وزارة التعليم العالي والبحث العلمي

Université Badji Mokhtar Annaba

Badji Mokhtar – Annaba University

جامعة باجي مختار

عنابة

Faculté des Sciences
Département de Mathématiques

THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

Option : Mathématiques Appliquées

Qualitative study of solutions for certain fractional differential equations

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Année : 2021/2022

Qualitative study of solutions for certain fractional differential equations

A Doctoral Thesis, By Adel Lachouri
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University of Annaba

Dedication

First of all, I would like to thank Allah who gave me the will and the courage to carry out this work.

My grandmother **Zakia**, my father **Tahar** and my dearest mother **Halima**, you have always sacrificed for me and held out your hand in difficult times.

To my dearest wife **Nesrine**, for the support and encouragement she has shown throughout the duration of this thesis.

My dear brother **Ali** and my dear sisters **Nawel** and **Warda**.

Acknowledgement

First of all, I would like to thank **Allah** who gave me the will and the courage to carry out this work.

I would like to express my deep gratitude to my supervisors, Prof. **Abdelouaheb Ardjouni** and Prof. **Ahcene Djoudi** for their guidance, valuable advice and support during the preparation of this work.

My sincere thanks also go to the members of my PhD thesis committee: Prof. **Smail Kelaiaia**, Prof. **Abdelouahab Salmi**, Dr. **Mohamed Amine Kerker** and Dr. **Rabah Khemis** for their valuable comments.

I would also like to heartily thank my friends, **K. Ali Khelil**, **F. Bouchelaghem**, **S. Aissa** and **F. Abdelhamid** for their precious help.

Finally, I would like to express my gratitude to my parents, my wife, my brother and my sisters for their support and encouragement.

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1) **A. Lachouri**, A. Ardjouni, A. Djoudi, The existence and Ulam-Hyers stability results for generalized Hilfer fractional integro-differential equations with nonlocal integral boundary conditions, Advances in the Theory of Nonlinear Analysis and its Application 6(1) (2022), 101–117.

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25) **A. Lachouri**, A. Ardjouni, A. Djoudi, Existence and uniqueness results for higher order fractional differential equations and inclusions with multi-strip Hadamard type fractional integral boundary conditions, (submitted).

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27) **A. Lachouri**, A. Ardjouni, N. Gouri, K. Ali Khelil, Caputo-Hadamard fractional differential equations and inclusions with integral boundary conditions via fixed point theory, (submitted).

ملخــــــــص

الهدف من هذه الأطروحة هو دراسة بعض الخصائص النوعية لحلول فئات مختلفة من المعادلات والانتماءات التفاضلية الكسرية غير الخطية التي تتضمن أنواع مختلفة من المشتقات الكسرية. لهذا الهدف، نقوم بتحويل المشكل المعطى إلى معادلة تكاملية مكافئة ثم نستخدم نظريات النقطة الثابتة المناسبة بحيث تكون النقاط الثابتة التي تم الحصول عليها هي حلول للمشكل المعطى. نقدم أيضا مثالا توضيحيا لكل مشكل مدروس لإظهار فعالية النتائج النظرية.

الكلمـات المفتاحيـة: المعـادلات التفاضلية الكسرية، المعـادلات التفاضلية الكسرية الهجينة، المعـادلات التكاملية التفاضلية الكسرية، الانتماءات التفاضلية الكسرية، المشتق الكسرية، المشاكل الحدية، فضاء بناخ، النقطة الثابتة، قياس كوراتوفسكي لعدم التراص، الوجود، الوحدانية، استقرار أولام.

Abstract

The objective of this thesis is to study some qualitative properties of solutions for various classes of nonlinear fractional differential equations and inclusions involving various kinds of fractional derivatives. For this aim, we convert the given problem into an equivalent integral equation and then use the appropriate fixed point theorems such that the fixed points obtained are the solutions of the given problem. We also provide an illustrative example to each of the considered problem to show the effectiveness of the theoretical results.

Keywords: Fractional differential equations, hybrid fractional differential equations, fractional integro-differential equations, fractional differential inclusions, fractional derivatives, boundary value problems, Banach space, fixed point, Kuratowski measure of noncompactness, existence, uniqueness, Ulam stability.

Mathematics Subject Classification: 26A33, 34A08, 34A12, 34B15, 34K20, 47H08.

Résumé

L'objectif de cette thèse est d'étudier certaines propriétés qualitatives des solutions pour diverses classes d'équations et d'inclusions différentielles fractionnaires non linéaires impliquant divers types de dérivées fractionnaires. Pour ce but, nous convertissons le problème donné en une équation intégrale équivalente puis utilisons les théorèmes de points fixes appropriés tels que les points fixes obtenus sont les solutions du problème donné. Nous fournissons également un exemple illustratif à chaque problème considéré pour montrer l'efficacité des résultats théoriques.

Mots-clés: Équations différentielles fractionnaires, équations différentielles fractionnaires hybrides, équations intégro-différentielles fractionnaires, inclusions différentielles fractionnaires, dérivées fractionnaires, problèmes aux limites, espace de Banach, point fixe, mesure de non-compactité de Kuratowski, existence, unicité, stabilité d'Ulam.

Mathematics Subject Classification: 26A33, 34A08, 34A12, 34B15, 34K20, 47H08.

Contents

Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. The fractional calculus appeared in the year 1695 [64, 65, 66], in an exchange of correspondence between L'Hopital and Leibniz, even during the construction of the classic differential and integral calculus. In these correspondences Leibniz urged the possible generalization of the whole-order derivative to an arbitrary order, L'Hopital then questioned him about the special case where the order of the derivative was $\frac{1}{2}$. In the reply letter, dated September 30, 1695, Leibniz presented a correct reflection, in which he affirmed that very important consequences would come from these developments [64, 65, 66]. This date is regarded as the exact birthday of the fractional calculus. Encouraged by this new perspective of fractional calculus application, several authors have developed definitions for fractional derivatives and integrals in subsequent decades, but some of these definitions have contradicted each other. One of these definitions, which emerged in the nineteenth century; is the proposal by Liouville, which was later reformulated by Riemann, more information about the definitions of the Riemann-Liouville derivative and integral can be found in [46, 79, 87]. In this sense, Caputo introduced the so-called Caputo fractional derivative, fundamental in the study of memory effects and also in the modeling of real problems by means of differential equations.

Among other applications that over time were justifying the relevance of the fractional derivative, the emergence of many definitions of fractional derivatives, among which we mention: Hadamard, Weyl, Caputo-Hadamard, Katugampola, Caputo-Katugampola, Caputo-Fabrizio, Hilfer, Hilfer-Hadamard, Hilfer-Katugampola, Jumarie, Erdélyi-Kober, Riesz, Caputo-Riesz, Cassar, Grünwald-Letnikov, each with its respective importance and application [9, 24, 36, 41, 43, 46, 78, 79, 87]. These integrals and fractional derivatives have a different kernel and this makes the number of definitions wide. Recently in 2017, Sousa and Oliviera [89] proposed interpolator of ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfer's sense of definition so-called ψ-Hilfer fractional derivative, ie, a fractional derivative of a function with respect to another ψ-function. With this fractional derivative, we recover a wide class of fractional derivatives and integrals. This work is undoubtedly one of the first to collect scattered results, a detailed historical account is given in the introduction of [81] and also can surveys of the history of the fractional theory derivative can be found in [32, 71, 77, 82, 87].

In the last two decades, fractional differential equations have received very broad regard because of their applicability in many scientific disciplines. Moreover, it is an excellent tool for the description of properties of various materials and processes such as chemistry, physics, biology, fractional dynamics, fitting of experimental data, signal and image processing, economics and control theory. See for instance [16, 27, 39, 45, 62, 68, 96, 104]. For the recent development of the topic, we refer the reader to a series of books and papers [2, 7, 13, 14, 22, 24, 42, 46, 69, 70, 73, 75, 79, 89, 100].

On the other hand, the theory of fixed point is one of the most powerful tools of modern mathematics, as it has been applied in such diverse fields as Biology, Chemistry, Economics, Engineering, Game Theory, and Physics. In particular, in obtaining existence results for a variety of mathematical problems. In addition, in most of the existed articles, Banach contraction principle, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem, etc. have been used to obtain the existence,uniqueness of solutions of various problems of fractional differential equations and inclusions with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions, and periodic boundary conditions, under suitable conditions. Some contributions around applications of fixed point theorems in fractional differential equations and inclusions to show the existence, uniqueness and stability of solution can be found in [1, 8, 11, 51, 53, 58, 60, 70, 75, 76, 84, 90, 91, 99] and the references therein. But, in the absence of compacity and the Lipschitz condition, the previously mentioned theorems are not applicable. In such cases, the measure of noncompactness (briefly, MNC) argument appears as the most convenient and useful in applications. It is a method which was first introduced by Kuratowski [49] in 1930 which was further extended to general Banach spaces by Banás and Goebel (see [17]). After, that many authors used this technique in study and solve different kind problems, as differential equations, integral equations, fractional differential equations and integro-differential equations, see [4, 19, 20, 37, 74, 85, 86, 92] and the references therein. We also refer the readers to the recent book [18], where several applications of the measure of noncompactness can be found.

This thesis is arranged as follows

In **Chapter** 1, we present some basic concepts, definitions and lemmas about fractional calculus, measures of noncompactness, multivalued analysis, Ulam stability and some fixed-point theorems that are used throughout this thesis.

In **Chapter** 2, we are concerned with the existence and uniqueness of solutions for two classes of nonlinear fractional differential equations, The desired results are based on fixed point theorems (Banach, Schaefer, Krasnoselskii). More specifically, in Section 2.1, we debate the existence and uniqueness of mild solutions for the following fractional boundary value problem with integral and anti-periodic conditions

$$\begin{aligned} {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}x(t) &= f(t,x(t)), \ t \in (1,T), \\ x(1)+x(T) &= b\int_{1}^{T} x(s)\frac{ds}{s}, \end{aligned}$$

where $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, ${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0<\alpha<1$ and $b\in\mathbb{R}$ such that $2-b\log(T)>0$. The main outcomes of this problem are published in [52].

In Section 2.2, we give similar results to another class of fractional differential equations, but this time with Riemann-Liouville fractional derivative and subject to nonlocal conditions of the form

$$\begin{cases} {}^{RL}D_{0+}^{\alpha}x(t) = f\left(t, x(t), {}^{RL}D_{0+}^{\alpha}x(t)\right), \ t\in(0,T], \\ t^{1-\alpha}x(t)\big|_{t=0} = x_0 - g(x), \ x_0\in\mathbb{R}, \end{cases}$$

where ${}^{RL}D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $0<\alpha<1$, $f:(0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:C((0,T],\mathbb{R})\to\mathbb{R}$ are continuous nonlinear functions. This problem has been studied in [57].

Finally, an example demonstrating the effectiveness of the theoretical results is presented at the end of each section.

In **Chapter** 3, we are interested to study some qualitative properties for certain classes of nonlinear hybrid fractional differential equations. More specifically, in Section 3.1, we discuss the existence and uniqueness of solutions and Ulam stability for the following nonlinear hybrid implicit Caputo fractional differential equations

$$\begin{cases} {}^{C}D_{0+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x(t), {}^{C}D_{0+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right), \ t\in(0,T], \\ x(0) = \theta g(0,x(0)) + f(0,x(0)), \ \theta\in\mathbb{R}, \end{cases}$$

where $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $h:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are nonlinear continuous functions and ${}^{C}D_{0+}^{\alpha}$ denotes the Caputo fractional derivative of order $0<\alpha<1$. We provide an example to illustrate our obtained results at the end of this section. The main results of this problem are published in [59].

In Section 3.2, we give some results about the existence, interval of existence, uniqueness and estimation of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations of the form

$$\begin{cases} {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x(t), {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right), \ t\in(1,T], \\ x(1) = \theta g(1,x(1)) + f(1,x(1)), \ \theta\in\mathbb{R}, \end{cases}$$

where $f:[1,T]\times\mathbb{R}\to\mathbb{R}$, $g:[1,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $h:[1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are nonlinear continuous functions and ${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order $0<\alpha<1$. This problem has been considered in the paper [10].

Chapter 4, is devoted to the existence of solutions for certain classes of fractional differential equations in Banach spaces. The desired findings are based on Mönch's fixed point theorem combined with the technique of Kuratowski measure of noncompactness. More specifically, in Section 4.1, we look into the existence of solutions for a nonlinear fractional differential equations involving Riemann-Liouville fractional derivative subject to integral boundary conditions

$$\begin{cases} {}^{RL}D_{0+}^{\alpha}x(t) - f(t,x(t)) = {}^{RL}D_{0+}^{\alpha-1}g(t,x(t)), \ t \in (0,1), \\ x(0) = 0, \ x(1) = \int_0^1 g(s,x(s))\,ds, \end{cases}$$

where ${}^{RL}D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$, $f, g : J \times X \to X$ are given functions satisfying some assumptions that will be specified later, and X be a Banach space with the norm $\|.\|$. The main results of this problem are published in [56].

As a second problem we debate in Section 4.2, the existence of solutions for the following nonlinear fractional differential equations involving Hadamard fractional derivative with two nonlinear terms

$$\begin{cases} {}^{H}\mathfrak{D}_{1+}^{\alpha}x(t) - f(t,x(t)) = {}^{H}\mathfrak{D}_{1+}^{\alpha-1}g(t,x(t)), \ t \in (1,e), \\ x(1) = 0, \ x(e) = \int_1^e g(s,x(s))\,\frac{ds}{s}, \end{cases}$$

where ${}^{H}\mathfrak{D}_{1+}^{\alpha}$ denotes the Hadamard fractional derivatives of order $1 < \alpha \leq 2$, $f, g : [1,e] \times X \to X$ are given functions satisfying some hypotheses that will be specified later.

Finally, an example is given at the end of each section to illustrate the theoretical results.

In **Chapter** 5, we are interested with the existence of solutions for certain classes of fractional differential inclusions involving convex and nonconvex multivalued maps. The results obtained are based on some fixed point theorems of multivalued analysis. More specifically, in Section 5.1. we study the existence of solutions for a nonlinear sequential Caputo and Caputo-Hadamard fractional differential inclusions with three-point boundary conditions of the form

$$\begin{cases} {}^{C}D_{0+}^{\beta}\left[{}_{H}^{C}\mathfrak{D}_{0+}^{\alpha}x(t)\right] \in \mathcal{F}(t,x(t)), \ t \in (a,b), \ a \geq 1, \\ x(a) = 0, \ x(b) = \lambda x(\eta), \ a < \eta < b, \end{cases}$$

where ${}_{H}^{C}\mathfrak{D}_{0+}^{\alpha}$ and ${}^{C}D_{0+}^{\beta}$ are the Caputo-Hadamard and Caputo fractional derivatives of orders α and β respectively, $0 < \alpha, \beta \leq 1$ and $\mathcal{F} : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map from $[a,b] \times \mathbb{R}$ to the family of $\mathcal{P}(\mathbb{R}) \subset \mathbb{R}$. This problem has been studied in [55].

In Section 5.2, we discuss the existence of solutions for the following nonlinear Hilfer fractional differential inclusion with nonlocal Erdélyi-Kober fractional integral boundary conditions

$$\begin{cases} {}^{\mathcal{H}}D_{0+}^{\alpha,\beta}x(t) \in \mathcal{F}(t,x(t)), \ t \in (0,T), \ T > 0, \\ x(0) = 0, \ x(T) = \sum\limits_{i=1}^{m} \theta_i \ {}^{EK}I_{0+,\gamma_i}^{\eta_i;\xi_i}x(\delta_i), \end{cases}$$

where ${}^{\mathcal{H}}D_{0+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha \in (1,2)$ and type $\beta \in [0,1]$, ${}^{EK}I_{0+,\gamma_i}^{\eta_i;\xi_i}$ is the Erdelyi-Kober fractional integral of order $\xi_i > 0$ with $\gamma_i > 0$ and $\eta_i \in \mathbb{R}$, $\mathcal{F} : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map from $[0,T] \times \mathbb{R}$ to the family of $\mathcal{P}(\mathbb{R}) \subset \mathbb{R}$, $\theta_i \in \mathbb{R}$ and $\delta_i \in (0,T)$, $i = 1,2,...,m$. This problem has been considered in the paper [54].

Finally, some pertinent examples demonstrating the effectiveness of the theoretical results are presented at the end of each section.

In **Chapter** 6, we study the existence and uniqueness of solutions for nonlinear ψ-Hilfer fractional integro-differential equations with nonlocal integral boundary conditions. Moreover, we discuss various kinds of stability of Ulam-Hyers for solutions to the given problem. The arguments are based on appropriate fixed point theorems together with generalized Gronwall inequality the desired results are proven.

$$\begin{cases} {}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi} x(t) = f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,x(\sigma))\, d\sigma\right), \ t \in (\mathfrak{a}, b), \\ x(\mathfrak{a}) = 0, \ I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi} x(b) = \sum\limits_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\psi} x(\delta_i), \end{cases}$$

where ${}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (1,2)$ and type $\beta \in [0,1]$, $I^{2-\mathfrak{v};\psi}$ and $I^{\eta_i;\psi}$ are the ψ-fractional integral of orders $2-\mathfrak{v}$, $\eta_i > 0$ respectively, $\mathfrak{v} = \alpha + \beta(2-\alpha) \in (1,2)$, $\infty < \mathfrak{a} < b < \infty$, $\theta_i \in \mathbb{R}$, $i = 1,2,...,m$, $0 \leq \mathfrak{a} \leq \delta_1 < \delta_2 < \delta_3 < ... < \delta_m \leq b$, $f : [\mathfrak{a}, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : [\mathfrak{a}, b] \times [\mathfrak{a}, b] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. This problem has been studied in [61].

Finally, some examples are presented to show the validity of the obtained results.

Chapter 1

Preliminaries and Background Materials

In this chapter, we introduce the necessary concepts for the good understanding of this thesis. We present some fundamental notions, definitions, and lemmas related to fractional calculus, measures of noncompactness, multivalued analysis, Ulam stability and some fixed point theorems which play an important role in the achievement of the desired results in this thesis.

1.1 Functional spaces

Let $J = [a, b]$, the compact intervals of $\mathbb{R}$. We present the following functional spaces:

Definition 1.1 Denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions $f : J \to \mathbb{R}$ endowed with the norm

$$\|f\|_{\infty} = \sup\{|f(t)| : t \in J\},$$

and $C^n(J, \mathbb{R})$ denotes the class of all real valued functions defined on J which have a continuous n th order derivative.

Definition 1.2 Denote by $L^1(J, \mathbb{R})$ the Banach space of measurable functions $f : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|f\|_{L^1} = \int_J |f(t)|\, dt,$$

and by $L^p(J, \mathbb{R})$ we denote the space of Lebesgue integrable functions on J where $|f|^P$ belongs to $L^1(J, \mathbb{R})$ endowed with the norm

$$\|f\|_{L^P} = \left(\int_J |f(t)|^P\, dt\right)^{\frac{1}{p}}.$$

Definition 1.3 A function $f : J \to \mathbb{R}$ is said absolutely continuous on J if for all $\epsilon > 0$ there exists a number δ_ϵ such that; for all finite partition $[a_i, b_i]$ in J, then $\sum_{i=1}^{p}(b_i - a_i) < \delta_\epsilon$

implies that $\sum_{i=1}^{p} |f(b_i) - f(a_i)| < \epsilon$.

Definition 1.4 Let $AC(J, \mathbb{R})$ be the space of absolutely continuous functions on J. For $n \in \mathbb{N}$, we denote by $AC^n(J, \mathbb{R})$ the space of functions $f : J \to \mathbb{R}$ which have continuous derivatives up to order $n-1$ on J such that $f^{(n-1)}$ belongs to $AC(J, \mathbb{R})$ defined by

$$AC^n(J, \mathbb{R}) = \left\{ f : J \to E : f, f', f'', \quad , f^{n-1} \in C(J, \mathbb{R}) \text{ and } f^{n-1} \in AC(J, \mathbb{R}) \right\}.$$

For more details about $AC(J, \mathbb{R})$ and $AC^n(J, \mathbb{R})$, see the book of Kolmogorov and Fomin ([48, pp.388]).

1.2 Special functions

In what follows, we recall three types of functions that are important in fractional calculus: the Gamma, Beta, and Mittag-Leffler functions. More details about these functions can be found in [33, 46, 79].

1.2.1 Gamma Function

Definition 1.5 (Gamma function [79]) The Gamma function, denoted by $\Gamma(z)$ is a generalization of the factorial function $n!$, i.e., $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, dt, \ \Re(z) > 0.$$

By analytic continuation the function is extended to the whole complex plane except for the points $0, -1, -2, -3, ...$, where it has simple poles. Thus, $\Gamma : \mathbb{C} \backslash \{0, -1, -2, ...\} \to \mathbb{C}$. Some of the most properties are

$$\begin{aligned} \Gamma(1) &= \Gamma(2) = 1, \ \Gamma(z+1) = z\Gamma(z), \\ \Gamma(n) &= (n-1)!, \ n \in \mathbb{N}, \ \Gamma\left(\frac{1}{2}\right) = \pi. \end{aligned} \tag{1.1}$$

The Gamma function is studied by many mathematicians. There is a long list of well known properties (see, for example [33]) but in this survey formulas (1.1) are sufficient.

1.2.2 Beta Function

Definition 1.6 (Beta function [79]) The Beta function is defined by the integral

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1}\, dt, \ \Re(z) > 0, \ \Re(w) > 0.$$

The functions $\Gamma(.)$ and $B(.,.)$ are related by the formula

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

To demonstrate this relationship, we use the Laplace transform, see [46].

1.2.3 Mittag-Leffler Function

Definition 1.7 (Mittag-Leffler function [79]) The Mittag-Leffler function in one parameter is defined by

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+1)},\ \alpha>0,\ z\in\mathbb{C}.$$

where it was introduced by Mittag-Leffler [72].

The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+\beta)},\ \alpha,\ \beta>0,\ z\in\mathbb{C}.$$

which is of great importance for the fractional calculus. In particular,

$$E_{1,1}(z)=\exp(z),\ E_{2,1}(z)=\cosh\left(\sqrt{z}\right),\ E_{\alpha,1}(z)=E_{\alpha}(z).$$

1.3 Elements from fractional calculus theory

The fractional calculus is a field of mathematical analysis that embraces the integrals and derivatives of functions of any real or complex order. For the past few decades, this field has been one of the handover fist sprawling fields of mathematics by the virtue of the amazing findings obtained when researchers enrolled the fractional operators in their attempts to construe some problems that arise in the nature. See [27, 30, 46, 68, 79, 87, 96].

At the beginning of the fractional calculus in 1695 [64], it was consisted of one main integral operator, namely the Riemann-Liouville fractional integral and two fractional derivatives, namely the Riemann-Liouville and Caputo derivatives. Because of penurious number of operators, researchers were compelled to discover and develop new fractional operators that allow them better comprehend the world around them. In this purpose, new derivatives and fractional integrals has been arising. The kernel of these integrals and fractional derivatives differs, resulting in a large number of definitions, see [36, 41, 43, 46, 78, 79, 87].

Due to the large number of integral and fractional derivative definitions, it was necessary to create a fractional derivative of a function f with respect to another function, which is called ψ-Riemann-Liouville, using the fractional derivative in the Riemann-Liouville sense, which given by [46]

$${}^{RL}D_{a+}^{\alpha;\psi}f(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\ I_{a+}^{(n-\alpha);\psi}f(t).$$

where $\alpha \in (n-1, n)$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. However, such a definition only encompasses the possible fractional derivatives that contain the differentiation operator acting on the integral operator. On the other hand, In subsection 1.3.2, We will mention a corresponding fractional integral which generalized the Riemann-Liouville fractional integrals and some special cases of this operator.

In the same way, recently, Almeida [9] using the idea of the fractional derivative in the Caputo sense, proposes a new fractional derivative called ψ-Caputo derivative with respect to another function ψ, which generalizes a class of fractional derivatives, whose definition is given by

$$^{C}D_{a+}^{\alpha;\psi} f(t) = I_{a+}^{(n-\alpha);\psi} f^{[n]}(t),$$

where $\alpha \in (n-1, n)$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

Despite that the ψ-Riemann-Liouville and ψ-Caputo definitions of fractional derivatives are very broad, there is the possibility of proposing a fractional differentiable operator that combines these operators and overcomes the wide range of definitions. Motivated by the Hilfer [39] fractional derivative definition, which includes the classical Riemann-Liouville and Caputo fractional derivatives as special cases. Depending on ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfer's sense of definition, Sousa and Oliviera [89] introduced a new fractional derivative of a function with respect to another ψ function so-called ψ-Hilfer derivative. Which unify a wide class of fractional derivatives. the definition of ψ-Hilfer fractional derivative, its relation with the ψ-Riemann-Liouville fractional integral and some special cases of this derivative are will presented in subsection 1.3.4.

The advantage of the fractional operator ψ-Hilfer proposed here is the freedom of choice of the classical differentiation operator and the choice of the function ψ, i.e., from the choice of the function ψ, the operator of classical differentiation, can act on the fractional integration operator or else the fractional integration operator can act on the classical differentiation operator. As a result, the properties of the two fractional operators mentioned above can be unified and obtained.

There are several definitions in fractional calculus that are widely used and important in showing different fractional calculus outcomes. In this section, We will present some definitions of classical fractional integrals and fractional derivatives and its properties. Next, we introduce a new class of fractional integrals and fractional derivatives, because there are so many different fractional operator definitions, the following definition is a special approach when the kernel is unknown, involving a function ψ, making this new operator a generalization of the fractional operators that we use throughout this thesis.

1.3.1 Fractional integrals and fractional derivatives

Let $J = [a, b]$, $(-\infty < a < b < \infty)$, be a finite interval on $\mathbb{R}$. In this subsection, we present some definitions of classical fractional integrals, fractional derivatives and its properties.

Definition 1.8 (Cauchy formula [46]) The Cauchy formula of nth integral of a locally integrable function f on $\mathbb{R}^+$ is given by

$$I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)\, ds.$$

Definition 1.9 (Riemann-Liouville fractional integral [46]) For $\alpha > 0$. The left-side (right-side resp.) of Riemann-Liouville fractional integral of the function $f \in L^1(J,\mathbb{R})$ of order α is defined by

$$\begin{aligned} {}^{RL}I^{\alpha}_{a+} f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\, ds,\ t\in J, \\ {}^{RL}I^{\alpha}_{b-} f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)\, ds,\ t\in J, \end{aligned}$$

resp., where $t \in J$.

Riemann-Liouville fractional derivative are defined depending on their fractional integral and integer order derivative as follows.

Definition 1.10 (Riemann-Liouville fractional derivative [46]) For $\alpha > 0$. The left-side (right-side resp.) of Riemann-Liouville fractional order derivative of order α of $f \in L^1(J,\mathbb{R})$, is given by

$$ {}^{RL}D^{\alpha}_{a+} f(t) = \left(\frac{d}{dt}\right)^n \left({}^{RL}I^{n-\alpha}_{a+} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds,$$

$$ {}^{RL}D^{\alpha}_{b-} f(t) = \left(-\frac{d}{dt}\right)^n \left({}^{RL}I^{n-\alpha}_{b-} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} f(s)\, ds,$$

resp., where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Definition 1.11 (Caputo fractional derivative [46]) For $\alpha > 0$. The left-side (right-side resp.) of Caputo fractional order derivative of order α of $f \in AC^n(J,\mathbb{R})$, is defined by

$$ {}^{C}D^{\alpha}_{a+} f(t) = {}^{RL}I^{n-\alpha}_{a+}\left(f^{(n)}(t)\right) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,\ t\in J,$$

$$ {}^{C}D^{\alpha}_{b-} f(t) = {}^{RL}I^{n-\alpha}_{b-}\left(f^{(n)}(t)\right) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\, ds,\ t\in J,$$

resp., where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

In what follows, we consider some properties of the Riemann-Liouville and Caputo fractional integral and derivatives. In particular, we are interested by the left-side fractional derivatives and integrals.

Lemma 1.1 (Relation between Riemann-Liouville and Caputo derivatives [46]) *Let $\alpha \in (n-1, n]$. If the function $f \in C^n(J)$, then*

$$^{C}D_{a+}^{\alpha} f(t) = \, ^{RL}D_{a+}^{\alpha} f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}.$$

Lemma 1.2 ([46]) *For α, $\beta > 0$ and $f \in L^1(J)$. Then, we have*

1) *The integral operator $^{RL}I_{a+}^{\alpha}$ is linear,*
2) $^{RL}I_{a+}^{\alpha} \, ^{RL}I_{a+}^{\beta} f(t) = ^{RL}I_{a+}^{\beta} \, ^{RL}I_{a+}^{\alpha} f(t) = \, ^{RL}I_{a+}^{\alpha+\beta} f(t)$,
3) $^{RL}D_{a+}^{\alpha} \, ^{RL}I_{a+}^{\alpha} f(t) = f(t)$,
4) $^{RL}D_{a+}^{\beta} \, ^{RL}I_{a+}^{\alpha} f(t) = \, ^{RL}I_{a+}^{\alpha-\beta} f(t)$.

Lemma 1.3 ([46]) *For $\alpha \geq 0$ and $\beta > 0$, we have*

$$\left(^{RL}I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}, \ \alpha > 0,$$

$$\left(^{RL}D_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \ \alpha \geq 0.$$

Lemma 1.4 ([46]) *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. and $f \in C(J)$, then, the Riemann-Liouville fractional differential equation*

$$^{RL}D_{a+}^{\alpha} f(t) = 0,$$

has a general solution

$$f(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3} + ... + c_n(t-a)^{\alpha-n}, \ c_i \in \mathbb{R}, \ i = 1, 2, ..., n,$$

From the above lemma, it follows that

$$^{RL}I_{a+}^{\alpha} \, ^{RL}D_{a+}^{\alpha} f(t) = f(t) - c_1(t-a)^{\alpha-1} - c_2(t-a)^{\alpha-2} - c_3(t-a)^{\alpha-3} - ... - c_n(t-a)^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n$.

Lemma 1.5 ([46]) *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. If $f \in AC^n(J)$, then the Caputo fractional differential equation*

$$^{C}D_{a+}^{\alpha} f(t) = 0,$$

has a general solution

$$f(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{n-1}(t-a)^{n-1},$$

From the above lemma, it follows that

$$^{RL}I_{0+}^{\alpha} \, ^{C}D_{0+}^{\alpha} f(t) = f(t) - c_0 - c_1(t-a) - c_2(t-a)^2 - ... - c_{n-1}(t-a)^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$.

Definition 1.12 (Hadamard fractional integral [46]) Let $a > 0$. The Hadamard fractional integral of order $\alpha > 0$ for a function $f \in L^1(J, \mathbb{R})$ is defined as

$$^{H}\mathfrak{I}_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{a-1} f(s) \frac{ds}{s}, \ t \in J.$$

Set $\delta = \left(t \frac{d}{dt}\right)$, $a, \alpha > 0$, $n = [\alpha] + 1$, where α denotes the integer part of α. Define the space

$$AC_{\delta}^{n}(J, \mathbb{R}) = \left\{ f : J \to \mathbb{R} : \delta^{n-1} f(t) \in AC(J, \mathbb{R}) \right\}.$$

Definition 1.13 (Hadamard fractional derivative [46]) Let $a > 0$. The Hadamard fractional derivative of order $\alpha > 0$ for a function $f \in AC_{\delta}^{n}(J, \mathbb{R})$ is defined as

$$^{H}\mathfrak{D}_{a+}^{\alpha} f(t) = \delta^{n} \left(^{H}\mathfrak{I}_{a+}^{n-\alpha} f \right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^{n} \int_a^t \left(\log \frac{t}{s} \right)^{n-a-1} f(s) \frac{ds}{s}.$$

Definition 1.14 (Caputo-Hadamard fractional derivative [41, 46]) Let $a > 0$. The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a function $f \in AC_{\delta}^{n}(J, \mathbb{R})$ is defined as

$$^{C}_{H}\mathfrak{D}_{a+}^{\alpha} f(t) = \left(^{H}\mathfrak{I}_{a+}^{n-\alpha} \delta^{n} f \right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{n-a-1} \delta^{n} f(s) \frac{ds}{s}.$$

Lemma 1.6 ([41, 46]) *Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$. Then, we have*

1) The integral operator $^{H}\mathfrak{I}_{a+}^{\alpha}$ is linear,
2) $^{H}\mathfrak{I}_{a+}^{\alpha} (\log t)^{\beta-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \left(\log \frac{x}{a} \right)^{\beta+\alpha-1}$,
3) $^{C}_{H}\mathfrak{D}_{a+}^{\alpha} (\log t)^{\beta-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \left(\log \frac{x}{a} \right)^{\beta-\alpha-1}$, $\beta > n$,
4) $^{C}_{H}\mathfrak{D}_{a+}^{\alpha} (\log t)^{k} = 0$, $k = 0, 1, ..., n-1$.

Lemma 1.7 ([46]) *let $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, the general solution of the fractional differential equation*

$$^{H}\mathfrak{D}_{a+}^{\alpha} f(t) = 0,$$

is given by

$$f(t) = \sum_{k=1}^{n} c_k \left(\log \frac{t}{a} \right)^{\alpha-k},$$

where $c_k \in \mathbb{R}$, $k = 1, 2, ..., n$ are arbitrary constants.

From the above lemma, it follows that

$$^{H}\mathfrak{I}_{a+}^{\alpha} \ ^{H}\mathfrak{D}_{a+}^{\alpha} f(t) = f(t) - \sum_{k=1}^{n} c_k \left(\log \frac{t}{a} \right)^{\alpha-k},$$

for some $c_k \in \mathbb{R}$, $k = 1, 2, ..., n$ are arbitrary constants.

Lemma 1.8 ([41, 46]) *Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$. If $f\in AC_{\delta}^{n}\left(J,\mathbb{R}\right)$, then the Caputo-Hadamard fractional differential equation*

$$ {}_{H}^{C}\mathfrak{D}_{a+}^{\alpha}f\left(t\right)=0, $$

has a solution

$$ f\left(t\right)=\sum_{k=0}^{n-1}c_{k}\left(\log\frac{t}{a}\right)^{k}, $$

and the following formula holds

$$ {}^{H}\mathfrak{J}_{a+}^{\alpha}\left({}_{H}^{C}\mathfrak{D}_{a+}^{\alpha}f\left(t\right)\right)=f\left(t\right)-\sum_{k=0}^{n-1}c_{k}\left(\log\frac{t}{a}\right)^{k}, $$

where $c_{k}\in\mathbb{R}$, $k=0,\,1,\,2,...,\,n-1$.

1.3.2 Fractional ψ-integral

Definition 1.15 ([89]) Let (a,b), $(-\infty\leq a<b\leq\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $\psi\left(\mathrm{t}\right)$ be an increasing and positive monotone function on (a,b), having a continuous derivative $\psi'\left(\mathrm{t}\right)$ on (a,b). The left sided fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by

$$ I_{a+}^{\alpha;\psi}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\psi'\left(s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right)^{\alpha-1}f\left(s\right)ds. \tag{1.2} $$

Lemma 1.9 ([46, 89]) *Let $\alpha,\beta,\delta>0$. Then, the left-sided ψ-fractional integral satisfies the following properties*

1) The integral operator $I_{a+}^{\alpha;\psi}$ is linear,

2) The semigroup property of the fractional integration operator $I_{a+}^{\alpha;\psi}$ is given by the following result

$$ I_{a+}^{\alpha;\psi}I_{a+}^{\beta;\psi}f\left(t\right)=I_{a+}^{\alpha+\beta;\psi}f\left(t\right), $$

holds almost everywhere if $f\in L^{1}\left(J,\mathbb{R}\right)$.

3) Commutativity

$$ I_{a+}^{\alpha;\psi}\left(I_{a+}^{\beta;\psi}f\left(t\right)\right)=I_{a+}^{\beta;\psi}\left(I_{a+}^{\alpha;\psi}f\left(t\right)\right). $$

Lemma 1.10 ([89]) *Let $\alpha,\beta>0$. Then*

$$ I_{a+}^{\alpha;\psi}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\beta-1}\left(t\right)=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\alpha+\beta-1}. $$

The fractional integral operator with respect to another function defined in (1.2) is a general operator, in the sense that it is enough to choose a function ψ and obtain an existing fractional integral operator. In the following, we present a class of fractional integrals, based on the choice of the arbitrary ψ function.

1) Choosing $\psi(\mathrm{t}) = t$ and replacing in equation (1.2), we get

$$I_{a+}^{\alpha;t} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds = {}^{RL}I_{a+}^{\alpha} f(t),$$

the Riemann-Liouville fractional integral.

2) If we consider $\psi(\mathrm{t}) = \log(t)$ and $a > 0$ in equation (1.2), we have

$$\begin{aligned} I_{a+}^{\alpha;t} h(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{1}{s} (\log t - \log s)^{\alpha-1} f(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{a-1} f(s) \frac{ds}{s} = {}^{H}\mathfrak{J}_{a+}^{\alpha} f(t), \end{aligned}$$

the Hadamard fractional integral.

3) Choosing $\psi(\mathrm{t}) = t^{\delta}$ and $g(t) = t^{\alpha\eta} f(t)$ and substituting in equation (1.2), we get

$$\begin{aligned} t^{-\delta(\alpha+\eta)} I_{a+}^{\alpha;t^{\delta}} g(t) &= t^{-\delta(\alpha+\eta)} I_{a+}^{\alpha;t^{\delta}} t^{\alpha\eta} f(t) \\ &= \frac{\delta t^{-\delta(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^t s^{\alpha\eta+\delta-1} \left(t^{\delta} - s^{\delta}\right)^{\alpha-1} f(s)\, ds \\ &= {}^{Ek}I_{a+,\delta}^{\eta,\alpha} f(t), \end{aligned}$$

the Erdélyi-Kober fractional integral.

1.3.3 Fractional ψ-derivative

We start by evoking two definitions of fractional derivatives with respect to another function, both of which are motivated by the fractional derivatives of Riemann-Liouville and Caputo, in that order, choosing a specific function ψ.

Definition 1.16 ([46]) Let $\psi'(\mathrm{t}) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivative of a function f with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by

$$\begin{aligned} {}^{RL}D_{a+}^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);\psi} f(t), \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} f(s)\, ds, \end{aligned}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Definition 1.17 ([9]) Let $\alpha > 0$, $n \in \mathbb{N}$, $J = [a, b]$ is the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n(J, \mathbb{R})$ two functions such that ψ is increasing and $\psi'(\mathrm{t}) \neq 0$, for any $t \in J$. The left sided ψ-Caputo fractional derivative of a function f of order α is given by

$${}^{C}D_{a+}^{\alpha;\psi} f(t) = I_{a+}^{(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t).$$

Lemma 1.11 ([42, 87]) *Let $\alpha, \beta > 0$. Then*

$$^{C}D_{a+}^{\alpha;\psi}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\beta-1}\left(t\right)=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha\right)}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\alpha+\beta-1}.$$

Lemma 1.12 ([89]) *If $f \in C^n\left(J, \mathbb{R}\right)$ and $\alpha \in (n-1, n)$, then*

$$I_{a+}^{\alpha;\psi}\,{}^{C}D_{a+}^{\alpha;\psi}\,f\left(\mathrm{t}\right)=f\left(t\right)-\sum_{k=0}^{n-1}\frac{f^{[n]}\left(a^{+}\right)}{k!}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{k}.$$

In particular, given $\alpha \in (0, 1)$, we have

$$I_{a+}^{\alpha;\psi}\,{}^{C}D_{a+}^{\alpha;\psi}\,f\left(\mathrm{t}\right)=f\left(t\right)-f\left(a\right).$$

1.3.4 Fractional ψ-Hilfer derivative

From the definition of fractional derivative in the Riemann-Liouville sense and the Caputo sense [46], was introduced the Hilfer fractional derivative [39], which combines both derivatives. Motivated by the definition of Hilfer, we present a new generalized operator the so-called ψ-Hilfer fractional derivative of a function f with respect to another function. From the fractional derivative ψ-Hilfer, we introduce some relations between the ψ-fractional integral and the fractional derivative ψ-Hilfer.

Definition 1.18 ([89]) Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, $J = [a, b]$ is the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n\left(J, \mathbb{R}\right)$ two functions such that ψ is increasing and $\psi'\left(\mathrm{t}\right) \neq 0$, for any $t \in J$. The left sided ψ-Hilfer fractional derivative $^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}\left(.\right)$ of function f of order α and type $\beta \in [0, 1]$ is defined by

$$^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}f\left(t\right)=I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'\left(t\right)}\frac{d}{dt}\right)^{n}I_{a+}^{(1-\beta)(n-\alpha);\psi}f\left(t\right),\ n\in\mathbb{N}. \tag{1.3}$$

Lemma 1.13 ([89]) *For $\delta > 0$, $\alpha \in (n-1, n)$ and $\beta \in [0, 1]$, we have*

$$^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\delta-1}=\frac{\Gamma\left(\delta\right)}{\Gamma\left(\delta-\alpha\right)}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{\delta-\alpha-1},\ \delta>n.$$

Lemma 1.14 ([89]) *In particular, given $n \leq k \in \mathbb{N}$ and as $\delta > n$ we have*

$$^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{k}=\frac{k!}{\Gamma\left(k+1-\alpha\right)}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{k-\alpha}.$$

On the other hand, for $n > k \in \mathbb{N}_0$, we have

$$^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}\left(\psi\left(t\right)-\psi\left(a\right)\right)^{k}=0.$$

Lemma 1.15 ([89]) *Let $n-1 < \alpha < n$, $\beta \in [0, 1]$ and $\gamma = \alpha + \beta\left(n-\alpha\right)$. If $f \in C^n\left(J, \mathbb{R}\right)$, then*

1) $I_{a+}^{\alpha;\psi}\ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}f\left(t\right)=f\left(t\right)-\sum\limits_{k=1}^{n}\frac{\left(\psi(t)-\psi(a)\right)^{\gamma-k}}{\Gamma(\gamma-k+1)}f_{\psi}^{[n-k]}I_{a+}^{(1-\beta)(n-\alpha);\psi}f\left(a\right)$ *where* $f_{\psi}^{[n-k]}f\left(t\right)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-k}f\left(t\right)$,

2) $^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}\ I_{a+}^{\alpha;\psi}f\left(t\right)=f\left(t\right)$.

In the following, using the ψ-Hilfer fractional derivative operator defined in equation (1.3), we can combine in this derivative a different types of fractional derivatives by changing the value for ψ and taking the limit of the parameter β. Some of them are presented below.

1) Consider the $\psi(t) = t$ and taking the limit $\beta \to 1$ on both sides of equation (1.3), we get

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,1;t} f(t) &= I_{a+}^{(n-\alpha);t} \left(\frac{d}{dt}\right)^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(s)\, ds \\ &= {}^{C}D_{a+}^{\alpha} f(t), \end{aligned}$$

the Caputo fractional derivative.

2) For $\psi(t) = t$ and taking the limit $\beta \to 0$ on both sides of equation (1.3), we have

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,0;t} f(t) &= \left(\frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);t} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds \\ &= {}^{RL}\, D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \end{aligned}$$

the Riemann-Liouville fractional derivative.

3) For $\psi(t) = \log t$, $a > 0$ and taking the limit $\beta \to 0$ on both sides of equation (1.3), we have

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,0;\log t} f(t) &= \left(t\frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);t} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-a-1} f(s) \frac{ds}{s} \\ &= {}^{H}\mathfrak{D}_{+a}^{\alpha} f(t), \end{aligned}$$

the Hadamard fractional derivative.

4) For $\psi(t) = \log t$, $a > 0$ and taking the limit $\beta \to 1$ on both sides of equation (1.3), we have

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,1;t} f(t) &= I_{a+}^{(n-\alpha);t} \left(t\frac{d}{dt}\right)^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{n-a-1} \left(s\frac{d}{ds}\right)^n f(s) \frac{ds}{s} \\ &= {}^{C}_{H}\mathfrak{D}_{a+}^{\alpha} f(t), \end{aligned}$$

the Caputo-Hadamard fractional derivative.

1.3. Elements from fractional calculus theory

5) For $\psi(t) = t$ and replacing in equation (1.3), we have

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;t} f(t) &= I_{a+}^{\beta(n-\alpha);t} \left(\frac{d}{dt}\right)^n I_{0+}^{(1-\beta)(n-\alpha);t} f(t) \\ &= {}^{RL}I_{a+}^{\beta(n-\alpha)} D^{[n]} \, {}^{RL}I_{a+}^{(1-\beta)(n-\alpha)} f(t) \\ &= {}^{\mathcal{H}}D_{a+}^{\alpha,\beta} f(t), \end{aligned}$$

the Hilfer fractional derivative.

1.4 Functional tools

In what follows, we present some concepts of functional analysis that will we use throughout this thesis.

Theorem 1.1 (Ascoli-Arzela Theorem [23]) *Let $A \subset C([0,T], \mathbb{R})$. A is relatively compact (i.e $\overline{A}$ is compact) if*

1) A is uniformly bounded i.e, there exists $M > 0$ such that

$$|f(t)| \leq M \textit{ for every } f \in A \textit{ and } t \in [0,T],$$

2) A is equicontinuous i.e, for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in [0,T]$, $|t_1 - t_2| \leq \delta$ implies $|f(t_1) - f(t_2)| \leq \epsilon$ for every $f \in A$.

Definition 1.19 ([28]) A map $f : [0,T] \times X \to X$ is said to be Carathéodory if

1) $t \to f(t,x)$ is measurable for each $x \in X$,

2) $x \to f(t,x)$ is continuous for almost all $t \in [0,T]$.

Moreover, f is called L^1-Carathéodory if $\forall \rho > 0$, there exists $\varphi_\rho \in L^1([0,T], \mathbb{R}^+)$ such that

$$|f(t,x)| \leq \varphi_\rho(t), \text{ for all } |x| \leq \rho \text{ and for } a.e. \ t \in [0,T].$$

Lemma 1.16 (Standard Gronwall inequality 01 [38]) *Let $f : [0,T] \to \mathbb{R}^+$ be real function and w is a nonnegative locally integrable function on $[0,T]$.*

Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$

$$f(t) \leq w(t) + a \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Then, there exist a constant $k = k(\alpha)$ such that

$$f(t) \leq w(t) + ka \int_0^t (t-s)^{\alpha-1} w(s) ds.$$

Lemma 1.17 (Standard Gronwall inequality 02 [67]) *Let $f : [1, T] \to [0, \infty)$ be a real function and w is a nonnegative locally integrable function on $[1, T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$*

$$f(t) \le w(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}.$$

Then, there exists a constant $k = k(\alpha)$ such that

$$f(t) \le w(t) + ka \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} w(s) \frac{ds}{s}.$$

for every $t \in [1, T]$.

Lemma 1.18 (Generalization of Gronwall inequality [102]) *Let f, g be two integrable functions and h be continuous with domain $[a, b]$. Let $\Psi \in C^1([a, b], \mathbb{R})$ be an increasing function such that $\Psi'(t) \neq 0$, $\forall t \in [a, b]$. Assume that*

1) f and g are nonnegative functions,

2) h is nonnegative and nondecreasing.

If

$$f(t) \le g(t) + h(t) \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} f(s)\, ds,$$

then

$$f(t) \le g(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(k\alpha)} \Psi'(s) (\Psi(t) - \Psi(s))^{k\alpha-1} g(s)\, ds.$$

Lemma 1.19 ([102]) *Under the hypotheses of Lemma (1.18), assume further that $g(t)$ is nondecreasing function for $t \in [a, b]$. Then*

$$f(t) \le g(t) E_\alpha \left(h(t) \Gamma(\alpha) (\Psi(t) - \Psi(s))^\alpha\right).$$

1.5 Background about measures of non-compactness

1.5.1 The general notion of a measure of noncompactness

Firstly, we need to fix the notation. In what follows, (E, d) will be a metric space, and $(X, \|.\|)$ a Banach space. Let Q is non-empty subset of X, then $\overline{Q}$ and $convQ$ denote the closure and the closed convex closure of Q, respectively. When Q is a bounded subset, Diam (Q) denotes the diameter of Q. Also, we denote by $\mathfrak{B}_E$ (resp.$\mathfrak{B}_X$) the class of nonempty and bounded subsets of E (resp. of X).

We begin with the following general definition.

Definition 1.20 ([15, 19]) A mapping $\mu : \mathfrak{B}_E \to \mathbb{R}^+$ will be called a measure of noncompactness in E if it satisfies the following conditions

1) Regularity : $\mu(Q) = 0$ if, and only if, Q is a precompact set.
2) Invariant under closure: $\mu(Q) = \mu(\overline{Q})$, for all $Q \in \mathfrak{B}_E$.
3) Semi-additivity : $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_1)\}$, for all Q_1 , $Q_2 \in \mathfrak{B}_E$.
To have a MNC in a Banach space X we need to add the two following additional properties
4) Semi homogeneity : $\mu(\lambda Q) = |\lambda|\mu(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathfrak{B}_X$.
5) Invariant under translations : $\mu(x + Q) = \mu(Q)$, for all $x \in X$ and $Q \in \mathfrak{B}_X$.

Three main and most frequently used MNCs: the Kuratowski MNC, the Hausdorff MNC, and the De Blasi Measure of Weak Noncompactness. In this thesis, we are interested by Kuratowski MNC.

1.5.2 The Kuratowski measure of noncompactness

Now we present some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 1.21 ([49, 50]) Let (E, d) be a metric vector space and Q be a bounded subset of E. Then the Kuratowski measure of noncompactness (the set-measure of noncompactness, k-measure) of Q, denoted by $\mu_k(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, i.e.,

$$\mu_k(Q) = \inf\{\epsilon > 0 : Q \subseteq \cup_{i=1}^{n} S_i,\ S_i \subset E,\ diam(S_i) < \epsilon,\ i = 1, 2, ..., n,\ n \in \mathbb{N}\}.$$

This measure of noncompactness satisfies the following properties
1) Regularity : $\mu_k(Q) = 0$ if and only if, Q is a precompact set.
2) Invariant under passage to the closure : $\mu_k(Q) = \mu_k(\overline{Q})$, for all $Q \in \mathfrak{B}_E$.
3) Semi-additivity : $\mu_k(Q_1 \cup Q_2) = \max\{\mu_k(Q_1), \mu_k(Q_1)\}$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.
4) Monotonicity : $Q_1 \subset Q_2 \Rightarrow \mu_k(Q_1) \leq \mu_k(Q_2)$.
5) Algebraic semi-additivity : $\mu_k(Q_1 + Q_2) \leq \mu_k(Q_1) + \mu_k(Q_2)$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.
6) Semi-homogeneity : $\mu_k(\lambda Q_1) = |\lambda| \mu_k(Q_1)$, for $\lambda \in \mathbb{R}$ and $Q_1 \in \mathfrak{B}_E$.
7) Invariant under passage to the convex hull : $\mu_k(convQ) = \mu_k(Q)$.
8) $\mu_k(Q_1 \cap Q_2) \leq \min\{\mu_k(Q_1), \mu_k(Q_1)\}$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.

The following lemma is important in order to attain the desired outcomes in this work.

Lemma 1.20 ([92]) *Let $J = [0, T]$ and D be a bounded, closed and convex subset of the Banach space $C(J, X)$. Let G be a continuous function on $J \times J$ and f a function from $J \times X \to X$, which satisfies the Carathéodory conditions, and assume there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset X$, we have*

$$\lim_{h \to 0^+} \mu_k(f(J_{t,h} \times B)) \leq p(t)\mu_k(B),\ \text{here } J_{t,h} = [t - h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu_k\left(\left\{\int_J G(s,t) f(s, y(s))\, ds : y \in V\right\}\right) \leq \int_J \|G(s,t)\| p(s)\mu_k(V(s))\, ds.$$

1.6 Multivalued Analysis

In this section, we introduce some definitions, notations, and preliminary facts for multivalued analysis, which are used throughout this thesis.

Definition 1.22 ([28]) Let $(X, \|.\|)$ and $(Y, \|.\|)$ two Banach spaces. A multivalued function $\mathcal{F}$ (or a set valued map, multivalued map) from X into $\mathcal{P}(Y)$ is a correspondence that associates to each element $x \in X$ a subset $\mathcal{F}(x)$ of Y.

We denote this correspondence by the symbol $\mathcal{F} : X \to \mathcal{P}(Y)$. We define

1) The effective domain $Dom\mathcal{F} = \{x \in X,\ \mathcal{F}(X) \neq \emptyset\}$.

2) The graph $Gr(\mathcal{F}) = \{(x, y) \in X \times Y,\ x \in Dom\mathcal{F},\ y \in \mathcal{F}(x)\}$.

3) The image of the set $A \in \mathcal{P}(X) : \mathcal{F}(A) = x \in \cup_{x \in A} \mathcal{F}(x)$.

4) The inverse image of the set $B \in \mathcal{P}(Y) : \mathcal{F}^{-1}(B) = \{x \in X : \mathcal{F}(x) \cap B = \emptyset\}$.

5) The multivalued map $\mathcal{F} : X \to \mathcal{P}(Y)$ is convex (closed, compact) valued if $\mathcal{F}(x)$ is convex (closed, compact) for all $x \in X$.

6) $\mathcal{F}$ is bounded on bounded sets if $\mathcal{F}(B) = \cup_{x \in B} \mathcal{F}(x)$ is bounded in Y for all bounded set B of X, i.e.,

$$\sup_{x \in B} \{\sup \{\|y\| : y \in \mathcal{F}(x)\}\} < \infty.$$

7) $\mathcal{F}$ is called upper semi-continuous (u.s.c for short) on X if $\mathcal{F}^{-1}(A)$ is closed in X whenever $A \subset X$ is closed.

8) $\mathcal{F}$ is said to be completely continuous if $\mathcal{F}(B)$ is relatively compact for every bounded subset B of X.

9) A multivalued map $\mathcal{F} : X \to \mathcal{P}_0(Y)$ (where $\mathcal{P}_0(Y) = \{A \in \mathcal{P}_0(Y), A \neq \emptyset\}$) is said to be measurable if for every open $U \subset X$ the set $\mathcal{F}^{-1}(U)$ is a measurable set.

10) $\mathcal{F}$ has a fixed point if there exists $x \in X$ such that $x \in \mathcal{F}(x)$. The fixed point set of the multivalued operator $\mathcal{F}$ will be denoted by $Fix\ \mathcal{F}$.

For each $y \in C([a, b], \mathbb{R})$, the set of selections of $\mathcal{F}$ at point y is defined by

$$S_{\mathcal{F},y} = \left\{v \in L^1([a, b], \mathbb{R}) : v(t) \in \mathcal{F}(t, y) \text{ for a.e. } t \in [a, b]\right\}.$$

In the following, by $\mathcal{P}_p$ we denote the set of all nonempty subsets of X which have the property "p" where "p" will be bounded (b), closed (cl), convex (c), compact (cp) etc. Thus $\mathcal{P}_b(X) = \{A \in \mathcal{P}(X) : A \text{ is bounded}\}$, $\mathcal{P}_{cl}(X) = \{A \in \mathcal{P}(X) : A \text{ is closed}\}$, $\mathcal{P}_{cp}(X) = \{A \in \mathcal{P}(X) : A \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{A \in \mathcal{P}(X) : A \text{ is compact and convex}\}$.

Definition 1.23 ([28]) A multivalued map $\mathcal{F} : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

1) $t \to \mathcal{F}(t, x)$ is measurable for each $x \in \mathbb{R}$,

2) $x \to \mathcal{F}(t, x)$ is uppe semi-continuous for almost all $t \in [a, b]$.

Further, a Carathéodory function $\mathcal{F}$ is called L^1-Carathéodory if

3) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([a,b], \mathbb{R}^+)$ such that

$$\|\mathcal{F}(t,x)\| = \sup\{|v| : v \in \mathcal{F}(t,x)\} \leq \varphi_\rho(t),$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [a,b]$.

Lemma 1.21 ([47]) *Let (E,d) be a metric space induced from the normed space $(X, \|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ given by*

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space.

Definition 1.24 ([28]) A multivalued operator $\mathcal{N} : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y) \text{ for each } x, y \in X.$$

(b) A contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 1.22 ([28], Proposition 1.2) *If $\mathcal{F} : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(\mathcal{F})$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in \mathcal{F}(x_n)$. then $y_* \in \mathcal{F}(x_*)$. Conversely, if $\mathcal{F}$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 1.23 ([63]) *Let X be a separable Banach space. Let $\mathcal{F} : [a,b] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([a,b], X)$ to $C([a,b], X)$. Then the operator*

$$\Theta \circ S_{\mathcal{F}} : C([a,b], X) \to \mathcal{P}_{cp,c}(C([a,b], X)), \ x \to (\Theta \circ S_{\mathcal{F}})(x) = \Theta(S_{\mathcal{F},x}),$$

is a closed graph operator in $C([a,b], X) \times C([a,b], X)$.

For more details on multivalued maps and the proof of the known results cited in this section, we refer the interested reader to the books by Deimling [28], Gorniewicz [35] and Hu and Papageorgiou [40].

1.7 Fixed point theorems

The theory of fixed points is one of the most powerful tools of modern mathematics, which used to show the existence and uniqueness of fixed points of various kinds of equations. Throughout this study, we convert the given problem into an equivalent integral equation and then use the appropriate fixed point theorem such that the fixed points obtained are thus the solutions of the given problem. In this section we collect the fixed point theorems which are used in the proofs of our main results. We start with the definition of a fixed point.

Definition 1.25 For a mapping Φ of a set E Into itself, an element x of E is a fixed point of Φ, if $\Phi(x) = x$.

Theorem 1.2 (Banach's fixed point theorem [88]) *Let Ω be a non-empty closed subset of a Banach space $(X, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 1.3 (Schauder's fixed point theorem [88]) *Let Ω be a nonempty closed bounded convex subset of a Banach space X and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then Φ has a fixed point in Ω.*

Theorem 1.4 (Schaefer's fixed point theorem [88]) *Let X be a Banach space, and $\Phi : X \to X$ is completely continuous operator. If the set $B_\lambda = \{x \in X : x = \lambda \Phi x,\ \lambda \in (0,1)\}$ is bounded. Then Φ has fixed point in X.*

Theorem 1.5 (Krasnoselskii's fixed point theorem [88]) *Let Ω be a non-empty closed bounded convex subset of a Banach space $(X, \|.\|)$. Suppose that F_1 and F_2 map Ω into X such that*

1) $F_1x + F_2y \in \Omega$ for all $x, y \in \Omega$,
2) F_1 is continuous and compact,
3) F_2 is a contraction with constant $l < 1$.
Then there is a $x \in \Omega$ with $F_1x + F_2x = x$.

Theorem 1.6 (Mönch's fixed point theorem [5]) *Let Ω be a bounded, closed and convex subset of the Banach space such that $0 \in \Omega$, and let Φ be a continuous mapping of Ω into itself. If the implication*

$$V = \overline{conv}\,\Phi(V) \ \ or\ V = \Phi(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every V of Ω, then Φ has a fixed point.

Theorem 1.7 (Nonlinear alternative of Kakutani maps [34]) *Let Ω be a closed convex subset of a Banach space X and $\mathcal{U}$ be an open subset of Ω with $0 \in \mathcal{U}$. Suppose that $N : \overline{\mathcal{U}} \to \mathcal{P}_{cp,c}(\Omega)$ is an upper semi-continuous compact map. Then either*

(i) N has a fixed point in $\overline{\mathcal{U}}$, or
(ii) there is a $x \in \partial\mathcal{U}$ and $\mu \in (0,1)$ with $x \in \mu N(x)$.

Theorem 1.8 (Covitz and Nadler fixed point theorem [26]) *Let (E, d) be complete metric space. If $N : E \to \mathcal{P}_{cl}(E)$ is a contraction, then $Fix\ N \neq \emptyset$.*

1.8 Ulam's stability

The stability of the Ulam can be viewed as a special kind of data dependence which was initiated by the Ulam in [97]. Rassias in [80] extended the concept of Ulam-Hyers stability.

In latest years, many authors discussed Ulam-Hyers stability problem for various types of fractional integral and fractional differential equation using different techniques, for more details see [2, 12, 21, 70, 98] and the references therein.

To define Ulam's stability, we consider the following fractional differential equation

$$^{H}D_{0+}^{\alpha,\beta;\psi}x\left(t\right)=f\left(t,x\left(t\right)\right),\ t\in\left[0,T\right]. \tag{1.4}$$

Definition 1.26 ([83]) The equation (1.4) is said to be Ulam-Hyers stable if there exists a real number $k>0$ such that for each $\epsilon>0$ and for each $y\in C\left(\left[0,T\right],\mathbb{R}\right)$ solution of the inequality

$$\left|^{H}D_{0+}^{\alpha,\beta;\psi}y\left(t\right)-f\left(t,y\left(t\right),\right)\right|\leq\epsilon,\ t\in\left[0,T\right], \tag{1.5}$$

there exists a solution $x\in C\left(\left[0,T\right],\mathbb{R}\right)$ of the equation (1.4) with

$$|y(t)-x\left(t\right)|\leq k\epsilon,\ t\in\left[0,T\right].$$

Definition 1.27 ([83]) Assume that $y\in C\left(\left[0,T\right],\mathbb{R}\right)$ satisfies the inequality in (1.5) and $x\in C\left(\left[0,T\right],\mathbb{R}\right)$ is a solution of the equation (1.4). If there is a function $\phi_f\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ with $\phi_f(0)=0$ satisfying

$$|y(t)-x\left(t\right)|\leq\phi_f(\epsilon),\ t\in\left[0,T\right].$$

Then the equation (1.4) is said to be generalized Ulam-Hyers stable.

Definition 1.28 ([83]) The equation (1.4) is said to be Ulam-Hyres-Rassias stable with respect to $\phi_f\in C\left(\left[0,T\right],\mathbb{R}^+\right)$ if there exists a real number $k>0$ such that for each $\epsilon>0$ and for each $y\in C\left(\left[0,T\right],\mathbb{R}\right)$ solution of the inequality

$$\left|^{H}D_{0+}^{\alpha,\beta;\psi}y\left(t\right)-f\left(t,y\left(t\right)\right)\right|\leq\epsilon\phi_f(t),\ t\in\left[0,T\right], \tag{1.6}$$

there exists a solution $x\in C\left(\left[0,T\right],\mathbb{R}\right)$ of the equation (1.4) with

$$|y(t)-x\left(t\right)|\leq k\phi_f(t)\epsilon,\ t\in\left[0,T\right].$$

Definition 1.29 ([83]) Assume that $y\in C\left(\left[0,T\right],\mathbb{R}\right)$ satisfies the inequality in (1.6) and $x\in C\left(\left[0,T\right],\mathbb{R}\right)$ is a solution of the equation (1.4). If there exists a constant $k>0$ such that

$$|y(t)-x\left(t\right)|\leq k\phi_f(t),\ t\in\left[0,T\right].$$

Then the equation (1.4) is said to be generalized Ulam-Hyres-Rassias stable.

Remark 1.1 If there is a function $v\in C\left(\left[0,T\right],\mathbb{R}\right)$ (dependent on y), such that

1) $|v\left(t\right)|\leq\epsilon$, for all $t\in[0,T]$,

2) $^{H}D_{0+}^{\alpha,\beta;\psi}y\left(t\right)=f\left(t,y\left(t\right)\right)+v\left(t\right)$, $t\in[0,T]$.

Then a function $y\in C\left(\left[0,T\right],\mathbb{R}\right)$ is a solution of the inequality (1.5).

Chapter 2

Existence and uniqueness results for two classes of nonlinear fractional differential equations

In this chapter, we are concerned with the existence and uniqueness of solutions for two classes of nonlinear fractional differential equations. In section 2.1, we study the existence and uniqueness of mild solutions of a boundary value problem for Caputo-Hadamard fractional differential equations with integral and anti-periodic conditions. In section 2.2, we establish sufficient conditions for the existence and uniqueness of solutions for nonlinear implicit Riemann-Liouville fractional differential equations with nonlocal conditions. our results are obtained via fixed point theorems. An example demonstrating the effectiveness of the theoretical results is presented at the end of each section.

2.1 Existence and uniqueness of mild solutions of boundary value problems for Caputo-Hadamard fractional differential equations with integral and anti-periodic conditions

In this section, we study the existence and uniqueness of mild solutions for a nonlinear fractional differential equation with integral and anti-periodic conditions as follows

$$\begin{aligned} {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}x(t) &= f(t,x(t)),\ t\in(1,T), && (2.1)\\ x(1)+x(T) &= b\int_{1}^{T}x(s)\frac{ds}{s}, && (2.2)\end{aligned}$$

where $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, ${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0<\alpha<1$ and $b\in\mathbb{R}$ such that $2-b\log(T)>0$. To show the existence and uniqueness of solutions, we transform the problem (2.1)-(2.2) into an integral equation and then use the Schaefer fixed point theorem to prove the existence result, while the uniqueness is demonstrated by using the Banach's contraction mapping principle.

At the end of each section, An example demonstrating the effectiveness of the theoretical results is presented.

2.1.1 Existence results

First, we start by defining what we mean by a solution of the boundary value problem (2.1)-(2.2).

Definition 2.1 Let $J = [1, T]$, A function $x \in C(J, \mathbb{R})$ is said to be a mild solution of the problem (2.1)-(2.2) if x satisfies the corresponding integral equation of (2.1)-(2.2).

For the existence of solutions for (2.1)-(2.2), we need the following auxiliary lemma.

Lemma 2.1 *Let $\Delta = 2 - b\log(T)$, $x \in C(J, \mathbb{R})$ and x' exists. If x is a solution of the boundary value problem (2.1)-(2.2), then x is a solution of the integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &+ \frac{b}{\Delta\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} f(\sigma, x(\sigma)) \frac{d\sigma}{\sigma}\right) \frac{ds}{s} \\ &- \frac{1}{\Delta\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \text{ for } t \in [1, T]. \end{aligned} \tag{2.3}$$

Proof. Suppose x satisfies the problem (2.1)-(2.2). Then, by applying ${}^H\mathfrak{J}_{1+}^{\alpha}$ to both sides of (2.1), we have

$${}^H\mathfrak{J}_{1+}^{\alpha}\left({}_H^C\mathfrak{D}_{1+}^{\alpha} x(t)\right) = {}^H\mathfrak{J}_{1+}^{\alpha}\left(f(t, x(t))\right).$$

In view of Lemma 1.8, we get

$$x(t) = x(1) + {}^H\mathfrak{J}_{1+}^{\alpha}\left(f(t, x(t))\right). \tag{2.4}$$

The condition (2.2) implies that

$$\begin{aligned} &2x(1) + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &= b\log(T)\, x(1) + \frac{b}{\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} f(\sigma, x(\sigma)) \frac{d\sigma}{\sigma}\right) \frac{ds}{s}, \end{aligned}$$

so

$$\begin{aligned} x(1) &= \frac{b}{\Delta\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} f(\sigma, x(\sigma)) \frac{d\sigma}{\sigma}\right) \frac{ds}{s} \\ &- \frac{1}{\Delta\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}. \end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.4) we get the integral equation (2.3). The proof is completed. ∎

Now, we transform the integral equation (2.3) to be applicable to fixed point theorems, we define the operator $\Phi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\begin{aligned}(\Phi x)(t) = & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} \\ & + \frac{b}{\Delta\Gamma(\alpha)}\int_1^T \left(\int_1^s \left(\log\frac{s}{\sigma}\right)^{\alpha-1} f(\sigma,x(\sigma))\frac{d\sigma}{\sigma}\right)\frac{ds}{s} \\ & - \frac{1}{\Delta\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}.\end{aligned}$$

Where figured fixed point must satisfy the identity operator equation $\Phi u = u$.

In the following, we prove existence, as well as existence and uniqueness results for the boundary value problem (2.1)-(2.2) by using a variety of fixed point theorems.

Existence and uniqueness results via Banach's fixed point theorem

Theorem 2.1 *Assume the following hypothesis*

(H1) There exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \le k|x-y|, \text{ for } t \in J \text{ and } x,y \in \mathbb{R}.$$

If

$$\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^{\alpha}}{\Delta\Gamma(\alpha+1)} < 1, \tag{2.6}$$

then the boundary value problem (2.1)-(2.2) has a unique mild solution on J.

Proof. Let Φ defined by (2.3). Clearly, the fixed points of operator Φ are mild solutions of the problem (2.1)-(2.2). Let $x, y \in C(J,\mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned} & |(\Phi x)(t) - (\Phi y)(t)| \\ & \le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,x(s)) - f(t,y(s))|\frac{ds}{s} \\ & + \frac{|b|}{\Delta\Gamma(\alpha)}\int_1^T \left(\int_1^s \left(\log\frac{s}{\sigma}\right)^{\alpha-1} |f(\sigma,x(\sigma)) - f(t,y(s))|\frac{d\sigma}{\sigma}\right)\frac{ds}{s} \\ & + \frac{1}{\Delta\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} |f(s,x(s)) - f(t,y(s))|\frac{ds}{s} \\ & \le \frac{k(\log t)^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{\infty} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)}\|x-y\|_{\infty} + \frac{k(\log T)^{\alpha}}{\Delta\Gamma(\alpha+1)}\|x-y\|_{\infty} \\ & \le \left(\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^{\alpha}}{\Delta\Gamma(\alpha+1)}\right)\|x-y\|_{\infty}.\end{aligned}$$

Therefore

$$\|\Phi x-\Phi y\|_{\infty}\leq\left(\frac{k\left(\log T\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{k\left|b\right|\left(\log T\right)^{\alpha+1}}{\Delta\Gamma\left(\alpha+2\right)}+\frac{k\left(\log T\right)^{\alpha}}{\Delta\Gamma\left(\alpha+1\right)}\right)\|x-y\|_{\infty}.$$

From (2.6), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is the unique mild solution of (2.1)-(2.2). ■

Existence results via Schaefer's fixed point theorem

Theorem 2.2 *Assume the following hypothesis*

(H2) There exists a constant $M>0$ such that

$$\left|f\left(t,x\right)\right|\leq M,\ \text{for } t\in J \text{ and each } x\in\mathbb{R}.$$

Then the boundary value problem (2.1)-(2.2) has at least one mild solution on J.

Proof. We shall use Schaefer's fixed point theorem to prove that Φ defined by (2.3) has a fixed point. The proof will be given in several steps.

Step 1. The continuity of f implies the continuity of the operator Φ defined by (2.3).

Step 2. Φ maps bounded sets into bounded sets in $C\left(J,\mathbb{R}\right)$.

Indeed, it is enough to show that for any $\eta>0$, there exists a positive constant l such that for each $x\in\Omega=\left\{x\in C\left(J,\mathbb{R}\right):\|x\|_{\infty}\leq\eta\right\}$, we have $\|\Phi x\|_{\infty}\leq l$. In fact, we have

$$\begin{aligned}\left|\left(\Phi x\right)\left(t\right)\right| &\leq \frac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left|f\left(s,x\left(s\right)\right)\right|\frac{ds}{s}\\ &+\frac{\left|b\right|}{\Delta\Gamma\left(\alpha\right)}\int_{1}^{T}\left(\int_{1}^{s}\left(\log\frac{s}{\sigma}\right)^{\alpha-1}\left|f\left(\sigma,x\left(\sigma\right)\right)\right|\frac{d\sigma}{\sigma}\right)\frac{ds}{s}\\ &+\frac{1}{\Delta\Gamma\left(\alpha\right)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\left|f\left(s,x\left(s\right)\right)\right|\frac{ds}{s}\\ &\leq\frac{M}{\Gamma\left(\alpha+1\right)}\left(\log t\right)^{\alpha}+\frac{M\left|b\right|}{\Delta\Gamma\left(\alpha+2\right)}\left(\log T\right)^{\alpha+1}+\frac{M}{\Delta\Gamma\left(\alpha+1\right)}\left(\log T\right)^{\alpha}\\ &\leq\left(\Delta+\frac{\left|b\right|\log T}{\alpha+1}+1\right)\frac{M}{\Delta\Gamma\left(\alpha+1\right)}\left(\log T\right)^{\alpha}.\end{aligned}$$

Thus

$$\|\Phi x\|_{\infty}\leq\left(\frac{\left(\Delta+1\right)\left(\alpha+1\right)+\left|b\right|\log T}{\alpha+1}\right)\frac{M}{\Delta\Gamma\left(\alpha+1\right)}\left(\log T\right)^{\alpha}=l.$$

Step 3. Φ maps bounded sets into equicontinuous sets of $C\left(J,\mathbb{R}\right)$.

Let $t_1,t_2\in J$ with $t_1<t_2$, Ω be a bounded set of $C\left(J,\mathbb{R}\right)$ as in Step 2, and let $x\in\Omega$.

Then

$$
\begin{aligned}
&\left|(\Phi x)(t_2)-(\Phi x)(t_1)\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right|\left|f(s,x(s))\right|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\left|f(s,x(s))\right|\frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha)}\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\frac{ds}{s}+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha+1)}\left((\log t_1)^{\alpha}+\left(\log\frac{t_2}{t_1}\right)^{\alpha}-(\log t_2)^{\alpha}+\left(\log\frac{t_2}{t_1}\right)^{\alpha}\right) \\
&\leq \frac{2M}{\Gamma(\alpha+1)}\left(\log\frac{t_2}{t_1}\right)^{\alpha}.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero and the convergence is independent of x in Ω. As consequence of Step 1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that Φ is completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$B_\lambda=\left\{x\in C(J,\mathbb{R}): x=\lambda\Phi x \text{ for some } 0<\lambda<1\right\},$$

is bounded. Let $x \in B_\lambda$, then $x=\lambda\Phi x$ for some $0<\lambda<1$. Thus, for each $t\in J$ we have

$$
\begin{aligned}
x(t)=\lambda\Bigg[&\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} \\
&+\frac{b}{\Delta\Gamma(\alpha)}\int_1^{T}\left(\int_1^{s}\left(\log\frac{s}{\sigma}\right)^{\alpha-1} f(\sigma,x(\sigma))\frac{d\sigma}{\sigma}\right)\frac{ds}{s} \\
&-\frac{1}{\Delta\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\Bigg].
\end{aligned}
$$

For $\lambda \in (0,1)$, we have

$$\begin{aligned}
|(\Phi x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\
&+ \frac{|b|}{\Delta\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} |f(\sigma,x(\sigma))| \frac{d\sigma}{\sigma}\right) \frac{ds}{s} \\
&+ \frac{1}{\Delta\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha+1)} (\log t)^\alpha + \frac{M|b|}{\Delta\Gamma(\alpha+2)} (\log T)^{\alpha+1} + \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^\alpha \\
&\leq \left(\Delta + \frac{|b| \log T}{\alpha+1} + 1\right) \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^\alpha = R.
\end{aligned}$$

Thus

$$\|\Phi x\|_\infty \leq \left(\frac{(\Delta+1)(\alpha+1) + |b|\log T}{\alpha+1}\right) \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^\alpha = R.$$

This implies that the set B_λ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Φ has a fixed point which is a mild solution of the problem (2.1)-(2.2). ■

2.1.2 Examples

In this subsection, we present some examples to illustrate our results of the previous subsection.

Example 2.1 We consider the problem for Caputo-Hadamard fractional differential equations of the form

$$\begin{aligned}
{}_H^C\mathfrak{D}_{1^+}^{\frac{1}{2}} x(t) &= \frac{\sin(x(t))}{5t}, \ t \in [1,e], &(2.7)\\
x(1) + x(e) &= \int_1^e x(s) \frac{ds}{s}, &(2.8)
\end{aligned}$$

where $\alpha = \frac{1}{2}$, $T = e$, $b = 1$ and $f(t,x) = \frac{\sin(x)}{5t}$. For any $x, y \in \mathbb{R}$ and $t \in [1,e]$, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{5} |x-y|.$$

Therefore, the condition $\frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^\alpha}{\Delta\Gamma(\alpha+1)} < 1$ holds with $k = \frac{1}{5}$ and $\Delta = 1$. Indeed, $\frac{2}{5\Gamma\left(\frac{1}{2}+1\right)} + \frac{1}{5\Gamma\left(\frac{1}{2}+2\right)} \simeq 0.60 < 1$. By Theorem 2.1, the problem (2.7)-(2.8) has a unique mild solution on $[1,e]$.

Example 2.2 We consider the following fractional boundary value problem

$$\begin{aligned}
{}_H^C\mathfrak{D}_{1^+}^{\frac{1}{2}} x(t) &= \frac{\cos(x(t))}{2\exp(-t)}, \ t \in [1,e], &(2.9)\\
x(1) + x(e) &= \frac{1}{2} \int_1^e x(s) \frac{ds}{s}, &(2.10)
\end{aligned}$$

where $\alpha = \frac{1}{2}$, $T = e$, $b = \frac{1}{2}$, $\Delta = \frac{3}{2}$ and $f(t,x) = \frac{\cos(x)}{2\exp(-t)}$. We have

$$|f(t,x)| \leq \frac{|\cos(x)|}{2\exp(-t)} \leq \frac{1}{2e^{-e}}. \; \forall (t,x) \in [1,e] \times \mathbb{R}.$$

Choosing $M = \frac{1}{2e^{-e}}$, then by Theorem 2.2, the problem (2.9)-(2.10) has a mild solution on $[1,e]$.

2.2 Existence and uniqueness results for nonlinear implicit Riemann-Liouville fractional differential equations with nonlocal conditions

In this section, we study the existence and uniqueness of solutions for the following fractional differential equation with nonlocal conditions

$$\begin{cases} {}^{RL}D_{0+}^{\alpha} x(t) = f\left(t, x(t), {}^{RL}D_{0+}^{\alpha} x(t)\right), \; t \in (0,T], \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0 - g(x), \; x_0 \in \mathbb{R}, \end{cases} \tag{2.11}$$

where ${}^{RL}D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $f : (0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C((0,T], \mathbb{R}) \to \mathbb{R}$ are continuous nonlinear functions. To prove the existence and uniqueness of solutions, we transform (2.11) into an integral equation and then use the Banach and Krasnoselskii fixed point theorems.

2.2.1 Existence of solutions

First, we start by defining what we mean by a solution of the problem (2.11).

Definition 2.2 A function $x \in C((0,T], \mathbb{R})$ is said to be a solution of (2.11) if x satisfies ${}^{RL}D_{0+}^{\alpha} x(t) = f\left(t, x(t), {}^{RL}D_{0+}^{\alpha} x(t)\right)$ for any $t \in (0,T]$ and $t^{1-\alpha} x(t)\big|_{t=0} = x_0 - g(x)$.

For the existence of solutions for the problem (2.11), we need the following auxiliary lemma.

Lemma 2.2 *The function x solves (2.11) if and only if it is a solution of the integral equation*

$$x(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(t, x(t), {}^{RL}D_{0+}^{\alpha} x(t)\right) ds, \; t \in (0,T]. \tag{2.12}$$

Proof. Suppose the function x satisfies the problem (2.11), then applying ${}^{RL}I_{a+}^{\alpha}$ to both sides of (2.11), we have

$${}^{RL}I_{a+}^{\alpha} \, {}^{RL}D_{0+}^{\alpha} x(t) = \; {}^{RL}I_{a+}^{\alpha} \, f\left(t, x(t), {}^{RL}D_{0+}^{\alpha} x(t)\right).$$

In view of Lemma 1.4, we get

$$x(t) = c_1 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(t, x(t), {}^{RL}D_{0+}^{\alpha} x(t)\right) ds. \tag{2.13}$$

The condition $t^{1-\alpha}x(t)|_{t=0} = x_0 - g(x)$ implies that

$$c_1 = x_0 - g(x). \tag{2.14}$$

Substituting (2.14) in (2.13) we get the integral equation (2.12). The converse can be proven by direct computations. The proof is completed. ■

In what follows, we show existence, as well as existence and uniqueness results, for the problem (2.11) by using a variety of fixed point theorems.

The following assumptions will be used in our results.

(H1) There exist constants $k_1 > 0$ and $k_2 \in (0,1)$ such that

$$|f(t,u,v) - f(t,u^*,v^*)| \leq k_1 |u - u^*| + k_2 |v - v^*|,$$

for $t \in (0,T]$, $u, v, u^*, v^* \in \mathbb{R}$ and $f(.,0,0) \in C_{1-\alpha}([0,T], \mathbb{R})$, where $C_{1-\alpha}([0,T], \mathbb{R})$ is the weighted space of continuous functions defined by

$$C_{1-\alpha}([0,T], \mathbb{R}) = \left\{x : (0,T] \to \mathbb{R} : t^{1-\alpha}x \in C([0,T], \mathbb{R})\right\},$$

with the norm

$$\|x\|_{C_{1-\alpha}} = \sup_{t\in[0,T]} \left|t^{1-\alpha}x(t)\right|.$$

(H2) There exist a constant $b \in (0,1)$ such that

$$|g(u) - g(u^*)| \leq b \|u - u^*\|_{C_{1-\alpha}},$$

for $u, u^* \in C_{1-\alpha}([0,T], \mathbb{R})$.

Existence and uniqueness results via Banach's fixed point theorem

Theorem 2.3 *Assume that the assumptions (H1) and (H2) are satisfied. If*

$$b + \frac{\Gamma(\alpha) k_1 T^{\alpha}}{\Gamma(2\alpha)(1-k_2)} < 1, \tag{2.15}$$

then there exists a unique solution for the problem (2.11) in the space $C_{1-\alpha}([0,T], \mathbb{R})$.

Proof. We define the operator $\Phi : C_{1-\alpha}([0,T], \mathbb{R}) \to C_{1-\alpha}([0,T], \mathbb{R})$ by

$$(\Phi x)(t) = t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds, \ t \in (0,T],$$

where $h : (0,T] \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, x(t), h(t)).$$

By Lemma 2.2, the fixed points of operator Φ are solutions of (2.11). The operator Φ is well define, i.e. for every $x \in C_{1-\alpha}([0,T],\mathbb{R})$ and $t>0$, the integral

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds, \tag{2.16}$$

belongs to $C_{1-\alpha}([0,T],\mathbb{R})$. Under the condition (H1),

$$\begin{aligned}|h(t)| &= |f(t,x(t),h(t))| \\ &\leq \frac{k_1}{1-k_2}|x(t)| + ct^{\alpha-1} \text{ for each } t\in(0,T],\end{aligned} \tag{2.17}$$

where $c = \frac{\sup_{t\in J}\left|t^{1-\alpha}f(t,0,0)\right|}{1-k_2}$. For every $x \in C_{1-\alpha}([0,T],\mathbb{R})$, we have

$$\begin{aligned}&\left|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds\right| \\ &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s)|\,ds \\ &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{k_1}{1-k_2}|x(s)| + cs^{\alpha-1}\right)ds \\ &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s^{\alpha-1}\left(\frac{k_1}{1-k_2}\left|s^{1-\alpha}x(s)\right| + c\right)ds \\ &\leq \left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}} + c\right) t\ {}^{RL}I_{a+}^{\alpha}\left(t^{\alpha-1}\right).\end{aligned}$$

By Lemma 1.3, we have

$$\begin{aligned}\left|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds\right| &\leq \left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}} + c\right)\frac{\Gamma(\alpha)\,t^{\alpha}}{\Gamma(2\alpha)} \\ &\leq \left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}} + c\right)\frac{\Gamma(\alpha)\,T^{\alpha}}{\Gamma(2\alpha)}.\end{aligned}$$

That is to say that the integral exists and belongs to $C_{1-\alpha}([0,T],\mathbb{R})$.

Let $x,y \in C_{1-\alpha}([0,T],\mathbb{R})$. Then for $t\in(0,T]$, we have

$$\begin{aligned}&|(\Phi x)(t) - (\Phi y)(t)| \\ &\leq t^{\alpha-1}|g(x)-g(y)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h_x(s)-h_y(s)|\,ds,\end{aligned}$$

where $h_x, h_y \in C_{1-\alpha}([0,T],\mathbb{R})$ be such that

$$h_x(t) = f(t,x(t),h_x(t)),$$

and

$$h_y(t) = f(t,y(t),h_y(t)).$$

By (H1) we have

$$\begin{aligned} |h_x(t) - h_y(t)| &= |f(t, x(t), h_x(t)) - f(t, y(t), h_y(t))| \\ &\leq k_1 |x(t) - y(t)| + k_2 |h_x(t) - h_y(t)|. \end{aligned}$$

Then

$$|h_x(t) - h_y(t)| \leq \frac{k_1}{1 - k_2} |x(t) - y(t)|.$$

Therefore, for each $t \in (0, T]$

$$\begin{aligned} &|(\Phi x)(t) - (\Phi y)(t)| \\ &\leq bt^{\alpha-1} \|x - y\|_{C_{1-\alpha}} + \frac{k_1}{\Gamma(\alpha)(1 - k_2)} \int_0^t (t - s)^{\alpha-1} |x(s) - y(s)| \, ds \\ &= t^{\alpha-1} b \|x - y\|_{C_{1-\alpha}} + \frac{k_1}{\Gamma(\alpha)(1 - k_2)} \int_0^t (t - s)^{\alpha-1} s^{\alpha-1} \left| s^{1-\alpha}(x(s) - y(s)) \right| ds \\ &\leq t^{\alpha-1} b \|x - y\|_{C_{1-\alpha}} + \frac{k_1}{1 - k_2} \, {}^{RL}I_{a+}^{\alpha} \left(t^{\alpha-1}\right) \|x - y\|_{C_{1-\alpha}}. \end{aligned}$$

By Lemma 1.3, we have

$$|(\Phi x)(t) - (\Phi y)(t)| \leq t^{\alpha-1} b \|x - y\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha) k_1 t^{2\alpha-1}}{\Gamma(2\alpha)(1 - k_2)} \|x - y\|_{C_{1-\alpha}},$$

which implies that

$$\begin{aligned} &\left| t^{1-\alpha} ((\Phi x)(t) - (\Phi y)(t)) \right| \\ &\leq b \|x - y\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha) k_1 t^{\alpha}}{\Gamma(2\alpha)(1 - k_2)} \|x - y\|_{C_{1-\alpha}} \\ &\leq b \|x - y\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha) k_1 T^{\alpha}}{\Gamma(2\alpha)(1 - k_2)} \|x - y\|_{C_{1-\alpha}}. \end{aligned}$$

Thus

$$\|\Phi x - \Phi y\|_{C_{1-\alpha}} \leq \left(b + \frac{\Gamma(\alpha) k_1 T^{\alpha}}{\Gamma(2\alpha)(1 - k_2)} \right) \|x - y\|_{C_{1-\alpha}}.$$

From (2.15), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem (2.11). ■

Existence results via Krasnoselskii's fixed point theorem

Theorem 2.4 *Assume (H1), (H2) and the following hypothesis*

(H3) There exist $p_1 \in C_{1-\alpha}([0, T], \mathbb{R}^+)$, $p_2, p_3 \in C([0, T], \mathbb{R}^+)$ *with* $p_3^* = \sup_{t \in [0,T]} p_3(t) < 1$ *such that*

$$|f(t, u, v)| \leq p_1(t) + p_2(t)|u| + p_3(t)|v|,$$

for $t \in (0, T]$ *and each* $u, v \in \mathbb{R}$.

If

$$\lambda = b + \frac{p_2^* \Gamma(\alpha) T^\alpha}{(1 - p_3^*) \Gamma(2\alpha)} < 1,$$

where $p_2^* = \sup_{t\in[0,T]} p_2(t)$. *Then the boundary value problem (2.11) has at least one solution in* Ω.

Proof. Set

$$R = \frac{1}{1-\lambda}, \; \Lambda = |x_0| + Q + \frac{T^\alpha p_1^* \Gamma(\alpha)}{(1 - p_3^*) \Gamma(2\alpha)},$$

where $p_1^* = \sup_{t\in[0,T]} \{t^{1-\alpha} p_1(t)\}$ and $Q = |g(0)|$. Let us fix

$$M \geq R\Lambda.$$

Consider the non-empty closed bounded convex subset

$$\Omega = \left\{ x \in C_{1-\alpha}([0,T], \mathbb{R}) : \|x\|_{C_{1-\alpha}} \leq M \right\},$$

and define two operators F_1 and F_2 on Ω, as follows

$$(F_1 x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

and

$$(F_2 x)(t) = t^{\alpha-1} (x_0 - g(x)),$$

where $h : (0,T] \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, x(t), h(t)).$$

We shall use the Krasnoselskii fixed point theorem to prove there exists at least one fixed point of the operator $F_1 + F_2$ in Ω. The proof will be given in several steps.

Step 1. We prove that $F_1 x + F_2 y \in \Omega$ for all $x, y \in \Omega$.

For any $x, y \in \Omega$ and $t \in (0,T]$, we have

$$\begin{aligned}
&|(F_1 x)(t) + (F_2 y)(t)| \\
&\leq \left| t^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \right| \\
&\leq t^{\alpha-1}|x_0| + t^{\alpha-1}|g(x) - g(0)| + t^{\alpha-1}|g(0)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left|s^{1-\alpha} h(s)\right| ds \\
&\leq t^{\alpha-1}|x_0| + t^{\alpha-1} b \|x\|_{C_{1-\alpha}} + t^{\alpha-1} Q \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left|s^{1-\alpha} h(s)\right| ds \\
&\leq t^{\alpha-1}|x_0| + t^{\alpha-1} bM + t^{\alpha-1} Q \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left|s^{1-\alpha} h(s)\right| ds.
\end{aligned} \tag{2.18}$$

By (H3), for each $t \in (0, T]$, we have

$$\begin{aligned} |h(t)| &= |f(t, x(t), h(t))| \\ &\leq p_1(t) + p_2(t)|x(t)| + p_3(t)|h(t)|. \end{aligned}$$

Hence, we get

$$\begin{aligned} \left|t^{1-\alpha}h(t)\right| &\leq t^{1-\alpha}p_1(t) + p_2(t)\left|t^{1-\alpha}x(t)\right| + p_3(t)\left|t^{1-\alpha}h(t)\right| \\ &\leq p_1^* + p_2^*M + p_3^*\left|t^{1-\alpha}h(t)\right|, \end{aligned}$$

then, we have

$$\left|t^{1-\alpha}h(t)\right| \leq \frac{p_1^* + p_2^*M}{1-p_3^*}. \tag{2.19}$$

Replacing (2.19) in the inequality (2.18) and with Lemma 1.3, we get

$$\begin{aligned} &|(F_1x)(t) + (F_2y)(t)| \\ &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}bM + t^{\alpha-1}Q \\ &+ \left(\frac{p_1^* + p_2^*M}{1-p_3^*}\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\ &\leq t^{\alpha-1}|x_0| + t^{\alpha-1}bM + t^{\alpha-1}Q + \left(\frac{p_1^* + p_2^*M}{1-p_3^*}\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}. \end{aligned}$$

Therefore

$$\begin{aligned} &\left|t^{1-\alpha}\left((F_1x)(t) + (F_2x)(t)\right)\right| \\ &\leq |x_0| + Q + \frac{T^\alpha p_1^*\Gamma(\alpha)}{(1-p_3^*)\Gamma(2\alpha)} + \left(b + \frac{p_2^*\Gamma(\alpha)T^\alpha}{(1-p_3^*)\Gamma(2\alpha)}\right)M. \end{aligned}$$

Thus

$$\begin{aligned} &\|F_1x + F_2x\|_{C_{1-\alpha}} \\ &\leq \Lambda + \lambda M \leq \frac{M}{R} + \left(1 - \frac{1}{R}\right)M = M. \end{aligned}$$

Hence $F_1x + F_2y \in \Omega$ for all $x, y \in \Omega$.

Step 2. We show that F_1 is continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in $C_{1-\alpha}([0,T], \mathbb{R})$, then for each $t \in (0, T]$, we have

$$|(F_1x_n)(t) - (F_1x)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h_n(s) - h(s)|\,ds, \tag{2.20}$$

where $h_n, h \in C_{1-\alpha}([0,T], \mathbb{R})$ be such that

$$h_n(t) = f(t, x_n(t), h_n(t)),$$

and

$$h(t) = f(t, x(t), h(t)).$$

By (H1) we have

$$\begin{aligned}|h_n(t) - h(t)| &= |f(t, x_n(t), h_n(t)) - f(t, x(t), h(t))| \\ &\leq k_1 |x_n(t) - x(t)| + k_2 |h_n(t) - h(t)|.\end{aligned}$$

Then

$$|h_n(t) - h(t)| \leq \frac{k_1}{1-k_2} |x_n(t) - x(t)|. \tag{2.21}$$

By replacing (2.21) in inequality (2.20), we find

$$\begin{aligned}&|(F_1 x_n)(t) - (F_1 x)(t)| \\ &\leq \frac{k_1}{(1-k_2)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_n(t) - x(t)| \, ds \\ &= \frac{k_1}{(1-k_2)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left| s^{1-\alpha} (x_n(t) - x(t)) \right| ds \\ &\leq \frac{k_1}{1-k_2} I_{0^+}^{\alpha} \left(t^{\alpha-1}\right) \|x_n - x\|_{C_{1-\alpha}}.\end{aligned}$$

By Lemma 1.3, we have

$$|(F_1 x_n)(t) - (F_1 x)(t)| \leq \frac{\Gamma(\alpha) k_1 t^{2\alpha-1}}{(1-k_2)\Gamma(2\alpha)} \|x_n - x\|_{C_{1-\alpha}},$$

which implies that

$$\begin{aligned}\left| t^{1-\alpha} ((F_1 x_n)(t) - (F_1 x)(t)) \right| &\leq \frac{\Gamma(\alpha) k_1 t^{\alpha}}{(1-k_2)\Gamma(2\alpha)} \|x_n - x\|_{C_{1-\alpha}} \\ &\leq \frac{\Gamma(\alpha) k_1 T^{\alpha}}{(1-k_2)\Gamma(2\alpha)} \|x_n - x\|_{C_{1-\alpha}}.\end{aligned}$$

Thus

$$\|F_1 x_n - F_1 x\|_{C_{1-\alpha}} \leq \frac{\Gamma(\alpha) k_1 T^{\alpha}}{(1-k_2)\Gamma(2\alpha)} \|x_n - x\|_{C_{1-\alpha}},$$

and hence

$$\|F_1 x_n - F_1 x\|_{C_{1-\alpha}} \to 0 \text{ as } n \to \infty.$$

Consequently, F_1 is continuous.

Step 3. We prove that F_1 is compact.

For all $x \in \Omega$ and $t \in (0, T]$, we have

$$|(F_1 x)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left| s^{1-\alpha} h(s) \right| ds. \tag{2.22}$$

Replacing (2.19) in the inequality (2.22) and with Lemma 1.3, we get

$$\left|\left(F_1 x\right)(t)\right| \leq \left(\frac{p_1^* + p_2^* M}{1 - p_3^*}\right) \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} t^{2\alpha-1}.$$

Therefore

$$\left|t^{1-\alpha}\left(F_1 x\right)(t)\right| \leq \left(\frac{p_1^* + p_2^* M}{1 - p_3^*}\right) \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha}.$$

Thus

$$\left\|F_1 x\right\|_{C_{1-\alpha}} \leq \left(\frac{p_1^* + p_2^* M}{1 - p_3^*}\right) \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha}.$$

Hence $F_1(\Omega)$ is uniformly bounded.

It remains to show that $F_1(\Omega)$ is equicontinuous, let $0 \leq t_1 < t_2 \leq T$ and $x \in \Omega$. Then

$$\begin{aligned}
&\left|t_2^{1-\alpha}\left(F_1 x\right)(t_2) - t_1^{1-\alpha}\left(F_1 x\right)(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} t_2^{1-\alpha}(t_2 - s)^{\alpha-1} h(s)\,ds + \int_{t_1}^{t_2} t_2^{1-\alpha}(t_2 - s)^{\alpha-1} h(s)\,ds\right. \\
&\left. - \int_0^{t_1} t_1^{1-\alpha}(t_1 - s)^{\alpha-1} h(s)\,ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|t_2^{1-\alpha}(t_2 - s)^{\alpha-1} s^{\alpha-1} - t_1^{1-\alpha}(t_1 - s)^{\alpha-1} s^{\alpha-1}\right| \left|s^{1-\alpha} h(s)\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{1-\alpha}(t_2 - s)^{\alpha-1} s^{\alpha-1} \left|s^{1-\alpha} h(s)\right| ds \\
&\leq \frac{p_1^* + p_2^* M}{1 - p_3^*}\left(\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|t_2^{1-\alpha}(t_2 - s)^{\alpha-1} - t_1^{1-\alpha}(t_1 - s)^{\alpha-1}\right| s^{\alpha-1} ds\right) \\
&+ \frac{p_1^* + p_2^* M}{1 - p_3^*}\left(\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{1-\alpha}(t_2 - s)^{\alpha-1} s^{\alpha-1} ds\right).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. That is to say that $F_1(\Omega)$ is equicontinuous, then by Ascoli-Arzela theorem, we can conclude that the operator F_1 is compact.

Step 4. We prove that $F_2 : \Omega \to C_{1-\alpha}([0, T], \mathbb{R})$ is a contraction mapping.

For all $x \in \Omega$ and from (H2), we have

$$\begin{aligned}
\left|\left(F_2 x\right)(t) - \left(F_2 y\right)(t)\right| &= \left|t^{\alpha-1}\left(g(x) - g(y)\right)\right| \\
&\leq t^{\alpha-1} b \left\|x - y\right\|_{C_{1-\alpha}}.
\end{aligned}$$

Therefore

$$\left|t^{1-\alpha}\left(\left(F_2 x\right)(t) - \left(F_2 y\right)(t)\right)\right| \leq b\left\|x - y\right\|_{C_{1-\alpha}}.$$

Thus

$$\left\|F_2 x - F_2 y\right\|_{C_{1-\alpha}} \leq b\left\|x - y\right\|_{C_{1-\alpha}}.$$

Hence, the operator F_2 is a contraction.

Clearly, all the hypotheses of the Krasnoselskii fixed point theorem are satisfied. Thus there a fixed point $x \in \Omega$ such that $x = F_1 x + F_2 x$, which is a solution of the problem (2.11). ■

2.2. Existence and uniqueness results for nonlinear implicit Riemann-Liouville fractional differential equations with nonlocal conditions

2.2.2 An example

We consider the following fractional initial value problem

$$\begin{cases} {}^{RL}D_{0^+}^{\frac{2}{3}} x(t) = \frac{1}{4\exp(-t+2)\left(1+|x(t)|+\left|{}^{RL}D_{0^+}^{\frac{2}{3}}x(t)\right|\right)} + \frac{1}{t^{\frac{1}{3}}},\ t \in (0,1], \\ t^{\frac{1}{3}} x(t)\Big|_{t=0} = \frac{1}{2} - \sum_{i=1}^{n} c_i t_i^{\frac{1}{3}} y(t_i), \end{cases} \tag{2.23}$$

where $0 < t_1 < ... < t_n < 1$ and c_i, $i = 1, ..., n$ are positive constants with

$$\sum_{i=1}^{n} c_i \leq \frac{1}{4}.$$

Set

$$f(t, u, v) = \frac{1}{4\exp(-t+2)(1+|u|+|v|)} + \frac{1}{t^{\frac{1}{3}}},\ t \in (0,1],\ u, v \in \mathbb{R},$$

We have

$$C_{1-\alpha}([0,1], \mathbb{R}) = C_{\frac{1}{3}}([0,1], \mathbb{R}) = \left\{h : (0,1] \to \mathbb{R} : t^{\frac{1}{3}} h \in C([0,1], \mathbb{R})\right\},$$

with $\alpha = \frac{2}{3}$. Clearly the functions f and g are continuous, $f(.,0,0) \in C_{\frac{1}{3}}([0,1], \mathbb{R})$. For each $u, u^*, v, v^* \in \mathbb{R}$ and $t \in (0,1]$, we have

$$\begin{aligned} & |f(t,u,v) - f(t,u^*,v^*)| \\ & = \left| \frac{1}{4\exp(-t+2)} \left(\frac{1}{(1+|u|+|v|)} - \frac{1}{(1+|u^*|+|v^*|)} \right) \right| \\ & \leq \frac{|u-u^*| + |v-v^*|}{4\exp(-t+2)(1+|u|+|v|)(1+|u^*|+|v^*|)} \\ & \leq \frac{1}{4e}(|u-u^*| + |v-v^*|), \end{aligned}$$

and

$$\begin{aligned} |g(u) - g(u^*)| & \leq \sum_{i=1}^{n} c_i t_i^{\frac{1}{3}} |u(t_i) - u^*(t_i)| \\ & \leq \sum_{i=1}^{n} c_i \|u-u^*\|_{C_{\frac{1}{3}}} \leq \frac{1}{4} \|u-u^*\|_{C_{\frac{1}{3}}}. \end{aligned}$$

Hence, conditions (H1) and (H2) are satisfied with $k_1 = k_2 = \frac{1}{4e}$and $b = \frac{1}{4}$. The condition

$$b + \frac{\Gamma(\alpha) k_1 T^{\alpha}}{\Gamma(2\alpha)(1-k_2)} = \frac{1}{4} + \frac{\frac{\Gamma\left(\frac{2}{3}\right)}{4e}}{\Gamma\left(\frac{4}{3}\right)\left(1-\frac{1}{4e}\right)} \simeq 0.4 < 1,$$

is satisfied with $T = 1$. It follows from Theorem 2.3 that the problem (2.23) has a unique solution in the space $C_{\frac{1}{3}}([0,1], \mathbb{R})$.

Chapter 3

Some qualitative properties for certain classes of nonlinear hybrid fractional differential equations

The area of differential equations where the terms in the equation are perturbed either linearly or quadratically or through the combination of first and second types is called hybrid differential equations. Perturbation taking place in the form of the sum or difference of terms in an equation is called linear. On the other hand, if the equation is perturbed through the product or quotient of the terms in it, then it is called quadratic perturbation. So the study of the hybrid differential equation is more general and covers several dynamic systems for some developments on the existence results of hybrid fractional differential equations. In latest years, the existence theory for solutions of boundary value problems of hybrid fractional differential equations has attracted the attention of many researchers, we refer to [7, 22, 29, 84, 91, 103] and the references therein for the recent development in this area.

In this chapter, we are interested to study some qualitative properties for certain classes of nonlinear hybrid fractional differential equations. First, In section 3.1, we study the existence and uniqueness of solutions and Ulam stability results for nonlinear hybrid implicit Caputo fractional differential equations. Next, In section 3.2, we present some results about the existence, interval of existence, uniqueness and estimation of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations.

3.1 Existence and Ulam stability results for nonlinear hybrid implicit Caputo fractional differential equations

In this section, we study the existence and uniqueness of solutions and the Ulam stability for the following nonlinear hybrid implicit Caputo fractional differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x\left(t\right), {}^{C}D_{0+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right), \ t \in (0, T], \\ x\left(0\right) = \theta g\left(0, x(0)\right) + f\left(0, x(0)\right), \ \theta \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $h : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions and ${}^{C}D_{0+}^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha < 1$. To show the existence, uniqueness and estimate of solutions of (3.1), we transform (3.1) into an integral equation and then use the contraction mapping principle and Gronwall's inequality. Further, we obtain Ulam-Hyers and Ulam-Hyers-Rassias stability results of (3.1). Finally, we provide an example to illustrate our obtained results.

3.1.1 Existence and estimate of solutions

First, we start by defining what we mean by a solution of the problem (3.1).

Definition 3.1 A function $x \in AC\left([0, T], \mathbb{R}\right)$ is said to be a solution of (3.1) if x satisfies ${}^{C}D_{0+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x\left(t\right), {}^{C}D_{0+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right)$ for any $t \in [0, T]$ and $x\left(0\right) = \theta g\left(0, x(0)\right) + f\left(0, x(0)\right)$.

To obtain our results, we need the following auxiliary lemma.

Lemma 3.1 *If the functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}\backslash\left\{0\right\}$ and $h : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, then the initial value problem (3.1) is equivalent to the nonlinear fractional Volterra integro-differential equation*

$$\begin{aligned} x(t) &= f\left(t, x(t)\right) + \theta g(t, x\left(t\right)) \\ &\quad + \frac{g\left(t, x\left(t\right)\right)}{\Gamma\left(\alpha\right)} \int_0^t \left(t-s\right)^{\alpha-1} h\left(s, x\left(s\right), {}^{C}D_{0+}^{\alpha}\left(\frac{x\left(s\right) - f\left(s, x(s)\right)}{g(s, x\left(s\right))}\right)\right) ds, \end{aligned}$$

for $t \in [0, T]$.

The proof of the above lemma is close to the proof of Lemma 6.2 given in [30].

Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.1 *Let $T > 0$. Assume that the continuous functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}\backslash\left\{0\right\}$ and $h : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions*

(H1) There exists $M_g \in \mathbb{R}^+$ *such that*

$$|g(t,u)| \leq M_g,$$

for all $u \in \mathbb{R}$ *and* $t \in [0,T]$.

(H2) There exists $M_h \in \mathbb{R}^+$ *such that*

$$|h(t,u,v)| \leq M_h,$$

for all $u, v \in \mathbb{R}$ *and* $t \in [0,T]$.

(H3) There exist $K_1, K_2, K_3 \in \mathbb{R}^+$, $K_4 \in (0,1)$ *with* $K_1 + K_2 |\theta| \in (0,1)$ *such that*

$$\begin{array}{rcl} |f(t,u) - f(t,u^*)| & \leq & K_1 |u-v|, \\ |g(t,u) - g(t,u^*)| & \leq & K_2 |u-v|, \end{array}$$

and

$$|h(t,u,v) - h(t,u^*,v^*)| \leq K_3 |u-u^*| + K_4 |v-v^*|,$$

for all $u, v, u^*, v^* \in \mathbb{R}$ *and* $t \in [0,T]$.

If

$$\beta = K_1 + K_2 |\theta| + M_h K_2 + \frac{M_g K_3}{(1-K_4)} \frac{T^\alpha}{\Gamma(\alpha+1)} < 1. \tag{3.2}$$

Then, the problem (3.1) has a unique solution $x \in C([0,T], \mathbb{R})$.

Proof. Let

$$^C D_{0+}^\alpha \left(\frac{x(t) - f(t,x(t))}{g(t,x(t))} \right) = z_x(t), \ x(0) = \theta g(0,x(0)) + f(0,x(0)),$$

then by Lemma 3.1, we have

$$x(t) = f(t,x(t)) + \theta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s)\,ds,$$

where

$$z_x(t) = h\Big(t, f(t,x(t)) + \theta g(t,x(t)) + g(t,x(t))^{RL} I_{0+}^\alpha z_x(t), z_x(t)\Big).$$

That is $x(t) = f(t,x(t)) + \theta g(t,x(t)) + g(t,x(t))^{RL} I_{0+}^\alpha z_x(t)$. Define the mapping $\Phi : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ as follows

$$(\Phi x)(t) = f(t,x(t)) + \theta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s)\,ds.$$

It is clear that the fixed points of Φ are solutions of (3.1). Let $x, y \in C([0,T], \mathbb{R})$, then we have

$$
\begin{aligned}
&\left|(\Phi x)(t)-(\Phi y)(t)\right| \\
&=\left|f(t,x(t))+\theta g(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z_x(s)\,ds\right. \\
&\left.-f(t,y(t))-\theta g(t,y(t))-\frac{g(t,y(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z_y(s)\,ds\right| \\
&\leq |f(t,x(t))-f(t,y(t))|+|\theta|\,|g(t,x(t))-g(t,y(t))| \\
&+|g(t,x(t))-g(t,y(t))|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z_x(s)|\,ds \\
&+|g(t,y(t))|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z_x(s)-z_y(s)|\,ds \\
&\leq K_1|x(t)-y(t)|+K_2|\theta|\,|x(t)-y(t)| \\
&+K_2|x(t)-y(t)|\frac{M_h}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds \\
&+\frac{M_g}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z_x(s)-z_y(s)|\,ds,
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
|z_x(t)-z_y(t)| &\leq |h(t,x(t),z_x(t))-h(t,y(t),z_y(t))| \\
&\leq K_3|x(t)-y(t)|+K_4|z_x(t)-z_y(t)| \\
&\leq \frac{K_3}{1-K_4}|x(t)-y(t)|.
\end{aligned} \tag{3.4}
$$

By replacing (3.4) in the inequality (3.3), we get

$$
\begin{aligned}
&\left|(\Phi x)(t)-(\Phi y)(t)\right| \\
&\leq K_1|x(t)-y(t)|+K_2|\theta|\,|x(t)-y(t)| \\
&+K_2|x(t)-y(t)|\frac{M_h}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds \\
&+\frac{M_g}{\Gamma(\alpha)}\frac{K_3}{1-K_4}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|\,ds \\
&\leq K_1\|x-y\|_\infty+K_2\left(|\theta|+\frac{M_h t^\alpha}{\Gamma(\alpha+1)}\right)\|x-y\|_\infty \\
&+\frac{M_g}{\Gamma(\alpha)}\frac{K_3}{1-K_4}\left(\int_0^t (t-s)^{\alpha-1}\,ds\right)\|x-y\|_\infty \\
&\leq \left(K_1+K_2|\theta|+\left(M_hK_2+\frac{M_gK_3}{1-K_4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\|x-y\|_\infty \\
&\leq \left(K_1+K_2|\theta|+\left(M_hK_2+\frac{M_gK_3}{1-K_4}\right)\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\|x-y\|_\infty.
\end{aligned}
$$

Then

$$\|\Phi x - \Phi y\|_\infty \leq \beta \|x - y\|_\infty .$$

By (3.2), the mapping Φ is a contraction in $C([0,T],\mathbb{R})$. Hence Φ has a unique fixed point $x \in C([0,T],\mathbb{R})$. Therefore (3.1) has a unique solution. ■

Estimate of solutions

Theorem 3.2 *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $g : [0,T] \times \mathbb{R} \to \mathbb{R}\backslash \{0\}$ and $h : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy (H1)-(H3) and (3.2) holds. Then (3.1) has a unique solution x and*

$$\begin{aligned} |x(t)| \leq & \left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1) + M_g K_3 K T^\alpha}{(1-K_4)(1-(K_1+K_2|\theta|))^2 \Gamma(\alpha+1)} \right) \\ & \times \left(Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1-K_4)\Gamma(\alpha+1)} \right), \end{aligned}$$

where $Q_1 = \sup\limits_{t\in[0,T]} |f(t,0)|$, $Q_2 = \sup\limits_{t\in[0,T]} |g(t,0)|$, $Q_3 = \sup\limits_{t\in[0,T]} |h(t,0,0)|$ and $K \in \mathbb{R}^+$ is a constant.

Proof. Theorem 3.1 shows that the problem (3.1) has a unique solution. Let

$$^C D^\alpha_{0+} \left(\frac{x(t) - f(t,x(t))}{g(t,x(t))} \right) = z_x(t), \ x(0) = \theta g(0,x(0)) + f(0,x(0)),$$

then by Lemma 3.1, $x(t) = f(t,x(t)) + \theta g(t,x(t)) + g(t,x(t))^{RL} I^\alpha_{0+} z_x(t)$. Then by (H1), (H2) and (H3), for any $t \in [0,T]$ we have

$$\begin{aligned} |x(t)| &\leq |f(t,x(t))| + |\theta| |g(t,x(t))| + |g(t,x(t))| \left|^{RL} I^\alpha_{0+} z_x(t)\right| \\ &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| \\ &\quad + |\theta| (|g(t,x(t)) - g(t,0)| + |g(t,0)|) + M_g \left|^{RL} I^\alpha_{0+} z_x(t)\right| \\ &\leq K_1 |x(t)| + Q_1 + |\theta| (K_2 |x(t)| + Q_2) + M_g \ ^{RL} I^\alpha_{0+} |z_x(t)| . \end{aligned}$$

On the other hand, for any $t \in [0,T]$ we get

$$\begin{aligned} |z_x(t)| &= |h(t,x(t),z_x(t))| \\ &\leq |h(t,x(t),z_x(t)) - h(t,0,0)| + |h(t,0,0)| \\ &\leq K_3 |x(t)| + K_4 |z_x(t)| + |h(t,0,0)| \\ &\leq \frac{K_3}{1-K_4} |x(t)| + \frac{Q_3}{1-K_4}. \end{aligned}$$

Therefore

$$\begin{aligned} |x(t)| \leq & K_1 |x(t)| + Q_1 + |\theta| (K_2 |x(t)| + Q_2) \\ & + M_g \ ^{RL} I^\alpha_{0+} \left(\frac{K_3}{1-K_4} |x(t)| + \frac{Q_3}{1-K_4} \right). \end{aligned}$$

Thus

$$\begin{aligned}
&(1-(K_1+K_2|\theta|))|x(t)| \\
&\leq Q_1+|\theta|Q_2+\frac{M_gQ_3T^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} \\
&+\left(\frac{M_gK_3}{(1-K_4)(1-(K_1+K_2|\theta|))}\right)\left({}^{RL}I_{a+}^{\alpha}\{(1-(K_1+K_2|\theta|))|x(t)|\}\right).
\end{aligned}$$

By Lemma 1.16, there is a constant $K=K(\alpha)$ such that

$$\begin{aligned}
&(1-(K_1+K_2|\theta|))|x(t)| \\
&\leq Q_1+|\theta|Q_2+\frac{M_gQ_3T^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} \\
&+\left(\frac{M_gK_3KT^{\alpha}}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)}\right) \\
&\times\left(Q_1+|\theta|Q_2+\frac{M_gQ_3T^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}\right) \\
&\leq\left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)+M_gK_3KT^{\alpha}}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)}\right) \\
&\times\left(Q_1+|\theta|Q_2+\frac{M_gQ_3T^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
|x(t)|\leq&\left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)+M_gK_3KT^{\alpha}}{(1-K_4)(1-(K_1+K_2|\theta|))^2\Gamma(\alpha+1)}\right) \\
&\times\left(Q_1+|\theta|Q_2+\frac{M_gQ_3T^{\alpha}}{(1-K_4)\Gamma(\alpha+1)}\right).
\end{aligned}$$

This completes the proof. ■

3.1.2 Ulam stability

In the following, we will study two types of Ulam stability of the hybrid implicit Caputo fractional differential equation (3.1) which are Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

Lemma 3.2 *Assume that g satisfies (H1). If $y\in C([0,T],\mathbb{R})$ is a solution of the fractional differential inequality for each $\epsilon>0$*

$$\left|{}^{C}D_{0+}^{\alpha}\left(\frac{y(t)-f(t,y(t))}{g(t,y(t))}\right)-h\left(t,y(t),{}^{C}D_{0+}^{\alpha}\left(\frac{y(t)-f(t,y(t))}{g(t,y(t))}\right)\right)\right|\leq\epsilon, \tag{3.5}$$

then, y is a solution of the following inequality

$$|y(t)-(\Phi y)(t)|\leq\frac{\epsilon M_gT^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.6}$$

Proof. Let $y \in C([0,T], \mathbb{R})$ be a solution of the inequality (3.5) for each $\epsilon > 0$. Then, from Remark 1.1 and Lemma 3.1 for some continuous function $v(t)$ such that $|v(t)| < \epsilon$, $t \in [0,T]$, we have

$$y(t) = f(t, y(t)) + \theta g(t, y(t)) + \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \left(h\left(s, y(s), {}^C D_{0+}^{\alpha}\left(\frac{y(s) - f(s, y(s))}{g(s, y(s))}\right)\right) + v(s)\right) ds.$$

Then, by Remark 1.1 and (H1), we obtain

$$\begin{aligned} |y(t) - (\Phi y)(t)| &= \left| \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \right| \\ &\leq \frac{|g(t, y(t))|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|\, ds \\ &\leq \frac{\epsilon M_g T^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

which is satisfied inequality (3.6). This completes the proof. ■

Theorem 3.3 *Assume that the assumptions (H1)–(H3) are fulfilled and (3.2) holds. Then the problem (3.1) is Ulam-Hyers stable.*

Proof. Under (H1)–(H3) and (3.2), the problem (3.1) has a unique solution in $C([0,T], \mathbb{R})$. Let $y \in C([0,T], \mathbb{R})$ be a solution of the inequality (3.5), then for each $t \in [0,T]$, we have

$$\begin{aligned} &|y(t) - x(t)| \\ &= \Bigg| y(t) - f(t, x(t)) + \theta g(t, x(t)) \\ &+ \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h\left(s, x(s), {}^C D_{0+}^{\alpha}\left(\frac{x(s) - f(s, x(s))}{g(s, x(s))}\right)\right) ds \Bigg| \\ &= |y(t) - \Phi y(t) + \Phi y(t) - \Phi x(t)| \\ &\leq |y(t) - \Phi y(t)| + |\Phi y(t) - \Phi x(t)| \\ &\leq \frac{\epsilon M_g T^{\alpha}}{\Gamma(\alpha+1)} + \beta \|y - x\|_{\infty}. \end{aligned}$$

Then

$$\|y - x\|_{\infty} \leq \frac{\epsilon M_g T^{\alpha}}{(1-\beta)\Gamma(\alpha+1)}.$$

By setting

$$k = \frac{M_g T^{\alpha}}{(1-\beta)\Gamma(\alpha+1)},$$

we obtain

$$|y(t) - x(t)| \leq k\epsilon.$$

Therefore, the problem (3.1) is Ulam-Hyers stable. This completes the proof. ■

3.1. Existence and Ulam stability results for nonlinear hybrid implicit Caputo fractional differential equations

In the next, we introduce the following function.

(H4) The function $\phi \in C([0,T],\mathbb{R}^+)$ is increasing and there exists a constant $\lambda_\phi > 0$ such that, for each $t \in [0,T]$, we have

$$^{RL}I\phi(t) \leq \lambda_\phi \phi(t).$$

Lemma 3.3 *Assume that (H1) and (H4) are satisfied. If $y \in C([0,T],\mathbb{R})$ is a solution of the fractional differential inequality for each $\epsilon > 0$*

$$\left| {}^C D_{0+}^{\alpha}\left(\frac{y(t) - f(t,y(t))}{g(t,y(t))}\right) - h\left(t, y(t), {}^C D_{0+}^{\alpha}\left(\frac{y(t) - f(t,y(t))}{g(t,y(t))}\right)\right)\right| \leq \epsilon\phi(t), \tag{3.7}$$

then, y is a solution of the following inequality

$$|y(t) - (\Phi y)(t)| \leq \epsilon M_g \lambda_\phi \phi(t). \tag{3.8}$$

Proof. Let $y \in C([0,T],\mathbb{R})$ be a solution of the inequality (3.7) for each $\epsilon > 0$. Then, from Remark 1.1 and Lemma 3.1 for some continuous function $v(t)$ such that $|v(t)| < \epsilon\phi(t)$, $t \in [0,T]$, we have

$$\begin{aligned} |y(t) - (\Phi y)(t)| &= \left|\frac{g(t,y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds\right| \\ &\leq \frac{|g(t,y(t))|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|\, ds \\ &\leq \frac{\epsilon M_g}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s)\, ds \\ &\leq \epsilon M_g \lambda_\phi \phi(t). \end{aligned}$$

which is satisfied inequality (3.8). This completes the proof. ■

Theorem 3.4 *Assume that the assumptions (H1)–(H4) are fulfilled and (3.2) holds. Then the problem (3.1) is Ulam-Hyers-Rassias stable.*

Proof. Under (H1)–(H4) and (3.2), the problem has a unique solution in $C([0,T],\mathbb{R})$. Let $y \in C([0,T],\mathbb{R})$ be a solution of the inequality (3.7), then for each $t \in [0,T]$, we have

$$\begin{aligned} &|y(t) - x(t)| \\ &= \Bigg| y(t) - f(t,x(t)) + \theta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\times h\left(s, x(s), {}^C D_{0+}^{\alpha}\left(\frac{x(s) - f(s,x(s))}{g(s,x(s))}\right)\right) ds \Bigg| \\ &= |y(t) - (\Phi y)(t) + (\Phi y)(t) - (\Phi x)(t)| \\ &\leq |y(t) - (\Phi y)(t)| + |(\Phi y)(t) - (\Phi x)(t)| \\ &\leq \epsilon M_g \lambda_\phi \phi(t) + \beta \|y - x\|_\infty. \end{aligned}$$

Then

$$|y(t) - x(t)| \leq \frac{\epsilon M_g \lambda_\phi \phi(t)}{1-\beta}.$$

By taking a constant

$$k_{\phi,f} = \frac{M_g \lambda_\phi}{1-\beta},$$

we obtain

$$|y(t) - x(t)| \leq k_{\phi,f} \epsilon \phi(t).$$

Therefore, the problem (3.1) is Ulam-Hyers-Rassias stable. This completes the proof. ■

3.1.3 An example

We consider the following fractional nonlinear hybrid implicit Caputo fractional differential equation

$$\begin{cases} {}^C D_{0+}^{\frac{3}{4}} \left(\frac{x(t) - \frac{1}{4} t \sin(x(t))}{1 + \frac{1}{10} \exp(-t^2) \cos(x(t))} \right) \\ = \frac{1}{5} t \cos(x(t)) + \frac{1}{(e^t + 24)\left(1 + \left| {}^C D_{0+}^{\frac{3}{4}} \left(\frac{x(t) - \frac{1}{4} t \sin(x(t))}{1 + \frac{1}{10} \exp(-t^2) \cos(x(t))} \right) \right| \right)}, \ t \in [0,1], \\ x(0) = g(0, x(0)) + f(0, x(0)). \end{cases} \tag{3.9}$$

where $\theta = 1$, $f(t, x(t)) = \frac{1}{4} t \sin(x(t))$, $g(t, x(t)) = 1 + \frac{1}{10} \exp(-t^2) \cos(x(t))$ and

$$\begin{aligned} & h\left(t, x(t), {}^C D_{0+}^{\frac{3}{4}} \left(\frac{x(t) - f(t, x(t))}{g(t, x(t))} \right) \right) \\ & = \frac{1}{5} t \cos(x(t)) + \frac{1}{(e^t + 24)\left(1 + \left| {}^C D_{0+}^{\frac{3}{4}} \left(\frac{x(t) - \frac{1}{4} t \sin(x(t))}{1 + \frac{1}{10} \exp(-t^2) \cos(x(t))} \right) \right| \right)}. \end{aligned}$$

Set $h(t, u, v) = \frac{1}{5} t \cos(u) + \frac{1}{(e^t + 24)(1 + |v|)}$. For any $u, v, u^*, v^* \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\begin{aligned} |f(t,u) - f(t,u^*)| &\leq \frac{1}{4} |u - u^*|, \\ |g(t,u) - g(t,u^*)| &\leq \frac{1}{10} |u - u^*|, \\ |h(t,u,v) - h(t,u^*,v^*)| &\leq \frac{1}{5} |u - u^*| + \frac{1}{25} |v - v^*|, \end{aligned}$$

and

$$|g(t,u)| \leq \frac{11}{10}, \ |h(t,u,v)| \leq \frac{e + 29}{5e + 120}.$$

Hence, all conditions of Theorem 3.1 are fulfilled and $\beta = 0.62309 < 1$, with

$$M_g = \frac{11}{10}, \ M_h = \frac{e+29}{5e+120}, \ K_1 = \frac{1}{4}, \ K_2 = \frac{1}{10}, \ K_3 = \frac{1}{5}, \ K_4 = \frac{1}{25}.$$

Then the problem (3.9) has a unique solution $x \in C([0,T], \mathbb{R})$. And from Theorem 3.3 we deduce that (3.9) is Ulam-Hyers stable.

Now, we choose $\phi(t) = t^2$ and in view of Lemma 1.3, we have

$$^{RL}I_{0+}^{\alpha}\phi(t) = \frac{\Gamma(3)}{\Gamma(3+\alpha)}t^{2+\alpha} \leq \frac{2}{\Gamma\left(\frac{7}{2}\right)}t^2 = \lambda_\phi \phi(t).$$

Thus condition (H4) is satisfied with $\phi(t) = t^2$ and $\lambda_\phi = \frac{2}{\Gamma\left(\frac{7}{2}\right)} = \frac{16}{15\sqrt{\pi}}$, it follows from Theorem 3.4 that the problem (3.9) is Ulam-Hyers-Rassias stable.

3.2 Existence and uniqueness results for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations

In this section, we study the existence, interval of existence and uniqueness of solution for the following nonlinear hybrid implicit Caputo-Hadamard fractional differential equation

$$\begin{cases} {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x(t), {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right), \ t \in (1,T], \\ x(1) = \theta g(1, x(1)) + f(1, x(1)), \ \theta \in \mathbb{R}, \end{cases} \tag{3.10}$$

where $f : [1,T] \times \mathbb{R} \to \mathbb{R}$, $g : [1,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $h : [1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions and ${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order $0 < \alpha < 1$. To show the existence, interval of existence and uniqueness of solutions, we transform (3.10) into an integral equation and then use the Banach fixed point theorem. Further, by Gronwall's inequality we obtain the estimate of solutions of (3.10).

3.2.1 Existence and estimate of solutions

First, we start by defining what we mean by a solution of the problem (3.10).

Definition 3.2 A function $x \in AC^1([1,T], \mathbb{R})$ is said to be a solution of (3.10) if x satisfies ${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h\left(t, x(t), {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right)$ for any $t \in [1,T]$ and $x(1) = \theta g(1, x(1)) + f(1, x(1))$.

To obtain our results, we need the following auxiliary lemma.

Lemma 3.4 *If the functions $f : [1,T] \times \mathbb{R} \to \mathbb{R}$, $g : [1,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $h : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, then the initial value problem (3.10) is equivalent to the nonlinear fractional Volterra integro-differential equation*

$$\begin{aligned} x(t) &= f(t, x(t)) + \theta g(t, x(t)) \\ &\quad + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h\left(s, x(s), {}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(s) - f(s, x(s))}{g(s, x(s))}\right)\right)\frac{ds}{s}, \end{aligned}$$

for $t \in [1,T]$.

The proof of the above lemma is close to the proof of Lemma 6.2 given in [30].

Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.5 *Let $T > 0$. Assume that the continuous functions $f : [1,T] \times \mathbb{R} \to \mathbb{R}$, $g : [1,T] \times \mathbb{R} \to \mathbb{R}\backslash \{0\}$ and $h : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the following conditions*

(H1) There exists $M_g \in \mathbb{R}^+$ such that

$$|g(t,u)| \leq M_g,$$

for all $u \in \mathbb{R}$ and $t \in [1,T]$.

(H2) There exists $M_h \in \mathbb{R}^+$ such that

$$|h(t,u,v)| \leq M_h,$$

for all $u,v \in \mathbb{R}$ and $t \in [1,T]$.

(H3) There exist $K_1, K_2, K_3 \in \mathbb{R}^+, K_4 \in (0,1)$ with $K_1 + K_2 |\theta| \in (0,1)$ such that

$$\begin{aligned} |f(t,u) - f(t,u^*)| &\leq K_1 |u - v|, \\ |g(t,u) - g(t,u^*)| &\leq K_2 |u - v|, \end{aligned}$$

and

$$|h(t,u,v) - h(t,u^*,v^*)| \leq K_3 |u - u^*| + K_4 |v - v^*|,$$

for all $u, v, u^*, v^* \in \mathbb{R}$ and $t \in [1,T]$.

Let

$$1 < b < \min\left\{ T, \exp\left(\frac{((1-(K_1 + K_2|\theta|))(1-K_4)\Gamma(\alpha+1))}{(M_h K_2 (1-K_4) + M_g K_3)} \right)^{\frac{1}{\alpha}} \right\}. \tag{3.11}$$

Then (3.10) has a unique solution $x \in C([1,b],\mathbb{R})$.

Proof. Let

$${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha}\left(\frac{x(t) - f(t,x(t))}{g(t,x(t))} \right) = z_x(t), \; x(1) = \theta g(1,x(1)) + f(1,x(1)),$$

then by Lemma 3.4,

$$x(t) = f(t,x(t)) + \theta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} z_x(s) \frac{ds}{s},$$

where

$$z_x(t) = h\left(t, f(t,x(t)) + \theta g(t,x(t)) + g(t,x(t))^{H} \mathfrak{I}_{1+}^{\alpha} z_x(t), z_x(t) \right).$$

That is $x(t) = f(t,x(t)) + \theta g(t,x(t)) + g(t,x(t))^{H} \mathfrak{I}_{1+}^{\alpha} z_x(t)$. Define the mapping $\Phi : C([1,b],\mathbb{R}) \to C([1,b],\mathbb{R})$ as follows

$$(\Phi x)(t) = f(t,x(t)) + \theta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} z_x(s) \frac{ds}{s}.$$

It is clear that the fixed points of Φ are solutions of (3.10). Let $x, y \in C([1, b], \mathbb{R})$, then we have

$$\begin{aligned}
&|(\Phi x)(t) - (\Phi y)(t)| \\
&= \left| f(t, x(t)) + \theta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} z_x(s) \frac{ds}{s} \right. \\
&\left. - f(t, y(t)) + \theta g(t, y(t)) - \frac{g(t, y(t))}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} z_y(s) \frac{ds}{s} \right| \\
&\leq |f(t, x(t)) - f(t, y(t))| + |\theta| \, |g(t, x(t)) - g(t, y(t))| \\
&+ |g(t, x(t)) - g(t, y(t))| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z_x(s)| \frac{ds}{s} \\
&+ |g(t, y(t))| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s} \\
&\leq K_1 |x(t) - y(t)| + K_2 |\theta| \, |x(t) - y(t)| \\
&+ K_2 |x(t) - y(t)| \frac{M_h}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{M_g}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z_x(s) - z_y(s)| \frac{ds}{s},
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
|z_x(t) - z_y(t)| &\leq |h(t, x(t), z_x(t)) - h(t, x(t), z_y(t))| \\
&\leq K_3 |x(t) - y(t)| + K_4 |z_x(t) - z_y(t)| \\
&\leq \frac{K_3}{1 - K_4} |x(t) - y(t)| .
\end{aligned} \tag{3.13}$$

By replacing (3.13) in the inequality (3.12), we get

$$\begin{aligned}
&|(\Phi x)(t) - (\Phi y)(t)| \\
&\leq K_1 |x(t) - y(t)| + K_2 |\theta| \, |x(t) - y(t)| \\
&+ K_2 |x(t) - y(t)| \frac{M_h}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{M_g}{\Gamma(\alpha)} \frac{K_3}{1 - K_4} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |x(s) - y(s)| \frac{ds}{s} \\
&\leq K_1 \|x - y\|_\infty + K_2 \left(|\theta| + \frac{M_h (\log t)^\alpha}{\Gamma(\alpha+1)}\right) \|x - y\|_\infty \\
&+ \frac{K_3}{1 - K_4} \left(\frac{M_g (\log t)^\alpha}{\Gamma(\alpha+1)}\right) \|x - y\|_\infty \\
&\leq (K_1 + K_2 |\theta| + \left(M_h K_2 + \frac{M_g K_3}{1 - K_4}\right) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}) \|x - y\|_\infty .
\end{aligned}$$

3.2. Existence and uniqueness results for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations

Since $t \in [1, b]$, Then
$$\|\Phi x - \Phi y\|_{\infty} \leq \beta \|x - y\|_{\infty},$$
where
$$\beta = K_1 + K_2 |\theta| + \left(M_h K_2 + \frac{M_g K_3}{1 - K_4} \right) \frac{(\log b)^{\alpha}}{\Gamma(\alpha + 1)}.$$
That is to say the mapping Φ is a contraction in $C([1, b], \mathbb{R})$. Hence, by the Banach fixed point theorem, Φ has a unique fixed point $x \in C([1, b], \mathbb{R})$. Therefore, (3.10) has a unique solution. ■

Estimate of solutions

Theorem 3.6 *Assume that $f : [1, T] \times \mathbb{R} \to \mathbb{R}$, $g : [1, T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $h : [1, T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy (H1), (H2) and (H3). If x is a solution of (3.10), then*
$$\begin{aligned} |x(t)| \leq &\left(\frac{(1 - K_4)(1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1) + M_g K_3 K (\log T)^{\alpha}}{(1 - K_4)(1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1)} \right) \\ &\times \left(Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^{\alpha}}{(1 - K_4) \Gamma(\alpha + 1)} \right), \end{aligned}$$
where $Q_1 = \sup_{t \in [1, T]} |f(t, 0)|$, $Q_2 = \sup_{t \in [1, T]} |g(t, 0)|$, $Q_3 = \sup_{t \in [1, T]} |h(t, 0, 0)|$ and $K \in \mathbb{R}^+$ is a constant.

Proof. Let
$${}_{H}^{C}\mathfrak{D}_{1+}^{\alpha} \left(\frac{x(t) - f(t, x(t))}{g(t, x(t))} \right) = z_x(t), \ x(1) = \theta g(1, x(1)) + f(1, x(1)),$$
then by Lemma 3.4, $x(t) = f(t, x(t)) + \theta g(t, x(t)) + g(t, x(t))^{H} \mathfrak{I}_{1+}^{\alpha} z_x(t)$. Then by (H1), (H2) and (H3), for any $t \in [1, T]$ we have
$$\begin{aligned} |x(t)| &\leq |f(t, x(t))| + |\theta| |g(t, x(t))| + |g(t, x(t))| \left| {}^{H}\mathfrak{I}_{1+}^{\alpha} z_x(t) \right| \\ &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &+ |\theta| (|g(t, x(t)) - g(t, 0)| + |g(t, 0)|) + M_g \left| {}^{H}\mathfrak{I}_{1+}^{\alpha} z_x(t) \right| \\ &\leq K_1 |x(t)| + Q_1 + |\theta| (K_2 |x(t)| + Q_2) + M_g \ {}^{H}\mathfrak{I}_{1+}^{\alpha} |z_x(t)|. \end{aligned}$$

On the other hand, for any $t \in [1, T]$ we get
$$\begin{aligned} |z_x(t)| &= |h(t, x(t), z_x(t))| \\ &\leq |h(t, x(t), z_x(t)) - h(t, 0, 0)| + |h(t, 0, 0)| \\ &\leq K_3 |x(t)| + K_4 |z_x(t)| + |h(t, 0, 0)| \\ &\leq \frac{K_3}{1 - K_4} |x(t)| + \frac{Q_3}{1 - K_4}. \end{aligned}$$

Therefore

$$|x(t)| \leq K_1 |x(t)| + Q_1 + |\theta| (K_2 |x(t)| + Q_2) + M_g\, {}^H\mathfrak{I}^{\alpha}_{1+} \left(\frac{K_3}{1-K_4} |x(t)| + \frac{Q_3}{1-K_4} \right).$$

Thus

$$\begin{aligned} & (1-(K_1+K_2|\theta|))\,|x(t)| \\ & \leq Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} + \left(\frac{M_g K_3}{(1-K_4)(1-(K_1+K_2|\theta|))} \right) \\ & \times \left({}^H\mathfrak{I}^{\alpha}_{1+} \{(1-(K_1+K_2|\theta|))\,|x(t)|\} \right). \end{aligned}$$

By Lemma 1.17, there is a constant $K = K(\alpha)$ such that

$$\begin{aligned} & (1-(K_1+K_2|\theta|))\,|x(t)| \\ & \leq Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} \\ & + \left(\frac{M_g K_3 K (\log T)^{\alpha}}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)} \right) \\ & \times \left(Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} \right) \\ & \leq \left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1) + M_g K_3 K (\log T)^{\alpha}}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)} \right) \\ & \times \left(Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} \right). \end{aligned}$$

Hence

$$\begin{aligned} |x(t)| \leq & \left(\frac{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1) + M_g K_3 K (\log T)^{\alpha}}{(1-K_4)(1-(K_1+K_2|\theta|))\Gamma(\alpha+1)} \right) \\ & \times \left(Q_1 + |\theta| Q_2 + \frac{M_g Q_3 (\log T)^{\alpha}}{(1-K_4)\Gamma(\alpha+1)} \right). \end{aligned}$$

This completes the proof. ■

Chapter 4

Existence of solutions for certain classes of nonlinear fractional differential equations via Kuratowski measure of noncompactness

In this chapter, we are interested with the existence of solutions for certain classes of fractional differential equations via Kuratowski measure of noncompactness. In section 4.1, we study the existence of solutions for a nonlinear fractional differential equations involving Riemann-Liouville fractional derivative subject to integral boundary conditions in Banach spaces. In section 4.2, we establish sufficient conditions for the existence of solutions for a nonlinear fractional differential equations involving Hadamard fractional derivative with two nonlinear terms in Banach spaces. The used approach is based on Mönch's fixed point theorem combined with the technique of Kuratowski measure of noncompactness. An example demonstrating the effectiveness of the theoretical results is given at the end of each section.

4.1 Existence results for integral boundary value problems of fractional differential equations in Banach spaces

In this section, we study the existence of solutions for the boundary value problem of a fractional differential equation with integral boundary conditions of the form

$$\begin{cases} {}^{RL}D_{0+}^{\alpha}x(t) - f(t,x(t)) = {}^{RL}D_{0+}^{\alpha-1}g(t,x(t)), \ t \in (0,1), \\ x(0) = 0, \ x(1) = \int_0^1 g(s,x(s))\,ds, \end{cases} \tag{4.1}$$

where ${}^{RL}D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$, $f, g : J \times X \to X$ are given functions satisfying some assumptions that will be specified later, and X be a Banach space with the norm $\|.\|$. In the case $X = \mathbb{R}$, Xu and Sun in [101] investigated the existence and uniqueness of a positive solution of (4.1) by using the method of the upper and lower solutions and the Schauder and Banach fixed point theorems. Then, the existence results obtained here extend the main results in [101].

4.1.1 Existence results

First, we start by defining what we mean by a solution of the problem (4.1).

Definition 4.1 Let $J=[0,1]$, A function $x\in C(J,X)$ is said to be a solution of problem (4.1) if x satisfies the equation ${}^{RL}D_{0+}^{\alpha}x(t)-f(t,x(t))={}^{RL}D_{0+}^{\alpha-1}g(t,x(t))$ on J and the conditions $x(0)=0$, $x(1)=\int_0^1 g(s,x(s))\,ds$.

For the existence of solutions for the problem (4.1), we need the following auxiliary lemma.

Lemma 4.1 *The function x solves the problem (4.1) if and only if it is a solution of the integral equation*

$$x(t)=\int_0^1 G(t,s)f(s,x(s))\,ds+\int_0^t g(s,x(s))\,ds,\ t\in J,$$

where G is the Green function given by

$$G(t,s)=\begin{cases}\frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\leq s\leq t\leq 1,\\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0\leq t\leq s\leq 1.\end{cases}$$

Proof. From Lemma 1.4, applying the Riemann-Liouville fractional integral ${}^{RL}I_{0+}^{\alpha}$ on both sides of (4.1), we have

$$\begin{aligned}x(t)-c_1t^{\alpha-1}-c_2t^{\alpha-2}+{}^{RL}I_{0+}^{\alpha}f(t,x(t)) &= {}^{RL}I_{0+}\left({}^{RL}I_{0+}^{\alpha-1}\,{}^{RL}D_{0+}^{\alpha-1}g(t,x(t))\right)\\ &= {}^{RL}I_{0+}\left(g(t,x(t))-c_3t^{\alpha-2}\right).\end{aligned}$$

That is,

$$\begin{aligned}x(t)&=c_1t^{\alpha-1}+c_2t^{\alpha-2}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))\,ds\\ &\quad+\int_0^t g(s,x(s))\,ds-\frac{c_3}{\alpha-1}t^{\alpha-1}.\end{aligned}$$

By the boundary conditions $x(0)=0$, $x(1)=\int_0^1 g(s,x(s))\,ds$, one has $c_2=0$ and

$$c_1=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s))\,ds+\frac{c_3}{\alpha-1}.$$

Therefore

$$\begin{aligned}x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}f(s,x(s))\,ds\\ &\quad-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))\,ds+\int_0^t g(s,x(s))\,ds\\ &=\int_0^1 G(t,s)f(s,x(s))\,ds+\int_0^t g(s,x(s))\,ds.\end{aligned}$$

This process is reversible. The proof is complete. ■

In the following, we prove existence results for the boundary value problem (4.1) by using a Mönch fixed point theorem.

The following assumptions will be used in our main results.

(H1) The functions $f, g : J \times X \to X$ satisfy the Carathéodory conditions.

(H2) There exist $p_f, p_g \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$ such that

$$\begin{aligned} \|f(t,x)\| &\leq p_f(t)\|x\|, \text{ for } t \in J \text{ and each } x \in X, \\ \|g(t,x)\| &\leq p_g(t)\|x\|, \text{ for } t \in J \text{ and each } x \in X. \end{aligned}$$

(H3) For each $t \in J$ and each bounded set $B \subset X$, we have

$$\begin{aligned} \lim_{h\to 0^+} \mu_k(f(J_{t,h} \times B)) &\leq p_f(t)\mu_k(B), \text{ here } J_{t,h} = [t-h,t] \cap J, \\ \lim_{h\to 0^+} \mu_k(g(J_{t,h} \times B)) &\leq p_g(t)\mu_k(B), \text{ here } J_{t,h} = [t-h,t] \cap J. \end{aligned}$$

Theorem 4.1 *Assume that the assumptions (H1)-(H3) hold. If*

$$\frac{2}{\Gamma(\alpha+1)}\|p_f\|_\infty + \|p_g\|_\infty < 1, \tag{4.2}$$

then the boundary value problem (4.1) has at least one solution.

Proof. We transform the problem (4.1) into a fixed point problem by defining an operator $\Phi : C(J,X) \to C(J,X)$ as

$$\begin{aligned} (\Phi x)(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} f(s,x(s))\,ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))\,ds + \int_0^t g(s,x(s))\,ds. \end{aligned}$$

Clearly, the fixed points of operator Φ are solutions of the problem (4.1). Let $R > 0$ and consider the set

$$\Omega = \{x \in C(J,X) : \|x\|_\infty \leq R\}.$$

Clearly, the subset Ω is closed, bounded, and convex. We will show that Φ satisfies the assumptions of Theorem 1.6. The proof will be given in three steps.

Step 1. Φ maps Ω into itself.

For each $x \in \Omega$, by (H2) and (4.2) we have for each $t \in J$

$$\begin{aligned} \|(\Phi x)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}\|f(s,x(s))\|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))\|\,ds + \int_0^t \|g(s,x(s))\|\,ds \\ &\leq R\left(\frac{2}{\Gamma(\alpha+1)}\|p_f\|_\infty + \|p_g\|_\infty\right) \\ &\leq R. \end{aligned}$$

Step 2. $\Phi(\Omega)$ is bounded and equicontinuous.

By Step 1, we have $\Phi(\Omega) = \{\Phi x : x \in \Omega\} \subset \Omega$. Thus, for each $x \in \Omega$, we have $\|\Phi x\|_\infty \leq R$, which means that $\Phi(\Omega)$ is bounded. For the equicontinuity of $\Phi(\Omega)$. Let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in \Omega$. Then

$$\begin{aligned}
&\|(\Phi x)(t_2) - (\Phi x)(t_1)\| \\
&\leq \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \|f(s, x(s))\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right| \|f(s, x(s))\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, x(s))\| \, ds + \int_{t_1}^{t_2} \|g(s, x(s))\| \, ds \\
&\leq \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p_f(s) \|x(s)\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) p_f(s) \|x(s)\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} p_f(s) \|x(s)\| \, ds + \int_{t_1}^{t_2} p_g(s) \|x(s)\| \, ds \\
&\leq \frac{\|p_f\|_\infty R}{\Gamma(\alpha+1)} \left(t_2^{\alpha-1} - t_1^{\alpha-1} + t_2^{\alpha} - t_1^{\alpha} \right) + \|p_g\|_\infty R (t_2 - t_1).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 3. Φ is continuous.

Let $\{x_n\}$ be sequence such that $x_n \to x$ in $C(J, X)$. Then, for each $t \in J$

$$\begin{aligned}
&\|(\Phi x_n)(t) - (\Phi x)(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| \, ds \\
&+ \int_0^t \|g(s, x_n(s)) - g(s, x(s))\| \, ds.
\end{aligned}$$

Since f and g are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|(\Phi x_n)(t) - (\Phi x)(t)\| \to 0 \text{ as } n \to \infty.$$

This shows that (Φx_n) converges pointwise to Φx on J. Moreover, the sequence (Φx_n) is equicontinuous by a similar proof of Step 2. Therefore (Φx_n) converges uniformly to Φx and hence Φ is continuous.

Now let V be a subset of Ω such that $V \subset \overline{conv}((\Phi V) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $v \to v(t) = \mu_k(V(t))$ is continuous on J. By assumption

(H3), Lemma 1.20 and the properties of the measure μ_k we have for each $t \in J$

$$\begin{aligned} v(t) &\leq \mu_k\left((\Phi V)(t) \cup \{0\}\right) \leq \mu_k\left((\Phi V)(t)\right) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} p_f(s)\, \mu_k(V(s))\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_f(s)\, \mu_k(V(s))\, ds + \int_0^t p_g(s)\, \mu_k(V(s))\, ds \\ &\leq \|v\|_\infty \left(\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty\right). \end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \left[\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty\right]\right) \leq 0.$$

By (4.2), it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in X. In view of the Ascoli-Arzela theorem, V is relatively compact in Ω. Applying now Theorem 1.6, we conclude that Φ has a fixed point, which is a solution of the problem (4.1). ■

4.1.2 An example

As an application of our results, we consider the following boundary value problem of a fractional differential equation

$$\begin{cases} {}^{RL}D_{0+}^{\frac{3}{2}} x(t) - \frac{1}{3+\exp(t)} x(t) = {}^{RL}D_{0+}^{\frac{1}{2}} \frac{1}{5+\exp(t^2)} x(t), \ t \in (0,1), \\ x(0) = 0, \ x(1) = \int_0^1 \frac{1}{5+\exp(s^2)} x(s)\, ds. \end{cases} \tag{4.3}$$

Let

$$X = l^1 = \left\{ x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

equipped with the norm

$$\|x\|_X = \sum_{n=1}^{\infty} |x_n|.$$

Set

$$x = (x_1, x_2, ..., x_n, ...), \ f = (f_1, f_2, ..., f_n, ...), \ g = (g_1, g_2, ..., g_n, ...),$$

and

$$\begin{aligned} f_n(t, x_n) &= \frac{1}{3+\exp(t)} x_n, \ t \in J, \\ g_n(t, x_n) &= \frac{1}{5+\exp(t^2)} x_n, \ t \in J. \end{aligned}$$

For each x_n and $t \in J$, we have

$$|f_n(t, x_n)| \leq \frac{1}{3+\exp(t)} |x_n|, \tag{4.4}$$

and

$$|g_n(t, x_n)| \leq \frac{1}{5+\exp(t^2)} |x_n|. \tag{4.5}$$

Hence conditions (H1) and (H2) are satisfied with $p_f(t) = \frac{1}{3+\exp(t)}$ and $p_g(t) = \frac{1}{5+\exp(t^2)}$. By (4.4) and (4.5), for any bounded set $B \subset l^1$, we have

$$\mu_k(f(t,B)) \leq \frac{1}{3+\exp(t)} \mu_k(B) \text{ for each } t \in J,$$
$$\mu_k(g(t,B)) \leq \frac{1}{5+\exp(t^2)} \mu_k(B) \text{ for each } t \in J.$$

Hence (H3) is satisfied. The condition

$$\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty \simeq 0.54 < 1,$$

is satisfied with $\|p_f\|_\infty = \frac{1}{4}$, $\|p_g\|_\infty = \frac{1}{6}$ and $\alpha = \frac{3}{2}$. Consequently, Theorem 4.1 implies that the problem (4.3) has a solution define on J.

4.2 Existence of solutions for Hadamard fractional differential equations with two nonlinear terms in Banach spaces

In this section, we prove the existence of solutions for nonlinear fractional differential equation involving Hadamard fractional derivative with integral boundary conditions of the type

$$\begin{cases} {}^H\mathfrak{D}_{1^+}^{\alpha} x(t) - f(t, x(t)) = {}^H\mathfrak{D}_{1^+}^{\alpha-1} g(t, x(t)), \ t \in (1, e), \\ x(1) = 0, \ x(e) = \int_1^e g(s, x(s)) \frac{ds}{s}. \end{cases} \tag{4.6}$$

where ${}^H\mathfrak{D}_{1^+}^{\alpha}$ denotes the Hadamard fractional derivatives of order $1 < \alpha \leq 2$, $f, g : [1, e] \times X \to X$ are given functions satisfying some assumptions that will be specified later, and X be a Banach space with the norm $\|.\|$.

4.2.1 Existence results

First, we start by defining what we mean by a solution of the problem (4.6).

Definition 4.2 Let $J = [1, e]$, A function $x \in C(J, X)$ is said to be a solution of problem (4.6) if x satisfies the equation ${}^H\mathfrak{D}_{1^+}^{\alpha} x(t) - f(t, x(t)) = {}^H\mathfrak{D}_{1^+}^{\alpha-1} g(t, x(t))$ on J and the conditions $x(1) = 0$, $x(e) = \int_1^e g(s, x(s)) \frac{ds}{s}$.

For the existence of solutions for the problem (4.6), we need the following auxiliary lemma.

Lemma 4.2 *The function x solves problem (4.6) if and only if is a solution of the integral equation*

$$x(t) = \int_1^e G(t,s) f(s,x(s)) \frac{ds}{s} + \int_1^t g(s,x(s)) \frac{ds}{s}, \ t \in J,$$

where G is the Green function given by

$$G(t,s) = \begin{cases} \frac{\left[(\log t)\left(\log \frac{e}{s}\right)\right]^{\alpha-1} - \left(\log \frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} & 1 \leq s \leq t \leq e, \\ \frac{\left[(\log t)\left(\log \frac{e}{s}\right)\right]^{\alpha-1}}{\Gamma(\alpha)}, & 1 \leq t \leq s \leq e. \end{cases}$$

Proof. From Lemma 1.7,applying the Hadamard fractional integral ${}^H\mathfrak{I}_{1+}^{\alpha}$ on both sides of equation (4.6), we have

$$\begin{aligned} x(t) - c_1 (\log t)^{\alpha-1} - c_2 (\log t)^{\alpha-2} +^H \mathfrak{I}_{1+}^{\alpha} f(t,x(t)) &= {}^H\mathfrak{I}_{1+} \left({}^H\mathfrak{I}_{1+}^{\alpha-1} \, {}^H\mathfrak{D}_{1+}^{\alpha-1} g(t,x(t))\right) \\ &= {}^H\mathfrak{I}_{1+} \left(g(t,x(t)) - c_3 (\log t)^{\alpha-2}\right), \end{aligned}$$

that is,

$$\begin{aligned} x(t) &= c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} \\ &+ \int_1^t g(s,x(s)) \frac{ds}{s} - \frac{c_3}{\alpha-1} (\log t)^{\alpha-1}. \end{aligned}$$

By boundary conditions $x(1) = 0$, $x(e) = \int_1^e g(s,x(s))\, ds$, one has $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} + \frac{c_3}{\alpha-1}.$$

Therefore

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} \\ &- \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,x(s)) \frac{ds}{s} + \int_1^t g(s,x(s)) \frac{ds}{s} \\ &= \int_1^e G(t,s) f(s,x(s)) \frac{ds}{s} + \int_1^t g(s,x(s)) \frac{ds}{s}. \end{aligned}$$

The converse follows by direct computation which completes the proof. ■

In the following we prove existence results for the boundary value problem (4.6) by using a Mönch fixed point theorem.

The following assumptions will be used in our main results.

(H1) The functions $f, g : J \times X \to X$ satisfy the Carathéodory conditions.

(H2) There exist p_f, $p_g \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$ such that

$$\begin{aligned} \|f(t,x)\| &\leq p_f(t) \|x\|, \text{ for } t \in J \text{ and each } x \in X, \\ \|g(t,x)\| &\leq p_g(t) \|x\|, \text{ for } t \in J \text{ and each } x \in X. \end{aligned}$$

(H3) For each $t \in J$ and each bounded set $B \subset X$, we have

$$\begin{aligned}\lim_{h\to 0^+} \mu_k\left(f\left(J_{t,h}\times B\right)\right) &\leq p_f(t)\,\mu_k(B)\,, \text{ here } J_{t,h}=[t-h,t]\cap J,\\ \lim_{h\to 0^+} \mu_k\left(g\left(J_{t,h}\times B\right)\right) &\leq p_g(t)\,\mu_k(B)\,, \text{ here } J_{t,h}=[t-h,t]\cap J.\end{aligned}$$

Theorem 4.2 *Assume that the assumptions (H1)-(H3) hold. If*

$$\frac{2}{\Gamma(\alpha+1)}\left\|p_f\right\|_\infty+\left\|p_g\right\|_\infty<1, \tag{4.7}$$

then the boundary value problem (4.6) has at least one solution.

Proof. We transform the problem (4.6) into a fixed point problem by defining an operator $\Phi : C(J,X) \to C(J,X)$ as

$$\begin{aligned}(\Phi x)(t) &= \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}\left(\log\frac{e}{s}\right)^{\alpha-1} f(s,x(s))\,\frac{ds}{s}\\ &\quad -\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\,\frac{ds}{s}+\int_1^t g(s,x(s))\,\frac{ds}{s}.\end{aligned}$$

Clearly, the fixed points of operator Φ are solutions of problem (4.6). Let $R>0$ and consider the set

$$\Omega=\left\{x\in C(J,X):\|x\|_\infty\leq R\right\}.$$

Clearly, the subset Ω is closed, bounded, and convex. We will show that Φ satisfies the assumptions of Theorem 1.6. The proof will be given in three steps.

Step 1: Φ maps Ω into itself.

For each $x\in\Omega$, by (H2) and (4.7) we have for each $t\in J$

$$\begin{aligned}\|(\Phi x)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}\left(\log\frac{e}{s}\right)^{\alpha-1}\|f(s,x(s))\|\,\frac{ds}{s}\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\|f(s,x(s))\|\,\frac{ds}{s}+\int_1^t\|g(s,x(s))\|\,\frac{ds}{s}\\ &\leq \frac{\|p_f\|_\infty}{\Gamma(\alpha)}\int_1^e(\log t)^{\alpha-1}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{ds}{s}+\frac{\|p_f\|_\infty}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\ &\quad+\int_1^e\|g(s,x(s))\|\,\frac{ds}{s}\\ &\leq R\left(\frac{2}{\Gamma(\alpha+1)}\left\|p_f\right\|_\infty+\left\|p_g\right\|_\infty\right)\\ &\leq R.\end{aligned}$$

Step 2: $\Phi(\Omega)$ is bounded and equicontinuous.

By Step 1, we have $\Phi(\Omega)=\{\Phi x : x\in\Omega\}\subset\Omega$. Thus, for each $x\in\Omega$, we have $\|\Phi x\|_\infty\leq R$, which means that $\Phi(\Omega)$ is bounded.

For the equicontinuity of $\Phi(\Omega)$. Let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in \Omega$. Then

$$\begin{aligned}
&\left\|(\Phi x)(t_2) - (\Phi x)(t_1)\right\| \\
&\leq \frac{\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \left\|f(s, x(s))\right\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left\| \left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right\| \left\|f(s, x(s))\right\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \left\|f(s, x(s))\right\| \frac{ds}{s} + \int_{t_1}^{t_2} \left\|g(s, x(s))\right\| \frac{ds}{s} \\
&\leq \frac{\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} p_f(s) \left\|x(s)\right\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right) p_f(s) \left\|x(s)\right\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} p_f(s) \left\|x(s)\right\| \frac{ds}{s} + \int_{t_1}^{t_2} p_g(s) \left\|x(s)\right\| \frac{ds}{s} \\
&\leq \frac{\left\|p_f\right\|_\infty R}{\Gamma(\alpha+1)} \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1} + (\log t_2)^{\alpha} - (\log t_1)^{\alpha}\right) \\
&+ \left\|p_g\right\|_\infty R\left((\log t_2) - (\log t_1)\right).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 3: Φ is continuous.

Let $\{x_n\}$ be sequence such that $x_n \to x$ in $C(J, X)$. Then, for each $t \in J$

$$\begin{aligned}
&\left\|(\Phi x_n)(t) - (\Phi x)(t)\right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} \left\|f(s, x_n(s)) - f(s, x(s))\right\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left\|f(s, x_n(s)) - f(s, x(s))\right\| \frac{ds}{s} \\
&+ \int_1^t \left\|g(s, x_n(s)) - g(s, x(s))\right\| \frac{ds}{s}.
\end{aligned}$$

Since f and g are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\left\|(\Phi x_n)(t) - (\Phi x)(t)\right\|_\infty \to 0 \text{ as } n \to \infty.$$

Now let V be a subset of Ω such that $V \subset \overline{conv}\left((\Phi V) \cup \{0\}\right)$. V is bounded and equicontinuous, and therefore the function $v \to v(t) = \mu_k(V(t))$ is continuous on J. By assumption

(H3), Lemma 1.20 and the properties of the measure μ_k we have for each $t \in J$

$$\begin{aligned} v(t) &\le \mu_k\left((\Phi V)(t) \cup \{0\}\right) \le \mu_k\left((\Phi V)(t)\right) \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} p_f(s)\, \mu_k(V(s)) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_f(s)\, \mu_k(V(s)) \frac{ds}{s} + \int_1^e p_g(s)\, \mu_k(V(s)) \frac{ds}{s} \\ &\le \|v\|_\infty \left(\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty\right). \end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \left[\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty\right]\right) \le 0.$$

By (4.7), it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in X. In view of the Ascoli-Arzela theorem, V is relatively compact in Ω. Applying now Theorem 1.6, we conclude that Φ has fixed point, which is a solution of the problem (4.6) ■

4.2.2 An example

To validate the existence results., we consider the following boundary value problem of a fractional differential equation

$$\begin{cases} {}^H\mathfrak{D}_{1+}^{\frac{3}{2}} x(t) - \frac{1}{\exp(t^2-1)+3}\left(\frac{|x|}{|x|+1}\right) = {}^H\mathfrak{D}_{1+}^{\frac{1}{2}} \frac{\cos(t)}{5+\exp(t^2)} x(t),\ t \in (1, e), \\ x(1) = 0,\ x(e) = \int_1^e \frac{\cos(s)}{5+\exp(s^2)} x(s) \frac{ds}{s}. \end{cases} \tag{4.8}$$

Here $\alpha = \frac{3}{2}$. Let

$$X = l^1 = \left\{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\right\},$$

equipped with the norm

$$\|x\|_X = \sum_{n=1}^{\infty} |x_n|.$$

Set

$$x = (x_1, x_2, ..., x_n, ...),\ f = (f_1, f_2, ..., f_n, ...),\ g = (g_1, g_2, ..., g_n, ...),$$

and

$$\begin{aligned} f_n(t, x_n) &= \frac{1}{\exp(t^2-1)+3} x_n,\ t \in J, \\ g_n(t, x_n) &= \frac{\cos(t)}{5+\exp(t^2)} x_n,\ t \in J. \end{aligned}$$

For each x_n and $t \in J$, we have

$$|f_n(t, x_n)| \leq \frac{1}{\exp(t^2 - 1) + 3} |x_n|, \tag{4.9}$$

and

$$|g_n(t, x_n)| \leq \frac{1}{5 + \exp(t^2)} |x_n|. \tag{4.10}$$

Hence conditions (H1) and (H2) are satisfied with

$$p_f(t) = \frac{1}{\exp(t^2 - 1) + 3} \text{ and } p_g(t) = \frac{1}{5 + \exp(t^2)}.$$

By (4.9) and (4.10), for any bounded set $B \subset l^1$, we have

$$\mu_k(f(t, B)) \leq \frac{1}{\exp(t^2 - 1) + 3} \mu_k(B) \text{ for each } t \in J,$$
$$\mu_k(g(t, B)) \leq \frac{1}{5 + \exp(t^2)} \mu_k(B) \text{ for each } t \in J.$$

Hence (H3) is satisfied. The condition

$$\frac{2}{\Gamma(\alpha + 1)} \|p_f\|_\infty + \|p_g\|_\infty \simeq 0.51 < 1,$$

is satisfied with

$$\|p_f\|_\infty = \frac{1}{4} \text{ and } \|p_g\|_\infty = \frac{1}{5 + e}.$$

Consequently,Theorem 4.2 implies that problem (4.8) has a solution defined on J.

Chapter 5

A study of certain classes of fractional differential inclusions

In this chapter, we are dealing with the existence of solutions for certain classes of fractional differential inclusions involving convex and nonconvex multivalued maps, In section 5.1. we prove the existence of solutions for a nonlinear sequential Caputo and Caputo-Hadamard fractional differential inclusions with three-point boundary conditions. In section 5.2, we debate the existence of solutions for a nonlinear Hilfer fractional differential inclusion with nonlocal Erdélyi-Kober fractional integral boundary conditions. The results obtained are based on some fixed point theorems of multivalued analysis. Some pertinent examples demonstrating the effectiveness of the theoretical results are presented at the end of each section.

5.1 Nonlinear sequential Caputo and Caputo-Hadamard fractional differential inclusions with three-point boundary conditions

In this section, we study the existence of solutions for nonlinear sequential Caputo and Caputo-Hadamard fractional differential inclusions with three-point boundary conditions as

$$\begin{cases} {}^{C}D^{\beta}_{a+}\left[{}^{C}_{H}\mathfrak{D}^{\alpha}_{a+}x\left(t\right)\right] \in \mathcal{F}\left(t, x\left(t\right)\right), \ t \in \left(a, b\right), \ a \geq 1, \\ x\left(a\right) = 0, \ x\left(b\right) = \lambda x\left(\eta\right), \ a < \eta < b, \end{cases} \tag{5.1}$$

where ${}^{C}_{H}\mathfrak{D}^{\alpha}_{a+}$ and ${}^{C}D^{\beta}_{a+}$ are the Caputo-Hadamard and Caputo fractional derivatives of orders α and β respectively, $0 < \alpha, \beta \leq 1$ and $\mathcal{F} : [a, b] \times \mathbb{R} \to \mathcal{P}\left(\mathbb{R}\right)$ is a multivalued map from $[a, b] \times \mathbb{R}$ to the family of $\mathcal{P}\left(\mathbb{R}\right) \subset \mathbb{R}$.

5.1.1 Existence results for multivalued problem

Let $J = [a, b]$. To obtain our desired results, we need the following auxiliary lemma.

Lemma 5.1 *Let* $\Lambda = \left(\lambda\frac{\left(\log\frac{\eta}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} - \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}\right) \neq 0$*. For any* $q \in C(J,\mathbb{R})$*, the unique solution of the boundary value problem*

$$\begin{cases} {}^{C}D^{\beta}_{a+}\left[{}^{C}_{H}\mathfrak{D}^{\alpha}_{a+}x(t)\right] = q(t), \ t \in (a,b), \\ x(a) = 0, \ x(b) = \lambda x(\eta), \ a < \eta < b, \end{cases} \tag{5.2}$$

is given by

$$\begin{aligned} x(t) = {} & {}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(t) + \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Lambda}\left({}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(b) - \lambda{}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(\eta)\right) \\ = {} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_a^s(s-\sigma)^{\beta-1}q(\sigma)\,d\sigma\right)\frac{ds}{s} \\ & + \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda}\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\left(\log\frac{b}{s}\right)^{\alpha-1}\left(\int_a^s(s-\sigma)^{\beta-1}q(\sigma)\,d\sigma\right)\frac{ds}{s}\right. \\ & \left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha-1}\left(\int_a^s(s-\sigma)^{\beta-1}q(\sigma)\,d\sigma\right)\frac{ds}{s}\right). \end{aligned} \tag{5.3}$$

Proof. Taking the Riemann-Liouville fractional integral of order β to the first equation of (5.2), we get

$${}^{C}_{H}\mathfrak{D}^{\alpha}_{a+}x(t) = {}^{RL}I^{\beta}_{a+}q(t) + c_0. \tag{5.4}$$

Again taking the Hadamard fractional integral of order α to the above equation, we obtain

$$x(t) = {}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(t) + \frac{\left[\log\left(\frac{t}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}c_0 + c_1. \tag{5.5}$$

Substituting $t = a$ in (5.5) and applying the first boundary condition of (5.2), it follows that $c_1 = 0$. For $t = b$ in (5.5) we get

$$x(b) = {}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(b) + \frac{\left[\log\left(\frac{b}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}c_0,$$

and for $t = \eta$, we have

$$x(\eta) = {}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(\eta) + \frac{\left[\log\left(\frac{\eta}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}c_0.$$

Using the second boundary condition of (5.2), we have

$${}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(b) + \frac{\left[\log\left(\frac{b}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}c_0 = \lambda{}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(\eta) + \lambda\frac{\left[\log\left(\frac{\eta}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}c_0. \tag{5.6}$$

By solving (5.6), we find that

$$\begin{aligned} c_0 &= \frac{1}{\left(\lambda\frac{\left[\log\left(\frac{\eta}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} - \frac{\left[\log\left(\frac{b}{a}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}\right)}\left({}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(b) - \lambda{}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(\eta)\right) \\ &= \frac{1}{\Lambda}\left({}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(b) - \lambda{}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}q\right)(\eta)\right). \end{aligned}$$

Replacing the values of c_0 and c_1 into (5.5), we get (5.3). The converse follows by direct computation which completes the proof. ■

5.1. Nonlinear sequential Caputo and Caputo-Hadamard fractional differential inclusions with three-point boundary conditions

Definition 5.1 A function $x \in AC(J,\mathbb{R})$ is said to be a solution of the problem (5.1) if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,x)$ for all $t \in J$ satisfying the boundary conditions

$$x(a) = 0,\ x(b) = \lambda x(\eta),\ a < \eta < b,$$

and

$$\begin{aligned} x(t) =\ & {}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}v\right)(t) + \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Lambda}\left({}^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}v\right)(b) - \lambda^{H}\mathfrak{J}^{\alpha}_{a+}\left({}^{RL}I^{\beta}_{a+}v\right)(\eta)\right) \\ =\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\,d\sigma\right)\frac{ds}{s} \\ & + \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda}\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\,d\sigma\right)\frac{ds}{s}\right. \\ & \left. -\frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)}\int_a^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\,d\sigma\right)\frac{ds}{s}\right). \end{aligned}$$

The upper semi-continuous case

Our first result, dealing with the convex valued $\mathcal{F}$, is based on Leray-Schauder nonlinear alternative for multivalued maps.

Theorem 5.1 *Set*

$$\Lambda_1 = \frac{(b-a)^{\beta}\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\left(1 + \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right), \tag{5.7}$$

and assume that

(A1) $\mathcal{F} : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory multivalued map,*

(A2) there exist a continuous nondecreasing function $Q : [0,\infty) \to (0,\infty)$ *and a function* $P \in C(J,\mathbb{R}^+)$ *such that*

$$\|\mathcal{F}(t,x)\|_{\mathcal{P}} = \sup\{|y| : y \in \mathcal{F}(t,x)\} \le P(t)\,Q(\|x\|_{\infty}),$$

for each $(t,x) \in J \times \mathbb{R}$,

(A3) there exists a constant $M > 0$ *such that*

$$\frac{M}{\Lambda_1 \|P\|_{\infty} Q(M)} > 1. \tag{5.8}$$

Then the boundary value problem (5.1) has at least one solution on J.

Proof. Firstly, we transform the problem (5.1) into a fixed point problem. Consider the multivalued map $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ defined by

$$N(x) = \left\{ \begin{array}{l} h \in C(J, \mathbb{R}), \\ h(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \\ + \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \right. \\ \left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right), \end{array} \right. \end{array} \right\}, \tag{5.9}$$

for $v \in S_{\mathcal{F},x}$. Clearly the fixed points of N are solutions of the problem (5.1). Now we proceed to show that the operator N satisfies all condition of Theorem 1.7. This is done in several steps.

Step 1. $N(x)$ is convex for each $x \in C(J, \mathbb{R})$.

Indeed, if h_1 and h_2 belong to $N(x)$, then there exist $v_1, v_2 \in S_{\mathcal{F},x}$ such that for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_i(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_i(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_i(\sigma)\, d\sigma\right) \frac{ds}{s}\right), \ i = 1, 2. \end{aligned}$$

Let $0 \leq \theta \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned} &[\theta h_1 + (1-\theta) h_2](t) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} [\theta v_1(\sigma) + (1-\theta) v_2(\sigma)]\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} [\theta v_1(\sigma) + (1-\theta) v_2(\sigma)]\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} [\theta v_1(\sigma) + (1-\theta) v_2(\sigma)]\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Since $\mathcal{F}$ has convex values, so $S_{\mathcal{F},x}$ is convex and $[\theta v_1(\sigma) + (1-\theta) v_2(\sigma)] \in S_{\mathcal{F},x}$. Thus, $\theta h_1 + (1-\theta) h_2 \in N(x)$.

Step 2. $N(x)$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

For a positive constant r, let $\Omega = \{x \in C(J, \mathbb{R}) : \|x\|_\infty \leq r\}$ be a bounded set in $C(J, \mathbb{R})$.

Then for each $h \in N(x)$, $x \in \Omega$, there exists $v \in S_{\mathcal{F},x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

In view of (A2), for each $t \in J$, we have

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s} \right. \\ &\left. + |\lambda| \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s}\right) \\ &\le \frac{\|P\|_\infty Q(r)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1}\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\|P\|_\infty Q(r)\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1}\, d\sigma\right) \frac{ds}{s} \right. \\ &\left. + \frac{|\lambda|}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1}\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Also, note that

$$\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1}\, d\sigma\right) \frac{ds}{s} \le \frac{(b-a)^{\beta}\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)}, \tag{5.10}$$

where we have used the fact that $(s-a)^{\beta} \le (b-a)^{\beta}$ for $0 < \beta \le 1$. Using the above arguments, we have

$$|h(t)| \le \frac{\|P\|_\infty Q(r)(b-a)^{\beta}\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} \left(1 + \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right).$$

Thus

$$\|h\|_\infty \le \Lambda_1 \|P\|_\infty Q(r).$$

Step 3. $N(x)$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let x be any element in Ω and $h \in N(x)$. Then there exists a function $v \in S_{\mathcal{F},x}$ such that for each $t \in J$, we have

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Let $t_1, t_2 \in J$, $t_1 < t_2$. Then

$$\begin{aligned} & |h(t_2) - h(t_1)| \\ & \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right) \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s} \\ & + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s} \\ & + \frac{\left(\log \frac{t_2}{a}\right)^{\alpha} - \left(\log \frac{t_1}{a}\right)^{\alpha}}{|\Lambda|\Gamma(\alpha+1)\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s}\right. \\ & \left. + |\lambda| \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} |v(\sigma)|\, d\sigma\right) \frac{ds}{s}\right) \\ & \leq \frac{\|P\|_\infty Q(r)(b-a)^\beta}{\Gamma(\beta+1)\Gamma(\alpha)} \left[\int_a^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right) \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s}\right] \\ & + \frac{\left(\log \frac{t_2}{a}\right)^{\alpha} - \left(\log \frac{t_1}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \frac{\|P\|_\infty Q(r)(b-a)^\beta \left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} (1+|\lambda|) \\ & \leq \frac{\|P\|_\infty Q(r)(b-a)^\beta}{\Gamma(\beta+1)\Gamma(\alpha+1)} \left(2\left(\log \frac{t_2}{t_1}\right)^{\alpha}\right) \\ & + \frac{\left(\log \frac{t_2}{a}\right)^{\alpha} - \left(\log \frac{t_1}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \frac{\|P\|_\infty Q(r)(b-a)^\beta \left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} (1+|\lambda|). \end{aligned}$$

The right hand side of the above inequality tends to zero independently of $x \in \Omega$ as $t_1 \to t_2$. As a consequence of Steps 1–3 together with Arzela-Ascoli theorem, we conclude that $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Since N is completely continuous, it is enough to show that it has a closed graph in view of Lemma 1.22, which will imply that N is u.s.c. This is done in the following step.

Step 4. N has a closed graph.

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in N(x_*)$. Observe that $h_n \in N(x_n)$ implies that there exists $v_n \in S_{\mathcal{F},x_n}$ such that for each $t \in J$,

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_n(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_n(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_n(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Therefore, we must show that there exists $v_* \in S_{\mathcal{F},x_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_*(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_*(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_*(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Consider the continuous linear operator $\Theta : L^1(J,X) \to C(J,X)$ defined by

$$\begin{aligned} v \to \Theta(v)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n - h_*\|_\infty \\ &= \left\| \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} (v_n(\sigma) - v_*(\sigma))\, d\sigma\right) \frac{ds}{s} \right. \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Lambda\Gamma(\alpha+1)\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} (v_n(\sigma) - v_*(\sigma))\, d\sigma\right) \frac{ds}{s}\right. \\ &\left.\left. - \lambda \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} (v_n(\sigma) - v_*(\sigma))\, d\sigma\right) \frac{ds}{s} \right\|_\infty \to 0, \end{aligned}$$

as $n \to \infty$. So it follows from Lemma 1.23, that $\Theta \circ S_{\mathcal{F},x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta\left(S_{\mathcal{F},x_n}\right).$$

Since $x_n \to x_*$, Lemma 1.23 implies that

$$h_*(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_*(\sigma)\, d\sigma\right) \frac{ds}{s}$$
$$+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_*(\sigma)\, d\sigma\right) \frac{ds}{s}\right.$$
$$\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_*(\sigma)\, d\sigma\right) \frac{ds}{s}\right),$$

for some $v_* \in S_{\mathcal{F},x_*}$.

Step 5. We show there exists an open set $\mathcal{U} \subseteq C(J,\mathbb{R})$ with $x \notin \mu N(x)$ for any $\mu \in (0,1)$ and all $x \in \partial\mathcal{U}$.

Let $\mu \in (0,1)$ and $x \in \mu N(x)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{\mathcal{F},x}$ such that, for $t \in J$, we have

$$x(t) = \frac{\mu}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}$$
$$+ \frac{\mu\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right.$$
$$\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right).$$

Using the method of computation employed in Step 2, for each $t \in J$, we get

$$|x(t)| \leq \Lambda_1 \|P\|_\infty Q(\|x\|_\infty),$$

which can alternatively be written as

$$\frac{\|x\|_\infty}{\Lambda_1 \|P\|_\infty Q(\|x\|_\infty)} \leq 1.$$

In view of (A3), there exists M such that $\|x\|_\infty \neq M$. Let us set

$$\mathcal{U} = \{x \in C(J,\mathbb{R}) : \|x\|_\infty < M\}.$$

Note that the operator $N : \overline{\mathcal{U}} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x \in \mu N(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type 1.7, we deduce that N has a fixed point $x \in \overline{\mathcal{U}}$ which is a solution of the boundary value problem (5.1). This completes the proof. ■

The Lipschitz case

Now we prove the existence of solutions for the boundary value problem (5.1) with nonconvex-valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler 1.8.

Theorem 5.2 *Assume that the following condition hold*

(A4) $\mathcal{F} : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $\mathcal{F}(.,x) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$,

(A5) $H_d(\mathcal{F}(t,x), \mathcal{F}(t,\overline{x})) \leq m(t)|x-\overline{x}|$ *for almost all* $t \in J$ *and* $x, \overline{x} \in \mathbb{R}$ *with* $m \in C(J,\mathbb{R}^+)$ *and* $d(0,\mathcal{F}(t,0)) \leq m(t)$ *for almost all* $t \in J$.

Then the boundary value problem (5.1) has at least one solution on J if

$$\Lambda_1 \|m\|_\infty < 1,$$

where Λ_1 *is given by (5.7).*

Proof. Observe that the set $S_{\mathcal{F},x}$ is nonempty for each $x \in C(J,\mathbb{R})$ by assumption (A4), so $\mathcal{F}$ has a measurable selection (see [25, Theorem III.6]). Now we show that the operator $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined in (5.9) satisfies the assumptions of Theorem 1.8. To show that $N(x)$ is closed for each $x \in C(J,\mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C(J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{\mathcal{F},x}$ such that, for each $t \in J$,

$$\begin{aligned} u_n(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_n(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log\frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_n(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_n(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

As $\mathcal{F}$ has compact values, we pass onto a subsequence to obtain that $\{v_n\}$ converges to v in $L^1(J,\mathbb{R})$. Thus $v \in S_{\mathcal{F},x}$ and for each $t \in J$, we have

$$\begin{aligned} u_n(t) \to u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log\frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

Hence $u \in N(x)$.

Next we show that there exists $0 < \tau < 1$, $(\tau = \Lambda_1 \|m\|_\infty)$ such that

$$H_d(N(x), N(\overline{x})) \leq \tau \|x-\overline{x}\|_\infty \text{ for each } x, \overline{x} \in C(J,\mathbb{R}).$$

Let $x, \overline{x} \in C(J,\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in \mathcal{F}(t, x(t))$ such that, for each

$t \in J$,

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_1(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_1(\sigma)\, d\sigma\right) \frac{ds}{s} \right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_1(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

By (A5), we have

$$H_d(\mathcal{F}(t, x(t)), \mathcal{F}(t, \overline{x}(t))) \le m(t)\,|x(t) - \overline{x}(t)|.$$

Therefore, there exists $w \in \mathcal{F}(t, \overline{x}(t))$ such that

$$|v_1(t) - w| \le m(t)\,|x(t) - \overline{x}(t)|, \ t \in J.$$

Define $\mathcal{U} : J \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{U}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)\,|x(t) - \overline{x}(t)|\}.$$

Since the multivalued operator $\mathcal{U}(t) \cap \mathcal{F}(t, \overline{x}(t))$ is measurable (see [25, Proposition III.4]), there exists a function v_2 which is a measurable selection for $\mathcal{U}$. So $v_2(t) \in \mathcal{F}(t, \overline{x}(t))$ and or each $t \in J$, we have $|v_1(t) - v_2(t)| \le m(t)\,|x(t) - \overline{x}(t)|$.

For each $t \in J$, let us define

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_2(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_2(\sigma)\, d\sigma\right) \frac{ds}{s} \right. \\ &\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} v_2(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned}$$

In consequence, we get

$$
\begin{aligned}
&\left|h_1(t)-h_2(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1}\left|v_1(\sigma)-v_2(\sigma)\right|d\sigma\right)\frac{ds}{s} \\
&+\frac{\left(\log\frac{t}{a}\right)^{\alpha}}{|\Lambda|\,\Gamma(\alpha+1)\Gamma(\alpha)\Gamma(\beta)}\left(\int_a^b\left(\log\frac{b}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1}\left|v_1(\sigma)-v_2(\sigma)\right|d\sigma\right)\frac{ds}{s}\right. \\
&\left.+|\lambda|\int_a^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1}\left|v_1(\sigma)-v_2(\sigma)\right|d\sigma\right)\frac{ds}{s}\right) \\
&\leq \|m\|_\infty\|x-\overline{x}\|_\infty\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1}d\sigma\right)\frac{ds}{s}\right. \\
&+\frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|}\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\left(\log\frac{b}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1}d\sigma\right)\frac{ds}{s}\right. \\
&\left.\left.+\frac{|\lambda|}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha-1}\left(\int_a^s (s-\sigma)^{\beta-1}d\sigma\right)\frac{ds}{s}\right)\right),
\end{aligned}
$$

by (5.10), we have

$$
\left|h_1(t)-h_2(t)\right| \leq \frac{\|m\|_\infty\|x-\overline{x}\|_\infty(b-a)^{\beta}\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\left[1+\frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right].
$$

Hence

$$
\|h_1-h_2\|_\infty \leq \Lambda_1\|m\|_\infty\|x-\overline{x}\|_\infty.
$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$
H_d(N(x),N(\overline{x})) \leq \Lambda_1\|m\|_\infty\|x-\overline{x}\|_\infty.
$$

Since N is a contraction, it follows from Theorem 1.8 that N has a fixed point x which is a solution of (5.1). This completes the proof. ■

5.1.2 Examples

In this part, we present two examples to validate the existence results.

Example 5.1 Consider the sequential fractional boundary value problem

$$
\begin{cases}
{}^{C}D_{1+}^{\frac{1}{2}}\left[{}_{H}^{C}\mathfrak{D}_{1+}^{\frac{1}{4}}x(t)\right]\in\mathcal{F}(t,x),\ t\in(1,2), \\
x(1)=0,\ x(2)=\frac{1}{8}x\left(\frac{3}{2}\right),
\end{cases}
\tag{5.11}
$$

where $a=1$, $b=2$, $\alpha=\frac{1}{4}$, $\beta=\frac{1}{2}$, $\lambda=\frac{1}{8}$, $\eta=\frac{3}{2}$ and $\mathcal{F}:[1,2]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$
x\to\mathcal{F}(t,x)=\left[\frac{1}{(t^3+4\exp(t))}\frac{x^2}{6(x^2+1)},\frac{1}{2\sqrt{t+3}}\frac{|x|}{|x|+1}\right].
$$

With these date we find $\Lambda = -0.89662 \neq 0$. Clearly the multivalued map $\mathcal{F}$ satisfies condition (A1) and that

$$\left\|\mathcal{F}(t,x)\right\|_{\mathcal{P}} = \sup\left\{|y| : y \in \mathcal{F}(t,x)\right\} \leq \frac{1}{2\sqrt{t+3}} = P(t)\,Q\left(\|x\|_{\infty}\right),$$

which yields $\|P\|_{\infty} = \frac{1}{4}$ and $Q\left(\|x\|_{\infty}\right) = 1$. Therefore, the condition (A2) is fulfilled. By the condition (A3), it found that $M > 0.64265$. Hence all assumptions of Theorem 5.1 hold. So there exists at least one solution of the problem (5.11) on $[1, 2]$.

Example 5.2 Consider the sequential fractional boundary value problem

$$\begin{cases} {}^{C}D_{1+}^{\frac{1}{2}}\left[{}_{H}^{C}\mathfrak{D}_{1+}^{\frac{2}{3}}x(t)\right] \in \mathcal{F}(t,x),\ t \in (1,2), \\ x(1) = 0,\ x(2) = \frac{1}{10}x\left(\frac{3}{2}\right). \end{cases} \tag{5.12}$$

Here $a = 1$, $b = 2$, $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$, $\lambda = \frac{1}{10}$, $\eta = \frac{3}{2}$ and $\mathcal{F} : [1,2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to \mathcal{F}(t,x) = \left[0, \frac{2\sin(x)}{(t^2+7)} + \frac{1}{12}\right].$$

With these date we find $\Lambda = -0.80691 \neq 0$. Clearly $H_d\left(\mathcal{F}(t,x), \mathcal{F}(t,\overline{x})\right) \leq m(t)\,|x - \overline{x}|$, where $m(t) = \frac{2}{t^2+7}$. Also $d\left(0, \mathcal{F}(t,0)\right) = \frac{1}{12} \leq m(t)$ for almost all $t \in [1,2]$. In addition, we get $\|m\|_{\infty} = \frac{1}{4}$ which leads to $\Lambda_1 \|m\|_{\infty} \approx 0.53 < 1$. As the hypothesis of Theorem 5.2 is satisfied, therefore we conclude that the multivalued problem (5.12) has at least one solution on $[1, 2]$.

5.2 Hilfer fractional differential inclusions with Erdélyi-Kober fractional integral boundary condition

In this section, we discuss the existence of solutions for a nonlinear Hilfer fractional differential inclusion with Erdélyi-Kober fractional integral boundary conditions as follows

$$\begin{cases} {}^{\mathcal{H}}D_{0+}^{\alpha,\beta}x(t) \in \mathcal{F}(t, x(t)),\ t \in (0,T),\ T > 0, \\ x(0) = 0,\ x(T) = \sum\limits_{i=1}^{m} \theta_i\ {}^{EK}I_{0+,\gamma_i}^{\eta_i;\xi_i}x(\delta_i), \end{cases} \tag{5.13}$$

where ${}^{\mathcal{H}}D_{0+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha \in (1,2)$ and type $\beta \in [0,1]$, ${}^{EK}I_{0+,\gamma_i}^{\eta_i;\xi_i}$ is the Erdelyi-Kober fractional integral of order $\xi_i > 0$ with $\gamma_i > 0$ and $\eta_i \in \mathbb{R}$, $\mathcal{F} : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map from $[0,T] \times \mathbb{R}$ to the family of $\mathcal{P}(\mathbb{R}) \subset \mathbb{R}$, $\theta_i \in \mathbb{R}$ and $\delta_i \in (0,T)$, $i = 1, 2, \ldots, m$.

Remark 5.1

i) In problem (5.13). If we replace Erdelyi-Kober fractional integral ${}^{EK}I^{\eta_i;\xi_i}_{0+,\gamma_i}$ with Riemann-Liouville fractional integral ${}^{RL}I^{\eta_i}_{o+}$, then inclusion problem (5.13) has been studied by Wongcharoen et al., in [99].

ii) If $\beta = 0$ in (5.13), then our problem (5.13) reduces to Riemann-Liouville inclusion problem considered by Ahmad and Ntouyas in [6].

iii) If $\beta = 1$ in (5.13), then our problem (5.13) reduces to Caputo inclusion problem.

5.2.1 Existence results for multivalued problem

Let $J = [0, T]$. To obtain our desired results, we need the following auxiliary lemma.

Lemma 5.2 ([1]) *Let*

$$\Lambda = T^{\mathfrak{v}-1} - \sum_{i=1}^{m} \frac{\theta_i \delta_i^{\mathfrak{v}-1} \Gamma\left(\eta_i + \frac{\mathfrak{v}-1}{\gamma_i} + 1\right)}{\Gamma\left(\eta_i + \frac{\mathfrak{v}-1}{\gamma_i} + \xi_i + 1\right)} \neq 0, \text{ where } \mathfrak{v} = \alpha + \beta(2-\alpha), \tag{5.14}$$

and for any $q \in C(J, \mathbb{R})$, then the solution of nonlocal boundary value problem

$$\begin{cases} {}^{\mathcal{H}}D^{\alpha,\beta}_{0+} x(t) = q(t), \ t \in (0,T), \ T > 0, \\ x(0) = 0, \ x(T) = \sum\limits_{i=1}^{m} \theta_i \ {}^{EK}I^{\eta_i;\xi_i}_{0+,\gamma_i} x(\delta_i), \end{cases} \tag{5.15}$$

is obtained as

$$x(t) = {}^{RL}I^{\alpha}_{0+} q(t) + \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i^{EK} I^{\eta_i;\xi_i}_{0+,\gamma_i} \ {}^{RL}I^{\alpha}_{0+} q(\delta_i) - {}^{RL}I^{\alpha}_{0+} q(T) \right). \tag{5.16}$$

Definition 5.2 A function $x \in C(J, \mathbb{R})$ is considered as a solution of (5.13), if there is an integrable function $v \in L^1(J, \mathbb{R})$ with $v(t) \in \mathcal{F}(t, x)$ for all $t \in J$ satisfying the nonlocal boundary conditions

$$x(0) = 0, \ x(T) = \sum_{i=1}^{m} \theta_i \ {}^{EK}I^{\eta_i;\xi_i}_{0+,\gamma_i} x(\delta_i),$$

and

$$\begin{aligned} x(t) &= {}^{RL}I^{\alpha}_{0+} v(t) + \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i^{EK} I^{\eta_i;\xi_i}_{0+,\gamma_i} \ {}^{RL}I^{\alpha}_{0+} v(\delta_i) - {}^{RL}I^{\alpha}_{0+} v(T) \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds \right). \end{aligned}$$

The upper semi-continuous case

The first outcome deals with the convex valued $\mathcal{F}$ relying on Leray-Schauder nonlinear alternative for set-valued maps.

Theorem 5.3 *Let*

$$\varrho = \frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\mathfrak{v}+\alpha-1}}{|\Lambda|} + \frac{T^{\mathfrak{v}-1}}{|\Lambda|}\left(\sum_{i=1}^{m}|\theta_i|\frac{\delta_i^{\alpha}\Gamma\left(\eta_i+\frac{\alpha}{\gamma_i}+1\right)}{\Gamma\left(\eta_i+\frac{\alpha}{\gamma_i}+\xi_i+1\right)}\right)\right), \tag{5.17}$$

and assume that:

(As1) $\mathcal{F} : J\times\mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a L^1-Carathéodory set-valued map,

(As2) There is a nondecreasing function $\vartheta \in C(\mathbb{R}^+,\mathbb{R}^+)$ and a continuous function $P : J \to \mathbb{R}^+$ such that

$$\|\mathcal{F}(t,x)\|_{\mathcal{P}} = \sup\{|\rho| : \rho\in\mathcal{F}(t,x)\} \leq P(t)\,\vartheta(\|x\|_{\infty}),\ \forall (t,x)\in J\times\mathbb{R}.$$

(As3) There is a constant $\mathcal{L} > 0$ such that

$$\frac{\mathcal{L}}{\varrho\,\|P\|_{\infty}\,\vartheta(\mathcal{L})} > 1. \tag{5.18}$$

Then the problem (5.13) has at least one solution on J.

Proof. Initially, to switch the problem (5.13) into a fixed point problem, we consider the operator $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ as

$$N(x) = \left\{\begin{array}{l} h\in C(J,\mathbb{R}), \\ h(t) = \left\{\begin{array}{l} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\,ds \\ +\frac{t^{\mathfrak{v}-1}}{\Lambda}\left(\sum\limits_{i=1}^{m}\theta_i\frac{\gamma_i\delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)}\int_0^{\delta_i}\frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{\left(\delta_i^{\gamma_i}-s^{\gamma_i}\right)}\left(\int_0^s (s-\sigma)^{\alpha-1}v(\sigma)\,d\sigma\right)ds \right.\\ \left.-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)\,ds\right), \end{array}\right. \end{array}\right\} \tag{5.19}$$

for $v\in\mathcal{S}_{\mathcal{F},x}$. Obviously, the solution of (5.13) is as a fixed point of the operator N. The proof steps will be presented as follows:

Step 1. The set-valued map $N(x)$ is convex for any $x\in C(J,\mathbb{R})$.

Let h_1, $h_2\in N(x)$. Then, there exist v_1, $v_2\in\mathcal{S}_{\mathcal{F},x}$ such that

$$\begin{aligned} h_j(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\,ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda}\left(\sum_{i=1}^{m}\theta_i\frac{\gamma_i\delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)}\int_0^{\delta_i}\frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i}-s^{\gamma_i})}\left(\int_0^s (s-\sigma)^{\alpha-1}v_j(\sigma)\,d\sigma\right)ds \right.\\ &\left.-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v_j(s)\,ds\right),\ j=1,2,\ \forall t\in J. \end{aligned}$$

Let $\lambda \in [0,1]$. Then for any $t \in J$, we have

$$\begin{aligned}
&[\lambda h_1 + (1-\lambda) h_2](t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\lambda v_1(s) + (1-\lambda) v_2(s)]\, ds \\
&+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} [\lambda v_1(\sigma) + (1-\lambda) v_2(\sigma)]\, d\sigma \right) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [\lambda v_1(s) + (1-\lambda) v_2(s)]\, ds \right).
\end{aligned}$$

Since $\mathcal{F}$ has convex values, $\mathcal{S}_{\mathcal{F},x}$ is convex and $[\lambda v_1(t) + (1-\lambda) v_2(t)] \in \mathcal{S}_{\mathcal{F},x}$. Thus, $\lambda h_1 + (1-\lambda) h_2 \in N(x)$.

Step 2. N is bounded on bounded sets of $C(J,\mathbb{R})$.

For a constant $r > 0$, let $\Omega = \{x \in C(J,\mathbb{R}) : \|x\|_\infty \le r\}$ be a bounded set in $C(J,\mathbb{R})$. Then for each $h \in N(x)$ and $x \in \Omega$, there exists $v \in \mathcal{S}_{\mathcal{F},x}$ such that

$$\begin{aligned}
h(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\
&+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v(\sigma)\, d\sigma \right) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds \right).
\end{aligned}$$

Under the hypothesis (As2) and for any $t \in J$, we attain

$$\begin{aligned}
&|h(t)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|\, ds \\
&+ \frac{t^{\mathfrak{v}-1}}{|\Lambda|} \left(\sum_{i=1}^{m} |\theta_i| \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} |v(\sigma)|\, d\sigma \right) ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v(s)|\, ds \right) \\
&\le \frac{\|P\|_\infty \vartheta(r)}{\Gamma(\alpha+1)} \left(T^\alpha + \frac{T^{\mathfrak{v}+\alpha-1}}{|\Lambda|} + \frac{T^{\mathfrak{v}-1}}{|\Lambda|} \left(\sum_{i=1}^{m} |\theta_i| \frac{\delta_i^\alpha \Gamma\left(\eta_i + \frac{\alpha}{\gamma_i} + 1\right)}{\Gamma\left(\eta_i + \frac{\alpha}{\gamma_i} + \xi_i + 1\right)} \right) \right).
\end{aligned}$$

Thus

$$\|h\|_\infty \le \varrho \|P\|_\infty \vartheta(r).$$

Step 3. N sends bounded sets of $C(J,\mathbb{R})$ into equicontinuous sets.

Let $x \in \Omega$ and $h \in N(x)$. Then there is a function $v \in \mathcal{S}_{\mathcal{F},x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds$$
$$+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v(\sigma)\, d\sigma \right) ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds \right).$$

Let $t_1, t_2 \in J$, $t_1 < t_2$. Then

$$|h(t_2) - h(t_1)|$$
$$\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right] |v(s)|\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |v(s)|\, ds$$
$$\frac{\left(t_2^{\mathfrak{v}-1} - t_1^{\mathfrak{v}-1}\right)}{|\Lambda|} \left(\sum_{i=1}^m |\theta_i| \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} |v(\sigma)|\, d\sigma \right) ds \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds \right)$$
$$\le \frac{\|P\|_\infty \vartheta(r)}{\Gamma(\alpha+1)} \left((t_2^\alpha - t_1^\alpha) + \frac{\left(t_2^{\mathfrak{v}-1} - t_1^{\mathfrak{v}-1}\right)}{|\Lambda|} \left(\sum_{i=1}^m |\theta_i| \frac{\delta_i^\alpha \Gamma\left(\eta_i + \frac{\alpha}{\gamma_i} + 1\right)}{\Gamma\left(\eta_i + \frac{\alpha}{\gamma_i} + \xi_i + 1\right)} + T^\alpha \right) \right).$$

As $t_1 \to t_2$, we obtain

$$|h(t_2) - h(t_1)| \to 0.$$

Hence $N(\Omega)$ is equicontinuous. From the above-mentioned steps $(2-3)$ along with theorem of Arzela-Ascoli, we infer that N is completely continuous.

Step 4. We prove that the graph of N is closed.

Let $x_n \to x_*$, $h_n \in N(x_n)$ and h_n tends to h_*. We show that $h_* \in N(x_*)$. Since $h_n \in N(x_n)$, there exists $v_n \in \mathcal{S}_{\mathcal{F},x_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)\, ds$$
$$+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v_n(\sigma)\, d\sigma \right) ds \right.$$
$$\left. - \tfrac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s)\, ds \right),\ t \in J.$$

Therefore, we have to prove that there exists $v_* \in \mathcal{S}_{\mathcal{F},x_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_*(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v_*(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_*(s)\, ds \right). \end{aligned}$$

Define the continuous linear operator $\Theta : L^1(J,x) \to C(J,x)$ as follows

$$\begin{aligned} v \to \Theta(v)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^T (T-s)^{\alpha-1} v(s)\, ds \right),\ t \in J. \end{aligned}$$

Notice that

$$\begin{aligned} &\|h_n - h_*\|_\infty \\ &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (v_n(s) - v_*(s))\, ds \right. \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} (v_n(\sigma) - v_*(\sigma))\, d\sigma \right) ds \right. \\ &\left.\left. - \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^T (T-s)^{\alpha-1} (v_n(s) - v_*(s))\, ds ds \right) \right\|_\infty \to 0, \end{aligned}$$

when $n \to \infty$. So in view of Lemma (1.23) that $\Theta \circ \mathcal{S}_{\mathcal{F},x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(\mathcal{S}_{\mathcal{F},x_n}).$$

Since $x_n \to x_*$, Lemma (1.23) gives

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_*(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v_*(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_*(s)\, ds \right). \end{aligned}$$

for some $v_* \in \mathcal{S}_{\mathcal{F},x_*}$.

Step 5. We show there exists an open set $\mathcal{U} \subseteq C(J,\mathbb{R})$ with $x \notin \mu N(x)$ for each $0 < \mu < 1$ and $\forall\ x \in \partial\mathcal{U}$.

5.2. Hilfer fractional differential inclusions with Erdélyi-Kober fractional integral boundary condition

Let $\mu \in (0,1)$ and $x \in \mu N(x)$. Then there exists $v \in \mathcal{S}_{\mathcal{F},x}$ such that

$$\begin{aligned}
|x(t)| = & \Bigg| \frac{\mu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\
& + \frac{\mu t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v(\sigma)\, d\sigma \right) ds \right. \\
& \left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds \right) \Bigg| \\
& \leq \varrho \|P\|_\infty \vartheta(\|x\|_\infty).
\end{aligned}$$

Thus, we have

$$|x(t)| \leq \varrho \|P\|_\infty \vartheta(\|x\|_\infty), \ \forall t \in J,$$

consequently, we obtain

$$\frac{\|x\|_\infty}{\varrho \|P\|_\infty \vartheta(\|x\|_\infty)} \leq 1.$$

Under the hypothesis (As3), there is a $\mathcal{L} > 0$ such that $\|x\|_\infty \neq \mathcal{L}$. We build the set $\mathcal{U}$ as follows

$$\mathcal{U} = \{x \in C(J,\mathbb{R}) : \|x\|_\infty < \mathcal{L}\}.$$

From the steps 1-4, the operator $N : \overline{\mathcal{U}} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous, and completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x \in \mu N(x)$ for some $\mu \in (0,1)$. So, by Leray-Schauder theorem for set-valued maps, we infer that problem (5.13) has at least one solution $x \in \overline{\mathcal{U}}$. ■

The Lipschitz case

For further existence investigation of problem (5.13). In this part, we deal with another existence criterion under new hypotheses. In what follows, we will demonstrate that our desired existence of solutions in the case of nonconvex-valued right-hand side follows by Covitz and Nadler theorem 1.8.

Theorem 5.4 *Suppose the following hypotheses are valid*

(As4) $\mathcal{F} : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $\mathcal{F}(.,x) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for any* $x \in \mathbb{R}$,

(As5) $\mathcal{H}_d(\mathcal{F}(t,x), \mathcal{F}(t,\overline{x})) \leq \varpi(t)|x-\overline{x}|$ *for (a.e.) all* $t \in J$ *and* $x, \overline{x} \in \mathbb{R}$ *with* $\varpi \in C(J,\mathbb{R}^+)$ *and* $d(0,\mathcal{F}(t,0)) \leq \varpi(t)$ *for (a.e.) all* $t \in J$.

Then, (5.13) has at least one solution on J if

$$\varrho \|\varpi\|_\infty < 1,$$

where ϱ *is defined in (5.17).*

Proof. By using the hypothesis (As4) and Theorem III.6 in [25], $\mathcal{F}$ has a measurable selection $v : J \to \mathbb{R}$, $v \in L^1(J, \mathbb{R})$ and so $\mathcal{F}$ is integrably bounded. Thus, $\mathcal{S}_{\mathcal{F},x} \neq \varnothing$. Now, we show that $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ defined in (5.19) satisfies the hypotheses of fixed point theorem of Nadler and Covitz. To prove that $N(x)$ is closed for any $x \in C(J, \mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $\mathfrak{u}_n \to \mathfrak{u}$ $(n \to \infty)$ in $C(J, \mathbb{R})$. Then $\mathfrak{u} \in C(J, \mathbb{R})$ and there is $v_n \in \mathcal{S}_{\mathcal{F},x_n}$ such that

$$\begin{aligned}\mathfrak{u}_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v_n(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s)\, ds \right), \ \forall t \in J.\end{aligned}$$

As $\mathcal{F}$ has compact values, so there exists a subsequence v_n converges to v in $L^1(J, \mathbb{R})$. Thus $v \in \mathcal{S}_{\mathcal{F},x}$ and we get

$$\begin{aligned}\mathfrak{u}_n(t) \to \mathfrak{u}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds \right), \ \forall t \in J.\end{aligned}$$

Hence $u \in N(x)$.

Next, we prove that there is a $\vartheta \in (0,1)$, $(\vartheta = \varrho \|\varpi\|_\infty)$ such that

$$H_d(N(x), N(\overline{x})) \leq \vartheta \|x - \overline{x}\|_\infty \text{ for each } x, \overline{x} \in C(J, \mathbb{R}).$$

Let $x, \overline{x} \in C(J, \mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in \mathcal{F}(t, x(t))$ such that

$$\begin{aligned}h_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s)\, ds \\ &+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^{m} \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v_1(\sigma)\, d\sigma \right) ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_1(s)\, ds \right).\end{aligned}$$

By (As5), we have

$$\mathcal{H}_d(\mathcal{F}(t,x), \mathcal{F}(t,\overline{x})) \leq \varpi(t) |x(t) - \overline{x}(t)|.$$

So, there exists $\tilde{w}(t) \in \mathcal{F}(t, \overline{x})$ such that

$$|v_1(t) - \tilde{w}| \leq \varpi(t) |x(t) - \overline{x}(t)|, \ t \in J.$$

We construct a set-valued map $\mathcal{E} : J \to \mathcal{P}(\mathbb{R})$ as follows

$$\mathcal{E}(t) = \{\tilde{w} \in \mathbb{R} : |v_1(t) - \tilde{w}| \leq \varpi(t) |x(t) - \overline{x}(t)|\}.$$

We see that v_1 and $\sigma = \varpi |x - \overline{x}|$ are measurable, therefore we can conclude that the set-valued map $\mathcal{E}(t) \cap \mathcal{F}(t, \overline{x})$ is measurable. Now, we choose the function $v_2(t) \in \mathcal{F}(t, \overline{x})$ such that

$$|v_1(t) - v_2(t)| \leq \varpi(t) |x(t) - \overline{x}(t)|, \ \forall t \in J.$$

We define

$$\begin{aligned}
h_2(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_2(s)\, ds \\
&+ \frac{t^{\mathfrak{v}-1}}{\Lambda} \left(\sum_{i=1}^m \theta_i \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} v_2(\sigma)\, d\sigma \right) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_2(s)\, ds \right), \ \forall t \in J.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
&|h(t) - h_2(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_1(s) - v_2(s)|\, ds \\
&+ \frac{T^{\mathfrak{v}-1}}{|\Lambda|} \left(\sum_{i=1}^m |\theta_i| \frac{\gamma_i \delta_i^{-\gamma_i(\xi_i+\eta_i)}}{\Gamma(\xi_i)\Gamma(\alpha)} \int_0^{\delta_i} \frac{s^{\gamma_i+\eta_i+\gamma_i-1}}{(\delta_i^{\gamma_i} - s^{\gamma_i})} \left(\int_0^s (s-\sigma)^{\alpha-1} |v_1(\sigma) - v_2(\sigma)|\, d\sigma \right) ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v_1(s) - v_2(s)|\, ds \right) \\
&\leq \frac{\|\varpi\|_\infty \|x - \overline{x}\|_\infty}{\Gamma(\alpha+1)} \left(T^\alpha + \frac{T^{\mathfrak{v}+\alpha-1}}{|\Lambda|} + \frac{T^{\mathfrak{v}-1}}{|\Lambda|} \left(\sum_{i=1}^m |\theta_i| \frac{\delta_i^\alpha \Gamma\left(\eta_i + \frac{\alpha}{\gamma_i} + 1\right)}{\Gamma\left(\eta_i + \frac{\alpha}{\gamma_i} + \xi_i + 1\right)} \right) \right).
\end{aligned}$$

Therefore

$$\|h_1 - h_2\|_\infty \leq \varrho \|\varpi\|_\infty \|x - \overline{x}\|_\infty.$$

Similarly, interchanging the roles of x and $\overline{x}$, we get

$$\mathcal{H}_d(N(x), N(\overline{x})) \leq \varrho \|\varpi\|_\infty \|x - \overline{x}\|_\infty.$$

Since N is a contraction, in the light of Covitz and Nadler theorem, we infer that N has a fixed point x which is a solution of (5.13). ■

5.2.2 Examples

In this portion, we consider the following fractional differential inclusion

$$\begin{cases} {}^{\mathcal{H}}D_{0+}^{\alpha,\beta} x(t) = q(t), \ t \in (0,T), \ T > 0, \\ x(0) = 0, \ x(T) = \sum\limits_{i=1}^m \theta_i \ {}^{EK}I_{0+,\gamma_i}^{\eta_i;\xi_i} x(\delta_i). \end{cases} \tag{5.20}$$

The next examples are special cases of fractional differential inclusion given by (5.20).

Example 5.3 Consider the fractional differential inclusion given by (5.20). Taking $\alpha = \frac{5}{4}$, $\beta = 0$, $T = 1$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{6}$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{5}{2}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{3}{2}$, $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{1}{8}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$ Then, the problem (5.20) reduce to

$$\begin{cases} {}^{\mathcal{H}}D_{0+}^{\frac{5}{4},0} x(t) \in \mathcal{F}(t,x),\ t \in (0,1), \\ x(0) = 0,\ x(1) = \frac{1}{4} I_{0+,\frac{1}{6}}^{\frac{1}{2};\frac{1}{2}} x\left(\frac{1}{4}\right) + \frac{1}{6} I_{\frac{1}{8}}^{\frac{5}{2};\frac{3}{2}} x\left(\frac{1}{2}\right), \end{cases} \tag{5.21}$$

which is fractional differential inclusion involving Riemann-Liouville fractional derivative. In this case $\mathfrak{v} = \frac{5}{4}$. Let $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a mapping such that

$$x \to \mathcal{F}(t,x) = \left[\frac{1}{6(t^2 + 4\exp(t^3))} \frac{x^2}{(x^2+1)}, \frac{1}{2\sqrt{t+9}} \frac{|x|}{|x|+1} \right]. \tag{5.22}$$

With these date we find $\Lambda \simeq 0.88343 \neq 0$. Obviously, $\mathcal{F}$ satisfies hypothesis (As1) and

$$\|\mathcal{F}(t,x)\|_{\mathcal{P}} = \sup\{|\rho| : \rho \in \mathcal{F}(t,x)\} \leq \frac{1}{2\sqrt{t+9}} = P(t)\,\vartheta(\|x\|_\infty),$$

where $\|P\|_\infty = \frac{1}{6}$ and $\vartheta(\|x\|_\infty) = 1$. Thus, the assumption (As2) is fulfilled, and by (As3), we get $\mathcal{L} > 0.31633$.

Therefore all hypotheses of Theorem (5.1) are valid. Hence the problem (5.21) with $\mathcal{F}$ given by (5.22) has at least one solution on $[0,1]$.

Example 5.4 Consider the fractional differential inclusion given by (5.20). Taking $\alpha = \frac{3}{2}$, $\beta = 1$, $T = 1$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{4}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{2}$, $\xi_1 = \frac{1}{6}$, $\xi_2 = \frac{3}{2}$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{8}$, $\delta_1 = \frac{1}{6}$, $\delta_2 = \frac{1}{4}$. Then, the problem (5.20) reduce to

$$\begin{cases} {}^{\mathcal{H}}D_{0+}^{\frac{5}{4},0} x(t) \in \mathcal{F}(t,x),\ t \in (0,1), \\ x(0) = 0,\ x(1) = \frac{1}{2} I_{\frac{1}{2}}^{\frac{1}{4};\frac{1}{6}} x\left(\frac{1}{6}\right) + \frac{1}{4} I_{\frac{1}{8}}^{\frac{3}{2};\frac{3}{2}} x\left(\frac{1}{4}\right), \end{cases} \tag{5.23}$$

which is fractional differential inclusion involving Caputo fractional derivative. In this case $\mathfrak{v} = 2$. Let $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a mapping with

$$x \to \mathcal{F}(t,x) = \left[\exp\left(-x^2\right) + t + 5, \frac{|x|}{|x|+1} + \frac{\sqrt{t+1}}{2} \right]. \tag{5.24}$$

With these date we find $\Lambda \simeq 0.92823 \neq 0$. Obviously $\mathcal{F}$ satisfies hypothesis (As1) and

$$\|\mathcal{F}(t,x)\|_{\mathcal{P}} = \sup\{|\rho| : \rho \in \mathcal{F}(t,x)\} \leq 7 = P(t)\,\vartheta(\|x\|_\infty),$$

where $\|P\|_\infty = 1$ and $\vartheta(\|x\|_\infty) = 7$. Thus, the assumption (As2) is fulfilled, and by (As3), we get $\mathcal{L} > 6.4976$.

Therefore all hypotheses of Theorem (5.1) are valid. Then, there exists at least one solution of (5.23) on $[0,1]$ with $\mathcal{F}$ given by (5.24).

5.2. Hilfer fractional differential inclusions with Erdélyi-Kober fractional integral boundary condition

Example 5.5 Consider the fractional differential inclusion given by (5.20). Taking $\alpha = \frac{5}{4}$, $\beta = \frac{1}{2}$, $T = 1$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{6}$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{5}{2}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{3}{2}$, $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{1}{8}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$. Then, the problem (5.20) reduce to

$$\begin{cases} {}^{H}D^{\frac{5}{4},\frac{1}{2}}x\left(t\right) \in \mathcal{F}\left(t,x\right), \; t \in \left(0,1\right), \\ x\left(0\right) = 0, \;\; x\left(1\right) = \frac{1}{4} I_{\frac{1}{6}}^{\frac{1}{2};\frac{1}{2}} x\left(\frac{1}{4}\right) + \frac{1}{6} I_{\frac{1}{8}}^{\frac{5}{2};\frac{3}{2}} x\left(\frac{1}{2}\right), \end{cases} \tag{5.25}$$

which is fractional differential inclusion involving Hilfer fractional derivative. In this case $\mathfrak{v} = \frac{13}{8}$. Let $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathcal{P}\left(\mathbb{R}\right)$ be given by

$$x \to \mathcal{F}\left(t,x\right) = \left[0, \frac{\sin\left(x\right)}{\left(\exp\left(t^{2}\right)+9\right)} + \frac{1}{15}\right]. \tag{5.26}$$

With these date we find $\Lambda \simeq 0.88343 \neq 0$. Clearly $H_d\left(\mathcal{F}\left(t,x\right), \mathcal{F}\left(t,\overline{x}\right)\right) \leq \varpi\left(t\right)\left|x - \overline{x}\right|$, where $\varpi\left(t\right) = \frac{1}{\exp(t^2)+9}$ and $d\left(0, \mathcal{F}\left(t,0\right)\right) = \frac{1}{15} \leq \varpi\left(t\right)$ for (a.e.) all $t \in [0,1]$. Besides, we obtain $\left\|\varpi\right\|_{\infty} = \frac{1}{10}$ which implies $\varrho\left\|\varpi\right\|_{\infty} \approx 0.19 < 1$. Therefore all assumptions of Theorem (5.2) are valid. Then, there exists at least one solution of (5.25) on $[0,1]$ with $\mathcal{F}$ given by (5.26).

Chapter 6

Existence and stability analysis for a class of ψ-Hilfer fractional integro-differential equations

In this chapter, we study the existence and uniqueness of solutions for nonlinear fractional integro-differential equations subject to nonlocal integral boundary conditions in the frame of a ψ-Hilfer fractional derivative. Further, we discuss different kinds of stability of Ulam-Hyers for mild solutions to the given problem. Using an appropriate fixed point theorems together with generalized Gronwall inequality the desired outcomes are proven. Examples are given which illustrate the effectiveness of the theoretical results.

6.1 Existence and stability results for a ψ-Hilfer fractional integro-differential equations with nonlocal integral boundary conditions

In this section, we study existence, uniqueness and Ulam stability of the following fractional integro-differential equation involving ψ-Hilfer fractional derivative with nonlocal integral boundary conditions

$$\begin{cases} {}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi}x\left(t\right)=f\left(t,x\left(t\right),\int_{\mathfrak{a}}^{t}h\left(t,\sigma,x\left(\sigma\right)\right)d\sigma\right),\ t\in\left(\mathfrak{a},b\right), \\ x\left(\mathfrak{a}\right)=0,\ I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi}x\left(b\right)=\sum\limits_{\mathrm{i}=1}^{\mathrm{m}}\theta_{i}I_{\mathfrak{a}+}^{\eta_{i};\psi}x\left(\delta_{i}\right), \end{cases} \tag{6.1}$$

where ${}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha\in(1,2)$ and type $\beta\in[0,1]$, $I^{2-\mathfrak{v};\psi}$ and $I^{\eta_i;\psi}$ are the ψ-fractional integral of orders $2-\mathfrak{v}$, $\eta_i>0$ respectively, $\mathfrak{v}=\alpha+\beta\left(2-\alpha\right)\in(1,2)$, $\infty<\mathfrak{a}<b<\infty$, $\theta_i\in\mathbb{R}$, $i=1,2,...,m$, $0\leq\mathfrak{a}\leq\delta_1<\delta_2<\delta_3<...<\delta_m\leq b$, $f:[\mathfrak{a},b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $h:[\mathfrak{a},b]\times[\mathfrak{a},b]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

6.1.1 Existence results

Let $J=[a,b]$. To obtain our desired results, we need the following auxiliary lemma.

Lemma 6.1 *Let*

$$\varpi=\frac{(\psi(b)-\psi(\mathfrak{a}))}{\Gamma(2)}-\sum_{\mathrm{i}=1}^{\mathfrak{m}}\frac{\theta_i}{\Gamma(\mathfrak{v}+\eta_i)}(\psi(\delta_i)-\psi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1}\neq 0, \tag{6.2}$$

and for any $q\in C(J,\mathbb{R})$, then the nonlocal boundary value problem

$$\begin{cases} {}^{\mathcal{H}}D^{\alpha,\beta;\psi}x(t)=q(t),\ t\in(\mathfrak{a},b),\\ x(\mathfrak{a})=0,\ I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi}x(b)=\sum\limits_{\mathrm{i}=1}^{\mathfrak{m}}\theta_i I_{\mathfrak{a}+}^{\eta_i;\psi}x(\delta_i), \end{cases} \tag{6.3}$$

has a unique solution given by

$$x(t)=\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})}\left(\sum_{\mathrm{i}=1}^{\mathfrak{m}}\theta_i I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi}q(\delta_i)-I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi}q(b)\right)+I_{\mathfrak{a}+}^{\alpha;\psi}q(t). \tag{6.4}$$

Proof. Taking ψ-fractional integral $I_{\mathfrak{a}+}^{\alpha;\psi}$ to the first equation of (6.3), and from Lemma 1.15, we obtain

$$x(t)-\sum_{k=1}^{2}\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-k}}{\Gamma(\mathfrak{v}-k+1)}h_{\psi}^{[2-k]}I_{\mathfrak{a}+}^{(1-\beta)(2-\alpha);\psi}x(\mathfrak{a})=I_{\mathfrak{a}+}^{\alpha;\psi}q(t),\ t\in J. \tag{6.5}$$

We have $(1-\beta)(2-\alpha)=2-\mathfrak{v}$. Therefore

$$\begin{aligned} x(t)=&\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi}x(t)\Big|_{t=\mathfrak{a}}\\ &+\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-2}}{\Gamma(\mathfrak{v}-1)}I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi}x(t)\Big|_{t=\mathfrak{a}}+I_{\mathfrak{a}+}^{\alpha;\psi}q(t)\\ =&\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}D^{\mathfrak{v}-1;\psi}x(t)\big|_{t=\mathfrak{a}}+\frac{(\psi(t)-\psi(\mathfrak{a}))^{\gamma-2}}{\Gamma(\mathfrak{v}-1)}I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi}x(t)\Big|_{t=a}+I_{\mathfrak{a}+}^{\alpha;\psi}q(t). \end{aligned}$$

Put

$$c_1=D^{\mathfrak{v}-1;\psi}x(t)\big|_{t=\mathfrak{a}},$$

and

$$c_2=I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi}x(t)\Big|_{t=\mathfrak{a}},\ t\in J.$$

Then

$$x(t)=\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})}c_1+\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-2}}{\Gamma(\mathfrak{v}-1)}c_2+I_{\mathfrak{a}+}^{\alpha;\psi}q(t).$$

Because $\lim\limits_{t=\mathfrak{a}}(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-2}=\infty$, in the view of boundary conditions $x(\mathfrak{a})=0$, we must have

$$c_2=0.$$

Replacing c_2 by their value in (6.5), we get

$$x(t) = \frac{(\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} c_1 + I_{\mathfrak{a}+}^{\alpha;\psi} q(t). \tag{6.6}$$

Next, we use the second boundary condition to determine the constant c_1. Applying $I_{\mathfrak{a}+}^{\eta_i;\psi}$ on both side of equation (6.6), we get

$$I_{\mathfrak{a}+}^{\eta_i;\psi} x(t) = \frac{c_1}{\Gamma(\mathfrak{v}+\eta_i)} (\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} + I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} q(t). \tag{6.7a}$$

From the condition $x(b) = \sum_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\psi} x(\delta_i)$ and (6.7a), we have

$$\begin{aligned} x(b) &= \sum_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\psi} x(\delta_i) \\ &= c_1 \sum_{\mathrm{i}=1}^{\mathfrak{m}} \frac{\theta_i}{\Gamma(\mathfrak{v}+\eta_i)} (\psi(\delta_i) - \psi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} + \sum_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} q(\delta_i). \end{aligned} \tag{6.8}$$

From equation (6.6) and (6.8), we have

$$\begin{aligned} I_{a+}^{2-\gamma;\psi} x(b) &= \frac{(\psi(b) - \psi(\mathfrak{a}))}{\Gamma(2)} c_1 + I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi} q(b) \\ &= c_1 \sum_{\mathrm{i}=1}^{\mathfrak{m}} \frac{\theta_i}{\Gamma(\mathfrak{v}+\eta_i)} (\psi(\delta_i) - \psi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} + \sum_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} q(\delta_i). \end{aligned}$$

Thus, we find

$$c_1 = \frac{1}{\varpi} \left(\sum_{i=1}^{m} \theta_i I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} q(\delta_i) - I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi} q(b) \right).$$

Replacing the value of c_1 into (6.6), we obtain (6.4). ■

In what follows, we apply some fixed point theorems to demonstrate the existence and uniqueness results for problem (6.1).

To obtain our findings, We need the following assumptions

(As1) There is a constants $l_i > 0$, $i = 1, 2, 3$ such that

$$\begin{aligned} &|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq l_1 |x_1 - x_2| + l_2 |y_1 - y_2|, \\ &\quad |h(t, \sigma, x_1) - h(t, \sigma, x_2)| \leq l_3 |x_1 - x_2|, \ \forall (t, \psi, x_j, y_j) \in J^2 \times \mathbb{R}^2, \ j = 1, 2. \end{aligned}$$

(As2) There is a function $w \in C(J, \mathbb{R}^+)$ such that

$$|f(t, x, y)| \leq w(t), \ \forall (t, x, y) \in J \times \mathbb{R} \times \mathbb{R}.$$

For the sake of convenience, we put

$$k_1 = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i| \frac{(\psi(b) - \psi(\mathfrak{a}))^{\alpha+\eta_i+\mathfrak{v}-1}}{|\varpi| \Gamma(\mathfrak{v}) \Gamma(\alpha + \eta_i + 1)}, \; k_2 = \frac{(\psi(b) - \psi(\mathfrak{a}))^{1+\alpha}}{|\varpi| \Gamma(\mathfrak{v}) \Gamma(3 + \alpha - \mathfrak{v})}, \; k_3 = \frac{(\psi(b) - \psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha + 1)},$$

$$A_x = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i| \left(I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t, \sigma, x(\sigma))\, d\sigma\right)\Big|_{t=\delta_i} - I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi} f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t, \sigma, x(\sigma))\, d\sigma\right)\Big|_{t=b} \right). \tag{6.9}$$

Existence and uniqueness results via Banach's fixed point theorem

Theorem 6.1 *Let (As1) valid. If*

$$(k_1 + k_2 + k_3)(l_1 + l_2 l_3 (b - \mathfrak{a})) < 1, \tag{6.10}$$

then, (6.1) has a unique solution on J, where k_1, k_2, k_3 are given by (6.9).

Proof. We switch the problem (6.1) into a fixed point problem, we consider the operator $\Phi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as

$$\begin{aligned}(\Phi x)(t) = \frac{(\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi \Gamma(\mathfrak{v})} &\left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i \, I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t, \sigma, x(\sigma))\, d\sigma\right)\Big|_{t=\delta_i} \right.\\
&\left. - I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi} f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t, \sigma, x(\sigma))\, d\sigma\right)\Big|_{t=b} \right)\\
&+ I_{\mathfrak{a}+}^{\alpha;\psi} f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t, \sigma, x(\sigma))\, d\sigma\right).\end{aligned}$$

Clearly, the solution of (6.1) is as a fixed point of the operator Φ.

By (As1), for any $x, y \in C(J, \mathbb{R})$ and $t \in J$, we get

$$\begin{aligned}&|(\Phi x)(t) - (\Phi y)(t)|\\
&\leq \frac{(\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{|\varpi| \Gamma(\mathfrak{v})} \left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \frac{|\theta_i|}{\Gamma(\alpha + \eta_i)} \int_{\mathfrak{a}}^{\delta_i} \psi'(s)(\psi(\delta_i) - \psi(s))^{\alpha+\eta_i-1} \right.\\
&\times \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s, \sigma, x(\sigma))\, d\sigma\right) - f\left(s, y(s), \int_{\mathfrak{a}}^{s} h(s, \sigma, y(\sigma))\, d\sigma\right) \right| ds\\
&+ \frac{1}{\Gamma(2 + \alpha - \mathfrak{v})} \int_{\mathfrak{a}}^{b} \psi'(s)(\psi(b) - \psi(s))^{1+\alpha-\mathfrak{v}}\\
&\left. \times \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s, \sigma, x(\sigma))\, d\sigma\right) - f\left(s, y(s), \int_{\mathfrak{a}}^{s} h(s, \sigma, y(\sigma))\, d\sigma\right) \right| ds \right)\\
&+ \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{\tau} \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}\\
&\times \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s, \sigma, x(\sigma))\, d\sigma\right) - f\left(s, y(s), \int_{\mathfrak{a}}^{s} h(s, \sigma, y(\sigma))\, d\sigma\right) \right| ds,\end{aligned}$$

so,

$$\begin{aligned}
&\left|\left(\Phi x\right)(t)-\left(\Phi y\right)(t)\right| \\
&\leq \sum_{i=1}^{\mathfrak{m}}\left|\theta_i\right| \frac{\left(\psi\left(b\right)-\psi\left(\mathfrak{a}\right)\right)^{\alpha+\eta_i+\mathfrak{v}-1}}{\left|\varpi\right|\Gamma\left(\mathfrak{v}\right)\Gamma\left(\alpha+\eta_i+1\right)}\left(l_1+l_2l_3\left(b-\mathfrak{a}\right)\right)\left\|x-y\right\|_{\infty} \\
&+\frac{\left(\psi\left(b\right)-\psi\left(\mathfrak{a}\right)\right)^{1+\alpha}}{\left|\varpi\right|\Gamma\left(\mathfrak{v}\right)\Gamma\left(3+\alpha-\gamma\right)}\left(l_1+l_2l_3\left(b-\mathfrak{a}\right)\right)\left\|x-y\right\|_{\infty} \\
&+\frac{\left(\psi\left(b\right)-\psi\left(\mathfrak{a}\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left(l_1+l_2l_3\left(b-\mathfrak{a}\right)\right)\left\|x-y\right\|_{\infty} \\
&\leq \left(k_1+k_2+k_3\right)\left(l_1+l_2l_3\left(b-\mathfrak{a}\right)\right)\left\|x-y\right\|_{\infty}.
\end{aligned}$$

Thus

$$\left\|\left(\Phi x\right)-\left(\Phi y\right)\right\|_{\infty} \leq \left(k_1+k_2+k_3\right)\left(l_1+l_2l_3\left(b-\mathfrak{a}\right)\right)\left\|x-y\right\|_{\infty}.$$

From (6.10), Φ is a contraction. As an outcome of Banach's fixed point theorem, Φ has a unique fixed point which is a unique solution of (6.1) on J. ∎

Existence results via Schauder's fixed point theorem

Theorem 6.2 *Suppose that the hypotheses (As1)-(As2) are satisfied. Then, (6.1) has at least one solution on J.*

Proof. Let $\Omega=\left\{x\in C\left(J,\mathbb{R}\right):\left\|x\right\|_{\infty}\leq M_0\right\}$ be a non-empty closed bounded convex subset of $C\left(J,\mathbb{R}\right)$, and M_0 is chosen such

$$M_0 \geq w^*\left(k_1+k_2+k_3\right),$$

where k_1, k_2, k_3 are given by (6.9). It is a known that continuity of the functions f and h implies that the operator Φ is continuous. It remains to demonstrate that the operator Φ is compact and will be given in the following steps.

Step 1. We show that $\Phi\left(\Omega\right)\subset\Omega$.

Let $w^*=\sup\left\{w(t):t\in J\right\}$. For $x\in\Omega$, we have

$$\begin{aligned}
\left|\left(\Phi x\right)(t)\right| \leq{}& \frac{\left(\psi\left(t\right)-\psi\left(\mathfrak{a}\right)\right)^{\mathfrak{v}-1}}{\varpi\Gamma\left(\mathfrak{v}\right)}\left(\sum_{i=1}^{\mathfrak{m}}\theta_i\, I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi}\left|f\left(t,x\left(t\right),\int_{\mathfrak{a}}^{t}h\left(t,\sigma,x\left(\sigma\right)\right)d\sigma\right)\right|\Big|_{t=\delta_i}\right. \\
&\left.+I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi}\left|f\left(t,x\left(t\right),\int_{\mathfrak{a}}^{t}h\left(t,\sigma,x\left(\sigma\right)\right)d\sigma\right)\right|\Big|_{t=b}\right) \\
&+I_{\mathfrak{a}+}^{\alpha;\psi}\left|f\left(t,x\left(t\right),\int_{\mathfrak{a}}^{t}h\left(t,\sigma,x\left(\sigma\right)\right)d\sigma\right)\right| \\
\leq{}& \sum_{i=1}^{\mathfrak{m}}\left|\theta_i\right|\frac{w^*\left(\psi\left(b\right)-\psi\left(\mathfrak{a}\right)\right)^{\alpha+\eta_i+\mathfrak{v}-1}}{\left|\varpi\right|\Gamma\left(\mathfrak{v}\right)\Gamma\left(\alpha+\eta_i+1\right)}+\frac{w^*\left(\psi\left(b\right)-\psi\left(\mathfrak{a}\right)\right)^{1+\alpha}}{\left|\varpi\right|\Gamma\left(\mathfrak{v}\right)\Gamma\left(3+\alpha-\mathfrak{v}\right)} \\
&+\frac{w^*\left(\psi\left(b\right)-\psi\left(\mathfrak{a}\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)} \\
\leq{}& w^*\left(k_1+k_2+k_3\right),
\end{aligned}$$

and consequently

$$\|\Phi x\|_{\infty} \leq M_0.$$

Hence, $\Phi(\Omega) \subset \Omega$ and the set $\Phi(\Omega)$ is uniformly bounded.

Step 2. Φ sends bounded sets of $C(J,\mathbb{R})$ into equicontinuous sets.

For $t_1, t_2 \in J$, $t_1 < t_2$ and for $x \in \Omega$, we have

$$\begin{aligned}
&|(\Phi x)(t_2) - (\Phi x)(t_1)| \\
&\leq \frac{(\psi(t2) - \psi(\mathfrak{a}))^{\mathfrak{v}-1} - (\psi(t_1) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{|\varpi| \Gamma(\mathfrak{v})} \\
&\times \left(\sum_{i=1}^{\mathfrak{m}} \frac{|\theta_i|}{\Gamma(\alpha+\eta_i)} \int_{\mathfrak{a}}^{\delta_i} \psi'(s)(\psi(\delta_i) - \psi(s))^{\alpha+\eta_i-1} \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s,\sigma,x(\sigma))\, d\sigma\right) \right| ds \right. \\
&\left. + \frac{1}{\Gamma(2+\alpha-\mathfrak{v})} \int_{\mathfrak{a}}^{b} \psi'(s)(\psi(b) - \psi(s))^{1+\alpha-\mathfrak{v}} \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s,\sigma,x(\sigma))\, d\sigma\right) \right| ds \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_{a}^{t_1} \psi'(s)\left((\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}\right) \\
&\times \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s,\sigma,x(\sigma))\, d\sigma\right) \right| ds + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \psi'(s)(\psi(\tau_2) - \psi(s))^{\alpha-1} \\
&\times \left| f\left(s, x(s), \int_{\mathfrak{a}}^{s} h(s,\sigma,x(\sigma))\, d\sigma\right) \right| ds \\
&\leq \frac{\left((\psi(t_2) - \psi(\mathfrak{a}))^{\mathfrak{v}-1} - (\psi(t_1) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}\right) w^*}{|\varpi| \Gamma(\mathfrak{v})} \\
&\times \left(\sum_{i=1}^{\mathfrak{m}} \frac{|\theta_i|}{\Gamma(\alpha+\eta_i)} \int_{\mathfrak{a}}^{\delta_i} \psi'(s)(\psi(\delta_i) - \psi(s))^{\alpha+\eta_i-1}\, ds \right. \\
&\left. + \frac{1}{\Gamma(2+\alpha-\mathfrak{v})} \int_{\mathfrak{a}}^{b} \psi'(s)(\psi(b) - \psi(s))^{1+\alpha-\mathfrak{v}}\, ds \right) \\
&+ \frac{w^*}{\Gamma(\alpha+1)} \left((\psi(t_2) - \psi(\mathfrak{a}))^{\alpha} - (\psi(t_1) - \psi(\mathfrak{a}))^{\alpha}\right).
\end{aligned}$$

As $t_1 \to t_2$, we obtain

$$|(\Phi x)(t_2) - (\Phi x)(t_1)| \to 0.$$

Hence $\Phi(\Omega)$ is equicontinuous. The Arzela-Ascoli theorem implies that Φ is compact. Thus by Schauder fixed point theorem, we prove that Φ has at least one fixed point $x \in \Omega$ that is in fact a solution of (6.1) on J. ■

6.1.2 Ulam stability results

In this portion, we discuss the various types of Ulam stability for the ψ-Hilfer problem (6.1).

Theorem 6.3 *Suppose that the hypothesis (As1) and condition (6.10) are satisfied. Then, the first equation of (6.1) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$. Let $y \in C(J, \mathbb{R})$ be any solution of the inequality

$$\left| {}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi} y(t) - f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right) \right| \le \epsilon,\ t \in J.$$

Then, there exists $v \in C(J, \mathbb{R})$ such that

$$ {}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi} y(t) = f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right) + v(t),\ t \in J, \tag{6.11}$$

and $|v(t)| \le \epsilon$, $t \in J$. In view of Lemma 6.1, we get

$$y(t) = \frac{(\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi \Gamma(\mathfrak{v})} A_y + I_{\mathfrak{a}+}^{\alpha;\psi} f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right) + I_{\mathfrak{a}+}^{\alpha;\psi} v(t), \tag{6.12}$$

is solution of equation (6.11), where

$$\begin{aligned} A_y = &\left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i|\, I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi} f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right)\Big|_{t=\delta_i} \right. \\ &\left. - I_{\mathfrak{a}+}^{2+\alpha-\mathfrak{v};\psi} f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right)\Big|_{t=b} \right). \end{aligned} \tag{6.13}$$

From equation (6.12), we have

$$\begin{aligned} &\left| y(t) - \frac{(\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi \Gamma(\mathfrak{v})} A_y - I_{\mathfrak{a}+}^{\alpha;\psi} f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right) \right| \\ &\le I_{\mathfrak{a}+}^{\alpha;\psi} |v(t)| \le \epsilon \frac{(\psi(t) - \psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{6.14}$$

Let $x \in C(J, \mathbb{R})$ be solution of the problem

$$\begin{cases} {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} y(t) = f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right), \\ x(\mathfrak{a}) = y(\mathfrak{a}),\ I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi} x(b) = I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi} y(b), \end{cases} \tag{6.15}$$

where $I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi} x(b) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\psi} x(\delta_i)$ and $I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi} y(b) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\psi} y(\delta_i)$. By Lemma 6.1, the equivalent fractional integral equation of (6.15) is

$$y(t) = \frac{(\psi(t) - \psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi \Gamma(\mathfrak{v})} A_y + I_{\mathfrak{a}+}^{\alpha;\psi} f\left(t, y(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,y(\sigma))\, d\sigma\right),$$

where A_y is given by (6.13).

Now, by using the assumption (As1), we obtain

$$
\begin{aligned}
&\left|A_x - A_y\right| \\
&\leq \frac{\left(\psi(t)-\psi(\mathfrak{a})\right)^{\mathfrak{v}-1}}{|\varpi|\,\Gamma(\mathfrak{v})}\left(\sum_{\mathrm{i}=1}^{\mathfrak{m}}\frac{|\theta_i|}{\Gamma(\alpha+\eta_i)}\int_{\mathfrak{a}}^{\delta_i}\psi'(s)\left(\psi(\delta_i)-\psi(s)\right)^{\alpha+\eta_i-1}\right.\\
&\times\left|f\left(s,x(s),\int_{\mathfrak{a}}^{s}h(s,\sigma,x(\sigma))\,d\sigma\right)-f\left(s,y(s),\int_{\mathfrak{a}}^{s}h(s,\sigma,y(\sigma))\,d\sigma\right)\right|ds\\
&+\frac{1}{\Gamma(2+\alpha-\mathfrak{v})}\int_{\mathfrak{a}}^{b}\psi'(s)\left(\psi(b)-\psi(s)\right)^{1+\alpha-\mathfrak{v}}\\
&\times\left.\left|f\left(s,x(s),\int_{\mathfrak{a}}^{s}h(s,\sigma,x(\sigma))\,d\sigma\right)-f\left(s,y(s),\int_{\mathfrak{a}}^{s}h(s,\sigma,y(\sigma))\,d\sigma\right)\right|ds\right)\\
&\leq \frac{\left(\psi(t)-\psi(\mathfrak{a})\right)^{\mathfrak{v}-1}}{|\varpi|\,\Gamma(\mathfrak{v})}\left(\sum_{\mathrm{i}=1}^{\mathfrak{m}}\frac{|\theta_i|\,(l_1+l_2l_3)}{\Gamma(\alpha+\eta_i)}\int_{\mathfrak{a}}^{\delta_i}\psi'(s)\left(\psi(\delta_i)-\psi(s)\right)^{\alpha+\eta_i-1}\right.\\
&\times\left|x(s)-y(s)\right|ds+\frac{l_1+l_2l_3}{\Gamma(2+\alpha-\mathfrak{v})}\int_{\mathfrak{a}}^{b}\psi'(s)\left(\psi(b)-\psi(s)\right)^{1+\alpha-\mathfrak{v}}\left|x(s)-y(s)\right|ds\\
&\leq \frac{\left(\psi(t)-\psi(\mathfrak{a})\right)^{\mathfrak{v}-1}(l_1+l_2l_3)}{|\varpi|\,\Gamma(\mathfrak{v})}\\
&\times\left(\sum_{\mathrm{i}=1}^{\mathfrak{m}}|\theta_i|\,I_{\mathfrak{a}+}^{\alpha+\eta_i;\psi}\left|x(\delta_i)-y(\delta_i)\right|+I_{\mathfrak{a}+}^{2+\alpha-\gamma;\psi}\left|x(b)-y(b)\right|\right).
\end{aligned}
\tag{6.16}
$$

Because $x(b)=y(b)$, we must have $x(\delta_i)=y(\delta_i)$, $i=1,2,...,m$. Therefore, from inequality (6.16), we obtain $A_x=A_y$. From (6.14) and (As1), we get

$$
\begin{aligned}
&\left|y(t)-x(t)\right|\\
&=\left|y(t)-\frac{\left(\psi(t)-\psi(\mathfrak{a})\right)^{\mathfrak{v}-1}}{\tilde{\mathfrak{f}}\Gamma(\mathfrak{v})}A_x-I_{\mathfrak{a}+}^{\alpha;\psi}f\left(t,x(t),\int_{\mathfrak{a}}^{t}h(t,\sigma,x(\sigma))\,d\sigma\right)\right|\\
&\leq\left|y(t)-\frac{\left(\psi(t)-\psi(\mathfrak{a})\right)^{\mathfrak{v}-1}}{\tilde{\mathfrak{f}}\Gamma(\mathfrak{v})}A_y-I_{\mathfrak{a}+}^{\alpha;\psi}f\left(t,x(t),\int_{\mathfrak{a}}^{t}h(t,\sigma,x(\sigma))\,d\sigma\right)\right|\\
&+\left|I_{\mathfrak{a}+}^{\alpha;\psi}f\left(t,y(t),\int_{\mathfrak{a}}^{t}h(t,\sigma,y(\sigma))\,d\sigma\right)-I_{\mathfrak{a}+}^{\alpha;\psi}f\left(t,x(t),\int_{\mathfrak{a}}^{t}h(t,\sigma,x(\sigma))\,d\sigma\right)\right|\\
&\leq\epsilon\frac{\left(\psi(b)-\psi(\mathfrak{a})\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_1+l_2l_3}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\tau}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left|y(s)-x(s)\right|ds.
\end{aligned}
$$

Applying Lemma 1.18 with $u(t)=\left|y(t)-x(t)\right|$, $v(t)=\epsilon\frac{(\psi(b)-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}$ and $z(t)=\frac{l_1+l_2l_3}{\Gamma(\alpha)}$, we obtain

$$\begin{aligned}
&|y(t)-x(t)| \\
&\leq \epsilon\frac{(\psi(b)-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}\left[1+\int_{\mathfrak{a}}^{t}\sum_{k=1}^{\infty}\frac{[l_1+l_2l_3]^k}{\Gamma(k\alpha)}\psi'(s)(\psi(t)-\psi(s))^{k\alpha-1}ds\right] \\
&\leq \epsilon\frac{(\psi(b)-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}\left[1+\sum_{k=1}^{\infty}\frac{[(l_1+l_2l_3)(\psi(b)-\psi(\mathfrak{a}))^{\alpha}]^k}{\Gamma(k\alpha+1)}\right] \\
&= \epsilon\frac{(\psi(b)-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}((l_1+l_2l_3)(\psi(b)-\psi(\mathfrak{a}))^{\alpha}).
\end{aligned}$$

By setting

$$k_f=\frac{(\psi(b)-\psi(\mathfrak{a}))^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}((l_1+l_2l_3)(\psi(b)-\psi(\mathfrak{a}))^{\alpha}).$$

we obtain

$$|y(t)-x(t)|\leq k_f\epsilon. \tag{6.17}$$

Therefore, the first equation of (6.1) is Ulam-Hyers stable. ■

Remark 6.1 Define $\phi_f:\mathbb{R}^+\to\mathbb{R}^+$ by $\phi_f(\epsilon)=k_f\epsilon$. Then, $\phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $\phi_f(0)=0$. Then inequality (6.17) can be written as

$$|y(t)-x(t)|\leq\phi_f(\epsilon).$$

Thus, the first equation of (6.1) is generalized Ulam-Hyers stable.

In the next, we introduce the following function

(As3) the function $\phi\in C([\mathfrak{a},b],\mathbb{R}^+)$ is increasing and there is a constant $\lambda_\phi>0$ such that

$$I_{\mathfrak{a}+}^{\alpha;\psi}\phi(t)\leq\lambda_\phi\phi(t),\ \forall t\in J.$$

Theorem 6.4 *Assume that the hypotheses (As1), (As3) and condition (6.10) are satisfied. Then, the first equation of (6.1) is Ulam-Hyres-Rassias stable.*

Proof. Let any $\epsilon>0$. Let $y\in C(J,\mathbb{R})$ be any solution of the inequality

$$\left|{}^{\mathcal{H}}D_{\mathfrak{a}+}^{\alpha,\beta;\psi}y(t)-f\left(t,x(t),\int_{\mathfrak{a}}^{t}h(t,\sigma,y(\sigma))d\sigma\right)\right|\leq\epsilon\phi(t),\ t\in J.$$

Then, proceeding as in the proof of Theorem 6.3. From Remark 1.1, for some continuous function v such that $|v(t)|<\epsilon\phi(t)$, we get

$$\begin{aligned}
&\left|y(t)-\frac{(\psi(t)-\psi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})}A_y-I_{\mathfrak{a}+}^{\alpha;\psi}f\left(t,y(t),\int_{\mathfrak{a}}^{t}h(t,\sigma,y(\sigma))d\sigma\right)\right| \\
&\leq I_{\mathfrak{a}+}^{\alpha;\psi}|v(t)|\leq\epsilon I_{\mathfrak{a}+}^{\alpha;\psi}|\phi(t)|\leq\epsilon\lambda_\phi\phi(t),\ t\in J.
\end{aligned}$$

6.1. Existence and stability results for a ψ-Hilfer fractional integro-differential equations with nonlocal integral boundary conditions

Taking $y \in C(J, \mathbb{R})$ as any solution of (6.15), and following same steps as in the proof of Theorem 6.3,we get

$$
\begin{aligned}
&|y(t) - x(t)| \\
&\leq \epsilon\lambda_\phi \phi(t) + \frac{l_1 + l_2 l_3}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |y(s) - x(s)| \, ds, \ t \in J.
\end{aligned}
$$

By applying Corollary 1.19, we obtain

$$
\begin{aligned}
|y(t) - x(t)| &\leq \epsilon\lambda_\phi \phi(t) E_\alpha\left((l_1 + l_2 l_3)(\psi(t) - \psi(\mathfrak{a}))^\alpha\right) \\
&\leq \epsilon\lambda_\phi \phi(t) E_\alpha\left((l_1 + l_2 l_3)(\psi(b) - \psi(\mathfrak{a}))^\alpha\right).
\end{aligned}
$$

By taking a constant

$$
k_{\phi,\, f} = \lambda_\phi \phi(t) E_\alpha\left((l_1 + l_2 l_3)(\psi(b) - \psi(\mathfrak{a}))^\alpha\right).
$$

We obtain

$$
|y(t) - x(t)| \leq k_{\phi,f}\epsilon\phi(t). \tag{6.18}
$$

Therefore, the first equation (6.1) is Ulam-Hyres-Rassias stable. ■

Remark 6.2 By putting $\epsilon = 1$ in the inequality (6.18), we deduce that first equation of (6.1) is generalized Ulam-Hyres-Rassias stable.

6.1.3 Examples

In this section, we consider some particular cases of the nonlinear fractional integro-differential equation to apply our results in the study of existence and Ulam stabilities, specifically, Ulam-Hyers and Ulam-Hyres-Rassias.

Consider the nonlinear fractional integro-differential equation of the form

$$
\begin{cases}
{}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} x(t) = f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,x(\sigma))\, d\sigma\right), \ t \in (\mathfrak{a}, b), \\
x(\mathfrak{a}) = 0, \ I_{\mathfrak{a}+}^{2-\mathfrak{v};\psi} x(b) = \sum\limits_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\psi} x(\delta_i).
\end{cases} \tag{6.19}
$$

The following examples are particular cases of the fractional integro-differential equation given by (6.19).

Example 6.1 Consider the fractional integro-differential equation given by (6.19). Taking $\psi(t) = \log t$, $\beta \to 0$, $\mathfrak{a} = 1$, $b = e$, $\alpha = \frac{3}{2}$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{10}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{2}$, $\delta_1 = \frac{3}{2}$, $\delta_2 = 2$ and f, h are continuous functions defined by

$$
f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t,\sigma,x(\sigma))\, d\sigma\right) = \frac{1}{4}x(t) + \frac{1}{10}\int_{1}^{t} \frac{1}{\sigma \exp(t^2-1)+4} x(\sigma)\, d\sigma,
$$

$$
h(t,\sigma,x(\sigma)) = \frac{1}{\sigma \exp(t^2-1)+4} x(\sigma).
$$

Then, the problem (6.19) reduce to the following

$$\begin{cases} {}^{Ha}D_{1+}^{\frac{3}{2},0;\log t}x(t)=\frac{1}{4}x(t)+\frac{1}{10}\int_1^t\frac{1}{\sigma\exp(t^2-1)+4}x(\sigma)\,d\sigma,\ t\in(1,e), \\ x(1)=0,\ I_{1+}^{\frac{1}{2};\log t}x(e)=\frac{1}{2}I_{1+}^{\frac{1}{4};\log t}x\left(\frac{3}{2}\right)+\frac{1}{10}I_{1+}^{\frac{5}{2};\log t}x(2), \end{cases} \tag{6.20}$$

which is nonlinear fractional integro-differential equation involving Hadamard fractional derivative. In this case $\mathfrak{v}=\frac{3}{2}$. Set

$$f(t,x,y)=\frac{1}{4}x+\frac{1}{10}y,\ \forall x,y\in\mathbb{R}.$$

For $x_i,y_i\in\mathbb{R}$, $i=1,2$ and $t\in[1,e]$, using the hypothesis (As1), we get

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq\frac{1}{4}|x_1-x_2|+\frac{1}{10}|y_1-y_2|,$$

and

$$\begin{aligned}|h(t,\sigma,x_1)-h(t,\sigma,x_2)| &\leq \frac{1}{\sigma\exp(t^2-1)+4}|x_1-x_2| \\ &\leq \frac{1}{5}|x_1-x_2|,\end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1=\frac{1}{4}$, $l_2=\frac{1}{10}$ and $l_3=\frac{1}{5}$ We will check that condition (6.10) is satisfied. Indeed

$$\begin{aligned}&(k_1+k_2+k_3)(l_1+l_2l_3(b-\mathfrak{a})) \\ &\simeq(0.5+0.79+0.75)\left(\frac{1}{4}+\frac{1}{50}\right) \\ &\simeq 0.55<1.\end{aligned}$$

Then by Theorem 6.1, (6.20) has a unique solution on $[1,e]$. Further, by Theorem 6.3 we conclude that the first equation of (6.20) is Ulam-Hyers stable with

$$k_f=\frac{1}{\Gamma\left(\frac{5}{2}\right)}E_{\frac{3}{2}}\left(\frac{27}{100}\right).$$

Define

$$\phi(t)=\log(t)^{\frac{3}{2}},\ t\in[1,e].$$

Then, ϕ is continuous increasing function such that

$$\begin{aligned}I_{1+}^{\frac{3}{2};\log t}\phi(t) &= \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{\frac{1}{2}}\log(t)^{\frac{3}{2}}\frac{ds}{s} \\ &\leq \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{\frac{1}{2}}\frac{ds}{s} \\ &\leq \frac{1}{\Gamma\left(\frac{5}{2}\right)}\log(t)^{\frac{3}{2}}.\end{aligned}$$

Therefore, for $\lambda_\phi=\frac{1}{\Gamma\left(\frac{5}{2}\right)}$ and $\phi(t)=\log(t)^{\frac{3}{2}}$, hypothesis (As3) is satisfied. Hence, by Theorem 6.4 the first equation of (6.20) is Ulam-Hyres-Rassias stable.

6.1. Existence and stability results for a ψ-Hilfer fractional integro-differential equations with nonlocal integral boundary conditions

Example 6.2 Consider the fractional integro-differential equation given by (6.19). Taking $\psi(t) = t$, $\beta \to 0$, $\mathfrak{a} = 0$, $b = 1$, $\alpha = \frac{5}{4}$, $\theta_1 = 3$, $\theta_2 = 5$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$and f, h are continuous functions defined by

$$f\left(t, x(t), \int_{\mathfrak{a}}^{t} h(t, \sigma, x(\sigma))\, d\sigma\right) = \frac{1}{8} x(t) + \frac{1}{6}\int_0^1 \frac{\sin(t)}{\exp(t^2)+9}\frac{|x(\sigma)|}{|x(\sigma)|+1} d\sigma,$$
$$h(t, \sigma, x(\sigma)) = \frac{\sin(t)}{\exp(t^2)+9}\frac{|x(\sigma)|}{|x(\sigma)|+1}.$$

Then, the problem (6.19) reduce to the following

$$\begin{cases} {}^{RL}D_{1+}^{\frac{5}{4},0;t} x(t) = \frac{1}{8} x(t) + \frac{1}{6}\int_0^1 \frac{\sin(t)}{\exp(t^2)+9}\frac{|x(\sigma)|}{|x(\sigma)|+1} d\sigma, \ t \in (0,1), \\ x(0) = 0, \ I_{0+}^{\frac{3}{4};t} x(1) = 3 I_{0+}^{\frac{1}{4};t} x\left(\frac{1}{4}\right) + 5 I_{0+}^{\frac{1}{2};t} x\left(\frac{1}{2}\right), \end{cases} \tag{6.21}$$

which is nonlinear fractional integro-differential equation involving Riemann-Liouville fractional derivative. In this case $\mathfrak{v} = \frac{5}{4}$. Set

$$f(t, x, y) = \frac{1}{8}x + \frac{1}{6}y, \ \forall x, y \in \mathbb{R}.$$

For $x_i, y_i \in \mathbb{R}$, $i = 1, 2$ and $t \in [0, 1]$, using the hypothesis (As1), we get

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \frac{1}{8}|x_1 - x_2| + \frac{1}{6}|y_1 - y_2|,$$

and

$$\begin{aligned} |h(t, \sigma, x_1) - h(t, \sigma, x_2)| &= \left| \frac{\sin(t)}{\exp(t^2)+9}\left(\frac{|x_1|}{|x_1|+1} - \frac{|x_2|}{|x_2|+1}\right)\right| \\ &\leq \frac{1}{\exp(t^2)+9}\left(\frac{|x_1 - x_2|}{(1+|x_1|)(1+|x_2|)}\right) \\ &\leq \frac{1}{10}(|x_1 - x_2|), \end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{8}$, $l_2 = \frac{1}{6}$ and $l_3 = \frac{1}{10}$ We will check that condition (6.10) is satisfied. Indeed

$$\begin{aligned} &(k_1 + k_2 + k_3)(l_1 + l_2 l_3 (b - \mathfrak{a})) \\ &\simeq (1.51 + 0.14 + 0.88)\left(\frac{1}{8} + \frac{1}{60}\right) \\ &\simeq 0.36 < 1. \end{aligned}$$

Then by Theorem 6.1, the (6.21) has a unique solution on $[0, 1]$. Moreover, by Theorem 6.3 we conclude that the first equation of (6.21) is Ulam-Hyers stable with

$$k_f = \frac{1}{\Gamma\left(\frac{9}{4}\right)} E_{\frac{5}{4}}\left(\frac{17}{120}\right).$$

6.1. Existence and stability results for a ψ-Hilfer fractional integro-differential equations with nonlocal integral boundary conditions

Define

$$\phi(t) = t^{\frac{5}{4}},\ t \in [0,1].$$

Then, ϕ is continuous increasing function such that

$$\begin{aligned} I_{0+}^{\frac{5}{4};t}\phi(t) &= \frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_0^t (t-s)^{\frac{1}{4}} t^{\frac{5}{4}} ds \\ &\leq \frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_0^t (t-s)^{\frac{1}{4}} ds \\ &\leq \frac{1}{\Gamma\left(\frac{9}{4}\right)} t^{\frac{5}{4}}. \end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{9}{4}\right)}$ and $\phi(t) = t^{\frac{5}{4}}$, hypothesis (As3) is satisfied. Hence, by Theorem 6.4 the first equation of (6.21) is Ulam-Hyres-Rassias stable.

Example 6.3 Consider the fractional integro-differential equation given by (6.19). Taking $\psi(t) = t$, $\beta \to \frac{1}{2}$, $\mathfrak{a} = 0$, $b = 1$, $\alpha = \frac{7}{4}$, $\theta_1 = 3$, $\theta_2 = 5$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$ and f, h are continuous functions defined by

$$f\left(t, x(t), \int_{\mathfrak{a}}^t h(t,\sigma,x(\sigma))\, d\sigma\right) = \frac{1}{9}x(t) + \frac{1}{30}\int_0^t \frac{\cos(t)}{\exp(t)+5}\frac{|x(\sigma)|}{|x(\sigma)|+1} d\sigma,$$
$$h(t,\sigma,x(\sigma)) = \frac{\cos(t)}{\exp(t)+5}\frac{|x(\sigma)|}{|x(\sigma)|+1}.$$

Then, the problem (6.19) reduce to the following

$$\begin{cases} {}^{\mathcal{H}}D_{0+}^{\frac{7}{4},\frac{1}{2};t} x(t) = \frac{1}{9}x(t) + \frac{1}{30}\int_0^t \frac{\cos(t)}{\exp(t)+5}\frac{|x(\sigma)|}{|x(\sigma)|+1} d\sigma,\ t \in (0,1), \\ x(0) = 0,\ I_{0+}^{\frac{1}{8};t} x(1) = 3 I_{0+}^{\frac{1}{4};t} x\left(\frac{1}{4}\right) + 5 I_{0+}^{\frac{1}{2};t} x\left(\frac{1}{2}\right), \end{cases} \tag{6.22}$$

which is nonlinear fractional integro-differential equation involving Hilfer fractional derivative. In this case $\mathfrak{v} = \frac{15}{8}$. Set

$$f(t,x,y) = \frac{1}{9}x + \frac{1}{30}y,\ \forall x,y \in \mathbb{R}.$$

For $x_i, y_i \in \mathbb{R}$, $i = 1,2$ and $t \in [0,1]$, using the hypothesis (As1), we get

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \frac{1}{9}|x_1 - x_2| + \frac{1}{30}|y_1 - y_2|,$$

and

$$\begin{aligned} |h(t,\sigma,x_1) - h(t,\sigma,x_2)| &= \left|\frac{\cos(t)}{\exp(t)+5}\left(\frac{|x_1|}{|x_1|+1} - \frac{|x_2|}{|x_2|+1}\right)\right| \\ &\leq \frac{1}{\exp(t)+5}\left(\frac{|x_1 - x_2|}{(1+|x_1|)(1+|x_2|)}\right) \\ &\leq \frac{1}{6}(|x_1 - x_2|), \end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{9}$, $l_2 = \frac{1}{30}$ and $l_3 = \frac{1}{6}$ We will check that condition (6.10) is satisfied. Indeed

$$\begin{aligned}
&(k_1 + k_2 + k_3)\left(l_1 + l_2 l_3 \left(b - \mathfrak{a}\right)\right) \\
&\simeq (3.07 + 0.5 + 0.62)\left(\frac{1}{9} + \frac{1}{180}\right) \\
&\simeq 0.49 < 1.
\end{aligned}$$

Then by Theorem 6.1, the (6.22) has a unique solution on $[0,1]$. Further, by Theorem 6.3 we conclude that the first equation of (6.22) is Ulam-Hyers stable with

$$k_f = \frac{1}{\Gamma\left(\frac{11}{4}\right)} E_{\frac{7}{4}}\left(\frac{7}{60}\right).$$

Define

$$\phi(t) = t^{\frac{7}{4}},\ t \in [0,1].$$

Then, ϕ is continuous increasing function such that

$$\begin{aligned}
I_{0+}^{\frac{7}{4};t}\phi(t) &= \frac{1}{\Gamma\left(\frac{7}{4}\right)} \int_0^t (t-s)^{\frac{1}{5}}\, t^{\frac{7}{4}} ds \\
&\leq \frac{1}{\Gamma\left(\frac{7}{4}\right)} \int_0^t (t-s)^{\frac{3}{4}}\, ds \\
&\leq \frac{1}{\Gamma\left(\frac{11}{5}\right)} t^{\frac{7}{4}}.
\end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{11}{5}\right)}$ and $\phi(t) = t^{\frac{7}{4}}$, hypothesis (As3) is satisfied. Hence, by Theorem 6.4 the first equation of (6.22) is Ulam-Hyres-Rassias stable.

Conclusion and perspective

In this thesis, we have studied some qualitative properties such as existence, uniqueness and stability of solutions for various classes of nonlinear fractional differential equations and inclusions involving different types of fractional derivatives like Riemann-Liouville, Caputo, Caputo-Hadamard, Hilfer and ψ-Hilfer. The results are based on the argument of the fixed point theorems. Some appropriate fixed point theorems have been used, in particular, Banach contraction, Schaefer's fixed point theorem, Schauder's fixed point theorem, Krasnoselskii's fixed point theorem, nonlinear alternative of Kakutani maps, Covitz and Nadler fixed point theorem and Mönch's fixed point theorem combined with the technique of measures of noncompactness.

For the perspective, it would be interesting to extend the results of the current thesis by considering the applied side of fractional differential equations due to their importance in the study of most natural phenomena and epidemics like transmission dynamics of COVID-19 [94, 95], Cancer [31], Langevin equations [3], pantograph equations [44], Mathieu equations [93], etc. Also, we will use the some powerful numerical methods such as Laplace transform and Adomian's decomposition method, Adams Bashforth method to find approximate solutions for these applications.

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