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THESE

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Option : Mathématiques appliquées

Etude de quelques classes d'équations différentielles d'ordre fractionnaire

Par:

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Sous la direction de

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Dedication

To my dear Mother, and To my father

To my wife and my daughter Aya Ismahane

To my sisters and brothers

To my friends and co-workers

ACKNOWLEDGMENTS

First of all, I thank God who enabled me to do this work.

Special and big thanks to to my supervisor Prof. Assia Guezane-Lakoud and my co-advisor Prof. Rabah Khaldi. I thank them for all the advice, guidance and knowledge they gave me. And I will not forget what they did for me, to make me a successful researcher.

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.To accept them to be on the jury.

Contents

ملخص:

في هذه الأطروحة ندرس بعض المسائل التفاضلية الكسرية التي تحوي على كل من المشتق الكسري من اليمين لكابوتو مع المشتق الكسري من اليسار لريمان-ليوفيل.

تم البرهان على وجود حلول في فضاءات تابعية مختلفة (فضاء التوابع المستمرة، فضاء لوبيغ، فضاءات سوبولاف الكسرية) باستعمال نظريات النقطة الصامدة.

تشمل هذه الدراسة عدة شروط مختلفة (متعددة القيم ، غير محلية، تأخير).

كلمات مفتاحية:

معادلة تفاضلية كسرية، نظرية النقطة الصامدة، فضاء لوبيغ، فضاء سوبولاف الكسري، الشروط غير المحلية.

ABSTRACT

In this thesis, we study nonlinear fractional boundary value problems involving both the right Caputo and the left Riemann-Liouville fractional derivatives, and also problems for nonlinear fractional differential equations with Riemann-Liouville fractional derivative. Several boundary conditions (multi-point, non-local, delay) are included.

The existence results are proven by using some fixed-point theorems, in different functional spaces (continuous functions space, Lebesgue space, fractional Sobolev Space).

Keywords:

Fractional differential equation, fixed-point theorem, Lebesgue space, Fractional Sobolev space, Non-local conditions.

Résumé

Dans cette thèse, nous étudions quelques problèmes aux limites contenant des dérivées fractionnaires à droite de Caputo et à gauche de Riemann-Liouville et des conditions aux imites non locales

Les résultats d'existence des solutions sont démontrés en utilisant des théorèmes de point fixe et dans différents espaces fonctionnels (espace des fonctions continues, espace de Lebesgue, espace de Sobolev fractionnaire)

Mots clés:

Equation aux dérivées fractionnaires, Théorème du point fixe, Espace de Lebesgue, Espace de Sobolev fractionnaire, Condition non locale.

.1 Introduction

Fractional calculus started with some attempts by Leibniz in 1695 and 1697 and was developed until recent years (see [61,75,77]), due to the fact that differential equations of noninteger order can represent the dynamics of various memory systems and arise from a variety of applications, including several fields of science and engineering such as geology, physics, optics, chemistry, biology, economics, signal and image processing,... Although the literature on fractional differential equations is now vast, more studies are needed. Recently, the investigation of the qualitative properties of solutions to fractional initial and boundary value problems has attracted the attention of many authors [5,73], and different tools are used in these researches, such as the method of upper and lower solutions, the variational method, the coincidence degree theory, the fixed point theorems ...

In the last few years, many researchers studied linear and nonlinear boundary value problems involving both the right Caputo and the left Riemann-Liouville fractional derivatives, and they used several methods. By the help of operational method and the successive approximations, some linear differential equations containing left and right fractional derivatives that may appear in fractional variational calculus, are studied in [26,32]. Recently, the method of upper and lower solutions is applied in [51,63,64] to solve nonlinear differential equations containing mixed fractional derivatives.
The main objective of this thesis is to study nonlinear boundary value problem with right and left fractional derivatives in different functional spaces

This thesis is divided into five chapters as follows:
In **Chapter 1**, we introduce definitions, basic properties of fractional calculus, functional spaces the L^p spaces, Sobolev spaces, and some fixed point theorems.

In **Chapter 2**, we the study of the existence of solutions for the following nonlinear boundary value problem involving both the right Caputo and the left Riemann-Liouville fractional derivatives:

$$-^C D^{\alpha}_{1^-} D^{\beta}_{0^+} u(t) + \omega^2 u(t) + f(t, u(t)) = 0 \ , \ t \in J = [0,\ 1]. \tag{1}$$

$$D^{\beta}_{0^+} u(1) = 0 \ , \ u(0) = 0 \tag{2}$$

where $0 < \alpha, \beta < 1, \alpha+\beta > 1, \omega \in \mathbb{R}$, ${}^C D^{\alpha}_{1^-}$ and $D^{\beta}_{0^+}$ denote respectively the right Caputo derivative and the left Riemann Liouville derivative, u is the unknown function and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function.
By Krasnoselskii fixed point theorem, we prove the existence of solution for problem (1)-(2).The results of this chapter are published:
H. Moffek, A. Guezane-Lakoud, Existence of solutions to a class of nonlinear boundary value problems with right and left fractional derivarives, AIMS Mathematics, 5(5): 4770-4780 (2020)

Chapter 3, we discuss the existence and uniqueness of solutions for fractional differential equations with multipoint boundary value conditions:

$$D^{\alpha}_{0^+} u(t) + f(t, u(t)) + g(t, D^{\alpha-1}_{0^+} u(t)) = 0 \ , \ t \in J = [0,\ 1]. \tag{3}$$

$$u(0) = 0, , \quad D^{\beta}_{0^+} u(1) = \sum_{k=1}^{m} \xi_k D^{\beta}_{0^+} u(\eta_k). \tag{4}$$

where $1 \leq \alpha \leq 2, 0 \leq \beta \leq 1,\ 0 < \alpha - \beta - 1,\ 0 < \xi_k, \eta_k < 1,\ \ k = 1..m-1$, denotes $D^{\beta}_{0^+}$ the left Riemann-Liouville , u is the unknown function and $f, g : J \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ are given continuous functions .
We get a result with Banachs fixed point theorem ,Krasnoselskii fixed point theorem and Leray-Schauder nonlinear alternative.
By Krasnoselskii fixed point theorem, we prove the existence of solutions for fractional differential equations with multipoint boundary value conditions in a fractional Sobolev space :

$$D^{\alpha}_{0^+}u(t) + f(t, u(t)) + g(t, D^{\beta}_{0^+}u(t)) = 0\ ,\ t \in J = [0,\ 1]. \tag{5}$$

$$D^{(\alpha-i)}_{0^+}u(0) = 0,\ \ i = 2...n,\quad\ D^{\beta}_{0^+}u(1) = \sum_{k=1}^{m} \xi_k D^{\beta}_{0^+}u(\eta_k). \tag{6}$$

where $n - 1 \leq \alpha \leq n,\ n \geq 4\ , 0 \leq \beta \leq 1, 0 < \xi_k, \eta_k < 1,\ \ k = 1..m-1$, denote $D^{\beta}_{0^+}$ the left Riemann-Liouville, u is the unknown function and $f, g : J \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ are given Caratheodory functions .

Chapter 4, this chapter investigates the existence of solutions for a nonlinear fractional oscillator equation with both left and right Caputo fractional derivatives subject to nonlocal conditions.

$$-\,^{C}D^{\alpha}_{1^-}\,^{C}D^{\beta}_{0^+}u(t) + \omega^2 u(t) + f(t, u(t), D^{\beta}_{0^+}u(t)) = 0\ ,\ t \in J = [0,\ 1]. \tag{7}$$

$$^{C}D^{\beta}_{0^+}u(1) = 0\ ,\ u(0) = g(u)\ ,\ u'(0) = h(u). \tag{8}$$

where $0 < \alpha < 1, 1 < \beta < 2, \omega \in \mathbb{R},\ ^{C}D^{\alpha}_{1^-},\ ^{C}D^{\beta}_{0^+}$ denote the right and left Caputo derivative respectively, denotes $D^{\beta}_{0^+}$ the left Riemann-Liouville, $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, and $g, h : C(J, \mathbb{R}) \longrightarrow \mathbb{R}$ are continuous functions.
We use the Krasnoselskiis fixed point theorem.

Chapter 5, Concerns the existence of solutions for a boundary value problem for a nonlinear fractional oscillator equation with both left Riemann-Liouville and right Caputo fractional derivatives, of the form:

$$-\,^{C}D^{\alpha}_{1^-}D^{\beta}_{0^+}u(t) + \omega^2 u(t) + f(t, u_t) = 0\ ,\ t \in J = [0,\ 1]. \tag{9}$$

$$^{C}D^{\beta}_{0^+}u(1) = 0\ ,\ u(t) = \phi(t),\ t \in [-d,\ 0]. \tag{10}$$

where $0 < \alpha < 1,\ 0 < \beta < 1,\ \omega \in \mathbb{R},\ ^{C}D^{\alpha}_{1^-}$ denotes the right Caputo derivative, denotes $D^{\beta}_{0^+}$ the left Riemann-Liouville, u is the unknown function, $f : J \times C([-d,\ 0], \mathbb{R}) \longrightarrow \mathbb{R}$ is a continuous function, and $\phi \in C([-d,\ 0], \mathbb{R})$ with $\phi(0) = 0$. For any continuous function u defined on $[-d,\ 1]$ and any $t \in J$, we denote by u_t the element of $C([-d,\ 0], \mathbb{R})$ defined by

$$u_t(\tau) = u(t + \tau),\ \tau \in [-d,\ 0].$$

Here $u_t(\cdot)$ represents the history of the state from time $(t - d)$ up to the present time t.
The Banach fixed point theorem is used to prove the existence and uniqueness of solutions of the problem (9)-(10), then we apply Leray-Schauder fixed point theorem to conclude the existence of nontrivial solutions.

Chapter I

Preliminaries

In this chapter, we present some notations, definitions, theorems and properties that will be used in the sequel.
This chapter is divided into 4 sections. in the first section, we introduced the special functions, in the second section, we focused on fractional calculus and in section 3, we introduce some functional spaces. Finally, the last section contains some fixed point theorems.

I.1 Special functions

We provide definitions and some properties of the gamma function and the beta function These two functions play a very important role in the theory of fractional calculus.(see [78]).

The Gamma function

Definition 1 *The Gamma function* $\Gamma(.)$ *is defined by*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \quad (Re(z) > 0)$$

This integral is convergent for any complex $z \in \mathbb{C}$ such that $(Re(z) > 0)$.

Proposition 2 *The Gamma function satisfies*
1) $\Gamma(z+1) = z\Gamma(z) \quad (Re(z) > 0)$ *and for any integer* $n \geq 0$, *we have*

$$\Gamma(n+1) = n! \sim \sqrt{2\pi n} n^n e^{-n} \ \ (Stirling's\ formula)$$

2) $\Gamma^{(n)}(z) = \int_0^{+\infty} e^{-t} t^{z-1} \log^n(t) dt,$

3) $\Gamma(x) \lim_{n \to +\infty} \frac{n! n^x}{x(x+1)...(x+n)}; \quad x > 0.$

4) $\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{+\infty} (1 + \frac{x}{n}) e^{\frac{-x}{n}}$, *(Weierstrass formula), where* $\gamma = 05772...$ *is Euler's constant*

5) $\frac{1}{\Gamma(x)\Gamma(1-x)} = x \prod_{n=1}^{+\infty} (1 - \frac{x^2}{n^2})$

6) $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

7) $\Gamma(x)\Gamma(x+\frac{1}{2})=\frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)$ *(Legendre formula)*

8) $\Gamma(x)\Gamma(x+\frac{1}{n})\Gamma(x+\frac{2}{n})...\Gamma(x+\frac{n-1}{n})=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\Gamma(nx)$, *(Gauss formula).*

Some special values of $\Gamma(.)$

1) $\Gamma(\frac{1}{2})=\sqrt{\pi}$

2) $\Gamma(n+\frac{1}{2})=\frac{1.3.5...(2n-1)}{2^n}\sqrt{\pi},\ \ n\in\mathbb{N},$

3) $\Gamma(n+\frac{1}{3})=\frac{1.4.7...(3n-2)}{3^n}\Gamma(\frac{1}{3}),\ \ n\in\mathbb{N},$

4) $\Gamma(n+\frac{1}{4})=\frac{1.5.9...(4n-3)}{4^n}\Gamma(\frac{1}{4}),\ \ n\in\mathbb{N},$

The Beta function

Definition 3 *The beta function is given by*

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt,\quad (Re(z)>0,\ Re(w)>0).$$

Proposition 4 *1)* $B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$

2) $B(z,w)=B(w,z),$

3) $B(z+1,w)=\frac{z}{z+w}B(z,w)$

4) $B(x,1-x)=\frac{\pi}{\sin(\pi x)},\ \ x>0$

5) $B(x,1)=\frac{1}{x}$

6) $B(x,n)=\frac{(n-1)!}{x(x+1)...(x+n-1)},\ \ n\geq 1.$

I.2 Fractional integrals and fractional derivatives

We introduce concepts about fractional calculus and will focus on the Riemann-Liouville integral, and Riemann-Liouville and Caputo derivatives and the relationship between them. We support this chapter with some examples (see[44,61,75,77])

Definition 5 ([61,75]) *Let $J=[a\ ,\ b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha\in\mathbb{R}_+$ are defined by*

$$(I^{\alpha}_{a^+})(f):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)dt}{(x-t)^{1-\alpha}},\qquad x>a,$$

$$(I^{\alpha}_{b^-})(f):=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{f(t)dt}{(t-x)^{1-\alpha}},\qquad x<b,$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals

Theorem 6 ([61,75]) *Let $f \in L^1[a\ ,\ b]$ and $\alpha > 0$. Then, the integral $I^{\alpha}_{a^+}$ exists for almost every $x \in [a\ ,\ b]$. Moreover, the function $I^{\alpha}_{a^+}$ itself is also an element of $L^1[a\ ,\ b]$.*

Proposition 7 ([61]) *Let $\alpha, \beta > 0$ and $f \in L^1[a\ ,\ b]$. Then*
1. $I^{\alpha}_{a^+} I^{\beta}_{a^+} f = I^{\beta}_{a^+} I^{\alpha}_{a^+} f = I^{\alpha+\beta}_{a^+} f$,
2. $I^{\alpha}_{b^-} I^{\beta}_{b^-} f = I^{\beta}_{b^-} I^{\alpha}_{b^-} f = I^{\alpha+\beta}_{b^-} f$,

Lemma 8 ([61]) *1. The fractional integration operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ with $\alpha > 0$ are bounded in $L^p[a\ ,\ b], 1 \le p \le +\infty$*

$$\parallel I^{\alpha}_{a^+} f \parallel_p \le K \parallel f \parallel_p, \ \ \parallel I^{\alpha}_{b^-} f \parallel_p \le K \parallel f \parallel_p, \qquad (K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}).$$

2. If $0 < \alpha < 1$ and $1 < p < \dfrac{1}{\alpha}$, then the operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are bounded from $L^p(a\ , b)$ into $L^q(a\ , b)$, where $q = \dfrac{p}{1-\alpha p}$.

Example 9 *Let $f(x) = (x-a)^{\beta}$ for some $\beta > -1$ and $\alpha > 0$. Then,*

$$I^{\alpha}_{a^+} f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}$$

Theorem 10 ([44]) *Let $\alpha > 0$. Assume that $(f_k)_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $[a\ ,\ b]$. Then we may interchange the fractional integral operator and the limit process, i.e.*

$$(I^{\alpha}_{a^+} \lim_{k \longrightarrow +\infty} f_k)(x) = (\lim_{k \longrightarrow +\infty} I^{\alpha}_{a^+} f_k)(x).$$

In particular, the sequence of functions $(I^{\alpha}_{a^+} f_k)_{k=1}^{\infty}$ is uniformly convergent.

Theorem 11 ([44]) *Let $1 \le p < \infty$ and let $(\alpha_k)_{k=1}^{\infty}$ be a convergent sequence of nonnegative numbers with limit α. Then, for every $f \in L^p[a\ ,\ b]$*

$$\lim_{k \longrightarrow +\infty} I^{\alpha_k}_{a^+} f = I^{\alpha}_{a^+} f.$$

where the convergence is in the sense of the $L^p[a\ ,\ b]$ norm.

Theorem 12 ([44]) *Let $f \in C[a\ ,\ b]$ and $\alpha \ge 0$. Moreover assume that $(\alpha_k)_{k=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{k \longrightarrow +\infty} \alpha_k = \alpha$. Then, for every $\varepsilon > 0$,*

$$\lim_{k \longrightarrow +\infty} \sup_{x \in [a+\varepsilon\ , b]} \mid I^{\alpha_k}_{a^+} f(x) - I^{\alpha}_{a^+} f(x) \mid = 0.$$

Definition 13 ([61,75]) *The Riemann-Liouville fractional derivative $D^{\alpha}_{a^+} f$ and $D^{\alpha}_{b^-}$ of order $\alpha \in \mathbb{R}_+$ are defined by*

$$(D^{\alpha}_{a^+}) f(x) := (\frac{d}{dx})^n (I^{n-\alpha}_{a^+}) f(x) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_a^x \frac{f(s)ds}{(x-s)^{\alpha-n+1}} \ \ (n = [\alpha]+1),$$

and

$$(D^{\alpha}_{b^-}) f(x) := (-\frac{d}{dx})^n (I^{n-\alpha}_{b^-}) f(x) = \frac{1}{\Gamma(n-\alpha)} (-\frac{d}{dx})^n \int_x^b \frac{f(s)ds}{(s-x)^{\alpha-n+1}} \ \ (n = [\alpha]+1),$$

respectively, where $[\alpha]$ means the integer part of α.

Example 14 *Let $f(x) = (x-a)^{\beta}$ for some $\beta > -1$ and $\alpha > 0$. Then,*

$$D^{\alpha}_{a^+} f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}.$$

Proposition 15 ([61,77]) *Let $\alpha \geq \beta > 0$, then for $f \in L^p[a\ ,\ b]$, $(1 \leq p \leq \infty)$, the relations*

$$(D^{\beta}_{a^+} I^{\alpha}_{a^+} f)(x) = I^{\alpha-\beta}_{a^+} f(x) \text{ and } (D^{\beta}_{b^-} I^{\alpha}_{b^-} f)(x) = I^{\alpha-\beta}_{b^-} f(x)$$

holds almost everywhere on $[a\ ,\ b]$. In particular if $\alpha = \beta$ we get

$$(D^{\beta}_{a^+} I^{\alpha}_{a^+} f)(x) = f(x) \text{ and } (D^{\beta}_{b^-} I^{\alpha}_{b^-} f)(x) = f(x).$$

Proposition 16 ([61,75]) *Let $\alpha \geq 0$, $m \in \mathbb{N}$ and $D = \dfrac{d}{dx}$ denotes the classical derivative. 1. If the fractional derivative $(D^{\alpha}_{a^+} f)(x)$ and $(D^{m+\alpha}_{a^+} f)(x)$ exist, then*

$$(D^m D^{\alpha}_{a^+} f)(x) = (D^{m+\alpha}_{a^+} f)(x)$$

2. If the fractional derivative $(D^{\alpha}_{b^-} f)(x)$ and $(D^{m+\alpha}_{b^-} f)(x)$ exist, then

$$(D^m D^{\alpha}_{b^-} f)(x) = (-1)^m (D^{m+\alpha}_{b^-} f)(x).$$

Remark 17 ([61,77]) *In the general case the Riemann-Liouville fractional derivative operators $D^{\alpha}_{a^+}$ and $D^{\beta}_{a^+}$, ($D^{\alpha}_{b^-}$ and $D^{\beta}_{b^-}$) do not commute, i.e.*

$$D^{\alpha}_{a^+} D^{\beta}_{a^+} f \neq D^{\beta}_{a^+} D^{\alpha}_{a^+} f \neq D^{\alpha+\beta}_{a^+} f, \;\; D^{\alpha}_{b^-} D^{\beta}_{b^-} f \neq D^{\beta}_{b^-} D^{\alpha}_{b^-} f \neq D^{\alpha+\beta}_{b^-} f, \;\; \alpha, \beta > 0.$$

Lemma 18 ([61,77]) *Let $f(x) \in L^1[a\ ,\ b]$ and $f_{n-\alpha}(x) \in AC^n[a\ ,\ b]$,then the equality*

$$(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f)(x) = f(x) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}.$$

holds almost everywhere on $[a\ ,\ b]$. In particular, if $0 < \alpha < 1$, then

$$(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f)(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}.$$

where $f_{n-\alpha} = I^{n-\alpha}_{a^+} f$ and $f_{1-\alpha} = I^{1-\alpha}_{a^+} f$

Lemma 19 ([61,77]) *Let $\alpha > 0$, then the fractional differential equation*

$$D^{\alpha}_{0^+} f(t) = 0$$

has $f(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, as solution, where $n = [\alpha] + 1$.

Definition 20 ([61,77]) *The left and right Caputo fractional derivatives of order $\alpha > 0$ of a function $f \in AC^n[a\ ,\ b]$ are defined respectively as*

$${}^{C}D^{\alpha}_{a^+} f(x) = I^{n-\alpha}_{a^+}(\frac{d^n}{dx^n} f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(s) ds}{(x-s)^{\alpha-n+1}}.$$

$${}^{C}D^{\alpha}_{b^-} f(x) = (-1)^n I^{n-\alpha}_{b^-}(\frac{d^n}{dx^n} f(x)) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(s) ds}{(s-x)^{\alpha-n+1}}.$$

where $n = [\alpha] + 1$.

Example 21 *Let* $f(x) = (x-a)^\beta$ *for some* $\beta > 0$ *and* $\alpha > 0, n = [\alpha]+1$. *Then,*

$$^{C}D^{\alpha}_{a^+}f(x) = \begin{cases} 0 & if\ \beta \in \{0,1,2,...,n-1\}, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha} & if\ \beta \in \mathbb{N}\ and\ \beta \geq n \\ & or\ \beta \notin \mathbb{N}\ and\ \beta > n-1. \end{cases}$$

Lemma 22 ([61,77]) *Let* $f \in C^n[a\ ,\ b]$, *then*

$$I^{\alpha}_{a^+}\,{}^{C}D^{\alpha}_{a^+}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

and

$$I^{\alpha}_{b^-}\,{}^{C}D^{\alpha}_{b^-}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k.$$

Proposition 23 ([61,77]) *Let* $\alpha > 0, n = [\alpha]+1$ *and* $f \in AC^n[a\ ,\ b]$, *then*

$$^{C}D^{\alpha}_{a^+}f(x) = D^{\alpha}_{a^+}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}$$

and

$$^{C}D^{\alpha}_{b^-}f(x) = D^{\alpha}_{b^-}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}$$

I.3 Functional spaces

This section contains notations, definitions, and properties of Lebesgue spaces and Sobolev spaces, we need this section in chapter 2, chapters 5 and 6 (see[3,38,45,58])

Definition 24 (L^p Spaces) *1) Let* $p \in \mathbb{R}$ *with* $1 < p < \infty$ *and let* Ω *be a domain in* $\mathbb{R}^n$

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f\ is\ measurable\ and\ \int_\Omega |\ f(t)\ |^p\ dt < \infty\}$$

with

$$\|\ f\ \|_{L^p} = \|\ f\ \|_p = \Big(\int_\Omega |\ f(t)\ |^p\ dt\Big)^{\frac{1}{p}}.$$

2) Si $p = \infty$, *we set*

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R};\quad f\ \textit{is measurable and}$$
$$\textit{there is a constant } C \textit{ such that} \quad |\ f(t)\ | \leq C \textit{ a.e. on } \Omega\}$$

with

$$\|\ f\ \|_{L^\infty} = \inf\{C; |\ f(x)\ | \leq C\ On\Omega\}.$$

Lemma 25 *Let* $1 \leq p \leq \infty$ *and let* p' *such that* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$
1) ***(Holder's inequality)*** *Assume that* $f \in L^p$ *and* $g \in L^{p'}$. *Then* $fg \in L^1$ *and*

$$\int_\Omega | fg | \leq \| f \|_p \| g \|_{p'} .$$

2) ***(Minkowski's inequality)*** *Assume that* $f \in L^p$ *and* $g \in L^p$. *Then* $f + g \in L^p$ *and*

$$\| f + g \|_p \leq \| f \|_p + \| g \|_p .$$

Definition 26 (Sobolev Space) *Let* $\Omega = (a, b)$ *be an open interval, possibly unbounded, and let* $p \in \mathbb{R}$ *with* $1 \leq p \leq \infty$.
The Sobolev space $W^{1,p}(\Omega)$ *is defined to be*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega); u' \in L^p(\Omega)\}$$

where u' *is the weak derivative.*
The $W^{1,p}(\Omega)$ *space is equipped with the norm* $\| u \|_{W^{1,p}} = \| u \|_p + \| u' \|_p$.

Definition 27 (Riemann-Liouville fractional Sobolev spaces) *Let* $p \in [1, +\infty]$ *and* $0 < s < 1$. *We define the left Riemann-Liouville fractional Sobolev space of order* s *and summability* p *as*

$$W^{s,p}_{RL,a^+}(\Omega) := \{u \in L^p(\Omega) : I^{1-s}_{a^+}(u) \in W^{1,p}(\Omega)\},$$

endowed with the norm $\| u \|_{W^{s,p}_{RL,a^+}} = \| u \|_p + \| I^{1-s}_{a^+}(u) \|_{W^{1,p}}$. *And we define the right Riemann-Liouville fractional Sobolev space of order* s *and summability* p *as*

$$W^{s,p}_{RL,b^-}(\Omega) := \{u \in L^p(\Omega) : I^{1-s}_{b^-}(u) \in W^{1,p}(\Omega)\}.$$

endowed with the norm $\| u \|_{W^{s,p}_{RL,b^-}} = \| u \|_p + \| I^{1-s}_{b^-}(u) \|_{W^{1,p}}$.

Remark 28 *1)* $(W^{s,p}_{RL,a^+}(\Omega), \| u \|_{W^{s,p}_{RL,a^+}})$ *is a Banach space.*
2) The norm $\| u \|_{W^{s,p}_{RL,a^+}})$ *is equivalent to the norm* $\| u \|_{RL}$, *where* $\| u \|_{RL} := \| u \|_p + \| D^s_{a^+}(u) \|_p$

Theorem 29 (Riesz compactness criteria ([38])) *Let* F *be a bounded set in* $L^p[0, 1], 1 \leq p < \infty$.
Assume that:
(i) $\lim_{h\to 0} \| \tau_h f - f \|_p = 0$ *uniformly on* F, *where* $\tau_h f(t) = f(t + h)$.
(ii) $\lim_{\varepsilon\to 0} \int_{1-\varepsilon}^1 | f(t) |^p \, dt = 0$ *uniformly on* F.
Then F *is relatively compact in* $L^p[0, 1]$.

I.4 Fixed point theorems

In this section, we cite some fixed point theorems.

Theorem 30 (Banach) *Let* A *be a contraction on a Banach space* E. *Then* A *has a unique fixed point*

Theorem 31 (Nonlinear Alternative of Leray-Schauder Type) *Let X be a Banach space,C a closed, convex subset of X, U an open subset of C and $0 \in U$.Suppose that $A : \overline{U} \to C$ is a continuous and compact map. Then either*
(i) A has a fixed point in $\overline{U}$, or
(ii) There exists $\lambda \in (0, 1)$ and $x \in \partial U$ (the boundary of U in C) with $x = \lambda A(x)$.

Theorem 32 (Krasnoselskii [65]) *Let M be a closed bounded convex nonempty subset of a Banach space E. Suppose that A and B map M into E such that*
(i) A is completely continuous,
(ii) B is a contraction mapping,
(iii) $x, y \in M$ implies $Ax + By \in M$
Then there exists $z \in M$ with $z = Az + Bz$.

Chapter II

Existence of solutions to a class of nonlinear boundary value problems with right and left fractional derivatives

II.1 Introduction

The aim of this chapter is the study of the existence of solutions, for the following nonlinear boundary value problem (P) involving both the right Caputo and the left Riemann-Liouville fractional derivatives:

$$-^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t)+\omega^2u(t)+f(t,u(t))=0\ ,\ t\in J=[0,\ 1]. \qquad \text{(P)}$$
$$D_{0^+}^{\beta}u(1)=0\ ,\ u(0)=0$$

where $0<\alpha,\beta<1,\alpha+\beta>1,\omega\in\mathbb{R}$, ${}^{C}D_{1^-}^{\alpha}$ and $D_{0^+}^{\beta}$ denote respectively the right Caputo derivative and the left Riemann Liouville derivative, u is the unknown function and $f:J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Caratheodory function.
Let us mention that if α and β tend to one, then problem (P) is a classical oscillator boundary value problem that is investigated in [6]. Note that problem (P) is studied in [51] by lower and upper solutions method, the authors proved the existence of solution under some specific conditions on the nonlinear term f. In the present study, we prove the existence of solution for problem (P) under Lipschitz type condition on the nonlinear term f and by using Krasnoselskiis fixed point theorem.
In [10], the authors considered a coupled system of nonlinear differential equations involving mixed type fractional derivatives

$$\begin{aligned}&-^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}x(t)=f(t,x(t),y(t)),\\&-^{C}D_{1^-}^{p}D_{0^+}^{q}x(t)=g(t,x(t),y(t)),\ 0<t<1,\end{aligned}$$

with nonlocal boundary conditions

$$\begin{aligned}&x(0)=x'(0)=0,\ \ x(1)=\gamma y(\eta),\ 0<\eta<1,\\&y(0)=y'(0)=0,\ \ y(1)=\delta x(\theta),\ 0<\theta<1.\end{aligned}$$

here $1<\alpha,p<2,0<\beta,q<1,\gamma,\delta\in\mathbb{R}$. The existence and uniqueness of solution is proved by the help of Leray-Schauder alternative and Banach fixed point theorem.
By Krasnoselskii's fixed point theorem, the authors in [48,52], investigated some boundary value problems involving mixed type fractional derivatives. In particular in [48], proved, under Lipschitz type condition on the nonlinear term, the existence of solution in a weighted space, for the following boundary value problem

$$\begin{aligned}&-^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t)=f(t,u(t)),\ 0<t<1\\&\lim_{t\to0^+}t^{1-\beta}u(t)=u(1)=u(\eta),\end{aligned}$$

where $0 < \alpha, \beta < 1, 1 < \alpha + \beta < 2$.
In [52], the authors studied by the help of Krasnoselskii's fixed point theorem and Arzela-Ascoli theorem, the existence of solution for the problem

$$\begin{array}{l} -^C D^{\alpha}_{1^-} D^{\beta}_{0^+} u(t) = f(t, u(t)),\ 0 < t < 1 \\ u(0) = u'(0) = u(1) = 0, \end{array}$$

where $0 < \alpha \leq 1, 1 < \beta \leq 2,^C D^{\alpha}_{1^-}$ denotes right Caputo derivative, $D^{\beta}_{0^+}$ denotes the left Riemann-Liouville and $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Lipschitz type condition.

II.2 Main results

To study the nonlinear problem (P), we consider first, the associated linear problem

$$-^C D^{\alpha}_{1^-} D^{\beta}_{0^+} u(t) + y(t) = 0\ ,\ t \in J = [0,\ 1]. \tag{II.1}$$

$$D^{\beta}_{0^+} u(1) = 0\ ,\ u(0) = 0. \tag{II.2}$$

Lemma 33 *Assume that $y \in L^p(J), p > 1$, then u is a solution to the linear boundary value problem $(II.1)(II.2)$ if and only if u satisfies the integral equation*

$$u(t) = \int_0^1 G(t, \tau) y(\tau) d\tau$$

where

$$G(t, \tau) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \begin{cases} \int_0^{\tau} (t - s)^{\beta - 1} (\tau - s)^{\alpha - 1} ds, & 0 \leq \tau \leq t \leq 1, \\ \int_0^{t} (t - s)^{\beta - 1} (\tau - s)^{\alpha - 1} ds, & 0 \leq t \leq \tau \leq 1. \end{cases} \tag{II.3}$$

Proof. Applying the right-hand side fractional integral $I^{\alpha}_{1^-}$ to equation $(II.1)$, we get

$$D^{\beta}_{0^+} u(t) = I^{\alpha}_{1^-} y(t) + a,\ \ a \in \mathbb{R}$$

The boundary condition $D^{\beta}_{0^+} u(1) = 0$, gives $a = 0$, then applying the fractional integral $I^{\beta}_{0^+}$, the obtained equation, it yields

$$u(t) = I^{\beta}_{0^+} I^{\alpha}_{1^-} y(t) + c t^{\beta - 1},\ \ c \in \mathbb{R} \qquad (2.5)$$

Multiplying the equation (2.5) by $t^{1-\beta}$, then using the condition $u(0) = 0$, we obtain $c = 0$, thus

$$\begin{aligned} u(t) &= I^{\beta}_{0^+} I^{\alpha}_{1^-} y(t) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \Big(\int_s^1 (\tau - s)^{\alpha - 1} y(\tau) d\tau \Big) ds. \end{aligned}$$

Finally, by Fubini theorem, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \Big(\int_0^{\tau} (t - s)^{\beta - 1} (\tau - s)^{\alpha - 1} ds \Big) y(\tau) d\tau \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^1 \Big(\int_0^{t} (t - s)^{\beta - 1} (\tau - s)^{\alpha - 1} ds \Big) y(\tau) d\tau \end{aligned}$$

.

■

Lemma 34 *The function G satisfies the following properties:*
(1) The function $G(t,\tau)$ is nonnegative.
(2) $G(t,\tau) < \dfrac{1}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}$ *for all $t,\tau \in J$*

Remark 35 *Let us mention the case $\alpha+\beta \to 1^+$. Since $\alpha+\beta > 1$ and $0<\alpha,\beta<1$, then $\alpha > \frac{1}{2}$ or $\beta > \frac{1}{2}$. If $\alpha > \frac{1}{2}$, then $\alpha+\beta \to 1^+$ implies ($\alpha \to 1^-$ and $\beta \to 0$) or ($\alpha \to \frac{1^+}{2}$ and $\beta \to \frac{1^-}{2}$), then the problem (P) is reduced respectively to*

$$\begin{aligned} u' + \omega^2 u(t) + f(t,u(t)) = 0, \;\; t \in J = [0,1]. \\ u(0) = 0 \end{aligned} \tag{P1}$$

and

$$\begin{aligned} -{}^C D^{\frac{1}{2}}_{1^-} D^{\frac{1}{2}}_{0^+} u(t) + \omega^2 u(t) + f(t,u(t)) = 0, \;\; t \in J = [0,1]. \\ u(0) = 0, \;\; D^{\frac{1}{2}}_{0^+} u(1) = 0. \end{aligned} \tag{P2}$$

For problem $(P2)$, let us fix $\alpha = \frac{1}{2}$, then we have,

$$\begin{aligned} G(1,1) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\beta+\alpha-2} ds = \frac{1}{\Gamma(\alpha)\Gamma(\beta)(\beta+\alpha-1)} \\ &= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\beta)} \int_0^1 (1-s)^{\beta-\frac{3}{2}} ds = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\beta)(\beta-\frac{1}{2})} \longrightarrow -\infty, \end{aligned}$$

as $\beta \to \frac{1^-}{2}$
thus the Green function is not bounded.

Lemma 36 *The function $u \in L^p(0,1)$ is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,\tau) f(\tau,u(\tau)) d\tau + \omega^2 \int_0^1 G(t,\tau) u(\tau) d\tau.$$

if and only if u is a solution of the fractional boundary value problem (P).

Now we define the operators A and B on $L^p(0,1)$ as

$$Au(t) = \int_0^1 G(t,\tau) f(\tau,u(\tau)) d\tau,$$

$$Bu(t) = \omega^2 \int_0^1 G(t,\tau) u(\tau) d\tau.$$

Obviously, the problem (P) has a solution if and only if the operator $A+B$ has a fixed point in $L^p(0,1)$.
Before stating and proving the main results, we introduce the following hypotheses.
(H1) $M = \sup_{0\le t\le 1} \mid f(t,0) \mid < \infty$, and there exists a constant k, $0 < \dfrac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \le \dfrac{1}{2}$, such that

$$\mid f(t,u) - f(t,v) \mid \le k \mid u - v \mid, \quad 0 \le t \le 1, \; u,v \in \mathbb{R}.$$

(H2) $\dfrac{\omega^2}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} < \dfrac{1}{2}$.

Theorem 37 *Assume that **(H1)-(H2)** hold, then the fractional boundary value problem (P) has a nontrivial solution in $L^p(0,1)$.*

To prove Theorem 37, we need the following lemmas.

Lemma 38 *Under the hypotheses **(H1)-(H2)**, the operator A is completely continuous on $L^p(0,1)$.*

Proof. Let

$$\Omega = \{u \in L^p(0,1), \| u \|_{L^p} \leq R\}$$

such that

$$R \geq \frac{M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)-(k+\omega^2)}. \tag{II.4}$$

Clearly, Ω is a nonempty, bounded and convex subset of the Banach space $L^p(0,1)$. We should prove that A is continuous and relatively compact on $L^p(0,1)$.

Claim 1. The mapping A is continuous on Ω. In fact, consider the sequence $(u_n)_n \in \Omega$, such that $u_n \longrightarrow u$ in $L^p(0,1)$, then from Lemma 34, hypothesis **(H1)** and Hölder inequality, we get

$$\begin{aligned}
\mid Au_n(t) - Au(t) \mid &\leq \int_0^1 G(t,\tau) \mid f(\tau,u_n(\tau)) - f(\tau,u(\tau)) \mid d\tau \\
&\leq \frac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mid u_n(\tau) - u(\tau) \mid d\tau \\
&\leq \frac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \| u_n(.) - u(.) \|_{L^p(0.1)} .
\end{aligned}$$

Hence

$$\| Au_n(.) - Au(.) \|_{L^p(0.1)} \leq \frac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \| u_n(.) - u(.) \|_{L^p(0.1)} \longrightarrow 0, \; as\; n \longrightarrow \infty.$$

Claim 2. (Au) is bounded in $L^p(0.1)$. Indeed, let $u \in \Omega$, then by condition **(H1)** and Hölder inequality, it yields

$$\begin{aligned}
\mid Au(t) \mid &\leq \frac{1}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mid f(\tau,u(\tau)) \mid d\tau \\
&\leq \frac{1}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \Big(k \Big(\int_0^1 \mid u(\tau) \mid d\tau \Big) + \int_0^1 \mid f(\tau,0) \mid d\tau \Big) \\
&\leq \frac{1}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \Big(k \Big(\int_0^1 \mid u(\tau) \mid^p d\tau \Big)^{\frac{1}{p}} + \int_0^1 \mid f(\tau,0) \mid d\tau \Big) \\
&\leq \frac{kR+M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)},
\end{aligned}$$

thus

$$\| Au \|_{L^p} \leq \frac{kR+M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}.$$

Claim 3. (Au) is relatively compact. In fact, let $u \in \Omega$, and $p > 1$, we have

$$\begin{aligned}
\mid Au(t+h) - Au(t) \mid &\leq \int_0^1 \mid G(t+h,\tau) - G(t,\tau) \mid\mid f(\tau,u(\tau)) \mid d\tau \\
&\leq \int_0^1 \mid G(t+h,\tau) - G(t,\tau) \mid (k \mid u(\tau) \mid + \mid f(\tau,0) \mid) d\tau \\
&\leq (kR+M) \Big(\int_0^1 \mid G(t+h,\tau) - G(t,\tau) \mid^p d\tau \Big)^{\frac{1}{p}}
\end{aligned}$$

$$\leq (kR+M)\Big(\int_0^t \mid G(t+h,\tau)-G(t,\tau)\mid^p d\tau+\int_t^{t+h}\mid G(t+h,\tau)-G(t,\tau)\mid^p d\tau$$
$$+\int_{t+h}^1 \mid G(t+h,\tau)-G(t,\tau)\mid^p d\tau\Big)^{\frac{1}{p}}$$
$$\leq \frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}\Big(\int_0^t\big(\int_0^\tau((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\big)^p d\tau$$
$$+\int_t^1\big(\int_0^t((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\big)^p d\tau$$
$$+\int_t^{t+h}\big(\int_0^\tau(t+h-s)^{\beta-1})ds\big)^p d\tau\Big)^{\frac{1}{p}}$$
$$=\frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}(I_1+I_2+I_3)^{\frac{1}{p}},$$

hence

$$\mid Au(t+h)-Au(t)\mid\leq \frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}(I_1+I_2+I_3)^{\frac{1}{p}}. \tag{II.5}$$

Let us calculate $I_i,\ i=1,2,3.$

$$I_1=\int_0^t\big(\int_0^\tau((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\big)^p d\tau$$
$$\leq (h(1-\beta))^p\int_0^t\big(\int_0^\tau(\tau-s)^{\alpha-1}ds\big)^p d\tau\leq\Big(\frac{h(1-\beta)}{\alpha(\alpha+1)}\Big)^p.$$

$$I_2=\int_t^1\big(\int_0^t((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\big)^p d\tau$$
$$\leq (h(1-\beta))^p\int_0^t((1-s)^\alpha-(t-s)^\alpha)^p ds\leq\frac{(h(1-\beta))^p}{\alpha p+1}.$$

$$I_3=\int_t^{t+h}\big(\int_0^\tau(t+h-s)^{\beta-1})ds\big)^p d\tau$$
$$\leq\frac{1}{\beta^p}\int_t^{t+h}(h^\beta-(t+h-\tau)^\beta)^p d\tau\leq\frac{h^{\beta p+1}}{\beta^p}.$$

Finally, we get

$$\parallel Au(.+h)-Au(.)\parallel_{L^p}\leq\frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}\Big(\Big(\frac{h(1-\beta)}{\alpha(\alpha+1)}\Big)^p+\frac{(h(1-\beta))^p}{\alpha p+1}+\frac{h^{\beta p+1}}{\beta^p}\Big)^{\frac{1}{p}} \tag{II.6}$$

By taking the limit in (II.6) as $h\longrightarrow 0$, we obtain that $\parallel Au(.+h)-Au(.)\parallel_{L^p}\longrightarrow 0$ for any $u\in\Omega$.
On the other hand we have by the help of claim 2

$$\int_{1-\varepsilon}^1\mid Au(t)\mid^p dt\leq\varepsilon\Big(\frac{kR+M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}\Big)^p\to 0\ \ as\ \varepsilon\to 0.$$

By Riesz compactness criteria Theorem , we conclude that A is relatively compact on Ω. From the above discussion we conclude that A completely continuous on $L^p(0,1)$. ■

Lemma 39 *Under the hypothesis **(H2)**, the mapping B is a contraction on Ω.*

Proof. Let $u \in \Omega$ and $t \in J$, we have

$$\begin{aligned} \mid Bu(t) - Bv(t) \mid &\leq \omega^2 \int_0^1 G(t,\tau) \mid u(\tau) - v(\tau) \mid d\tau \\ &\leq \frac{\omega^2}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \parallel u - v \parallel_{L^p}, \end{aligned}$$

hence

$$\parallel Bu - Bv \parallel_{L^p} \leq \frac{\omega^2}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \parallel u - v \parallel_{L^p},$$

by hypothesis **(H2)**, we conclude that B is a contraction. ■

Lemma 40 *Assume that hypotheses (H1) and (H2) hold, then $Au + Bv \in \Omega$ for all $u, v \in \Omega$.*

Proof. Let $u, v \in \Omega$ then taking (II.4) into account, it yields

$$\begin{aligned} \parallel Au + Bv \parallel_{L^p} &\leq \parallel Au \parallel_{L^p} + \parallel Bv \parallel_{L^p} \\ &\leq \frac{R(\omega^2 + k) + M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} \leq R, \end{aligned}$$

hence $Au + Bv \in \Omega$. ■

Proof of Theorem 37. By Lemmas 38, 39 and 40, we conclude respectively that the mapping A is completely continuous, the mapping B is a contraction and $Au + Bv \in \Omega$ for all $u, v \in \Omega$, then all hypotheses of Theorem 32 are satisfied. Hence, there exists a nontrivial solution $u \in \Omega$ for problem (P) such that $u = Au + Bu$. The proof is complete.

Now, we give an example to illustrate the usefulness of the obtained results.

Example 1. Consider the problem (P) with

$$f(t,x) = \frac{e^{-t}x}{9 + e^t(1+x^2)} + e^t, \;\; (t,x) \in J \times \mathbb{R},$$
$$\omega = 0.5, \;\alpha = 0.5, \;\beta = 0.8,$$
$$M = \sup_{0 \leq t \leq 1} \mid f(t,0) \mid = e = 2.7183.$$

Let us check hypotheses (H1)-(H2). We have for all $(t,x) \in J \times \mathbb{R}$

$$\mid f(t,x) - f(t,y) \mid \leq \frac{e^{-t}}{9 + e^t} \mid x - y \mid \leq \frac{1}{10} \mid x - y \mid,$$

then $k = \frac{1}{10}$, $0 < k = 0.1 \leq \frac{1}{2}(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta) = 0.30953$. By Theorem 37, we conclude that the problem (P) has a nontrivial solution $u \in L^p(0,1)$, such that $\parallel u \parallel_{L^p} \leq R$ where $R \geq 10.103$ and $u = Au + Bu$.

Example 2. Consider the problem (P) with

$$f(t,x) = \frac{t^{\frac{1}{3}} \sin x + t^3}{15}, \;\; (t,x) \in J \times \mathbb{R},$$
$$\omega = \frac{1}{10}, \;\alpha = \frac{1}{3}, \;\beta = \frac{3}{4},$$
$$M = \sup_{0 \leq t \leq 1} \mid f(t,0) \mid = \frac{1}{15}.$$

We have for all $(t,x) \in J \times \mathbb{R}$

$$| f(t,x) - f(t,y) | \leq \frac{t^{\frac{1}{3}}}{15} | \sin(x) - \sin(y) | \leq \frac{1}{15} | x - y |,$$

and $k = \frac{1}{15}$, $\frac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} = 0..24369 \leq \frac{1}{2}$, $\frac{\omega^2}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} = 3.6554 \times 10^{-2} < \frac{1}{2}$. Thus hypotheses (H1) and (H2) are satisfied.

By Theorem 37, we conclude that the problem (P) has a nontrivial solution $u \in L^p(0,1)$, such that $\| u \|_{L^p} \leq R$ where $R = 1 > \frac{M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta) - (k+\omega^2)} = 0.33858$ and $u = Au + Bu$.

Chapter III

Existence Solutions Of Multi-Point Boundary Value Problems For Nonlinear Fractional Differential Equations

III.1 Introduction

In this chapter, we study the existence of solutions for fractional differential equations with multi-point boundary value conditions. We get a result with Banach fixed point theorem, Krasnoselskii fixed point theorem and Leray-Schauder nonlinear alternative.
In [54], the authors studied the existence of positive solutions in a Sobolev space for a fractional boundary value problem:

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t), D^{\gamma}_{0^+}u(t)), & 0 < t < 1, \\ \lim_{t\to 0} t^{i-\alpha}u(t) = 0, & i = 2, ..., n, \\ u(1) = \sum_{k=0}^{m} \lambda_k I^{\beta}_{0^+}u(\eta_k). \end{cases}$$

where $n - 1 \leq \alpha < n, n \geq 4, 0 < \gamma < 1, \beta > 0, \lambda_k > 0, 0 < \eta_k < 1, k = 0, ..., m$ and $f : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ is Caratheodory function By utilizing the method of the lower and upper solution and Schauder fixed-point theorem,the authors got the existence of a solution.

In [68], by using the Schauder fixed point theorem, the author proved the positivity of solutions for the following multi-point boundary value problem (BVP)

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = 0,\ D^{\beta}_{0^+}u(1) = \sum_{i=1}^{m} \xi_i D^{\beta}_{0^+}u(\eta_i). \end{cases}$$

where $1 \leq \alpha \leq 2, 0 \leq \beta \leq 1,\ 0 < \alpha - \beta - 1,\ 0 < \xi_i, \eta_i < 1,\ \ i = 1..m, \sum_{i=1}^{m} \xi_i \eta_i^{\alpha-\beta-1} \neq 1$, denotes $D^{\beta}_{0^+}$ the left Riemann-Liouville, and $f : [0, 1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous.

The aim of this chapter is to study of existence of solutions for fractional differential equations with multipoint boundary value conditions :

$$D^{\alpha}_{0^+}u(t) + f(t, u(t)) + g(t, D^{\alpha-1}_{0^+}u(t)) = 0\ ,\ t \in J = [0,\ 1]. \tag{III.1}$$

$$u(0) = 0, ,\quad D^{\beta}_{0^+}u(1) = \sum_{k=1}^{m} \xi_k D^{\beta}_{0^+}u(\eta_k). \tag{III.2}$$

where $1 \leq \alpha \leq 2, 0 \leq \beta \leq 1,\ 0 < \alpha - \beta - 1,\ 0 < \xi_k, \eta_k < 1,\ \ k = 1..m$, denotes D^{β}_{0+} the left Riemann-Liouville , u is the unknown function and $f, g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions.

To study the nonlinear problem (III.1)(III.2), we first consider the associated linear problem

$$D^{\alpha}_{0+}u(t) + h(t) = 0\ ,\ t \in J = [0,\ 1]. \tag{III.3}$$

$$u(0) = 0, \quad D^{\beta}_{0+}u(1) = \sum_{k=1}^{m} \xi_k D^{\beta}_{0+}u(\eta_k). \tag{III.4}$$

Lemma 41 *Assume that $h \in C(J)$ and $\delta > 0$, then u is a solution to the linear boundary value problem $(III.3)(III.4)$ if and only if u satisfies the integral equation*

$$u(t) = \int_0^1 G(t,\tau)h(\tau)d\tau + \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k \int_0^1 H(\eta_k,\tau)h(\tau)d\tau$$

where $\delta = 1 - \sum_{k=1}^{m} \xi_k \eta_k^{\alpha-\beta-1}$,

$$G(t,\tau) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-\tau)^{\alpha-\beta-1} - (t-\tau)^{\alpha-1}, & 0 \leq \tau \leq t \leq 1, \\ t^{\alpha-1}(1-\tau)^{\alpha-\beta-1}, & 0 \leq t \leq \tau \leq 1, \end{cases} \tag{III.5}$$

and

$$H(t,\tau) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-\beta-1}(1-\tau)^{\alpha-\beta-1} - (t-\tau)^{\alpha-\beta-1}, & 0 \leq \tau \leq t \leq 1, \\ t^{\alpha-\beta-1}(1-\tau)^{\alpha-\beta-1}, & 0 \leq t \leq \tau \leq 1. \end{cases} . \tag{III.6}$$

Proof. we apply the left-hand side fractional integral I^{α}_{0+} to equation (III.3). We get

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau$$

Using the boundary conditions $u(0) = 0$,, we get

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau$$

then using the condition $D^{\beta}_{0+}u(1) = \sum_{k=1}^{m} \xi_k D^{\beta}_{0+}u(\eta_k)$, we obtain

$$D^{\beta}_{0+}u(1) = \frac{-1}{\Gamma(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-\beta-1}h(\tau)d\tau + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}$$

$$\begin{aligned} D^{\beta}_{0+}u(\eta_k) \ &= \ \frac{-1}{\Gamma(\alpha-\beta)}\sum_{k=1}^{m}\xi_k\int_0^{\eta_k}(\eta_k-\tau)^{\alpha-\beta-1}h(\tau)d\tau + \\ & \quad c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\sum_{k=1}^{m}\xi_k\eta_k^{\alpha-\beta-1} \end{aligned}$$

So

$$c_1 = \frac{1}{\delta\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-\beta-1}h(\tau)d\tau - \sum_{k=1}^{m}\xi_k\int_0^{\eta_k}(\eta_k-\tau)^{\alpha-\beta-1}h(\tau)d\tau\Big)$$

Then

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau + \frac{t^{\alpha-1}}{\delta\Gamma(\alpha)}\Big(\int_0^1 (1-\tau)^{\alpha-\beta-1}h(\tau)d\tau \\ &-\sum_{k=1}^{m}\xi_k\int_0^{\eta_k}(\eta_k-\tau)^{\alpha-\beta-1}h(\tau)d\tau\Big) \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau + \\ &\frac{t^{\alpha-1}\Big(1-\sum_{k=1}^{m}\xi_k\eta_k^{\alpha-\beta-1}+\sum_{k=1}^{m}\xi_k\eta_k^{\alpha-\beta-1}\Big)}{\delta\Gamma(\alpha)}\times \\ &\Big(\int_0^1 (1-\tau)^{\alpha-\beta-1}h(\tau)d\tau - \sum_{k=1}^{m}\xi_k\int_0^{\eta_k}(\eta_k-\tau)^{\alpha-\beta-1}h(\tau)d\tau\Big) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t^{\alpha-1}(1-\tau)^{\alpha-\beta-1}-(t-\tau)^{\alpha-1})h(\tau)d\tau \\ &+\int_t^1 (t^{\alpha-1}(1-\tau)^{\alpha-\beta-1}h(\tau)d\tau \\ &+\frac{1}{\delta\Gamma(\alpha)}\sum_{k=1}^{m}\xi_k\eta_k^{\alpha-\beta-1}\int_0^{\eta_k} t^{\alpha-1}(1-\tau)^{\alpha-\beta-1}h(\tau)d\tau \\ &+\frac{1}{\delta\Gamma(\alpha)}\sum_{k=1}^{m}\xi_k\eta_k^{\alpha-\beta-1}\int_{\eta_k}^1 t^{\alpha-1}(1-\tau)^{\alpha-\beta-1}h(\tau)d\tau \\ &-\sum_{k=1}^{m}\xi_k\int_0^{\eta_k} t^{\alpha-1}(\eta_k-\tau)^{\alpha-\beta-1}h(\tau)d\tau \\ &= \int_0^1 G(t,\tau)h(\tau)d\tau + \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau)h(\tau)d\tau. \end{aligned}$$

■

Lemma 42 *The functions G and H are continuous and satisfy*

$$0 \le G(t,\tau) \le \frac{1}{\Gamma(\alpha)}, \quad 0 \le H(t,\tau) \le \frac{1}{\Gamma(\alpha)}, \ \ t,\tau \in J.$$

Proof. If $0 < \tau \le t < 1$,we have

$$t^{\alpha-1}(1-\tau)^{\alpha-\beta-1} - (t-\tau)^{\alpha-1} = (t-t\tau)^{\alpha-1}(1-\tau)^{-\beta} - (t-\tau)^{\alpha-1} > 0,$$

if $0 < t \le \tau < 1$, we have $t^{\alpha-1}(1-\tau)^{\alpha-\beta-1} > 0$,so $0 < G(t,\tau)$
then $G(t,\tau) \le \frac{1}{\Gamma(\alpha)}(1-\tau)^{\alpha-\beta-1} \le \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-\tau)^{\alpha-\beta-1} \le \frac{1}{\Gamma(\alpha)}$
Accordingly,, we have

$$(t-\tau)^{\alpha-\beta-1} = t^{\alpha-\beta-1}(1-\frac{\tau}{t})^{\alpha-\beta-1} \le t^{\alpha-\beta-1}(1-\tau)^{\alpha-\beta-1}$$

Similary, we prove the properties for $H(t,\tau)$. ■

Define the space

$$\mathbb{X}=\{u \mid u\in C(J),\ D_{0^+}^{\alpha-1}u\in C(J)\}$$

endowed with the norm $\| u \|_{\mathbb{X}}=\max_{t\in J}|u(t)|+\max_{t\in J}\left|D_{0^+}^{\alpha-1}u(t)\right|$. It is clear that $(\mathbb{X},\| \,.\, \|_{\mathbb{X}})$ is a Banach space.

III.2 Uniqueness result via Banach fixed point theorem

Define the operator T on $\mathbb{X}$ by

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,\tau)(f(\tau,u(\tau))+g(\tau,D_{0^+}^{\alpha-1}u(\tau)))d\tau \\ &+ \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau)(f(\tau,u(\tau))+g(\tau,D_{0^+}^{\alpha-1}u(\tau)))d\tau. \end{aligned}$$

Theorem 43 *Assume that:*

(H1) *For each $t\in J$ and all $u,v\in\mathbb{R}$. There exists a constants $L_1,L_2>0$ such that*

$$| f(t,u)-f(t,v) |\leq L_1 \mid u-v \mid,$$

$$| g(t,u)-g(t,v) |\leq L_2 \mid u-v \mid$$

(H2)

$$\frac{\max(L_1,L_2)}{\Gamma(\alpha)}\Big(1+2\Gamma(\alpha)+\frac{(1+\Gamma(\alpha))\sum_{k=1}^{m}\xi_k}{\delta}\Big)<1.$$

If conditions (H1)-(H2) hold, then the fractional boundary value problem (III.1)-(III.2) has a unique solutions in $\mathbb{X}$.

Proof. Let $u,v\in\mathbb{X}$, then

$$\begin{aligned} | Tu(t)-Tv(t) | &\leq \int_0^1 G(t,\tau)(| f(\tau,u(\tau))-f(\tau,v(\tau)) | \\ &+ | g(\tau,D_{0^+}^{\alpha-1}u(\tau))-g(\tau,D_{0^+}^{\alpha-1}v(\tau)) |)d\tau \\ &+ \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(t,\eta_k)(| f(\tau,u(\tau))-f(\tau,v(\tau)) | \\ &+ | g(\tau,D_{0^+}^{\alpha-1}u(\tau))-g(\tau,D_{0^+}^{\alpha-1}v(\tau)) |)d\tau \\ &\leq \frac{1}{\Gamma(\alpha)}\Big(1+\frac{\sum_{k=1}^{m}\xi_k}{\delta} \\ &\Big)\int_0^1(L_1 \mid u(\tau)-v(\tau) \mid +L_2 \mid D_{0^+}^{\alpha-1}u(\tau)-D_{0^+}^{\alpha-1}v(\tau) \mid)d\tau \\ &\leq \Big(1+\frac{\sum_{k=1}^{m}\xi_k}{\delta}\Big)\frac{\max(L_1,L_2)}{\Gamma(\alpha)} \| u-v \|_{\mathbb{X}}, \end{aligned}$$

and

$$\begin{aligned}
&\mid D_{0^+}^{\alpha-1}Tu(t) - D_{0^+}^{\alpha-1}Tv(t) \mid \leq I_{0^+}^{1}\Big(\mid f(\tau,u(\tau)) - f(\tau,v(\tau)) \mid \\
&+ \mid g(\tau, D_{0^+}^{\alpha-1}u(\tau)) - g(\tau, D_{0^+}^{\alpha-1}v(\tau)) \mid \Big) \\
&+ \int_0^1 (1-\tau)^{\alpha-\beta-1}(\mid f(\tau,u(\tau)) - f(\tau,v(\tau)) \mid \\
&+ \mid g(\tau, D_{0^+}^{\alpha-1}u(\tau)) - g(\tau, D_{0^+}^{\alpha-1}v(\tau)) \mid \Big)d\tau \\
&+ \frac{\Gamma(\alpha)}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau)(\mid f(\tau,u(\tau)) - f(\tau,v(\tau)) \mid \\
&+ \mid g(\tau, D_{0^+}^{\alpha-1}u(\tau)) - g(\tau, D_{0^+}^{\alpha-1}v(\tau)) \mid \Big)d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq 2\int_0^1 \Big(L_1 \mid u(\tau) - v(\tau) \mid + L_2 \mid D_{0^+}^{\alpha-1}u(\tau) - D_{0^+}^{\alpha-1}v(\tau) \mid \Big)d\tau \\
&+ \frac{1}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 \Big(L_1 \mid u(\tau) - v(\tau) \mid + L_2 \mid D_{0^+}^{\alpha-1}u(\tau) - D_{0^+}^{\alpha-1}v(\tau) \mid \Big)d\tau \\
&\leq \Bigg(2 + \frac{\sum_{k=1}^{m}\xi_k}{\delta}\Bigg)\max(L_1,L_2) \parallel u - v \parallel_{\mathbb{X}} .
\end{aligned}$$

So

$$\parallel Tu - Tv \parallel_{\mathbb{X}} \leq \Bigg(1 + 2\Gamma(\alpha) + \frac{\Big(1+\Gamma(\alpha)\Big)\sum_{k=1}^{m}\xi_k}{\delta}\Bigg)\frac{\max(L_1,L_2)}{\Gamma(\alpha)} \parallel u - v \parallel_{\mathbb{X}}$$

Consequently T is a contraction. Therefore, by Banach fixed point theorem, we deduce that T has a unique fixed point which is the unique solution of problem (III.1)-(III.2). ■

III.3 Existence result via Krasnoselskii fixed point theorem

Now we define the operators A and B on $\mathbb{X}$ as

$$Au(t) = \int_0^1 G(t,\tau)(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau,$$

$$Bu(t) = \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau)(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau$$

Obviously, problem (III.1)-(III.2) has a solution if and only if $A+B$ has a fixed point. We introduce the following hypothesis.

(H'1) For each $t \in J$ and all $u \in \mathbb{R}$, there exist constants $L'_1 > 0$ and $L'_2 > 0$ such that:

$$\mid f(t,u) \mid \leq L'_1 \mid u \mid, \quad \mid g(t,u) \mid \leq L'_2 \mid u \mid,$$

(H'2)

$$\frac{(1+\Gamma(\alpha))\max(L_1,L_2)}{\delta\Gamma(\alpha)}\Big(\sum_{k=1}^{m}\xi_k\Big) < 1$$

Theorem 44 *Assume that (H'1) and (H'2) hold, then the fractional boundary value problem (III.1)-(III.2) has at least one solution.*

Proof. Let $B_r = \{u \in \mathbb{X} : \| u \|_{\mathbb{X}} \leq r\}$. Clearly, B_r is a nonempty, bounded and convex subset of the Banach space $\mathbb{X}$.

Step 1. The mapping A is continuous on B_r. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ in B_r, the hypothesis (H'1) we get

$$| Au_n(t) - Au(t) | \leq$$
$$\int_0^1 G(t,\tau)(| f(\tau, u_n(\tau)) - f(\tau, u(\tau)) | + | g(\tau, D_{0^+}^{\alpha-1}u_n(\tau)) - g(\tau, D_{0^+}^{\alpha-1}u(\tau)) |)d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^1 \sup_{\tau\in J} | f(\tau, u_n(\tau)) - f(\tau, u(\tau)) | + \sup_{\tau\in J} | g(\tau, D_{0^+}^{\alpha-1}u_n(\tau)) - g(\tau, D_{0^+}^{\alpha-1}u(\tau)) |)d\tau$$
$$\leq \frac{1}{\Gamma(\alpha)}(\| f(\tau, u_n(\tau)) - f(\tau, u(\tau)) \|_\infty + \| g(\tau, D_{0^+}^{\alpha-1}u_n(\tau)) - g(\tau, D_{0^+}^{\alpha-1}u(\tau)) \|_\infty$$

and

$$\begin{aligned}
&| D_{0^+}^{\alpha-1}Au_n(t) - D_{0^+}^{\alpha-1}Au(t) | \\
&\leq I_{0^+}^1(| f(\tau, u_n(\tau)) - f(\tau, u(\tau)) | + | g(\tau, D_{0^+}^{\alpha-1}u_n(\tau)) - g(\tau, D_{0^+}^{\alpha-1}u(\tau)) |) \\
&\leq \| f(., u_n(.)) - f(., u(.)) \|_\infty + \| g(., D_{0^+}^{\alpha-1}u_n(.)) - g(., D_{0^+}^{\alpha-1}u(.)) \|_\infty
\end{aligned}$$

So obtained

$$\begin{aligned}
&\| \quad Au_n - Au \|_{\mathbb{X}} \leq (\frac{1}{\Gamma(\alpha)} + 1) \\
&(\quad \| \quad f(., u_n(.)) - f(., u(.)) \|_\infty + \| g(., D_{0^+}^{\alpha-1}u_n(.)) - g(., D_{0^+}^{\alpha-1}u(.)) \|_\infty),
\end{aligned}$$

Since f and g are continuous, then $\| Au_n - Au \|_{\mathbb{X}} \to 0$ as $n \to \infty$.

Step 2. The mapping A is uniformly bounded on B_r. Let $u \in B_r$, then by condition (H'1) it yields

$$\begin{aligned}
| Au(t) | &\leq \int_0^1 G(t,\tau)(| f(\tau, u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)) |)d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (| f(\tau, u(\tau)) | + | g(\tau, D_{0^+}^{\alpha-1}u(\tau)) | d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (L_1' | u(\tau) | + L_2' | D_{0^+}^{\alpha-1}u(\tau) |)d\tau \leq \frac{\max(L_1', L_2')}{\Gamma(\alpha)} r,
\end{aligned}$$

and we have

$$| D_{0^+}^{\alpha-1}Au\,(t) | \leq I_{0^+}^1(| f(\tau, u(\tau)) | + | g(\tau, D_{0^+}^{\alpha-1}u(\tau)) |) \leq \max(L_1', L_2')r,$$

and consequently

$$\| Au \|_{\mathbb{X}} \leq (\frac{1}{\Gamma(\alpha)} + 1) \max(L_1', L_2')r$$

thus A is uniformly bounded.

Step 3.(Au) is equicontinuous on B_r. We have, for $u \in B_r$, $0 \leq t_1 \leq t_2 \leq 1$.

$$
\begin{aligned}
&\mid Au(t_2) - Au(t_1) \mid = \mid \int_0^1 (G(t_2,\tau) - G(t_1,\tau))(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \mid \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_2}\Big(t_2^{\alpha-1}(1-\tau)^{\alpha-\beta-1} - (t_2-\tau)^{\alpha-1}\Big)(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \\
&- \int_0^{t_1}\Big(t_1^{\alpha-1}(1-\tau)^{\alpha-\beta-1} - (t_1-\tau)^{\alpha-1}\Big)(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \\
&+ \int_{t_2}^1 \Big(t_2^{\alpha-1}(1-\tau)^{\alpha-\beta-1}\Big)(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \\
&- \int_{t_1}^1 \Big(t_1^{\alpha-1}(1-\tau)^{\alpha-\beta-1}\Big)(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \mid \Big) \\
&\leq \frac{\max(L_1', L_2')r}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-\beta}\Big(t_2^{\alpha-1}(1-(1-t_2)^{\alpha-\beta}) - t_1^{\alpha-1}(1-(1-t_1)^{\alpha-\beta})\Big) \\
&+ \frac{t_1^{\alpha} - t_2^{\alpha}}{\alpha} + \frac{1}{\alpha-\beta}\Big(t_2^{\alpha-1}(1-t_2)^{\alpha-\beta} - t_1^{\alpha-1}(1-t_1)^{\alpha-\beta}\Big)\Big) \\
&\longrightarrow 0 \quad \text{when } \; t_2 \to t_1
\end{aligned}
$$

and

$$
\begin{aligned}
&\mid D_{0^+}^{\alpha-1}Au(t_2) - D_{0^+}^{\alpha-1}Au(t_1) \mid = \mid I_{0^+}^1(f(t_2,u(t_2)) + g(t_2, D_{0^+}^{\alpha-1}u(t_2)) \\
&- I_{0^+}^1(f(t_1,u(t_1)) + g(t_1, D_{0^+}^{\alpha-1}u(t_1))) \mid \\
&= \mid \int_0^{t_2}(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau - \int_0^{t_1}(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \mid \\
&\leq ((L_1' + L_2)r)(t_2 - t_1) \longrightarrow 0 \quad \text{when } \; t_2 \to t_1
\end{aligned}
$$

thus (Au) is equicontinuous. Finally, by Arzela-Ascoli theorem, it follows that A is a completely continuous mapping on B_r.

Step 4. B is a contraction on B_r. Let $u, v \in B_r$, then

$$
\begin{aligned}
&\mid Bu(t) - Bv(t) \mid = \mid \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^m \xi_k \int_0^1 H(\eta_k,\tau)[(f(\tau,u(\tau) - f(\tau,v(\tau)) \\
&+ g(\tau, D_{0^+}^{\alpha-1}u(\tau)) - g(\tau, D_{0^+}^{\alpha-1}v(\tau)))]d\tau \mid \\
&\leq \frac{1}{\delta\Gamma(\alpha)}\sum_{k=1}^m \xi_k \int_0^1 (L_1 \mid u(\tau) - v(\tau) \mid) + L_2 \mid D_{0^+}^{\alpha-1}u(\tau) - D_{0^+}^{\alpha-1}v(\tau) \mid)d\tau \\
&\leq \frac{\max(L_1, L_2)}{\delta\Gamma(\alpha)}\sum_{k=1}^m \xi_k \parallel u - v \parallel_{\mathbb{X}}
\end{aligned}
$$

and

$$
\mid D_{0^+}^{\alpha-1}Bu(t) - D_{0^+}^{\alpha-1}Bv(t) \mid \leq \mid \frac{\Gamma(\alpha)\sum_{k=1}^m \xi_k}{\delta} \times
$$

$$
\int_0^1 H(\eta_k,\tau)(\mid f(\tau,u(\tau) - f(\tau,v(\tau) \mid + \mid g(\tau, D_{0^+}^{\alpha-1}u(\tau)) - g(\tau, D_{0^+}^{\alpha-1}v(\tau) \mid)d\tau
$$

$$
\leq \frac{\max(L_1, L_2)}{\delta}\Big(\sum_{k=1}^m \xi_k\Big) \parallel u - v \parallel_{\mathbb{X}}
$$

Hence

$$\| Bu - Bv \|_{\mathbb{X}} \leq \frac{(1+\Gamma(\alpha))\max(L_1, L_2)}{\delta\Gamma(\alpha)}\Big(\sum_{k=1}^{m}\xi_k\Big) \| u - v \|_{\mathbb{X}}$$

taking Hypothesis (H'2) into account, we conclude that B is a contraction.
Step 5. $Au + Bv \in B_r$ for all $u, v \in B_r$, in fact

$$\| Bu \|_{\mathbb{X}} \leq r\frac{(1+\Gamma(\alpha))\max(L_1', L_2')}{\delta\Gamma(\alpha)}\sum_{k=1}^{m}\xi_k$$

and

$$\begin{aligned}\| Au + Bv \|_{\mathbb{X}} &\leq \| Au \|_{\mathbb{X}} + \| Bv \|_{\mathbb{X}} \leq \\ &\leq r\frac{(1+\Gamma(\alpha))\max(L_1', L_2')}{\Gamma(\alpha)}\left(1 + \frac{\sum_{k=1}^{m}\xi_k}{\delta}\right) < r\end{aligned}$$

thus $Au + Bv \in B_r$.
Then all the hypotheses of Theorem 32 are satisfied. Thus there exists at least one solution $u \in B_r$ for problem (III.1)-(III.2). ∎

III.4 Existence result via Leray-Schauder nonlinear alternative

Theorem 45 *Assume that the following conditions are satisfied*
(H"1) *There exist a function $p \in C(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

$$| g(t, u) | \leq p(t)\Phi(| u |), \quad for\ \ each\ \ (t, u) \in J \times \mathbb{R}$$

(H"2) *There exists a constant $N > 0$ such that:*

$$\frac{(M + p\Phi(N))}{\Gamma(\alpha)}\Big(1 + 2\Gamma(\alpha) + \frac{\Big(1+\Gamma(\alpha)\Big)\sum_{k=1}^{m}\xi_k}{\delta}\Big) < N$$

where $p = \max_{t\in J}|p(t)|$ and $M = \max\{|f(t,u)|,\ t \in J,\ |u| \leq N\}$. Then the problem (III.1)-(III.2) has at least one solution.

Proof. Claim 1. The mapping T is continuous since f and g are continuous.
Claim 2. Set

$$U = \{u \in \mathbb{X} : \| u \|_{\mathbb{X}} < N\}$$

then U is an open in $\mathbb{X}$ and $0 \in U$. Then $T(U)$ is uniformly bounded. In fact, let $u \in U$, then by conditions (H"1), it yields

$$\| Tu \|_{\mathbb{X}} \leq \frac{(M + p\Phi(N))}{\Gamma(\alpha)}\Big(1 + 2\Gamma(\alpha) + \frac{\Big(1+\Gamma(\alpha)\Big)\sum_{k=1}^{m}\xi_k}{\delta}\Big)$$

Claim 3.(Tu) is equicontinuous on U. We have, for $u \in U, 0 \leq t_1 \leq t_2 \leq 1$.

$$\begin{aligned}
&\mid Tu(t_2) - Tu(t_1) \mid = \mid \int_0^1 (G(t_2,\tau) - G(t_1,\tau))(f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)))d\tau \mid \\
&+ \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k,\tau) \mid f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)) \mid d\tau \\
&\leq \frac{(M + p\Phi(N))}{\Gamma(\alpha)} \Bigg(\frac{1}{\alpha-\beta}\Big(t_2^{\alpha-1}(1-(1-t_2)^{\alpha-\beta}) - t_1^{\alpha-1}(1-(1-t_1)^{\alpha-\beta})\Big) \\
&+ \frac{t_1^{\alpha} - t_2^{\alpha}}{\alpha} + \frac{1}{\alpha-\beta}\Big(t_2^{\alpha-1}(1-t_2)^{\alpha-\beta} - t_1^{\alpha-1}(1-t_1)^{\alpha-\beta}\Big) \\
&+ \frac{\Gamma(\alpha)(t_2^{\alpha-1} - t_1^{\alpha-1})}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k,\tau) \mid f(\tau,u(\tau)) + g(\tau, D_{0^+}^{\alpha-1}u(\tau)) \mid d\tau \Bigg) \\
&\longrightarrow 0 \text{ when } t_2 \to t_1
\end{aligned}$$

and we have

$$\begin{aligned}
&\mid D_{0^+}^{\alpha-1}Tu(t_2) - D_{0^+}^{\alpha-1}Tu(t_1) \mid \leq \mid \int_{t_1}^{t_2} (\mid f(\tau,u(\tau)) \mid + \mid g(\tau, D_{0^+}^{\alpha-1}u(\tau) \mid) d\tau \\
&\leq (M + \Phi(N))(t_2 - t_1) \longrightarrow 0 \text{ when } t_2 \to t_1
\end{aligned}$$

thus (Tu) is equicontinuous. Finally, by Arzela-Ascoli theorem, it follows that T is a completely continuous mapping on B_R.

Claim 4. Assume that there exists $u \in \partial U$ such that $u = \lambda T(u)$,for some $0 < \lambda < 1$. Then

$$\begin{aligned}
N &= \parallel u \parallel_{\mathbb{X}} = \lambda \parallel Tu \parallel_{\mathbb{X}} \leq \parallel Tu \parallel_{\mathbb{X}} \\
&\leq \frac{(M + p\Phi(N))}{\Gamma(\alpha)} \left(1 + 2\Gamma(\alpha) + \frac{\Big(1+\Gamma(\alpha)\Big)\sum_{k=1}^{m}\xi_k}{\delta}\right)
\end{aligned}$$

that contradicts hypothesis **(H''2)**. Then the statement (ii) in Theorem 31 does not hold. As consequence of the nonlinear alternative of Leray-Schauder, we deduce that the operator T has at least one fixed point $u^* \in \overline{U}$, which is the solution of the problem (III.1)-(III.2). ■

III.5 Boundary Value Problems With Fractional Derivatives in a Fractional Sobolev Space

The aim of this section is to study of existence of solutions in the Riemann-Liouville fractional Sobolev space for the nonlinear boundary value problem (III.1)-(III.2).

Definition 46 *The Riemann-Liouville fractional Sobolev space is defined by*

$$W_{RL,a^+}^{s,p} = \{u \in L^p(a,b), I_{a^+}^{1-s}u \in W^{p,1}(a,b), 0 < s < 1\}.$$

where

$$W^{p,1}(a,b) = \{u \in L^p(a,b), u' \in L^p(a,b)\}.$$

$W_{RL,a^+}^{s,p}$ *is a Banach space endowed with the norm*

$$\parallel u \parallel_{W_{RL,a^+}^{s,p}} = \parallel u \parallel_{L^p} + \parallel I_{a^+}^{1-s}u \parallel_{W^{p,1}} .$$

Denote $\mathbb{E} = W^{\beta,p}_{RL,0^+}$, $0 \leq \beta \leq 1$, then the norm is

$$\| u \|_{\mathbb{E}} = \| u \|_{L^p} + \| I^{1-\beta}_{a^+} u \|_{L^p} + \| D^{\beta}_{0^+} u \|_{L^p}$$

Define the operators Q_1 and Q_2 on $\mathbb{E}$ as

$$\begin{aligned} Q_1 u(t) = & \int_0^1 G(t,\tau)(g(\tau, D^{\beta}_{0^+} u(\tau)))d\tau \\ & + \frac{t^{\alpha-1}}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k,\tau)(g(\tau, D^{\beta}_{0^+} u(\tau)))d\tau \end{aligned}$$

$$\begin{aligned} Q_2 u(t) = & \int_0^1 G(t,\tau)(f(\tau, u(\tau)))d\tau \\ & + \frac{t^{\alpha-1}}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k,\tau)(f(\tau, u(\tau)))d\tau \end{aligned}$$

Obviously, problem (III.1)-(III.2) has a solution if and only if $Q_1 + Q_2$ has a fixed point. We introduce the following hypotheses.

(C1) $f, g : J \times \mathbb{R} \to \mathbb{R}$ are Caratheordory functions.

(C2) There exists a function $\psi \in L^1(J, \mathbb{R}_+)$ such for any $t \in J$ and any $u, v \in \mathbb{R}$

$$| f(t,u) - f(t,v) | \leq \psi(t) \mid u - v \mid$$

$$\frac{\psi^*}{\Gamma(\alpha)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta}\right) \left(1 + \frac{1}{\Gamma(2-\beta)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right) < 1$$

where $\psi^* = \int_0^1 \mid \psi(t) \mid dt$.

(C3) There exists a function $\varphi \in L^1(J, \mathbb{R}_+)$ such for any $t \in J$ and any $u \in \mathbb{R}$

$$| g(t,u) | \leq \varphi(t)$$

(C4) There exists $\rho > 0$ such

$$(\varphi^* + \rho\psi^* + M) \left(2 + \frac{\sum_{k=1}^{m} \xi_k)}{\delta}\right) \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha-\beta)}\right) \leq \rho$$

where $\varphi^* = \int_0^1 \mid \varphi(t) \mid dt$, $M = \sup_{t \in J} \mid f(t,0) \mid$.

Theorem 47 *Assume that (C1)-(C4) hold. then the boundary value problem (III.1)-(III.2) has at least one solution*

Proof. We will use Krasnoselski fixed point theorem. Let $\Omega = \{u \in \mathbb{E} : \| u \|_{\mathbb{E}} \leq \rho\}$.

Claim 1. Q_1 is continuous and relatively compact. In fact let $u \in \Omega$, we have

$$\begin{aligned}
\mid Q_1(t) \mid &\leq \int_0^1 G(t,\tau) \mid g(\tau, D_{0^+}^{\beta} u(\tau)) \mid d\tau \\
&+ \frac{t^{\alpha-1}}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k,\tau) \mid g(\tau, D_{0^+}^{\beta} u(\tau)) \mid d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} (1 + \frac{1}{\delta} \sum_{k=1}^{m} \xi_k) \int_0^1 \varphi(\tau) d\tau \\
&\leq \frac{\varphi^*}{\Gamma(\alpha)} (1 + \frac{1}{\delta} \sum_{k=1}^{m} \xi_k)
\end{aligned}$$

So

$$\parallel Q_1 \parallel_p \leq \frac{\varphi^*}{\Gamma(\alpha)} (1 + \frac{1}{\delta} \sum_{k=1}^{m} \xi_k)$$

and

$$\begin{aligned}
&\mid I_{0^+}^{1-\beta} Q_1(t) \mid = \mid I_{0^+}^{1-\beta} \left(t^{\alpha-1}\right) \left(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-\beta-1} g(\tau, D_{0^+}^{\beta} u(\tau)) d\tau \right) - I_{0^+}^{1-\beta} I_{0^+}^{\alpha} (g(t, D_{0^+}^{\beta} u(t)) \\
&+ I_{0^+}^{1-\beta} \left(t^{\alpha-1}\right) \left(\frac{1}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k,\tau) g(\tau, D_{0^+}^{\beta} u(\tau)) d\tau \right) \mid \\
&\leq \mid I_{0^+}^{1+\alpha-\beta} g(t, D_{0^+}^{\beta} u(t)) \mid + \frac{t^{\alpha-\beta} \varphi^*}{\Gamma(\alpha-\beta+1)} \left(1 + \frac{\sum_{k=1}^{m} \xi_k}{\delta} \right) \\
&\leq \frac{\varphi^*}{\Gamma(\alpha-\beta+1)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta} \right)
\end{aligned}$$

to obtain

$$\parallel I_{0^+}^{1-\beta} Q_1 \parallel_p \leq \frac{\varphi^*}{\Gamma(\alpha-\beta+1)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta} \right)$$

and

$$
\begin{aligned}
&| D^{\beta}_{0^+}Q_1(t) |=| -D^{\beta}_{0^+}I^{\alpha}_{0^+}(g(t, D^{\beta}_{0^+}u(t)) + D^{\beta}_{0^+}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-\beta-1}g(\tau, D^{\beta}_{0^+}u(\tau))d\tau \\
&+ D^{\beta}_{0^+}\frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau)g(\tau, D^{\beta}_{0^+}u(\tau))d\tau \;| \\
&\leq \left|I^{\alpha-\beta}_{0^+}(g(t, D^{\beta}_{0^+}u(t))\right| + \frac{\varphi^* t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big(1+\frac{\sum_{k=1}^{m}\xi_k}{\delta}\Big) \\
&\leq \frac{\varphi^*}{\Gamma(\alpha-\beta)}\Big(2+\frac{\sum_{k=1}^{m}\xi_k}{\delta}\Big)
\end{aligned}
$$

Hence

$$
\| D^{\beta}_{0^+}Q_1 \|_p \leq \frac{\varphi^*}{\Gamma(\alpha-\beta)}\Big(2+\frac{\sum_{k=1}^{m}\xi_k}{\delta}\Big)
$$

and consequently

$$
\| Q_1 \|_{\mathbb{E}} \leq \varphi^* \left(2+\frac{\sum_{k=1}^{m}\xi_k}{\delta}\right) \times \left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha-\beta)}\right)
$$

thus Q_1 is uniformly bounded.

Claim 2. The mapping Q_1 is continuous in $W^{\beta,p}_{RL,0^+}$. consider the sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ in $W^{\beta,p}_{RL,0^+}$, then by condition (C1), Hölder inequality and taking into account that g is a Caratheordory function, it yields

$$
\begin{aligned}
&| Q_1u_n(t) - Q_1u(t) | \leq \int_0^1 G(t,\tau) \mid g(\tau, D^{\beta}_{0^+}u_n(\tau)) - g(\tau, D^{\beta}_{0^+}u(\tau)) \mid d\tau \\
&+ \frac{t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau) \mid g(\tau, D^{\beta}_{0^+}u_n(\tau)) - g(\tau, D^{\beta}_{0^+}u(\tau)) \mid d\tau \\
&\leq \frac{1}{\Gamma(\alpha)}(1+\frac{1}{\delta}\sum_{k=1}^{m}\xi_k) \parallel g(, D^{\beta}_{0^+}u_n(.)) - g(., D^{\beta}_{0^+}u(.)) \parallel_p
\end{aligned}
$$

So

$$
\begin{aligned}
\| \quad & Q_1u_n - Q_1u \|_p \leq \frac{1}{\Gamma(\alpha)}(1+\frac{1}{\delta}\sum_{k=1}^{m}\xi_k) \parallel g(., D^{\beta}_{0^+}u_n(.)) - g(., D^{\beta}_{0^+}u(.)) \parallel_p \\
\to \quad & 0 \text{ as } n \to \infty.
\end{aligned}
$$

And we have

$$| I_{0^+}^{1-\beta}Q_1u_n(t) - I_{0^+}^{1-\beta}Q_1u(t) | \leq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta} \mid Q_1u_n(\tau) - Q_1u(\tau) \mid d\tau$$

$$\leq \frac{1}{\Gamma(2-\beta)}\frac{1}{\Gamma(\alpha)}(1+\frac{1}{\delta}\sum_{k=1}^{m}\xi_k) \parallel g(,D_{0^+}^{\beta}u_n(.)) - g(.,D_{0^+}^{\beta}u(.)) \parallel_p$$

thus

$$\parallel I_{0^+}^{1-\beta}Q_1u_n - I_{0^+}^{1-\beta}Q_1u \parallel_p \leq \frac{1}{\Gamma(2-\beta)}\frac{1}{\Gamma(\alpha)}(1+\frac{1}{\delta}\sum_{k=1}^{m}\xi_k)$$
$$\parallel g(,D_{0^+}^{\beta}u_n(.)) - g(.,D_{0^+}^{\beta}u(.)) \parallel_p \to 0 \text{ as } n\to\infty.$$

hence

$$\parallel D_{0^+}^{\beta}Q_1u_n - D_{0^+}^{\beta}Q_1u \parallel_p \leq \frac{1}{\Gamma(\alpha-\beta+1)}(2+\frac{\sum_{k=1}^{m}\xi_k(1+\eta_k^{\alpha-\beta-1})}{\delta})$$
$$\parallel g(.,D_{0^+}^{\beta}u_n(.)) - g(.,D_{0^+}^{\beta}u(.)) \parallel_p \to 0 \text{ as } n\to\infty.$$

Finally, we get $\parallel Q_1u_n - Q_1u \parallel_{\mathbb{E}} \to 0$ as $n\to\infty$.
Claim 3. Q_1 is relatively compact, let $u\in\Omega$

$$| Q_1u(t+h) - Q_1u(t) | \leq \int_0^1 \mid G(t+h,\tau) - G(t,\tau) \mid\mid g(\tau,D_{0^+}^{\beta}u(\tau)) \mid d\tau$$
$$+\frac{(t+h)^{\alpha-1}-t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\int_0^1 H(\eta_k,\tau) \mid g(\tau,D_{0^+}^{\beta}u(\tau)) \mid d\tau$$
$$\leq \frac{\varphi^*}{\Gamma(\alpha)}\Big(\int_0^1((t+h)^{\alpha-1}-t^{\alpha-1})(1-\tau)^{\alpha-\beta-1}d\tau$$
$$+\int_0^t(t-\tau)^{\alpha-1}d\tau - \int_0^{t+h}(t+h-\tau)^{\alpha-1}d\tau \mid\Big)$$
$$+\frac{(t+h)^{\alpha-1}-t^{\alpha-1}}{\delta}\sum_{k=1}^{m}\xi_k\varphi^*\int_0^1 H(\eta_k,\tau)d\tau$$
$$\leq \big((t+h)^{\alpha-1}-t^{\alpha-1}\big)\left(\frac{2\varphi^*}{(\alpha-\beta)\,\Gamma(\alpha)}+\frac{\varphi^*\sum_{k=1}^{m}\xi_k}{\Gamma(\alpha)\delta}\right)$$

So

$$\parallel Q_1u(.+h) - Q_1u(.) \parallel_p \leq \left(\frac{2\varphi^*}{(\alpha-\beta)\,\Gamma(\alpha)}+\frac{\varphi^*\sum_{k=1}^{m}\xi_k}{\delta\Gamma(\alpha)}\right)\left(\int_0^1\big((t+h)^{\alpha-1}-t^{\alpha-1}\big)^p dt\right)^{\frac{1}{p}}$$
$$\to 0 \text{ when } h\to 0$$

and

$$| I_{0^+}^{1-\beta}Q_1u(t+h) - I_{0^+}^{1-\beta}Q_1u(t) | =$$
$$\frac{1}{\Gamma(1-\beta)} \mid \int_0^{t+h}(t+h-\tau)^{-\beta}Q_1u(\tau)d\tau - \int_0^t(t-\tau)^{-\beta}Q_1u(\tau)d\tau \mid$$

$$\leq \frac{1}{\Gamma(1-\beta)} \mid \int_0^t \left((t+h-\tau)^{-\beta} - (t-\tau)^{-\beta}\right) Q_1u(\tau)d\tau + \int_t^{t+h} (t+h-\tau)^{-\beta} Q_1u(\tau)d\tau \mid$$
$$\leq \frac{((t+h)^{1-\beta} - t^{1-\beta})}{\Gamma(2-\beta)} \Big(\frac{\varphi^*}{\Gamma(\alpha)} (1 + \frac{1}{\delta} \sum_{k=1}^m \xi_k \Big)$$

that implies

$$\| \quad I_{0^+}^{1-\beta} Q_1u(.+h) - I_{0^+}^{1-\beta} Q_1u(.) \|_p \leq \frac{\Big(\frac{\varphi^*}{\Gamma(\alpha)} (1 + \frac{1}{\delta} \sum_{k=1}^m \xi_k \Big)}{\Gamma(2-\beta)} \left(\int_0^1 \left((t+h)^{\alpha-1} - t^{\alpha-1}\right)^p dt \right)^{\frac{1}{p}}$$
$$\rightarrow \quad 0 \text{ when } h \rightarrow 0$$

and

$$\mid D_{0+}^{\beta} Q_1u(t+h) - D_{0+}^{\beta} Q_1u(t) \mid \leq \mid I_{0^+}^{\alpha-\beta} g(t+h, D_{0+}^{\beta}u(t+h)) - I_{0^+}^{\alpha-\beta} g(t, D_{0+}^{\beta}u(t)) \mid$$
$$+ \frac{(t+h)^{\alpha-\beta-1} - t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \Bigg(\int_0^1 (1-\tau)^{\alpha-\beta-1} g(\tau, D_{0+}^{\beta}u(\tau))d\tau$$
$$+ \frac{\sum_{k=1}^m \xi_k}{\delta\Gamma(\alpha)} \int_0^1 g(\tau, D_{0+}^{\beta}u(\tau))d\tau \Bigg)$$
$$\leq \frac{\varphi^*}{\Gamma(\alpha-\beta+1)} \big(t^{\alpha-\beta} + 2h^{\alpha-\beta} - (t+h)^{\alpha-\beta}$$
$$+ (1 + \frac{\sum_{k=1}^m \xi_k}{\delta\Gamma(\alpha)})((t+h)^{\alpha-\beta-1} - t^{\alpha-\beta-1})\big)$$
$$\rightarrow 0 \text{ when } h \rightarrow 0$$

from the above it follows that $\| Q_1u(.+h) - Q_1u(.) \|_{\mathbb{E}} \rightarrow 0$ as $h \rightarrow 0$.
On the other hand we have

$$\int_{1-\varepsilon}^1 \mid Q_1u(\tau) \mid d\tau + \int_{1-\varepsilon}^1 \mid I_{0^+}^{1-\beta} Q_1u(\tau) \mid d\tau + \int_{1-\varepsilon}^1 \mid D_{0+}^{\beta} Q_1u(\tau) \mid d\tau$$
$$\leq \varepsilon\varphi^* \left(2 + \frac{\sum_{k=1}^m \xi_k)}{\delta} \right) \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta+1)} \frac{1}{\Gamma(\alpha-\beta)} \right)$$
$$\rightarrow 0 \ \ as \ \ \varepsilon \rightarrow 0.$$

By Theorem 29, we conclude that Q_1 is relatively compact on Ω. From the above discussion we conclude that Q_1 is completely continuous on $\mathbb{E}$.

Claim 4. the mapping Q_2 is a contraction on Ω. In fact for $u, v \in \Omega$ and $t \in J$, we have

$$\begin{aligned}&| Q_2u(t) - Q_2v(t) | \leq \int_0^1 G(t,\tau) | f(\tau, u(\tau)) - f(\tau, v(\tau)) | d\tau \\ &+ \frac{t^{\alpha-1}}{\delta} \sum_{k=1}^{m} \xi_k \int_0^1 H(\eta_k, \tau)(| f(\tau, u(\tau)) - f(\tau, v(\tau)) | d\tau \\ &\leq \frac{\psi^*}{\Gamma(\alpha)}(1 + \frac{1}{\delta}\sum_{k=1}^{m} \xi_k) \parallel u - v \parallel_p\end{aligned}$$

hence

$$\parallel Q_2u - Q_2v \parallel_p \leq \frac{\psi^*}{\Gamma(\alpha)}(1 + \frac{1}{\delta}\sum_{k=1}^{m} \xi_k) \parallel u - v \parallel_p.$$

And we have

$$\begin{aligned}&| I_{0^+}^{1-\beta}Q_2v(t) - I_{0^+}^{1-\beta}Q_2u(t) | \leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} | Q_2v(\tau) - Q_2u(\tau) | d\tau \\ &\leq \frac{1}{\Gamma(2-\beta)} \frac{\psi^*}{\Gamma(\alpha)}(1 + \frac{1}{\delta}\sum_{k=1}^{m} \xi_k) \parallel u - v \parallel_p\end{aligned}$$

thus

$$\parallel I_{0^+}^{1-\beta}Q_2v - I_{0^+}^{1-\beta}Q_2u \parallel_p \leq \frac{1}{\Gamma(2-\beta)} \frac{\psi^*}{\Gamma(\alpha)}(1 + \frac{1}{\delta}\sum_{k=1}^{m} \xi_k) \parallel u - v \parallel_p$$

and

$$\begin{aligned}&| D_{0^+}^{\beta}Q_2v(t) - D_{0^+}^{\beta}Q_2u(t) | \leq I_{0^+}^{\alpha-\beta} | f(t, v(t)) - f(t, u(t)) | \\ &+ \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-\tau)^{\alpha-\beta-1} | f(\tau, v(\tau)) - f(\tau, u(\tau)) | d\tau \\ &+ \frac{t^{\alpha-\beta-1}}{\delta\Gamma(\alpha-\beta)} \sum_{k=1}^{m} \xi_k \int_0^1 | f(\tau, v(\tau)) - f(\tau, u(\tau)) | d\tau \\ &\leq \frac{\psi^*}{\Gamma(\alpha-\beta)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta}\right) \parallel u - v \parallel_p\end{aligned}$$

so

$$\parallel D_{0^+}^{\beta}Q_2v - D_{0^+}^{\beta}Q_2u \parallel_p \leq \frac{\psi^*}{\Gamma(\alpha-\beta)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta}\right) \parallel u - v \parallel_p$$

Finally, we get

$$\begin{aligned}&\parallel Q_2u - Q_2v \parallel_{\mathbb{E}} \\ &\leq \frac{\psi^*}{\Gamma(\alpha)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta}\right) \left(1 + \frac{1}{\Gamma(2-\beta)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right) \parallel u - v \parallel_p \\ &\leq \frac{\psi^*}{\Gamma(\alpha)} \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta}\right) \left(1 + \frac{1}{\Gamma(2-\beta)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right) \parallel u - v \parallel_{\mathbb{E}}\end{aligned}$$

by hypothesis (C2), we conclude that Q_2 is a contraction.

Claim 5. $Q_1u + Q_2v \in \Omega$, for $u, v \in \Omega$, indeed, let $u \in \Omega$ and $t \in J$, by (C2), we have

$$\parallel Q_2 \parallel_{\mathbb{E}} \leq \left(\rho\psi^* + M\right) \left(2 + \frac{\sum_{k=1}^{m} \xi_k}{\delta}\right) \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(2-\beta)} + \frac{1}{\Gamma(\alpha-\beta)}\right)$$

then

$$\| Q_1u + Q_2v \|_{\mathbb{E}} \leq \| Q_1u \|_{\mathbb{E}} + \| Q_2v \|_{\mathbb{E}} \leq$$
$$(\varphi^* + \rho\psi^* + M) \left(2 + \frac{\sum_{k=1}^{m} \xi_k)}{\delta} \right) \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha - \beta)} \right)$$
$$\leq \rho.$$

Consequently all the conditions of Krasnoselskii's fixed point theorem are satisfied, we deduce that the problem (III.1)-(III.2) has at least one solution in $\mathbb{E}$. ■

III.6 Exemples

Exemple 1. Let us consider the following boundary value problem:

$$D_{0^+}^{1.8}u(t) + f(t, u(t)) + g(t, D_{0^+}^{0.8}u(t)) = 0 \ , \ t \in J = [0, \ 1]. \tag{III.7}$$

$$u(0) = 0, \quad D_{0^+}^{0.4}u(1) = \sum_{k=1}^{3} \xi_k D_{0^+}^{0.4}u(\eta_k). \tag{III.8}$$

Here $\alpha = 1.8, \ \beta = 0.4, \ \xi_1 = 0.21, \xi_2 = 0.01, \xi_3 = 0.13, \eta_1 = 0.32, \eta_2 = 0.17, \eta_3 = 0.31$ and

$$f(t, x) = \frac{e^{-7t}}{100} \frac{1}{x^2 + 1}, \quad t \in J, x \in \mathbb{R}$$

and

$$g(t, y) = \frac{1 - t^2 \sin(y + 1)}{120}, \quad t \in J, y \in \mathbb{R}$$

Let $u, v \in \mathbb{R}$, then we have

$$| \ f(t, u) - f(t, v) \ | \leq \frac{1}{100} \ | \ u - v \ | \ .$$

and

$$| \ g(t, u) - g(t, v) \ | \leq \frac{t^2}{120} \left|\sin(u + 1) - \sin(v + 1)\right| \leq \frac{1}{120} \ | \ u - v \ |$$

thus $L_1 = \frac{1}{100}$, $L_2 = \frac{1}{60}$, then, some computations give us $\delta = 0.781 > 0$

$$\frac{\max(L_1, L_2)}{\Gamma(\alpha)} \Big(1 + 2\Gamma(\alpha) + \frac{(1 + \Gamma(\alpha)) \sum_{k=1}^{m} \xi_k}{\delta} \Big) = 0.53591 < 1.$$

Thus, by Theorem 43, the boundary value problem ((III.7)-(III.8) has a unique solution.

Exemple 2. Let us consider the following boundary value problem:

$$D_{0^+}^{1.7}u(t) + \frac{\sin(tu(t))}{100} + \frac{D_{0^+}^{0.7}u(t)}{100e^{-2t}} = 0 \ , \ t \in J = [0, \ 1]. \tag{III.9}$$

$$u(0) = 0, \quad D_{0^+}^{0.5}u(1) = \sum_{k=1}^{3} \xi_k D_{0^+}^{0.5}u(\eta_k). \tag{III.10}$$

Here $\alpha = 1.7,\ \beta = 0.5,\ \xi_1 = 0.21, \xi_2 = 0.01, \xi_3 = 0.13, \eta_1 = 0.32, \eta_2 = 0.17, \eta_3 = 0.31$, and $\delta = 0.781$

$$f(t,x) = \frac{\sin(tx)}{100}, \quad t \in J, x \in \mathbb{R}_+$$

and

$$g(t,y) = \frac{e^{-2t}y}{100}, \quad t \in J, y \in \mathbb{R}_+$$

Let $x, y \in \mathbb{R}$, then we have

$$\begin{aligned} &\mid\ f(t,x) - g(t,y \mid = \frac{1}{100} \mid x - y \mid \leq \frac{1}{100} \mid x \mid, \\ &\mid\ f(t,x) \mid \leq \frac{1}{100} \mid x \mid \\ &\mid\ g(t,x) - g(t,y) \mid = \frac{1}{100} \mid x - y \mid, \\ &\mid\ g(t,x) \mid \leq \frac{1}{100} \mid x \mid \end{aligned}$$

So $L_1 = L_2 = L_1' = L_2' = 0.01$, and we can show that

$$\frac{\max(L_1, L_2)}{\Gamma(\alpha)} \left(1 + 2\Gamma(\alpha) + \frac{(1+\Gamma(\alpha))\sum_{k=1}^{m}\xi_k}{\delta}\right) = 4.957\,8 \times 10^{-2} < 1.$$

$$\frac{(1+\Gamma(\alpha))\max(L_1', L_2')}{\Gamma(\alpha)} \left(\frac{\sum_{k=1}^{m}\xi_k}{\delta}\right) = 9.413 \times 10^{-3} < 1$$

As all assumptions of Theorem 44 are satisfied, we conclude that the problem (III.9)-(III.10) has at least one solution u

Exemple 3. Let us consider the following boundary value problem:

$$D_{0^+}^{1.2}u(t) + f(t, u(t)) + g(t, D_{0^+}^{0.2}u(t)) = 0,\ t \in J = [0,1]. \tag{III.11}$$

$$u(0) = 0, \qquad D_{0^+}^{0.1}u(1) = \sum_{k=1}^{3} \xi_k D_{0^+}^{0.1}u(\eta_k). \tag{III.12}$$

Here $\alpha = 1.2,\ \beta = 0.1,\ \ \xi_1 = 0.21,\ \xi_2 = 0.01,\ \xi_3 = 0.13,\ \eta_1 = 0.32,\ \eta_2 = 0.17, \eta_3 = 0.31$, then $\delta = 0.781$. Choose

$$f(t,x) = \frac{tx}{10}, \quad t \in J, x \in \mathbb{R}$$

and

$$g(t,y) = \frac{t}{10}(\frac{y^2}{1+|y|}), \quad t \in J,\ y \in \mathbb{R}.$$

We have f is a continuous and

$$\mid g(t,y) \mid \leq \frac{t\,|y|}{10}$$

Thus $p(t) = \dfrac{t}{10} \in C(J, \mathbb{R}_+)$ $p = 0.1$ and $\Phi(x) = x$ is continuous and nondecreasing on $\mathbb{R}_+$, $M = \dfrac{N}{10}$, then for $N = 0.5$ we get

$$\frac{(M + p\Phi(N))}{\Gamma(\alpha)}\Big(1 + 2\Gamma(\alpha) + \frac{\Big(1+\Gamma(\alpha)\Big)\displaystyle\sum_{k=1}^{m}\xi_k}{\delta}\Big) = 0.74N < N$$

Since all conditions of Theorem (45) are satisfied then the problem (III.11)-(III.12) has at least one solution.

Exemple 4. Consider the problem (III.1)-(III.2) with
$\alpha = 1.2,\ \beta = 0.3,\ \xi_1 = 0.21,\ \xi_2 = 0.01,\ \xi_3 = 0.13,\ \eta_1 = 0.32,\ \eta_2 = 0.17, \eta_3 = 0.31,$ then $\delta = 0.781.$

$$f(t,u) = \frac{e^{-3t}}{100}\sin\left(u+t\right), \qquad t \in J,\ u \in \mathbb{R}.$$

and

$$g(t,x) = 2t\arctan x \qquad t \in J,\ x \in \mathbb{R}.$$

Let $t \in J, u, v \in \mathbb{R}_+$, then we have

$$\begin{aligned} &| \ \ f(t,u) - f(t,v) \mid\leq \frac{e^{-3t}}{100} \mid u-v \mid \\ &| \ \ g(t,x) \mid\leq 2t \mid \arctan x \mid\leq \pi t \end{aligned}$$

hence $\psi(t) = \dfrac{e^{-3t}}{100}$, $\varphi(t) = \pi t$, so $\psi^* = 3.1674 \times 10^{-2}$ and $\varphi^* = \frac{\pi}{2}$, $M = \sup_{t\in J} \mid f(t,0) \mid = \sup_{t\in J} \dfrac{e^{-3t}}{100}\sin t = 0.001.$

$$\frac{\psi^*}{\Gamma(\alpha)}\left(2 + \frac{\sum_{k=1}^{m}\xi_k}{\delta}\right)\left(1 + \frac{1}{\Gamma(2-\beta)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right) = 0.24996 < 1$$

and for $\rho \geq 1.6232$ the condition $(C4)$ holds thus by Theorem 47, twe conclude that the boundary value problem (III.1)-(III.2) has at least one solution in the sobolev space $W^{0.3,4}_{RL,0^+}$.

Chapter IV

Boundary Value Problem of Fractional Oscillator Equation

IV.1 Introduction

In this chapter ,we study the existence of solutions for a nonlinear fractional oscillator equation with both left and right Caputo fractional derivatives subject to nonlocal conditions. We use the Krasnoselskiis fixed point theorem.
Recently, much attention has been focused on the study of fractional differential equations with nonlocal conditions. For some recent works on the existence of solutions for fractional differential equations with non-local conditions see [8,11,12,15,16,24,25,28,29,47,56,72]
In [37], the authors studied by means of lower and upper solutions method and Schauder fixed point theorem the existence of positive solutions

$$\begin{cases} D^{\alpha}_{0^+}y(t)+f(t,y(t))=0, \qquad n-1\leq\alpha<n,\ 0<t<1\\ y^{(i)}(0)=0,\ \ i=0,1,...,n-2\\ y(1)=\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}y(s)ds. \end{cases}$$

Where $f\in C([0,1]\times\mathbb{R},\mathbb{R}_+)$ is a given function,$n\in\mathbb{N},n\geq 2,\lambda_k>0,\ 0<\eta_k<1,\ \forall k=0,...,m.$
The aim of this chapter is the study of existence of solutions for a nonlinear boundary value problem involving both the right Caputo and the left Caputo fractional derivatives:

$$-\,{}^{C}D^{\alpha}_{1^-}\,{}^{C}D^{\beta}_{0^+}u(t)+\omega^2u(t)+f(t,u(t),D^{\beta}_{0^+}u(t))=0,\ t\in J=[0,\ 1]. \tag{IV.1}$$

$${}^{C}D^{\beta}_{0^+}u(1)=0\ ,\ u(0)=g(u)\ ,\ u'(0)=h(u). \tag{IV.2}$$

where $0<\alpha<1,\ 1<\beta<2,\ \omega\in\mathbb{R},\ {}^{C}D^{\alpha}_{1^-},{}^{C}D^{\beta}_{0^+}$ denotes the right and left Caputo derivative respectively,and denotes $D^{\beta}_{0^+}$ the left Riemann-Liouville , u is the unknown function and $f:J\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is a continuous function, and $g,h:C(J,\mathbb{R})\longrightarrow\mathbb{R}$ are continuous functions.

IV.2 Main results

We consider the following boundary value problem:

$$-^{C}D^{\alpha}_{1^-}\,{}^{C}D^{\beta}_{0^+}u(t)+K(t,u(t),D^{\beta}_{0^+}u(t))=0,\ t\in J=[0,1]. \tag{IV.3}$$

$${}^{C}D^{\beta}_{0^+}u(1)=0,\ u(0)=g(u),\ u'(0)=h(u). \tag{IV.4}$$

Where $K(t,u(t),D^{\beta}_{0^+}u(t))=\omega^2u(t)+f(t,u(t),D^{\beta}_{0^+}u(t))$, $0<\alpha<1,\ 1<\beta<2$. If u is a solution of problem (IV.3)-(IV.4), then u is solution of problem (IV.1)-(IV.2).

To study the nonlinear problem (IV.3)-(IV.4), we first consider the associated linear problem

$$-\,{}^{C}D_{1^-}^{\alpha}\,{}^{C}D_{0^+}^{\beta}u(t)+y(t)=0,\ t\in J=[0,1]. \tag{IV.5}$$

$${}^{C}D_{0^+}^{\beta}u(1)=0,\ u(0)=g(u),\ u'(0)=h(u). \tag{IV.6}$$

Lemma 48 *Assume that $y\in L_1(J)$, then u is a solution to the linear boundary value problem $(IV.5)-(IV.6)$ if and only if u satisfies the integral equation*

$$u(t)=\int_0^1 G(t,\tau)y(\tau)d\tau+g(u)+th(u)$$

where

$$G(t,\tau)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\begin{cases}\displaystyle\int_0^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds, & 0\le\tau\le t\le 1,\\ \displaystyle\int_0^{t}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds, & 0\le t\le\tau\le 1.\end{cases} \tag{IV.7}$$

Proof. we apply the right-hand side fractional integral $I_{1^-}^{\alpha}$ to equation (IV.5).We get

$${}^{C}D_{0^+}^{\beta}u(t)=I_{1^-}^{\alpha}y(t)+{}^{C}D_{0^+}^{\beta}u(1)$$

Using the boundary conditions ${}^{C}D_{0^+}^{\beta}u(1)=0$, we get

$${}^{C}D_{0^+}^{\beta}u(t)=I_{1^-}^{\alpha}y(t)$$

then we apply the fractional integral $I_{0^+}^{\beta}$, we get

$$u(t)=I_{0^+}^{\beta}\Big(I_{1^-}^{\alpha}y(t)\Big)+u(0)+tu'(0).$$

Using the conditions nonlocal $u(0)=g(u),\ u'(0)=h(u)$, so

$$u(t)=I_{0^+}^{\beta}\Big(I_{1^-}^{\alpha}y(t)\Big)+g(u))+th(u).$$

$$u(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\Big(\int_s^1(\tau-s)^{\alpha-1}y(\tau)d\tau\Big)ds+g(u))+th(u).$$

Finally, by using the Fubini theorem, we get

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\Big(\int_0^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\Big)y(\tau)d\tau\\&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^1\Big(\int_0^t(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\Big)y(\tau)d\tau\\&+g(u))+th(u).\end{aligned}$$ ■

Lemma 49 *The function G satisfy the following properties:*
(1) The function $G(t,\tau)$ is nonnegative.
(2) $G(t,\tau)<\dfrac{1}{\Gamma(\alpha+1)\Gamma(\beta)}$ *for all $t,\tau\in J$*

Let $C(J,\mathbb{X})$ be the space of all continuous functions defined on J.
Define the space $\mathbb{X} = \{u \mid u \in C(J),\ D^{\beta}_{0^+}u \in C(J)\}$ endowed with the norm $\| u \|_{\mathbb{X}}= \max\limits_{t\in J}|u(t)| + \max\limits_{t\in J}\left|D^{\beta}_{0^+}u(t)\right|$ It is clear that $(\mathbb{X}, \| \,.\, \|_{\mathbb{X}})$ is a Banach space.

Lemma 50 *Let $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function. A function $u \in \mathbb{X}$ is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,\tau)f(\tau, u(\tau), D^{\beta}_{0^+}u(\tau))d\tau$$
$$+ \omega^2 \int_0^1 G(t,\tau)u(\tau)d\tau + g(u)) + th(u).$$

if and only if u is a solution of the fractional boundary value problem (IV.1)-(IV.2)

Now we define the operators A and B on $\mathbb{X}$ as

$$Au(t) = \int_0^1 G(t,\tau)f(\tau, u(\tau), D^{\beta}_{0^+}u(\tau))d\tau$$

$$Bu(t) = \omega^2 \int_0^1 G(t,\tau)u(\tau)d\tau + g(u)) + th(u)$$

Obviously, problem (IV.1)-(IV.2) has a solution if and only if $A + B$ has a fixed point.
Before stating and proving the main results, we introduce the following hypotheses.
(H1) $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function,
(H2) For each $t \in J$ and all $u, v \in \mathbb{R}$. There exists a constant $L > 0$ such that

$$\mid f(t,u,v) \mid \leq L(1+ \mid u \mid + \mid v \mid),$$

(H3) There exist constants $M_1, M_2, M_3 > 0$ such that $u, \in \mathbb{X},\ \ t \in J$ we have

$$\mid g(u) \mid \leq M_1, \quad \mid h(u) \mid \leq M_2, \quad \mid D^{\beta}_{0^+}(g(u) + th(u)) \mid \leq M_3.$$

(H4) There exists a constants $k_1, k_2, k_3 > 0$ such that for $u, v \in \mathbb{X},\ \ t \in J$ we have

$$\begin{aligned}
&\mid g(u) - g(v) \mid \leq k_1 \mid u - v \mid, \\
&\mid h(u) - h(v) \mid \leq k_2 \mid u - v \mid, \\
&\mid D^{\beta}_{0^+}(g(u) + th(u)) - D^{\beta}_{0^+}(g(v) + th(v)) \mid \leq k_3 \mid u - v \mid,
\end{aligned}$$

and

$$N := \left[\frac{\omega^2(1 + \Gamma(\beta))}{\Gamma(\alpha+1)\Gamma(\beta)} + k_1 + k_2 + k_3\right] < 1$$

and there exists $R > 0$ such that

$$\left[\frac{1 + \Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}\Big(R(\omega^2 + L) + L\Big) + M_1 + M_2 + M_3\right] \leq R$$

Theorem 51 *Assume that (H1)-(H4) hold, then the fractional boundary value problem (IV.1)-(IV.2) has at least one solution in $\mathbb{X}$.*

To prove Theorem 51, we have to prove that all the assumptions of Krasnoselskii's fixed point theorem are satisfied, for this we need the following lemmas.

Lemma 52 *Under the hypothesis (H1)-(H2), the mapping A is completely continuous on Ω.*

Proof. Set $\Omega = \{u \in C(J, \mathbb{X}) :\| u \|_{\mathbb{X}} \leq R\}$. Clearly, Ω is a nonempty, bounded and convex subset of the Banach space $\mathbb{X}$.
The proof will be done in three steps.
Step 1. The mapping A is continuous on Ω. Consider the sequence $(u_n)_n \in \Omega$ such that $u_n \longrightarrow u$ in Ω, then from Lemma 49 and the hypothesis (H1) we get

$$\begin{aligned}
&\mid Au_n(t) - Au(t) \mid \leq \int_0^1 G(t,\tau) \mid f(\tau, u_n(\tau), D^{\beta}_{0^+}u_n(\tau)) - f(\tau, u(\tau), D^{\beta}_{0^+}u(\tau)) \mid d\tau \\
&\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 \sup_{\tau\in J} \mid f(\tau, u_n(\tau), D^{\beta}_{0^+}u_n(\tau)) - f(\tau, u(\tau), D^{\beta}_{0^+}u(\tau)) \mid d\tau \\
&\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \sup_{\tau\in J} \mid f(\tau, u_n(\tau), D^{\beta}_{0^+}u_n(\tau)) - f(\tau, u(\tau), D^{\beta}_{0^+}u(\tau)) \mid
\end{aligned}$$

And

$$\begin{aligned}
&\mid D^{\beta}_{0^+}(Au_n(t)) - D^{\beta}_{0^+}(Au(t)) \mid = \mid I^{\alpha}_{1^-} f(t, u_n(t), D^{\beta}_{0^+}u_n(t)) - I^{\alpha}_{1^-} f(t, u(t), D^{\beta}_{0^+}u(t)) \mid \\
&\leq \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} \sup_{s\in J} \mid f(s, u_n(s), D^{\beta}_{0^+}u_n(s)) - f(s, u(s), D^{\beta}_{0^+}u(s)) \mid ds \\
&\leq \frac{\sup_{s\in J} \mid f(s, u_n(s), D^{\beta}_{0^+}u_n(s)) - f(s, u(s), D^{\beta}_{0^+}u(s)) \mid}{\Gamma(\alpha+1)} (1-t)^{\alpha} \\
&\leq \frac{\sup_{s\in J} \mid f(s, u_n(s), D^{\beta}_{0^+}u_n(s)) - f(s, u(s), D^{\beta}_{0^+}u(s)) \mid}{\Gamma(\alpha+1)}
\end{aligned}$$

Thus, we get

$$\| Au_n - Au \|_{\mathbb{X}} \leq \frac{1+\Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \sup_{t\in J} \mid f(t, u_n(t), D^{\beta}_{0^+}u_n(t)) - f(t, u(t), D^{\beta}_{0^+}u(t)) \mid$$

Since f is continuous, then $\| Au_n - Au \|_{\mathbb{X}} \longrightarrow 0$ as $n \longrightarrow \infty$.
Step 2. (Au) is uniformly bounded on Ω. Let $u \in \Omega$, then by condition (H2) it yields

$$\begin{aligned}
&\mid Au(t) \mid \leq \int_0^1 G(t,\tau) \mid f(\tau, u(\tau), D^{\beta}_{0^+}u(\tau)) \mid d\tau \\
&\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} L(1+ \| u \|_{\mathbb{X}}) \leq \frac{L(1+R)}{\Gamma(\alpha+1)\Gamma(\beta)}.
\end{aligned}$$

And

$$\begin{aligned}
&\mid D^{\beta}_{0^+}(Au(t)) \mid = \mid I^{\alpha}_{1^-} f(t, u(t), D^{\beta}_{0^+}u(t)) \mid \\
&\leq \frac{L(1+ \| u \|_{\mathbb{X}})}{\Gamma(\alpha+1)} (1-t)^{\alpha} \leq \frac{L(1+R)}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence, we get

$$\| Au \|_{\mathbb{X}} \leq \frac{1+\Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} L(1+R).$$

Step 3. (Au) is equicontinuous on Ω. We have, for $u \in \Omega,\ 0 \leq t_1 \leq t_2 \leq 1$.

$$
\begin{aligned}
&\mid Au(t_1) - Au(t_2) \mid \leq \int_0^{t_1} \mid G(t_1,\tau) - G(t_2,\tau) \mid\mid f(\tau, u(\tau), D_{0^+}^{\beta} u(\tau)) \mid d\tau \\
&+ \int_{t_1}^{t_2} \mid G(t_1,\tau) - G(t_2,\tau) \mid\mid f(\tau, u(\tau), D_{0^+}^{\beta} u(\tau)) \mid d\tau \\
&+ \int_{t_2}^{1} \mid G(t_1,\tau) - G(t_2,\tau) \mid\mid f(\tau, u(\tau), D_{0^+}^{\beta} u(\tau)) \mid d\tau \\
&\leq \frac{L(1+R)}{\Gamma(\alpha)\Gamma(\beta)} \Big[\int_0^{t_1} \Big(\int_0^{\tau} ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1})(\tau - s)^{\alpha-1} ds \Big) d\tau \\
&+ \int_{t_1}^{1} \Big(\int_0^{t_1} ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1})(\tau - s)^{\alpha-1} ds \Big) d\tau \\
&+ \int_{t_1}^{t_2} \Big(\int_{t_1}^{\tau} (t_2 - s)^{\beta-1}(\tau - s)^{\alpha-1} ds \Big) d\tau \\
&+ \int_{t_2}^{1} \Big(\int_{t_1}^{t_2} (t_2 - s)^{\beta-1}(\tau - s)^{\alpha-1} ds \Big) d\tau \Big] \\
&\leq \frac{L(1+R)}{\Gamma(\alpha)\Gamma(\beta)} \Big[(\beta - 1)(t_2 - t_1) \Big(\frac{1 - (1 - t_1)^{\alpha+1}}{\alpha(\alpha+1)} \Big) \\
&+ \frac{(1 - t_1)^{\alpha+1} - (1 - t_2)^{\alpha+1}}{\alpha(\alpha+1)} \Big] \to 0 \text{ when } t_1 \longrightarrow t_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mid D_{0^+}^{\beta}(Au(t_1)) - D_{0^+}^{\beta}(Au(t_2)) \mid = \mid I_{1^-}^{\alpha} f(t_1, u(t_1), D_{0^+}^{\beta} u(t_1)) - I_{1^-}^{\alpha} f(t_2, u(t_2), D_{0^+}^{\beta} u(t_2)) \mid \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} \mid f(s, u(s), D_{0^+}^{\beta} u(s)) \mid ds \\
&+ \int_{t_2}^{1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \mid f(s, u(s), D_{0^+}^{\beta} u(s)) \mid ds \\
&\leq \frac{L(1+R)}{\Gamma(\alpha+1)} \mid (t_1 - s)^{\alpha} - (t_2 - s)^{\alpha} \mid \to 0 \text{ when } t_1 \longrightarrow t_2
\end{aligned}
$$

Hence (Au) is equicontinuous. Finally, by Arzela-Ascoli's theorem, it follows that A is a completely continuous mapping on Ω. ■

Lemma 53 *Under the hypothesis (H3)-(H4), the mapping B is a contraction on Ω.*

Proof. **Step 1.** For $u \in \Omega$ and $t \in J$, we get

$$
\begin{aligned}
&\mid Bu(t) \mid = \mid \omega^2 \int_0^1 G(t,\tau) u(\tau) d\tau + g(u)) + t h(u) \mid \\
&\leq \frac{\omega^2 \mid u \mid}{\Gamma(\alpha+1)\Gamma(\beta)} + M_1 + t M_2 \\
&\leq \frac{\omega^2 R}{\Gamma(\alpha+1)\Gamma(\beta)} + M_1 + M_2.
\end{aligned}
$$

And

$$\begin{aligned}&\mid D^{\beta}_{0^+}Bu(t)\mid=\mid \omega^2 I^{\alpha}_{1^-}u(t)+D^{\beta}_{0^+}(g(u)+th(u))\mid\\ &\leq \omega^2 I^{\alpha}_{1^-}\mid u(t)\mid+\mid D^{\beta}_{0^+}(g(u)+th(u))\mid\leq \frac{\omega^2 R}{\Gamma(\alpha+1)}+M_3\end{aligned}$$

So

$$\parallel Bu\parallel_{\mathbb{X}}\leq \frac{\omega^2 R(1+\Gamma(\beta))}{\Gamma(\alpha+1)\Gamma(\beta)}+M_1+M_2+M_3$$

Step 2. Let $u, v\in\Omega$, then

$$\begin{aligned}&\mid Bu(t)-Bv(t)\mid\leq \omega^2\int_0^1 G(t,\tau)\mid u(\tau)-v(\tau)\mid d\tau+\mid g(u))-g(v)\mid+\mid th(u)-th(v)\mid\\ &\leq\left[\frac{\omega^2}{\Gamma(\alpha+1)\Gamma(\beta)}+k_1+k_2\right]\mid u-v\mid\\ &\leq\left[\frac{\omega^2}{\Gamma(\alpha+1)\Gamma(\beta)}+k_1+k_2\right]\parallel u-v\parallel_{\mathbb{X}}.\end{aligned}$$

And

$$\begin{aligned}&\mid D^{\beta}_{0^+}Bu(t)-D^{\beta}_{0^+}Bv(t)\mid\leq\omega^2\mid I^{\alpha}_{1^-}u(t)-I^{\alpha}_{1^-}v(t)\mid+\mid D^{\beta}_{0^+}(g(u)+th(u))-D^{\beta}_{0^+}(g(v)+th(v))\mid\\ &\leq\left[\frac{\omega^2}{\Gamma(\alpha+1)}+k_3\right]\mid u-v\mid\leq\left[\frac{\omega^2}{\Gamma(\alpha+1)}+k_3\right]\parallel u-v\parallel_{\mathbb{X}}.\end{aligned}$$

So

$$\parallel Bu-Bv\parallel_{\mathbb{X}}\leq\left[\frac{\omega^2(1+\Gamma(\beta))}{\Gamma(\alpha+1)\Gamma(\beta)}+k_1+k_2+k_3\right]\parallel u-v\parallel_{\mathbb{X}}.$$

Thus, B is a contraction. ■

Lemma 54 *Under the hypothesis (H1)-(H4),* $Au+Bv\in\Omega$ *for all* $u, v\in\Omega$

Proof. Let $u, v\in\Omega$

$$\begin{aligned}&\parallel Au+Bv\parallel_{\mathbb{X}}\leq\parallel Au\parallel_{\mathbb{X}}+\parallel Bv\parallel_{\mathbb{X}}\\ &\leq\frac{1+\Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}\Big(R(\omega^2+L)+L\Big)+M_1+M_2+M_3\leq R\end{aligned}$$

so, $Au+Bv\in\Omega$. ■

Proof of Theorem 51. Since the mapping A is completely continuous by Lemma 52, the mapping B is a contraction by Lemma 53 and $Au+Bv\in\Omega$ for all $u, v\in\Omega$ by Lemma 54, then all the hypotheses of Theorem 32 are satisfied. Thus there exists at least one solution $u^*\in\Omega$ for problem (IV.1)-(IV.2) such that $u^*=Au^*+Bu^*$. The proof is complete.

IV.3 An Example

Let us consider the following boundary value problem:

$$-{}^{C}D^{\frac{1}{2}}_{1^-}\,{}^{C}D^{\frac{3}{2}}_{0^+}u(t)+10^{-4}u(t)+f(t,u(t),D^{\frac{3}{2}}_{0^+}u(t))=0\ ,\ t\in J=[0,\ 1].\qquad\text{(IV.8)}$$

$$^{C}D_{0^+}^{\frac{3}{2}}u(1) = 0\ ,\ u(0) = g(u)\ ,\ u'(0) = h(u). \tag{IV.9}$$

Where $f(t,u,v) = \dfrac{e^{-t}(u+v)}{(1+9e^t)(1+u+v)}$, and $g(u) = I_{0^+}^{\frac{3}{2}}\left(\sum_{i=0}^{n} C_i u(t_i)\right) = \dfrac{\sum_{i=0}^{n} C_i u(t_i)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}}$,

where $C_i, i = 0, \cdots, n$ are given constants such that $\sum_{i=0}^{n} C_i \leq \dfrac{1}{5}$ and $0 < t_1 < \cdots < t_n < 1$,

and $h(u) = \dfrac{\sum_{j=0}^{m} \lambda_j u(t_j)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}}$ where $\lambda_j, j = 0, \cdots, m$ are given constants such that $\sum_{j=0}^{m} \lambda_j \leq \dfrac{1}{50}$ and $0 < t_1 < \cdots < t_m < 1$.

Let $u, v \in [0,\ \infty)$ and $t \in J$. Then we have

$$\begin{aligned} \mid f(t,u,v) \mid &= \mid \frac{e^{-t}(u+v)}{(1+9e^t)(1+u+v)} \mid \leq \frac{e^{-t}}{1+9e^t} \mid u+v \mid \\ &\leq \frac{e^{-t}}{1+9e^t} \mid 1+u+v \mid \leq \frac{1}{10}(1+ \mid u \mid + \mid v \mid). \end{aligned}$$

Hence the condition (H2) holds with $L = \dfrac{1}{10}$.

Set $\delta = \max\{u(t_i), u(t_j)\ : 0 \leq i \leq n\ ;\ 0 \leq j \leq m\}$, then we have

$$\mid g(u) \mid = \mid \frac{\sum_{i=0}^{n} C_i u(t_i)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} \mid \leq \frac{\sum_{i=0}^{n} C_i \mid u(t_i) \mid}{\Gamma(\frac{5}{2})} \leq \frac{\delta \sum_{i=0}^{n} C_i}{\Gamma(\frac{5}{2})} = M_1.$$

and, we have

$$\mid h(u) \mid = \mid \frac{\sum_{j=0}^{m} \lambda_j u(t_j}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} \mid \leq \frac{\sum_{j=0}^{m} \lambda_j \mid u(t_j) \mid}{\Gamma(\frac{5}{2})} \leq \frac{\delta \sum_{j=0}^{m} \lambda_j}{\Gamma(\frac{5}{2})} = M_2.$$

$$\begin{aligned} \mid D_{0^+}^{\frac{3}{2}}(g(u) + th(u)) \mid &= \mid \sum_{i=0}^{n} C_i u(t_i) + t\frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j u(t_j) \mid \\ &\leq \sum_{i=0}^{n} C_i \mid u(t_i) \mid + \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j \mid u(t_j) \mid \\ &\leq \delta\left(\sum_{i=0}^{n} C_i + \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j\right) = M_3. \end{aligned}$$

Hence the condition (H3) holds, and

$$\begin{aligned} \mid g(u) - g(v) \mid &= \mid \frac{\sum_{i=0}^{n} C_i (u(t_i) - v(t_i))}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} \mid \\ &\leq \frac{\sum_{i=0}^{n} C_i \mid u(t_i) - v(t_i) \mid}{\Gamma(\frac{5}{2})} \leq \frac{\sum_{i=0}^{n} C_i \mid u - v \mid}{\Gamma(\frac{5}{2})}. \end{aligned}$$

therefore $k_1 = \dfrac{\sum_{i=0}^{n} C_i}{\Gamma(\frac{5}{2})}$,and we have

$$\begin{aligned} \mid h(u) - h(v) \mid &= \mid \frac{t^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j (u(t_j) - v(t_j)) \mid \\ &\leq \frac{\sum_{j=0}^{m} \lambda_j \mid u(t_j) - v(t_j) \mid}{\Gamma(\frac{5}{2})} \leq \frac{\sum_{j=0}^{m} \lambda_j}{\Gamma(\frac{5}{2})} \mid u - v \mid . \end{aligned}$$

we get $k_2 = \dfrac{\sum_{j=0}^{m} \lambda_j}{\Gamma(\frac{5}{2})}$, then we have

$$\begin{aligned} &\mid D_{0^+}^{\frac{3}{2}}(g(u) - g(v)) + D_{0^+}^{\frac{3}{2}} t(h(u) - h(v)) \mid \leq \mid D_{0^+}^{\frac{3}{2}}(g(u) - g(v)) \mid + \mid D_{0^+}^{\frac{3}{2}} t(h(u) - h(v)) \mid \\ &\leq \sum_{i=0}^{n} C_i \mid u - v \mid + \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j \mid u - v \mid \leq \left(\sum_{i=0}^{n} C_i + \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j \right) \mid u - v \mid . \end{aligned}$$

we get $k_3 = \sum_{i=0}^{n} C_i + \dfrac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \sum_{j=0}^{m} \lambda_j$. So

$$N = \left(10^{-4} \frac{1 + \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \right) + \sum_{i=0}^{n} C_i \left(1 + \frac{1}{\Gamma(\frac{5}{2})} \right) + \sum_{j=0}^{m} \lambda_j \left(\frac{1}{\Gamma(\frac{5}{2})} + \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2})} \right) < 0.42 < 1.$$

Hence the condition (H4) holds.
Thus, by Theorem 51, the boundary value problem (IV.8 -IV.9) has at least one solution.

Chapter V

On a Fractional Oscillator Equation With Finite Delay

V.1 Introduction

This chapter deals with the existence of solutions for initial value problems for nonlinear fractional oscillator equation with both left Riemann-Liouville and right Caputo fractional derivatives. The Banach theorem about the fixed point are used to prove the existence and uniqueness of solutions of the problem considered, then we apply Leray-Schauder fixed point theorem to conclude the existence of nontrivial solutions
In [60], the following boundary value problem of the fractional differential equation is considered

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = f(t, u(t), u(\delta(t)), {}^{C}D^{\beta}_{0^+}u(t)), & t \in [0, b] \\ u(t) = \varphi(t), \quad t \in [-a, 0]. \end{cases}$$

where $0 < \beta < \alpha < 1,\ \delta : [0, b] \longrightarrow \mathbb{R}$ is continuous, nondecreasing, $\delta(t) \leq t$, $a = \inf_{0 \leq t \leq b} \delta(t)$, and f, φ are continuous functions.
The nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\sigma}y(t) = f(t, y_t), & t \in [0, \xi] \\ y(0) = y'(0) = 0,\ y''(\xi) = 1. \end{cases}$$

has been studied In [81], where $2 < \sigma \leq 3, {}^{C}D^{\sigma}$ denotes the standard Caputo fractional derivative, the function $f : \Omega \times C([-r, 0]) \longrightarrow \mathbb{R}, 0 < r < \xi$ and the y_t devote $y_t(\theta) = y(t + \theta), \theta \in [-r, 0]$. By utilizing the Banach fixed point theorem, Schauder fixed point theorem and the nonlinear alternative theorem, the previous problem has a solution. For some recent works on the existence of solutions for fractional differential equations with finite delay see [20,30,60,67,69,74,79,84].
The aim of this chapter is the study of the existence of solutions for the following nonlinear boundary value problem:

$$-{}^{C}D^{\alpha}_{1^-}D^{\beta}_{0^+}u(t) + \omega^2 u(t) + f(t, u_t) = 0\ ,\ t \in J = [0, 1]. \tag{V.1}$$

$${}^{C}D^{\beta}_{0^+}u(1) = 0,\ u(t) = \phi(t),\ t \in [-d, 0]. \tag{V.2}$$

where $0 < \alpha < 1,\ 0 < \beta < 1,\ \omega \in \mathbb{R}, u$ is the unknown function and $f : J \times C([-d, 0], \mathbb{R}) \longrightarrow \mathbb{R}$ is a continuous function, and $\phi \in C([-d,\ 0], \mathbb{R})$ with $\phi(0) = 0$. For any continuous function u defined on $[-d, 1]$ and any $t \in J$, we denote by u_t the element of $C([-d,\ 0], \mathbb{R})$ defined by

$$u_t(\tau) = u(t + \tau),\ \tau \in [-d, 0]$$

here $u_t(\cdot)$ represents the history of the state from time $(t-d)$ up to the present time t.
By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\| u \|_\infty := \max\{| u(t) |: t \in J\}.$$

Also, $C([-d,0],\mathbb{R})$ is endowed with the norm $\| \cdot \|_C$ defined by

$$\| u \|_C := \max\{| u(\tau) |: -d \leq \tau \leq 0\}.$$

Set $E = C([-d,1],\mathbb{R})$ endowed with the norm

$$\| u \| := \max\{\| u \|_\infty, \| u \|_C\}.$$

V.2 Existence of solutions

Lemma 55 *Let $f : J \times C([-d,0],\mathbb{R}) \longrightarrow \mathbb{R}$ be a continuous function. A function u is a solution of the integral equation*

$$u(t) = \begin{cases} \phi(t), & t \in [-d,\ 0], \\ \displaystyle\int_0^1 G(t,\tau) f(\tau, u_\tau) d\tau + \omega^2 \int_0^1 G(t,\tau) u(\tau) d\tau, & t \in J. \end{cases}$$

if and only if u is a solution of the fractional boundary value problem (V.1)-(V.2)

Theorem 56 *Assume that:*
(H) There exists a constant $L > 0$ such that for $t \in J$ and $u, v \in E$:

$$| \quad f(t,u_t) - f(t,v_t) \, | \leq L \, \|u_t - v_t\|_C \, ,$$
$$\frac{L+\omega^2}{\Gamma(\alpha+1)\Gamma(\beta)} \quad < \quad 1,$$

then the problem (V.1)-(V.2) has solution unique on $[-d,1]$.

Proof. We shall prove that the operators A is a contraction. Indeed, let $u, v \in E$, then from condition (H) and Lemma(49) we have for $t \in [-d,1]$:

$$A(u)(t) - A(v)(t) = 0, \ \ t \in [-d,\ 0]$$
$$|A(u)(t) - A(v)(t)| \leq \int_0^1 |G(t,\tau)| \, |f(\tau,u_\tau) - f(\tau,v_\tau)| \, d\tau + \omega^2 \int_0^1 |G(t,\tau)| \, |u-v| \, d\tau$$
$$\leq \frac{L+\omega^2}{\Gamma(\alpha+1)\Gamma(\beta)} \, \| u - v \|, \quad t \in J$$

We conclude

$$\| A(u) - A(v) \| \leq \frac{l+\omega^2}{\Gamma(\alpha+1)\Gamma(\beta)} \, \| u - v \|$$

Therefore A is a contraction mapping. As a consequence of the Banach fixed point theorem, we deduce that A has a unique fixed point which is the unique solution of the problem $(V.1) - (V.2)$.
■

Theorem 57 *Assume that:*
$(H1)$ $f : J \times C\big([-d\ ,\ 0],\ \mathbb{R}\big) \longrightarrow \mathbb{R}$ *is continuous function.*
$(H2)$ *There exist a nonnegative function* $k \in C(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* $\chi : [0, +\infty) \longrightarrow [0, +\infty)$ *such that for* $t \in J, u \in C\big([-d, 0],\ \mathbb{R}\big)$ *we have:*

$$| f(t, u_t) | \leq k(t)\chi(\| u \|).$$

$(H3)$ *There exists a constant* $M > \| \varphi \|_C$ *such that:*

$$\frac{M\Gamma(\alpha+1)\Gamma(\beta)}{\| k \| \chi(M) + \omega^2 M} > 1.$$

Then the problem $(V.1) - (V.2)$ *has at least one solution.*

Proof. The proof will be given in several steps.
Step 1**:** A is continuous. In fact, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in E. Then for each $t \in J$, we have

$$\begin{aligned} | A(u_n)(t) - A(u)(t) | &\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \times \\ \int_0^1 \Big(| f(\tau, u_{n\tau}) - f(\tau, u_\tau) | &+ \omega^2 | u_n(\tau) - u(\tau) | \Big) d\tau \\ &\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \times \\ \int_0^1 \Big(\sup_{\tau\in[0,1]} | f(\tau, u_{n\tau}) - f(\tau, u_\tau) | &+ \omega^2 \sup_{\tau\in[0,1]} | u_n(\tau) - u(\tau) | \Big) d\tau \\ &\leq \frac{\| f(\cdot, u_{n\cdot}) - f(\cdot, u_\cdot) \|_\infty + \omega^2 \| u_n(\cdot) - u(\cdot) \|_\infty}{\Gamma(\alpha+1)\Gamma(\beta)} \end{aligned}$$

Since f is a continuous function, we have

$$\| A(u_n) - A(u) \| \leq \frac{\| f(\cdot, u_{n\cdot}) - f(\cdot, u_\cdot) \|_\infty + \omega^2 \| u_n(\cdot) - u(\cdot) \|_\infty}{\Gamma(\alpha+1)\Gamma(\beta)} \to 0, \qquad as\ n \to \infty.$$

Step 2 : A maps bounded sets into bounded sets in E. In fact let $u \in \Omega = \{u \in E, \| u \| \leq R\}$, then by condition $(H2)$ it yields

$$\begin{aligned} | A(u)(t) | &\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 \Big(| f(\tau, u_\tau) | + \omega^2 | u(\tau) | \Big) d\tau \\ &\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \bigg(\int_0^1 k(\tau)\chi(\| u_\tau \|_C) d\tau + \omega^2 \| u \|_\infty \bigg) \\ &\leq \frac{\| k \|_\infty \chi(R) + \omega^2)R}{\Gamma(\alpha+1)\Gamma(\beta)}. \end{aligned}$$

Then

$$\| A(u) \|_\infty \leq \frac{\| k \|_\infty \chi(R) + \omega^2)R}{\Gamma(\alpha+1)\Gamma(\beta)}.$$

For $t \in [-d, 0]$ we get

$$\| Au \|_C = \| \varphi \|_C$$

hence

$$\| A(u) \| \leq \max\left\{ \| \varphi \|_C, \frac{\| k \|_\infty \chi(R) + \omega^2) R}{\Gamma(\alpha+1)\Gamma(\beta)} \right\}$$

Step 3 : (Au) is equicontinuous. We have, for $u \in E$, $0 \leq t_1 \leq t_2 \leq 1$,

$$\begin{aligned}
&| Au(t_1) - Au(t_2) | \leq \int_0^{t_1} | G(t_1,\tau) - G(t_2,\tau) || f(\tau,u_\tau) + \omega^2 u(\tau) | \, d\tau \\
&+ \int_{t_1}^{t_2} | G(t_1,\tau) - G(t_2,\tau) || f(\tau,u_\tau) + \omega^2 u(\tau) | \, d\tau \\
&+ \int_{t_2}^{1} | G(t_1,\tau) - G(t_2,\tau) || f(\tau,u_\tau) + \omega^2 u(\tau) | \, d\tau \\
&\leq \frac{\| k \|_\infty \chi(\| u \|) + \omega^2 \| u \|}{\Gamma(\alpha)\Gamma(\beta)} \\
&\times \Big[\int_0^{t_1} \Big(\int_0^{\tau} ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1})(\tau - s)^{\alpha-1} ds \Big) d\tau \\
&+ \int_{t_1}^{1} \Big(\int_0^{t_1} ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1})(\tau - s)^{\alpha-1} ds \Big) d\tau \\
&+ \int_{t_1}^{t_2} \Big(\int_{t_1}^{\tau} (t_2 - s)^{\beta-1} (\tau - s)^{\alpha-1} ds \Big) d\tau \\
&+ \int_{t_2}^{1} \Big(\int_{t_1}^{t_2} (t_2 - s)^{\beta-1} (\tau - s)^{\alpha-1} ds \Big) d\tau \Big] \\
&\leq \frac{\| k \|_\infty \chi(\| u \|) + \omega^2 \| u \|}{\Gamma(\alpha)\Gamma(\beta)} \times \\
&\Big[(\beta - 1)(t_2 - t_1) \Big(\frac{1 - (1 - t_1)^{\alpha+1}}{\alpha(\alpha+1)} \Big) \\
&+ \frac{(1 - t_1)^{\alpha+1} - (1 - t_2)^{\alpha+1}}{\alpha(\alpha+1)} \Big].
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\| Au(t_1) - Au(t_2) \| \leq \frac{\| k \|_\infty \chi(\| u \|) + \omega^2 \| u \|}{\Gamma(\alpha)\Gamma(\beta)} \times \\
&\Big[(\beta - 1)(t_2 - t_1) \Big(\frac{1 - (1 - t_1)^{\alpha+1}}{\alpha(\alpha+1)} \Big) \\
&+ \frac{(1 - t_1)^{\alpha+1} - (1 - t_2)^{\alpha+1}}{\alpha(\alpha+1)} \Big] \to 0 \text{ as } t_1 \longrightarrow t_2,
\end{aligned}$$

thus (Au) is equicontinuous. Finally, by Arzela-Ascoli's theorem, it follows that A is a completely continuous mapping on Ω.

Step 4 : (A priori bounds). Let us set

$$U = \{ u \in E : \| u \| < M \}.$$

Assume that there exists $u \in \partial U$ such that $u = \lambda A(u)$, for some $0 < \lambda < 1$. Then

$$\begin{aligned}
&\| \ u \|_\infty = \lambda \| A(u) \|_\infty \leq \| A(u) \|_\infty \leq \frac{\| k \| \chi(\| u \|) + \omega^2 \| u \|}{\Gamma(\alpha+1)\Gamma(\beta)}, \\
&\| \ u \|_C \leq \| \varphi \|_C
\end{aligned}$$

thus

$$\| u \| \leq \max \left\{ \| \varphi \|_C, \frac{\| k \| \chi(M) + \omega^2 M}{\Gamma(\alpha+1)\Gamma(\beta)} \right\} = \frac{\| k \| \chi(M) + \omega^2 M}{\Gamma(\alpha+1)\Gamma(\beta)}$$

This contradicts condition $(H3)$.

Then the statement (ii) in Theorem 31 does not hold. As consequence of the nonlinear alternative of Leray-Schauder the statement (i) holds, we deduce that the operator A has at least one fixed point $u^* \in \overline{U}$, which is the the solution of the problem (V.1)-(V.2). ■

V.3 Exemples

Exemple 1 :

Let us consider the following fractional boundary value problem:

$$-^C D_{1^-}^{\frac{1}{2}} D_{0^+}^{\frac{1}{3}} u(t) + \frac{1}{100} u(t) + \frac{u_t \ln(t+1)}{1+u_t} = 0, \ t \in J = [0,1]. \tag{V.3}$$

$$D_{0^+}^{\frac{1}{3}} u(1) = 0, \ u(t) = \phi(t), \ t \in [-d, 0]. \tag{V.4}$$

Set

$$f(t,u) = \frac{u \ln(t+1)}{1+u}, \ (t,u) \in [0,1] \times [0, +\infty[$$

Let $t \in J$, then we have

$$\begin{aligned} &| f(t,u) - f(t,v) | = \ln(t+1) \mid \frac{u}{1+u} - \frac{v}{1+v} \mid \\ &\leq \ln(t+1) \frac{| u - v |}{|1+u| \, |1+v|} \leq \ln(t+1) \mid u - v \mid \leq \ln(2) \mid u - v \mid . \end{aligned}$$

Choose $L = \ln(2)$ then

$$\frac{L + \omega^2}{\Gamma(\alpha+1)\Gamma(\beta)} = \frac{\ln(2) + 10^{-2}}{\Gamma(\frac{3}{2})\Gamma(\frac{31}{3})} \approx 0.296171 < 1.$$

Thus all the assumptions in Theorem (56)are satisfied, then theproblem (V.3- V.4) has a unique solution in E.

Exemple 2 :

Let us consider the following fractional boundary value problem:

$$-^C D_{1^-}^{\alpha} D_{0^+}^{\beta} u(t) + \omega^2 u(t) + \frac{e^t t \sqrt{u_t^2 + 1}}{t+1} = 0, \ t \in J = [0,1]. \tag{V.5}$$

$$D_{0^+}^{\beta} u(1) = 0, \ u(t) = \phi(t), \ t \in [-d, 0]. \tag{V.6}$$

Here, $\alpha = 0.3, \ \beta = 0.4, \ \omega = 0.1, \phi(t) = e^t,$ and

$$f(t,u) = \frac{e^t t \sqrt{u^2+1}}{t+1}, \ (t,u) \in J \times \mathbb{R}$$

Then we have

$$\begin{aligned} &| f(t,u) | = | \frac{te^t}{t+1} \sqrt{u^2+1} \mid \\ &\leq \frac{te^t}{t+1} \sqrt{u^2+1} \\ &\leq \frac{te^t}{t+1} (|u| + 1) . \end{aligned}$$

We get $k(t) = \dfrac{te^t}{t+1}$ and $\chi(u) = u + 1$. We have $\| \ k \ \|_\infty = \dfrac{e}{2}$, If we choose $M > 2$ then $M > \| \ \phi \ \| = 1$ and

$$\frac{M\Gamma(\alpha+1)\Gamma(\beta)}{\| \ k \ \|_\infty \ \chi(M) + \omega^2 M} > 1$$

hence the condition $(H3)$ is satisfied. Since all assumptions of Theorem (57) hold, we conclude that the problem (V.5-V.6) has at least one solution u such $\|u\| \leq 2$.

Conclusion

In this thesis, we discussed the existence of solutions for some problems for nonlinear fractional oscillator equation containing both left Riemann-Liouville and right fractional derivatives, different boundary conditions and finite delay, in the functional spaces $L^p(0,1)$, $1 \leq p \leq \infty$, and in the space of continuous functions.
We also studied the existence solutions of multi-point boundary value problems for nonlinear fractional differential equations in the Riemann-Liouville fractional Sobolev space $W^{s,p}_{RL,a^+}$, $0 < s < 1$. To prove the existence results, some fixed point theorems are used, such Banach fixed point theorem, Leray-Schauder nonlinear alternative and Krasnoselskii fixed point theorem.

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