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Résolution de quelques systèmes des équations Différentielles fractionnaires

Option
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Dedication

I dedicate this manuscript:

To my dear mother and to my dear father,
who thanks to their sacrifices have given me the best
of their lives and thanks to them I have reached
where I am now
To the soul of my dear father
Who missed me
To my husband: ***Allouche ouahem Azz Eddine***
who was a support for me in difficult times and always
provides me strength and encourages me
I wish him all the best
Thank you
Azz Eddine
To my children, my heart: ***Basma, Ranim*** *and* ***Yahya***
Who were with me in hard times
I wish them all the happiness in the world.
To my sisters and my brothers.
To all my family and my friends.

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Thank you all...

ملخص

هذه الأطروحة مخصصة لدراسة بعض أنظمة المعادلات التفاضلية الكسرية التي تحتوي على المشتقات الكسرية من اليمين و من اليسار. في الجزء الأول قمنا بدراسة وجود حلول لبعض أنظمة المعادلات التفاضلية غير خطية بمشتقات كسرية مختلطة وشروط حدية غير محلية، حيث تم الحصول على النتائج باستخدام نظرية كراسنوسلسكي للنقطة الصامدة.

في الجزء الثاني من هذه الأطروحة قمنا بمناقشة الوجود، وحدانية و ايجابية الحلول لنظام من معادلات تفاضلية مختلفة التي تحوي على المؤثر p-لابلاسيا ومشتقات كسرية مختلطة. تم الحصول على النتائج باستخدام بعض نظريات النقطة الصامدة مثل نظرية جيو-كراسنوسلسكي في المخروط ، نظرية شاودر و نظرية التقليص لبناخ.

لضمان فائدة النتائج المحصل عليها قدمنا بعض الأمثلة التوضيحية.

نعتقد أن النتائج المحصل عليها جديدة و ستساهم في تطوير الدراسات حول المعادلات التفاضلية الكسرية

الكلمات المفتاحية:

مشتق كسري، مسالة القيمة الحدية، شرط التكامل،جمل معادلات تفاضلية كسرية، وجود حلول، الحل الموجب، وحدانية الحل، نظرية النقطة الصامدة.

Abstract

This thesis is devoted to the study of some systems for fractional differential equations containing both left and right fractional derivatives. In the first part, we study a system of nonlinear differential equations with mixed fractional derivatives and nonlocal boundary conditions. Using Krasnoselskii's fixed point theorem, the existence of solutions is established.

In the second part, we discuss the existence, uniqueness and positivity of solutions for a system of differential equations containing the p-Laplacian operator and mixed fractional derivatives. The proofs are obtained by the help of some fixed point theorems such Guo-Krasnoselski'is fixed point theorem on cones, Schauder fixed point theorem and Banach fixed point theorem.

To guarantee the usefulness of the obtained results some illustrative examples are given.

We believe that the obtained results are new and will contribute to the development of the studies on fractional differential equations.

Keywords: Fractional derivative, Boundary value problem, Integral condition, System of fractional differential equations, Existence of solutions, Positive solution, uniqueness of a solution, Fixed point theorem.

Résumé

Cette thèse est consacrée à l'étude de certains systèmes d'équations différentielles fractionnaires contenant à la fois des dérivées fractionnaires gauche et droite. Dans la première partie, nous étudions un système d'équations différentielles non linéaires avec des dérivées fractionnaires mixtes et des conditions aux limites non locales. En utilisant le théorème du point fixe de Krasnoselskii, l'existence de solutions est établie.

Dans la deuxième partie, nous discutons l'existence, l'unicité et la positivité des solutions pour un système d'équations différentielles contenant l'opérateur p-Laplacien et des dérivées fractionnaires mixtes. Les démonstrations sont obtenues à l'aide de quelques théorèmes de point fixe tels que le théorème de point fixe de Guo-Krasnoselski sur les cônes, le théorème de point fixe de Schauder et le théorème de point fixe de Banach.

Pour garantir la validité des résultats obtenus, quelques exemples illustratifs sont donnés.

Nous pensons que les résultats obtenus sont nouveaux et contribueront au développement des études sur les équations différentielles fractionnaires.

Mots-clés : Dérivée fractionnaire, Problème aux limites, Condition intégrale, Système d'équations différentielles fractionnaires, Existence de solutions, Solution positive, Unicité de la solution, Théorème du point fixe.

CONTENTS

Introduction

The theory of derivatives of fractional order dates back to Leibniz's letter to the Hospital in 1695 where he raised the question of the meaning of the derivative of noninteger order. Since then many mathematicians have contributed to this theory, including Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov. Abel in 1823, was the first who use fractional operators in the solution of tautochrone problems. The first major study of fractional calculus was made by Liouville in 1832, where he applied his definitions to some problems.

Fractional calculus theory is a branch of mathematics that studies the properties of derivatives and integrals of non-integer order.

Recently, fractional calculus has become a very attractive subject for mathematicians, and many different forms of fractional differential operators have appeared as fractional derivatives of Grunwald - Letnikow, Riemann - Liouville, Hadamard, Caputo, Riesz ...For more historical details, see [66, 68, 74]

Furthermore, fractional order calculus plays an important role in several fields of science such as in physics, electrical engineering, control systems, robotics, signal processing, chaos theory, etc, [12, 25, 45, 67].

Various techniques and methods are applied in the study of fractional differential equations. We mention some of them such Mawhin theory, decomposition method, variational iteration method, the homotopy method, lower and upper solutions method.... Some contributions concerning the applications of fixed point theorems in fractional differential equations to show the existence, uniqueness and stability of the solution can be found in [1, 14, 16, 19, 46, 50, 53, 65, 79, 86]. In [17, 42], the authors have investigated the existence of one and two solutions by applying the fixed point index.

The monotonic iterative techniques jointed to the upper and lower solutions method is a powerful tool to prove the existence of solutions of differential equations of fractional order, this kind of work can be found in [8, 12, 51, 64, 71, 81].

Moreover, the existence and multiplicity of positive solutions for the nonlinear fractional differential equations have been investigated in [4, 7, 16, 17, 48, 80, 83]. The existence of positive solutions to fractional boundary value problems is discussed in [28, 34, 35, 52, 59, 75].

Recently, differential equations containing both left and right fractional derivatives are discussed in several papers.

. In physics, if the left fractional derivative is interpreted as a past state of the process, in which memory effects occur, then the right fractional derivative is interpreted as a future state of this process. Since the evolution of a certain phenomena depends on both their past and future, the differential equations studied in this thesis contain a combination of left and right fractional derivatives of Caputo and Riemann-Liouville types in order to represent their evolution. There are many papers that have studied differential equations of fractional order using fixed point theory, but few of them have studied these equations by fixed point theory [9, 36, 52], by the critical point theory and the variational methods [10, 47], also by using the Min-Max Theorem In [49, 77].

The existence and uniqueness of the solutions of some systems of nonlinear fractional differential equations have been studied using various methods such as the fixed point theory, the method of lower and upper solutions, the theory of degrees of coincidence, see [5, 23, 35, 38, 42, 44, 81]. On this point, we can cite the following works.

Using fixed point theory or coincidence degree theory, the existence and uniqueness of some systems for nonlinear fractional differential equations have been studied in [1, 29, 40, 91].

In [3], the authors derived the existence and uniqueness results for a system of coupled three-point Caputo fractional differential equations:

$$\begin{aligned}
&\sum_{i=1}^{2} a_i D_{0^+}^{\beta+i} x(t) = f(t,x,y),\ 0<t<1,\ 0\leq\beta<1\\
&\sum_{i=1}^{2} b_i D_{0^+}^{\alpha+i} y(t) = g(t,x,y),\ 0\leq\alpha<1\\
&\qquad x(0)=x'(0)=0, x(1)=ay(\eta),\\
&\qquad y(0)=y'(0)=0, y(1)=bx(\sigma).
\end{aligned}$$

The existence of solutions is established by the nonlinear alternative of Leray-Schauder and the uniqueness result is proved by Banach's contraction principle.

In [29], the authors used coincidence degree theory to prove the existence results for the following resonant boundary value problem:

$$\begin{cases} D_{1^-}^{\theta} D_{0^+}^{v} x(t) = f(t, x(t)), \ 0 < t < 1, \\ x(0) = 0, D_{0^+}^{v} x(1) = D_{0^+}^{v} x(0), \end{cases}$$

where $0 < \theta, v < 1$, $\theta + v > 1$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $D_{1^-}^{\theta}$ and $D_{0^+}^{v}$ denote respectively the right and left Caputo fractional derivatives.

In [85], the author applied a fixed point theorem in cones to prove the existence of positive solutions as well as multiplicity and nonexistence of solutions for the following system involving singular nonlinear higher order fractional differential equations subject to nonlocal boundary conditions:

$$\begin{gathered} D_{0+}^{\alpha} u(t) + h_1(t) f_1(t, u(t), v(t)) = 0, \quad t \in [0,1], \\ D_{0+}^{\beta} v(t) + h_2(t) f_2(t, u(t), v(t)) = 0, \\ u^{(i)}(0) = 0, \qquad v^{(i)}(0) = 0, 1 \le i \le n-2, \\ D_{0+}^{\mu} u(1) = \eta_1 D_{0+}^{\mu} u(\xi_1), \qquad D_{0+}^{\nu} u(1) = \eta_2 D_{0+}^{\nu} u(\xi_2), \end{gathered}$$

where D_{0+}^{α}, D_{0+}^{β}denote the Riemann–Liouville fractional derivatives, $n-1 < \alpha, \beta \le n$, $1 \le \mu, \nu \le n-3$ and $n > 3$, $\xi_i \in (0,1)$, $0 < \eta_1 \xi_1^{\alpha-\mu-1} < 1$, $0 < \eta_2 \xi_2^{\beta-\nu-1} < 1$, $f_i \in C\left([0,1] \times \mathbb{R}_+^2, \mathbb{R}_+\right)$, $h_i \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$.

In [35], the authors used the upper and lower solutions method and Schauder fixed point theorem to prove the existence of positive solutions for a system of multi-order fractional differential equations with nonlocal boundary conditions, that is

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, 0 < t < 1 \\ u(0) = u'(0) = 0 \\ Au(1) - Bu'(1) = 0 \end{gathered}$$

where the function $u = (u_1, u_2, .., u_n)$, $u_i : [0,1] \to \mathbb{R}$,

$$D_{0+}^{\alpha} u(t) = \left(D_{0+}^{\alpha_1} u_1(t), D_{0+}^{\alpha_2} u_2(t), ..., D_{0+}^{\alpha_n} u_n(t)\right),$$

$D_{0+}^{\alpha_i}$ denote the Riemann–Liouville fractional derivatives, $2 < \alpha_i < 3, i \in \{1, 2, .., n\}$, $n \geq 2$. The function f is such that

$$f(t, u) = (f_1(t, u), ..., f_n(t, u)),$$

$f_i \in C([0, 1] \times \mathbb{R}^n, \mathbb{R}_+)$, $A = (a_1, ..., a_2)$, $B = (b_1, ..., b_n) \in \mathbb{R}^n$.

Furthermore, the existence of solutions for coupled systems of fractional differential equations is discussed in [5, 11, 65, 72, 75, 79, 81, 86], and systems with fractional differential equations subject to various types of boundary conditions such Riemann–Stieltjes integral conditions or multi-point conditions are studied in [6, 21, 61, 63, 89, 90].

In [11, 23, 26, 40, 58, 60, 76, 86, 88], the authors investigated the existence and multiplicity of positive solutions of systems for nonlinear fractional differential equations with nonlocal boundary conditions.

On the other hand, the p-Laplacian operator is widely applied in mechanics, physics and dynamic systems, and the related fields of mathematical modeling. Leibenson [57] is the first who introduce the p-Laplacian operator when studying a mechanics problem that is the turbulent flow in porous media. Various methods are applied to investigate this kind of problems such fixed point theory, the coincidence degree theory, lower and upper solutions method....

In [18, 22], a coupled system of fractional differential equations involving the p-Laplacian operator at resonance is studied by using the coincidence degree theory.

In [32], the authors discussed, by the help of the lower and upper solutions method and Schauder's fixed point theorem, the existence of solutions for fractional p-Laplacian differential equations containing mixed type of fractional derivatives:

$$\begin{cases} -{}^C D_{1^-}^{\beta} \left(\phi_p \left(D_{0+}^{\alpha} u(t)\right)\right) + f(t, u(t)) = 0, 0 \leq t \leq 1, \\ u(0) = u'(0) = 0, D_{0+}^{\alpha} u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $0 < \beta < 1$, ${}^C D_{1^-}^{\beta}$ and D_{0+}^{α} denote respectively the right Caputo derivative and the left Riemann-Liouville derivative.

In [62], by the help of Guo–Krasnosel'skii fixed-point theorem, the authors investigated the existence and nonexistence of positive solutions for the following couple system of nonlinear Riemann–Liouville fractional differential equations with

r_1-Laplacian and r_2-Laplacian operators:

$$\begin{aligned} D_{0+}^{\alpha_1}\phi_{r_1}\left(D_{0^+}^{\beta_1}u(t)\right)+\lambda f(u(t),v(t)) &= 0,\ t\in(0,1), \\ D_{0+}^{\alpha_2}\phi_{r_2}\left(D_{0^+}^{\beta_2}v(t)\right)+\mu g(u(t),v(t)) &= 0,\ t\in(0,1), \end{aligned}$$

and multi-point boundary conditions:

$$\begin{gathered} u^{(j)}(0)=0, j=1,...n-2, \\ D_{0^+}^{\beta_1}u(0)=0, \\ D_{0^+}^{p_1}u(1)=\sum_{i=1}^{N}a_iD_{0^+}^{q_1}u(\xi_i) \\ v^{(j)}(0)=0, j=1,...m-2, \\ D_{0^+}^{\beta_2}v(0)=0, \\ D_{0^+}^{p_2}v(1)=\sum_{i=1}^{M}b_iD_{0^+}^{q_2}v(\eta_i), \end{gathered}$$

where $\alpha_1,\alpha_2\in(0,1]$, $\beta_1,\in(n-1,n]$, $\beta_2\in(m-1,m]$, $n,m\geq 3$, $p_1\in(1,n-2]$, $p_2\in(1,m-2]$, $q_1\in(0,p_1]$, $q_2\in(0,p_2]$, $a_i,b_i,\xi_i,\eta_i\in\mathbb{R}$, $0<\xi_i<....<\xi_N<1$, $0<\eta_1<...<\eta_M<1$, $r_1,r_2>1$, $\lambda,\mu>0$, $f,g\in C([0,1]\times\mathbb{R}^2,\mathbb{R}_+)$. D_{0+}^{k} denotes the Riemann–Liouville derivative of order k.

The main objective of this thesis is to prove the existence, uniqueness and positivity results for certain systems of nonlinear fractional differential equations involving mixed type fractional derivatives. To this end, we use various fundamental concepts of fractional calculus as well as some fixed point theorems. We use Schauder's fixed point theorem, Krasnoselskii fixed point theorem, Guo-Krasnoselskii fixed point theorem, for the existence and positivity of solutions, as well as Banach's contraction principle for the uniqueness result.

Let us give the review of each chapter of the thesis.

In Chapter 1, we recall the definitions of certain fundamental functional spaces, special functions, fractional derivatives and integrals, such as Riemann-Liouville fractional integrals, Riemann-Liouville fractional derivatives, Caputo fractional derivatives, certain tools of functional analysis, the p-Laplacian operator, and then we present some fixed point theorems.

In Chapter 2, using Krasnoselskii fixed point theorem, we study the existence of solutions for the following system of fractional differential equations involving left and right Riemann-Liouville fractional derivatives.

$$\begin{gathered}
D_{1^-}^{\alpha}\left(D_{0^+}^{\beta}u\left(t\right)\right)=-f\left(t,u\left(t\right)\right),\ 0<t<1,\\
D_{0^+}^{\beta}u\left(0\right)=D_{0^+}^{\beta}u\left(1\right)=0,\\
u'\left(1\right)=u\left(0\right)=0.
\end{gathered}$$

where $u=\left(u_1,u_2,...,u_n\right)^T$ is the unknown function with, $u_i:[0,1]\rightarrow\mathbb{R}$,

$$D_{1^-}^{\alpha}\left(D_{0^+}^{\beta}u\left(t\right)\right)=\left(D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_1}u_1\left(t\right)\right),D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_2}u_2\left(t\right)\right),...,D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_n}u_n\left(t\right)\right)\right).$$

Denote $D_{1^-}^{\alpha}$ the left Riemann-Liouville fractional derivative and $D_{0^+}^{\beta_i}$ the right Reimann-Liouville fractional derivative of order β_i, $1<\alpha,\beta_i<2$, $\beta=\left(\beta_1,\beta_2,..,\beta_n\right)$, $i\in\{1,..,n\}$, $n\geq 2$, $f:[0,1]\times\mathbb{R}^n\rightarrow\mathbb{R}_+^n$,

$$f\left(t,u\right)=\left(f_1\left(t,u_1,u_2,...,u_n\right),f_2\left(t,u_1,u_2,...,u_n\right),..,f_n\left(t,u_1,u_2,...,u_n\right)\right)$$

where $f_i\in C\left([0,1]\times\mathbb{R}^n,\mathbb{R}_+\right)$.

The results of this chapter are published in:

[37] A. Guezane-Lakoud, S. Ramdane, Existence of solutions for a system of mixed fractional differential equations, Journal of Taibah University for Science, Volume 12, 2018, Issue 4, (2018).

Chapter 3 concerns the existence, uniqueness and positivity of solutions for p-Laplacian systems with integral conditions involving left and right fractional derivatives:

$$\begin{aligned}
{}^{R}D_{1^-}^{\alpha}\phi_p\left({}^{C}D_{0^+}^{\beta_1}u\left(t\right)\right)+a_1\left(t\right)f_1\left(u\left(t\right),v\left(t\right)\right)=0,\ t\in[0,1],\\
{}^{R}D_{1^-}^{\alpha}\phi_p\left({}^{C}D_{0^+}^{\beta_2}v\left(t\right)\right)+a_2\left(t\right)f_2\left(u\left(t\right),v\left(t\right)\right)=0,\ t\in[0,1],
\end{aligned}$$

$$\phi_p\left({}^{C}D_{0^+}^{\beta_1}u(1)\right) = 0, u'(0) = 0,$$
$$\eta_1 u(1) - u(0) = \int_0^1 g_1(s, u(s), v(s))\, ds,$$
$$\phi_p\left({}^{C}D_{0^+}^{\beta_2}v(1)\right) = 0, v'(0) = 0,$$
$$\eta_2 v(1) - v(0) = \int_0^1 g_2(s, u(s), v(s))\, ds.$$

Where $0 < \alpha < 1$, $\beta = (\beta_1, \beta_2)$, such that $1 < \beta_i < 2$, $\eta_i > 1$, $(i = 1, 2)$ and $\phi_p(s) = |s|^{p-2} s$, $p > 1$, $\phi_q = \left(\phi_p\right)^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, ${}^{R}D_{1^-}^{\alpha}$ the right Riemann-Liouville fractional derivative,${}^{C}D_{0^+}^{\beta_i}$ denotes the left Caputo fractional derivative of order β_i, the functions $a_i \in C([0,1], \mathbb{R}^+)$, $f_i \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $g_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ for $i = 1, 2$.

We prove the existence of at least one solution by the help of Schauder fixed point theorem. The existence of a unique solution is established by means of Banach contraction principle, while the existence of positive solutions is obtained by applying Guo-Krasnosel'skiĭ's fixed point theorem. Moreover, we give sufficient conditions to have no positive solutions.

The results of this Chapter are published in:

[70] S. Ramdane, A. Guezane-Lakoud: Existence of positive solutions for p-Laplacian systems involving left and right fractional derivatives, Arab Journal of Mathematical Sciences, (2021), DOI 10.1108/AJMS-10-2020-0086.

CHAPTER 1

Preliminaries

In this chapter, we give some basic notations, definitions, properties, some necessary concepts on the theory of fractional calculus and some fixed point theorems, which are useful for studying the next chapters. For more details, we refer to the books of Kilbas [54], Kolmogorov [55], Podlubny [68, 69], Samko [73] and Zeidler [87].

1.1 Functional spaces and tools

We present in this Section, some definitions, lemmas and properties of certain spaces that will be used later. Let $I = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$.

1.1.1 Spaces of continuous functions

Definition 1.1.1 *Let $C^m(I, \mathbb{R})$, $m \in \mathbb{N}$, is the Banach space of functions $x\colon I \to \mathbb{R}$ where x is m time continuously differentiable on I with the norm*

$$\|x\|_{C^m} = \sum_{k=0}^{m} \left\|x^{(k)}\right\|_\infty = \sum_{k=0}^{m} \max_{t\in I} \left|x^{(k)}(t)\right|,$$

We denote in particular, by $C = C(I, \mathbb{R}) = C([a, b])$, when $m = 0$ the Banach space of continuous functions $x\colon I \to \mathbb{R}$, equipped with the norm

$$\|x\|_\infty = \max_{t\in I} |x(t)|.$$

1.1.2 Spaces of absolutely continuous functions

Definition 1.1.2 *A function $x : I \to \mathbb{R}$ is said absolutely continuous on I if for all $\varepsilon > 0$, there exists a number $\delta > 0$ such that; for all finite partition $[a_k, b_k]_{k=1}^{q}$ in I , then*

$$\sum_{k=1}^{q} (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{q} (x(b_k) - x(a_k)) < \varepsilon$$

Definition 1.1.3 *[55]*

1- Let $AC(I,\mathbb{R}) = AC[a,b]$ be the space of functions absolutely continuous on $[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:

$$x \in AC[a,b] \Leftrightarrow x(t) = c + \int_a^t \varphi(s)\,ds \qquad (\varphi \in L_1(a,b)), \tag{1.1}$$

2- For $n \in \mathbb{N}$, we denote by $AC^n[a,b]$ the space of real-valued functions x that have continuous derivatives up to order $(n-1)$ on $[a,b]$ i.e.. $x^{(n-1)} \in AC[a,b]$:

$$AC^n[a,b] = \left\{x : [a,b] \to \mathbb{R}, x^{(n-1)} \in AC[a,b]\right\}.$$

The space $AC^n[a,b]$ consists of those and only those functions x which can be represented in the form

$$x(t) = \left(I_{a+}^n \varphi\right)(t) + \sum_{i=0}^{n-1} c_i (t-a)^i, \tag{1.2}$$

where $\varphi \in L_1[a,b]$, c_i, $i \in \{1,2,...,n-1\}$ are arbitrary constants.

For more details about $AC(I,\mathbb{R})$ and $AC^n(I,\mathbb{R})$ see Samko [73].

1.1.3 Spaces of integral functions

Definition 1.1.4 *1- We denote by $L_p(I,\mathbb{R})$, $1 < p < \infty$, the set of all Lebesgue measurable functions x, real valued in general for which*

$$\int_I |x(t)|^p\,dt < \infty.$$

equipped with the norm

$$\|x\|_{L_p} = \left(\int_I |x(t)|^p\,dt\right)^{\frac{1}{p}}.$$

2- For $p = 1$, *the space* $L_1(I,\mathbb{R})$ *is defined as all Lebesgue measurable functions with a finite norm*

$$\|x\|_{L_1} = \int_I |x(t)|\,dt.$$

3- For $p = \infty$, $L_\infty(I,\mathbb{R})$ *is the space of all functions* x *that are essentially bounded on* I *with essential supremum*

$$\|x\|_{L_\infty} = \underset{t\in I}{esssup}\,|x(t)| = \inf\{C \geq 0 : \ |x(t)| \leq C \text{ for a.e. } t\}.$$

Definition 1.1.5 *Let* X *and* Y *two Banach spaces and* T *be a mapping defined on* X *in* Y. *We say that* T *is completely continuous if it is continuous and transforms any bounded set of* X *into a relatively compact set in* Y.

Remark 1.1.1 $T : X \to Y$ *is called compact if* $T(B)$ *is relatively compact in* Y, $\left(\overline{T(B)}\right.$ *is compact in* $\left.Y\right)$, *for all bounded subset* B *of* X.

Theorem 1.1.1 *(Arzela-Ascoli Theorem)[54]*

Let Ω *be a bounded subset of* $C[a,b]$ *equipped with the uniform norm. Then* Ω *is relatively compact in* $C[a,b]$ *if and only if,* Ω *is uniformly bounded and equicontinuous.*

Let us recall,

a) Ω is uniformly bounded i.e,

$$\exists M > 0 \ for\ all\ x \in \Omega, \|x\| \leq M.$$

b) Ω is equicontinuous, i.e

$$\begin{aligned} \forall \varepsilon &> 0,\ \exists \delta > 0, \text{ s.t.} \\ \forall\ x &\in \Omega \text{ and } \forall t,t' \in [a,b] \text{ with } |t-t'| < \delta \ \Rightarrow |x(t) - x(t')| < \varepsilon. \end{aligned}$$

Definition 1.1.6 *[24] A function* $\chi\colon I \times \mathbb{R} \to \mathbb{R}$ *is said to be generalized Lipschitz if there exists a function* $k \in L_1(I,\mathbb{R})$ *such that*

$$|\chi(t,x) - \chi(t,y)| \leq k(t)\,|x-y| \ \text{ a.e. } t \in I \text{ for all } x,y \in \mathbb{R}.$$

The function k is called the Lipschitz function of χ.

Definition 1.1.7 *[24] Let X be a normed linear space and let $\varphi : X \to X$. φ is called Lipschitz if there exists a constant $h > 0$ such that*

$$\|\varphi x - \varphi y\| \leq h \|x - y\| \text{ for all } x, y \in X.$$

The constant h is called a Lipschitz constant of φ on X.

Remark 1.1.2 *Further if $h < 1$, then φ is called a contraction on X with contraction constant k.*

1.1.4 Gamma function

We introduce the Gamma function which play an important role in the theory of fractional differential equations.

Definition 1.1.8 *[27, 54]The **Euler** Gamma function $\Gamma(.)$ is defined by*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt = \int_0^1 \left(\log \frac{1}{t}\right)^{z-1} dt, (\text{Re}(z) > 0), \tag{1.3}$$

*which is the **Euler** integral of second kind and converges in the right half of the complex plane $Re(z) > 0$. Here $t^{z-1} = e^{(z-1)\log t}$.*

The Gamma function $\Gamma(z)$ can be defined by the following expression

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1) \ldots\ldots (z+n)}. \tag{1.4}$$

One of the basic properties of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z), \quad \text{Re}(z) > 0. \tag{1.5}$$

1.1.5 p-Laplacian operator

Definition 1.1.9 *[90]The p-Laplacian operator ϕ_p, $p \in (1, +\infty)$ is defined on $\mathbb{R}$ as*

$$\phi_p(x) = \begin{cases} |x|^{p-2} x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Lemma 1.1.1 *[90]The p-Laplacian operator $\phi_p : \mathbb{R} \to \mathbb{R}$ is an homeomorphism and strictly monotone increasing, and $\phi_p^{-1}(.)$ is continuous, sends bounded sets to bounded sets, and is defined by*

$$\phi_p^{-1}(x) = \phi_q(x) = \begin{cases} |x|^{q-2} x, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.1.2 *[17] Let $c > 0$, $\nu > 0$. for any $x, y \in [0, c]$ we have*

(i) if $\nu > 1$, then

$$|x^\nu - y^\nu| \leq \nu c^{\nu-1} |x - y|.$$

(ii) if $0 < \nu \leq 1$, then

$$|x^\nu - y^\nu| \leq |x - y|^\gamma.$$

Lemma 1.1.3 *Let $\phi_p : \mathbb{R} \to \mathbb{R}$ be a p-Laplacian operator. Then we have following inequalities*

(i) If $1 < p < 2$, $a, b > 0$, $|a|, |b| \geq M > 0$, then

$$\left|\phi_p(a) - \phi_p(b)\right| \leq (p-1) M^{p-2} |a - b|,$$

(ii) If $p \geq 2$, $|a|, |b| \leq c$, then

$$\left|\phi_p(a) - \phi_p(b)\right| \leq (p-1) c^{p-2} |a - b|.$$

1.2 Fractional integrals and fractional derivatives

The integral and differential operators of fractional order are nonlocal in nature and allow a better understanding of the past and future histories of the associated phenomena.

In this Section we present the definitions of fractional integrals operators of Riemann-Liouville and fractional derivatives of Riemann-Liouville and Caputo types on a finite interval of the real line, then we expose some of their properties, for more details see [54, 68, 73].

Definition 1.2.1 *[54, 68, 73] The Riemann-Liouville fractional integrals $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha \in \mathbb{R}^{+}$ are defined by*

$$\left(I_{a+}^{\alpha}f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a \tag{1.6}$$

and

$$\left(I_{b-}^{\alpha}f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} \frac{f(s)}{(s-t)^{1-\alpha}} ds, \qquad t < b, \tag{1.7}$$

these integrals are called the left and right fractional integrals respectively. Provided the right-hand sides are pointwise defined on $I = [a, b]$, $-\infty < a < b < \infty$.

In particular, when $\alpha = n$ (1.6) and (1.7) coincide with the nth integrals of the form

$$\left(I_{a+}^{\alpha}f\right)(t) = \frac{1}{(n-1)!} \int_{a}^{t} \frac{f(s)}{(t-s)^{1-n}} ds, \quad t > a, \quad n \in \mathbb{N},$$

and

$$\left(I_{b-}^{\alpha}f\right)(t) = \frac{1}{(n-1)!} \int_{t}^{b} \frac{f(s)}{(s-t)^{1-n}} ds, \qquad t < b, \quad n \in \mathbb{N}.$$

Lemma 1.2.1 *The fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ with $\alpha \in \mathbb{R}^{+}$ are* *bounded in* $L_p[a,b]$, $1 \le p \le \infty$,

$$\left\| I_{a+}^{\alpha} f \right\| \le K \left\| f \right\|_{L_p},$$
$$\left\| I_{b-}^{\alpha} f \right\| \le K \left\| f \right\|_{L_p},$$
$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Properties.

Let $\alpha > 0,\ \alpha > m > 0, m \in \mathbb{N}$, $D = \frac{d}{dt}$ the classical derivative and $f \in L_p[a,b]$, $(1 \le p \le \infty)$. Then the following relations hold:

$$\left(D^m I_{a+}^{\alpha} f\right)(t) = \left(I_{a+}^{\alpha-m} f\right)(t), \tag{1.8}$$

$$\left(D^m I_{b-}^{\alpha} f\right)(t) = (-1)^m \left(I_{b-}^{\alpha-m} f\right)(t). \tag{1.9}$$

If $m = 1$, then

$$\left(D I_{a+}^{\alpha} f\right)(t) = \left(I_{a+}^{\alpha-1} f\right)(t), \tag{1.10}$$
$$\left(D I_{b-}^{\alpha} f\right)(t) = -\left(I_{b-}^{\alpha-1} f\right)(t).$$

Definition 1.2.2 *[54, 68, 73] The left and right Riemann-Liouville fractional derivatives* $D_{a+}^{\alpha} f$ *and* $D_{b-}^{\alpha} f$ *of order* $\alpha \in \mathbb{R}^+$ *are defined by*

$$\begin{aligned} D_{a+}^{\alpha} f(t) &= \left(\frac{d}{dt}\right)^n \left(I_{a+}^{n-\alpha} f\right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds \quad , \ t > a \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} D_{b-}^{\alpha} f(t) &= \left(-\frac{d}{dt}\right)^n \left(I_{b-}^{n-\alpha} f\right)(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b \frac{f(s)}{(s-t)^{\alpha-n+1}} ds \quad , \ t < b \end{aligned} \tag{1.12}$$

where $n = [\alpha] + 1$. $[\alpha]$ *denotes the integer part of* α.

In particular, if $0 < \alpha < 1$ *and* $n = 1$ *then,*

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad t > a,$$

and

$$D_{b-}^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}} ds, \quad t < b.$$

Properties.

We have the following properties for $\alpha \geq 0, \ \beta > 0$:

$$\begin{aligned}
\left(I_{a+}^{\alpha} (t-a)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (x-a)^{\beta+\alpha-1}, \\
\left(D_{a+}^{\alpha} (t-a)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-a)^{\beta-\alpha-1}, \\
\left(I_{b-}^{\alpha} (b-t)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-x)^{\beta+\alpha-1}, \\
\left(D_{b-}^{\alpha} (b-t)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-x)^{\beta-\alpha-1}.
\end{aligned}$$

Moreover, the Riemann-Liouville fractional derivative of a constant is in general not equal to zero,

Example 1.2.1 *if* $\beta = 1$ *and* $0 \leq \alpha < 1$ *we have*

$$\left(D_{a+}^{\alpha} 1\right)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)},$$

$$\left(D_{b-}^{\alpha} 1\right)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Corollary 1.2.1 *For* $\alpha > 0$, *and* $n = [\alpha] + 1$ *we have*

a) $\left(D_{a+}^{\alpha} f\right)(t) = 0$ *if, and only if,*

$$f(t) = \sum_{i=1}^{n} c_i (t-a)^{\alpha-i}. \qquad (1.13)$$

b) $\left(D_{b-}^{\alpha}f\right)(t)=0$ *if, and only if,*

$$f(t)=\sum_{i=1}^{n}d_i\,(b-t)^{\alpha-i}, \tag{1.14}$$

where $c_i, d_i\in\mathbb{R}$, $i=1,2,...n$, *are arbitrary constants.*

In particular, when $0<\alpha<1$, *then (1.13) and (1.14) take the following forms*

$$f(t)=c\,(t-a)^{\alpha-1},$$

and

$$f(t)=d\,(b-t)^{\alpha-1},$$

where $c,d\in\mathbb{R}$ *are arbitrary constants.*

Lemma 1.2.2 *[54, 73]Assume that* $f\in L_1[a,b]$ *and* $\alpha>0$, *then*

$$I_{a+}^{\alpha}D_{a+}^{\alpha}f(t)=f(t)+\sum_{i=1}^{n}c_i\,(t-a)^{\alpha-i}, \tag{1.15}$$

$$I_{b-}^{\alpha}D_{b-}^{\alpha}f(t)=f(t)+\sum_{i=1}^{n}d_i\,(b-t)^{\alpha-i}, \tag{1.16}$$

where $c_i, d_i\in\mathbb{R}\,(i=1,2,...n)$ *are arbitrary constants and* $n=[\alpha]+1$.

In particular, when $0<\alpha<1$, *then the relations (1.15) and (1.16) take the following forms*

$$I_{a+}^{\alpha}D_{a+}^{\alpha}f(t)=f(t)+c\,(t-a)^{\alpha-1}, \tag{1.17}$$

and

$$I_{b-}^{\alpha}D_{b-}^{\alpha}f(t)=f(t)+d\,(b-t)^{\alpha-1}, \tag{1.18}$$

where $c,d\in\mathbb{R}$ *are arbitrary constants.*

Definition 1.2.3 *[54, 73]The left and right Caputo derivative* ${}^{C}D_{a+}^{\alpha}$ *and* ${}^{C}D_{b-}^{\alpha}$ *of order* $\alpha\in\mathbb{R}^{+}$ *of the function* f *can be defined via the above Riemann-Liouville* *fractional derivatives by*

$$ {}^{C}D_{a+}^{\alpha}f(t) = \left(D_{a+}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^{k} \right] \right)(t) \tag{1.19}$$

and

$$ {}^{C}D_{b-}^{\alpha}f(t) = \left(D_{b-}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(b-x)^{k} \right] \right)(t) \tag{1.20}$$

respectively, where $n = [\alpha] + 1$.

Lemma 1.2.3 *[54, 73] For* $\alpha > 0$*, and* $n = [\alpha] + 1$ *we have*

$$ {}^{C}D_{a^+}^{\alpha}f(t) = D_{a^+}^{\alpha}f(t), $$

if

$$ f(a) = f'(a) = f^{(2)}(a) = ... = f^{(n)}(a) = 0, $$

and

$$ {}^{C}D_{b-}^{\alpha}f(t) = D_{b-}^{\alpha}f(t), $$

if

$$ f(b) = f'(b) = f^{(2)}(b) = ... = f^{(n)}(b) = 0. $$

Theorem 1.2.1 *[54, 73] Let* $f \in AC^{n}[a,b]$*, then the Caputo fractional derivatives* ${}^{C}D_{a^+}^{\alpha}f$ *and* ${}^{C}D_{b-}^{\alpha}f$ *exist a.e. on* $[a,b]$ *and are represented by*

$$ \begin{aligned} {}^{C}D_{a+}^{\alpha}f(t) &= \left(I_{a+}^{n-\alpha}D^{(n)}f \right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \qquad t > a, \end{aligned} $$

$$ \begin{aligned} {}^{C}D_{b-}^{\alpha}f(t) &= (-1)^{n}\left(I_{b-}^{n-\alpha}D^{(n)}f \right)(t) \\ &= \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{t}^{b} (s-t)^{n-\alpha-1} f^{(n)}(s)\,ds, \qquad t < b. \end{aligned} $$

respectively, where $n = [\alpha] + 1$.

Properties. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$\begin{aligned}\left({}^C D^{\alpha}_{a+} (t-a)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-a)^{\beta-1}, \\ \left({}^C D^{\alpha}_{b-} (b-t)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-x)^{\beta-1}\end{aligned}$$

In particular, if $\beta = 1$ we get

$$\begin{aligned}\left({}^C D^{\alpha}_{a+} 1\right)(x) &= 0, \\ \left({}^C D^{\alpha}_{b-} 1\right)(x) &= 0.\end{aligned}$$

Lemma 1.2.4 *Let* $\alpha > 0$.

i) The fractional differential equation

$${}^C D^{\alpha}_{a+} f(t) = 0$$

has

$$f(t) = \sum_{i=1}^{n} c_i (t-a)^i, c_i \in \mathbb{R},$$

as solution.

ii) The fractional differential equation

$${}^C D^{\alpha}_{b-} f(t) = 0$$

has

$$f(t) = \sum_{i=1}^{n} d_i (b-t)^i, d_i \in \mathbb{R},$$

as solution.

Lemma 1.2.5 *Let* $\alpha > 0$. *If* $f \in C^n [a, b]$, *then*

$$I^{\alpha}_{a+} D^{\alpha}_{a+} f(t) = f(t) + \sum_{k=0}^{n-1} c_k (t-a)^k, c_k \in \mathbb{R}, \tag{1.21}$$

$$I_{b-}^{\alpha} D_{b-}^{\alpha} f(t) = f(t) + \sum_{k=0}^{n-1} d_k (b-t)^k, d_k \in \mathbb{R}. \quad (1.22)$$

1.3 Fixed point theorems

Fixed point theorem states that a mapping A has at least one fixed point, i.e. $A(x) = x$, under certain conditions on A. In a wide range of mathematics, the existence of a solution to a problem is equivalent to the existence of a fixed point for a suitable operator. Fixed points are therefore of importance in many fields of mathematics, science and engineering.

Many situations in the study of nonlinear equations can be formulated in the term of a fixed point problem. Therefore, fixed point theorems are useful mathematical tools for discussing the existence, uniqueness and positivity of solutions for differential equations. In this section, we recall some fixed point theorems that will be used later.

Definition 1.3.1 *For a mapping T from a set X into itself, an element x of X is a fixed point of T if $T(x) = x$.*

Definition 1.3.2 *[39] Let X be a Banach space. A nonempty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:*

a) $x \in P$, $\lambda \geq 0$, implies $\lambda x \in P$,

b) $x \in P$, $-x \in P$, implies $x = 0$.

Theorem 1.3.1 *[73, 78, 87]* ***(Banach's fixed point Theorem).***

Let T be a contraction on a Banach space X. Then T has a unique fixed point.

Theorem 1.3.2 *[78, 87]* ***(Schauder's fixed point Theorem)***

Let M be a closed convex subset of a Banach space E. If $A : M \to M$ is continuous and the set $\overline{A(M)}$ is compact, then A has a fixed point in M.

Theorem 1.3.3 *[56, 78, 87]* ***(Krasnoselskii fixed point Theorem)***

Let Ω be a closed bounded convex nonempty subset of a Banach space X. Suppose that A and B map Ω into X such that

(i) $x, y \in \Omega$ implies $Ax + By \in \Omega$.

(ii) B is a contraction mapping.

(iii) A is completely continuous.

Then there exists $z \in \Omega$ such $z = Az + Bz$.

Theorem 1.3.4 *[73, 69, 78, 87]* ***(Guo-Krasnoselskii Theorem)***

Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $T : K \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to K$, be a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, or

(ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$.

CHAPTER 2

Existence of solutions for a system of mixed fractional differential equations

2.1 Introduction

Fractional differential equations are gaining more and more attention, this is due to their several applications in different scientific disciplines such as physics, chemistry, viscoelasticity, aerodynamics, electromagnetic see [54, 68, 73] and the references therein.

In [23], the author studied the existence of solutions for a system of multi-order fractional differential equations with nonlocal boundary conditions, here the order of each equation may be different from the order of the other equations:

$$\begin{cases} D_{0^+}^{\alpha_i} u_i = f_i(t, u_1, u_2, ..., u_n), u_i(0) = 0, \quad 0 < \alpha_i < 1, \ 1 \leq i \leq n. \\ u_i(0) = 0. \end{cases}$$

where $0 \leq t \leq T$. $D_{0^+}^{\alpha_i}$ denote the standard Riemann–Liouville fractional derivatives and $f_i : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}_+$.

In [48], the authors discussed the existence of positive solutions of the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u + a(t) f(u) = 0, 0 < t < 1, 1 < \alpha < 2 \\ u(0) = 0, u'(1) = 0 \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville derivative , $f \in C(\mathbb{R}, \mathbb{R}_+)$ and a is a positive and continuous function on $[0, 1]$.

Fractional differential equations involving both left and right fractional derivatives attract much attention recently, as they appear in Euler-Lagrange equations when studying variational principles.

The existence results for such type of differential equations are obtained by means of different methods such as fixed point theorems, lower and upper solution method, variational methods, ...we refer to [7, 9, 13, 15, 34, 52].

In [34], the authors established by using the lower and upper solutions method the existence of solutions for fractional oscillator equation involving mixed type fractional derivatives with an initial condition and a natural boundary condition:

$$\begin{aligned} -{}^{C}D_{1^-}^{p}D_{0^+}^{q}u(t)+\omega^2 u(t) &= f(t,u(t)),0\leq t\leq 1,\omega\in\mathbb{R},\omega\neq 0,\\ u(0) &= 0,\\ D_{0^+}^{q}u(1) &= 0 \end{aligned}$$

where $0<p,q<1$, ${}^{C}D_{1^-}^{p}$ and $D_{0^+}^{q}$ denote the right Caputo derivative and the left Riemann–Liouville respectively and $f\in C([0,1]\times\mathbb{R},\mathbb{R})$.

In [36], the authors discussed the existence of solutions for the following boundary value problem containing a mixed type of fractional derivatives:

$$\begin{cases} {}^{C}D_{1^-}^{\alpha}\left(D_{0^+}^{\beta}u(t)\right)+f(t,u(t))=0,0<t<1\\ u(0)=u'(0)=u(1)=0 \end{cases}$$

Where $0<\alpha\leq 1, 1<\beta\leq 2, f:[0,1]\times\mathbb{R}\to\mathbb{R}$. The proofs are based on Krasnoselskii's fixed point theorem.

This chapter is devoted to the study of the following system of fractional differential equations with boundary conditions:

$$(S)\begin{cases} D_{1^-}^{\alpha}\left(D_{0^+}^{\beta}u(t)\right)+f(t,u(t))=0,\ 0<t<1,\\ D_{0^+}^{\beta}u(0)=D_{0^+}^{\beta}u(1)=0,\\ u'(1)=u(0)=0. \end{cases} \tag{2.1}$$

where the function $u=(u_1,u_2,...,u_n)^T$ is an unknown function, $u_i:[0,1]\to\mathbb{R}$,

$$D_{1^-}^{\alpha}\left(D_{0^+}^{\beta}u(t)\right)=\left(D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_1}u_1(t)\right),D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_2}u_2(t)\right),...,D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_n}u_n(t)\right)\right)^T,$$

$D_{0^+}^{\beta_i}$and$D_{1^-}^{\alpha}$ denote the left and right Riemann-Liouville fractional derivatives respectively, $1<\alpha,\ \beta_i<2,\ i\in\{1,..,n\},\ n\geq 2,\ f:[0,1]\times\mathbb{R}^n\to\mathbb{R}^n$,

$$f(t,u)=(f_1(t,u_1,u_2,...,u_n),...,f_n(t,u_1,u_2,...,u_n))^T,$$

$f_i\in C([0,1]\times\mathbb{R}^n,\mathbb{R})$.

This chapter is organized as follows. In Section 2, we establish the existence of a unique solution for the corresponding linear system (S), then we present some properties of the Green functions. In Section 3, we convert the system (S) to a sum of a contraction and a compact operator, then we use Krasnoselskii fixed point theorem to prove the existence of at least one solution to system (S). In Section 4, two examples are constructed to validate the results.

2.2 Auxiliary results

We shall transform the system (2.1) to an equivalent system of integral equations. Consider the corresponding linear system:

$$D_{1^-}^{\alpha}\left(D_{0^+}^{\beta_i}u_i(t)\right)=-y_i(t), \qquad 0\leq t\leq 1, \tag{2.2}$$

$$D_{0^+}^{\beta_i}u_i(0)=D_{0^+}^{\beta_i}u_i(1)=0, \tag{2.3}$$

$$u_i'(1)=u_i(0)=0. \tag{2.4}$$

here $i\in\{1,2,...n\}$.

Lemma 2.2.1 *Assume that $y_i\in C[0,1]$, $i\in\{1,...,n\}$, then the system (2.2)-(2.4), has a unique solution $u=(u_1,...,u_n)$ given by*

$$u_i(t)=\int_0^1 G_i(t,r)\,y_i(r)\,dr+g_i(t)\int_0^1 s^{\alpha-1}y_i(s)\,ds, \tag{2.5}$$

where

$$G_i(t,r)=\frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\begin{cases}\int_0^r\left(t^{\beta_i-1}(1-s)^{\beta_i-2}-(t-s)^{\beta_i-1}\right)(r-s)^{\alpha-1}ds,\\ 0\leq r\leq t\leq 1,\\ t^{\beta_i-1}\int_0^r(1-s)^{\beta_i-2}(r-s)^{\alpha-1}ds-\int_0^t(t-s)^{\beta_i-1}(r-s)^{\alpha-1}ds,\\ 0\leq t\leq r\leq 1.\end{cases}$$

$$g_i(t)=\frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^t(t-s)^{\beta_i-1}(1-s)^{\alpha-1}ds-\frac{t^{\beta_i-1}}{\alpha+\beta_i-2}\right).$$

Proof Applying the integral operator $I_{1^-}^{\alpha}$ to equation (2.2), it yields

$$D_{0^+}^{\beta_i} u_i(t) = -I_{1^-}^{\alpha} y_i(t) + c_1 (1-t)^{\alpha-1} + c_2 (1-t)^{\alpha-2} \tag{2.6}$$

where $c_1, c_2 \in \mathbb{R}$.

Conditions (2.3), implies

$$c_2 = 0,\ c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1} y_i(s)\, ds. \tag{2.7}$$

Substituting c_1 and c_2 in (2.6), we get

$$D_{0^+}^{\beta_i} u_i(t) = \frac{1}{\Gamma(\alpha)} \left((1-t)^{\alpha-1} \int_0^1 s^{\alpha-1} y_i(s)\, ds - \int_t^1 (s-t)^{\alpha-1} y_i(s)\, ds \right). \tag{2.8}$$

Now, we obtain by applying the operator $I_{0^+}^{\beta_i}$ to equation (2.8):

$$\begin{aligned} u_i(t) &= -I_{0^+}^{\beta_i} I_{1^-}^{\alpha} y_i(t) + \frac{1}{\Gamma(\alpha)} \left(I_{0^+}^{\beta_i} (1-t)^{\alpha-1} \right) \int_0^1 s^{\alpha-1} y_i(s)\, ds \\ &\quad + c_3 t^{\beta_i-1} + c_4 t^{\beta_i-2}, \end{aligned} \tag{2.9}$$

it's easy to get $c_4 = 0$ by the boundary conditions (2.4). Differentiating the obtained equation, we obtain

$$\begin{aligned} u_i'(t) &= -I_{0^+}^{\beta_i-1} I_{1^-}^{\alpha} y_i(t) + \frac{1}{\Gamma(\alpha)} \left(I_{0^+}^{\beta_i-1} (1-t)^{\alpha-1} \right) \int_0^1 s^{\alpha-1} y_i(s)\, ds \\ &\quad + (\beta_i - 1) c_3 t^{\beta_i-2}. \end{aligned}$$

Using the initial conditions (2.4), we obtain

$$\begin{aligned} c_3 &= \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\beta_i-2}\left(\int_s^1 (r-s)^{\alpha-1} y_i(r)\,dr\right)\right. \\ &\quad \left. -\left(\int_0^1 (1-s)^{\beta_i-2}(1-s)^{\alpha-1}\,ds\right)\times\left(\int_0^1 s^{\alpha-1}y_i(s)\,ds\right)\right) \\ &\quad \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\beta_i-2}\left(\int_s^1 (r-s)^{\alpha-1} y_i(r)\,dr\right)-\right. \\ &\quad \left.\frac{1}{\alpha+\beta_i-2}\left(\int_0^1 s^{\alpha-1}y_i(s)\,ds\right)\right). \end{aligned}$$

Substituting c_3 and c_4 in (2.9) yields

$$\begin{aligned} u_i(t) &= -\frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^t (t-s)^{\beta_i-1}\left(\int_s^1 (r-s)^{\alpha-1} y_i(r)\,dr\right)ds \\ &\quad +\frac{t^{\beta_i^{-1}}}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^1 (1-s)^{\beta_i-2}\left(\int_s^1 (r-s)^{\alpha-1} y_i(r)\,dr\right)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\left(\left(I_{0^+}^{\beta_i}(1-t)^{\alpha-1}\right)-\frac{t^{\beta_i^{-1}}}{(\alpha+\beta-2)\Gamma(\beta_i)}\right) \\ &\quad \times\int_0^1 s^{\alpha-1}y_i(s)\,ds. \end{aligned}$$

Thanks to Fubini theorem, we get

$$\begin{aligned} u_i(t) &= -\frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^t\left(\int_0^r (t-s)^{\beta_i-1}(r-s)^{\alpha-1}\,ds\right)y_i(r)\,dr \\ &\quad -\frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_t^1\left(\int_0^t (t-s)^{\beta_i-1}(r-s)^{\alpha-1}\,ds\right)y_i(r)\,dr \\ &\quad +\frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^1\left(\int_0^r (1-s)^{\beta_i-2}(r-s)^{\alpha-1}\,ds\right)y_i(r)\,dr \\ &\quad -\frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\frac{t^{\beta_i-1}}{\alpha+\beta_i-2}-\left(\int_0^t (t-s)^{\beta_i-1}(1-s)^{\alpha-1}\,ds\right)\right) \\ &\quad \times\left(\int_0^1 s^{\alpha-1}y_i(s)\,ds\right), \end{aligned}$$

hence (2.5) holds. ∎

Let us present the properties of the functions g_i and G_i, $i = 1, ..., n$.

Lemma 2.2.2 *The functions G_i and g_i, $i = 1, ..., n$ are continuous and satisfy the following properties:*

$$0 \le G_i(t,r) \le \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}, \quad 0 \le t, r \le 1 \tag{2.10}$$

$$g_i(t) \le 0, \quad |g_i(t)| \le \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}, \quad 0 \le t \le 1. \tag{2.11}$$

Proof G_i are nonnegative. In fact, if $0 \le r \le t \le 1$, then

$$\begin{aligned} G_i(t,r) &= \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^r \left(t^{\beta_i-1}(1-s)^{\beta_i-2}-(t-s)^{\beta_i-1}\right) \\ &\quad \times (r-s)^{\alpha-1}\,ds \\ &\ge \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^r \left(t^{\beta_i-1}-(t-s)^{\beta_i-1}\right)(r-s)^{\alpha-1}\,ds \ge 0, \end{aligned}$$

and if $0 \leq t \leq r \leq 1$, then

$$\begin{aligned}
G_i(t,r) &= \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(t^{\beta_i-1}\int_0^r (1-s)^{\beta_i-2}(r-s)^{\alpha-1}\,ds - \right. \\
&\left. \int_0^t (t-s)^{\beta_i-1}(r-s)^{\alpha-1}\,ds\right) \\
&\geq \frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^r (1-s)^{\beta_i-2}(r-s)^{\alpha-1}\,ds - \int_0^t (r-s)^{\alpha-1}\,ds\right) \\
&\geq \frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^r (r-s)^{\alpha-1}\,ds - \int_0^t (r-s)^{\alpha-1}\,ds\right) \geq 0.
\end{aligned}$$

On the other hand, we have if $0 \leq r \leq t \leq 1$

$$\begin{aligned}
G_i(t,r) &\leq \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^r t^{\beta_i-1}(1-s)^{\beta_i-2}(r-s)^{\alpha-1}\,ds \\
&\leq \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^r (r-s)^{\beta_i+\alpha-3}\,ds \\
&\leq \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}
\end{aligned}$$

and if $0 \leq t \leq r \leq 1$, then

$$\begin{aligned}
G_i(t,r) &\leq \frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}t^{\beta_i-1}\int_0^r (1-s)^{\beta_i-2}(r-s)^{\alpha-1}\,ds \\
&\leq \frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}\int_0^r (r-s)^{\beta_i+\alpha-3}\,ds \\
&\leq \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}.
\end{aligned}$$

Similarly, we prove that the functions g_i are non positive. Indeed, we have

$$\begin{aligned} g_i(t) &= \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^t (t-s)^{\beta_i-1}(1-s)^{\alpha-1}\,ds - \frac{t^{\beta_i-1}}{\alpha+\beta_i-2}\right) \\ &\leq \frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\alpha-1}\,ds - \frac{1}{\alpha+\beta_i-2}\right) \\ &\leq \frac{t^{\beta_i-1}}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\frac{\beta_i-2}{\alpha+\beta_i-2}\right) \leq 0. \end{aligned}$$

Moreover, we have,

$$\begin{aligned} |g_i(t)| &= -g_i(t) \\ &= \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\frac{t^{\beta_i-1}}{\alpha+\beta_i-2} - \int_0^t (t-s)^{\beta_i-1}(1-s)^{\alpha-1}\,ds\right) \\ &\leq \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}. \end{aligned}$$

■

2.3 Existence of solutions

We consider the Banach space X of all functions

$$x = (x_1, x_2, ..., x_n) \in \underbrace{C[0,1] \times ... \times C[0,1]}_{n\ times}$$

with the norm $\|.\|$ defined by

$$\|x\| = \sum_{i=1}^{n} \max_{t\in[0,1]} |x_i(t)|.$$

Define the integral operators A and B on X by

$$\begin{aligned} Ax(t) &= (A_1x_1(t), A_2x_2(t), ..., A_nx_n(t)) \\ Bx(t) &= (B_1x_1(t), B_2x_2(t), ..., B_nx_n(t)) \end{aligned}$$

where

$$\begin{aligned} A_ix_i(t) &= \int_0^1 G_i(t,r) f_i(r, x(r))\, dr \\ B_ix_i(t) &= g_i(t) \int_0^1 s^{\alpha-1} f_i(s, x(s))\, ds \end{aligned}$$

Lemma 2.3.1 *The function $x = (x_1, x_2, ..., x_n) \in X$ is a solution of the system (S) if and only if $A_ix_i(t) + B_ix_i(t) = x_i(t)$ for all $t \in [0,1]$ and $i = 1, ..., n$.*

Consequently, to prove the existence of a solution for the system (S) it suffices to prove that the operator $A + B$ has a fixed point, that is

$$Ax(t) + Bx(t) = x(t), \qquad t \in [0,1].$$

Now, let us make the necessary hypotheses to prove the existence results for the system (S).

H_1) There exist nonnegative functions $K_i \in L_1(0,1)$, such that:

$$|f_i(t,x) - f_i(t,y)| \leq K_i(t) \sum_{j=1}^{n} |x_j - y_j|,$$
$$t \in [0,1], x, y \in \mathbb{R}, \quad i \in \{1, ..., n\},$$

where

$$\sum_{i=1}^{n} \frac{\|K_i\|_{L_1}}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)} < \frac{1}{4}. \tag{2.12}$$

H_2) The functions $f_i(t,0)$ are continuous and not identically null on $[0,1]$, $\forall i \in \{1, ..., n\}$.

Theorem 2.3.1 *Under hypotheses* (H_1) *and* (H_2) *the system* (S) *has at least one nontrivial solution.*

Proof Let $\Omega = \{x \in X, \|x\| \leq R\}$, here R is chosen such

$$R \geq 4 \sum_{i=1}^{n} \frac{L_i}{(\alpha + \beta_i - 2)\,\Gamma(\beta_i)\,\Gamma(\alpha)}, \tag{2.13}$$

and set

$$L_i = \max_{t \in [0,1]} |f_i(t, 0)|.$$

Clearly, Ω is a nonempty, bounded and convex subset of X.

We will use Krasnoselskii's fixed point theorem to prove that the operator $A+B$ has a fixed point, to this end, the proof will be done in three steps.

Step 1: $Ax + By \in \Omega$ for all $x, y \in \Omega$. In fact, taking into account hypothesis (H_2) and the properties of the functions G_i, we get for all $i = 1, ..., n$,

$$\begin{aligned} |A_i x_i(t)| &\leq \int_0^1 G_i(t, r)\,|f_i(r, x(r))|\,dr \\ &\leq \frac{1}{(\alpha + \beta_i - 2)\,\Gamma(\beta_i)\,\Gamma(\alpha)} \int_0^1 (|f_i(r, x(r)) - f_i(r, 0)| + |f_i(r, 0)|)\,dr \\ &\leq \frac{1}{(\alpha + \beta_i - 2)\,\Gamma(\beta_i)\,\Gamma(\alpha)} \int_0^1 \left(|K_i(r)| \sum_{i=1}^{n} |x_i(r)| + L_i \right) dr \\ &\leq \frac{R\,\|K_i\|_{L_1} + L_i}{(\alpha + \beta_i - 2)\,\Gamma(\beta_i)\,\Gamma(\alpha)} \end{aligned}$$

Taking the maximum over $t \in [0, 1]$, it yields

$$\|A_i x_i\| \leq \frac{\|K_i\|_{L_1} R + L_i}{(\alpha + \beta_i - 2)\,\Gamma(\beta_i)\,\Gamma(\alpha)}. \tag{2.14}$$

Summing the n inequalities in (2.14), then in view of (2.12) and (2.13), we obtain

$$\|Ax\| \leq \sum_{i=1}^{n} \left(\frac{\|K_i\|_{L_1} R + L_i}{(\alpha + \beta_i - 2)\,\Gamma(\beta_i)\,\Gamma(\alpha)} \right) < \frac{R}{2}. \tag{2.15}$$

Thanks to hypothesis (H_2) and the properties of the functions g_i, we get

$$\begin{aligned}
|B_i y_i(t)| &\leq |g_i(t)| \int_0^1 s^{\alpha-1} |f_i(s, y(s))| \, ds \\
&\leq \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} \int_0^1 (|f_i(r, y(r)) - f_i(r, 0)| + |f_i(r, 0)|) \, dr \\
&\leq \frac{||K_i||_{L_1} R + L_i}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}, \ \forall i = 1, ..., n.
\end{aligned}$$

Taking the supremum over $[0,1]$, then summing the n obtained inequalities according to i from 1 to n, we get by the help of (2.12) and (2.13),

$$\|By\| \leq \sum_{i=1}^{n} \left(\frac{\|K_i\|_{L_1} R + L_i}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} \right) < \frac{R}{2}.$$

Hence

$$\|Ax + By\| \leq \|Ax\| + \|By\| < R.$$

So, $Ax + By \in \Omega$ for all $x, y \in \Omega$.

Step 2: The mapping B is a contraction on Ω. Indeed let $x, y \in \Omega$, then by hypothesis (H_1) it yields

$$\begin{aligned}
|B_i x_i(t) - B_i y_i(t)| &\leq |g_i(t)| \int_0^1 s^{\alpha-1} |f_i(s, x(s)) - f_i(s, y(s))| \, ds \\
&\leq \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} \int_0^1 |K_i(s)| \sum_{i=1}^{n} |x_i - y_i| \, ds \\
&\leq \frac{\|K_i\|_{L_1} ||x-y||}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}, \quad i = 1, ..., n.
\end{aligned}$$

Taking the maximum over $t \in [0,1]$, we get

$$\|B_i x_i - B_i y_i\| \leq \frac{\|K_i\|_{L_1} ||x-y||}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} \tag{2.16}$$

Summing the n inequalities in (2.16), then taking (2.12) into account, we obtain:

$$\begin{aligned}\|Bx-By\| &\leq \sum_{i=1}^{n}\frac{\|K_i\|_{L_1}\|x-y\|}{(\alpha+\beta_i-2)\,\Gamma(\beta_i)\,\Gamma(\alpha)}\\ &< \frac{\|x-y\|}{4}.\end{aligned}$$

Step 3: The operator A is completely continuous on Ω. In fact,

i) A is continuous on Ω. Let $(x_k)_k = (x_k^1, x_k^2, ...x_k^n)_k$ be a sequence such that $x_k \to x = (x^1, ..., x^n)$ in Ω, $x_k^i \to x^i$ as $k \to \infty$. Taking into account hypothesis (H_1) and the properties of the functions G_i, we get

$$\begin{aligned}\left|A_i x_k^i(t) - A_i x^i(t)\right| &\leq \int_0^1 G_i(t,r)\,|f_i(r,x_n(r)) - f_i(r,x(r))|\,dr\\ &\leq \frac{\|K_i\|_{L_1}\|x_k-x\|}{(\alpha+\beta_i-2)\,\Gamma(\beta_i)\,\Gamma(\alpha)}\\ &< \frac{\|x_k-x\|}{4} \to 0,\ as\ \ k\to\infty.\end{aligned}$$

Hence, $\|Ax_k - Ax\| \to 0$, when k tends to ∞.

ii) $A(\Omega) \subset \Omega$. Indeed, let $x \in \Omega$. From (2.15) we get

$$\|Ax\| < \frac{R}{2}$$

iii) (Ax) is equicontinuous on Ω. Let $x \in \Omega$, $0 \leq t_1 \leq t_2 \leq 1$,

$$\begin{aligned}\left|A_i x^i(t_1) - A_i x^i(t_2)\right| &\leq \int_0^{t_1} |G_i(t_1,r) - G_i(t_2,r)|\,|f_i(r,x(r))|\,dr\\ &+ \int_{t_1}^{t_2} |G_i(t_1,r) - G_i(t_2,r)|\,|f_i(r,x(r))|\,dr\\ &+ \int_{t_2}^{1} |G_i(t_1,r) - G_i(t_2,r)|\,|f_i(r,x(r))|\,dr\end{aligned}$$

$$\leq \frac{L}{\Gamma(\alpha)\Gamma(\beta_i)}\left[\frac{3\left(t_2^{\beta_i-1}-t_1^{\beta_i-1}\right)}{\beta_i-1}\right.$$
$$\left.+\frac{2\left(t_2^{\beta_i}-t_1^{\beta_i}\right)-(t_2-t_1)^{\beta_i}}{\beta_i}+3(t_2-t_1)\right]\to 0,$$
as $t_1 \rightarrow t_2,\ i=1,...,n.$

Consequently (Au) is equicontinuous on Ω.

From the above steps, it follows by Arzela-Ascoli's theorem that A is completely continuous mapping on Ω.

Finally, we conclude by Krasnoselskii fixed point theorem that the operator $A+B$ has at least one fixed point in Ω, and consequently the system (S) has at least one solution in Ω. ■

2.4 Examples

Now, we give two examples to illustrate the usefulness of our main results.

Example 2.4.1 *Consider the following two-dimensional fractional order system*

$$(S_1)\begin{cases} D_{1^-}^{1.2}\left(D_{0^+}^{1.9}u_1(t)\right)=\frac{(1-2t)}{10}\left(u_2-\frac{1}{2\left(1+u_2^2\right)}\right) \\ D_{1^-}^{1.2}\left(D_{0^+}^{1.5}u_2(t)\right)=\frac{e^{-t}}{60}\left(tu_2+\frac{1}{2}\left(3u_1-\frac{1}{1+u_2^2}\right)\right) \\ D_{0^+}^{1.9}u_1(0)=D_{0^+}^{1.9}u_1(1)=0, \\ D_{0^+}^{1.5}u_2(0)=D_{0^+}^{1.5}u_2(1)=0 \\ u_1'(1)=u_1(0)=0, u_2'(1)=u_2(0)=0. \end{cases}$$

Here we have $\alpha=1.2,\ \beta_1=1.9,\ \beta_2=1.5,$

$$f_1(t,u)=\frac{(1-2t)}{10}\left(u_2-\frac{1}{2\left(1+u_2^2\right)}\right)$$

and

$$f_2(t,u)=\frac{e^{-t}}{60}\left(tu_2+\frac{1}{2}\left(3u_1-\frac{1}{1+u_2^2}\right)\right).$$

Hypotheses (H_1) *and* (H_2) *hold, in fact*

$$\begin{aligned} f_1(t,0) &= -\frac{(1-2t)}{20}, \\ f_2(t,0) &= -\frac{e^{-t}}{120}, \end{aligned}$$

$$\begin{aligned} |f_1(t,u) - f_1(t,v)| &\leq \frac{3}{20}(1-t)|u_2 - v_2| \\ &= K_1(t)|u_2 - v_2| \end{aligned}$$

and

$$\begin{aligned} |f_2(t,u) - f_2(t,v)| &\leq \frac{e^{-t}}{40}\sum_{i=1}^{2}|u_i - v_i| \\ &= K_2(t)\sum_{i=1}^{2}|u_i - v_i|. \end{aligned}$$

Moreover, we get by computations,

$$\|K_1\|_{L_1} = \int_0^1 \frac{3}{20}(1-t)\,dt = 0.075,$$

$$\|K_2\|_{L_1} = \int_0^1 \frac{e^{-t}}{40}dt = 1.580\,3 \times 10^{-2},$$

$$\sum_{i=1}^{2}\frac{\|K_i\|_{L_1}}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} = 0.10495 < \frac{1}{4},$$

$$L_1 = \max_{t\in[0,1]}|f_1(t,0)| = \frac{1}{20},\ L_2 = \max_{t\in[0,1]}|f_2(t,0)| = \frac{1}{120}.$$

Then R *can be chosen as*

$$R = 0.5 \geq 4\sum_{i=1}^{2}\frac{L_i}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)},$$

We conclude by Theorem 2.3.1 that the system (S_1) has at least one non-trivial solution u such that $\|u\| \leq 0.5$.

Example 2.4.2 *Consider the system*

$$(S_2)\begin{cases} D_{1^-}^{1.5}\left(D_{0^+}^{1.5}u_1(t)\right) = \frac{e^{-t}}{10}(u_2-u_1) - \frac{t}{4} \\ D_{1^-}^{1.5}\left(D_{0^+}^{1.5}u_2(t)\right) = \frac{e^{-t}}{60}\left(t(u_1+u_3)+\frac{1}{2}\right) \\ D_{1^-}^{1.5}\left(D_{0^+}^{1.5}u_3(t)\right) = \frac{\sin^2 t}{3}\left(u_2 - \frac{t}{2(1+u_2^2)}\right) \\ D_{0^+}^{1.5}u_1(0) = D_{0^+}^{1.5}u_1(1) = 0, \\ D_{0^+}^{1.5}u_2(0) = D_{0^+}^{1.5}u_2(1) = 0, \\ D_{0^+}^{1.5}u_3(0) = D_{0^+}^{1.5}u_3(1) = 0, \\ u_i'(1) = u_i = 0, i = 1,2,3. \end{cases}$$

here $\alpha = \frac{3}{2}$, $\beta_i = \frac{3}{2}$, $i = 1,2,3$, $t \in [0,1]$, $u \in \mathbb{R}^3$,

$$\begin{aligned} f_1(t,u) &= \frac{e^{-t}}{10}(u_2-u_1) - \frac{t}{100}, \\ f_2(t,u) &= \frac{e^{-t}}{60}\left(t(u_1+u_3)+\frac{1}{2}\right), \\ f_3(t,u) &= \frac{\sin^2 t}{3}\left(u_2 - \frac{t}{2(1+u_2^2)}\right) \end{aligned}$$

and

$$\begin{aligned} f_1(t,0) &= -\frac{t}{100}, \\ f_2(t,0) &= \frac{e^{-t}}{120}, \\ f_3(t,0) &= -\frac{t\sin^2 t}{6}. \end{aligned}$$

Hypotheses (H_1) *and* (H_1) *are satisfied. Indeed,*

$$\begin{aligned} |f_1(t,u) - f_1(t,v)| &\le \frac{e^{-t}}{10}\left(|u_1 - v_1| + |u_2 - v_2|\right) \\ &= K_1(t)\sum_{i=1}^{3}|u_i - v_i|, \end{aligned}$$

$$\begin{aligned} |f_2(t,u) - f_2(t,v)| &\le \frac{e^{-t}}{60}\sum_{i=1}^{3}|u_i - v_i| \\ &= K_2(t)\sum_{i=1}^{3}|u_i - v_i| \end{aligned}$$

$$\begin{aligned} |f_3(t,u) - f_3(t,v)| &\le \frac{2\sin^2 t}{3}|u_2 - v_2| \\ &= K_3(t)\sum_{i=1}^{3}|u_i - v_i|. \end{aligned}$$

Some computations yield,

$$\|K_1\|_{L_1} = 0.063212, \quad \|K_2\|_{L_1} = 0.010535, \quad \|K_3\|_{L_1} = 0.181\,78,$$

$$\begin{aligned} L_1 &= \max_{t\in[0,1]}|f_1(t,0)| = \frac{1}{100}, \\ L_2 &= \max_{t\in[0,1]}|f_2(t,0)| = \frac{1}{120}, \\ L_3 &= \max_{t\in[0,1]}|f_3(t,0)| = \frac{\sin^2 1}{6} = 0.11801, \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{3}\frac{\|K_i\|_{L_1}}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} &= 0.11796 < \frac{1}{4}, \\ \sum_{i=1}^{3}\frac{Li}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} &= 0.23555. \end{aligned}$$

Let us choose $R = 1 \geq 0.9422$. *Hence Theorem 2.3.1 implies that the problem* (S_2) *has a nontrivial solution* u *satisfying* $\|u\| < 1$.

CHAPTER 3

Existence of positive solutions for p-Laplacian systems involving left and right fractional derivatives

3.1 Introduction

This Chapter concerns the study of the existence, uniqueness and positivity of solutions for the following system of a coupled nonlinear differential equations involving the p-Laplacian operator and a mixed type of fractional derivatives:

$$(S)\begin{cases} {}^{R}D_{1^-}^{\alpha}\phi_p\left({}^{C}D_{0^+}^{\beta_1}u(t)\right)+a_1(t)\,f_1(u(t),v(t))=0,\ 0<t<1,\\ {}^{R}D_{1^-}^{\alpha}\phi_p\left({}^{C}D_{0^+}^{\beta_2}v(t)\right)+a_2(t)\,f_2(u(t),v(t))=0,\ 0<t<1,\\ \phi_p\left({}^{C}D_{0^+}^{\beta_1}u(1)\right)=0,\quad u'(0)=0,\\ \eta_1u(1)-u(0)=\int_0^1 g_1(s,u(s),v(s))\,ds,\\ \phi_p\left({}^{C}D_{0^+}^{\beta_2}v(1)\right)=0,\quad v'(0)=0,\\ \eta_2v(1)-v(0)=\int_0^1 g_2(s,u(s),v(s))\,ds.\end{cases}$$

Where $0<\alpha<1$, $1<\beta_i<2$, $\eta_i>1$, $i=1,2$ and $\phi_p(s)=|s|^{p-2}s$, $p>1$. Denote by ${}^{R}D_{1^-}^{\alpha}$ the right Riemann-Liouville fractional derivative. ${}^{C}D_{0^+}^{\beta_i}$ denotes the left Caputo fractional derivative of order β_i. The functions $a_i\in C([0,1],\mathbb{R}^+)$, $f_i\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $g_i\in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $i=1,2$.

The uniqueness of the solution is obtained by means of Banach contraction principle, while the existence of positive solutions is proved by the help of Guo-Krasnoselskii fixed point theorem in cones. Furthermore, under some conditions on the nonlinear terms, we prove the nonexistence of positive solutions.

The p-Laplacian operator was introduce for the first time by Leibenson [57] when studying the turbulent flow in porous media. Thenceforward, the p-Laplacian operator was widely introduced in different fields of mathematical modeling, such in mechanics, physics, dynamic systems, ...Moreover, several methods are applied to study differential equations involving the p-Laplacian operator such upper and lower solutions method, fixed point theory, the coincidence degree theory, critical point theory, variational methods, see [6, 20, 21, 41, 80, 82].

In [17] the author discussed the existence of positive solutions for p-Laplacian fractional differential equations with nonlocal boundary conditions

$$\begin{cases} D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}u\left(t\right)\right)+f\left(t,u\left(t\right)\right)=0,\ 0<t<1, \\ D_{0^+}^{\alpha}u\left(0\right)=0, \\ D_{0^+}^{\alpha}u\left(1\right)+\sigma D_{0^+}^{\gamma}u\left(1\right)=0, \\ u\left(0\right)=0. \end{cases}$$

Where $1<\alpha<2, 0<\beta<1, \phi_p\left(s\right)=\left|s\right|^{p-2}s,\ p>1,\ D_{0^+}^{\alpha}, D_{0^+}^{\beta}$ are the Riemann-Liouville fractional derivatives, $0<\gamma\leq 1$, the function $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous.

Recently, more attention are paid to the investigation of initial and boundary value problems involving different types of fractional derivatives. In particular, the existence results for differential equations involving both left and right fractional derivatives is discussed in several articles, see [4, 16, 26, 35, 36, 37, 52, 50, 84, 85]. Let us recall that the left fractional derivative is interpreted as the past state of the process, in which memory effects occur, while the right fractional derivative is interpreted as the future state of this process. In physics, the evolution of many phenomenon depends on both the past and future, then the presence of left and right fractional derivatives in differential equations may appear naturally to represent the evolution of the process.

In [2], the authors discussed by means of fixed point theorems the existence and uniqueness of solutions for a system of coupled differential equations involving mixed type Caputo fractional derivatives

$$\begin{aligned} {}^{C}D_{1^-}^{\alpha}\left({}^{C}D_{0^+}^{\beta}x\left(t\right)\right)&=f\left(t,x,y\right),\ 0<t<1, \\ {}^{C}D_{1^-}^{p}\left({}^{C}D_{0^+}^{q}y\left(t\right)\right)&=g\left(t,x,y\right), \\ x\left(0\right)&=x'\left(0\right)=0,\quad x\left(1\right)=ay\left(\eta\right), \\ y\left(0\right)&=y'\left(0\right)=0, y\quad\left(1\right)=bx\left(\sigma\right). \end{aligned}$$

Here $0\leq\beta,q<1$, $1<\alpha,p<2$, ${}^{C}D_{1^-}^{\alpha}$ and ${}^{C}D_{0^+}^{\beta}$ denote respectively the right and left Caputo derivatives.

This Chapter is structured as follows. Section 2, we study the solvability of the corresponding linear system and we present some properties of the associated Green functions.

In Section 3, first, we prove, by Guo-Krasnosel'skii fixed-point theorem, the existence of positive solutions for the nonlinear system (S). Second, we prove the existence of nonnegative solutions under some conditions on the nonlinear terms and by the help of Schauder's fixed point theorem.

In Section 4, we establish by Banach fixed-point theorem the uniqueness of a solution.

In section 5, we study the nonexistence of positive solutions for the system (S) and in Section 6, some examples are also given to illustrate the obtained results.

3.2 Solvability of an auxiliary system

Let us consider the linear boundary value problem

$$ {}^{C}D_{0+}^{\beta_i} u(t) + y(t) = 0, \; 0 < t < 1, 1 < \beta_i < 2 \tag{3.1} $$

$$ u'(0) = 0, \tag{3.2} $$

$$ \eta_i u(1) - u(0) = \int_0^1 g_i(s)\, ds, \tag{3.3} $$

Lemma 3.2.1 *Let $y \in C([0,1])$, then the unique solution of the boundary value problem (3.1),(3.2) and (3.3) is given by*

$$ u(t) = \int_0^1 G_i(t,s)\, y(s)\, ds + \frac{1}{\eta_i - 1} \int_0^1 g_i(s)\, ds \tag{3.4} $$

where

$$ G_i(t,s) = \frac{1}{\Gamma(\beta_i)} \begin{cases} \frac{\eta_i}{\eta_i - 1}(1-s)^{\beta_i - 1} - (t-s)^{\beta_i - 1}, 0 \le s \le t \le 1 \\ \frac{\eta_i}{\eta_i - 1}(1-s)^{\beta_i - 1}, 0 \le t \le s \le 1. \end{cases} \tag{3.5} $$

Proof We apply (1.21) to equation (3.1), it yields

$$u(t) = -I_{0^+}^{\beta_i} y(t) + a_1 + a_2 t. \tag{3.6}$$

Differentiating (3.6), we get

$$u'(t) = -I_{0^+}^{\beta_i - 1} y(t) + a_2,$$

by the boundary condition (3.2) we obtain $a_2 = 0$, and by (3.3)

$$a_1 = \frac{1}{\eta_i - 1}\left[\frac{\eta_i}{\Gamma(\beta_i)}\int_0^1 (1-s)^{\beta_i - 1} y(s)\,ds + \int_0^1 g_i(s)\,ds\right].$$

Substituting a_1 and a_2 in (3.6), then

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\beta_i)}\left[-\int_0^t (t-s)^{\beta_i - 1} y(s)\,ds + \frac{\eta_i}{\eta_i - 1}\int_0^1 (1-s)^{\beta_i - 1} y(s)\,ds\right] \\ &+ \frac{1}{\eta_i - 1}\int_0^1 g_i(s)\,ds \\ &= \int_0^1 G_i(t,s)\, y(s)\,ds + \frac{1}{\eta_i - 1}\int_0^1 g_i(s)\,ds. \end{aligned}$$

■

Lemma 3.2.2 *Let $y \in C([0,1])$. Then the boundary value problem*

$${}^{R}D_{1^-}^{\alpha} \phi_p\left({}^{C}D_{0^+}^{\beta_i} u(t)\right) + y(t) = 0, 0 \le t \le 1 \tag{3.7}$$

$$\phi_p\left({}^{C}D_{0^+}^{\beta_i} u(1)\right) = 0 \tag{3.8}$$

$$\begin{aligned} u'(0) &= 0 \\ \eta_i u(1) - u(0) &= \int_0^1 g_i(s)\,ds \end{aligned} \tag{3.9}$$

has a unique solution

$$u(t) = \int_0^1 G_i(t,s)\,\phi_q\left(\int_s^1 \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds + \frac{1}{\eta_i - 1}\int_0^1 g_i(s)\,ds$$

where $G_i(t,s)$ are defined in (3.5).

Proof Applying the fractional integral $I_{1^-}^{\alpha}$ to equation (3.7), we obtain

$$\phi_p\left({}^C D_{0^+}^{\beta_i} u(t)\right) = -I_{1^-}^{\alpha} y(t) + a_1 (1-t)^{\alpha-1}, \quad a_1 \in \mathbb{R}, \tag{3.10}$$

then the boundary condition (3.8) implies $a_1 = 0$, hence equation (3.10) becomes

$${}^C D_{0^+}^{\beta_i} u(t) = \phi_q\left(-I_{1^-}^{\alpha} y(t)\right)$$

thus

$${}^C D_{0^+}^{\beta_i} u(t) + \phi_q\left(\frac{1}{\Gamma(\alpha)}\int_t^1 (s-t)^{\alpha-1} y(s)\,ds\right) = 0. \tag{3.11}$$

Consequently the problem (3.7)-(3.9) is equivalent to

$$\begin{gathered} {}^C D_{0^+}^{\beta_i} u(t) + \phi_q\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (s-t)^{\alpha-1} y(s)\,ds\right) = 0, t \in [0,1] \\ u'(0) = 0, \\ \eta_i u(1) - u(0) = \int_0^1 g_i(s)\,ds. \end{gathered}$$

Now, thanks to Lemma 3.2.1, we conclude that the fractional boundary value problem (3.7), (3.8) and (3.9) has a unique solution given by

$$u(t) = \int_0^1 G_i(t,s)\,\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau - s)^{\alpha-1}\,y(\tau)\,d\tau\right)ds + \frac{1}{\eta_i - 1}\int_0^1 g_i(s)\,ds.$$

■

Lemma 3.2.3 *The functions $G_i(t,s)$, $i = 1,2$ are continuous, nonnegative for $t,s \in [0,1]$ and satisfy*

$$\frac{1}{\eta_i}G_i(s,s) \le G_i(t,s) \le G_i(s,s),\ t,s \in [0,1],\ i = 1,2. \tag{3.12}$$

Proof It is easy to show that $G_i(t,s)$ are continuous and nonnegative for $t,s \in [0,1]$, $i = 1,2$. Now we shall show the inequalities in (3.12). For $s \le t \le 1$, we have

$$\begin{aligned} G_i(t,s) &= \frac{\eta_i}{(\eta_i - 1)\Gamma(\beta_i)}(1-s)^{\beta_i - 1} - \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} \\ &\le \frac{\eta_i}{(\eta_i - 1)\Gamma(\beta_i)}(1-s)^{\beta_i - 1} \le G_i(s,s), \end{aligned}$$

moreover, since $G_i(t,s)$ is decreasing with respect to t then $G_i(t,s) \ge G_i(1,s)$, hence

$$G_i(t,s) \ge G_i(1,s) = \frac{(1-s)^{\beta_i - 1}}{(\eta_i - 1)\Gamma(\beta_i)} = \frac{1}{\eta_i}G_i(s,s).$$

Now, let $t \le s$, we have

$$G_i(t,s) = \frac{\eta_i}{(\eta_i - 1)\Gamma(\beta_i)}(1-s)^{\beta_i - 1} = G_i(s,s),$$

remarking that $G_i(t,s)$ is independent of t and $\eta_i > 1$, then

$$G_i(t,s) = \frac{\eta_i}{(\eta_i - 1)\Gamma(\beta_i)}(1-s)^{\beta_i - 1} \ge \frac{(1-s)^{\beta_i - 1}}{(\eta_i - 1)\Gamma(\beta_i)} = \frac{1}{\eta_i}G_i(s,s).$$

■

3.3 Existence of positive solutions

We need to introduce the functional tools and notations for the forthcoming discussion. Let $Y = C[0,1]$ and $X = C[0,1] \times C[0,1]$ be the Banach spaces endowed respectively with the norms

$$\begin{aligned} \|u\|_\infty &= \max_{t\in[0,1]} |u(t)|, \qquad u \in Y, \\ \|(u_1,u_2)\| &= \max_{i=1,2} \|u_i\|_\infty, \qquad (u_1,u_2) \in X. \end{aligned}$$

Define the cones

$$P_1 = \left\{ u \in Y, \min_{t\in[0,1]} u(t) \geq \frac{1}{\eta_1} \|u\|_\infty \right\} \subset Y,$$
$$P_2 = \left\{ u \in Y, \min_{t\in[0,1]} u(t) \geq \frac{1}{\eta_2} \|u\|_\infty \right\} \subset Y,$$

then $P = P_1 \times P_2 \subset X$.

We need the following assumptions.

$$\begin{aligned} &H_1)\ f_i \in C\left(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+\right), \quad i = 1,2. \\ &H_2)\ a_i \in C\left([0,1], \mathbb{R}^+\right), \quad i = 1,2. \\ &H_3)\ g_i \in C\left([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+\right), \quad i = 1,2 \end{aligned}$$

Set

$$\begin{aligned} \Delta_i &= \int_0^1 G_i(s,s)\,ds\ , \\ \Lambda_i &= \frac{a_i^{q-1}}{(\Gamma(\alpha))^{q-1}} \int_0^1 G_i(s,s) \left(\int_s^1 (\tau - s)^{\alpha-1} d\tau \right)^{q-1} ds, \end{aligned}$$

where

$$a_i = \max_{t\in[0.1]} a_i(t)$$

Simple calculations give

$$\Delta_i = \frac{\eta_i}{(\eta_i - 1)\,\Gamma(\beta_i + 1)},$$
$$\Lambda_i = \frac{\eta_i a_i^{q-1}}{(\eta_i - 1)\,(\Gamma(\alpha + 1))^{q-1}\,\Gamma(\beta_i)\,[\alpha(q-1)(\beta_i - 1) + 1]}.$$

Lemma 3.3.1 *(u,v) is a solution for the coupled system (S) if and only if (u,v) is a solution for the following system of integral equations:*

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau - s)^{\alpha-1} a_1(\tau)\, f_1(u(\tau), v(\tau))\, d\tau\right)^{q-1} ds \\ \qquad + \frac{1}{\eta_1 - 1}\int_0^1 g_1(s, u(s), v(s))\, ds \\ v(t) = \int_0^1 G_2(t,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau - s)^{\alpha-1} a_2(\tau)\, f_2(u(\tau), v(\tau))\, d\tau\right)^{q-1} ds \\ \qquad + \frac{1}{\eta_2 - 1}\int_0^1 g_2(s, u(s), v(s))\, ds \end{cases} \tag{3.13}$$

Proof The proof is immediately obtained by Lemma 3.2.2. ■

Define the operator

$$\begin{aligned} \digamma &: X \to X \\ \digamma(u,v) &= (\digamma_1(u,v), \digamma_2(u,v)), \end{aligned} \tag{3.14}$$

where

$$\digamma_i : X \to Y$$

$$\begin{aligned} \digamma_i(u,v) &= \int_0^1 G_i(t,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) \times \right. \\ &\quad \left. f_i(u(\tau), v(\tau))\, d\tau\right)^{q-1} ds + \frac{1}{\eta_i - 1}\int_0^1 g_i(s, u(s), v(s))\, ds. \end{aligned} \tag{3.15}$$

Thanks to Lemma 3.3.1, the system (S) is equivalent to a fixed point problem, that is to prove the existence of solutions for the system (S) it suffices to prove that the operator $\mathcal{F}$ has a fixed point, i.e. $\mathcal{F}(u,v) = (u,v)$.

Lemma 3.3.2 *The operator $\mathcal{F}$ is completely continuous and $\mathcal{F}(P) \subset P$.*

Proof First, let us show that $\mathcal{F}(P) \subset P$. Let $t \in [0,1]$, then taking (3.2.3) into account, we get

$$\begin{aligned}\left|\mathcal{F}_i(u(t),v(t))\right| &\leq \int_0^1 G_i(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_i(\tau)\right. \\ &\qquad f_i(u(\tau),v(\tau))\,d\tau)^{q-1} ds + \frac{1}{\eta_i - 1}\int_0^1 g_i(s,u(s),v(s))\,ds,\end{aligned}$$

that implies by taking the supremum over $[0,1]$

$$\begin{aligned}\left\|\mathcal{F}_i(u,v)\right\|_\infty \leq \int_0^1 G_i(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_i(\tau) f_i(u(\tau),v(\tau))\,d\tau\right)^{q-1} ds \\ + \frac{1}{\eta_i - 1}\int_0^1 g_i(s,u(s),v(s))\,ds.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathcal{F}_i(u(t),v(t)) &\geq \frac{1}{\eta_i}\int_0^1 G_i(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a(\tau)\,\times\right. \\ &\qquad f_i(u(\tau),v(\tau))\,d\tau)^{q-1} ds + \frac{1}{\eta_i - 1}\int_0^1 g_i(s,u(s),v(s))\,ds,\end{aligned}$$

thus,

$$\mathcal{F}_i(u(t),v(t)) \geq \frac{1}{\eta_i}\left\|\mathcal{F}_i(u,v)\right\|_\infty,$$

which implies $\mathcal{F}(P) \subset P$.

Second, we shall prove that $\mathcal{F}$ is completely continuous. Let Ω be an open bounded set in P.

Set

$$L_i = \max_{(u,v)\in\overline{\Omega}} f_i\left(u\left(t\right), v\left(t\right)\right) < \infty, \quad l_i = \max_{(t,u,v)\in[0,1]\times\overline{\Omega}} g_i\left(t, u\left(t\right), v\left(t\right)\right).$$

The proof will be done in two steps.

Step 1. The operator $\mathcal{F}$ is uniformly bounded and equicontinuous on Ω. Indeed, let $(t, u, v) \in [0,1] \times \Omega$, we have

$$\begin{aligned}
\left|\mathcal{F}_i\left(u\left(t\right), v\left(t\right)\right)\right| &\leq \int_0^1 G_i\left(s,s\right)\left(\frac{1}{\Gamma\left(\alpha\right)}\int_s^1 \left(\tau - s\right)^{\alpha-1} a_i\left(\tau\right) f_i\left(u\left(\tau\right), v\left(\tau\right)\right) d\tau\right)^{q-1} ds \\
&+ \frac{1}{\eta_i - 1}\int_0^1 g_i\left(s, u\left(s\right), v\left(s\right)\right) ds \\
&\leq \left[\frac{L_i a_i}{\left(\Gamma\left(\alpha+1\right)\right)}\right]^{q-1}\int_0^1 G_i\left(s,s\right) ds + \frac{l_i}{\left(\eta_i - 1\right)} \\
&= \left[\frac{L_i a_i}{\Gamma\left(\alpha+1\right)}\right]^{q-1} E_i + \frac{l_i}{\eta_i - 1} < \infty,
\end{aligned}$$

thus $\mathcal{F}\left(\Omega\right)$ is uniformly bounded.

Now, let $(u, v) \in \Omega, 0 \leq t_1 \leq t_2 \leq 1$. We have

$$\begin{aligned}
&\left|\mathcal{F}_i\left(u\left(t_1\right), v\left(t_1\right)\right) - \mathcal{F}_i\left(u\left(t_2\right), v\left(t_2\right)\right)\right| \\
\leq &\int_0^{t_1} \left|G_i\left(t_2, s\right) - G_i\left(t_1, s\right)\right| \times \left(\frac{1}{\Gamma\left(\alpha\right)}\right. \\
&\left.\int_s^1 \left(\tau - s\right)^{\alpha-1} a_i\left(\tau\right) f_i\left(u\left(\tau\right), v\left(\tau\right)\right) d\tau\right)^{q-1} ds \\
&+ \int_{t_1}^{t_2} \left|G_i\left(t_2, s\right) - G_i\left(t_1, s\right)\right| \times \left(\frac{1}{\Gamma\left(\alpha\right)}\right. \\
&\left.\int_s^1 \left(\tau - s\right)^{\alpha-1} a_i\left(\tau\right) f_i\left(u\left(\tau\right), v\left(\tau\right)\right) d\tau\right)^{q-1} ds
\end{aligned}$$

$$
\begin{aligned}
&.. + \int_{t_2}^{1} |G_i(t_2,s) - G_i(t_1,s)| \\
&\times \left(\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u(\tau), v(\tau)) \, d\tau \right)^{q-1} ds \\
&\leq \left[\frac{L_i a_i}{\Gamma(\alpha+1)} \right]^{q-1} \frac{|t_2 - t_1|^{\beta_i}}{\Gamma(\beta_i + 1)} \to 0, \quad \text{as } t_2 \to t_1
\end{aligned}
$$

Thus $\mathcal{F}(\Omega)$ is equicontinuous. We conclude by Arzela–Ascoli's theorem that the operator $\mathcal{F}$ is compact on Ω.

Step 2. $\mathcal{F}$ is continuous. In fact, let (u_n, v_n) be an arbitrary convergent sequence in P such $(u_n, v_n) \to (u, v) \in P$. Since f_i are continuous, then

$$
0 \leq f_i(u_n(\tau), v_n(\tau)) \leq L_i, \ \tau \in I, \ n \geq 0,
$$

so,

$$
\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u_n(\tau), v_n(\tau)) \, d\tau \leq \frac{a_i L_i}{\Gamma(\alpha+1)} = c_i. \tag{3.16}
$$

Taking into account that f_i are uniformly continuous, then there exists $N \geq 1$ such that for $n \geq N$, we have

$$
|f_i(u_n(\tau), v_n(\tau)) - f_i(u(\tau), v(\tau))| < \varepsilon,
$$

$$
|g_i(s, u_n(s), v_n(\tau)) - g_i(s, u(s), v(\tau))| < \varepsilon.
$$

According to the values of p and then of q, we have the following.

i) If $1 < q \leq 2$, then by the help of Lemma 1.1.2, it yields

$$\left| \left(\int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u_n(\tau), v_n(\tau)) \, d\tau \right)^{q-1} \right.$$
$$\left. - \left(\int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u(\tau), v(\tau)) \, d\tau \right)^{q-1} \right|$$
$$\leq \left(\int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) \left| f_i(u_n(\tau), v_n(\tau)) - f_i(u(\tau), v(\tau)) \right| d\tau \right)^{q-1}$$
$$< \left[\frac{\varepsilon}{\alpha} a_i \right]^{q-1}.$$

Consequently,

$$\begin{aligned} \left| \mathcal{F}_i(u_n, v_n) - \mathcal{F}_i(u, v) \right| &< \frac{a_i^{q-1} \varepsilon^{q-1}}{(\Gamma(\alpha+1))^{q-1}} \int_0^1 G_i(s, s) \, ds + \frac{\varepsilon}{\eta_i - 1} \\ &= \frac{a_i^{q-1} \varepsilon^{q-1}}{(\Gamma(\alpha+1))^{q-1}} \Delta_i + \frac{1}{\eta_i - 1} \varepsilon \end{aligned}$$

and then

$$\left\| \mathcal{F}_i(u_n, v_n) - \mathcal{F}_i(u, v) \right\|_\infty \leq \left(\frac{a_i^{q-1} \Delta_i}{(\Gamma(\alpha+1))^{q-1}} + \frac{1}{\eta_i - 1} \right) \varepsilon^{q-1}. \tag{3.17}$$

ii) If $q > 2$, then by the help of Lemma 1.1.2, we obtain

$$\left| \left(\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u_n(\tau), v_n(\tau)) \, d\tau \right)^{q-1} \right.$$
$$\left. - \left(\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u(\tau), v(\tau)) \, d\tau \right)^{q-1} \right|$$

$$\leq \frac{(q-1)(c_i)^{q-2}}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) |f_i(u_n(\tau), v_n(\tau)) - f_i(u(\tau), v(\tau))| d\tau$$
$$< \frac{(q-1) c_i^{q-2} a_i}{\Gamma(\alpha+1)} \varepsilon.$$

Hence,

$$|F_i(u_n, v_n) - F_i(u, v)| < \left(\frac{(q-1) c_i^{q-2} a_i}{\Gamma(\alpha+1)} \int_0^1 G_i(s, s) ds + \frac{1}{\eta_i - 1} \right) \varepsilon,$$

consequently

$$\|F_i(u_n, v_n) - F_i(u, v)\|_\infty < \left(\frac{(q-1) c_i^{q-2} a_i}{\Gamma(\alpha+1)} \Delta_i + \frac{1}{\eta_i - 1} \right) \varepsilon. \tag{3.18}$$

In view of (3.17) and (3.18) we conclude the continuity of F. Finally, we deduce from the above discussing that the operator F is completely continuous on P. ■

Now we give an existence result.

Theorem 3.3.1 *Assume that hypotheses* $(H_1) - (H_3)$ *hold and*

H_4) There exist two nonnegative functions $c_1, c_2 \in L^1[0,1]$ and two constants $b_1, b_2 > 0$ such that for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\begin{aligned} g_i(t, u, v) &\leq b_i c_i(t)(u+v), \\ \|c_i\|_{L^1} &\leq \frac{\eta_i - 1}{2b_i}, i = 1, 2. \end{aligned}$$

Then the system (S) has at least one positive solution (u, v), in the case $D_{0,i} = 0$ and $D_{\infty,i} = \infty$, $i = 1, 2$, where

$$D_{\delta,i} = \lim_{(|u|+|v|)\to\delta} \frac{f_i(u, v)}{(|u| + |v|)^{p-1}}, \left(\delta = 0^+ \text{ or } +\infty\right),$$

Proof Since $D_{0,i}=0$, $i=1,2$, then for

$$0<\varepsilon\leq\min_{i=1,2}\left\{\left(\frac{1}{2\Lambda_i}\right)^{\frac{1}{q-1}}\right\},$$

there exists $\rho_1>0$, such that if $0<u+v\leq\rho_1$, then

$$f_i(u,v)\leq\varepsilon(|u|+|v|)^{p-1}.$$

Let

$$\Omega_1=\{(u,v)\in X,\ \ \|(u,v)\|<\rho_1\},$$

and $(u,v)\in P\cap\partial\Omega_1$, then,

$$\begin{aligned}
\mathcal{F}_i(u(t),v(t)) &\leq \int_0^1 G_i(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1(\tau-s)^{\alpha-1}a_i(\tau)\varepsilon(|u|+|v|)^{p-1}d\tau\right)^{q-1}ds\\
&+\frac{1}{\eta_i-1}\int_0^1 b_ic_i(s)(|u|+|v|)\,ds,\\
&\leq\left(\frac{\varepsilon}{\Gamma(\alpha)}\right)^{q-1}\int_0^1 G_i(s,s)\left(\int_s^1(\tau-s)^{\alpha-1}a_i(\tau)\times\right.\\
&\left.(\|u\|_\infty+\|v\|_\infty)^{p-1}d\tau\right)^{q-1}ds+\frac{b_i}{\eta_i-1}\int_s^1 c_i(s)(\|u\|_\infty+\|v\|_\infty)\,ds\\
&\leq\left((\varepsilon)^{q-1}\Lambda_i+\frac{1}{2}\right)\|(u,v)\|.
\end{aligned}$$

Hence

$$\|\mathcal{F}(u,v)\|\leq\|(u,v)\|,\ \text{for}\ (u,v)\in\partial\Omega_1\cap P$$

On the other hand since $D_{\infty,i}=\infty$, $i=1,2$, then for

$$\mu^{q-1}\geq\max_{i=1,2}\left\{\frac{\eta_i}{\xi_i}\right\}(\Gamma(\alpha))^{q-1}$$

where

$$\xi_i = \int_0^1 G_i(s,s) \left(\int_s^1 (\tau - s)^{\alpha-1} a_i(\tau)\, d\tau \right)^{q-1} ds,$$

there exists $\rho > 0$, such that if $u + v \geq \rho$, then

$$f_i(u,v) \geq \mu (|u| + |v|)^{p-1}.$$

Setting $\rho_2 = \max_{i=1,2} \left(\frac{3}{2}\rho_1, \eta_i \rho \right)$ and

$$\Omega_2 = \{(u,v) \in X, \ \|(u,v)\| < \rho_2\},$$

then $\bar{\Omega}_1 \subset \Omega_2$. Let $(u,v) \in P \cap \partial\Omega_2$, then

$$\begin{aligned}
\mathcal{F}_i(u(t), v(t)) &\geq \frac{1}{\eta_i} \int_0^1 G_i(s,s) \times \\
&\quad \left(\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau)\, \mu (|u| + |v|)^{p-1} d\tau \right)^{q-1} ds \\
&\geq \frac{1}{\eta_i} \left(\frac{\mu}{\Gamma(\alpha)} \right)^{q-1} \int_0^1 G_i(s,s) \times \\
&\quad \left(\int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) (\|u\|_\infty + \|v\|_\infty)^{p-1} d\tau \right)^{q-1} ds \\
&\geq \frac{1}{\eta_i} \left(\frac{\mu}{\Gamma(\alpha)} \right)^{q-1} \xi_i \|(u,v)\| \geq \|(u,v)\|
\end{aligned}$$

thus

$$\|\mathcal{F}(u,v)\| \geq \|(u,v)\|, (u,v) \in \partial\Omega_2 \cap P.$$

By the help of Guo-Krasnoselskii fixed point Theorem we deduce that $\mathcal{F}$ has a fixed point $(u,v) \in P \cap \left(\bar{\Omega}_2 \setminus \Omega_1 \right)$, i.e. the system (S) has at least one positive solution (u,v). ■

Remark 3.3.1 *The case $D_{0,i} = 0$ and $D_{\infty,i} = \infty$ is called superlinear case and* *the case* $D_{0,i} = \infty$ *and* $D_{\infty,i} = 0$ *is called sublinear case.*

Theorem 3.3.2 *Assume that hypotheses* $(H_1) - (H_3)$ *are satisfied and*

(H_5) *There exist constants* $c_i, d_i > 0,\ \ 0 < \theta_i, \rho_i < 1,$ *such*

$$0 < f_1(u,v) \leq \left(c_1 |u|^{\theta_1} + c_2 |v|^{\theta_2}\right)^{p-1},$$
$$0 < f_2(u,v) \leq \left(d_1 |u|^{\rho_1} + d_2 |v|^{\rho_2}\right)^{p-1}.$$

(H_6) *There exist two nonnegative functions* $h(t), k(t) \in L_1[0,1]$ *such that*

$$\begin{aligned} g_1(t,u,v) &\leq h(t) + c_1 |u|^{\theta_1} + c_2 |v|^{\theta_2}, \\ g_2(t,u,v) &\leq k(t) + d_1 |u|^{\rho_1} + d_2 |v|^{\rho_2}. \end{aligned}$$

Then the fractional boundary value problem (S) *has at least one positive solution.*

Proof We shall use Schauder fixed-point Theorem. From lemma 3.3.2, we know that $\mathcal{F}$ is completely continuous. Let

$$M = \left\{(u,v) \in P, \|(u,v)\|_X < R\right\},$$

$$R > \max\left\{[3c_1(\Lambda_1 + A_1)]^{\frac{1}{1-\theta_1}}, [3c_2(\Lambda_1 + A_1)]^{\frac{1}{1-\theta_2}}, [3d_1(\Lambda_2 + A_2)]^{\frac{1}{1-\rho_1}}, [3d_2(\Lambda_2 + A_2)]^{\frac{1}{1-\rho_2}}, 3H, 3K\right\},$$

where

$$A_1 = \frac{1}{\eta_1 - 1}, \qquad A_2 = \frac{1}{\eta_2 - 1},$$

and

$$H = A_1 \|h\|_{L_1}, \quad K = A_2 \|k\|_{L_1}.$$

We shall prove that $\mathcal{F}(M) \subset M$. Let $(u,v) \in M$, then

$$\begin{aligned}
|\mathcal{F}_1(u(t),v(t))| &\leq \int_0^1 G_1(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_1(\tau)\times \right.\\
&\left.\left(c_1|u|^{\theta_1}+c_2|v|^{\theta_2}\right)^{p-1} d\tau\right)^{q-1} ds+\\
&\frac{1}{\eta_1-1}\int_0^1 \left(h(s)+c_1|u|^{\theta_1}+c_2|v|^{\theta_2}\right) ds\\
&\leq \int_0^1 G_1(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_1(\tau)\times \right.\\
&\left.\left(c_1R^{\theta_1}+c_2R^{\theta_2}\right)^{p-1} d\tau\right)^{q-1} ds+\\
&\frac{1}{\eta_1-1}\int_0^1 \left(h(s)+c_1R^{\theta_1}+c_2R^{\theta_2}\right) ds.\\
&\leq \Lambda_1\left(c_1R^{\theta_1}+c_2R^{\theta_2}\right)+A_1\left[\|h\|_{L_1}+c_1R^{\theta_1}+c_2R^{\theta_2}\right]\\
&= (\Lambda_1+A_1)c_1R^{\theta_1}+(\Lambda_1+A_1)c_2R^{\theta_2}+H,
\end{aligned}$$

thus,

$$\|\mathcal{F}_1(u,v)\| < \frac{R}{3}+\frac{R}{3}+\frac{R}{3} = R.$$

Similarly, we get

$$\begin{aligned}
\|\mathcal{F}_2(u,v)\| &\leq (\Lambda_2+A_2)d_1R^{\rho_1}+(\Lambda_2+A_2)d_2R^{\rho_2}+K\\
&< \frac{R}{3}+\frac{R}{3}+\frac{R}{3} = R,
\end{aligned}$$

that implies

$$\|\mathcal{F}(u,v)\| < R.$$

Thus, we have $\mathcal{F}(M) \subset M$.

Finally, we conclude by Schauder fixed-point theorem that the operator $\mathcal{F}$ has at least one fixed point $(u,v) \in M$, that implies the system (S) has at least one positive solution in $M \subset P$. ■

3.3.1 Examples

Example 3.3.1 *Consider the system* (S)*, with*

$$\begin{aligned}
f_1(u,v) &= (u+v)^3, \\
a_1(t) &= e^{-2t}, \\
f_2(u,v) &= e^{(u+v)^2}-1, \\
a_2(t) &= 1, \\
g_1(t,u,v) &= \frac{(1-t)(u+v)^2}{3u+4v}, \\
g_2(t,u,v) &= \frac{t}{9}u.
\end{aligned}$$

where $\alpha=\frac{1}{2}$, $\beta_1=\beta_2=\frac{4}{3}$, $p=2$, $\eta_1=\frac{3}{2}$, $\eta_2=\frac{5}{4}$.

Easily we get $D_{0,i}=0$, $D_{\infty,i}=\infty$, $i=1,2$.

$$\begin{aligned}
g_1(t,u,v) &\leq \frac{1-t}{3}(u+v), \\
g_2(t,u,v) &\leq \frac{t}{5}(u+v).
\end{aligned}$$

Since hypotheses (H_1)*-*(H_1) *and* (H_4) *hold, then by Theorem 3.3.1, it follows that the system* (S) *has at least one positive solution.*

Example 3.3.2 *Consider the system (S) with* $\alpha=0.5$, $\beta_1=\beta_2=1.7$, $\eta_1=16$, $\eta_2=100$, $p=2$,

$$\begin{aligned}
f_1(u(t),v(t)) &= \sqrt[3]{v(t)}, \\
f_2(u(t),v(t)) &= \frac{\sqrt[3]{v(t)}}{1+\sqrt[3]{u(t)+v(t)}}, \\
g_1(t,u(t),v(t)) &= 3+\left(t-\frac{1}{3}\right)^5\sqrt[3]{u(s)}, \\
g_1(t,u(t),v(t)) &= \left(t-\frac{1}{3}\right)^5+t\left(\sqrt[3]{u(s)}+\sqrt[3]{v(s)}\right).
\end{aligned}$$

By computation we get

$$\begin{aligned} f_1(u(t), v(t)) &\le \sqrt[3]{u(t)} + \sqrt[3]{v(t)}, \\ f_2(u(t), v(t)) &\le \sqrt[3]{u(t)} + \sqrt[3]{v(t)}, \\ g_1(t, u(t), v(t)) &\le 3 + \sqrt[3]{u(t)} + \sqrt[3]{v(t)}, \\ g_1(t, u(t), v(t)) &\le 1 + \sqrt[3]{u(t)} + \sqrt[3]{v(t)}. \end{aligned}$$

So, we have

$$\begin{aligned} \theta_i &= \rho_i = \frac{1}{3}, \\ c_i &= d_i = 1, \\ h(t) &= 3, \quad k(t) = 1 \end{aligned}$$

Then, all assumptions of Theorem 3.3.2, consequently, the system (S) has at least one solution $(u, v) \in P$.

3.4 Uniqueness results

In this section, we state and prove uniqueness results for the system (S) by using Banach fixed point theorem.

Theorem 3.4.1 *Assume* $1 < p < 2$*, hypotheses* $(H_1) - (H_3)$ *are satisfied, and*

(H_7) *There exist constants* $\mu_i, \xi_i > 0$*, such that for* $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+$*, we have*

$$|f_i(u_1, u_2) - f_i(v_1, v_2)| \le \mu_i \sum_{j=1}^{2} |u_j - v_j|, \quad i = 1, 2,$$

and

$$\frac{1}{\Gamma(\alpha)} \int_0^1 a_i(\tau) f_i(u_1(\tau), u_2(\tau)) d\tau \le \xi_i, \quad i = 1, 2.$$

(H_8) *There exist functions* $K_i \in L^1[0,1]$ *and* $\xi_i > 0$ *such the estimate*

$$|g_i(t,u_1,u_2) - g_i(t,v_1,v_2)| \le K_i(t) \sum_{j=1}^{2} |u_j - v_j|, \quad i = 1,2,$$

holds for all $(t,u_1,u_2), (t,v_1,v_2) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$,

and

$$B = \max_{i=1,2} \left[\left(\frac{(q-1)\,\xi_i^{q-2}}{\Gamma(\alpha+1)} \Delta_i \mu_i a_i + \frac{\|K_i\|_{L^1[0,1]}}{\eta_i - 1} \right) \right] < 1.$$

then the system (S) *has a unique solution.*

Proof Taking into account the properties of the function G_i, we get

$$\begin{aligned} |F_i u(t) - F_i v(t)| \le & \int_0^1 G_i(s,s) \left| \left(\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(u_1,u_2)\, d\tau \right)^{q-1} \right. \\ & \left. - \left(\frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) f_i(v_1,v_2)\, d\tau \right)^{q-1} \right| ds \\ & + \frac{1}{\eta_i - 1} \int_0^1 |g_i(s,u_1,u_2) - g_i(s,v_1,v_2)|\, ds. \end{aligned}$$

Since $1 < p < 2$ then $q > 2$. In view of Lemma 1.1.3, we get

$$\begin{aligned}
\left|\digamma_i u(t) - \digamma_i v(t)\right| &\leq \frac{(q-1)\,\xi_i^{q-2}}{\Gamma(\alpha)} \int_0^1 G_i(s,s) \int_s^1 (\tau - s)^{\alpha-1} a_i(\tau) \times \\
&\quad \left|f_i(u_1,u_2) - f_i(v_1,v_2)\right| d\tau\, ds + \\
&\quad \frac{1}{\eta_i - 1} \int_0^1 \left|K_i(s)\right| \sum_{j=1}^{2} \left|u_j(\tau) - v_j(\tau)\right| \\
&\leq \frac{(q-1)\,\xi_i^{q-2}}{\Gamma(\alpha)} a_i \mu_i \int_0^1 G_i(s,s) \\
&\quad \times \int_s^1 (\tau - s)^{\alpha-1} \sum_{j=1}^{2} \left|u_j(\tau) - v_j(\tau)\right| d\tau ds \\
&\quad + \frac{1}{\eta_i - 1} \int_0^1 \left|K_i(s)\right| \sum_{j=1}^{2} \left|u_j(s) - v(s)_j\right| ds \\
&\leq \sum_{j=1}^{2} \left\|u_j - v_j\right\|_\infty \left(\frac{(q-1)\,c_i^{q-2}}{\Gamma(\alpha)} a_i \mu_i \int_0^1 G_i(s,s) \int_s^1 (\tau - s)^{\alpha-1} d\tau ds \right. \\
&\quad \left. + \frac{1}{\eta_i - 1} \int_0^1 \left|K_i(s)\right| ds \right) \\
&\leq \left[\frac{(q-1)\,\xi_i^{q-2}}{\Gamma(\alpha+1)} \Delta_i \mu_i a_i + \frac{\left\|K_i\right\|_{L^1[0,1]}}{\eta_i - 1} \right] \left\|u - v\right\|.
\end{aligned}$$

Taking the maximum over $t \in [0,1]$, we obtain

$$\left\|\digamma_i u - \digamma_i v\right\|_\infty \leq B \left\|u - v\right\|, \quad i = 1, 2,$$

consequently

$$\left\|\digamma u - \digamma v\right\| \leq B \left\|u - v\right\|.$$

By Banach contraction principle, we deduce the existence of a unique solution for the system (S). ■

Theorem 3.4.2 *Assume $p > 2$ and hypotheses $(H_1) - (H_3)$, (H_7) hold and*

(H_9) *There exist functions $K_i \in L^1[0,1]$, such that for all $(t, u_1, u_2), (t, v_1, v_2) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$, the estimate*

$$|g_i(t, u_1, u_2) - g_i(t, v_1, v_2)| \leq K_i(t) \sum_{j=1}^{2} |u_j - v_j|, \quad i = 1, 2,$$

holds and

$$C = \max_{i=1,2} \left[\left(\frac{\Delta_i \mu_i a_i}{\Gamma(\alpha+1)} + \frac{\|K_i\|_{L^1[0,1]}}{\eta_i - 1} \right) \right] < 1.$$

then the system (S) has a unique solution.

Proof The proof follows easily by remarking that $1 < q < 2$, then using Lemma 1.1.3 and reasoning as in the proof of Theorem 3.4.1. ■

3.4.1 Example

We consider the system (S) with $\alpha = 0.5$, $\beta_1 = \beta_2 = \frac{3}{2}$, $\eta_1 = \eta_2 = 4$, $p = 3$,

$$\begin{aligned}
f_1(u_1, u_2) &= \frac{u_1}{u_1+1} + \frac{u_2}{2u_1 + u_2}, \\
f_2(u_1, u_2) &= \frac{u_2}{u_2 + e^{u_1+u_2}} + 1, \\
a_1(t) &= \frac{1}{44}, \ a_2(t) = \frac{e^{-t}}{44}, \\
g_1(t, u_1, u_2) &= \frac{t u_1}{56(u_1+1)}, \\
g_2(t, u_1, u_2) &= \frac{t(u_1+u_2)}{120(u_1+u_2+e^{u_1})}.
\end{aligned}$$

Some calculations give

$$\begin{aligned}
\left|f_1(u_1,u_2)-f_1(v_1,v_2)\right| &\leq \sum_{j=1}^{2}\left|u_j-v_j\right|,\\
\left|f_2(u_1,u_2)-f_2(v_1,v_2)\right| &\leq \left|u_2-v_2\right| \leq \sum_{j=1}^{2}\left|u_j-v_j\right|,\\
\left|g_1(t,u_1,u_2)-g_1(t,v_1,v_2)\right| &\leq \frac{t}{140}\left|u_1-v_1\right|,\\
\left|g_1(t,u_1,u_2)-g_1(t,v_1,v_2)\right| &\leq \frac{t}{44}\sum_{j=1}^{2}\left|u_j-v_j\right|,
\end{aligned}$$

$$\begin{aligned}
\mu_i &= 1,\ a_i=\frac{1}{44},\ \Delta_i=\frac{4}{3\Gamma\left(\frac{5}{2}\right)},\\
L_i &= 2,\ c_i=\frac{2}{\Gamma(0.5)},
\end{aligned}$$

$$\begin{aligned}
K_1(t) &= \frac{t}{140},\ \|K_1\|_{L^1[0,1]}=\frac{1}{280},\\
K_2(t) &= \frac{t}{44},\ \|K_2\|_{L^1[0,1]}=\frac{1}{88},
\end{aligned}$$

$$A_1=\frac{\Delta_1\mu_1a_1}{\Gamma(\alpha+1)}+\frac{\|K_1\|_{L^1[0,1]}}{\eta_1-1}=2.6912\times10^{-2}<1,$$

$$A_2=\frac{\Delta_2\mu_2a_2}{\Gamma(\alpha+1)}+\frac{\|K_2\|_{L^1[0,1]}}{\eta_2-1}=2.9510\times10^{-2}<1,$$

$$C=\max_{i=1,2}\left[\left(\frac{\Delta_i\mu_ia_i}{\Gamma(\alpha+1)}+\frac{\|K_i\|_{L^1[0,1]}}{\eta_i-1}\right)\right]=2.9510\times10^{-2}<1.$$

Hence all assumptions of Theorem 3.4.2 are satisfied and then the system (S) has a unique solution.

3.5 Nonexistence of positive solutions

In this section, we give sufficient conditions for the system (S) to have no positive solutions.

Theorem 3.5.1 *Assume that hypotheses* $(H_1)-(H_3)$ *are satisfied and that there exist four positive numbers* m_1, m_2, M_1, M_2 *such that*

$$f_1(u,v) \leq m_1\phi_p(u+v), \tag{3.19}$$

$$f_2(u,v) \leq m_2\phi_p(u+v), \tag{3.20}$$

$$g_1(t,u,v) \leq M_1(u+v), \tag{3.21}$$

$$g_2(t,u,v) \leq M_2(u+v), \tag{3.22}$$

for $t \in [0,1]$, $(u,v) \in X$ *with*

$$J_i = \phi_q\left(\frac{m_i}{\Gamma(\alpha+1)}a_i\right)\Delta_i + \frac{M_i}{\eta_i - 1} < \frac{1}{2},\ i = 1,2. \tag{3.23}$$

Then the system (S) *has no positive solution.*

Proof Set

$$D = \max(J_1, J_2) < \frac{1}{2}. \tag{3.24}$$

Assume the contrary, i.e. the system (S) has a positive solution $(u,v)\in P$, then for $t\in[0,1]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_1(\tau) f_1(u(\tau),v(\tau))\,d\tau\right)^{q-1} ds \\ &\quad + \frac{1}{\eta_1-1}\int_0^1 g_1(s,u(s),v(s))\,ds \\ &\le \int_0^1 G_1(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_1(\tau) f_1(u(\tau),v(\tau))\,d\tau\right)^{q-1} ds \\ &\quad + \frac{1}{\eta_1-1}\int_0^1 g_1(s,u(s),v(s))\,ds. \end{aligned}$$

In view of (3.19) and (3.21) of Theorem 3.5.1, we obtain

$$\begin{aligned} u(t) &\le \int_s^1 G_1(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_1(\tau)\, m_1 \times (u(\tau)+v(\tau))^{p-1}\,d\tau\right)^{q-1} ds \\ &+ \frac{1}{\eta_1-1}\int_s^1 M_1(u(s)+v(s))\,ds \\ &\le \left[\left(\frac{m_1}{\Gamma(\alpha+1)}a_1\right)^{q-1}\int_s^1 G_1(s,s)\,ds + \frac{M_1}{\eta_1-1}\right]\left(\|u\|_\infty+\|v\|_\infty\right) \\ &< 2D\,\|(u,v)\|\,,\ \forall t\in[0,1]\,. \end{aligned}$$

Similarly, by (3.20) and (3.22), it yields

$$\begin{aligned} v(t) &\le \int_0^1 G_2(s,s)\left(\frac{1}{\Gamma(\alpha)}\int_s^1 (\tau-s)^{\alpha-1} a_2(\tau)\, m_2 (u(\tau)+v(\tau))^{p-1}\,d\tau\right)^{q-1} ds \\ &\quad + \frac{1}{\eta_2-1}\int_0^1 M_2(u(s)+v(s))\,ds \end{aligned}$$

$$\begin{aligned} v(t) &\leq \int_0^1 G_2(s,s)\left(\frac{a_2 m_2}{\Gamma(\alpha)}\int_s^1 (\tau - s)^{\alpha-1} d\tau\right)^{q-1} (\|u\|_\infty + \|v\|_\infty)\, ds \\ &\quad + \frac{1}{\eta_2 - 1}\int_0^1 M_2 (\|u\|_\infty + \|v\|_\infty)\, ds \\ &< 2\left[\left(\frac{a_2 m_2}{\Gamma(\alpha+1)}\right)^{q-1}\int_0^1 G_2(s,s)\, ds + \frac{M_2}{\eta_2 - 1}\right]\|(u,v)\| \\ &= 2D\|(u,v)\|, \forall t \in [0,1]. \end{aligned}$$

Thus

$$\|u\|_\infty < 2D\|(u,v)\| \text{ and } \|v\|_\infty < 2D\|(u,v)\|,$$

taking into (3.24) account it yields

$$\|(u,v)\| = \max(\|u\|_\infty, \|v\|_\infty) < 2D\|(u,v)\| < \|(u,v)\|$$

which is impossible, and then the system (S) has no positive solution. ■

3.5.1 Example

Example 3.5.1 *We consider the system (S) with* $\alpha = 0.5$, $\beta_1 = \beta_2 = 1.7$, $\eta_1 = 16$, $\eta_2 = 100$, $p = 2$ *and*

$$\begin{aligned} f_1(u,v) &= \left((u+v)^2 - \frac{1}{u}\right), \\ f_2(u,v) &= \left[\frac{v^2}{(u+v)^2 + 5}\right], \\ a_1(t) &= \frac{e^{-t}}{10},\ a_2(t) = \sin^2 t, \\ g_1(u,v) &= \left(2u + t^2 v\right), \\ g_2(u,v) &= \frac{u}{3u + 2tv}, \end{aligned}$$

we check easily that

$$\begin{aligned} f_1(u,v) &\le (u+v)^2, \\ f_2(u,v) &\le (u+v)^2, \end{aligned}$$

$$\begin{aligned} g_1(u,v) &\le 2(u+v), \\ g_2(u,v) &\le (u+v). \end{aligned}$$

By calculation it yields

$$\begin{aligned} a_1 &= \frac{1}{10}, a_2 = \frac{1}{24}, m_1 = m_2 = 1, \\ M_1 &= 2, M_2 = 1, \Delta_1 = \Delta_2 = \frac{4}{3\Gamma\left(\frac{5}{2}\right)}. \end{aligned}$$

$$\begin{aligned} J_1 &= \frac{4}{3\Gamma\left(\frac{5}{2}\right)} \times \frac{1}{10\Gamma(1.5)} + \frac{2}{15} = 0.24651 < 0.5 \\ J_2 &= \frac{4}{3\Gamma\left(\frac{5}{2}\right)} \times \frac{1}{24\Gamma(1.5)} + \frac{1}{99} = 0.05725 < 0.5 \end{aligned}$$

Thanks to Theorem 3.5.1, the system (S) has no positive solution.

Conclusion

In this thesis, we have proved several new and different results for the existence and uniqueness of solutions for certain types of systems for fractional differential equations and p-Laplacian fractional differential equations, involving both left and right fractional derivatives. The main tools used in these studies are fixed point theorems, such as Banach's fixed-point theorem, Schauder's fixed-point theorem, Krasnoselski's fixed-point theorem, and Guo-Krasnoselski's fixed-point theorem in cones. The results presented in this thesis are an important contribution in the field of fractional differential equations.

This work opens the way to new developments on fractional nonlinear systems. Many extensions can be made to our work. In particular, we can study the existence of solutions for similar systems with other types of fractional derivatives such as the derivatives of Hadamards, Grunwald -Letnikov, Erdelyi Kober.... Another perspective is to establish the necessary and sufficient conditions for the existence of solutions for fractional singular systems.

These perspectives constitute possible orientations for future work which will find their place both in a theoretical and numerical frameworks of fractional differential equations.

Finally, it would be interesting to get similar results presented in this thesis under other conditions on the nonlinear terms and by applying other methods from nonlinear analysis.

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