at (2.29). □

Lemma 2.8. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.6)-(2.8). Then the functional*

$$F_5(t) := \rho_2\int_0^1 w_t(\psi-\varphi_x)\,dx+\rho_2\int_0^1 w_t\varphi_x dx-\frac{D\rho_1}{G}\int_0^1(w_x\varphi_t-w_{xt}\varphi)\,dx$$

satisfies, for any $\varepsilon_3>0$, the estimate

$$\begin{aligned} F_5'(t) \le\ & -\frac{G}{2}\int_0^1(\psi-\varphi_x)^2\,dx+\varepsilon_3\int_0^1(3w_t-\psi_t)^2\,dx+\frac{16\gamma^2}{9G}\int_0^1 w^2dx \\ & +C_6\int_0^1 w_x^2dx+C_7(\varepsilon_3)\int_0^1 w_t^2dx+\frac{D\mu_0}{2G}\int_0^1\varphi_t^2dx \\ & +\frac{D}{2G}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,s,t)\,dsdx, \end{aligned} \tag{2.31}$$

where

$$C_6=\frac{D\mu_0}{2G}+\frac{D}{2G}\int_{\tau_1}^{\tau_2}\mu(s)\,ds,\ C_7(\varepsilon_3)=\frac{16\beta^2}{9G}+\frac{\rho_2^2}{2\varepsilon_3}+9\varepsilon_3.$$

Proof. Using the first and the third equations in (2.6), and integrating by parts, we obtain

$$\begin{aligned} & \frac{d}{dt}\left\{\rho_2\int_0^1 w_t(\psi-\varphi_x)\,dx\right\} \\ =\ & \frac{D\rho_1}{G}\left\{\frac{d}{dt}\int_0^1(w_x\varphi_t-w_{xt}\varphi)\,dx-\int_0^1 w_{tt}\varphi_x dx\right\}+\frac{\mu_0 D}{G}\int_0^1 w_x\varphi_t dx \\ & +\frac{D}{G}\int_0^1 w_x\int_{\tau_1}^{\tau_2}\mu(s)\varphi_t(x,t-s)\,dsdx-G\int_0^1(\psi-\varphi_x)^2\,dx \\ & -\frac{4\gamma}{3}\int_0^1 w(\psi-\varphi_x)\,dx-\frac{4\beta}{3}\int_0^1 w_t(\psi-\varphi_x)\,dx+\rho_2\int_0^1 w_t\psi_t dx \\ & -\frac{d}{dt}\left\{\rho_2\int_0^1 w_t\varphi_x dx\right\}+\rho_2\int_0^1 w_{tt}\varphi_x dx. \end{aligned}$$

We conclude for

$$\begin{aligned} F_5'(t) =\ & D\left(\frac{\rho_2}{D}-\frac{\rho_1}{G}\right)\int_0^1 w_{tt}\varphi_x dx+\frac{\mu_0 D}{G}\int_0^1 w_x\varphi_t dx \\ & +\frac{D}{G}\int_0^1 w_x\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)\,dsdx-G\int_0^1(\psi-\varphi_x)^2\,dx \\ & -\frac{4\gamma}{3}\int_0^1 w(\psi-\varphi_x)\,dx-\frac{4\beta}{3}\int_0^1 w_t(\psi-\varphi_x)\,dx+\rho_2\int_0^1 w_t\psi_t dx. \end{aligned}$$

Using Young's inequality and $\frac{\rho_2}{D}=\frac{\rho_1}{G}$, we obtain (2.31). □

Lemma 2.9. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.6)-(2.8) and (2.5). Then the functional*

$$F_6(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\mu(s)z^2(x,\rho,s,t)\,dsd\rho dx$$

satisfies, for some positive constant n, the following estimate

$$\begin{aligned} F_6^{'}(t) &\leq -n\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)z^2(x,\rho,s,t)\,dsd\rho dx \\ &\quad -n\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,s,t)\,dsdx+\mu_0\int_0^1\varphi_t^2dx. \end{aligned} \tag{2.32}$$

Proof. By differentiating $F_6(t)$ with respect to t, and using the equation (2.5), we obtain

$$\begin{aligned} F_6^{'}(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\mu(s)z(x,\rho,s,t)z_\rho(x,\rho,s,t)\,dsd\rho dx \\ &= -\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\left[e^{-s}z^2(x,1,s,t)-z^2(x,0,s,t)\right]dsdx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\mu(s)z^2(x,\rho,s,t)\,dsd\rho dx. \end{aligned} \tag{2.33}$$

Using the fact that $z(x,0,s,t)=\varphi_t$ and $e^{-s}\leq e^{-s\rho}\leq 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} F_6^{'}(t) &\leq -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\mu(s)z^2(x,1,s,t)\,dsdx+\int_{\tau_1}^{\tau_2}\mu(s)\,ds\int_0^1\varphi_t^2dx \\ &\quad -n_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,\rho,s,t)\,dsd\rho dx. \end{aligned} \tag{2.34}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\leq -e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $n=e^{-\tau_2}$ and recalling (2.4), we obtain (2.32). □

Next, we define a Lyapunov functional $L(t)$ and show that it is equivalent to the energy functional $E(t)$.

Lemma 2.10. *Let $N,N_2,N_3,N_4,N_5,N_6>0$ and $\frac{\rho_1}{G}=\frac{\rho_2}{D}$, we define*

$$L(t):=NE(t)+F_1(t)+\sum_{i=2}^{i=6}N_iF_i(t) \tag{2.35}$$

For two positive constants β_1 and β_2, we have

$$\beta_1E(t)\leq L(t)\leq\beta_2E(t),\forall t\geq 0. \tag{2.36}$$

Proof. Now, let

$$\mathcal{L}(t)=F_1(t)+\sum_{i=2}^{i=6}N_iF_i(t)$$

$$\begin{aligned}
|\mathcal{L}(t)| \leq\ & \rho_1 \int_0^1 |\varphi\varphi_t|\,dx + N_2\rho_2 \int_0^1 \left|(3w-\psi)(3w-\psi)_t\right| dx \\
& + N_3\rho_2 \int_0^1 |ww_t|\,dx + N_4 \frac{k\rho_2}{\sigma} \int_0^1 \left|(3w-\psi)_t \int_0^x \theta dy\right| dx \\
& + N_5\rho_2 \int_0^1 \left|w_t(\psi-\varphi_x)\right| dx + N_5 \frac{D\rho_1}{G} \int_0^1 |(w_x\varphi_t - w_{xt}\varphi)|\,dx \\
& + N_5\rho_2 \int_0^1 |w_t\varphi_x|\,dx \\
& + N_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \left|se^{-s\rho}\mu(s)z^2(x,\rho,s,t)\right| ds\,d\rho\,dx.
\end{aligned}$$

Exploiting Young's, Poincaré and Cauchy-Schwarz inequalities, (2.16), and the fact that $e^{-s\rho} \leq 1$ for all $\rho \in [0,1]$, we obtain

$$\begin{aligned}
|\mathcal{L}(t)| \leq c\int_0^1 \left[\varphi_t^2 + (3w_t-\psi_t)^2 + w_t^2 + (\psi-\varphi_x)^2 + (3w_x-\psi_x)^2 + w_x^2 + w^2 + \theta^2\right] dx \\
+ c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)z^2(x,\rho,z,t)\,ds\,d\rho\,dx \leq cE(t).
\end{aligned}$$

Consequently, $|L(t) - NE(t)| \leq cE(t)$, which yields

$$(N-c)E(t) \leq L(t) \leq (N+c)E(t).$$

Choosing such that $(N-c) > 0$, we obtain estimate (2.36). □

Now, we are ready to state and prove the main result of this section.

Proof. (Of Theorem 2.2)

By differentiating (2.35) and recalling (2.25), (2.26), (2.28), (2.29), (2.31) and

(2.32), we obtain

$$\begin{aligned}
L'(t) \leq & -\left[\left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)\,ds\right)N + \frac{\rho_1}{2} - \frac{D\mu_0}{2G}N_5 - \mu_0 N_6\right]\int_0^1 \varphi_t^2 dx \\
& -\left[\frac{4\gamma}{3}N_3 - \varepsilon_1 N_3 - \frac{16\gamma^2}{9G}N_5\right]\int_0^1 w^2 dx \\
& -\left[\tau N - \frac{\tau\rho_2}{2\sigma^2}N_4\right]\int_0^1 \theta_x^2 dx \\
& -\left[DN_3 - C_3 - C_6 N_5\right]\int_0^1 w_x^2 dx + \left[\frac{\sigma^2}{D}N_2 + C_5(\varepsilon_2)N_4\right]\int_0^1 \theta^2 dx \\
& -\left[\frac{G}{2}N_5 - C_1 - \frac{G^2}{D}N_2 - \frac{G^2}{2\varepsilon_1}N_3 - \varepsilon_2 N_4\right]\int_0^1 (\psi - \varphi_x)^2 dx \\
& -\left[\frac{\rho_2}{2}N_4 - \rho_2 N_2 - \varepsilon_3 N_5\right]\int_0^1 (3w_t - \psi_t)^2 dx \\
& -\left[4\beta N - C_4(\varepsilon_1)N_3 - C_7(\varepsilon_3)N_5\right]\int_0^1 w_t^2 dx \\
& -\left[\frac{D}{2}N_2 - C_2 - \varepsilon_2 N_4\right]\int_0^1 (3w_x - \psi_x)^2 dx \\
& -\left[nN_6\right]\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\
& -\left[nN_6 - \frac{1}{2} - \frac{D}{2G}N_5\right]\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s,t)\,ds\,dx. \qquad (2.37)
\end{aligned}$$

At this point, we need to choose our constants very carefully. First, we take N_2 large enough, such that

$$\frac{D}{2}N_2 - C_2 \geq 0.$$

Then, we choose N_4 and N_5 large enough, so that

$$\frac{\rho_2}{2}N_4 - \rho_2 N_2 \geq 0, \quad \frac{G}{2}N_5 - C_1 - \frac{G^2}{D}N_2 \geq 0.$$

Next, we pick ε_1 small and choose N_3 large enough, such that

$$DN_3 - C_3 - C_6 N_5 \geq 0, \quad \frac{4\gamma}{3}N_3 - \varepsilon_1 N_3 - \frac{16\gamma^2}{9G}N_5 \geq 0.$$

Then, we select N_3 even smaller (if needed) and ε_2, ε_3 small enough, so that

$$\frac{D}{2}N_2 - C_2 - \varepsilon_2 N_4 \geq 0, \quad \frac{\rho_2}{2}N_4 - \rho_2 N_2 - \varepsilon_3 N_5 \geq 0,$$

$$\frac{G}{2}N_5 - C_1 - \frac{G^2}{D}N_2 - \frac{G^2}{2\varepsilon_1}N_3 - \varepsilon_2 N_4 \geq 0.$$

Furthermore, we choose N_6 large enough, so that

$$nN_6 - \frac{D}{2G}N_5 - \frac{1}{2} \geq 0.$$

Finally, we choose N so large such that

$$\begin{aligned}\left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)\,ds\right)N + \frac{\rho_1}{2} - \frac{D\mu_0}{2G}N_5 - \mu_0 N_6 &\geq 0, \\ 4\beta N - C_4(\varepsilon_1)N_3 - C_7(\varepsilon_3)N_5 &\geq 0.\end{aligned}$$

Thus, we deduce that there exist positive constants α_1 and α_2 such that (2.37) becomes

$$\begin{aligned}L'(t) &\leq -\alpha_1 E(t) - \left[\tau N - \frac{\tau\rho_2}{2\sigma^2}N_4\right]\int_0^1 \theta_x^2 dx + \alpha_2 \int_0^1 \theta^2 dx \\ &\quad -\left[nN_6 - \frac{1}{2} - \frac{D}{2G}N_5\right]\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s,t)\,ds\,dx \\ &\leq -\alpha_1 E(t) + \alpha_2 \int_0^1 \theta_x^2 dx.\end{aligned}$$

By (2.17), we obtain

$$L'(t) \leq -\alpha_1 E(t) - \alpha_3 E'(t), \tag{2.38}$$

for some $\alpha_3 > 0$. It is obvious that

$$\mathfrak{L}(t) = L(t) + \alpha_3 E(t) \sim E(t).$$

Next, exploiting (2.38), we get

$$\mathfrak{L}'(t) = L'(t) + \alpha_3 E'(t) \leq -\alpha_1 E(t) \leq -c_1 \mathfrak{L}(t), \tag{2.39}$$

for some $c_1 > 0$. Integration (2.39) over $(0,t)$, leads to

$$\mathfrak{L}(t) \leq \mathfrak{L}(0) e^{-c_1 t}, \qquad \forall t \geq 0. \tag{2.40}$$

It gives the desired result theorem 2.2 when combined with the equivalence of $L(t)$ and $E(t)$. □

Chapter 3

A laminated beam in thermoelasticity of type III with delay term

Contents

3.1 Presentation of the problem

In the present chapter, we are concerned onedimensional laminated beam system in thermoelasticity of type III with delay term (See [13]), which has the form

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi - \varphi_x)_x + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0, \\ \rho_2(3\omega - \psi)_{tt} - G(\psi - \varphi_x) - D(3\omega - \psi)_{xx} + \sigma\theta_{tx} = 0, \\ \rho_2\omega_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t - D\omega_{xx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma(3\omega - \psi)_{tx} - k\theta_{txx} = 0, \end{cases} \tag{3.1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), & x \in [0,1], \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau), & x \in (0,1),\ t \in (0,\tau), \\ \varphi_x(0,t) = \psi(0,t) = \omega(0,t) = \theta(0,t) = 0, & \forall t \geq 0, \\ \varphi_x(1,t) = \psi(1,t) = \omega(1,t) = \theta_x(1,t) = 0, & \forall t \geq 0. \end{cases} \tag{3.2}$$

Here $\varphi = \varphi(x,t)$ denotes the transverse displacement of the beam which departs from its equilibrium position, $\psi = \psi(x,t)$ represents the rotation angle, $\omega = \omega(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, $\theta = \theta(x,t)$ is the differential temperature, and ρ_1, ρ_2, ρ_3, G, D, σ, γ, β, δ, k, μ_1 are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay.

We will assume that

$$\mu_1 > |\mu_2|, \tag{3.3}$$

and show the well-posedness of the problem and that this condition is sufficient to prove the uniform decay of the solution energy.

The purpose of this chapter is to study the well-posedness and asymptotic behaviour of solutions to the laminated beam (3.1)-(3.2) in thermoelasticity of type III with delay term appearing in the control term in the first equation. Introducing the delay term $\mu_2\varphi_t(x,t-\tau)$ makes the problem different from those considered in the literature. In Section 3.2, we prove the well-posedness of the system. In Section 3.3, we prove that the system is exponentially stable in the case of equal wave speeds.

3.2 Well-posedness of the problem

In this Section, we prove the well-posedness of problem (3.1)-(3.2) by using Lumer-Phillips theorem. We introduce as in [44] the new variable

$$z(x,\rho,t) = \varphi_t(x,t-\tau\rho), \quad (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Therefore, system (3.1) takes the form

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi-\varphi_x)_x + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0, \\ \rho_2(3\omega-\psi)_{tt} - G(\psi-\varphi_x) - D(3\omega-\psi)_{xx} + \sigma\theta_{tx} = 0, \\ \rho_2\omega_{tt} + G(\psi-\varphi_x) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t - D\omega_{xx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma(3\omega-\psi)_{tx} - k\theta_{txx} = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \end{cases} \tag{3.4}$$

where $(x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty)$, with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), & x \in [0,1], \\ z(x,\rho,0) = f_0(x,-\tau\rho), & x \in (0,1),\ \rho \in (0,1), \\ z(x,0,t) = \varphi_t(x,t), & x \in (0,1),\ t \in (0,\infty), \\ \varphi_x(0,t) = \psi(0,t) = \omega(0,t) = \theta(0,t) = 0, & \forall t \geq 0, \\ \varphi_x(1,t) = \psi(1,t) = \omega(1,t) = \theta_x(1,t) = 0, & \forall t \geq 0. \end{cases} \tag{3.5}$$

In order to be able to use Poincaré's inequality for θ, we introduce

$$\overline{\theta}(x,t) := \theta(x,t) - \int_0^1 \theta_0(x)\,dx - t\int_0^1 \theta_1(x)\,dx.$$

Then by $(3.4)_4$ we have

$$\int_0^1 \overline{\theta}(x,t)\,dx = 0,\ \forall t > 0.$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$, furthermore, $\left(\varphi,\psi,\omega,\overline{\theta},z\right)$ satisfies the same equations and boundary conditions. In what follows, we will work with $\overline{\theta}$. For convenience, we write θ instead of $\overline{\theta}$.

From now on, we let

$$U = (\varphi, \varphi_t, 3\omega - \psi, 3\omega_t - \psi_t, \omega, \omega_t, \theta, \theta_t, z)^T,$$

then (3.4) and (3.5) can be written as an evolutionary equation

$$\begin{cases} \dfrac{dU(t)}{dt} = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0 = (\varphi_0, \varphi_1, 3\omega_0 - \psi_0, 3\omega_1 - \psi_1, \omega_0, \omega_1, \theta_0, \theta_1, f_0)^T, \end{cases} \tag{3.6}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ \varphi_t \\ 3\omega-\psi \\ 3\omega_t-\psi_t \\ \omega \\ \omega_t \\ \theta \\ \theta_t \\ z \end{pmatrix} = \begin{pmatrix} \varphi_t \\ -\frac{G}{\rho_1}(\psi-\varphi_x)_x - \frac{\mu_1}{\rho_1}\varphi_t(x,t) - \frac{\mu_2}{\rho_1}z(x,1,t) \\ 3\omega_t-\psi_t \\ \frac{G}{\rho_2}(\psi-\varphi_x) + \frac{D}{\rho_2}(3\omega-\psi)_{xx} - \frac{\sigma}{\rho_2}\theta_{tx} \\ \omega_t \\ -\frac{G}{\rho_2}(\psi-\varphi_x) - \frac{4\gamma}{3\rho_2}\omega - \frac{4\beta}{3\rho_2}\omega_t + \frac{D}{\rho_2}\omega_{xx} \\ \theta_t \\ \frac{\delta}{\rho_3}\theta_{xx} - \frac{\sigma}{\rho_3}(3\omega-\psi)_{tx} + \frac{k}{\rho_3}\theta_{txx} \\ -\frac{1}{\tau}z_\rho \end{pmatrix}.$$

We consider the following spaces:

$$\begin{aligned} L_*^2(0,1) &= \left\{ w \in L^2(0,1) : \int_0^1 w(s)\,ds = 0 \right\}, \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\ H_*^2(0,1) &= \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\}, \end{aligned}$$

and the energy space:

$$\begin{aligned} \mathcal{H} = {} & H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \\ & \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times L^2\big((0,1), L^2(0,1)\big). \end{aligned}$$

The inner product on Hilbert space $\mathcal{H}$ is defined by

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \rho_1\int_0^1 \varphi_t\tilde{\varphi}_t dx + G\int_0^1 (\psi-\varphi_x)\left(\tilde{\psi}-\tilde{\varphi}_x\right)dx + 4\gamma\int_0^1 \omega\tilde{\omega}dx \\ &+\rho_2\int_0^1 (3\omega-\psi)_t\left(3\tilde{\omega}-\tilde{\psi}\right)_t dx + 3\rho_2\int_0^1 \omega_t\tilde{\omega}_t dx \\ &+D\int_0^1 (3\omega-\psi)_x\left(3\tilde{\omega}-\tilde{\psi}\right)_x dx + 3D\int_0^1 \omega_x\tilde{\omega}_x dx \\ &+\rho_3\int_0^1 \theta_t\tilde{\theta}_t dx + \delta\int_0^1 \theta_x\tilde{\theta}_x dx + \lambda\int_0^1\int_0^1 z\tilde{z}d\rho dx,\end{aligned}$$

where λ is the positive constant satisfying

$$\begin{cases} \tau\left|\mu_2\right| < \lambda < \tau\left(2\mu_1 - \left|\mu_2\right|\right), & \text{if} \quad \left|\mu_2\right| < \mu_1, \\ \lambda = \tau\mu_1, & \text{if} \quad \left|\mu_2\right| = \mu_1. \end{cases} \tag{3.7}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} \mid \varphi,\theta \in H_*^2(0,1)\cap H_*^1(0,1), \\ \omega,\psi \in H^2(0,1)\cap H_0^1(0,1),\ \psi_t,\omega_t \in H_0^1(0,1), \\ \varphi_t,\theta_t \in H_*^1(0,1),\ \ \delta\theta + k\theta_t \in H_*^2(0,1), \\ z, z_\rho \in L^2\left((0,1),L^2(0,1)\right),\ \ z(x,0) = \varphi_t(x) \end{array} \right\},$$

and it is dense in $\mathcal{H}$.

The well-posedness of problem (3.6) is ensured by

Theorem 3.1. *Assume that $U_0 \in \mathcal{H}$ and (3.3) holds. Then there exists a unique solution $U \in C(\mathbb{R}^+;\mathcal{H})$ of problem (3.6). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C\left(\mathbb{R}^+; D(\mathcal{A}) \cap C^1(\mathbb{R}^+;\mathcal{H})\right).$$

Proof. To obtain the above result, we need to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is dissipative.

For any $U \in D(\mathcal{A})$, by using the inner product and integration by parts, we can imply that

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} &= -4\beta\int_0^1 \omega_t^2 dx - k\int_0^1 \theta_{tx}^2 dx - \mu_1\int_0^1 \varphi_t^2 dx \\ &-\mu_2\int_0^1 \varphi_t z(x,1,t)\,dx - \frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)\,d\rho dx. \end{aligned} \tag{3.8}$$

By using Young's inequality, the fourth term on the right-hand side of Equation (3.8) gives

$$-\mu_2 \int_0^1 \varphi_t z(x,1,t)\,dx \le \frac{|\mu_2|}{2}\int_0^1 \varphi_t^2 dx + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx.$$

Also, using integration by parts and the fact that $z(x,0)=\varphi_t(x)$, the last term in the right-hand side of (3.8) gives

$$\begin{aligned} -\frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)\,d\rho dx &= -\frac{\lambda}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho dx \\ &= \frac{\lambda}{2\tau}\int_0^1\left(\varphi_t^2 - z^2(x,1,t)\right)dx. \end{aligned}$$

Consequently, (3.8) yields

$$\begin{aligned} \langle \mathcal{A}U,U\rangle_{\mathcal{H}} \le\; & -4\beta\int_0^1 \omega_t^2 dx - k\int_0^1 \theta_{tx}^2 dx - \left(\mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 \varphi_t^2 dx \\ & -\left(\frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 z^2(x,1,t)\,dx. \end{aligned}$$

Keeping in mind condition (3.7), we observe that

$$\mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \ge 0 \qquad , \qquad \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \ge 0.$$

Consequently, $\mathcal{A}$ is a dissipative operator. Next, we prove that the operator $Id-\mathcal{A}$ is surjective. Given $F=(f_1,\dots,f_9)^T \in \mathcal{H}$, we prove that there exists a unique $U=\left(\varphi,\varphi_t,3\omega-\psi,(3\omega-\psi)_t,\omega,\omega_t,\theta,\theta_t,z\right)\in D(\mathcal{A})$ such that

$$(Id-\mathcal{A})\,U = F,$$

which is equivalent to

$$\begin{cases} \varphi - \varphi_t = f_1, \\ \rho_1\varphi_t + G(\psi-\varphi_x)_x + \mu_1\varphi_t + \mu_2 z(x,1,t) = \rho_1 f_2, \\ (3\omega-\psi) - (3\omega-\psi)_t = f_3, \\ \rho_2(3\omega-\psi)_t - G(\psi-\varphi_x) - D(3\omega-\psi)_{xx} + \sigma\theta_{tx} = \rho_2 f_4, \\ \omega - \omega_t = f_5, \\ 3\rho_2\omega_t + 3G(\psi-\varphi_x) + 4\gamma\omega + 4\beta\omega_t - 3D\omega_{xx} = 3\rho_2 f_6, \\ \theta - \theta_t = f_7, \\ \rho_3\theta_t - \delta\theta_{xx} + \sigma(3\omega-\psi)_{tx} - k\theta_{txx} = \rho_3 f_8, \\ \tau z(x,\rho,t) + z_\rho(x,\rho,t) = \tau f_9. \end{cases} \tag{3.9}$$

The last equation in (3.9) and the fact that $z(x,0) = \varphi_t(x,t)$, we get

$$z(x,\rho) = \varphi(x)e^{-\tau\rho} - e^{-\tau\rho} f_1 + \tau e^{-\tau\rho} \int_0^\rho e^{\tau s} f_9(x,s)\,ds, \tag{3.10}$$

$(3.9)_1$, $(3.9)_3$, $(3.9)_5$ and $(3.9)_7$ give

$$\begin{cases} \varphi_t = \varphi - f_1, \\ (3\omega - \psi)_t = (3\omega - \psi) - f_3, \\ \omega_t = \omega - f_5, \\ \theta_t = \theta - f_7. \end{cases} \tag{3.11}$$

Inserting (3.11) into $(3.9)_2$, $(3.9)_4$, $(3.9)_6$ and $(3.9)_8$, we get

$$\begin{cases} (\rho_1 + \mu_1 + \mu_2 e^{-\tau})\varphi - G\varphi_{xx} - G(3\omega - \psi)_x + 3G\omega_x \\ = (\rho_1 + \mu_1 + \mu_2 e^{-\tau}) f_1 + \rho_1 f_2 - \mu_2 \tau e^{-\tau} \int_0^1 e^{\tau s} f_9\,ds, \\ (\rho_2 + G)(3\omega - \psi) + G\varphi_x - 3G\omega - D(3\omega - \psi)_{xx} + \sigma\theta_x \\ = \rho_2 (f_3 + f_4) + \sigma\partial_x f_7, \\ (3\rho_2 + 4\beta + 4\gamma + 9G)\omega - 3G(3\omega - \psi) - 3G\varphi_x - 3D\omega_{xx} \\ = (3\rho_2 + 4\beta) f_5 + 3\rho_2 f_6, \\ \rho_3\theta - (\delta + k)\theta_{xx} + \sigma(3\omega - \psi)_x = \rho_3(f_7 + f_8) + \sigma\partial_x f_3 - k\partial_{xx} f_7. \end{cases} \tag{3.12}$$

Multiplying the forth equation of system (3.12) by $\tilde{\varphi}, \left(3\tilde{\omega} - \tilde{\psi}\right), \tilde{\omega}$ and $\tilde{\theta}$ respectively, and integrating over $(0,1)$, we arrive

$$\begin{cases} (\rho_1 + \mu_1 + \mu_2 e^{-\tau}) \int_0^1 \varphi\tilde{\varphi}\,dx + G\int_0^1 \varphi_x\tilde{\varphi}_x\,dx - G\int_0^1 (3\omega - \psi)_x \tilde{\varphi}\,dx \\ +3G\int_0^1 \omega_x\tilde{\varphi}\,dx = (\rho_1 + \mu_1 + \mu_2 e^{-\tau}) \int_0^1 f_1\tilde{\varphi}\,dx + \rho_1 \int_0^1 f_2\tilde{\varphi}\,dx \\ -\mu_2\tau e^{-\tau} \int_0^1\int_0^1 e^{\tau s} f_9\tilde{\varphi}\,ds\,dx, \\ \\ (\rho_2 + G)\int_0^1 (3\omega - \psi)\left(3\tilde{\omega} - \tilde{\psi}\right)dx + D\int_0^1 (3\omega - \psi)_x \left(3\tilde{\omega} - \tilde{\psi}\right)_x dx \\ +G\int_0^1 \varphi_x\left(3\tilde{\omega} - \tilde{\psi}\right)dx - 3G\int_0^1 \omega\left(3\tilde{\omega} - \tilde{\psi}\right)dx + \sigma\int_0^1 \theta_x\left(3\tilde{\omega} - \tilde{\psi}\right)dx \\ = \rho_2\int_0^1 (f_3 + f_4)\left(3\tilde{\omega} - \tilde{\psi}\right)dx + \sigma\int_0^1 \partial_x f_7\left(3\tilde{\omega} - \tilde{\psi}\right)dx, \\ \\ (3\rho_2 + 4\beta + 4\gamma + 9G)\int_0^1 \omega\tilde{\omega}\,dx - 3G\int_0^1 (3\omega - \psi)\tilde{\omega}\,dx - 3G\int_0^1 \varphi_x\tilde{\omega}\,dx \\ +3D\int_0^1 \omega_x\tilde{\omega}_x\,dx = (3\rho_2 + 4\beta)\int_0^1 f_5\tilde{\omega}\,dx + 3\rho_2\int_0^1 f_6\tilde{\omega}\,dx, \\ \\ \rho_3\int_0^1 \theta\tilde{\theta}\,dx + (\delta + k)\int_0^1 \theta_x\tilde{\theta}_x\,dx + \sigma\int_0^1 (3\omega - \psi)_x\tilde{\theta}\,dx \\ = \rho_3\int_0^1 (f_7 + f_8)\tilde{\theta}\,dx + \sigma\int_0^1 \partial_x f_3\tilde{\theta}\,dx - k\int_0^1 \partial_{xx} f_7\tilde{\theta}\,dx. \end{cases} \tag{3.13}$$

The sum of the equations in (3.13) gives the following variational formulation

$$\begin{aligned} B\Big((\varphi,3\omega-\psi,\omega,\theta)^T,\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\Big) &= L\Big(\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\Big), \\ \forall\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T \in H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1), \end{aligned} \tag{3.14}$$

where $B:\left(H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1)\right)^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} & B\Big((\varphi,3\omega-\psi,\omega,\theta)^T,\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\Big) \\ = & \int_0^1 \Big[G(\psi-\varphi_x)\left(\tilde{\psi}-\tilde{\varphi}_x\right)+(\rho_1+\mu_1+\mu_2e^{-\tau})\varphi\tilde{\varphi}+3D\omega_x\tilde{\omega}_x \\ & +\rho_2(3\omega-\psi)\left(3\tilde{\omega}-\tilde{\psi}\right)+D(3\omega-\psi)_x\left(3\tilde{\omega}-\tilde{\psi}\right)_x+\rho_3\theta\tilde{\theta} \\ & +(3\rho_2+4\beta+4\gamma)\omega\tilde{\omega}+\sigma(3\omega-\psi)_x\tilde{\theta}+(\delta+k)\theta_x\tilde{\theta}_x \\ & +\sigma\theta_x\left(3\tilde{\omega}-\tilde{\psi}\right)\Big]dx, \end{aligned}$$

and $L:\left(H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1)\right)\longrightarrow \mathbb{R}$ is the linear functional given by

$$\begin{aligned} & L\Big(\left(\tilde{\varphi},3\tilde{\omega}-\tilde{\psi},\tilde{\omega},\tilde{\theta}\right)^T\Big) \\ = & \int_0^1 \Big[(\rho_1+\mu_1+\mu_2e^{-\tau})f_1\tilde{\varphi}+\rho_1f_2\tilde{\varphi}-\mu_2\tau e^{-\tau}\int_0^1 e^{\tau s}f_9\tilde{\varphi}ds \\ & +\rho_2(f_3+f_4)\left(3\tilde{\omega}-\tilde{\psi}\right)+\sigma(\partial_xf_7)\left(3\tilde{\omega}-\tilde{\psi}\right)+(3\rho_2+4\beta)f_5\tilde{\omega} \\ & +3\rho_2f_6\tilde{\omega}+\rho_3(f_7+f_8)\tilde{\theta}+\sigma(\partial_xf_3)\tilde{\theta}+k(\partial_xf_7)\tilde{\theta}_x\Big]dx. \end{aligned}$$

Now, for

$$V = H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_*^1(0,1),$$

equipped with the norm

$$\begin{aligned} \left\|\varphi,3\omega-\psi,\omega,\theta\right\|_V^2 &= \left\|-\varphi_x-(3\omega-\psi)+3\omega\right\|_2^2+\|\varphi\|_2^2+\left\|(3\omega-\psi)_x\right\|_2^2 \\ &\quad +\|\omega_x\|_2^2+\|\theta\|_2^2+\|\theta_x\|_2^2. \end{aligned}$$

It is clear that B and L are bounded. Furthermore, using integration by parts, we have

$$\begin{aligned} & B\Big((\varphi,3\omega-\psi,\omega,\theta)^T,(\varphi,3\omega-\psi,\omega,\theta)^T\Big) \\ = & \int_0^1 \Big[G(\psi-\varphi_x)^2+(\rho_1+\mu_1+\mu_2e^{-\tau})\varphi^2+\rho_2(3\omega-\psi)^2+D(3\omega-\psi)_x^2 \\ & +(3\rho_2+4\beta+4\gamma)\omega^2+3D\omega_x^2+\rho_3\theta^2+(\delta+k)\theta_x^2\Big]dx \\ \geq & \; m\left\|\varphi,3\omega-\psi,\omega,\theta\right\|_V^2, \end{aligned}$$

for some m Thus, B is coercive.

Hence, we assert that $B(\cdot,\cdot)$ is a bilinear continuous coercive form on $V \times V$, and $L(\cdot)$ is a linear continuous form on V. Applying the Lax-Milgram theorem [47], we obtain that (3.14) has a unique solution

$$(\varphi, 3\omega - \psi, \omega, \theta) \in H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1).$$

The substitution of $\varphi, 3\omega - \psi, \omega$ and θ into (3.11) yields

$$(\varphi_t, 3\omega_t - \psi_t, \omega_t, \theta_t) \in H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1).$$

Next, it remains to show that

$$\begin{aligned} \varphi &\in \left(H_*^2(0,1) \cap H_*^1(0,1)\right), \ (3\omega - \psi), \omega \in \left(H^2(0,1) \cap H_0^1(0,1)\right), \\ \theta &\in \left(H_*^2(0,1) \cap H_*^1(0,1)\right). \end{aligned}$$

Furthermore, if $\left(3\tilde{\omega} - \tilde{\psi}, \tilde{\omega}, \tilde{\theta}\right) = (0,0,0) \in H_0^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$, then (3.14) reduces to

$$\begin{aligned} & B\left((\varphi, 3\omega - \psi, \omega, \theta)^T, (\tilde{\varphi}, 0,0,0)^T\right) \\ =& \int_0^1 \left[-G(3\omega - \psi)_x \tilde{\varphi} - G\varphi_{xx}\tilde{\varphi} + 3G\omega_x\tilde{\varphi} + (\rho_1 + \mu_1 + \mu_2 e^{-\tau})\varphi\tilde{\varphi}\right] dx \\ =& \int_0^1 \left[(\rho_1 + \mu_1 + \mu_2 e^{-\tau}) f_1 \tilde{\varphi} + \rho_1 f_2 \tilde{\varphi} - \mu_2 \tau e^{-\tau} \int_0^1 e^{\tau s} f_9 \tilde{\varphi} ds\right] dx, \end{aligned} \tag{3.15}$$

for all $\forall \tilde{\varphi} \in H_*^1(0,1)$,which implies

$$\begin{aligned} G\varphi_{xx} &= (\rho_1 + \mu_1 + \mu_2 e^{-\tau})\varphi + 3G\omega_x - G(3\omega - \psi)_x \\ & \quad -(\rho_1 + \mu_1 + \mu_2 e^{-\tau}) f_1 - \rho_1 f_2 + \mu_2 \tau e^{-\tau} \int_0^1 e^{\tau s} f_9 ds. \end{aligned} \tag{3.16}$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\varphi \in H^2(0,1) \cap H_*^1(0,1).$$

Moreover, (3.15) is also true for any $\phi \in C^1[0,1] \subset H_*^1(0,1)$. Hence, we have

$$\begin{aligned} & \int_0^1 G\varphi_x \phi_x dx + \int_0^1 \left[(\rho_1 + \mu_1 + \mu_2 e^{-\tau})\varphi - G(3\omega - \psi)_x + 3G\omega_x \right. \\ & \left. -(\rho_1 + \mu_1 + \mu_2 e^{-\tau}) f_1 - \rho_1 f_2 + \mu_2 \tau e^{-\tau} \int_0^1 e^{\tau s} f_9 ds\right] \phi dx = 0, \end{aligned}$$

for all $\phi \in C^1[0,1]$. Thus, using integration by parts and bearing in mind (3.16), we obtain

$$\varphi_x(1)\phi(1) - \varphi_x(0)\phi(0) = 0, \forall \phi \in C^1[0,1].$$

Therefore, $\varphi_x(0) = \varphi_x(1) = 0$. Consequently, we obtain

$$\varphi \in H_*^2(0,1) \cap H_*^1(0,1).$$

Similarly, we obtain

$$(3\omega - \psi),\ \omega \in H^2(0,1) \cap H_0^1(0,1).$$

Also, if we take $\left(\tilde{\varphi}, 3\tilde{\omega} - \tilde{\psi}, \tilde{\omega}\right) = (0,0,0) \in H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$ in (3.14), then using $(3.9)_3$ and $(3.9)_7$, we get

$$\delta\theta_{xx} + k\theta_{txx} = \rho_3\theta_t - \rho_3 f_8 + \sigma(3\omega - \psi)_{tx},$$

and we conclude that

$$\delta\theta + k\theta_t \in H^2(0,1).$$

Furthermore, it is obvious from

$$\delta\theta_x + k\theta_{tx} = \rho_3 \int_0^x \theta_t dx - \rho_3 \int_0^x f_8 dx + \sigma(3\omega - \psi)_t,$$

that

$$(\delta\theta_x + k\theta_{tx})(0) = (\delta\theta_x + k\theta_{tx})(1) = 0,$$

then, we get

$$\delta\theta + k\theta_t \in H_*^2(0,1).$$

Finally, it follows, from (3.10), that

$$z(x,0) = \varphi_t(x) \text{ and } z, z_\rho \in L^2\left((0,1), L^2(0,1)\right).$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (3.14) is satisfied, the operator $Id - \mathcal{A}$ is surjective. Moreover, it is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Consequently, the result of Theorem 3.1 follows from Lumer-Phillips theorem. □

3.3 Exponential stability of solution

In this section, we show that, under the assumption $|\mu_2| \leq \mu_1$ and $\dfrac{\rho_1}{G} = \dfrac{\rho_2}{D}$ for the solution of problem (3.4)-(3.5) decays exponentially to the study state. To achieve our goal we use the perturbed energy method to produce a suitable Lyapunov functional. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) \; : \; &= \frac{1}{2}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2(3\omega-\psi)_t^2 + 3\rho_2\omega_t^2 + \rho_3\theta_t^2 + G(\psi-\varphi_x)^2 \\ &+ D(3\omega-\psi)_x^2 + 4\gamma\omega^2 + 3D\omega_x^2 + \delta\theta_x^2 + \frac{\lambda\tau}{2}\int_0^1 z^2(x,\rho,t)\,ds\Big]dx. \end{aligned}$$

If the wave speeds are equal, we have the following exponentially stable result.

Theorem 3.2. *Assume that $\frac{\rho_1}{G} = \frac{\rho_2}{D}$ and (3.3) holds. Let $U^0 \in \mathcal{H}$, then there exists positive constants c_0, c_1 such that the energy $E(t)$ associated with problem (3.4)-(3.5) satisfies*

$$E(t) \leq c_0 e^{-c_1 t}, \; t \geq 0.$$

To prove our this result, we will state and prove some useful lemmas in advance.

Lemma 3.3. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of (3.4)-(3.5) with (3.7). Then the energy functional satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) &\leq -4\beta\int_0^1 \omega_t^2 dx - k\int_0^1 \theta_{tx}^2 dx - C_1\int_0^1 \varphi_t^2 dx - C_2\int_0^1 z^2(x,1,t)\,dx \\ &\leq 0, \end{aligned}$$

where

$$C_1 = \mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \geq 0 \quad , \quad C_2 = \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2} \geq 0.$$

Proof. First, multiplying $(3.4)_1$ by φ_t, integrating over $(0,1)$, using integration by parts and the boundary conditions in (3.5), we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\rho_1\int_0^1 \varphi_t^2 dx\right) - G\int_0^1 (\psi-\varphi_x)\varphi_{tx}dx \\ &= -\mu_1\int_0^1 \varphi_t^2 dx - \mu_2\int_0^1 \varphi_t z(x,1,t)\,dx, \end{aligned} \tag{3.17}$$

note that

$$\begin{aligned} G\int_0^1 (\psi-\varphi_x)\varphi_{tx}dx &= -G\int_0^1 (\psi-\varphi_x)(\psi-\varphi_x-\psi)_t\, dx \\ &= \frac{d}{dt}\left(-\frac{1}{2}G\int_0^1 (\psi-\varphi_x)^2\, dx\right)+G\int_0^1 (\psi-\varphi_x)\psi_t dx. \end{aligned}$$

Hence, equation (3.17) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_1\int_0^1 \varphi_t^2 dx+G\int_0^1 (\psi-\varphi_x)^2\, dx\right) \\ =\ & G\int_0^1 (\psi-\varphi_x)\psi_t dx-\mu_1\int_0^1 \varphi_t^2 dx-\mu_2\int_0^1 \varphi_t z(x,1,t)\, dx. \end{aligned} \tag{3.18}$$

Similarly, multiplying $(3.4)_2$, $(3.4)_3$, $(3.4)_4$ by $(3\omega-\psi)_t$, $3\omega_t$, θ_t and integrating over $(0,1)$, using integration by parts and the boundary conditions in (3.5), we can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_2\int_0^1 (3\omega-\psi)_t^2\, dx+D\int_0^1 (3\omega-\psi)_x^2\, dx\right) \\ =\ & G\int_0^1 (\psi-\varphi_x)(3\omega-\psi)_t\, dx-\sigma\int_0^1 \theta_{tx}(3\omega-\psi)_t\, dx, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(3\rho_2\int_0^1 \omega_t^2 dx+4\gamma\int_0^1 \omega^2 dx+3D\int_0^1 \omega_x^2 dx\right) \\ =\ & -3G\int_0^1 (\psi-\varphi_x)\omega_t dx-4\beta\int_0^1 \omega_t^2 dx, \end{aligned} \tag{3.20}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_3\int_0^1 \theta_t^2 dx+\delta\int_0^1 \theta_x^2 dx\right)=\sigma\int_0^1 (3\omega-\psi)_t\,\theta_{tx} dx-k\int_0^1 \theta_{tx}^2 dx. \tag{3.21}$$

Now, multiplying $(3.4)_5$, by $\frac{\lambda}{\tau}z$ and integrating over $(0,1)\times(0,1)$, using integration by parts and the boundary conditions in (3.5), we can get

$$\frac{\lambda}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)\, d\rho dx=-\frac{\lambda}{2\tau}\int_0^1 \left(z^2(x,1,t)-\varphi_t^2\right)dx. \tag{3.22}$$

Finally, adding (3.18), (3.19), (3.20), (3.21) and (3.22), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\rho_1\int_0^1 \varphi_t^2 dx + G\int_0^1 (\psi-\varphi_x)^2 dx + D\int_0^1 (3\omega-\psi)_x^2 dx\right.\\
&+\rho_2\int_0^1 (3\omega-\psi)_t^2 dx + \rho_3\int_0^1 \theta_t^2 dx + \delta\int_0^1 \theta_x^2 dx + 4\gamma\int_0^1 \omega^2 dx\\
&\left.+3\rho_2\int_0^1 \omega_t^2 dx + 3D\int_0^1 \omega_x^2 dx\right] + \frac{\lambda}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx\\
= \; & -4\beta\int_0^1 \omega_t^2 dx - k\int_0^1 \theta_{tx}^2 dx - \mu_1\int_0^1 \varphi_t^2 dx - \mu_2\int_0^1 \varphi_t z(x,1,t)\,dx\\
&-\frac{\lambda}{2\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{\lambda}{2\tau}\int_0^1 \varphi_t^2 dx.
\end{aligned}$$

Meanwhile, using Young's inequality, we have

$$-\mu_2\int_0^1 \varphi_t z(x,1,t)\,dx \le \frac{|\mu_2|}{2}\int_0^1 \varphi_t^2 dx + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx.$$

Hence,

$$\begin{aligned}
\frac{d}{dt}E(t) \;\le\; & -4\beta\int_0^1 \omega_t^2 dx - k\int_0^1 \theta_{tx}^2 dx - \left(\mu_1 - \frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 \varphi_t^2 dx\\
&-\left(\frac{\lambda}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^1 z^2(x,1,t)\,dx,
\end{aligned}$$

using (3.7), we obtain the result. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.4. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of (3.4)-(3.5). Then the functional*

$$I_1(t) := \rho_2\int_0^1 (3\omega-\psi)(3\omega-\psi)_t\,dx$$

satisfies the estimate

$$\begin{aligned}
I_1'(t) \;\le\; & -\frac{D}{2}\int_0^1 (3\omega-\psi)_x^2 dx + \rho_2\int_0^1 (3\omega-\psi)_t^2 dx\\
&+\frac{\sigma^2}{D}\int_0^1 (\psi-\varphi_x)^2 dx + \frac{G^2}{D}\int_0^1 \theta_t^2 dx, \qquad (3.23)
\end{aligned}$$

Proof. Taking the derivative of $I_1(t)$ with respect to t, using $(3.4)_2$ and integrating by parts, we get

$$\begin{aligned} I_1'(t) &= \rho_2 \int_0^1 (3\omega-\psi)_t^2\,dx + G\int_0^1 (\psi-\varphi_x)(3\omega-\psi)\,dx \\ &\quad -D\int_0^1 (3\omega-\psi)_x^2\,dx + \sigma\int_0^1 \theta_t (3\omega-\psi)_x\,dx. \end{aligned}$$

Using Young's and Poincaré inequalities, we arrive at (3.23). □

Lemma 3.5. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of (3.4)-(3.5). Then the functional*

$$I_2(t) := \rho_2 \int_0^1 \omega\omega_t dx$$

satisfies the estimate

$$I_2'(t) \le -\frac{2}{3}\gamma \int_0^1 \omega^2 dx - D\int_0^1 \omega_x^2 dx + C_3\int_0^1 \omega_t^2 dx + \frac{3G^2}{4\gamma}\int_0^1 (\psi-\varphi_x)^2\,dx, \quad (3.24)$$

where

$$C_3 = \rho_2 + \frac{4\beta^2}{3\gamma}.$$

Proof. By differentiating I_2 with respect to t with respect to t, using $(3.4)_3$ and integrating by parts, we obtain

$$\begin{aligned} I_2'(t) &= \rho_2\int_0^1 \omega_t^2 dx - G\int_0^1 (\psi-\varphi_x)\,\omega dx - \frac{4}{3}\gamma\int_0^1 \omega^2 dx - \frac{4}{3}\beta\int_0^1 \omega_t\omega dx \\ &\quad -D\int_0^1 \omega_x^2 dx. \end{aligned}$$

Using Young's inequality, we establish (3.24). □

Lemma 3.6. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of (3.4)-(3.5). Then the functional*

$$I_3(t) := \rho_2\rho_3\int_0^1 (3\omega-\psi)_t \int_0^x \theta_t(y,t)\,dydx - \rho_2\delta\int_0^1 \theta_x(3\omega-\psi)\,dx$$

satisfies the estimate

$$\begin{aligned} I_3'(t) &\le -\frac{\rho_2\sigma}{2}\int_0^1 (3\omega-\psi)_t^2\,dx + \varepsilon_1\int_0^1 (\psi-\varphi_x)^2\,dx + C_4(\varepsilon_1)\int_0^1 \theta_{tx}^2 dx \\ &\quad +\varepsilon_1\int_0^1 (3\omega-\psi)_x^2\,dx, \end{aligned} \quad (3.25)$$

for any $\varepsilon_1 > 0$, where

$$C_4(\varepsilon_1) = \sigma\rho_3 + \frac{\rho_3^2 G^2}{4\varepsilon_1} + \frac{\delta^2\rho_2^2}{2\varepsilon_1} + \frac{\rho_2 k^2}{2\sigma} + \frac{D^2\rho_3^2}{2\varepsilon_1}.$$

Proof. Taking the derivative of $I_3(t)$ with respect to t, using $(3.4)_2$, $(3.4)_4$ and integrating by parts, we get

$$\begin{aligned}
I_3'(t) &= \rho_3 \int_0^1 \left[G(\psi-\varphi_x)+D(3\omega-\psi)_{xx}-\sigma\theta_{tx}\right]\int_0^x \theta_t(y,t)\,dydx \\
&\quad +\rho_2 \int_0^1 (3\omega-\psi)_t \int_0^x \left[\delta\theta_{xx}+k\theta_{txx}-\sigma(3\omega-\psi)_{tx}\right]dydx \\
&\quad -\delta\rho_2 \int_0^1 \theta_{xt}(3\omega-\psi)\,dx-\delta\rho_2\int_0^1 \theta_x(3\omega-\psi)_t\,dx \\
&= \rho_3\int_0^1 G\psi\int_0^x \theta_t(y,t)\,dydx \\
&\quad +\rho_2\int_0^1 (3\omega-\psi)_t\left[\delta\theta_x+k\theta_{tx}-\sigma(3\omega-\psi)_t\right]dx \\
&\quad +\rho_3\int_0^1\left[-G\varphi_x+D(3\omega-\psi)_{xx}-\sigma\theta_{tx}\right]\int_0^x\theta_t(y,t)\,dydx \\
&\quad -\delta\rho_2 \int_0^1 \theta_{xt}(3\omega-\psi)\,dx-\delta\rho_2\int_0^1 \theta_x(3\omega-\psi)_t\,dx \\
&= \rho_3\int_0^1 G(\psi-\varphi_x)\int_0^x\theta_t(y,t)\,dydx-\delta\rho_2\int_0^1\theta_{xt}(3\omega-\psi)\,dx \\
&\quad +\left[\rho_3\left(-G\varphi+D(3\omega-\psi)_x-\sigma\theta_t\right)\int_0^x\theta_t(y,t)\,dy\right]_{x=0}^{x=1} \\
&\quad +\sigma\rho_3\int_0^1\theta_t^2dx-\rho_2\sigma\int_0^1(3\omega-\psi)_t^2\,dx+\rho_2 k\int_0^1(3\omega-\psi)_t\theta_{tx}dx \\
&\quad -D\rho_3\int_0^1\theta_t(3\omega-\psi)_x\,dx.
\end{aligned}$$

Note that

$$\int_0^1 \theta_t(y,t)\,dy=\frac{d}{dt}\int_0^1\theta(y,t)\,dy=0,$$

then, by Young's and Poincaré inequalities with $\varepsilon_1>0$ to obtain (3.25). □

Lemma 3.7. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of problem (3.4)-(3.5). the functional*

$$I_4(t):=\int_0^1\left[\rho_3\theta_t\theta+\frac{k}{2}\theta_x^2+\sigma(3\omega-\psi)_x\theta\right]dx$$

satisfies the estimate

$$I_4'(t)\leq-\delta\int_0^1\theta_x^2dx+C_5(\varepsilon_2)\int_0^1\theta_t^2dx+\varepsilon_2\int_0^1(3\omega-\psi)_x^2\,dx, \tag{3.26}$$

for any $\varepsilon_2>0$, where

$$C_5(\varepsilon_2)=\rho_3+\frac{\sigma^2}{4\varepsilon_2}.$$

Proof. By differentiating I_4 with respect to t, using $(3.4)_4$ and integrating by parts, we obtain

$$\begin{aligned} I_4'(t) &= \int_0^1 \rho_3\theta_{tt}\theta dx + \int_0^1 \rho_3\theta_t^2 dx + \int_0^1 \frac{k}{2}(\theta_{xt}\theta_x + \theta_x\theta_{xt})dx \\ &\quad + \int_0^1 \sigma(3\omega-\psi)_{xt}\theta dx + \int_0^1 \sigma(3\omega-\psi)_x\theta_t dx \\ &= \int_0^1 \left[\delta\theta_{xx} + k\theta_{txx} - \sigma(3\omega-\psi)_{tx}\right]\theta dx + \int_0^1 \rho_3\theta_t^2 dx \\ &\quad - \int_0^1 k\theta_{xxt}\theta dx + \int_0^1 \sigma(3\omega-\psi)_{xt}\theta dx + \int_0^1 \sigma(3\omega-\psi)_x\theta_t dx \\ &= \int_0^1 \delta\theta_{xx}\theta dx + \int_0^1 \rho_3\theta_t^2 dx + \int_0^1 \sigma(3\omega-\psi)_x\theta_t dx. \end{aligned}$$

Using Young's inequality with $\varepsilon_2 > 0$, we establish (3.26). □

Lemma 3.8. *Let $(\varphi, \psi, \omega, \theta, z)$ be the solution of (3.4)-(3.5). Then the functional*

$$I_5(t) := \rho_2 \int_0^1 (3\omega-\psi)_t(\varphi_x-\psi)dx + \frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_t dx$$

satisfies the estimate

$$\begin{aligned} I_5'(t) &\leq -\frac{G}{2}\int_0^1 (\varphi_x-\psi)^2 dx + \frac{\sigma^2}{2G}\int_0^1 \theta_{tx}^2 dx + \frac{9\rho_2^2}{4\varepsilon_3}\int_0^1 \omega_t^2 dx \\ &\quad + (\rho_2+\varepsilon_3)\int_0^1 (3\omega-\psi)_t^2 dx + \left(\frac{D\rho_1}{G} - \rho_2\right)\int_0^1 (3\omega-\psi)_{xt}\varphi_t dx \\ &\quad + \varepsilon_4\int_0^1 (3\omega-\psi)_x^2 dx + \frac{D^2\mu_1^2}{2G^2\varepsilon_4}\int_0^1 \varphi_t^2 dx + \frac{D^2\mu_2^2}{2G^2\varepsilon_4}\int_0^1 z^2(x,1,t)dx, \end{aligned} \tag{3.27}$$

for any $\varepsilon_3, \varepsilon_4 > 0$.

Proof. By differentiating I_5 with respect to t, using $(3.4)_1$, $(3.4)_2$ and integrating by parts, we obtain

$$\begin{aligned}
I_5'(t) &= \rho_2\int_0^1 (3\omega-\psi)_{tt}(\varphi_x-\psi)\,dx+\rho_2\int_0^1 (3\omega-\psi)_t(\varphi_x-\psi)_t\,dx\\
&\quad +\frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_{xt}\varphi_t dx+\frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_{tt}dx\\
&= -\int_0^1 G(\varphi_x-\psi)^2\,dx+\int_0^1 D(3\omega-\psi)_{xx}(\varphi_x-\psi)\,dx\\
&\quad -\int_0^1 \sigma\theta_{tx}(\varphi_x-\psi)\,dx+\rho_2\int_0^1 (3\omega-\psi)_t(\varphi_x-\psi)_t\,dx\\
&\quad +\frac{D\rho_1}{G}\int_0^1 (3\omega-\psi)_{xt}\varphi_t dx-D\int_0^1 (3\omega-\psi)_x(\psi-\varphi_x)_x\,dx\\
&\quad -\frac{D\mu_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_t dx-\frac{D\mu_2}{G}\int_0^1 (3\omega-\psi)_x z(x,1,t)\,dx\\
&= -G\int_0^1 (\varphi_x-\psi)^2\,dx-\sigma\int_0^1 \theta_{tx}(\varphi_x-\psi)\,dx\\
&\quad -\rho_2\int_0^1 (3\omega-\psi)_t\psi_t dx+\left(\frac{D\rho_1}{G}-\rho_2\right)\int_0^1 (3\omega-\psi)_{xt}\varphi_t dx\\
&\quad -\frac{D\mu_1}{G}\int_0^1 (3\omega-\psi)_x\varphi_t dx-\frac{D\mu_2}{G}\int_0^1 (3\omega-\psi)_x z(x,1,t)\,dx.
\end{aligned}$$

Using Young's inequality with $\varepsilon_3,\varepsilon_4>0$, we establish (3.27). □

Lemma 3.9. *Let $(\varphi,\psi,\omega,\theta,z)$ be the solution of (3.4)-(3.5). Then the functional*

$$I_6(t):=\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx$$

satisfies the estimate

$$I_6'(t)\le -m\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx-\frac{c}{\tau}\int_0^1 z^2(x,1,t)\,dx+\frac{1}{\tau}\int_0^1 \varphi_t^2 dx, \tag{3.28}$$

for any $m,c>0$.

Proof. By differentiating I_6 with respect to t, using $(3.4)_5$ and integrating by parts, we obtain

$$\begin{aligned}
I_6'(t) &= -\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho dx\\
&= -2\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx-\frac{1}{\tau}\int_0^1\int_0^1\frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx\\
&\le -m\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx-\frac{c}{\tau}\int_0^1 z^2(x,1,t)\,dx+\frac{1}{\tau}\int_0^1 \varphi_t^2 dx.
\end{aligned}$$

This gives (3.28). □

Proof. (of Theorem 3.2)

To finalize the proof, we assume $\frac{G}{\rho_1} = \frac{D}{\rho_2}$ and define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := NE(t) + N_1F_1(t) + F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t) + F_6(t),$$

where N, N_1, N_3, N_5 are positive constants to be chosen properly later. Using Cauchy-Schwarz inequality and the Poincare inequality, one can easily see that all $F_i(t)$, $i = 1,2,3,4,5,6$ are bounded by an expression with the existing terms in the energy $E(t)$. This leads to the equivalence of $\mathcal{L}(t)$ and $E(t)$. Gathering the estimates in the previous lemmas and using

$$\int_0^1 \theta_t^2 dx \leq \int_0^1 \theta_{tx}^2 dx,$$

we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[C_1N - \frac{D^2\mu_1^2}{2G^2\varepsilon_4}N_5 - \frac{1}{\tau}\right]\int_0^1 \varphi_t^2 dx - D\int_0^1 \omega_x^2 dx \\
& -\left[4\beta N - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}N_5\right]\int_0^1 \omega_t^2 dx - \delta\int_0^1 \theta_x^2 dx \\
& -\left[kN - \frac{G^2}{D}N_1 - C_4(\varepsilon_1)N_3 - C_5(\varepsilon_2) - \frac{\sigma^2}{2G}N_5\right]\int_0^1 \theta_{tx}^2 dx \\
& -\left[\frac{G}{2}N_5 - \frac{\sigma^2}{D}N_1 - \frac{3G^2}{4\gamma} - \varepsilon_1N_3\right]\int_0^1 (\varphi_x - \psi)^2 dx \\
& -\left[\frac{D}{2}N_1 - \varepsilon_1N_3 - \varepsilon_2 - \varepsilon_4N_5\right]\int_0^1 (3\omega - \psi)_x^2 dx - \frac{2}{3}\gamma\int_0^1 \omega^2 dx \\
& -\left[\frac{\rho_2\sigma}{2}N_3 - \rho_2N_1 - (\rho_2 + \varepsilon_3)N_5\right]\int_0^1 (3\omega - \psi)_t^2 dx \\
& -\left[C_2N + \frac{c}{\tau} - \frac{D^2\mu_2^2}{2G^2\varepsilon_4}N_5\right]\int_0^1 z^2(x,1,t)dx \\
& -m\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (3.29)
\end{aligned}$$

At this point, we choose our constants carefully. First, we take N_1 large enough and ε_2 small, such that

$$\frac{D}{2}N_1 - \varepsilon_2 > 0.$$

Then, we choose N_5 large enough, so that

$$\frac{G}{2}N_5 - \frac{\sigma^2}{D}N_1 - \frac{3G^2}{4\gamma} > 0.$$

Next, we pick ε_3 small and choose N_3 large enough such that

$$\frac{\rho_2\sigma}{2}N_3 - \rho_2 N_1 - (\rho_2 + \varepsilon_3) N_5 > 0.$$

Furthermore, we select ε_1 and ε_4 so small that

$$\frac{G}{2}N_5 - \frac{\sigma^2}{D}N_1 - \frac{3G^2}{4\gamma} - \varepsilon_1 N_3 > 0\ ,\quad \frac{D}{2}N_1 - \varepsilon_1 N_3 - \varepsilon_2 - \varepsilon_4 N_5 > 0.$$

Finally, we choose N so large such that

$$C_1 N - \frac{D^2\mu_1^2}{2G^2\varepsilon_4}N_5 - \frac{1}{\tau} > 0\ ,\quad 4\beta N - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}N_5 > 0,$$

$$kN - \frac{G^2}{D}N_1 - C_4(\varepsilon_1) N_3 - C_5(\varepsilon_2) - \frac{\sigma^2}{2G}N_5 > 0.$$

From the above, we deduce that for some positive constants α_1, α_2 one has

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t).$$

Therefore, (3.29) becomes

$$\mathcal{L}'(t) \le -cE(t).$$

For $c_1 = \dfrac{c}{\alpha_2}$, we get

$$\mathcal{L}'(t) \le -c_1 \mathcal{L}(t), \forall t \ge 0. \tag{3.30}$$

A simple integration of (3.30) over $(0, t)$ leads to

$$\mathcal{L}(t) \le \mathcal{L}(0) e^{-c_1 t}, \forall t \ge 0.$$

It gives the desired result Theorem 3.2 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. □

Part II

A flexible structure system

Chapter 4

A flexible structure system with Fourier's type heat conduction and distributed delay term

Contents

4.1 Presentation of the problem

In the present chapter, we consider a coupled system of a flexible structure with Fourier's type heat conduction and distributed delay. The system is written as

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \gamma\theta_x + \mu_0 u_t \\ + \int_{\tau_1}^{\tau_2} \mu(s)u_t(x,t-s)\,ds = 0, \\ \theta_t - \theta_{xx} + \gamma u_{xt} = 0, \end{cases} \tag{4.1}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(.,0) = \theta_0(x),\ \forall x \in [0,L], \\ u(0,t) = u(L,t) = 0,\ \theta(0,t) = \theta(L,t) = 0,\ \forall t \geq 0, \\ u_t(x,-t) = f_0(x,t),\ \ 0 < t \leq \tau_2, \end{cases} \tag{4.2}$$

where $u = u(x,t)$ is the displacement of a particle at position $x \in (0,L)$ and time $t > 0$. $\theta = \theta(x,t)$ is the temperature difference and γ is a constant known as coupling coefficient. u_0, u_1, θ_0 are initial data, and f_0 is the history function. The parameters $m(x)$, $\delta(x)$ and $p(x)$ is responsible for the non-uniform structure of the body, where $m(x)$ denote mass per unit length of structure, $\delta(x)$ coefficient of internal material damping and $p(x)$ a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m(x), \delta(x)$ and $p(x)$ in [3] such that

$$m, \delta, p \in W^{1,\infty}(0,L), \quad m(x), \delta(x), p(x) > 0, \quad \forall x \in [0,L].$$

The coefficients μ_0 is positive constants, and $\mu : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption

$$\mu_0 > \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| ds. \tag{4.3}$$

The rest of the chapter is organized as follows. In Section 4.2, we state and prove the well-posedness of system (4.1)-(4.2) by using semigroup method and Lumer-Phillips theorem. In Section 4.3, we establish an exponential stability by using the perturbed energy method and construct some Lyapunov functionals.

4.2 Well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (4.1)-(4.2) using the semigroup theory [47]. As in [45], we introduce the new variable

$$z(x,\rho,t,s) = u_t(x,t-\rho s),\ x\in(0,L),\rho\in(0,1),s\in(\tau_1,\tau_2),t>0. \tag{4.4}$$

It is straight forward to check that z satisfies

$$sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0 \text{ in } (0,L)\times(0,1)\times(0,\infty)\times(\tau_1,\tau_2). \tag{4.5}$$

Therefore, problem (4.1) takes the form

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\gamma\theta_x+\mu_0 u_t \\ +\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds=0, \\ \theta_t-\theta_{xx}+\gamma u_{xt}=0, \\ sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0, \end{cases} \tag{4.6}$$

with the following initial and boundary conditions

$$\begin{cases} u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \theta(.,0)=\theta_0(x),\ \forall x\in[0,L], \\ u(0,t)=u(L,t)=0,\ \theta(0,t)=\theta(L,t)=0,\ \forall t\geq 0, \\ z(x,0,t,s)=u_t(x,t) \text{ on } (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z(x,\rho,0,s)=f_0(x,\rho s) \text{ on } (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{4.7}$$

Introducing the vector function $U=(u,v,\theta,z)^T$, where $v=u_t$, system (4.6)-(4.7) can be written as

$$\begin{cases} \dfrac{dU(t)}{dt}=\mathcal{A}U(t),\quad t>0, \\ U(0)=U_0=(u_0,u_1,\theta_0,f_0)^T, \end{cases} \tag{4.8}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U=\begin{pmatrix} v \\ \frac{1}{m(x)}\left[(p(x)u_x+2\delta(x)v_x)_x-\gamma\theta_x-\mu_0 v-\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds\right] \\ \theta_{xx}-\gamma v_x \\ -s^{-1}z_\rho \end{pmatrix}.$$

Let

$$\mathcal{H}=H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),$$

be the Hilbert space equipped with the inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^L p(x)u_x\widetilde{u}_x dx+\int_0^L m(x)v\widetilde{v}dx+\int_0^L \theta\widetilde{\theta}dx \\ &+\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z(x,\rho,s)\widetilde{z}(x,\rho,s)\,dsd\rho dx.\end{aligned}$$

Then, the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{c} U\in\mathcal{H}\mid u,\theta\in H^2(0,L)\cap H_0^1(0,L),\ v\in H_0^1(0,L), \\ p(x)u_x+2\delta(x)v_x\in H^1(0,L),\ \theta_x+\gamma v\in H^1(0,L), \\ z,z_\rho\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),\ z(x,0,s)=v(x)\end{array}\right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 4.1. *Assume that $U_0\in\mathcal{H}$ and (4.3) holds, then problem (4.8) exists a unique solution $U\in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C(\mathbb{R}^+;D(\mathcal{A}))\cap C^1(\mathbb{R}^+;\mathcal{H}).$$

Proof. We use the semigroup approach to prove that $\mathcal{A}$ is a maximal monotone operator, which means $\mathcal{A}$ is dissipative and $Id-\mathcal{A}$ is surjective. First, we prove that $\mathcal{A}$ is dissipative. For any $U=(u,v,\theta,z)^T\in D(\mathcal{A})$, by using the inner product and integrating by parts, we obtain

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}} &= -2\int_0^L\delta(x)v_x^2dx-\left(\mu_0-\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|ds\right)\int_0^L v^2dx \\ &-\int_0^L\theta_x^2dx-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,s)\,dsdx \\ &-\int_0^L v\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx. \end{aligned}\tag{4.9}$$

Using Young's inequality, the last term in (4.9), we can estimate

$$\begin{aligned}&-\int_0^L v\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx \\ \leq\ &\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|ds\int_0^1 v^2dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,s)\,dsdx.\end{aligned}\tag{4.10}$$

Substituting (4.10) in (4.9), and using (4.3), we obtain

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}} &\leq -2\int_0^L\delta_1(x)v_x^2dx-\left(\mu_0-\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|ds\right)\int_0^L v^2dx-\int_0^L\theta_x^2dx \\ &\leq 0.\end{aligned}$$

Hence, $\mathcal{A}$ is a dissipative operator.

Next, we prove that the operator $Id-\mathcal{A}$ is surjective.

Given $F=(f_1,f_2,f_3,f_4)^T\in\mathcal{H}$, we prove that there exists $U=(u,v,\theta,z)^T\in D(\mathcal{A})$ satisfying

$$(Id-\mathcal{A})U=F, \tag{4.11}$$

that is

$$\begin{cases} u-v=f_1, \\ (m(x)+\mu_0)v-(p(x)u_x+2\delta(x)v_x)_x+\gamma\theta_x+\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds \\ =m(x)f_2, \\ \theta-\theta_{xx}+\gamma v_x=f_3, \\ sz+z_\rho=sf_4. \end{cases} \tag{4.12}$$

Suppose that we have found u. Then, equation $(4.12)_1$ yield

$$v=u-f_1, \tag{4.13}$$

it is clear that $v\in H_0^1(0,L)$.

Equation $(4.12)_4$ with (4.13) and recall $z(x,0,t,s)=v$ yield

$$z(x,\rho,s)=u(x)e^{-\rho s}-f_1(x)e^{-\rho s}+se^{-\rho s}\int_0^\rho f_4(x,\tau,s)e^{\tau s}d\tau, \tag{4.14}$$

clearly, $z,z_\rho\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$. Inserting (4.13) and (4.14) into $(4.12)_2$, and inserting (4.13) into $(4.12)_3$, we get

$$\begin{cases} \eta_1 u-(p(x)u_x+2\delta(x)v_x)_x+\gamma\theta_x=g_1, \\ -\theta_{xx}+\theta+\gamma u_x=g_2, \\ u_x-v_x=g_3, \end{cases} \tag{4.15}$$

where

$$\begin{aligned} \eta_1 &= m(x)+\mu_0+\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds, \\ g_1 &= \eta_1 f_1+m(x)f_2-\int_{\tau_1}^{\tau_2}se^{-s}\mu(s)\int_0^1 f_4(x,\tau,s)e^{\tau s}d\tau ds, \\ g_2 &= f_3+\gamma f_{1x}, \\ g_3 &= f_{1x}. \end{aligned}$$

The variational formulation corresponding to Equation (4.15) takes the form

$$B\left((u,\theta)^T,\left(\widetilde{u},\widetilde{\theta}\right)^T\right)=G\left(\widetilde{u},\widetilde{\theta}\right)^T, \tag{4.16}$$

where $B : \left[H_0^1(0,L) \times L^2(0,L)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form given by

$$\begin{aligned} B\left((u,\theta)^T,\left(\widetilde{u},\widetilde{\theta}\right)^T\right) &= \eta_1 \int_0^L u\tilde{u}dx + \int_0^L (p(x)+2\delta(x))u_x\tilde{u}_x dx \\ &+\gamma \int_0^L \theta_x \tilde{u} dx + \int_0^L \theta_x \widetilde{\theta}_x dx + \int_0^L \theta\widetilde{\theta}dx \\ &+\gamma \int_0^L u_x \widetilde{\theta} dx, \end{aligned}$$

and $G : \left[H_0^1(0,L) \times L^2(0,L)\right] \longrightarrow \mathbb{R}$ is the linear form defined by

$$G\left(\widetilde{u},\widetilde{\theta}\right)^T = \int_0^L g_1\tilde{u}dx + \int_0^L g_2\widetilde{\theta}dx + \int_0^L 2\delta(x)g_3\tilde{u}_x dx.$$

Now, we introduce the Hilbert space $V = H_0^1(0,L) \times L^2(0,L)$, equipped with the norm

$$\|(u,\theta)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2.$$

It is clear that $B(.,.)$ and $G(.)$ are bounded. Furthermore, we can obtain that there exists a positive constant κ such that

$$\begin{aligned} B\left((u,\theta)^T,(u,\theta)^T\right) &= \eta_1 \int_0^L u^2 dx + \int_0^L (p(x)+2\delta(x))u_x^2 dx \\ &+ \int_0^L \theta^2 dx + \int_0^L \theta_x^2 dx, \\ &\geq \kappa \|(u,\theta)\|_V^2, \end{aligned}$$

which implies that $B(.,.)$ is coercive. Consequently, by the Lax–Milgram theorem, problem (4.16) has a unique solution $(u,\theta) \in H_0^1(0,L) \times L^2(0,L)$. Applying the classical elliptic regularity, it follows from (4.15) that $(u,\theta) \in \left(H_0^1(0,L) \cap H^2(0,L)\right)^2$. Hence, there exists a unique $U = (u,v,\theta,z)^T \in D(\mathcal{A})$ such that (4.11) is satisfied, the operator $Id - \mathcal{A}$ is surjective. At last, the result of Theorem 5.1 follows from the Lumer-Phillips theorem. □

4.3 Exponential stability of solution

In this section, we prove the exponential decay for system (4.6)-(4.7). It will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$E(t) = \frac{1}{2}\int_0^L \left[m(x)u_t^2 + p(x)u_x^2 + \theta^2\right]dx$$
$$+\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,t,s)\,dsd\rho dx. \quad (4.17)$$

We have the following exponentially stable result.

Theorem 4.2. *Let (u,v,θ,z) be the solution of (4.6)-(4.7) and assume (4.3) holds. Then there exists positive constants λ_0 and λ_1 such that the energy $E(t)$ satisfies*

$$E(t) \le \lambda_0 e^{-\lambda_1 t},\ t \ge 0. \quad (4.18)$$

To prove our this result, we will state some useful lemmas in advance.

Lemma 4.3. *(**Poincaré-type Scheeffer's inequality**, [39]): Let $h \in H_0^1(0,L)$. Then it holds*

$$\int_0^L |h|^2\,dx \le \frac{L^2}{\pi^2}\int_0^L |h_x|^2\,dx. \quad (4.19)$$

Lemma 4.4. *(**Mean value theorem,** [3]): Let (u,v,θ,z) be the solution to system (4.1)-(4.2), with an initial datum in $D(\mathcal{A})$. Then, for any $t>0$, there exists a sequence of real numbers (depending on t), denoted by $\zeta_i \in [0,L](i=1,...,6)$, such that*

$$\int_0^L p(x)u_x^2dx = p(\zeta_1)\int_0^L u_x^2dx, \quad \int_0^L m(x)u_t^2dx = m(\zeta_2)\int_0^L u_t^2dx,$$

$$\int_0^L m(x)u^2dx = m(\zeta_3)\int_0^L u^2dx, \quad \int_0^L \delta(x)u^2dx = \delta(\zeta_4)\int_0^L u^2dx,$$

$$\int_0^L \delta(x)u_x^2dx = \delta(\zeta_5)\int_0^L u_x^2dx, \quad \int_0^L \delta(x)u_{xt}^2dx = \delta(\zeta_6)\int_0^L u_{xt}^2dx.$$

Lemma 4.5. *Let (u,v,θ,z) be the solution of (4.6)-(4.7) and assume (4.3) holds. Then the energy functional defined by (4.17), satisfies the estimate*

$$E'(t) \le -2\int_0^L \delta(x)u_{xt}^2dx - \int_0^L \theta_x^2dx - \left(\mu_0 - \int_{\tau_1}^{\tau_2}\left|\mu(s)\right|ds\right)\int_0^L u_t^2dx \le 0, \quad (4.20)$$

for all $t \ge 0$.

Proof. A simple multiplication of Equations $(4.6)_1$ and $(4.6)_2$ by u_t and θ, respectively, and integrating over $(0,L)$, using integration by parts and the boundary conditions in (4.7), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \left[m(x)u_t^2+p(x)u_x^2+\theta^2\right]dx\\
= &-2\int_0^L \delta(x)u_{xt}^2dx-\int_0^L \theta_x^2dx-\mu_0\int_0^L u_t^2dx\\
&-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx. \quad (4.21)
\end{aligned}$$

On the other hand, multiplying $(4.6)_3$ by $\left|\mu(s)\right| z$, integrating the product over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $z(x,0,t,s)=u_t$, yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right|z^2(x,\rho,t,s)\,dsd\rho dx\\
= &-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,t,s)\,dsdx+\frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|dsdx. \quad (4.22)
\end{aligned}$$

A combination of (4.21) and (4.22) gives

$$\begin{aligned}
E'(t) = &-2\int_0^L \delta(x)u_{xt}^2dx-\int_0^L\theta_x^2dx-\left(\mu_0-\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|ds\right)\int_0^L u_t^2dx\\
&-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,t,s)\,dsdx\\
&-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx. \quad (4.23)
\end{aligned}$$

Now, using Young's inequality, the last term in (4.23) and using (4.3) give (4.20), which concludes the proof. □

Before defining a Lyapunov functional, we need some lemmas as follows.

Lemma 4.6. *Let (u,v,θ,z) be the solution of (4.6)-(4.7). Then the functions*

$$I_1(t):=\int_0^L \delta(x)u_x^2dx+\int_0^L m(x)u_tu\,dx, \quad (4.24)$$

satisfies, for all $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, the estimate

$$\begin{aligned}
I_1'(t) \le &-\left(p(\zeta_1)-\frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1-\gamma\varepsilon_2-\frac{L^2\varepsilon_3}{\pi^2}\right)\int_0^L u_x^2dx+\frac{\gamma}{\varepsilon_2}\int_0^L\theta^2dx\\
&+\left(m(\zeta_2)+\frac{1}{2\varepsilon_1}\right)\int_0^L u_t^2dx+\frac{\mu_0}{4\varepsilon_3}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|z^2(x,1,t,s)\,dsdx.
\end{aligned}$$

(4.25)

Proof. By differentiating $I_1(t)$ with respect to t, using $(4.7)_1$ and integrating by parts, we obtain

$$\begin{aligned} I_1'(t) &= -\int_0^L p(x)u_x^2dx - \mu_0\int_0^L u_t u dx + \gamma\int_0^L \theta u_x dx \\ &\quad -\int_0^L u\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx + \int_0^L m(x)u_t^2dx. \end{aligned}$$

By using Young's inequality, lemma 5.3 and (4.3), we get for $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$

$$-\mu_0\int_0^L u_t u dx \leq \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1\int_0^L u_x^2dx + \frac{1}{2\varepsilon_1}\int_0^L u_t^2dx, \tag{4.26}$$

$$\gamma\int_0^L \theta u_x dx \leq \gamma\varepsilon_2\int_0^L u_x^2dx + \frac{\gamma}{\varepsilon_2}\int_0^L \theta^2dx, \tag{4.27}$$

$$\begin{aligned} &-\int_0^L u\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)dsdx \\ \leq\ & \frac{L^2\varepsilon_3}{\pi^2}\int_0^L u_x^2dx + \frac{\mu_0}{4\varepsilon_3}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)dsdx. \end{aligned} \tag{4.28}$$

Consequently, using lemma 5.4, (4.26), (4.27) and (4.28), we establish (4.25). □

Lemma 4.7. *Let (u,v,θ,z) be the solution of (4.6)-(4.7). Then the functions*

$$I_2(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx, \tag{4.29}$$

satisfies, for some positive constant n_1, the estimate

$$\begin{aligned} I_2'(t) &\leq -n_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx \\ &\quad -n_1\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)dsdx + \mu_0\int_0^L u_t^2dx. \end{aligned} \tag{4.30}$$

Proof. By differentiating $I_2(t)$ with respect to t, and using the equation $(4.6)_3$, we obtain

$$\begin{aligned} I_2'(t) &= -2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu(s)|z(x,\rho,t,s)z_\rho(x,\rho,t,s)dsd\rho dx \\ &= -\frac{d}{d\rho}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx \\ &\quad -\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx. \end{aligned}$$

Hence

$$\begin{aligned} I_2'(t) &= -\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| [e^{-s} z^2(x,1,t,s) - z^2(x,0,t,s)] ds dx \\ &\quad - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} \left|\mu(s)\right| z^2(x,\rho,t,s) ds d\rho dx. \end{aligned}$$

Using the fact that $z(x,0,t,s) = u_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $\rho \in [0,1]$, we obtain

$$\begin{aligned} I_2'(t) &\leq -\int_0^L \int_{\tau_1}^{\tau_2} e^{-s} \left|\mu(s)\right| z^2(x,1,t,s) ds dx + \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| ds \int_0^L u_t^2 dx \\ &\quad - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} \left|\mu(s)\right| z^2(x,\rho,t,s) ds d\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$. Finally, setting $n_1 = e^{-\tau_2}$ and recalling (4.3), we obtain (4.30). □

Now, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

Lemma 4.8. *Let $N, N_2 > 0$, the functional defined by*

$$L(t) := NE(t) + I_1(t) + N_2 I_2(t). \tag{4.31}$$

For two positive constants α and β, we have

$$\alpha E(t) \leq L(t) \leq \beta E(t), \forall t \geq 0. \tag{4.32}$$

Proof. Now, let

$$\mathcal{L}(t) := I_1(t) + N_2 I_2(t).$$

Then

$$\begin{aligned} |\mathcal{L}(t)| &\leq \int_0^L \delta(x) u_x^2 dx + \int_0^L m(x) |u_t u| dx \\ &\quad + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu(s) e^{-s\rho}\right| z^2(x,\rho,s,t) ds d\rho dx. \end{aligned}$$

Exploiting Cauchy-Schwarz inequality, lemma 5.3, lemma 5.4, (4.17) and the fact that $e^{-s\rho} \leq 1$ for all $\rho \in [0,1]$, we obtain

$$|\mathcal{L}(t)| \leq c_0 E(t),$$

where

$$c_0 = 1 + \frac{L^2 m(\zeta_3)}{\pi^2 p(\zeta_1)} + \frac{2\delta(\zeta_5)}{p(\zeta_1)} + 2N_2.$$

Consequently, $|L(t) - NE(t)| \leq c_0 E(t)$, which yields

$$(N - c_0)E(t) \leq L(t) \leq (N + c_0)E(t).$$

Choosing N large enough, we obtain estimate (4.32). □

Now, we prove our main result in this section.

Proof. (Of Theorem 5.2)

By differentiating (4.31) and recalling (4.20), (4.25) and (4.30), we obtain

$$\begin{aligned} L'(t) \leq & -\left[\left(\mu_0 - \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| ds\right) N - \left(m(\zeta_2) + \frac{1}{2\varepsilon_1}\right) - \mu_0 N_2\right] \int_0^L u_t^2 dx \\ & -\left[\left(p(\zeta_1) - \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1 - \gamma\varepsilon_2 - \frac{L^2\varepsilon_3}{\pi^2}\right)\right] \int_0^L u_x^2 dx - N \int_0^L \theta_x^2 dx \\ & -2N \int_0^L \delta(x) u_{xt}^2 dx - n_1 N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,t,s) ds d\rho dx \\ & + \frac{\gamma}{\varepsilon_2} \int_0^L \theta^2 dx - \left[n_1 N_2 - \frac{\mu_0}{4\varepsilon_3}\right] \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| z^2(x,1,t,s) ds dx, \end{aligned}$$

using lemma 5.3 and lemma 5.4 gives

$$\begin{aligned} L'(t) \leq & -\left[\eta N - \frac{L^2}{\pi^2}\left(m(\zeta_2) + \frac{1}{2\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2} N_2\right] \int_0^L u_{tx}^2 dx \\ & -\left[\left(p(\zeta_1) - \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1 - \gamma\varepsilon_2 - \frac{L^2}{\pi^2}\varepsilon_3\right)\right] \int_0^L u_x^2 dx \\ & -\left[n_1 N_2 - \frac{\mu_0}{4\varepsilon_3}\right] \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| z^2(x,1,t,s) ds dx \\ & -n_1 N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,t,s) ds d\rho dx \\ & -\left(N - \frac{L^2\gamma}{\pi^2\varepsilon_2}\right) \int_0^L \theta_x^2 dx, \end{aligned} \tag{4.33}$$

where

$$\eta = \frac{L^2}{\pi^2}\left(\mu_0 - \int_{\tau_1}^{\tau_2} \left|\mu(s)\right| ds\right) + 2\delta(\zeta_6) > 0.$$

At this point, we need to choose our constants very carefully. First, we choose $\varepsilon_1 < \frac{\pi^2}{2L^2\mu_0^2} p(\zeta_1)$ and $\varepsilon_3 < \frac{\pi^2}{4L^2} p(\zeta_1)$ so that $p(\zeta_1) - \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1 - \frac{L^2}{\pi^2}\varepsilon_3 > \frac{p(\zeta_1)}{2}$.

Next, we select N_2 large enough so that $n_1 N_2 - \dfrac{\mu_0}{4\varepsilon_3} > 0$.

Then, we choose ε_2 small enough, satisfies $\dfrac{p(\zeta_1)}{2} - \gamma\varepsilon_2 > 0$.

Finally, we then choose N large enough so that

$$\eta N - \frac{L^2}{\pi^2}\left(m(\zeta_2) + \frac{1}{2\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2}N_2 > 0, \quad N - \frac{L^2\gamma}{\pi^2\varepsilon_2} > 0.$$

By (4.17), we deduce that there exist positive constant c_1 such that (4.33) becomes

$$L'(t) \leq -c_1 E(t), \ \forall t \geq 0, \tag{4.34}$$

using (4.32), we have

$$L'(t) \leq -\lambda_1 L(t), \ \forall t \geq 0, \tag{4.35}$$

where $\lambda_1 = \dfrac{c_1}{\beta}$. Then, a simple integration of (4.35) over $(0,t)$ leads to

$$L(t) \leq L(0)e^{-\lambda_1 t}, \ \forall t \geq 0. \tag{4.36}$$

Combining (4.32) and (4.36) we obtain (4.18) with $\lambda_0 = \dfrac{\beta E(0)}{\alpha}$. Hence, the proof is complete. $\square$

Chapter 5

Coupled flexible structure system with distributed delay

Contents

5.1 Presentation of the problem

In the present chapter, we consider the coupled flexible structure system with distributed delay in the two equations (See [12]). The system is written as

$$\begin{cases} m_1(x)u_{tt} - (p_1(x)u_x + 2\delta_1(x)u_{xt})_x + \mu_0 u_t + \int_{\tau_1}^{\tau_2} \mu_1(s)u_t(x,t-s)\,ds = 0, \\ m_2(x)v_{tt} - (p_2(x)v_x + 2\delta_2(x)v_{xt})_x + \mu_0' v_t + \int_{\tau_1}^{\tau_2} \mu_2(s)v_t(x,t-s)\,ds = 0, \end{cases} \tag{5.1}$$

where $(x,t) \in (0,L)\times(0,+\infty)$, with the following initial and boundary conditions:

$$\begin{aligned} & u(.,0) = u_0(x),\ u_t(.,0) = u_1(x),\quad \forall x \in [0,L] \\ & u(0,t) = u(L,t) = 0,\ \forall t \geq 0, \\ & v(.,0) = v_0(x),\ v_t(.,0) = v_1(x),\quad \forall x \in [0,L] \\ & v(0,t) = v(L,t) = 0,\ \forall t \geq 0, \\ & u_t(x,-t) = f_0(x,t),\quad 0 < t \leq \tau_2, \\ & v_t(x,-t) = g_0(x,t),\quad 0 < t \leq \tau_2, \end{aligned} \tag{5.2}$$

where $u = u(x,t)$, $v = v(x,t)$ are the displacements of a particle at position $x \in (0,L)$ and time $t > 0$. u_0, v_0 are initial data, and f_0, g_0 are the history function. The parameters $m_i(x)$, $\delta_i(x)$ and $p_i(x)$ (for $i = 1,2$) are responsible for the non-uniform structure of the body, where $m_i(x)$ denote mass per unit length of structure, $\delta_i(x)$ coefficient of internal material damping and $p_i(x)$ a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m_i(x), \delta_i(x)$ and $p_i(x)$ in [3] such that:

$$m_i, \delta_i, p_i \in W^{1,\infty}(0,L),\quad m_i(x), \delta_i(x), p_i(x) > 0,\quad \forall x \in [0,L],\ \text{for } i = 1,2.$$

The coefficients μ_0, μ_0' are positive constants, and $\mu_1, \mu_2 : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption:

$$\begin{cases} \mu_0 > \int_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds, \\ \mu_0' > \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds. \end{cases} \tag{5.3}$$

The rest of the chapter is organized as follows. In Section 5.2, we state and prove the well-posedness of system (5.1)-(5.2) by using semigroup method. In Section 5.3, we establish an exponential stability by using the perturbed energy method and construct some Lyapunov functionals.

5.2 Well-posedness of the problem

In this section, we give a brief idea about the existence and uniqueness of solutions for (5.1)-(5.2) using the semigroup theory [47]. We introduce as in [45] the new variable

$$z_1(x,\rho,t,s) = u_t(x,t-\rho s),\ x \in (0,L), \rho \in (0,1), s \in (\tau_1,\tau_2), t > 0. \tag{5.4}$$

$$z_2(x,\rho,t,s) = v_t(x,t-\rho s),\ x \in (0,L), \rho \in (0,1), s \in (\tau_1,\tau_2), t > 0. \tag{5.5}$$

Then, we have

$$sz_{it}(x,\rho,t,s) + z_{i\rho}(x,\rho,t,s) = 0 \text{ in } (0,L)\times(0,1)\times(0,\infty)\times(\tau_1,\tau_2),\ \text{for } i = 1,2. \tag{5.6}$$

Therefore, problem (5.1) takes the form

$$\begin{cases} m_1(x)u_{tt} - (p_1(x)u_x + 2\delta_1(x)u_{xt})_x + \mu_0 u_t + \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)ds = 0, \\ sz_{1t}(x,\rho,t,s) + z_{1\rho}(x,\rho,t,s) = 0, \\ m_2(x)v_{tt} - (p_2(x)v_x + 2\delta_2(x)v_{xt})_x + \mu_0' v_t + \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)ds = 0, \\ sz_{2t}(x,\rho,t,s) + z_{2\rho}(x,\rho,t,s) = 0, \end{cases} \tag{5.7}$$

with the following initial and boundary conditions

$$\begin{cases} u(.,0) = u_0(x),\ u_t(.,0) = u_1(x),\ \ \forall x \in [0,L], \\ u(0,t) = u(L,t) = 0,\ \forall t \geq 0, \\ v(.,0) = v_0(x),\ v_t(.,0) = v_1(x),\ \ \forall x \in [0,L], \\ v(0,t) = v(L,t) = 0,\ \forall t \geq 0, \\ z_1(x,0,t,s) = u_t(x,t) \text{ on } (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z_2(x,0,t,s) = v_t(x,t) \text{ on } (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z_1(x,\rho,0,s) = f_0(x,\rho s) \text{ on } (0,L)\times(0,1)\times(\tau_1,\tau_2), \\ z_2(x,\rho,0,s) = g_0(x,\rho s) \text{ on } (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{5.8}$$

Introducing the vector function $U = (u,\varphi,z_1,v,\psi,z_2)^T$, where $\varphi = u_t$ and $\psi = v_t$, system (5.7)-(5.8) can be written as

$$\begin{cases} \dfrac{dU(t)}{dt} + \mathcal{A}U(t) = 0,\ \ t > 0, \\ U(0) = U_0 = (u_0,u_1,f_0,v_0,v_1,g_0)^T, \end{cases} \tag{5.9}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -\varphi \\ -\frac{1}{m_1(x)}(p_1(x)u_x + 2\delta_1(x)\varphi_x)_x + \frac{\mu_0}{m_1(x)}\varphi + \frac{1}{m_1(x)}\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,t,s)ds \\ s^{-1}z_{1\rho} \\ -\psi \\ -\frac{1}{m_2(x)}(p_2(x)v_x + 2\delta_2(x)\psi_x)_x + \frac{\mu_0'}{m_2(x)}\psi + \frac{1}{m_2(x)}\int_{\tau_1}^{\tau_2}\mu_2(s)z_2(x,1,t,s)ds \\ s^{-1}z_{2\rho} \end{pmatrix}.$$

Next, we define the energy space as

$$\begin{aligned} \mathcal{H} &= H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)) \\ &\quad \times H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned}$$

equipped with the inner product

$$\begin{aligned} \left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^L p_1(x)u_x\widetilde{u}_x dx + \int_0^L m_1(x)\varphi\widetilde{\varphi}dx \\ &+ \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_1(s)\right| z_1(x,\rho,s)\widetilde{z}_1(x,\rho,s)\,dsd\rho dx \\ &+ \int_0^L p_2(x)v_x\widetilde{v}_x dx + \int_0^L m_2(x)\psi\widetilde{\psi}dx \\ &+ \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z_2(x,\rho,s)\widetilde{z}_2(x,\rho,s)\,dsd\rho dx. \end{aligned}$$

Then, the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U\in\mathcal{H} \mid u,v\in H^2(0,L)\cap H_0^1(0,L),\ \varphi,\psi\in H_0^1(0,L), \\ z_1,z_{1\rho},z_2,z_{2\rho}\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)), \\ z_1(x,0,s)=\varphi(x), z_2(x,0,s)=\psi(x) \end{array} \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

The well-posedness of problem (5.9) is ensured by

Theorem 5.1. *Assume that $U_0 \in \mathcal{H}$ and (5.3) holds, then problem (5.9) exists a unique solution $U \in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}^+;D(\mathcal{A}))\cap C^1(\mathbb{R}^+;\mathcal{H}).$$

Proof. The result follows from Lumer-Phillips theorem provided we prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $U = (u, \varphi, z_1, v, \psi, z_2)^T \in D(\mathcal{A})$, by using the inner product and integrating by parts, we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= 2\int_0^L \delta_1(x)\varphi_x^2 dx + \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)dsdx \\ &+\mu_0 \int_0^L \varphi^2 dx + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1(x,\rho,s) z_{1\rho}(x,\rho,s)\, ds d\rho dx \\ &+2\int_0^L \delta_2(x)\psi_x^2 dx + \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)dsdx \\ &+\mu_0' \int_0^L \psi^2 dx + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2(x,\rho,s) z_{2\rho}(x,\rho,s)\, ds d\rho dx.\end{aligned} \tag{5.10}$$

Integrating by parts in ρ, we have

$$\begin{aligned}&\int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_i(s)| z_i(x,\rho,s) z_{i\rho}(x,\rho,s)\, d\rho ds dx \\ = &\ \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_i(s)| \left[z_i^2(x,1,s) - z_i^2(x,0,s)\right] dsdx, \text{ for } i = 1,2.\end{aligned}$$

Using the fact that $z_1(x,0,s,t) = \varphi$ and $z_2(x,0,s,t) = \psi$, we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= 2\int_0^L \delta_1(x)\varphi_x^2 dx + \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)dsdx \\ &+\left(\mu_0 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_1(s)| ds\right)\int_0^L \varphi^2 dx \\ &+\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s)\, dsdx \\ &+2\int_0^L \delta_2(x)\psi_x^2 dx + \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)dsdx \\ &+\left(\mu_0' - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right)\int_0^L \psi^2 dx \\ &+\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s)\, dsdx.\end{aligned} \tag{5.11}$$

Now, using Young's inequality, we can estimate

$$\begin{aligned}&\int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)dsdx \\ \geq &\ -\frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_1(s)| ds \int_0^1 \varphi^2 dx - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s)\, dsdx,\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}&\int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\,ds\,dx \\ \geq\ & -\frac{1}{2}\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds \int_0^1 \psi^2 dx - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s)\,ds\,dx. \end{aligned} \tag{5.13}$$

Substituting (5.12) and (5.13) in (5.11), and using (5.3), we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\geq 2\int_0^L \delta_1(x)\varphi_x^2 dx + \left(\mu_0 - \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| ds\right)\int_0^L \varphi^2 dx \\ &\quad +2\int_0^L \delta_2(x)\psi_x^2 dx + \left(\mu_0' - \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds\right)\int_0^L \psi^2 dx \\ &\geq 0.\end{aligned}$$

Hence, $\mathcal{A}$ is monotone. Next, we prove that the operator $I+\mathcal{A}$ is surjective, that is, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, there exists $U = (u, \varphi, z_1, v, \psi, z_2)^T \in D(\mathcal{A})$ satisfying

$$(I+\mathcal{A})\,U = F \tag{5.14}$$

which is equivalent to

$$\begin{cases} u - \varphi = f_1, \\ \left(m_1(x) + \mu_0\right)\varphi - \left(p_1(x)u_x + 2\delta_1(x)\varphi_x\right)_x + \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\,ds = m_1(x) f_2, \\ s z_1 + z_{1\rho} = s f_3, \\ v - \psi = f_4, \\ \left(m_2(x) + \mu_0'\right)\psi - \left(p_2(x)v_x + 2\delta_2(x)\psi_x\right)_x + \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\,ds = m_2(x) f_5, \\ s z_2 + z_{2\rho} = s f_6. \end{cases} \tag{5.15}$$

Suppose that we have found u and v. Then, equations $(5.15)_1$ and $(5.15)_4$ yield

$$\begin{cases} \varphi = u - f_1, \\ \psi = v - f_4. \end{cases} \tag{5.16}$$

It is clear that $\varphi \in H_0^1(0,L)$ and $\psi \in H_0^1(0,L)$. Equations $(5.15)_3$ and $(5.15)_6$ with (5.16) and recall $z_1(x,0,t,s) = \varphi$, $z_2(x,0,t,s) = \psi$ yield

$$z_1(x,\rho,s) = u(x)e^{-\rho s} - f_1(x)e^{-\rho s} + s e^{-\rho s}\int_0^\rho f_3(x,\tau,s)e^{\tau s}\,d\tau, \tag{5.17}$$

and

$$z_2(x,\rho,s) = v(x)e^{-\rho s} - f_4(x)e^{-\rho s} + s e^{-\rho s}\int_0^\rho f_6(x,\tau,s)e^{\tau s}\,d\tau. \tag{5.18}$$

Clearly, $z_1, z_{1\rho}, z_2, z_{2\rho} \in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$. Inserting $(5.16)_1$ and (5.17) into $(5.15)_2$, and inserting $(5.16)_2$ and (5.18) into $(5.15)_5$, we get

$$\begin{cases} \eta_1 u - (p_1(x)u_x + 2\delta_1(x)\varphi_x)_x = g_1, \\ \eta_2 v - (p_2(x)v_x + 2\delta_2(x)\psi_x)_x = g_2, \\ u_x - \varphi_x = g_3, \\ v_x - \psi_x = g_4, \end{cases} \tag{5.19}$$

where

$$\begin{aligned} \eta_1 &= m_1(x) + \mu_0 + \textstyle\int_{\tau_1}^{\tau_2} \mu_1(s)e^{-s}ds, \\ \eta_2 &= m_2(x) + \mu_0' + \textstyle\int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}ds, \\ g_1 &= \eta_1 f_1 + m_1(x) f_2 - \textstyle\int_{\tau_1}^{\tau_2} s\mu_1(s)e^{-s} \int_0^1 f_3(x,\tau,s)e^{\tau s}d\tau ds, \\ g_2 &= \eta_2 f_4 + m_2(x) f_5 - \textstyle\int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s} \int_0^1 f_6(x,\tau,s)e^{\tau s}d\tau ds, \\ g_3 &= f_{1x}, \\ g_4 &= f_{4x}. \end{aligned}$$

The variational formulation corresponding to Equation (5.19) takes the form

$$B\left((u,v)^T, (\widetilde{u},\widetilde{v})^T\right) = G(\widetilde{u},\widetilde{v})^T, \tag{5.20}$$

where

$$B : \left[H_0^1(0,L) \times H_0^1(0,L)\right]^2 \longrightarrow \mathbb{R}$$

is the bilinear form given by

$$\begin{aligned} B\left((u,v)^T, (\widetilde{u},\widetilde{v})^T\right) &= \eta_1 \int_0^L u\tilde{u}dx + \int_0^L (p_1(x) + 2\delta_1(x))u_x\tilde{u}_x dx \\ &\quad + \eta_2 \int_0^L v\tilde{v}dx + \int_0^L (p_2(x) + 2\delta_2(x))v_x\tilde{v}_x dx, \end{aligned}$$

and

$$G : \left[H_0^1(0,L) \times H_0^1(0,L)\right] \longrightarrow \mathbb{R}$$

is the linear form defined by

$$G(\widetilde{u},\widetilde{v})^T = \int_0^L g_1\tilde{u}dx + \int_0^L g_2\tilde{v}dx + \int_0^L 2\delta_1(x)g_3\tilde{u}_x dx + \int_0^L 2\delta_2(x)g_4\tilde{v}_x dx.$$

Now, we introduce the Hilbert space $V = H_0^1(0,L) \times H_0^1(0,L)$, equipped with the norm

$$\|(u,v)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|v\|_2^2 + \|v_x\|_2^2.$$

It is clear that $B(.,.)$ and $G(.)$ are bounded. Furthermore, we can obtain that there exists a positive constant α such that

$$\begin{aligned} & B\Big((u,v)^T,(u,v)^T\Big) \\ = \ & \eta_1\int_0^L u^2dx+\int_0^L (p_1(x)+2\delta_1(x))u_x^2dx+\eta_2\int_0^L v^2dx+\int_0^L (p_2(x)+2\delta_2(x))v_x^2dx \\ \geq \ & \alpha\|(u,v)\|_V^2. \end{aligned}$$

Which implies that $B(.,.)$ is coercive.

Consequently, applying the Lax–Milgram lemma, we obtain that (5.20) has a unique solution $(u,v)^T \in V$.

Then, by substituting u,v into (5.16), we obtain

$$\varphi,\psi \in H_0^1(0,L).$$

Next, it remains to show that

$$u,v \in H^2(0,L)\cap H_0^1(0,L).$$

Furthermore, if $\widetilde{v}\equiv 0 \in H_0^1(0,L)$, then (5.20) reduces to

$$-\int_0^L [(p_1(x)+2\delta_1(x))u_x]_x\,\tilde{u}dx = \int_0^L g_1\tilde{u}dx-\int_0^L 2(\delta_1(x)g_3)_x\,\tilde{u}dx-\eta_1\int_0^L u\tilde{u}dx,$$

for all $\widetilde{u}$ in $H_0^1(0,L)$, which implies

$$[(p_1(x)+2\delta_1(x))u_x]_x = \eta_1 u - g_1 + 2(\delta_1(x)g_3)_x \in L^2(0,L).$$

Thus, by the L^2 theory for the linear elliptic equations, we obtain that

$$u \in H^2(0,L)\cap H_0^1(0,L).$$

By a similar way, we have

$$v \in H^2(0,L)\cap H_0^1(0,L).$$

Finally, the application of the classical regularity theory for the linear elliptic equations guarantees the existence of unique solution $U\in D(\mathcal{A})$ which satisfies (5.14). Therefore, the operator $\mathcal{A}$ is maximal.

Hence, the result of Theorem 5.1 follows. □

5.3 Exponential stability of solution

In this section, we prove the exponential decay for problem (5.7)-(5.8). It will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$\begin{aligned} &E(t) = E_1(t) + E_2(t), \\ &E_1(t) = \tfrac{1}{2}\int_0^L \left[m_1(x) u_t^2 + p_1(x) u_x^2 \right] dx + \tfrac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_1(s)\right| z_1^2(x,\rho,z,t)\, ds d\rho dx, \\ &E_2(t) = \tfrac{1}{2}\int_0^L \left[m_2(x) u_t^2 + p_2(x) u_x^2 \right] dx + \tfrac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_2(s)\right| z_2^2(x,\rho,z,t)\, ds d\rho dx. \end{aligned} \tag{5.21}$$

We have the following exponentially stable result.

Theorem 5.2. *Let (u,u_t,z_1,v,v_t,z_2) be the solution of (5.7)-(5.8) and assume (5.3) holds. Then there exists positive constants λ_0 and λ_1 such that the energy $E(t)$ associated with problem (5.7)-(5.8) satisfies*

$$E(t) \le \lambda_0 e^{-\lambda_1 t},\ t \ge 0. \tag{5.22}$$

To prove our this result, we will state and prove some useful lemmas in advance.

Lemma 5.3. *(**Poincaré-type Scheeffer's inequality**, [39]): Let $h \in H_0^1(0,L)$. Then it holds*

$$\int_0^L |h|^2\, dx \le \frac{L^2}{\pi^2} \int_0^L |h_x|^2\, dx. \tag{5.23}$$

Lemma 5.4. *(**Mean value theorem,** [3]): Let (u,u_t,v,v_t) be the solution to system (5.1)-(5.2), with an initial datum in $D(\mathcal{A})$. Then, for any $t>0$, there exists a sequence of real numbers (depending on t), denoted by $\zeta_i, \xi_i \in [0,L] (i=1,...,6)$, such that:*

$$\int_0^L p_1(x) u_x^2 dx = p_1(\zeta_1) \int_0^L u_x^2 dx, \quad \int_0^L m_1(x) u_t^2 dx = m_1(\zeta_2) \int_0^L u_t^2 dx,$$

$$\int_0^L m_1(x) u^2 dx = m_1(\zeta_3) \int_0^L u^2 dx, \quad \int_0^L \delta_1(x) u^2 dx = \delta_1(\zeta_4) \int_0^L u^2 dx,$$

$$\int_0^L \delta_1(x) u_x^2 dx = \delta_1(\zeta_5) \int_0^L u_x^2 dx, \quad \int_0^L \delta_1(x) u_{xt}^2 dx = \delta_1(\zeta_6) \int_0^L u_{xt}^2 dx,$$

$$\int_0^L p_2(x) v_x^2 dx = p_2(\xi_1) \int_0^L v_x^2 dx, \quad \int_0^L m_2(x) v_t^2 dx = m_2(\xi_2) \int_0^L v_t^2 dx,$$

$$\int_0^L m_2(x)v^2dx = m_2(\xi_3)\int_0^L v^2dx, \quad \int_0^L \delta_2(x)v^2dx = \delta_2(\xi_4)\int_0^L v^2dx,$$
$$\int_0^L \delta_2(x)v_x^2dx = \delta_2(\xi_5)\int_0^L v_x^2dx, \quad \int_0^L \delta_2(x)v_{xt}^2dx = \delta_2(\xi_6)\int_0^L v_{xt}^2dx.$$

Lemma 5.5. *Let (u,u_t,z_1,v,v_t,z_2) be the solution of (5.7)-(5.8). Then the energy functional satisfies*

$$\begin{aligned} &E'(t) = E_1'(t) + E_2'(t) \le 0, \ \ \forall t \ge 0, \\ &E_1'(t) \le -2\int_0^L \delta_1(x)u_{xt}^2dx + \left(\int_{\tau_1}^{\tau_2}\left|\mu_1(s)\right|ds - \mu_0\right)\int_0^L u_t^2dx \le 0, \\ &E_2'(t) \le -2\int_0^L \delta_2(x)v_{xt}^2dx + \left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds - \mu_0'\right)\int_0^L v_t^2dx \le 0. \end{aligned} \tag{5.24}$$

Proof. Multiplying $(5.7)_1$ and $(5.7)_3$ by u_t and v_t, respectively, and integrating over $(0,L)$, using integration by parts and the boundary conditions in (5.8), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L\left[m_1(x)u_t^2 + p_1(x)u_x^2\right]dx \\ = \ &-2\int_0^L \delta_1(x)u_{xt}^2dx - \mu_0\int_0^L u_t^2dx - \int_0^L u_t\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,t,s)dsdx, \end{aligned} \tag{5.25}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L\left[m_2(x)v_t^2 + p_2(x)v_x^2\right]dx \\ = \ &-2\int_0^L \delta_2(x)v_{xt}^2dx - \mu_0'\int_0^L v_t^2dx - \int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z_2(x,1,t,s)dsdx. \end{aligned} \tag{5.26}$$

On the other hand, multiplying $(5.7)_2$ and $(5.7)_4$ by $\left|\mu_1(s)\right| z_1$ and $\left|\mu_2(s)\right| z_2$, respectively, and integrating over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, and recall $z_1(x,0,t,s) = u_t$ and $z_2(x,0,t,s) = v_t$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_1(s)\right|z_1^2(x,\rho,s,t)\,dsd\rho dx \\ = \ &-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu_1(s)\right|z_1^2(x,1,s,t)\,dsdx + \frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2}\left|\mu_1(s)\right|dsdx, \end{aligned} \tag{5.27}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|z_2^2(x,\rho,s,t)\,dsd\rho dx \\ = \ &-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|z_2^2(x,1,s,t)\,dsdx + \frac{1}{2}\int_0^L v_t^2\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|dsdx. \end{aligned} \tag{5.28}$$

A combination of (5.25) and (5.27) gives

$$\begin{aligned} E_1'(t) &= -2\int_0^L \delta_1(x)\, u_{xt}^2 dx - \mu_0 \int_0^L u_t^2 dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) ds dx \\ &\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s,t)\, ds dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| ds \int_0^L u_t^2 dx. \end{aligned} \tag{5.29}$$

Also, (5.26) and (5.28) gives

$$\begin{aligned} E_2'(t) &= -2\int_0^L \delta_2(x)\, v_{xt}^2 dx - \mu_0' \int_0^L v_t^2 dx - \int_0^L v_t \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s) ds dx \\ &\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s,t)\, ds dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds \int_0^L v_t^2 dx. \end{aligned} \tag{5.30}$$

Now, using Young's inequality, we obtain

$$\begin{aligned} & -\int_0^L u_t \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) ds dx \\ \leq\ & \frac{1}{2}\int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| ds \int_0^1 u_t^2 dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s)\, ds dx, \end{aligned} \tag{5.31}$$

$$\begin{aligned} & -\int_0^L v_t \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s) ds dx \\ \leq\ & \frac{1}{2}\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds \int_0^1 v_t^2 dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s)\, ds dx. \end{aligned} \tag{5.32}$$

Substitution of (5.35) into (5.29), (5.36) into (5.34) and using (5.3) give (5.24), which concludes the proof. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 5.6. *Let $(u, u_t, z_1, v, v_t, z_2)$ be the solution of (5.7)-(5.8). Then the functions*

$$I_1(t) := \int_0^L \delta_1(x)\, u_x^2 dx + \int_0^L m_1(x)\, u_t u\, dx, \tag{5.33}$$

$$F_1(t) := \int_0^L \delta_2(x)\, v_x^2 dx + \int_0^L m_2(x)\, v_t v\, dx, \tag{5.34}$$

satisfies, for all $\varepsilon_1, \varepsilon_2 > 0$ *and* $\varepsilon_1', \varepsilon_2' > 0$, *the estimates*

$$\begin{aligned} I_1'(t) &\leq -\left(p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1 - \frac{L^2\varepsilon_2}{\pi^2}\right)\int_0^L u_x^2 dx + \left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right)\int_0^L u_t^2 dx \\ &\quad + \frac{\mu_0}{4\varepsilon_2}\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s,t)\,dsdx, \end{aligned} \tag{5.35}$$

$$\begin{aligned} F_1'(t) &\leq -\left(p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\varepsilon_1' - \frac{L^2\varepsilon_2'}{\pi^2}\right)\int_0^L v_x^2 dx + \left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right)\int_0^L v_t^2 dx \\ &\quad + \frac{\mu_0'}{4\varepsilon_2'}\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s,t)\,dsdx. \end{aligned} \tag{5.36}$$

Proof. By differentiating $I_1(t)$ with respect to t, using $(5.7)_1$ and integrating by parts, we obtain

$$\begin{aligned} I_1'(t) &= -\int_0^L p_1(x)u_x^2 dx - \mu_0\int_0^L u_t u\,dx - \int_0^L u\int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,s,t)\,dsdx \\ &\quad + \int_0^L m_1(x)u_t^2 dx. \end{aligned}$$

By using Young's inequality, lemma 5.3 and $(5.3)_1$, we get for $\varepsilon_1, \varepsilon_2 > 0$

$$-\mu_0\int_0^L u_t u\,dx \leq \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1\int_0^L u_x^2 dx + \frac{1}{4\varepsilon_1}\int_0^L u_t^2 dx, \tag{5.37}$$

$$\begin{aligned} &-\int_0^L u\int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,s,t)\,dsdx \\ &\leq \frac{L^2\varepsilon_2}{\pi^2}\int_0^L u_x^2 dx + \frac{\mu_0}{4\varepsilon_2}\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s,t)\,dsdx. \end{aligned} \tag{5.38}$$

Consequently, using Lemma (5.4), (5.37) and (5.38), we establish (5.35).

Similarly, we get (5.36). □

Lemma 5.7. *Let* $(u, u_t, z_1, v, v_t, z_2)$ *be the solution of (5.7)-(5.8). Then the functions*

$$I_2(t) := \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu_1(s)\right| z_1^2(x,\rho,s,t)\,dsd\rho dx, \tag{5.39}$$

$$F_2(t) := \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu_2(s)\right| z_2^2(x,\rho,s,t)\,dsd\rho dx, \tag{5.40}$$

satisfies, for some positive constants n_1 *and* n_2, *the estimates*

$$\begin{aligned} I_2'(t) &\leq -n_1\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_1(s)\right| z_1^2(x,\rho,s,t)\,dsd\rho dx \\ &\quad -n_1\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s,t)\,dsdx + \mu_0\int_0^L u_t^2 dx, \end{aligned} \tag{5.41}$$

$$\begin{aligned} F_2'(t) \leq & -n_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_2(s)\right| z_2^2(x,\rho,s,t) ds d\rho dx \\ & -n_2 \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s,t) ds dx + \mu_0' \int_0^L v_t^2 dx. \end{aligned} \tag{5.42}$$

Proof. By differentiating $I_2(t)$ with respect to t, and using the equation $(5.7)_2$, we obtain

$$\begin{aligned} I_2'(t) &= -2\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} \left|\mu_1(s)\right| z_1(x,\rho,s,t) z_{1\rho}(x,\rho,s,t) ds d\rho dx \\ &= -\frac{d}{d\rho} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} \left|\mu_1(s)\right| z_1^2(x,\rho,s,t) ds d\rho dx \\ & \quad - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} \left|\mu_1(s)\right| z_1^2(x,\rho,s,t) ds d\rho dx \\ &= -\int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| [e^{-s} z_1^2(x,1,s,t) - z_1^2(x,0,s,t)] ds dx \\ & \quad - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} \left|\mu_1(s)\right| z_1^2(x,\rho,s,t) ds d\rho dx. \end{aligned}$$

Using the fact that $z_1(x,0,s,t) = u_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} I_2'(t) \leq & -\int_0^L \int_{\tau_1}^{\tau_2} e^{-s} \left|\mu_1(s)\right| z_1^2(x,1,s,t) ds dx + \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| ds \int_0^L u_t^2 dx \\ & - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} \left|\mu_1(s)\right| z_1^2(x,\rho,s,t) ds d\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$.

Finally, setting $n_1 = e^{-\tau_2}$and recalling $(5.3)_1$, we obtain (5.41).

Similarly, we get (5.36). □

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

Lemma 5.8. *Let $N, N_1, N_2 > 0$, the functional defined by*

$$L(t) := NE(t) + I_1(t) + N_1 I_2(t) + F_1(t) + N_2 F_2(t). \tag{5.43}$$

For two positive constants c_1 and c_2, we have

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \forall t \geq 0. \tag{5.44}$$

Proof. Now, let

$$\mathcal{L}(t) := I_1(t) + N_1 I_2(t) + F_1(t) + N_2 F_2(t).$$

Then

$$\begin{aligned}
|\mathcal{L}(t)| \leq\ & \int_0^L \delta_1(x)\, u_x^2 dx + \frac{1}{2}\int_0^L m_1(x)\, u_t^2 dx + \frac{1}{2}\int_0^L m_1(x)\, u^2 dx \\
& + N_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_1(s)\right| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx \\
& + \int_0^L \delta_2(x)\, v_x^2 dx + \frac{1}{2}\int_0^L m_2(x)\, v_t^2 dx + \frac{1}{2}\int_0^L m_2(x)\, v^2 dx \\
& + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z_2^2(x,\rho,s,t)\, ds\, d\rho\, dx \\
\leq\ & c' E_1(t) + c'' E_2(t) \leq c_0 E(t),
\end{aligned}$$

where $c_0 = \max\{c', c''\}$, with

$$c' = 1 + \frac{L^2 m_1(\zeta_3)}{\pi^2 p_1(\zeta_1)} + \frac{2\delta_1(\zeta_5)}{p_1(\zeta_1)} + 2N_1,\ c'' = 1 + \frac{L^2 m_2(\xi_3)}{\pi^2 p_2(\xi_1)} + \frac{2\delta_2(\xi_5)}{p_2(\xi_1)} + 2N_2.$$

Consequently, $|L(t) - NE(t)| \leq c_0 E(t)$, which yields

$$(N - c_0) E(t) \leq L(t) \leq (N + c_0) E(t).$$

Choosing N large enough, we obtain estimate (5.44). □

Now, we prove our main result in this section.

Proof. (Of Theorem 5.2)

By differentiating (5.43) and recalling (5.24), (5.35), (5.36), (5.41) and (5.42), we obtain

$$\begin{aligned}
L'(t) \leq & \left[\left(\int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| ds - \mu_0\right) N + \left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right) + N_1 \mu_0\right] \int_0^L u_t^2 dx \\
& - \left[p_1(\zeta_1) - \frac{L^2 \mu_0^2}{\pi^2} \varepsilon_1 - \frac{L^2}{\pi^2} \varepsilon_2\right] \int_0^L u_x^2 dx - 2N \int_0^L \delta_1(x) u_{xt}^2 dx \\
& - n_1 N_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_1(s)\right| z_1^2(x,\rho,s,t) ds d\rho dx \\
& - \left[n_1 N_1 - \frac{\mu_0}{4\varepsilon_2}\right] \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s,t) ds dx \\
& + \left[\left(\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds - \mu_0'\right) N + \left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right) + N_2 \mu_0'\right] \int_0^L v_t^2 dx \\
& - \left[p_2(\xi_1) - \frac{L^2 \mu_0'^2}{\pi^2} \varepsilon_1' - \frac{L^2}{\pi^2} \varepsilon_2'\right] \int_0^L v_x^2 dx - 2N \int_0^L \delta_2(x) v_{xt}^2 dx \\
& - n_2 N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_2(s)\right| z_2^2(x,\rho,s,t) ds d\rho dx \\
& - \left[n_2 N_2 - \frac{\mu_0'}{4\varepsilon_2'}\right] \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s,t) ds dx,
\end{aligned}$$

using Lemma (5.3) and Lemma (5.4) gives

$$\begin{aligned}
L'(t) \leq & -\left[\gamma_1 N - \frac{L^2}{\pi^2}\left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right) - \frac{L^2 \mu_0}{\pi^2} N_1\right] \int_0^L u_{tx}^2 dx \\
& - \left[p_1(\zeta_1) - \frac{L^2 \mu_0^2}{\pi^2} \varepsilon_1 - \frac{L^2}{\pi^2} \varepsilon_2\right] \int_0^L u_x^2 dx \\
& - n_1 N_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_1(s)\right| z_1^2(x,\rho,s,t) ds d\rho dx \\
& - \left[n_1 N_1 - \frac{\mu_0}{4\varepsilon_2}\right] \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| z_1^2(x,1,s,t) ds dx \\
& - \left[\gamma_2 N - \frac{L^2}{\pi^2}\left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right) - \frac{L^2 \mu_0'}{\pi^2} N_2\right] \int_0^L v_{tx}^2 dx \\
& - \left[p_2(\xi_1) - \frac{L^2 \mu_0'^2}{\pi^2} \varepsilon_1' - \frac{L^2}{\pi^2} \varepsilon_2'\right] \int_0^L v_x^2 dx \\
& - n_2 N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_2(s)\right| z_2^2(x,\rho,s,t) ds d\rho dx \\
& - \left[n_2 N_2 - \frac{\mu_0'}{4\varepsilon_2'}\right] \int_0^L \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z_2^2(x,1,s,t) ds dx, \qquad (5.45)
\end{aligned}$$

where

$$\gamma_1 = 2\delta_1(\zeta_6) - \frac{L^2}{\pi^2}\left(\int_{\tau_1}^{\tau_2} \left|\mu_1(s)\right| ds - \mu_0\right) > 0,$$
$$\gamma_2 = 2\delta_2(\xi_6) - \frac{L^2}{\pi^2}\left(\int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| ds - \mu_0'\right) > 0.$$

At this point, we need to choose our constants very carefully.

First, we choose $\varepsilon_2 < \frac{\pi^2}{2L^2}p_1(\zeta_1)$ and $\varepsilon_2' < \frac{\pi^2}{2L^2}p_2(\xi_1)$ so that

$$p_1(\zeta_1) - \frac{L^2}{\pi^2}\varepsilon_2 > \frac{p_1(\zeta_1)}{2}, \;\; p_2(\xi_1) - \frac{L^2}{\pi^2}\varepsilon_2' > \frac{p_2(\xi_1)}{2}.$$

Next, we choose N_1 and N_2 large enough so that

$$n_1 N_1 - \frac{\mu_0}{4\varepsilon_2} > 0, \;\; n_2 N_2 - \frac{\mu_0'}{4\varepsilon_2'} > 0.$$

Then, we choose ε_1 and ε_1' small enough, satisfies

$$\frac{p_1(\zeta_1)}{2} - \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1 > 0, \;\; \frac{p_2(\xi_1)}{2} - \frac{L^2\mu_0'^2}{\pi^2}\varepsilon_1' > 0.$$

Finally, we then choose N large enough so that

$$\gamma_1 N - \frac{L^2}{\pi^2}\left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2}N_1 > 0,$$

$$\gamma_2 N - \frac{L^2}{\pi^2}\left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right) - \frac{L^2\mu_0'}{\pi^2}N_2 > 0.$$

By (5.21), we deduce that there exist positive constant c_3 such that (5.45) becomes

$$L'(t) \leq -c_3 E(t), \;\; \forall t \geq 0, \tag{5.46}$$

A combination of (5.44) and (5.46) gives

$$L'(t) \leq -\lambda_1 L(t), \;\; \forall t \geq 0, \tag{5.47}$$

where $\lambda_1 = \dfrac{c_3}{c_2}$. Then, a simple integration of (5.47) over $(0,t)$ yields

$$c_1 E(t) \leq L(t) \leq L(0)e^{-\lambda_1 t}, \;\; \forall t \geq 0. \tag{5.48}$$

Finally, by combining (5.44) and (5.48) we obtain (5.22) with $\lambda_0 = \dfrac{c_2 E(0)}{c_1}$, which completes the proof. □

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