وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR-ANNABA UNIVERSITY
UNIVERSITE BADJI MOKHTAR-ANNABA

جامعة باجي مختار – عنابة –

Faculté des Sciences

Département de Mathématiques

Année : 2020/2021

THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

Option
Modélisation mathématique

ETUDE D'EXISTENCE ET NON EXISTENCES DE SOLUTIONS DE CERTAINES CLASSES D'EQUATIONS ET DES SYSTEMES ELLIPTIQUES SEMI-LINEAIRES

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Acknowledgements

First of all, I want to thank Almighty God, for giving me the courage and patience to be able to finish this thesis.

It is my foremost privilege to express sincere gratitude to my supervisor **Pr. Kamel.Akrout**, for his immense support, invaluable guidance and motivation throughout my research work, without which this thesis would have been inconceivable.

I would like to thank my thesis director. I first thank him for the interesting subject he proposed to me. I thank him again, for his help, his patience, his advice, his encouragement, his great availability, and his open-mindedness which helped me to carry out this work. I feel privileged to have been mentored by this person. I also thank him for his help and advice in writing the thesis. Without his ideas and expertise, the realization of this thesis would not have been possible. It is impossible to express my gratitude to him in just a few lines.

I address my warmest thanks to my co-supervisor, **Pr. Brahim Khodja**, for his help, his advice, and his encouragement.

I would like to warmly thank **Pr. Kouche Mahieddine** for the honor he did me by agreeing to chair my thesis jury.

My thanks then go to **Pr. Abderahman Zarai** who agreed to participate in the jury for this thesis.

Dedication

I am dedicating this modest thesis to both my parents, who taught me and made everything is well known in my mind, whose constant help and encouragement allowed me to continue my studies in the best conditions. I thank them very much for all. Wishing that God grant them long life, full of happiness and prosperity, to my dear sisters Chaima, Chahinez, Ouboud, to my husband Mohamed Ayad Merdaci whom I am proud of his support and love, to my angel Tasnim.

Résumé

Le travail présenté dans cette thèse est consacré à l'étude de l'existence et la multiplicité de solutions positives non triviales d'un système de type elliptique fractionnaire singulier associé à l'opérateur p-Laplacien fractionnaire dans un domaine borné de $\mathbb{R}^N$. Les résultats sont obtenus en utilisant certaines techniques variationnelle, la variété de Nehari, la méthode des fibering (F.M), et le principe variationnelle d'Ekeland.

Mots-clés: Opérateur fractionnaire, la variété de Nehari, méthode des fibering, principe d'Ekeland.

Abstract

The work presented in this thesis is devoted to studying the existence and multiplicity of non-trivial positive solutions of a singular fractional elliptic type system associated with the fractional p-Laplacian operator in a bounded domain $\mathbb{R}^N$. The results are obtained using some of the variational techniques, Nehari Manifold, Fibering method (F.M), and applying the Ekeland variational principle.

Keywords: Fractional operator, Nehari Manifold, Fibering method, Ekeland's variational principle.

ملخــــص

تم تخصيص العمل المقدم في هذه الأطروحة لدراسة وجود وتعدد الحلول الإيجابية غير التافهة لنظام من النوع البيضاوي الكسري. المفرد المرتبط بالمؤثر p-Laplacian الكسري في مجال محدود من $\mathbb{R}^N$.
تم الحصول على النتائج باستخدام بعض تقنيات التنويع ، Nehari Manifold، Fibreing method (F.M)، وتطبيق مبدأ التباين ل Ekeland

الكلمات المفتاحية: المعامل الكسري ، Nehari Manifold ، طريقة Fibreing ، مبدأ التباين ل Ekeland

Notation

To achieve the understanding of what will follow, we give an interpretation of the notations and abbreviations used in this thesis.

Ω	Bounded domain in $\mathbb{R}^N$.
$\mathbb{R}^N$	Euclidean space provided with its usual denoted norm $\|.\|$.
$\mathbb{R}^N \backslash \Omega$	complementary to Ω.
E, E'	Banach space with dual E'.
$\langle .,. \rangle : E \times E' \to \mathbb{R}$	Dual pairing, occasionally also used to denote scalar product in $\mathbb{R}^N$.
p_s^*	Fractional critical Sobolev exponent.
J_λ	Energy functional.
$\|.\|_E$	Norm in E.
$\|.\|_{E'}$	Induced norm in E'.
$\rightharpoonup$	Weak convergence.
$\longrightarrow$	Strong Convergence .
$a.e$	almost everywhere.
$C(\Omega)$	Set of continuous functions on Ω.
$L^p(\Omega)$	Standard Lebesgue space on Ω of exponent p.
$L^\infty(\Omega)$	Standard Lebesgue space on Ω of exponent ∞.
$W^{s,p}(\Omega)$	Fractional Sobolev space.
$\|u\|_{L^p(\Omega)}$	Norm of u on $L^p(\Omega)$ defined by $\|u\|_{L^p(\Omega)} = \left(\int_\Omega \|u\|^p \, dx\right)^{\frac{1}{p}}$.
$\|u\|_{L^\infty(\Omega)}$	Norm of u on $L^\infty(\Omega)$ defined by $\|u\|_{L^\infty(\Omega)} = ess \sup_{x \in \Omega} \|u(x)\|$.

Introduction

In this thesis, we focus on the variational case to prove the existence and multiplicity of nontrivial positive solutions for singular fractional elliptic system via the Nehari manifold, fibering method, and applying Ekeland's variational principl.

We study the existence and multiplicity of nontrivial positive solutions for the following singular fractional elliptic system

$$(P_\lambda) \qquad \begin{cases} (-\Delta)_p^s u = a(x)u^{-\gamma} + \lambda f(x,u,v) \text{ in } \Omega, \\ (-\Delta)_p^s v = b(x)v^{-\gamma} + \lambda g(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Where Ω is a smooth bounded set in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, λ is a positive parameter and $0 < \gamma < 1 < p < r < p_s^* - 1$, where $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent. Where a and b are positive functions of class $L^\infty(\Omega)$. Let $f, g \in C(\Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ be positively homogeneous of degree $(r-1)$ that is

$$(0.\ 0.\ 1) \qquad \forall t > 0, \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R} : \begin{cases} f(x,tu,tv) = t^{r-1} f(x,u,v), \\ g(x,tu,tv) = t^{r-1} g(x,u,v). \end{cases}$$

Our aim in the present work is to show how the variational method can establish the existence and multiplicity of nontrivial positive solutions combine with Nehari Manifold, fibering maps introduced by Pohozaev (i.e., maps of the form $t \to J_\lambda(tu)$ where J_λ is the Euler function associated with the equation) and

Ekeland's variational principle, they gave an interesting explanation. For more detail, we refer the reader to see [**2,4,5,11**].

This method has been applied to partial differential equations, systems nonlinear, semilinear and quasilinear involving Laplacian and P-Laplacian fractional among the many studies have been published, we refer the reader to see [**1,9,17,19,20,32**]. At this point, we briefly recall literature concerning related singular problems and systems. In [**23**], K. Saoudi & A. Ghanmi studied a singular problem

$$\begin{cases} (-\Delta)_p^s u = \frac{a(x)}{u^\gamma} + \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain, $a \in L^\infty(\Omega)$, λ is a positive parameter, $p \geq 2$ such that $N \geq ps$ and $0 < \gamma < 1 < p < r < p_s^*$ where $p_s^* = \frac{Np}{N-ps}$ and r is the homogeneity degree of the function f. Under appropriate assumptions on the function f, the authors employ the method of Nehari manifold combined with the fibering maps in order to show the existence of $T_{p,r,\gamma}$ such that for all$\lambda \in (0, T_{p,r,\gamma})$. Then, the problem has at least two positives solutions.

The equations of this type have growing interest since they arise in many fields of sciences, physical phenomena, probability, stochastic calculus, and finance. For more details, one can see **[6,7,16,22]**.

Moreover in **[2] Adimurthi and Giacomoni** proved the multiplicity of positive solutions for a singular and critical elliptic problem in $\mathbb{R}^2$. In **[15,26] Mohammed**, **Coclite** and **Palmieri** established the existence of Positive solutions of the p-Laplace equation with singular nonlinearity. In **[13] Caffarelli** and **Silvestre** studied the fractional Laplacian through extension theory. In **[14] Chen**, **Hajaiej** and **Wang**, established the existence, non-existence and uniqueness of positive weak solutions of the semilinear fractional equation.

Some other results dealing with the existence of solutions concerning the singular problem has been treated in Ghanmi, Saoudi and Kratou (**see[24,27,28]**) they studied the existence and multiplicity of solutions of the semilinear singular elliptic equations involving the fractional Laplace and fractional p-Laplacian operator using Nehari manifold, the fibering method and applying Ekeland's variational principle with boundary conditions displaying more interests on semilinear problems involving fractional Laplace operator, see for instance **[29–30]** and the references therein.

In the first chapter, we give a brief some basic notations and preliminaries about the subject used in subsequent chapters.

In the second chapter, we present everything devoted to the theory of Nehari manifold, the fibering method, and Ekeland's variational principl.

In the third chapter, we give the main results of this thesis.

Chapter 1

Preliminaries

Contents

In this chapter, we briefly recall the essential definitions on space of continuous functions, $L^p(\Omega)$ spaces , Sobolev and fractional Sobolev spaces, will be useful to us later and we let us give, at the same time, the critical points theory, and recall the Lagrange multiply theorem, Implicit function theorem, Maximum principle. The chapter ends with some definitions.

1.1 Functional spaces

1.1.1 Space of continuous functions

Definition 1.1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open and $u : \Omega \to \mathbb{R}$ a function. We say that u is continuous if*

$$\forall x_0 \in \Omega, \forall \varepsilon > 0, \exists \delta > 0,$$

such that

$$x \in E, \|x - x_0\| < \delta \Longrightarrow |u(x) - u(y)| < \varepsilon,$$

where the norm in $\mathbb{R}^N$ is the Euclidean norm.

Definition 1.1.2. *Let Ω be an open in $\mathbb{R}$. We define :*

$$C(\Omega) := \{u : \Omega \to \mathbb{R} \; u \text{ is continuous}\},$$

$$C\left(\overline{\Omega}\right) := \left\{u : \Omega \to \mathbb{R} \; u \text{ is continous and extends continuously to } \overline{\Omega}\right\}.$$

Let

$$\|.\|_C : C\left(\overline{\Omega}\right) \rightarrow \mathbb{R},$$

$$u \longmapsto \sup_{x\in\Omega} |u(x)| \textit{ is a norm.}$$

1.1.2 $L^p(\Omega)$ Spaces

Let $p \in \mathbb{R}$ with $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$, we set

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \backslash\ f \text{ is measurable and } \int |f|^p \, d\mu < \infty \right\},$$

we define the L^p norm of f by

$$\|f\|_{L^p} = \|f\|_p = \left(= \int_\Omega |f|^p \, d\mu \right)^{1/p}.$$

If $p = \infty$, the space $L^\infty(\Omega)$ satisfy

$$L^\infty(\Omega) = \{ f : \Omega \longrightarrow \mathbb{R} /\ f \text{ is measurable and } \exists C > 0 \text{ such that } |f(x)| \leq C \text{ a.e on } \Omega \},$$

we define the L^∞ norm of f by

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C ; |f| \leq C \text{ a.e on } \Omega \},$$

$L^\infty(\Omega)$ is a Banach space.

If $p = 2$, the space $L^2(\Omega)$ is a Hilbert space for scalar product

$$(f, g) = \int_{\Omega} f(x)g(x)dx.$$

We denote by $L^1_{loc}(\Omega)$ the set of locally integrable functions on Ω and we write

$$L^1_{loc}(\Omega) = \left\{u : u \in L^1(K) \text{ for all compact } K \text{ of } \Omega\right\}.$$

Remark 1.1.1.
- *- If $f \in L^\infty(\Omega)$ then we have $|f| \leq \|f\|_{L^\infty}$ a.e. on Ω.*
- *$L^p(\Omega) \subset L^1_{loc}(\Omega)$ for all $1 \leq p \leq \infty$.*
- *$\left(L^p(\Omega), \|.\|_p\right)$ is Banach space for $1 \leq p \leq \infty$, separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.*

Theorem 1.1.1. *[10] **(Hölder's inequality)***

Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

Lemma 1.1.1. ***(Brezis–Lieb's).***[11] *Let Ω be a bounded open in $\mathbb{R}^n$ and $1 < p < +\infty$, $(f_n)_n \to f$ a.e. in $L^p(\Omega)$, then*

$$f \in L^p(\Omega) \text{ and } \|f\|_p^p = \|f_n\|_p^p - \|f_n - f\|_p^p + \sigma(1).$$

1.1.3 Sobolev Space $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.1.3. *The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \exists g_1, ..., g_N \in L^p(\Omega) \text{ such that} \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \ \forall \varphi \in C_c^\infty(\Omega), \ \forall i = \overline{1, N} \right\}$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \in W^{1,p}(\Omega)$ we define $\frac{\partial u}{\partial x_i} = g_i$,and we write

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}.$$

Proposition 1.1.1. *$W^{1,p}(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$. $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, and it is separable for $1 \leq p < \infty$.*

Corollary 1.1.1. *Let* $1 \leq p \leq \infty$. *We have*

- $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, *if* $p < N$,
- $W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [p, +\infty)$, *if* $p = N$,
- $W^{1,p}(\Omega) \subset L^\infty(\Omega)$, *if* $p > N$,

and all these injections are continuous. Moreover, if $p > N$ *we have, for all* $u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}} |x - y|^\alpha \text{ a.e. } x, y \in \Omega,$$

with $\alpha = 1 - (N/p)$ *and* C *is a constant depends only on* $\Omega, p,$ *and* N. *In particular* $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Theorem 1.1.2. *[10]* ***(Rellich–Kondrachov)***

Suppose that Ω *is bounded and of class* C^1. *Then we have the following compact injections:*

- $W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [1, p^*) W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [1, p^*)$, *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, *if* $p < N$,
- $W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty)$, *if* $p = N$,
- $W^{1,p}(\Omega) \subset C(\overline{\Omega}), \forall q \in [p, +\infty)$, *if* $p > N$.

In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$, *with compact injection for all* p *(and all* N*).*

1.1.4 Fractional Sobolev spaces $W^{s,p}(\Omega)$

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, we introduce fractional Sobolev space

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Omega) \right\},$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

Consider the space

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R}, u \in L^p(\Omega) \text{ and } \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Lambda) \right\},$$

with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_{\Lambda} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \right)^{\frac{1}{p}}.$$

Proposition 1.1.2. *The space $W^{s,p}(\Omega)$ is of local type, that is, for every u in $W^{s,p}(\Omega)$ and for every $\varphi \in D(\Omega)$, the product φu belongs to $W^{s,p}(\Omega)$.*

Proposition 1.1.3. *The space $D(\mathbb{R}^N)$ is dense in $W^{s,p}(\Omega)$.*

Theorem 1.1.3. *[10]*

Let $s \in]0,1[$ and let $p \in]1,\infty[$. We have

- *If $sp < N$, then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for every $q \leq Np/(N-sp)$.*

- *If* $N = sp$*, then* $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *for every* $q < \infty$.
- *If* $sp > N$*, then* $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ *and, more precisely,*

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow C_b^{0,s-N/p}(\mathbb{R}^N).$$

Proposition 1.1.4. *Let* $s \in [0, 1[$ *and let* $p > 1$*. Let* Ω *be an open set that admits an* (s, p)*-extension; then* $D(\overline{\Omega})$*, the space of restrictions to* Ω *of functions in* $D(\mathbb{R}^N)$*, is dense in* $W^{s,p}(\mathbb{R}^N)$*.*

Corollary 1.1.2. *Let* $s \in]0, 1[$ *and let* $p \in]1, \infty[$*. Let* Ω *be a Lipschitz open set. We then have:*

- *If* $sp < N$*, then* $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for every* $q \leq Np/(N - sp)$.
- *If* $N = sp$*, then* $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for every* $q < \infty$.
- *If* $sp > N$*, then* $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ *and, more precisely,*

$$W^{s,p}(\Omega) \hookrightarrow C_b^{0,s-N/p}(\Omega).$$

Theorem 1.1.4. *[10]*

Let Ω *be a bounded Lipschitz open subset of* $\mathbb{R}^N$*. Let* $s \in [0, 1[$*, let* $p > 1$*, and let* $N \geq 1$*. We then have*

- *If* $sp < N$*, then the embedding of* $W^{s,p}(\Omega)$ *into* L^k *is compact for every*

$$k < Np/(N - sp).$$

- *If $sp = N$, then the embedding of $W^{s,p}(\Omega)$ into L^q is compact for every $q < \infty$.*
- *If $sp > N$, then the embedding of $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}(\Omega)$ is compact for $\lambda < s - N/p$.*

1.2 Convergence criteria

Theorem 1.2.1. *[10]* ***(Lebesgue's dominated convergence)***

Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy

- *$f_n(x) \longrightarrow f$ a.e, on Ω,*
- *There is a function $g \in L^1(\Omega)$ such that for all n,*

$$|f_n(x)| \leq g(x), \quad a.e. \text{ on } \Omega.$$

Then

$$f \in L^1(\Omega) \text{ and } \|f_n - f\|_{L^1} \longrightarrow 0.$$

Theorem 1.2.2. ***(Vitali's convergence theorem)***

Let $f_1, f_2, ...$be L^p-integrable functions on some measure space, for $1 \leq p < \infty$. The sequence $\{f_n\}$ converges in L^p to a measurable function f if and and only if

- *The sequence $\{f_n\}$ converges to f in measure.*
- *The functions $\{|f_n|^p\}$are uniformly integrable.*

- *For every* $\epsilon > 0$*, there exists a set* E*of finite measure, such that* $\int\limits_{E^c} |f_n|^p < \epsilon$*for all* n.

Theorem 1.2.3. *[10] Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^p(\Omega)$ *and* $f \in L^p(\Omega)$ *such that*

$$\|f_n - f\|_p \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Then, there exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ *and a function* $h \in L^p(\Omega)$ *such that*

- $f_{n_k}(x) \longrightarrow f(x)$ $a.e$ *on* Ω,
- $|f_{n_k}(x)| \leq h(x)$ $\forall k$, $a.e.$ *on* Ω.

Lemma 1.2.1. *[10]* ***(Fatou's Lemma)***

Let (f_n) *a sequence of functions in* $L^1(\Omega)$ *that satisfy*

- *For all* n, $f_n \geq 0$ *a.e.*
- $\sup\limits_n \int f_n < \infty$.
- *For almost all* $x \in \Omega$ *we set* $f(x) = \liminf\limits_{n\to\infty} f_n(x) \leq +\infty$*. Then* $f \in L^1(\Omega)$ *and*

$$\int_\Omega f(x)dx \leq \lim_{n\to\infty} \inf \int_\Omega f_n(x)dx.$$

1.3 Maximum principle

A very large number of results of regularity, uniqueness or existence of solutions in second order elliptical problems can be established using (one might say only) the maximum principle.

Let Ω be an open set of $\mathbb{R}^N$, $a(\cdot) := (a_{ij}(\cdot))_{1\leq i,j\leq N}$ a matrix, $b(\cdot) := (b_i(\cdot))_{1\leq i\leq N}$ a vector field and c a function.

We consider second-order L elliptic differential operators of the form

(1. 3. 1) $$Lu := -\sum_{i,j=1}^{N} a_{ij}\partial_{ij}u + \sum_{i=1}^{N} b_i\partial_i u + cu.$$

As a general rule, we will suppose that the matrix $a(.)$ satisfies the condition of coercivity (or ellipticity):

(1. 3. 2) $$\begin{cases} \forall\alpha > 0, \forall\varsigma \in \mathbb{R}^N, p.p. \text{ on } \Omega, \\ a(.)\varsigma.\varsigma = \sum_{i,j=1}^{N} a_{ij}(x)\varsigma_j\varsigma_i \geq \alpha\left|\varsigma\right|^2. \end{cases}$$

Where $|\varsigma|$ denote the Euclidean norm of ς in $\mathbb{R}$.

Theorem 1.3.1. *[10]* ***(Classical maximum principle)***

Let Ω be a connexe bounded open, and L as in (1. 3. 1). We assume $c \geq 0$, that (1. 3. 2) is satisfied and that $a_{ij}, b_i, c \in C(\overline{\Omega})$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ verifies $Lu \leq 0$

then we have

$$\sup_{x\in\overline{\Omega}} u(x) \leq \sup_{\sigma\in\partial\Omega} u^+(\sigma), \text{ or } u^+(\sigma) = \max(u(\sigma), 0).$$

Theorem 1.3.2. *[10]* ***(Principle of the Hopf maximum)***

Let Ω be a connexe bounded open, and L as in (1. 3. 1). We assume $c \geq 0$, that (1. 3. 2) is satisfied and that $a_{ij}, b_i, c \in C(\overline{\Omega})$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ verifies $Lu \leq 0$ and if u attains a maximum ≥ 0 inside Ω, then u is constant over Ω.

Definition 1.3.1. ***(Directional derivative)***

Let w be a part of a Banach space X and $F : w \to \mathbb{R}$ a real valued function. If $u \in w$ and $z \in X$ we have $u + tz \in w$, we say that F admits (at the point u) a derivative in the direction z if the limit

$$\lim_{t\to \overset{+}{0}} \frac{F(u+tz) - F(u)}{t}, \text{ for all } t > 0 \text{ small enough.}$$

We will denote this limit $F'_z(u)$.

The Gateaux differential generalizes the idea of a directional derivative.

Definition 1.3.2. ***(Gateaux derivative)***

Let w be a part of a Banach space X and $F : w \to \mathbb{R}$. If $u \in w$, we say that F is Gateaux differentiable in u, if there exists $l \in X'$ or $F(u + tz)$ for $t > 0$ small *enough. The Gateaux differential is defined*

$$\langle l, z \rangle = \lim_{t \to \overset{+}{0}} \frac{F(u+tz) - F(u)}{t},$$

where $F'(u) := l$.

Definition 1.3.3. ***(Frechet derivative)***

Let X be a Banach space, w an open space in X and F a function. If $u \in w$, we say that F is differentiable (or derivable) in u (in the sense of Frechet) if there exists $l \in X'$, such that

$$\forall v \in w \; F(v) - F(u) = \langle l, v - u \rangle + \sigma(v - u).$$

If F is differentiable, l is unique and we denote by $F'(u) := l$. The set of differentiable functions of $w \to \mathbb{R}$ will be denoted by $C^1(w, \mathbb{R})$.

1.4 Notions on operators

Let $(X, \|.\|)$ be a real Banach space and let X' be topological dual.

Definition 1.4.1. *Let $A : X \to X'$, we say that*

- *A is bounded by the image by A of any born of X is bound of X.*
- ***Continuous** if $\|Ax_n - Ax\|_{X'} \to 0$ when $\|x_n - x\|_X \to 0$.*
- ***Compact** if $A(\overline{B}_X)$ is relatively compact in X', where B_X denotes the ball unit in X.*

- ***Coercive*** *if*

$$\lim_{\|x\|\to+\infty} \frac{\langle A(x), x\rangle}{\|x\|} = +\infty.$$

- ***Monotonous*** *if*

$$\langle Au - Av, u - v\rangle \geq 0, \forall u, v \in X \text{ with } u \neq v.$$

- ***Strictly monotonous*** *if*

$$\langle Au - Av, u - v\rangle > 0, \forall u, v \in X \text{ with } u \neq v.$$

- ***Bounded*** *if the image by A of any bounded of X is a bounded of X'.*
- ***Semi-continuous*** *(continuous from strong X in X'weak)*

$$\text{if } u_n \to u \text{ when } \infty \text{ implies } Au_n \rightharpoonup Au \text{ when } n \to \infty.$$

- ***Strongly continuous***

$$\text{if } u_n \rightharpoonup u \text{ when } \infty \text{ implies } Au_n \to Au \text{ when } n \to \infty.$$

Chapter 2

Nehari manifold and Fibering method

Contents

2.1 Introduction

The aim of this chapter is to present some variational methods used in nonlinear analysis. Apply a variational method means, that rather than directly search a solution of a PDE or a Hamiltonian system, we consider the equivalent system of finding a critical point of a functional. We begin by presenting a unified approach and a generalization to the method of Nehari manifold for functionals that have a local minimum at 0 of finding positive solutions and multiple solutions.

2.2 Fractional p-Laplacian operator

Let $(-\Delta)_p^s u$ be the fractional p-Laplacian operator defined on smooth functions by

$$(-\Delta)_p^s u = 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+ps}}dy, x\in\mathbb{R}^N.$$

If $p=2$, $(-\Delta)_p^s$coincides with the usual fractional Laplacian operator $(-\Delta)^s$, defined as follows

$$\begin{aligned}(-\triangle)^s u(x) &= C(n,s)P.V.\int\limits_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\left(|x-y|\right)dy,\\ &= C(n,s)\lim_{\epsilon\to 0^+}\int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\left(|x-y|\right)dy.\end{aligned}$$

Here $P.V.$ is a commonly used abbreviation for “in the principal value sense" (as defined by the latter equation) and $C(n,s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n,s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\varsigma_1)}{|\varsigma_1|^{N+2s}} d\varsigma_1 \right)^{-1}.$$

2.3 Critical point theory

Definition 2.3.1. ***(Homogeneous function of degree k)***

Let f be a function of n variables defined on a set S for which $(tx_1, ..., tx_n) \in S$ whenever $t > 0$ and $(tx_1, ..., tx_n) \in S$. Then f is homogeneous of degree k if

$$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n) \text{ for all } (x_1, ..., x_n) \in S \text{ and all } t > 0.$$

Definition 2.3.2. ***(Variationnal system)***

In general, we distinguish two types of variational elliptical systems

(1) *Lagrangian system: where the non-linearity is the gradient of a function H*

$$\frac{\partial H(x,u,v)}{\partial u} = f(x,u,v) \text{ and } \frac{\partial H(x,u,v)}{\partial v} = g(x,u,v).$$

(2) *Hamiltonian system: where ther exist H verifie*

$$\frac{\partial H(x,u,v)}{\partial u} = g(x,u,v) \text{ and } \frac{\partial H(x,u,v)}{\partial v} = f(x,u,v).$$

Let E be a reflexive Banach space, $J_\lambda \in C^1(E, \mathbb{R})$. There are several notions of derivatives for functions defined on Banach spaces.

Definition 2.3.3. ***(Critical point)***

A point $(u, v) \in E$ is critical if $J'_\lambda(u, v) = 0$, otherwise (u, v) is regular. If $J_\lambda(u, v)$ $= c$ for some critical point $(u, v) \in E$ of J_λ, the value c is critical, otherwise c is regular.

Lagrange multiplier

Let E be a Banach space, $\Phi \in C^1(E, \mathbb{R})$ is a set of constraints:

$$\mathcal{N} = \{v \in E : \Phi(v) = 0\},$$

Definition 2.3.4. *we suppose that for all $u \in \mathcal{N}$, we have $\Phi'(u) \neq 0$. If J $\in C^1(E, \mathbb{R})$ we say that $c \in \mathbb{R}$ is critical value of J on $\mathcal{N}$, if there exists $u \in \mathcal{N}$, and $\lambda \in \mathbb{R}$ such that*

$$J(u) = c \text{ and } J'(u) = \lambda \Phi'(u).$$

The point u is a critical point of J on $\mathcal{N}$ and the real λ is called the Lagrange multiplier for the critical value c (or the critical point u).

When X is a functional space and the equation $J'(u) = \lambda \Phi'(u)$ corresponds to a partial derivative equation, we say that $J'(u) = \lambda \Phi'(u)$ is the Euler-Lagrange equation (or the Euler's equation) satisfied by the critical point u on the constraint $\mathcal{N}$.

Theorem 2.3.1. *Let $(E, \|.\|)$ be a Banach space, Ω an open in E and $J : \Omega \to \mathbb{R}$ a differentiable function on Ω and $\Phi \in C^1(\Omega, \mathbb{R}^n)$ of components $\Phi_1, ..., \Phi_n$. Given a point in $\mathbb{R}^n$, we set $K = \Phi^{-1}(a)$ which we assume not empty, if at a point $u_0 \in K$*

$$J(u_0) = \inf_{x \in K} J(u),$$

and if moreover the differential $\Phi'(u_0) \in L(E, \mathbb{R}^n)$ is surjective then there exist real numbers $\lambda_1, ..., \lambda_n$ for which

$$J'(u_0) = \sum_{i=1}^{n} \lambda_i \Phi_i'(u_0).$$

2.4 The Nehari Manifold

Nehari has introduced a variational method very useful in critical point theory and eventually came to bear his name. He considered a boundary value problem for a certain nonlinear second-order ordinary differential equation in an interval $[a, b]$ and proved that it has a nontrivial solution which may be obtained by constrained minimization. To describe Nehari's method in an abstract setting, let E be a Banach space and $J \in C^1(E, \mathbb{R})$ a functional. The Frechet derivative of J at u, $J'(u)$, is an element of the dual space E'. Suppose $u \neq 0$ is a critical point of J, i.e., $J'(u) = 0$. Then necessarily u is contained in the set

$$\mathcal{N} = \left\{ u \in E \backslash \{0\} : \left\langle J'(u), u \right\rangle = 0 \right\}.$$

So $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of $J(u)$ by minimizing the energy functional J on the constraint $\mathcal{N}$ is called the Nehari manifold. Set

$$c := \inf_{u \in \mathcal{N}} J(u).$$

Under appropriate conditions on J one hopes that c is attained at some $u_0 \in \mathcal{N}$ and that u_0 is a critical point.

2.5 Fibering method

At the end of the 1990s, the fibering method or the decomposition method introduced by Pohozaev for investigating some variational problems, and its applications to nonlinear elliptic equations.

Let X and Y be Banach spaces, and let A be an nonlinear operator acting from X to Y. We consider the equation

$$A(u) = h. \tag{2. 5. 1}$$

The fibering method is based on representation of solutions of equation (2. 5. 1) in the form

$$u = tu.$$

Where t is a real parameter, $t \neq 0$ in some open $J \subseteq \mathbb{R}$. Now, we give a complete description of the fibering method, we begin by defining the fibre map of the following

$$\phi(t) : \mathbb{R}^{+} \to \mathbb{R} \text{ such that } \phi(t) = J(tu),$$

then, we calculate $\phi'(t), \phi''(t)$ the first and second derivative of $\phi(t)$. We decompose $\mathcal{N}$ into three parts $\mathcal{N}^{+}, \mathcal{N}^{-}$, and $\mathcal{N}^{0}$ corresponding respectively, to local minima, local maxima and points of inflection of ϕ defined as follows

$$\begin{aligned} \mathcal{N}^{+} &= \{u \in \mathcal{N} : \phi''(1) > 0\}, \\ \mathcal{N}^{-} &= \{u \in \mathcal{N} : \phi''(1) < 1\}, \\ \mathcal{N}^{0} &= \{u \in \mathcal{N} : \phi''(1) = 0\}, \end{aligned}$$

and it is $\phi''(1)$ which is used for these definitions, since it is clear that if u is a local minimum for J, then u has a local minimum at $t = 1$.

The method of decomposition $(F.M)$ makes it possible to find solutions to the noncoercive problems and in the absence of the continuity of the operator A.

Example 2.5.1. *We consider the following problem:*

$$(P) \qquad \begin{cases} -\Delta u(x) = f(x, u(x)) \text{ in } \Omega, \\ \\ u(x) = 0 \text{ on } x \in \partial\, \Omega. \end{cases}$$

Let $E = W_0^{1,2}(\Omega)$ be the Banach space. The energy functional $J : E \to \mathbb{R}$ corresponding to the problem (P) defined as follows

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega F(x, u(x))dx.$$

Where $F(x, u(x)) = \int_0^u f(x, s)dx$. Obviously, the functional J may not be bounded on all the space but can be on some parts of E (called the Nehari manifold $\mathcal{N}$) defined as follows

$$\mathcal{N} = \left\{u \in E : \left\langle J'(u), u\right\rangle = 0\right\}.$$

Theorem 2.5.1. *Let $u \in E\backslash\{0\}$ and $t > 0$. Then $tu \in \mathcal{N}$ if and only if*

$$\phi_u'(t) = 0$$

where

$$\phi_u(t) = J(tu).$$

Proof. By definition, one has

$$\phi_u(t) = J(tu).$$

Therefore

$$\phi_u'(t) = \langle J'(u), u\rangle = \frac{1}{t}\langle J'(tu), tu\rangle.$$

If $\phi_u'(t) = 0$, then $\langle J'(tu), tu\rangle = 0$ i.e $tu \in \mathcal{N}$. In other terms, the points of the manifold $\mathcal{N}$ correspond to the stationary points of the maps $\phi_u(t)$. On the other hand, we decompose $\mathcal{N}$ into three parts $\mathcal{N}^+, \mathcal{N}^-, \mathcal{N}^0$ corresponding to local minima, local maxima and points of inflection of $\phi_u(t)$. For that, we calculate the second derivative of $\phi_u(t)$

$$\begin{aligned} \phi_u'(t) &= \langle J'(tu), u\rangle \\ &= \int_\Omega |\nabla(tu)|\,|\nabla u|\,dx - \lambda\int_\Omega f(x,tu)u dx \\ &= t\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f(x,tu)u dx, \end{aligned}$$

So

$$\begin{aligned} \phi_u''(t) &= \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega (f_u'(x,tu)u)u dx \\ &= \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f_u'(x,tu)u^2 dx. \end{aligned}$$

Thus, we conclude $\mathcal{N}^+, \mathcal{N}^-$, and $\mathcal{N}^0$defined as follows

$$\mathcal{N}^0 = \left\{ u \in \mathcal{N}, \int_\Omega (|\nabla u|^2 - \lambda(f'_u(x,u)u^2)dx = 0 \right\},$$

$$\mathcal{N}^+ = \left\{ u \in \mathcal{N}, \int_\Omega (|\nabla u|^2 - \lambda(f'_u(x,u)u^2)dx > 0 \right\},$$

$$\mathcal{N}^- = \left\{ u \in \mathcal{N}, \int_\Omega (|\nabla u|^2 - \lambda(f'_u(x,u)u^2)dx < 0 \right\},$$

and it is $\phi''_u(1)$ which is used for these definitions, since it is clear that if u is a local minimum for J, then u has a local minimum at $t = 1$. ■

Theorem 2.5.2. *Let $u \in \mathcal{N}$. Then*

- (i) $\phi'_u(1) = 0.,$
- (ii)

$$\begin{cases} u \in \mathcal{N}^+ \text{ if } \phi''_u(1) > 0, \\ u \in \mathcal{N}^- \text{ if } \phi''_u(1) < 0, \\ u \in \mathcal{N}^0 \text{ if } \phi''_u(1) = 0. \end{cases}$$

Proof. Let $u \in \mathcal{N}$ if and only if

$$\left\langle J'(u), u \right\rangle = 0,$$

which is equivalent to : $\phi'_u(1) = 0$ hence (i).

For (ii), there are three cases:

case $1 : u \in \mathcal{N}^+$, then

$$\int_\Omega (|\nabla u|^2 - \lambda f_u'(x,u)u^2)dx > 0$$

which is equivalent to $\phi_u^{''}(1) > 0$.

case $2 : u \in \mathcal{N}^-$, then

$$\int_\Omega (|\nabla u|^2 - \lambda f_u'(x,u)u^2)dx < 0$$

which is equivalent to $\phi_u^{''}(1) < 0$.

case $3 : u \in \mathcal{N}^0$, then

$$\int_\Omega (|\nabla u|^2 - \lambda f_u'(x,u)u^2)dx = 0$$

which is equivalent to $\phi_u^{''}(1) = 0$. ■

The following theorem attests that the minimizers of J on the manifold $\mathcal{N}$ are true, in general, critical points of J.

Theorem 2.5.3. *Suppose u_0 is a local minimizer for J on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then*

$$J'(u_0) = 0.$$

Proof. According to Lagrange's multiplier theorem

$$\exists \eta \in \mathbb{R} : J'(u_0) = \eta \xi'(u_0),$$

so

$$\langle J'(u_0), u_0\rangle = \eta \langle \xi'(u_0), u_0\rangle .$$

The constraint ξ defined as follows

$$\xi(u) = \langle J'(u), u\rangle = \int_{\Omega} (|\nabla u|^2 - \lambda f(x,u)u)dx.$$

For all $u_0 \in \mathcal{N}$, we have

$$\langle J'(u_0), u_0\rangle = \eta \langle \xi'(u_0), u_0\rangle = 0.$$

Therefore

$$\int_{\Omega} (|\nabla u_0|^2 - \lambda f(x,u_0)u_0)dx = 0,$$

then

$$\int_{\Omega} (|\nabla u_0|^2 \, dx = \lambda \int_{\Omega} f(x,u_0)u_0 dx,$$

thus

$$\begin{aligned}
\langle \xi'(u_0), u_0\rangle &= \int_{\Omega} (2\,|\nabla u_0|^2 - \lambda f'_u(x,u_0)u_0^2)dx - \lambda \int_{\Omega} f(x,u_0)u_0 dx \\
&= \int_{\Omega} (|\nabla u_0|^2 - \lambda f'_u(x,u_0)u_0^2)dx \\
&= \phi''_{u_0}(1) \neq 0.
\end{aligned}$$

Which implies that $\eta = 0$, then $J'(u_0) = 0$. ■

2.6 Ekeland's variational principle

In general, it is not clear that a bounded and lower semi-continuous functional E actually attains its infimum. The analytic function $f(x) = \arctan x$, for example, neither attains its infimum nor its supremum on the real line.

A variant due to Ekeland of Dirichlet's principle, however, permits one to construct minimizing sequences for such functionals E whose elements u_m each minimize a functional E_m, for a sequence of functionals $\{E_m\}$ converging locally uniformly to E.

Theorem 2.6.1. *[31] Let E be a reflexive Banach space with norm $\|.\|$, and $J : E \to \mathbb{R}$ is coercive and weakly lower semi-continuous on E , that is, suppose the following conditions are fullfilled:*

- $J(u, v) \to \infty$ as $\|(u, v)\| \to \infty$, $(u, v) \in E$.
- For any $(u, v) \in E$, any sequence (u_n, v_n) in E such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E there holds

$$J(u, v) \leq \liminf_{n\to\infty} J(u_n, v_n).$$

- Then J is bounded from below on E and attains its infimum in E such that

$$J(u_0, v_0) = \inf_E J.$$

Theorem 2.6.2. *[31] Let M be a complete metric space with metric d, and let $J : M \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, bounded from below, and $\neq \infty$.Then for any $\epsilon, \delta > 0$, any $u \in M$ with*

$$J(u) \leq \inf_M J(u) + \epsilon,$$

there is an element $v \in M$ strictly minimizing the functional

$$J_v(w) \leq J(w) + \frac{\epsilon}{\delta} d(v, w).$$

Moreover, we have

$$J(v) \leq J(u), \; d(u, v) \leq \delta.$$

Chapter 3

Singular fractional elliptic system

Contents

3.1 Introduction

In this chapter, we apply the Nehari Manifold, Fibering method, and Ekeland's variational principle to establish the existence and multiplicity results of nontrivial positive solutions for the system (p_λ). We consider the following singular fractional elliptic system

$$(P_\lambda) \qquad \begin{cases} (-\Delta)_p^s u = a(x)u^{-\gamma} + \lambda f(x,u,v) \text{ in } \Omega, \\ \\ (-\Delta)_p^s v = b(x)v^{-\gamma} + \lambda g(x,u,v) \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Where Ω is a smooth bounded set in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, λ is a positive parameter and $0 < \gamma < 1 < p < r < p_s^* - 1$, where $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent. Where a and b are positive functions of class $L^\infty(\Omega)$.

We assume there exists a function $H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$H_u\left(u,v\right) = f\left(u,v\right) \text{ and } H_v\left(u,v\right) = g\left(u,v\right).$$

From $(0.\ 0.\ 1)$, we can easily deduce that H is homogeneous of degree r which satisfies the following assumptions

$H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} H(x, tu, tv) &= t^r H(x, u, v), t > 0, \\ rH(x, u, v) &= uf(x, u, v) + vg(x, u, v) \text{ (Euler identity)}, \\ |H(x, u, v)| &\leq K\left(|u|^r + |v|^r\right), \text{ for some constant } K > 0. \qquad (3.\ 1.\ 1) \end{aligned}$$

(3. 1. 2) $\qquad H^{\pm}(x, u, v) = \max(\pm H(x, u, v), 0) \neq 0$ for all $(u, v) \neq (0, 0)$.

Let $\Lambda = \mathbb{R}^{2N} \backslash ((\mathbb{R}^N \backslash \Omega) \times (\mathbb{R}^N \backslash \Omega))$, X denote the usual space defined as followX_0 denote the usual space define as follow

$$X_0 = \left\{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\right\},$$

with the norm

$$\|u\|_{X_0} = (T(u, u))^{\frac{1}{p}} = \left(\int_\Lambda \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy\right)^{\frac{1}{p}},$$

where

$$T(u, z) = \int_\Lambda \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(z(x) - z(y))}{|x - y|^{N+ps}} dxdy.$$

Setting $E = X_0 \times X_0$ a reflexive Banach space, with the norm

$$\|(u,v)\|_E = \left(\|u\|_{X_0}^p + \|v\|_{X_0}^p\right)^{\frac{1}{p}} = (T(u,u) + T(v,v))^{\frac{1}{p}},$$

where $\mu_{s,p}$ is the best Sobolev constant of the embedding from X_0 into $L^{p_s^*}(\Omega)$ given by

$$\mu_{s,p} = \inf_{u \in X_0} \left(\frac{\|u\|_{X_0}}{\|u\|_{L^{p_s^*}}} \right)^p. \tag{3. 1. 3}$$

Now, we give the definition of the weak solution.

3.2 Main results

Let us start by defining the notion of weak solution of the system (P_λ).

Definition 3.2.1. ***(Weak solution)****We say that $(u,v) \in E$ is a weak solution of system (P_λ) if for every $(z,w) \in E$ we have*

$$\begin{aligned} T(u,z) + T(v,w) &= \int_\Omega \left(a(x)u^{-\gamma} z dx + b(x)v^{-\gamma} w\right) dx \\ &\quad + \lambda \int_\Omega (z f(x,u,v) + w g(x,u,v))\, dx, \end{aligned} \tag{3. 2. 1}$$

where

$$T(u,z) = \int_{\Lambda} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(z(x)-z(y))}{|x-y|^{N+ps}}dxdy,$$

$$T(v,w) = \int_{\Lambda} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+ps}}dxdy.$$

The following theorem show the existence and multiplicity of positive nontrivial solutions to the system (p_λ) for all $\lambda \in (0,\Gamma)$.

Theorem 3.2.1. *Suppose that* $0<\gamma<1<p<r<p_s^*-1$ *and the assumptions* $(0.0.1)-(3.\ 1.2)$ *holds. Then, there exists* $\Gamma>0$. *Such that, system* (P_λ) *has at least two positive nontrivial solutions for all* $\lambda \in (0,\Gamma)$.

Let Γ be a constant define by

(3. 2. 2)

$$\Gamma = \frac{1}{rK}\left(\frac{p+\gamma-1}{r+\gamma-1}\right)\left[\left(\frac{r+\gamma-1}{r-p}\right)\mu_{s,p}^{\frac{\gamma-1}{p}}\max(\|a\|_\infty,\|b\|_\infty)\right]^{\frac{p-r}{p-1+\gamma}}|\Omega|^{\frac{(r+\gamma-1)(p-p_s^*)}{p_s^*(p+\gamma-1)}}.$$

3.3 The Nehari Manifold and the Fibering maps

We consider the energy functional $J_\lambda : E \to \mathbb{R}$ corresponding to the system (P_λ) defined as

$$J_\lambda(u,v) = \frac{1}{p}\|(u,v)\|_E^p - \frac{1}{1-\gamma}\int_\Omega (a(x)u^{1-\gamma}+b(x)v^{1-\gamma})dx - \lambda\int_\Omega H(x,u,v)dx.$$

Then, the functional J_λ is Fréchet derivative and the critical points are the weak solutions of (P_λ).

Notations

To simplify the calculus, we put

$$\begin{aligned} Q &= Q\left(u,v\right) = \|(u,v)\|_E^p = T(u,u)+T(v,v), \\ R &= R\left(u,v\right) = \int_\Omega (a(x)u^{1-\gamma}+b(x)v^{1-\gamma})dx, \\ S &= S\left(u,v\right) = \int_\Omega H(x,u,v)dx. \end{aligned}$$

We consider a sequence $\{(u_n,v_n)\} \in E$ such that $(u_n,v_n) \to (u,v)$ in E, as $n \to +\infty$. Therfore, one has

$$Q_n = Q\left(u_n,v_n\right), R_n = R\left(u_n,v_n\right) \text{ and } S_n = S\left(u_n,v_n\right),$$

Furthermore, $(u_0,v_0) \in \mathcal{N}_\lambda^0$ is a critical point of J_λ such that

$$Q_0 = Q\left(u_0,v_0\right), R_0 = R\left(u_0,v_0\right) \text{ and } S_0 = S\left(u_0,v_0\right).$$

Thus, we can write J_λ as follows

$$J_\lambda(u,v) = \frac{1}{p}Q - \frac{1}{1-\gamma}R - \lambda S.$$

Also, $J_\lambda \in C^1(E,\mathbb{R})$ and $J_\lambda^{'} : E \to E^{'}$ for every $(u,v) \in E$, we have that

$$\left\langle J_\lambda^{'}(u,v),(u,v)\right\rangle = Q - R - \lambda r S.$$

To find the critical points of J_λ , we will minimize the energy functional J_λ on the constraint of the Nehari Manifold defined as follows

$$\mathcal{N}_\lambda = \left\{(u,v) \in E\backslash\{(0,0)\} : \left\langle J_\lambda^{'}(u,v),(u,v)\right\rangle = 0\right\}.$$

Then, $(u,v) \in \mathcal{N}_\lambda$ if and only if

(3. 3. 1) $$Q - R - \lambda r S = 0 \text{ for all } (u,v) \in E \setminus (0,0),$$

we define the fiber map $\phi_{u,v}(t) : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\phi_{u,v}(t) = J_\lambda(tu,tv) = \frac{1}{p}Qt^p - \frac{1}{1-\gamma}Rt^{1-\gamma} - \lambda S t^r,$$

the first and the second derivative of the map $\phi_{u,v}(t)$ is given by

$$\phi_{u,v}^{'}(t) = Qt^{p-1} - Rt^{-\gamma} - \lambda r S t^{r-1},$$

and

$$\phi_{u,v}^{''}(t) = (p-1)Qt^{p-2} + \gamma R t^{-\gamma-1} - \lambda r(r-1)St^{r-2}.$$

It is easy to see that

$$(tu, tv) \in \mathcal{N}_\lambda \text{ if and only if } \phi'_{u,v}(t) = 0,$$

and

$$(u, v) \in \mathcal{N}_\lambda \text{ if and only if } \phi'_{u,v}(1) = 0.$$

Hence for $(u, v) \in \mathcal{N}_\lambda$, we obtain

$$\begin{aligned}
\phi''_{u,v}(1) &= (p-1)Q + \gamma R - \lambda r(r-1)S, \\
&= (r+\gamma-1)R - (r-p)Q, \\
&= (p+\gamma-1)Q - \lambda r(r+\gamma-1)S, \\
&= (p+\gamma-1)R - \lambda r(r-p)S.
\end{aligned}$$

Now, we decompose $\mathcal{N}_\lambda$ into three parts $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-, \mathcal{N}_\lambda^0$ corresponding to local minima, local maxima and points of inflection of $\phi_{u,v}$ defined as follows

$$\begin{aligned}
\mathcal{N}_\lambda^0 &= \left\{(u, v) \in \mathcal{N}_\lambda, \phi''_{u,v}(1) = 0\right\}, \\
\mathcal{N}_\lambda^+ &= \left\{(u, v) \in \mathcal{N}_\lambda, \phi''_{u,v}(1) > 0\right\}, \\
\mathcal{N}_\lambda^- &= \left\{(u, v) \in \mathcal{N}_\lambda, \phi''_{u,v}(1) < 0\right\}.
\end{aligned}$$

It's clear that, if $(u,v) \in \mathcal{N}_\lambda^+$, we have

$$R > \frac{r-p}{r+\gamma-1} Q > 0 \text{ then } R > 0. \tag{3. 3. 2}$$

Now, we give some very interesting lemmas.

Lemma 3.3.1. *Suppose that (u_0, v_0) is a local minimizer of J_λ on $\mathcal{N}_\lambda$ and that $(u_0, v_0) \notin \mathcal{N}_\lambda^0$. Then (u_0, v_0) is a critical point of J_λ.*

Proof. According to the theorem of Lagrange multiplier

$$\exists \eta \in \mathbb{R} : J_\lambda'(u_0, v_0) = \eta \xi'(u_0, v_0),$$

$$(u_0, v_0) \in \mathcal{N}_\lambda : \langle J_\lambda'(u_0, v_0), (u_0, v_0)\rangle = \eta \langle \xi'(u_0, v_0), (u_0, v_0)\rangle = 0.$$

Where the constraint is

$$\xi(u,v) = Q - R - \lambda r S.$$

For all $(u,v) \in \mathcal{N}_\lambda$ and $(u_0, v_0) \notin \mathcal{N}_\lambda^0$, we have

$$\begin{aligned} \langle \xi'(u,v), (u,v)\rangle &= pQ - (1-\gamma)R - \lambda r^2 S \\ &= (p-1)Q + \gamma R - \lambda r(r-1)S \\ &= \phi''_{u,v}(1) \neq 0. \end{aligned}$$

Which implies that $\eta = 0$, then $J_\lambda'(u_0, v_0) = 0$. ■

Lemma 3.3.2. *$J_\lambda$is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. Let $(u,v)\in\mathcal{N}_\lambda$. From (3. 1. 3) and the Hölder's inequality, we obtain

$$\begin{aligned} R &\leq \mu_{s,p}^{\frac{\gamma-1}{p}}\,|\Omega|^{\frac{p_s^*+\gamma-1}{p_s^*}}\max(\|a\|_\infty,\|b\|_\infty)\,\|(u,v)\|_E^{1-\gamma}.\\ &= \mu_{s,p}^{\frac{\gamma-1}{p}}\,|\Omega|^{\frac{p_s^*+\gamma-1}{p_s^*}}\max(\|a\|_\infty,\|b\|_\infty)Q^{\frac{1-\gamma}{p}}. \end{aligned} \tag{3. 3. 3}$$

and

$$\begin{aligned} S &\leq K\int_\Omega(|u|^r+|v|^r)\,dx \leq K\,|\Omega|^{\frac{p_s^*-r}{p_s^*}}\left(\|u\|_{p_s^*}^r+\|v\|_{p_s^*}^r\right)\\ &\leq K\,|\Omega|^{\frac{p_s^*-r}{p_s^*}}\,\mu_{s,p}^{\frac{-r}{p}}\,\|(u,v)\|_E^r = K\,|\Omega|^{\frac{p_s^*-r}{p_s^*}}\,\mu_{s,p}^{\frac{-r}{p}}Q^{\frac{r}{p}}. \end{aligned} \tag{3. 3. 4}$$

From the assumptions, (3. 3. 1) and (3. 3. 3) we have that, for all $(u,v)\in\mathcal{N}_\lambda$,

$$\begin{aligned} J_\lambda(u,v) &= \frac{1}{p}Q-\frac{1}{1-\gamma}R-\lambda S\\ &= \left(\frac{r-p}{rp}\right)Q-\left(\frac{\gamma+r-1}{r(1-\gamma)}\right)R\\ &\geq \left(\frac{r-p}{rp}\right)Q-\left(\frac{\gamma+r-1}{r(1-\gamma)}\right)\mu_{s,p}^{\frac{\gamma-1}{p}}\,(\|a\|_\infty+\|b\|_\infty)\,|\Omega|^{\frac{p_s^*+\gamma-1}{p_s^*}}\,Q^{\frac{1-\gamma}{p}}, \end{aligned}$$

since $0<\gamma<1$ and $r>p>1>1-\gamma$, the functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. ■

Lemma 3.3.3. *Let $\lambda\in(0,\Gamma)$. Then, there exist two number denoted t_1 and t_2 such that*

$$\phi'_{u,v}(t_1)=\phi'_{u,v}(t_2)=0,$$

and

$$\phi''_{u,v}(t_1) < 0 < \phi''_{u,v}(t_1),$$

that is $(t_1u, t_1v) \in \mathcal{N}_\lambda^+$ *and* $(t_2u, t_2v) \in \mathcal{N}_\lambda^-$.

Proof. We consider $\psi_{u,v}(t) : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_{u,v}(t) = Qt^{p-r} - Rt^{1-\gamma-r} - \lambda r S.$$

The first derivative $\psi_{u,v}(t)$ is given by

$$\begin{aligned} \psi'_{u,v}(t) &= (r+\gamma-1)Rt^{-\gamma-r} - (r-p)Qt^{p-r-1} \\ &= (r+\gamma-1)Rt^{-\gamma-r} - (r-p)Qt^{p-r-1}, \end{aligned}$$

then $\psi'_{u,v}(t) = 0$ if and only if $\psi_{u,v}(t)$ has a unique critical point at

$$t_{\max} = \left(\frac{(r+\gamma-1)R}{(r-p)Q}\right)^{\frac{1}{p+\gamma-1}}, \tag{3. 3. 5}$$

and

$$\lim_{t\to+\infty} \psi_{u,v}(t) = -\lambda r S \text{ and } \lim_{t\to 0^+} \psi_{u,v}(t) = -\infty.$$

Moreover, $\psi'_{u,v}(t) > 0$ for all $0 < t < t_{\max}$ and $\psi'_{u,v}(t) < 0$ for all $t > t_{\max}$. Then

$$\begin{aligned} \psi_{u,v}(t_{\max}) &= Qt_{\max}^{p-r} - Rt_{\max}^{-\gamma-r+1} - \lambda r S \\ &= \left(\frac{\gamma+r-1}{r-p}\right)^{\frac{p-r}{p-1+\gamma}} \left(\frac{p+\gamma-1}{\gamma+r-1}\right) Q^{\frac{\gamma+r-1}{p+\gamma-1}} R^{\frac{p-r}{p+\gamma-1}} - \lambda r S \end{aligned} \tag{3. 3. 6}$$

From $(3.\,2.\,2)$, $(3.\,3.\,3)$ and $(3.\,3.\,4)$, one sees that

$$\begin{aligned}
\psi(t_{\max}) &\geq \left(\frac{\gamma+r-1}{r-p}\right)^{\frac{p-r}{p+\gamma-1}}\left(\frac{p+\gamma-1}{r+\gamma-1}\right)\left(\frac{Q^{r+\gamma-1}}{\left(\mu_{s,p}^{\frac{\gamma-1}{p}}\left|\Omega\right|^{\frac{p_s^*+\gamma-1}{p_s^*}}\max(\|a\|_\infty,\|b\|_\infty)Q^{\frac{1-\gamma}{p}}\right)^{r-p}}\right)^{\frac{1}{p+\gamma-1}} \\
&\quad -\lambda rK\left|\Omega\right|^{\frac{p_s^*-r}{p_s^*}}\mu_{s,p}^{\frac{-r}{p}}Q^{\frac{r}{p}} \\
&\geq \left[\left(\frac{p+\gamma-1}{r+\gamma-1}\right)\frac{\left(\left(\frac{r+\gamma-1}{r-p}\right)\mu_{s,p}^{\frac{\gamma-1}{p}}\max(\|a\|_\infty,\|b\|_\infty)\right)^{\frac{p-r}{p-1+\gamma}}}{rK\left|\Omega\right|^{\frac{(r+\gamma-1)(p_s^*-p)}{p_s^*(p+\gamma-1)}}}-\lambda\right]rK\mu_{s,p}^{\frac{-r}{p}}\left|\Omega\right|^{\frac{p_s^*-r}{p_s^*}}Q^{\frac{r}{p}} \\
&= (\Gamma-\lambda)\,rK\mu_{s,p}^{\frac{-r}{p}}\left|\Omega\right|^{\frac{p_s^*-r}{p_s^*}}\|(u,v)\|_E^r.
\end{aligned}$$

Then, for all $\lambda\in(0,\Gamma)$,we obtain

$$\psi(t_{\max})\geq 0.$$

As a consequence, there exist t_1 and t_2 such that $0<t_1<t_{\max}<t_2$ verifies

$$\psi_{u,v}(t_1)=\psi_{u,v}(t_2)=0,$$

and

$$\psi_{u,v}^{'}(t_1)<0<\psi_{u,v}^{'}(t_2).$$

We conclude that $(t_1u,t_1v)\in\mathcal{N}_\lambda^+$ and $(t_2u,t_2v)\in\mathcal{N}_\lambda^-$. ■

Corollary 3.3.1. *For all $\lambda \in (0,\Gamma)$, then $\mathcal{N}_\lambda^{\pm} \neq \emptyset$ and $\mathcal{N}_\lambda^{0} = \{(0,0)\}$. Moreover $\mathcal{N}_\lambda^{-}$ is a closed set in $E-$topology.*

Proof. First, according to the **Lemma** 3. 3. 3, $\mathcal{N}_\lambda^{\pm}$ are non-empty for all $\lambda \in (0,\Gamma)$. Now, we proceed by contradiction to establish that $\mathcal{N}_\lambda^{0} = \{(0,0)\}$ for all $\lambda \in (0,\Gamma)$. For this, let us suppose that there exists $(0,0) \neq (u_0,v_0) \in \mathcal{N}_\lambda^{0}$. Then we get

$$(p+\gamma-1)Q_0 - \lambda r(r+\gamma-1)S_0 = 0, \tag{3. 3. 7}$$

which implies that

$$0 = Q_0 - R_0 - \lambda r S_0 = \left(\frac{r-p}{r+\gamma-1}\right) Q_0 - R_0. \tag{3. 3. 8}$$

From (3. 3. 7) and (3. 3. 8), we obtain

$$\begin{aligned}
0 &< \psi_{u_0,v_0}(t_{\max}) - \lambda r S_0\\
&= \left(\frac{\gamma+r-1}{r-p}\right)^{\frac{p-r}{p-1+\gamma}} \left(\frac{p-1+\gamma}{\gamma+r-1}\right) \frac{(Q_0)^{\frac{\gamma+r-1}{p+\gamma-1}}}{(R_0)^{\frac{r-p}{p+\gamma-1}}} - \lambda r S_0\\
&\leq \left(\frac{r+\gamma-1}{r-p}\right)^{\frac{p-r}{p-1+\gamma}} \left(\frac{p+\gamma-1}{r+\gamma-1}\right) \left(\frac{r+\gamma-1}{r-p}\right)^{\frac{r-p}{p+\gamma-1}} Q_0 - \left(\frac{p+\gamma-1}{r+\gamma-1}\right) Q_0 = 0,
\end{aligned}$$

for all $\lambda \in (0,\Gamma)$, which is impossible. Thus $\mathcal{N}_\lambda^{0} = \{(0,0)\}$.

Now, to prove that $\mathcal{N}_\lambda^{-}$ is closed for all $\lambda \in (0,\Gamma)$, we consider a sequence $\{(u_n,v_n)\} \subset \mathcal{N}_\lambda^{-}$ such that $(u_n,v_n) \to (u,v)$ in E as $n \to +\infty$, then $(u,v) \subset \mathcal{N}_\lambda^{-}$.

Using the definition of $\mathcal{N}_{\lambda}^{-}$, we get

$$Q_n - R_n - \lambda r S_n = 0,$$

and

$$(p-1)Q_n + \gamma R_n - \lambda r(r-1)S_n < 0.$$

Hence, we get that

$$Q - R - \lambda r S = 0,$$

and

$$(p-1)Q + \gamma R - \lambda r(r-1)S \leq 0.$$

Therefore, $(u,v) \in \mathcal{N}_{\lambda}^{0} \cup \mathcal{N}_{\lambda}^{-} = \mathcal{N}_{\lambda}^{-}$. ■

Lemma 3.3.4. *Let $(u,v) \in \mathcal{N}_{\lambda}^{+}$ (respectively $\mathcal{N}_{\lambda}^{-}$), with $u,v \geq 0$ then for any $(z,w) \in E$ with $z,w \geq 0$, there exists a number $\varepsilon > 0$ and a continuous function $g : B(0,\varepsilon) \to \mathbb{R}$ such that*

$$g(0) = 1 \text{ and } g(s)(u+sz, v+sw) \in \mathcal{N}_{\lambda}^{+} \text{ (respectively } \mathcal{N}_{\lambda}^{-}).$$

Proof. We give the proof only for the case $(u,v) \in \mathcal{N}_{\lambda}^{+}$, the case $\mathcal{N}_{\lambda}^{-}$ may be preceded exactly. We define $\Psi : \mathbb{R}^{+} \times \mathbb{R}^{+} \to \mathbb{R}$ as follows

$$\begin{aligned}\Psi(t,s) &= Q(u+sz, v+sw)\, t^{p+\gamma-1} - \lambda r S(u+sz, v+sw) t^{r+\gamma-1} \\ &\quad - R(u+sz, v+sw)\,.\end{aligned}$$

The first derivative of the function Ψ is given by

$$\begin{aligned}\Psi_t(t,s) &= (p+\gamma-1)Q\left(u+sz,v+sw\right)t^{p+\gamma-2}\\ &\quad -\lambda r(r+\gamma-1)S(u+sz,v+sw)t^{r+\gamma-2}.\end{aligned}$$

It is clear that the function $\Psi_t(t,s)$ is continuous in $\mathbb{R}^+\times\mathbb{R}^+$. Moreover, since $(u,v)\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$, we obtain

$$\Psi(1,0)=Q-R-\lambda rS=0,$$

and

$$\Psi_t(1,0)=(p+\gamma-1)Q-\lambda r(r-1)S>0.$$

Thus, Using the implicit function theorem at the point $(1,0)$, we have that there exists $\delta>0$ and a positive continuous function $g:B(0,\varepsilon)\to\mathbb{R}$ such that

$$g(0)=1, g(s)(u+sz,v+sw)\in\mathcal{N}_\lambda,\forall s\in B(0,\delta).$$

Hence, putting $\varepsilon>0$ smaller enough, we get

$$g(s)(u+sz,v+sw)\in\mathcal{N}_\lambda^+,\forall s\in B(0,\varepsilon).$$

The proof of the **Lemma** 3.3.4 is completed. ■

3.4 Existence and multiplicity results

Now, we prove the theorem of existence and multiplicity of positive nontrivial solutions to the system (p_λ) for all $\lambda \in (0, \Gamma)$ in $\mathcal{N}_\lambda^{\pm}$.

Since $J_\lambda(u, v) = J_\lambda(|u|, |v|)$ for all $\lambda \in (0, \Gamma)$, we can assume that all the price elements in $\mathcal{N}_\lambda$ are nonnegative and from **Lemma** 3. 3. 2 and **Lemma** 3. 3. 3 we can found α^+ and α^- such that

$$\alpha^+ = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u, v) \text{ and } \alpha^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v).$$

3.4.1 Positive solutions in $\mathcal{N}_\lambda^+$

For all $(u, v) \in \mathcal{N}_\lambda^+$, and consequently since $r > p > 1 > 1 - \gamma$ and $0 < \gamma < 1$, we have that

$$J_\lambda(u, v) = \frac{1}{p}Q - \frac{1}{1-\gamma}R - \lambda S = -\frac{(p+\gamma-1)(r-p)}{rp(1-\gamma)}Q < 0.$$

Which means that

(3. 4. 1) $$\alpha^+ = \inf_{(u,v)\in N_\lambda^+} J_\lambda(u, v) < 0 \text{ for all } \lambda \in (0, \Gamma).$$

Proof of Theorem 3. 2. 1. The proof is split into two steps.

Proof. Step 1: Let us consider the minimizing sequences $\{(u_n, v_n)\}$ and applying Ekeland's variational principle, we obtain

$$\begin{cases} i)\ J_\lambda(u_n, v_n) < \alpha^+ + \frac{1}{n}, \\ \\ ii)\ J_\lambda(u, v) \geq J_\lambda(u_n, v_n) - \frac{1}{n}\left\|(u - u_n, v - v_n)\right\|_E \text{ for all } (u, v) \in \mathcal{N}_\lambda^+. \end{cases}$$

We can assume that $u_n, v_n \geq 0$. Clearly, as J_λ is coercive on $\mathcal{N}_\lambda$, $\{(u_n, v_n)\}$ is bounded in E. so there exists a sub-sequence denoted by $\{(u_n, v_n)\}$ and $u_0, v_0 \geq 0$ such that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E, strongly in $L^q(\Omega) \times L^q(\Omega), 1 < q < p_s^*$, and $u_n(x) \rightarrow u_0(x), v_n(x) \rightarrow v_0(x)$, a.e in Ω, Therefore, from (3. 4. 1) and by using the weak lower semi-continuity of norm, we obtain

$$J_\lambda(u_0, v_0) \leq \lim_{n\to\infty} \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u, v) < 0. \tag{3. 4. 2}$$

Claim 1. $u_0(x)$, $v_0(x) > 0$ **a. e. in** Ω.

Firstly, we start by observing that, since $(u_n, v_n) \in \mathcal{N}_\lambda^+$, one has

$$(p + \gamma - 1)Q_n - \lambda r(r + \gamma - 1)S_n > 0, \tag{3. 4. 3}$$

equivalent to

$$(p + \gamma - 1)R_n - \lambda r(r - p)S_n > 0. \tag{3. 4. 4}$$

By Vitali's convergence theorem, we have that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\int_{\Omega} H(x,u_n,v_n)dx = \int_{\Omega} H(x,u_0,v_0)dx = S_0. \tag{3. 4. 5}$$

Moreover, using Hölder inequality, we obtain that, as $n \to \infty$

$$\begin{aligned}\int_{\Omega} a(x)\, u_n^{1-\gamma}dx &\leq \int_{\Omega} a(x)\, u_0^{1-\gamma}dx + \|a\|_{\infty}\int_{\Omega}|u_n-u_0|^{1-\gamma}dx\\ &\leq \int_{\Omega} a(x)\, u_0^{1-\gamma}dx + |\Omega|^{\frac{p+\gamma-1}{p}}\|a\|_{\infty}\|u_n-u_0\|_{L^p(\Omega)}^{1-\gamma},\\ &= \int_{\Omega} a(x)\, u_0^{1-\gamma}dx + o(1)\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega} a(x)\, u_0^{1-\gamma}dx &\leq \int_{\Omega} a(x)\, u_n^{1-\gamma}dx + \|a\|_{\infty}\int_{\Omega}|u_n-u_0|^{1-\gamma}dx\\ &\leq \int_{\Omega} a(x)\, u_n^{1-\gamma}dx + |\Omega|^{\frac{p+\gamma-1}{p}}\|a\|_{\infty}\|u_n-u_0\|_{L^p(\Omega)}^{1-\gamma},\\ &= \int_{\Omega} a(x)\, u_n^{1-\gamma}dx + o(1).\end{aligned}$$

Then

$$\int_{\Omega} a(x)\, u_n^{1-\gamma}dx = \int_{\Omega} a(x)\, u_0^{1-\gamma}dx + o(1).$$

Similarly, we can obtain

$$\int_{\Omega} b(x)\, v_n^{1-\gamma}dx = \int_{\Omega} b(x)\, v_0^{1-\gamma}dx + o(1).$$

Thus

$$R_n = R_0 + o(1). \tag{3. 4. 6}$$

From the Brézis-Lieb lemma, we obtain that

$$\|(u_n, v_n)\|_E^p = \|(u_0, v_0)\|_E^p + \|u_n - u_0\|_{X_0}^p + \|v_n - v_0\|_{X_0}^p + o(1).$$

Thus

$$Q_n = Q_0 + o(1). \tag{3. 4. 7}$$

Therefore, it follows from (3. 4. 5) and (3. 4. 6) that

$$\begin{aligned} & \lim_{n\to\infty} [(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n] \\ = & \lim_{n\to\infty} [(p+\gamma-1)R_n - \lambda(r-p)S_n] \\ = & (p-1+\gamma)R_0 - \lambda r(r-p)S_0 \geq 0. \end{aligned}$$

Moreover, by contradiction, assume that

$$(p+\gamma-1)R_0 - \lambda r(r-p)S_0 = 0. \tag{3. 4. 8}$$

Hence, Using (3. 4. 5) – (3. 4. 8), and the weakly lower semi continuity of the norme, obtain

$$0 \geq Q_0 - R_0 - \lambda r S_0 = Q_0 - \lambda r \left(\frac{r+\gamma-1}{p+\gamma-1} \right) S_0.$$

From (3. 4. 3), one has a contradiction, thus

(3. 4. 9) $$(p+\gamma-1)R_0 - \lambda r(r-p)S_0 > 0.$$

Now, let us consider the functions $0 < z, w \in E$. From **Lemma 3.3.4**, there exits a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ such that $g_n(s)(u_n + sz, v_n + sw) \in \mathcal{N}_\lambda^+$ and $g_n(0) = 1$, that is

$$Q\left(g_n(s)(u_n+sz, v_n+sw)\right) - R\left(g_n(s)(u_n+sz, v_n+sw)\right) - \lambda r S\left(g_n(s)(u_n+sz, v_n+sw)\right) = 0,$$

since

(3. 4. 10) $$Q_n - R_n - \lambda r S_n = 0.$$

For s small enough, it follows that

$$\begin{aligned}
0 &= \left(g_n^p(s)-1\right)Q\left(u_n+sz,v_n+sw\right)+\left(Q\left(u_n+sz,v_n+sw\right)-Q_n\right)\\
&\quad -\left(g_n^{1-\gamma}(s)-1\right)R\left(u_n+sz,v_n+sw\right)-\left(R\left(u_n+sz,v_n+sw\right)-R_n\right)\\
&\quad -\lambda r\left(g_n^r(s)-1\right)S\left(u_n+sz,v_n+sw\right)-\lambda r\left(S\left(u_n+sz,v_n+sw\right)-S_n\right)\\
&\leq \left(g_n^p(s)-1\right)Q\left(u_n+sz,v_n+sw\right)+\left(Q\left(u_n+sz,v_n+sw\right)-Q_n\right)\\
&\quad -\left(g_n^{1-\gamma}(s)-1\right)R\left(u_n+sz,v_n+sw\right)\\
&\quad -\lambda r\left(g_n^r(s)-1\right)S\left(u_n+sz,v_n+sw\right)-\lambda r\left(S\left(u_n+sz,v_n+sw\right)-S_n\right),
\end{aligned}$$

dividing by $s>0$ and passing to the limit as $s\to 0$, we get

$$\begin{aligned}
0 &\leq g_n'(0)\left(pQ_n-(1-\gamma)R_n-\lambda r^2 S_n\right)+p\left(T(u_n,z)+T(v_n,w)\right)\\
&\quad -\lambda r\int_\Omega \left(zf(x,u_n,v_n)+wg(x,u_n,v_n)\right)dx\\
&\leq g_n'(0)\left(\left(p-r\right)Q_n+\left(r+\gamma-1\right)R_n\right)+p\left(T(u_n,z)+T(v_n,w)\right),
\end{aligned}$$

then, by $(3.\ 4.\ 3)$ and $(3.\ 4.\ 10)$, we obtain

$$g_n'(0)\geq -\frac{p\left(T(u_n,z)+T(v_n,w)\right)}{\left(p-r\right)Q_n+\left(r+\gamma-1\right)R_n} \tag{3. 4. 11}$$

where $g_n'(0)\in[-\infty,\infty]$ denotes the right derivative of $g_n(s)$ at zero and since $(u_n,v_n)\in\mathcal{N}_\lambda^+$, $g_n'(0)\neq-\infty$. For simplicity, we assume that the right derivative of g_n at $s=0$ exists. Moreover, from $(3.\ 4.\ 11)$ $g_n'(0)$ is uniformly bounded from below. Now, using the condition (ii), we have

$$\begin{aligned}
& |g_n(s)-1| \frac{\|(u_n,v_n)\|}{n} + s g_n(s) \frac{\|(z,w)\|}{n} \\
\geq \;& J_\lambda(u_n,v_n) - J_\lambda(g_n(s)(u_n+sz, v_n+sw)) \\
= \;& -\frac{p+\gamma-1}{p(1-\gamma)} Q_n + \lambda\left(\frac{r+\gamma-1}{1-\gamma}\right) S_n \\
& + \frac{p+\gamma-1}{p(1-\gamma)} g_n^p(s) Q(u_n+sz, v_n+sw) - \lambda\left(\frac{r+\gamma-1}{1-\gamma}\right) g_n^r(s) S(u_n+sz, v_n+sw) \\
= \;& \frac{p+\gamma-1}{p(1-\gamma)}\left[Q(u_n+sz, v_n+sw) - Q_n + (g_n^p(s)-1) Q(u_n+sz, v_n+sw)\right] \\
& -\lambda \frac{r+\gamma-1}{1-\gamma}\left[S(u_n+sz, v_n+sw) - S_n + (g_n^r(s)-1) S(u_n+sz, v_n+sw)\right],
\end{aligned}$$

dividing by $s>0$ and passing to the limit as $s \to 0$, we obtain

$$\begin{aligned}
& \frac{1}{n}\left(|g_n'(0)| \|(u_n,v_n)\| + \|(z,w)\|\right) \\
\geq \;& \frac{g_n'(0)}{1-\gamma}\left((p-r) Q_n + (r+\gamma-1) R_n\right) \\
& + \frac{p(p+\gamma-1)}{1-\gamma}\left(T(u_n,z) + T(v_n,w)\right) \\
& -\lambda\left(\frac{r+\gamma-1}{1-\gamma}\right) \int_\Omega \left(z f(x,u_n,v_n) + w g(x,u_n,v_n)\right) dx
\end{aligned}$$

which implies that

$$\begin{aligned}
&|g_n'(0)|\left(-\frac{\|(u_n,v_n)\|}{n}-((r-p)\,Q_n-(r+\gamma-1)\,R_n)\right)\\
\leq\;& \frac{1}{n}\|(z,w)\|-\frac{p\,(p+\gamma-1)}{1-\gamma}\,(T(u_n,z)+T(v_n,w))\\
&+\lambda\left(\frac{r+\gamma-1}{1-\gamma}\right)\int_\Omega (f(x,u_n,v_n)z+g(x,u_n,v_n)w)\,dx
\end{aligned}$$

Hence, there exists a positive constant L such that

$$-\frac{\|(u_n,v_n)\|}{n}-((p-r)\,Q_n+(r+\gamma-1)\,R_n)\geq L>0,$$

then

$$\begin{aligned}
|g_n'(0)| \;\leq\;& L^{-1}\left(\frac{1}{n}\|(z,w)\|+\frac{p\,(p+\gamma-1)}{1-\gamma}\,(T(u_n,z)+T(v_n,w))\right.\\
&\left.+\lambda\left(\frac{r+\gamma-1}{1-\gamma}\right)\int_\Omega (f(x,u_n,v_n)z+g(x,u_n,v_n)w)\,dx\right)
\end{aligned} \tag{3. 4. 12}$$

Thus, according to $(3.\ 4.\ 12)$, $g_n'(0)$ is uniformly bounded from above. Consequently,

(3. 4. 13) $\qquad g_n'(0)$ is uniformly bounded for n large enough.

Thus from condition (ii) it follows that for $s>0$ small enough, one has

$$\begin{aligned}
& \frac{1}{n}\left(\left|g_n(s)-1\right|\left\|(u_n,v_n)\right\|+s g_n(s)\left\|(z,w)\right\|\right) \\
\geq\ & \frac{1}{n}\left(\left\|g_n(s)\left(u_n+sz,v_n+sw\right)-(u_n,v_n)\right\|\right) \\
\geq\ & J_\lambda\left(u_n,v_n\right)-J_\lambda\left(g_n(s)\left(u_n+sz,v_n+sw\right)\right) \\
=\ & -\frac{g_n^p(s)-1}{p}Q_n+\frac{g_n^{1-\gamma}(s)-1}{1-\gamma}R_n+\lambda\left(g_n^r(s)-1\right)S_n+\frac{g_n^p(s)}{p}\left(Q_n-Q\left(u_n+sz,v_n+sw\right)\right) \\
& +\frac{g_n^{1-\gamma}(s)}{1-\gamma}\left(R\left(u_n+sz,v_n+sw\right)-R_n\right)+\lambda g_n^r(s)\left(S\left(u_n+sz,v_n+sw\right)-S_n\right),
\end{aligned}$$

dividing by $s>0$ and passing to the limit as $s\to 0$, we obtain

$$\begin{aligned}
& \frac{1}{n}\left(\left|g_n'(0)\right|\left\|(u_n,v_n)\right\|+\left\|(z,w)\right\|\right) \\
\geq\ & -g_n'(0)\left(Q_n-R_n-\lambda r S_n\right)-\left(T(u_n,z)+T(v_n,w)\right) \\
& +\lambda\int_\Omega\left(f(x,u_n,v_n)z+g(x,u_n,v_n)w\right)dx \\
& +\frac{1}{1-\gamma}\liminf_{s\to 0^+}\left(\frac{R\left(u_n+sz,v_n+sw\right)-R_n}{s}.\right) \qquad (3.\ 4.\ 14)
\end{aligned}$$

From (3. 4. 14), we deduce that

$$\begin{aligned}
& \frac{1}{1-\gamma}\liminf_{s\to 0^+}\left(\frac{R\left(u_n+sz,v_n+sw\right)-R_n}{s}\right) \\
\leq\ & T(u_n,z)+T(v_n,w)-\lambda\int_\Omega\left(f(x,u_n,v_n)z+g(x,u_n,v_n)w\right)dx \\
& +\frac{\left|g_n'(0)\right|\left\|(u_n,v_n)\right\|+\left\|(z,w)\right\|}{n}, \qquad (3.\ 4.\ 15)
\end{aligned}$$

since

$$R\left(u_n+sz,v_n+sw\right)-R_n\geq 0,\forall t>0,\forall x\in\Omega.$$

Using Fatou's Lemma, we get

$$\int_{\Omega}\left(a\left(x\right)u_n^{-\gamma}z+b\left(x\right)v_n^{-\gamma}w\right)dx\leq\frac{1}{1-\gamma}\liminf_{s\to 0^+}\left(\frac{R\left(u_n+sz,v_n+sw\right)-R_n}{s}\right),$$

hence, using $(3.\ 4.\ 15)$, it follow that

$$\begin{aligned}&\int_{\Omega}\left(a\left(x\right)u_n^{-\gamma}z+b\left(x\right)v_n^{-\gamma}w\right)dx\\ \leq\ & T(u_n,z)+T(v_n,w)-\lambda\int_{\Omega}\left(f(x,u_n,v_n)z+g(x,u_n,v_n)w\right)dx\\ &+\frac{\left|g_n'\left(0\right)\right|\left\|(u_n,v_n)\right\|+\left\|(z,w)\right\|}{n},\end{aligned}$$

for n large enough. By using $(3.\ 4.\ 14)$ and Fatou's Lemma again, we conclude that

$$\begin{aligned}&T(u_0,z)+T(v_0,w)\\ \geq\ &\int_{\Omega}\left(a\left(x\right)u_0^{-\gamma}z+b\left(x\right)v_0^{-\gamma}w\right)dx+\lambda\int_{\Omega}\left(f(x,u_0,v_0)z+g(x,u_0,v_0)w\right)dx,\end{aligned}\tag{3. 4. 16}$$

for all $(z,w)\in X$,with $z,w\geq 0$.Then, the maximum principal theorem implies that $u_0\left(x\right),v_0\left(x\right)>0$ a.e. in Ω and.

Now, we prove that $(u_0, v_0) \in \mathcal{N}_\lambda^+$ for all $\lambda \in (0, \Gamma)$. We start by choosing $(z, w) = (u_0, v_0)$ in (3. 4. 16) , we obtain

$$Q_0 - R_0 - \lambda r S_0 \geq 0.$$

On the other hand, from the weakly lower semi continuity of the norm, we get

$$Q_0 - R_0 - \lambda r S_0 \leq 0,$$

thus

$$Q_0 - R_0 - \lambda r S_0 = 0. \tag{3. 4. 17}$$

This implies that $(u_0, v_0) \in \mathcal{N}_\lambda$. Hence, combining (3. 4. 9) and (3. 4. 17), we conclude that $(u_0, v_0) \in \mathcal{N}_\lambda^+$.

Claim 2. Now, we prove (u_0, v_0) is positive solution to the system (p_λ) for all $\lambda \in (0, \Gamma)$. Let $(z, w) \in X$ and $\varepsilon > 0$. We define $(\omega_1, \omega_2) \in X$ by $(\omega_1, \omega_2) = (u_0 + \varepsilon z, v_0 + \varepsilon w)$and $\omega_1^+ = \max\{\omega_1, 0\}$ and $\omega_2^+ = \max\{\omega_2, 0\}$. Let

$$\begin{aligned}
\Omega_\varepsilon^+ &= \{(u_0 + \varepsilon z, v_0 + \varepsilon w) : u_0 + \varepsilon z > 0 \text{ and } v_0 + \varepsilon w > 0\}, \\
\Omega_\varepsilon^- &= \{(u_0 + \varepsilon z, v_0 + \varepsilon w) : u_0 + \varepsilon z \leq 0 \text{ and } v_0 + \varepsilon w \leq 0\},
\end{aligned}$$

$$\begin{aligned}
T(u_0,\omega_1^+) &= \int_Q \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))(\omega_1^+(x)-\omega_1^+(y))}{|x-y|^{N+ps}}dxdy \\
&= \int_Q \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy \\
&\quad -\int_{\Omega_\varepsilon^-\times\Omega_\varepsilon^-} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy \\
&= T(u_0,u_0)+\varepsilon T(u_0,z) \\
&\quad -\int_{\Omega_\varepsilon^-\times\Omega_\varepsilon^-} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy.
\end{aligned}$$

By Claim 1, the measure of the domain of integration $\Omega_\varepsilon^-\times\Omega_\varepsilon^-$ tends to 0 as $\varepsilon\to 0^+$ that

$$\int_{\Omega_\varepsilon^-\times\Omega_\varepsilon^-} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy \underset{\varepsilon\to 0^+}{\longrightarrow} 0,$$

then

(3. 4. 18) $$T(u_0,\omega_1^+) \underset{\varepsilon\to 0^+}{\longrightarrow} T(u_0,u_0).$$

By the same manner, we obtain

(3. 4. 19) $$T(v_0,\omega_2^+) \underset{\varepsilon\to 0^+}{\longrightarrow} T(v_0,v_0).$$

On the other hand, we have

$$\begin{aligned}\int_{\Omega} a(x)\, u_0^{-\gamma}\omega_1^{+}dx &= \int_{\Omega_\varepsilon^{+}} a(x)\, u_0^{-\gamma}\omega_1 dx\\ &= \int_{\Omega} a(x)\, u_0^{-\gamma}(u_0+\varepsilon z)\, dx - \int_{\Omega_\varepsilon^{-}} a(x)\, u_0^{-\gamma}(u_0+\varepsilon z)\, dx\\ &= \int_{\Omega} a(x)\, u_0^{1-\gamma}dx + \varepsilon\int_{\Omega} a(x)\, u_0^{-\gamma}z dx - \int_{\Omega_\varepsilon^{-}} a(x)\, u_0^{-\gamma}(u_0+\varepsilon z)\, dx\end{aligned}$$

$$\geq \int_{\Omega} a(x)\, u_0^{1-\gamma}dx + \varepsilon\int_{\Omega} a(x)\, u_0^{-\gamma}z dx. \tag{3. 4. 20}$$

By the same manner, we obtain

$$\int_{\Omega} b(x)\, v_0^{-\gamma}\omega_2^{+}dx \geq \int_{\Omega} b(x)\, u_0^{1-\gamma}dx + \varepsilon\int_{\Omega} b(x)\, v_0^{-\gamma}z dx. \tag{3. 4. 21}$$

Now,

$$\begin{aligned}\int_{\Omega} f(x,u_0,v_0)\omega_1^{+}dx &= \int_{\Omega_\varepsilon^{+}} f(x,u_0,v_0)\omega_1 dx\\ &= \int_{\Omega} f(x,u_0,v_0)\,(u_0+\varepsilon z)\, dx - \int_{\Omega_\varepsilon^{-}} f(x,u_0,v_0)\,(u_0+\varepsilon z)\, dx\\ &= \int_{\Omega} f(x,u_0,v_0)u_0 dx + \varepsilon\int_{\Omega} f(x,u_0,v_0)z dx - \int_{\Omega_\varepsilon^{-}} f(x,u_0,v_0)\,(u_0+\varepsilon z)\, dx\end{aligned}$$

$$\geq \int_{\Omega} f(x,u_0,v_0)u_0 dx + \varepsilon\int_{\Omega} f(x,u_0,v_0)z dx, \tag{3. 4. 22}$$

and

$$\text{(3. 4. 23)} \qquad \int_\Omega g(x,u_0,v_0)\omega_2^+ dx \geq \int_\Omega g(x,u_0,v_0)v_0 dx + \varepsilon \int_\Omega g(x,u_0,v_0)z dx,$$

combining (3. 4. 18) – (3. 4. 23), we get

$$\begin{aligned}
0 &\leq T(u_0,\omega_1^+) + T(v_0,\omega_2^+) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma}\omega_1^+ + b\left(x\right)v_0^{-\gamma}\omega_2^+\right)dx \\
&\quad -\lambda \int_\Omega \left(f(x,u_0,v_0)\omega_1^+ + g(x,u_0,v_0)\omega_2^+\right)dx \\
&\leq T(u_0,u_0) + T(v_0,v_0) - \int_\Omega \left(a\left(x\right)u_0^{1-\gamma} + b\left(x\right)u_0^{1-\gamma}\right)dx \\
&\quad - \left(\int_\Omega f(x,u_0,v_0)u_0 + \int_\Omega g(x,u_0,v_0)v_0\right)dx \\
&\quad +\varepsilon \left(T(u_0,z) + T(u_0,z) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma}z + b\left(x\right)v_0^{-\gamma}w\right)dx\right. \\
&\quad \left. - \int_\Omega \left(f(x,u_0,v_0)z + g(x,u_0,v_0)w\right)dx\right) \\
&= \varepsilon \left(T(u_0,z) + T(u_0,z) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma}z + b\left(x\right)v_0^{-\gamma}w\right)dx\right. \\
&\quad \left. - \int_\Omega \left(f(x,u_0,v_0)z + g(x,u_0,v_0)w\right)dx\right).
\end{aligned}$$

then

$$T(u_0,z)+T(u_0,z)-\int_\Omega \left(a\left(x\right)u_0^{-\gamma}z + b\left(x\right)v_0^{-\gamma}w\right)dx - \int_\Omega \left(f(x,u_0,v_0)z + g(x,u_0,v_0)w\right)dx \geq 0.$$

Since the equality holds if we replace (z,w) by $(-z,-w)$ which implies that (u_0,u_0) is a positive week solution of the problem (P_λ). ■

3.4.2 Positive solutions in $\mathcal{N}_\lambda^-$

Similarly to the arguments in $\mathcal{N}_\lambda^+$, applying Ekeland's variational principle to the minimization problem $\alpha^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u,v)$ there exists a minimizing sequences $\{(u_n,v_n)\} \subset \mathcal{N}_\lambda^-$ such that

$$\begin{cases} J_\lambda(u_n,v_n) < \alpha^- + \frac{1}{n}, \\ \\ J_\lambda(u,v) \geq J_\lambda(u_n,v_n) - \frac{1}{n}\left\|(u-u_n, v-v_n)\right\|_E \text{ for all } (u,v) \in \mathcal{N}_\lambda^+. \end{cases}$$

Firstly, we may assume that $(u_n,v_n) \geq 0$, Clearly $\{(u_n,v_n)\}$ is a bounded sequences in E. So, there exists a subsequences denoted by $\{(u_n,v_n)\}$ and $\tilde{u}_0, \tilde{v}_0 \geq 0$ such that $(u_n,v_n) \rightharpoonup (\tilde{u}_0,\tilde{v}_0)$ weakly in E, strongly in $L^{1-\gamma}(\Omega)$ and $(u_n(x),v_n(x)) \to (\tilde{u}_0(x),\tilde{v}_0(x))$ in a.e Ω, as $n \to \infty$. Moreover, using the weak lower semi continuity of norm, which means that

$$J_\lambda(\tilde{u}_0,\tilde{v}_0) \leq \lim_{n\to\infty} \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u_n,v_n) < 0. \tag{3. 4. 24}$$

we see that $(\tilde{u}_0,\tilde{v}_0) \neq (0,0)$ in a.e Ω. Now, we prove that $\tilde{u}_0, \tilde{v}_0 > 0$ in a.e Ω. Similarly to the arguments, for $(u_n,v_n) \in \mathcal{N}_\lambda^-$, one has

$$(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n < 0, \tag{3. 4. 25}$$

equivalent to

$$(p+\gamma-1)R_n - \lambda r(r-p)S_n < 0. \tag{3. 4. 26}$$

Moreover, From (3. 4. 25) and (3. 4. 26), we obtain that

$$\begin{aligned}
& \lim_{n\to\infty} [(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n] \\
= & \lim_{n\to\infty} [(p+\gamma-1)R_n - \lambda(r-p)S_n] \\
= & (p-1+\gamma)R_0 - \lambda r(r-p)S_0 \leq 0.
\end{aligned}$$

Now, repeating the same arguments as in Claim 1, we see that, for all $\lambda \in (0,\Gamma)$

$$(p+\gamma-1)R_0 - \lambda r(r-p)S_0 < 0. \tag{3. 4. 27}$$

Let us consider the functions $0 < z, w \in E$. Then, there exits a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ such that $g_n(s)(u_n + sz, v_n + sw) \in \mathcal{N}_\lambda^-$ and $g_n(0) = 1$.

Therefore, repeating the same arguments as in claim 3.4.1, we have that $g_n^{'}(0)$ is uniformly bounded for n large enough.

We conclude that $\tilde{u}_0(x), \tilde{v}_0(x) > 0$ a.e. in Ω and

$$\begin{aligned}
T(\tilde{u}_0, z) + T(\tilde{v}_0, w) \quad \leq & \quad -\int_\Omega \left(a(x)\tilde{u}_0^{-\gamma} z + b(x)\tilde{v}_0^{-\gamma} w\right) dx \\
& -\lambda \int_\Omega (f(x,\tilde{u}_0,\tilde{v}_0)z + g(x,\tilde{u}_0,\tilde{v}_0)w)\, dx.
\end{aligned} \tag{3. 4. 28}$$

for all $(z, w) \in E$. Finally, we obtain that $(\tilde{u}_0, \tilde{v}_0)$ positive nontrivial solutions of system (P_λ). The proof of the Theorem (3. 2. 1) is completed.

Conclusion et Perspectives

In this thesis, we have studied the existence and multiplicity of nontrivial positive solutions of a singular elliptic system associated with the fractional p-Laplacian operator. the results are obtained using the Nehari manifold, fibering method, and Ekland's variational principle.

The results of this thesis can be generalized to singular elliptic systems involving the fractional $p(x)$-Laplacian operator or $\varphi\left(x\right)$-Laplacian in Orlisz Sobolev spaces.

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