وزارة التعليم العالي والبحث العلمي

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THÈSE

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Etude d'une classe d'équations différentielles fractionnaires impulsives

Option
Mathématiques Appliquées

Présentée par :

KENEF Esma

DIRECTEUR DE THÈSE : GUEZANE-LAKOUD Assia Prof. U.B.M. ANNABA

Devant le jury

PRESIDENT :	KHALDI	Rabah	Prof	U.B.M. ANNABA
EXAMINATEUR :	YAROU	Mustapha Fateh	Prof	UNIV. JIJEL
EXAMINATEUR :	FRIOUI	Assia	M.C.A	UNIV. GUELMA
EXAMINATEUR :	AFFANE	Doria	M.C.A	UNIV. JIJEL
EXAMINATEUR :	KERKER	Mohamed Amine	M.C.A	UNIV. ANNABA

Dedication

I dedicate this thesis:

To my parents, who with their sacrifices allowed me to give best of myself.

To my husband who shared the moments with me the most painful.

To my daughter, my heart Ranim to whom I wish all the happiness in the world

To my sisters and brothers.

To all my family, and my friends.

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In the name of Allah, the most Gracious, most Merciful
First and foremost, I thank Allah for who gave me the desire and strength to try and do this work..

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At last, I wish to thank all my colleagues and my friends for their encouragement.

Thank you all...

ملخص

الهدف من هذه الأطروحة هو دراسة وحدانية ووجود الحلول والحلول الموجبة لبعض المسائل المتدبدبة ذات الشروط الإبتدائية والشروط الحدية، أين يمكن أن تحوي المعادلات التفاضلية على مشتق كسري من نوع كابوتو، التأخير، مشتقات كابوتو من اليمين ومن اليسار والمؤثر p-لابلاس. النتائج الرئيسية تمّ الحصول عليها باستخدام نظريات النقطة الصامدة كنظرية التقليص لبناخ، نظرية النقطة الصامدة لشودر، نظرية كراسنوسلسكي للنقطة الصامدة و نظرية النقطة الصامدة في المخروط.

وجود التدبدب في الزمن مع المشتقات الكسرية من اليمين ومن اليسارفي المسائل يجعلها أكثر تعقيدًا و ذات أهمية. يمكن إعتبار هذه الأطروحة كمساهمة في تطوير دراسة المسائل ذات المعادلات التفاضلية الكسرية المتدبدبة.

الكلمات المفتاحية:

المعادلات التفاضلية الكسرية المتدبدبة، وجود الحل، وحدانية الحل، المعادلات التفاضلية بتأخير، الحل الموجب، المؤثر p-لابلاس، المشتقات الكسرية، نظرية النقطة الصامدة.

Abstract

The aim of this thesis is the study of the uniqueness, existence and positivity of solutions for some classes of impulsive initial and boundary value problems, where the differential equations may involves Caputo fractional derivative, delay, right and left Caputo derivatives and p-Laplacian operator. The main results are obtained using fixed point theorems such as Banach's contraction principle, Schauder's fixed point theorem, Krasnoselski fixed point theorem and a fixed point theorem in cones. The presence of impulse moments with left and right fractional derivatives in the problems makes it more complicated and interesting. The results of this thesis can be considered as a contribution to the development of the study of impulsive fractional differential equations.

Keywords: Impulsive differential equation, Existence of solution, Uniqueness of solution, Delay differential equation, Positive solution, p-Laplacian operator, Fractional derivative, Fixed point theorem.

Resumé

Le but de cette thèse est l'étude de l'unicité, l'existence et la positivité des solutions pour certaines classes de problèmes impulsifs initials et aux limites, où les équations différentielles peuvent contenir la dérivée fractionnaire de Caputo, le retard, les dérivées de Caputo à droite et à gauche et l'opérateur p-Laplacien . Les principaux résultats sont obtenus en utilisant des théorèmes de point fixe tels que le principe de contraction de Banach, le théorème de point fixe de Schauder, le théorème de point fixe de Krasnoselski et un théorème de point fixe sur le cône. La présence des moments d'impulsion avec des dérivées fractionnaires gauche et droite dans les problèmes les rend plus compliqué et intéressant. Les résultats de cette thèse peuvent être considérés comme une contribution au développement de l'étude des équations différentielles fractionnaires impulsives.

Mots-clés: Equation différentielle impulsive, Existence de la solution, Unicité de la solution, Equation differentielle à retard , Solution positive, Operateur p-Laplacien, Dérivée fractionnaire, Théorème du point fixe.

CONTENTS

Introduction

The theory of fractional calculus which as a branch of mathematics studies the properties of integrals and derivatives of non-integer order which we generally call fractional integrals and fractional derivatives that are generalization of differentiation and integration to a non-integer order.

The theory of fractional calculus began almost at the same time as that of classical calculus theory. In his letter to L'Hopital in 1695, Leibniz mentionned the fractional order derivation by asking about the meaning of $\frac{d^\alpha f}{dt^\alpha}$, $\alpha = \frac{1}{2}$. Subsequently many relevant scientists have contributed to the development of fractional calculus, a simple chronology in the period from 1650 to 1970 is shown in figure 1.

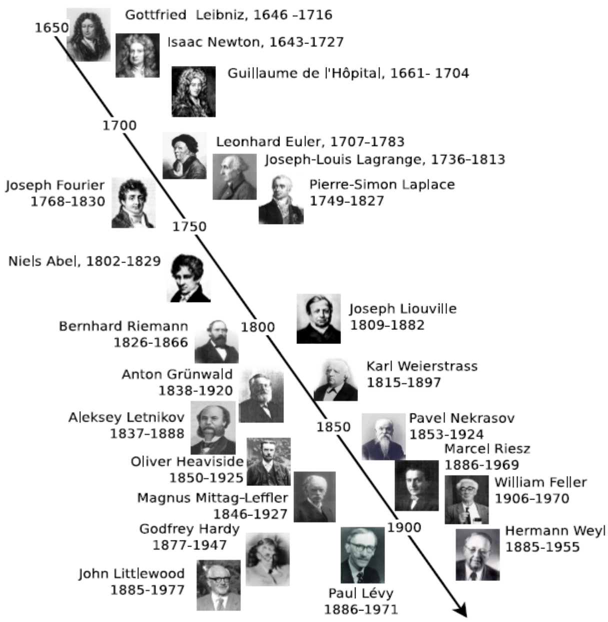

Fig. 1 Timeline of main scientists in the area of Fractional Calculus.

The development of the theory of fractional calculus is due to many mathematicians such as Spanier, Oldham, Ross, Nishimito, Marichev, Kilbas, Srivastava, Bagley, Podlubny, Miller, Caputo, ...

Nowadays, fractional mathematical theory is applied in all branches where differential equations are used to describe physical phenomena. There are several approaches to fractional calculus, such as Riemann-Liouville, Hadamard, Grunwald-Letnikov, Weyl, Caputo ...

Due to the fact that the formulation of initial value problems with fractional derivative of Caputo type is more similar to the formulation with a classical derivative, the Caputo fractional derivative is more widely used in the literature. In fact, the applied problems require definitions of fractional derivative allowing the use of physically interpretable initial conditions, and then the fractional derivative of Caputo, initially introduced by Caputo [23] and then adopted in the theory of linear viscoelasticity, satisfies this demand.

For a good bibliography on fractional calculus and their applications, we refer to [15, 27, 54, 66, 73, 74, 75, 78, 79, 83] and the references therein.

The fractional derivative being a non-local operator, the fractional differential equations model situations in which it is important to consider the history and the future of the phenomenon studied.

Different methods have been applied in the study of boundary value problems for fractional differential equations such as fixed point theorems, upper and lower solutions method, critical point theorem... see [31, 32, 33, 34, 37, 43, 44, 45, 53, 56, 58, 88, 96].

For example, in [45], the authors studied the existence of one and two solutions by applying the fixed point index theory. In [58], the authors obtained the existence of positive solutions by the help of lower and upper solutions method and Schauder fixed point theorem. In [36], the authors applied Krasnoselskii fixed point theorem to prove the existence of solutions.

Impulsive differential equations appear as a natural description of many evolutionary phenomena undergoing sudden changes and which are often of very short duration and are therefore produced instantaneously in the form of impulses, like biological systems, optimal control models in economics, population dynamics, natural disasters....

Impulsive differential equations were introduced in 1960 by Milman and Myshkis in their paper [67]. Since then, this type of differential equations has gained popularity and significant consideration and several monographs have been published by many authors such a like Samoilenko and Perestyuk [80, 81], Lakshmikantham, Bainov and Simeonov [56], Bainov and Simeonov [14], Bainov and Covachev [13], and Benchohra, Henderson and Ntouyas [17].

In [2], the authors launched the study of impulsive fractional differential equations. In [19], the authors studied impulsive initial value problems for Caputo fractional differential equations. Since then, the investigation of the existence of solutions for impulsive initial or boundary value problems for fractional differential equations was the subject of several papers [2, 3, 5, 8, 13, 14, 18, 21, 22, 39, 46, 59, 60, 72, 76, 86, 95]

Different types of impulsive fractional boundary and initial value problems are studied such, in [7, 28, 95] where the authors investigated impulsive anti-periodic boundary value problems.

In [59, 65, 84, 93], impulsive periodic boundary value problems are considered. Impulsive initial value problems, have been treated in [19, 68]. One-point, two-point, three-point or multi-point impulsive boundary value problems have been studied in [12, 43, 44, 58, 77, 87, 90, 94].

For impulsive boundary value problems on infinite intervals, we refer to [60, 92] and for impulsive mixed fractional differential equations with left and right Caputo fractional derivatives see [71].

In addition, when these processes deal with hereditary phenomena or delay that can lead to undesirable performance in the system, it is necessary to analyze the effect of delay on the dynamic behaviors of the impulsive fractional differential equations. For more results on the impulsive fractional differential equations with delay, we refer to [1, 3, 8, 18].

On the other hand, the p-Laplacian operator is widely applied in the mathematical modeling of several real world phenomena in physics, mechanics, dynamical systems, etc. This operator was introduced in 1945 by Leibenson [57] during his studies of a fundamental problem of turbulent flow in porous media. Moreover, different methods such as critical point theorems, variational methods and energy functional, are used to obtain the existence and multiplicity of the solutions of this class of problems, For more results on nonlinear fractional differential equations involving p-Laplacian operator, we refer the reader to a series of papers [4, 24, 25, 26, 43, 44, 61, 62, 63, 82, 85, 90, 91, 94].

Motivated by the above works, we are interested in the existence of solutions for some classes of impulsive initial and boundary value problems for fractional order with delay, p-Laplacian operator, single and multiple base points and involving both left and right Caputo fractional derivatives. In physics, the right fractional derivative is interpreted as a future state of the process, while the left fractional derivative is interpreted as a past state of the process, in which intervene memory effects. Since the evolution of a certain resource depends on its past and its future, the differential equations studied in this thesis contain a combination of left and right Caputo fractional derivatives to model their evolution.

This thesis is devoted to the study nonlinear impulsive mixed fractional differential equations with delay and p-Laplacian operator. We are interested in the question of the existence of solutions and positive solutions of certain nonlinear impulsive fractional differential equations with the left and right Caputo fractional derivatives. The results obtained are based on some fixed point theorems.

The remaining part of the thesis is divided into four chapters followed by a bibliography. In what follows, we give a brief outline of each chapter.

In Chapter 1, we provide the basic preliminaries, definitions, theorems and other auxiliary results required for proving the main results. In Section 1.1, we give some generalities about functional analysis. In Sections 1.2 and 1.3, we give a brief review of fractional calculus, such as left and right Riemann-Liouville integral and derivatives of fractional-order, left and right Caputo derivative of fractional-order. Section 1.4 is devoted to some fixed point theorems.

In Chapter 2, we consider the question of existence and uniqueness of a solution for a class of initial value problems for impulsive fractional differential equations involving Caputo fractional derivative in a Banach space with single base point:

$$\begin{cases} \left({}^{C}D_{0^+}^{\alpha}u\right)(t) = f(t,u)\,,\ t \in J' = J \setminus \{t_1,...,t_p\}\,;\ J = [0,1]\,, \\ u(0) = u_0, \\ \Delta u(t_j) = h_j\left(u(t_j^-)\right),\ j = 1,2...p \end{cases}$$

and with multi base points

$$\begin{cases} \left({}^{C}D_{*}^{\alpha}u\right)(t) = f(t,u(t))\,,\ 0 < t < 1,\ t \neq t_j;\ j = 0,1...p, \\ u(0) = u_0, \\ \Delta u(t_j) = h_j\left(u(t_j^-)\right),\ j = 1,2...p. \end{cases}$$

where ${}^{C}D^{\beta}_{0^+}$ and ${}^{C}D^{\alpha}_{*}$ denote the single and multiple base points Caputo fractional derivatives respectively, u is the unknown function, the functions f, and h_i, are given for $i = 0, ..., m$ and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$.

Chapter 3, Concerns the existence of solutions for a boundary value problem for impulsive fractional differential equations with delay and involving multi-base points right and left fractional derivatives

$$\begin{cases} {}^{C}D^{\alpha}_{t^-_{j+1}}\left({}^{C}D^{\beta}_{t^+_j}u(t)\right) = f\left(t, u_t\right) & , \ 0 < t < 1, \ t \neq t_j; \ j = 0, 1...p, \\ u\left(t\right) = \varphi(t) & , \ t \in \left[-r, \ 0\right], \ r > 0, \\ u'\left(0\right) = 0 & \\ \left(D^{\beta}_{t^+_j}u\right)_{|_{t=t^-_{j+1}}} = g_j\left(t^-_{j+1}, u\left(t^-_{j+1}\right) - u\left(t^+_j\right)\right) & , \ j = 0, 1, ..., p \\ \Delta u\left(t_j\right) = h_j\left(t^-_j, u(t^-_j)\right) & , \ j = 1, 2...p \\ \Delta u'\left(t_j\right) = \tilde{h}_j\left(t^-_j, u'(t^-_j)\right) & , \ j = 1, 2...p \end{cases}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, such that $\alpha + \beta > 2$, ${}^{C}D^{\alpha}_{t^-_{j+1}}$ and ${}^{C}D^{\beta}_{t^+_j}$ are respectively the left and the right Caputo fractional derivatives, f, g_j, h_j and $\tilde{h}_j$, for $j = 0, \cdots, p$, are given functions satisfying some assumptions that will be specified later.

The results of this chapter are accepted for publication:

E. Kenef and A. Guezane-Lakoud, Impulsive Mixed Fractional Differential Equations with Delay, Prog. Frac. Diff. Appl., 7 (3) (2021), 1–13.

Chapter 4, is devoted to the study of a boundary value problem for nonlinear impulsive mixed fractional differential equation with p-Laplacian operator:

$$\begin{cases} {}^{C}D_{1^-}^{\beta}\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)(t)=f\left(t,u\left(t\right)\right),\ t\in\left[0,\ 1\right], & t\neq t_i,\ i=1,\ldots,m,\\ \Delta\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)(t_i)=I_i\left(u\left(t_i\right)\right), & i=1,\ldots,m,\\ \Delta\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)'(t_i)=J_i\left(u\left(t_i\right)\right), & i=1,\ldots,m,\\ a\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)(0)-b\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)(1)=0, & \\ a\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)'(0)-b\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)'(1)=0, & \\ \Delta u\left(t_i\right)=\tilde{I}_i\left(u\left(t_i\right)\right),\ \Delta u'\left(t_i\right)=\tilde{J}_i\left(u\left(t_i\right)\right), & i=1,\ldots,m,\\ \gamma u\left(0\right)-\eta u\left(1\right)=0,\ \gamma u'\left(0\right)-\eta u'\left(1\right)=0. & \end{cases}$$

where ${}^{C}D_{1^-}^{\alpha}$ and ${}^{C}D_{0^+}^{\beta}$ denote the left and the right Caputo fractional derivatives, u is the unknown function, the functions f, I_i, J_i, $\tilde{I}_i$, and $\tilde{J}_i$, for $i=0,...,m$ and $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, are given. We end this chapter with some examples.

The results of this chapter are accepted for publication by the journal of Nonlinear Studies:

E. Kenef and A. Guezane-Lakoud, Positive solutions for impulsive mixed fractional differential equations with p-Laplacian operator, Nonlinear Studies, 28 (2) (2021), 357–373.

CHAPTER 1

Preliminaries

We provide the basic notations, definitions, theorems, lemmas and results from functional analysis, fractional calculus and theory of fixed points in Banach spaces, for later use. For more details we refer to the books
[27, 40, 41, 47, 54, 55, 66, 79].

1.1 Generalities

In this section, we give some tools and preliminary notions of analysis which are used throughout this thesis.

Let $J = [a, b]$, $(-\infty < a < b < \infty)$ be a finite interval of the real axis $\mathbb{R} = (-\infty, +\infty)$.

1.1.1 Some functional spaces

Let $C(J, \mathbb{R}) = C(J)$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the Chebyshev norm

$$\|u\|_{C(J)} = \sup_{t\in J} |u(t)|.$$

We denote by $C^m([a, b], \mathbb{R})$ the space of functions f that are m times continuously differentiable on J with the norm

$$\|u\|_{C^m(J)} = \sum_{k=0}^{m} \left\|u^{(k)}\right\|_{C(J)}, \quad m \in \mathbb{N}.$$

Let $L_p(J, \mathbb{R}) = L_p(J)$ $(1 \leq p \leq \infty)$ denotes the set of Lebesgue real-valued measurable functions f on J for which $\|f\|^p_{L_p(J)} < \infty$, where

$$\|f\|^p_{L_p(J)} = \int_a^b |f(t)|^p \, dt$$

and

$$\|f\|_{L_\infty([a,b])} = ess \sup_{t\in J} |f(t)| = \inf \{M > 0, |f(t)| < M, \ a.e \text{ on } J\}$$

In particular, for $p=1,\ L_1(J,\mathbb{R})=L_1(J)$.

Denote by $AC(J,\mathbb{R})=AC(J)$, the space of all functions f which are absolutely continuous on J. It is known that $AC(J)$ coincides with the space of primitives of Lebesgue summable functions .i.e:

$$f\in AC(J)\Longleftrightarrow f(x)=c+\int_a^x\varphi(t)\,dt,\varphi\in L_1(J),$$

For $n\in\mathbb{N}$, we define $AC^n(J)$ by

$$AC^n(J)=\left\{f:J\to\mathbb{R},\ \text{and } f^{(n-1)}\in AC(J)\right\},$$

For more details, see [54].

Let $J_0=[0,t_1],\ J_k=(t_k,t_{k+1}],\ J_p=(t_p,1]$ where $k=1,\cdots,p-1$ and $0=t_0<t_1<\cdots<t_p<t_{p+1}=1$. Denote by $PC([0,1],\mathbb{R})$, $PC^1([0,1],\mathbb{R})$ and $PC([-r,1],\mathbb{R})$ the Banach spaces defined respectively by

$$\begin{aligned}PC([0,1],\mathbb{R}) &= PC([0,1])\\ &= \{u:[0,1]\to\mathbb{R},\ u\in C(J_k,\ \mathbb{R}),\ \text{for } k=0,\cdots,p\\ &\quad \text{and there exist } u\left(t_k^+\right) \text{ and } u\left(t_k^-\right) \text{ with } u\left(t_k^-\right)=\ u(t_k)\right\},\end{aligned}$$

$$\begin{aligned}PC^1([0,1],\mathbb{R}) &= PC^1([0,1])\\ &= \left\{u:[0,1]\to\mathbb{R},\ u\in C^1(J_k,\mathbb{R}),\ u'\left(t_k^+\right)\right.\\ &\quad \left.\text{and } u'\left(t_k^-\right),\ k=0,\cdots,p,\ \text{exists and } u'\left(t_k^-\right)=\ u'(t_k)\right\}.\end{aligned}$$

and

$$\begin{aligned}PC([-r,1],\mathbb{R}) &= PC([-r,1])\\ &= \left\{u:[-r,1]\to\mathbb{R},\ u\in C(J_k,\ \mathbb{R})\cup C([-r,0],\ \mathbb{R}),\ u\left(t_k^+\right)\right.\\ &\quad \left.\text{and } u\left(t_k^-\right),\ k=0,\cdots,p,\ \text{exists and } u\left(t_k^-\right)=\ u(t_k)\right\},\end{aligned}$$

equiped with the normes

$$\|u\|_{PC([0,1])} = \max_{t\in[0,1]} |u(t)|$$

$$\|u\|_{PC^1([0,1])} = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u'(t)|$$

and

$$\|u\|_{PC([-r,1])} = \max_{t\in[-r,1]} |u(t)|$$

Denote by u_t the element of $C([-r,0],\mathbb{R})$ defined by:

$$u_t(\theta) = u(t+\theta),\ \theta \in [-r,0],\ t \in J,$$

$u_t(.)$ represent the evolution history of equation state from time $t-r$ to time t.

We refer to [1, 3, 8, 18, 54] for more details.

Definition 1.1.1 *[40]Let E be a Banach space. A nonempty closed set $C \subset E$ is called a cone of E if it satisfies the following two conditions:*

1. $x \in C$, $\lambda \geq 0$, *implies* $\lambda x \in C$,
2. $x \in C$, $-x \in C$, *implies* $x = 0$.

Example 1.1.1 *Let $E = C(J)$ and let P_1, P_2 be two subsets of E defined as*

$$P_1 = \{x \in C(J),\ x(t) \geq 0, t \in J\}$$

$$P_2 = \left\{x \in C(J),\ x(t) \geq 0, t \in J \text{ and } \min_{t\in J} x(t) \geq \varepsilon_0 \|x\|_{C(J)}\right\}$$

where ε_0 is a given number satisfying $0 < \varepsilon_0 < 1$. It is easy to show that P_1 and P_2 are cones in $C(J)$.

1.1.2 Functional analysis tools

Let E be a Banach space.

Definition 1.1.2 *(Nemytskii operator)[10] Let $f : J \times E \to E$. The Nemytskii* *operator or induced Nemytskii N_f is defined by*

$$N_f: \quad \begin{array}{l} E^J \to E^J \\ u \to f(., u(.)) \end{array}$$

where $E^J = \{u,\ u: J \to E\}$. This means N_f is the map that associates to every function $u: J \to E$ the function $N_f(u)$ defined as follow

$$N_f(u): \quad \begin{array}{l} J \to E \\ t \to f(t, u(t)). \end{array}$$

Definition 1.1.3 *A function $f: J \times E \to E$ is said to be generalized Lipshitz, if there exists a positive function $\psi: J \to \mathbb{R}_+$, such that*

$$|N_f(x)(t) - N_f(y)(t)| \leq \psi(t)\, |x(t) - y(t)|;$$

for all $t \in J$; $x, y \in E$. ψ is called the Lipschitz function of f, and N_f the induced Nemytskii of f.

Remark 1.1.1 *If $\psi(t) = k$, for all $t \in J$ where $k > 0$, N_f is a Lipschitz function with a Lipschitzian constant k ($k-$Lipschitzian). In this case if $0 < k < 1$ then N_f is called a contraction function with a contraction constant k.*

Definition 1.1.4 *[40] An operator $A: E \to E$ is called compact if the image of each bounded set $B \subset E$ is relatively compact i.e ($\overline{A(B)}$ is compact).*

Definition 1.1.5 *[40] An operator $A: E \to E$ is called completely continuous operator if it is continuous and compact.*

We recall that a family H of continuous functions on J into $\mathbb{R}$ is called:

1. Uniformly bounded if there exists a constant $M > 0$ such that

$$\|f\|_{C(J)} = \max_{t \in J} |f(t)| \leq M, \ \forall f \in H.$$

2. Equicontinuous on J, if, for every $\varepsilon > 0$, there exists some $\eta > 0$ such that for all $f \in H$ and all t_1, $t_2 \in J$, with $|t_1 - t_2| < \eta$, we have

$$|f(t_1) - f(t_2)| < \varepsilon.$$

The criteria for compactness for sets in the space of continuous functions $C(J)$ is the following.

Theorem 1.1.1 *(Arzela-Ascoli theorem)[54]. Let H be a subset of $C(J)$ equipped with the Chebyshev norm. Then H is relatively compact in $C(J)$ if and only if, H is equicontinuous and uniformly bounded.*

We present now, another version of Ascoli-Arzela Theorem in $PC(J)$ space.

Lemma 1.1.1 *(PC-type Ascoli-Arzela theorem)[42, 61, 89]. Let $\Omega \subset PC(J)$. Suppose the following conditions are satisfied:*
(i) Ω is uniformly bounded subset of $PC(J)$;
(ii) Ω is equicontinuous in $J_k, k = 0, 1, \ldots, p$.
Then Ω is relatively compact in $PC(J)$.

Proof See [89]. ■

Definition 1.1.6 *(p-Laplacian operator)[24, 61] The p-Laplacian operator ϕ_p is defined on $\mathbb{R}$ as*

$$\phi_p(u) = |u|^{p-2} u, \ u \neq 0, \ \phi_p(0) = 0,$$

where $p > 1$.

Some basic properties of the p-Laplacian operator are given in the following lemmas.

Lemma 1.1.2 *[82] The p-Laplacian operator ϕ_p is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Moreover, $(\phi_p)^{-1}$ is continuous, sends bounded sets to bounded sets, and is defined as follow*

$$(\phi_p)^{-1}(u) = \phi_q(u), u \neq 0, (\phi_p)^{-1}(0) = 0,$$

such that $\frac{1}{p} + \frac{1}{q} = 1$.

We need the following inequalities:

Lemma 1.1.3 *[24] If $x,\ y>0,\ \gamma>0,$ then*

$$(x+y)^{\gamma}\leq \max\left\{2^{\gamma-1},\ 1\right\}(x^{\gamma}+y^{\gamma}).$$

Lemma 1.1.4 *[24] Let $c>0,\ \gamma>0$. For any $x,\ y\in[0,\ c]$, we have that*
i) If $\gamma>1$, then $|x^{\gamma}-y^{\gamma}|\leq\gamma c^{\gamma-1}\ |x-y|$;
ii) If $0<\gamma\leq 1$, then $|x^{\gamma}-y^{\gamma}|\leq|x-y|^{\gamma}$.

Lemma 1.1.5 *[24, 61] Let $\phi_p:\mathbb{R}\rightarrow\mathbb{R}$ be a p-Laplacian operator.*
(i) If $1<p\leq 2,\ xy>0,\ |x|,|y|\geq m>0,$ then

$$\left|\phi_p(x)-\phi_p(y)\right|\leq(p-1)\,m^{p-2}\,|x-y|;$$

(ii) If $p>2,\ |x|,|y|\leq M,$ then

$$\left|\phi_p(x)-\phi_p(y)\right|\leq(p-1)\,M^{p-2}\,|x-y|.$$

1.2 Some fixed point theorems

Fixed point theorems are the basic mathematical tools that help to establish the existence of solutions of differential equations. The main results of this thesis are proved by the help of fixed point method that consists of transforming the given problem into a fixed point problem, and then the fixed points of the transformed problem are thus the solutions of the given problem.

In this section we recall the famous fixed point theorems that we used to obtain the existence results. We start with the Krasnoselskii fixed point theorem which only gives the existence of a fixed point without its uniqueness.

Theorem 1.2.1 *(Krasnoselskii fixed point theorem) [55]. Let H be a closed bounded and convex nonempty subset of a Banach space X. Suppose that A and B map H into X such that*
(i) A is continuous and compact;
(ii) B is a contraction mapping;

(iii) $x, y \in H$, *implies* $Ax + By \in H$;
Then there exists $x \in H$ *with* $x = Ax + Bx$.

Banach's fixed point theorem guarantees the existence of a unique fixed point for a contraction in a Banach space. This theorem, proved in 1922 by Stefan Banach, is based essentially on the notion of a contraction mapping.

Theorem 1.2.2 *(Banach's fixed point theorem)[29]. Let* Ω *be a non-empty closed subset of a Banach space* E*, then any contraction mapping* A *of* Ω *into itself has a unique fixed point.*

The third fixed point theorem that we will state is that Schauder fixed point theorem that gives the existence of at least one fixed point.

Theorem 1.2.3 *(Schauder fixed point theorem)[29] Let* E *a Banach space and* Ω *be a closed convex subset of* E *and* $A : \Omega \to \Omega$ *is compact, and continuous map. Then* A *has at least one fixed point in* Ω.

Denote $\overline{H}$ and ∂H the closure of H and the boundary of H, respectively. Let E be a Banach space and let $C \subset E$ be a cone. The following lemma is often called the fixed point theorem in a cone.

Lemma 1.2.1 *[40, 87]. Suppose* H_1 *and* H_2 *are open subsets of* E *such that* $0 \in H_1 \subset \overline{H_1} \subset H_2$ *and suppose*

$$A : C \cap \left(\overline{H_2} \diagdown H_1\right) \to C$$

is completely continuous operator such that

- $\inf\limits_{u \in C \cap \partial H_1} \|Au\| > 0$, *and* $\mu Au \neq u$ *for every* $u \in C \cap \partial H_1$ *and* $\mu \geq 1$, *and* $\mu Au \neq u$ *for every* $u \in C \cap \partial H_2$ *and* $0 < \mu \leq 1$; *or*
- $\inf\limits_{u \in C \cap \partial H_2} \|Au\| > 0$, *and* $\mu Au \neq u$ *for every* $u \in C \cap \partial H_2$ *and* $\mu \geq 1$, *and* $\mu Au \neq u$ *for every* $u \in C \cap \partial H_1$ *and* $0 < \mu \leq 1$;

then A *has a fixed point in* $C \cap \left(\overline{H_2} \diagdown H_1\right)$.

1.3 Riemann-Liouville Fractional Integrals

Definition 1.3.1 *[54, 75, 79] The left and the right Riemann-Liouville fractional integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha \in \mathbb{R}^+$ are defined by*

$$\left(I_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \ t > a$$

and

$$\left(I_{b^-}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds, \ t < b$$

respectively. Provided the right-hand sides are pointwise defined on J.

Here $\Gamma(.)$ is the Euler gamma function defined by

Definition 1.3.2 *[54, 75, 79] The Euler gamma function $\Gamma(.)$ is defined by the so-called Euler integral of the second kind:*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt, \ \ (\alpha > 0), \tag{1.1}$$

where $t^{\alpha-1} = e^{(\alpha-1)\log(t)}$. This integral is convergent in the right half of the real line, that is, $\alpha > 0$.

For this function the reduction formula

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \ \ (\alpha > 0), \tag{1.2}$$

holds. We have

$$\Gamma(n+1) = n!, \ \ (n \in \mathbb{N}).$$

Property. If α, $\beta > 0$, then the following relations hold:

$$\left(I_{a^+}^{\alpha} (x-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1},$$

and

$$\left(I_{b^-}^{\alpha}\left(b-x\right)^{\beta-1}\right)(t)=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta+\alpha\right)}\left(b-t\right)^{\beta+\alpha-1}.$$

In particular, for $k\in\mathbb{N}$ and $\alpha>0$, then

$$\left(D^k I_{a^+}^{\alpha}f\right)(t)=I_{a^+}^{\alpha-k}f\left(t\right)$$

and

$$\left(D^k I_{b^-}^{\alpha}f\right)(t)=\left(-1\right)^k I_{b^-}^{\alpha-k}f\left(t\right).$$

Lemma 1.3.1 *[54, 79] If $\alpha>0$ and $\beta>0$, then the equations*

$$\left(I_{a^+}^{\alpha}I_{a^+}^{\beta}f\right)(t)=\left(I_{a^+}^{\alpha+\beta}f\right)(t) \tag{1.3}$$

and

$$\left(I_{b^-}^{\alpha}I_{b^-}^{\beta}f\right)(t)=\left(I_{b^-}^{\alpha+\beta}f\right)(t) \tag{1.4}$$

are satisfied at almost every point $x\in J$ for $f\in Lp(J)$ $(1\le p\le\infty)$. If $\alpha+\beta>1$, then the relations in (1.3) and (1.4) hold at any point of J.

1.4 Fractional derivatives

Differentiation of fractional order is a generalization of the classical differentiation. There are many mathematical definitions for the fractional derivative. We will present two approaches that of Riemann-Liouville and the other of Caputo.

1.4.1 Approach of Riemann-Liouville derivative

Historically, the first definition of the noninteger order derivative was given by Riemann and Liouville as a consequence of Abel's solution to integral equations. This derivative has been designated in their honor as the operators Riemann-Liouville fractional derivative.

Definition 1.4.1 *[54] The left and the right Riemann-Liouville fractional deriv-*

atives $D^{\alpha}_{a^+}$ *and* $D^{\alpha}_{b^-}$ *of order* $\alpha \in \mathbb{R}^+$ *of the function* f *are defined by*

$$\begin{aligned}\left(D^{\alpha}_{a^+} f\right)(t) &:= \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{a^+} f)(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int\limits_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \ t > a,\end{aligned}$$

and

$$\begin{aligned}\left(D^{\alpha}_{b^-} f\right)(t) &:= \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{b^-} f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int\limits_t^b (s-t)^{n-\alpha-1} f(s)\, ds, \ t < b.\end{aligned}$$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ *is the integer part of* α. *In particular, when* $\alpha = n \in \mathbb{N}$, *then*

$$\left(D^{n}_{a^+} f\right)(t) = D^n f(t) = f^{(n)}(t)$$

and

$$\left(D^{n}_{b^-} f\right)(t) = (-1)^n D^n f(t) = (-1)^n f^{(n)}(t),$$

where $f^{(n)}$ *is the usual derivative of* f *of order* n.

Property. If $\alpha > \beta > 0$, then for $f \in L_p(J)$ $(1 \le p \le \infty)$, the relations hold almost everywhere on J :

$$\left(D^{\beta}_{a^+} I^{\alpha}_{a^+} f\right)(t) = I^{\alpha-\beta}_{a^+} f(t),$$

$$\left(D^{\beta}_{b^-} I^{\alpha}_{b^-} f\right)(t) = I^{\alpha-\beta}_{b^-} f(t).$$

$$\left(D^{\alpha}_{a^+} (x-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1},$$

$$\left(D^{\alpha}_{b^-} (b-x)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1},$$

On the other hand, for $k = 1, 2, ..., n$, we have

$$\begin{aligned}\left(D_{a^+}^{\alpha}\left(x-a\right)^{\alpha-k}\right)(t) &= 0,\\ \left(D_{b^-}^{\alpha}\left(b-x\right)^{\alpha-k}\right)(t) &= 0,\end{aligned}$$

In particular, the Riemann-Liouville fractional derivative of a constant is in general not equal to zero, in fact

$$\left(D_{a^+}^{\alpha}1\right)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)},\ 0<\alpha<1.$$

$$\left(D_{b^-}^{\alpha}1\right)(t) = \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)},\ 0<\alpha<1.$$

1.4.2 Approach of Caputo derivative

To compute the Riemann-Liouville fractional derivative of order α, we must calculate the integral of order $(n-\alpha)$ and then derive the result obtained. For the definition of Caputo, a different construction is used with a reverse order of operation, that is first the function is derived n times and then the integral of order $(n-\alpha)$ is determined.

Definition 1.4.2 *[54] The left and the right Caputo derivatives ${}^{C}D_{a^+}^{\alpha}$ and ${}^{C}D_{b^-}^{\alpha}$ of order $\alpha \in \mathbb{R}^+$ of the function f are defined by*

$$\left({}^{C}D_{a^+}^{\alpha}f\right)(t) = \left(D_{a^+}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right]\right)(t),\ t>a,$$

and

$$\left({}^{C}D_{b^-}^{\alpha}f\right)(t) = \left(D_{b^-}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(b-x)^k\right]\right)(t),\ t<b.$$

respectively, where $n = [\alpha]+1$, for $\alpha \notin \mathbb{N}$; $n = \alpha$ for $\alpha \in \mathbb{N}$.

If $\alpha = n \in \mathbb{N}$ and the classical derivative $f^{(n)}$ of order n exists then

$$\left({}^{C}D_{a^+}^{n}f\right)(t) = f^{(n)}(t), (n \in \mathbb{N}),$$

and

$$\left({}^{C}D^{n}_{b^-}f\right)(t) = (-1)^n f^{(n)}(t), (n \in \mathbb{N}).$$

Theorem 1.4.1 *[54] Let $\alpha > 0$ and let $n = [\alpha] + 1$, for $\alpha \notin \mathbb{N}$; $n = \alpha$ for $\alpha \in \mathbb{N}$. If $f \in AC^n(J)$, then the Caputo fractional derivatives ${}^{C}D^{n}_{a^+}f(t)$ and ${}^{C}D^{n}_{b^-}f(t)$ exist almost every where on J.*

1. *If $\alpha \notin \mathbb{N}$, ${}^{C}D^{\alpha}_{a^+}f$ and ${}^{C}D^{\alpha}_{b^-}f$ are represented by*

$$\begin{aligned}\left({}^{C}D^{\alpha}_{a^+}f\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds \\ &= :\left(I^{n-\alpha}_{a^+}D^n f\right)(t),\ t > a,\end{aligned}$$

and

$$\begin{aligned}\left({}^{C}D^{\alpha}_{b^-}f\right)(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\,ds \\ &= :(-1)^n\left(I^{n-\alpha}_{b^-}D^n f\right)(t),\ t < b.\end{aligned}$$

respectively where $D = \frac{d}{dx}$ and $n = [\alpha] + 1$.

2. *If $\alpha \in \mathbb{N}$, ${}^{C}D^{\alpha}_{a^+}f$ and ${}^{C}D^{\alpha}_{b^-}f$ are represented by*

$$\left(D^{n}_{a^+}f\right)(t) = \left({}^{C}D^{n}_{a^+}f\right)(t) = f^{(n)}(t),$$

and

$$\begin{aligned}\left(D^{n}_{b^-}f\right)(t) &= \left({}^{C}D^{n}_{b^-}f\right)(t) \\ &= (-1)^n f^{(n)}(t), (n \in \mathbb{N}),\end{aligned}$$

In particular

$$\left({}^{C}D^{0}_{a^+}f\right)(t) = \left({}^{C}D^{0}_{b^-}f\right)(t) = f(t).$$

Property. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$\left({}^{C}D^{\alpha}_{a^+}(x-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-1},\ \beta > n,$$

$$\left({}^{C}D_{b^-}^{\alpha}(b-x)^{\beta-1}\right)(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-1},\ \beta>n.$$

On the other hand, for $k=1,2,...,n$, we have

$$\left({}^{C}D_{a^+}^{\alpha}(x-a)^{k}\right)(t)=0,\qquad \left({}^{C}D_{b^-}^{\alpha}(x-a)^{k}\right)(t)=0.$$

In particular, the Caputo fractional derivative of a constant is zero, i.e.

$$\left({}^{C}D_{a^+}^{\alpha}1\right)(t)=\left({}^{c}D_{b^-}^{\alpha}1\right)(t)=0.$$

For more details see [54, 75, 79].

Lemma 1.4.1 *[54] Let $\alpha\in\mathbb{R}^+$. If $f\in AC^n(J)$ or $f\in C^n(J)$, then*

$$I_{a^+}^{\alpha C}D_{a^+}^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^{k},$$

and

$$I_{b^-}^{\alpha C}D_{b^-}^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{(-1)^{k}f^{(k)}(b)}{k!}(b-t)^{k}.$$

where $n=[\alpha]+1$, for $\alpha\notin\mathbb{N}$; $n=\alpha$ for $\alpha\in\mathbb{N}$.

In particular if $0<\alpha\leq 1$ and $f\in AC(J)$ or $f\in C(J)$, then

$$I_{a^+}^{\alpha C}D_{a^+}^{\alpha}f(t)=f(t)-f(a),$$

and

$$I_{b^-}^{\alpha C}D_{b^-}^{\alpha}f(t)=f(t)-f(b).$$

Lemma 1.4.2 *For $\alpha>0$, the solution of the fractional differential equation ${}^{C}D_{a^+}^{\alpha}u(t)=0$ is given by*

$$u(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}$$

where $c_i\in\mathbb{R}$, $t\in J$, $i=0,...,n-1$, $n=[\alpha]+1$.

CHAPTER 2

Initial value problems for Impulsive Fractional Differential Equations with Single and Multiple Base Points

2.1 Introduction and motivation

The states of many evolutionary processes are often subject to instantaneous disturbances and undergo abrupt changes at certain points of time. The duration of the changes is very short and negligible compared to the duration of the considered process, and can be considered as impulses. This type of differential equations becomes an interesting field of research and still attract more attention in many fields of nonlinear science. For example, in population ecology, communications security, neural networks, electronics, automatic control systems, computer networks, artificial intelligence, robotics and telecommunications. Many sudden and abrupt changes occur instantaneously in these systems, in the form of impulses which cannot be well described by a continuous or pure discrete time model.

The purpose of this Chapter, is to establish existence and uniqueness results to the following initial value problems IVP:

$$\left({}^{C}D_{0^{+}}^{\alpha}u\right)(t)=f\left(t,u\right),\ t\in J'=J\setminus\left\{t_{1},...,t_{p}\right\},\ J=\left[0,1\right], \tag{2.1}$$

$$u\left(0\right)=u_{0}, \tag{2.2}$$

$$\Delta u\left(t_{j}\right)=h_{j}\left(u(t_{j}^{-})\right),\ j=1,2...p \tag{2.3}$$

and

$$\left({}^{C}D_{*}^{\alpha}u\right)(t)=f\left(t,u\left(t\right)\right),\ 0<t<1,\ t\neq t_{j},\ j=0,1...p, \tag{2.4}$$

$$u\left(0\right)=u_{0} \tag{2.5}$$

$$\Delta u\left(t_{j}\right)=h_{j}\left(u(t_{j}^{-})\right),\ j=1,2...p \tag{2.6}$$

where $0<\alpha<1$, ${}^{C}D_{0^{+}}^{\alpha}$, is the standard Caputo fractional derivative at the base point $t=0$, then equation (2.1) is called single base fractional differential equation. ${}^{C}D_{*}^{\alpha}u$ is the standard Caputo fractional derivative at the base points $0,t_{1},...,t_{p}$, that is $\left({}^{C}D_{*}^{\alpha}u\right)(t)=\left({}^{C}D_{t_{j}^{+}}^{\alpha}u\right)(t)$, for all $t\in\left(t_{j},\ t_{j+1}\right],j=1,2,...,p$, then equation (2.4) is called multi-base fractional differential equation. u is the unknown function , $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is measurable on $[0,1]$ with respect to the t for any u. The functions $h_{j}:[0,1]\to\mathbb{R}$, for $j=1,...,p$ are given. The impulsive moments t_{k} are such, $0=t_{0}<t_{1}<\cdots<t_{p}<t_{p+1}=1$, $\Delta u\left(t_{j}\right)=u\left(t_{j}^{+}\right)-u\left(t_{j}^{-}\right)$,

$u\left(t_j^+\right) = \lim\limits_{h\to 0^+} u\left(t_j+h\right)$, and $u\left(t_j^-\right) = \lim\limits_{h\to 0^-} u\left(t_j+h\right)$ are the right and left limits of $u\left(t\right)$ at the point $t = t_j$, $j = 1, \ldots, p$ respectively.

Recent work on the existence of solutions for the boundary value problems of Caputo-type impulsive fractional differential equations can be found in a series of papers [2, 5, 46, 60].

In [46] the authors considered the following multi-base points initial value problem for impulsive fractional differential equations:

$$\begin{cases} {}^{C}D_*^{\alpha}u\left(t\right) = f\left(t, u\left(t\right)\right),\ t \in (0, b] \setminus \{t_1, ..., t_m\}, \alpha \in (0, 2], \\ u\left(t_k^+\right) = I_k\left(u\left(t_k^-\right)\right),\ k = 1, 2, ..., m, \\ u'\left(t_k^+\right) = J_k\left(u\left(t_k^-\right)\right),\ k = 1, 2, ..., m, \\ u\left(0\right) = a, u'\left(0\right) = c, \end{cases}$$

and

$$\begin{cases} {}^{C}D_*^{\alpha}u\left(t\right) = f\left(t, u\left(t\right)\right),\ t \in (0, b] \setminus \{t_1, ..., t_m\}, \alpha \in (0, 1], \\ u\left(t_k^+\right) = I_k\left(u\left(t_k^-\right)\right),\ k = 1, 2, ..., m, \\ u\left(0\right) = a, \end{cases}$$

where $0 = t_0 < t_1 < \cdots < t_m < b$, t_i are the impulsive points $i = 0, 1, 2, ..., m$, ${}^{C}D_*^{\alpha}$ is the standard Caputo fractional derivative at the base points $0, t_1, ..., t_m$, $f \in C\left([0, b] \times \mathbb{R}, \mathbb{R}\right)$, and $I_j, J_j \in C\left(\mathbb{R}, \mathbb{R}\right)\left(j = 1, 2, ..., m\right)$. The existence results are proved by means of Leray–Schauder nonlinear alternative.

Thanks to Schauder fixed point theorem, the existence of solutions are discussed for both single base point and multi-base points fractional initial value problems with impulses on the half line in [60]:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u\left(t\right) = q\left(t\right) f\left(t, u\left(t\right), \left({}^{c}D_{0^+}^{\beta}u\right)\left(t\right)\right),\ t > 0, \\ u\left(0\right) = u_0 \\ \Delta u\left(t_j\right) = I_j\left(t_j^-, u\left(t_j^-\right)\right),\ j = 1, 2, ... \end{cases}$$

$$\begin{cases} {}^{C}D_*^{\alpha}u\left(t\right) = q\left(t\right) f\left(t, u\left(t\right), \left({}^{c}D_*^{\beta}u\right)\left(t\right)\right),\ t > 0, \\ u\left(0\right) = u_0 \\ \Delta u\left(t_j\right) = I_j\left(t_j^-, u\left(t_j^-\right)\right),\ j = 1, 2, ... \end{cases}$$

here $u_0 \in \mathbb{R}$, $0 < \beta < \alpha < 1$, $0 = t_0 < t_1 < \cdots < t_j$, $\lim_{j\to+\infty} t_j = +\infty$, ${}^{C}D_{0^+}^{\alpha}$

is the standard Caputo fractional derivative at the base point $t = 0$, and ${}^{C}D_{*}^{\alpha}$ is the standard Caputo fractional derivative at the base points t_j, $q : \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$, and $I_j : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}\,(j = 1, 2, ...)$, that is for all $t \in (t_{j,}\ t_{j+1}]$ we have ${}^{C}D_{*}^{\alpha} = {}^{C}D_{t_j^+}^{\alpha}$.

In this chapter, we transform the problems (2.1)-(2.3) and (2.4)-(2.6) to equivalent integral equtions, then we prove the existence of a unique solution by the help of Banach contraction principal while the existence of at least on solution is obtained by Schauder fixed point theorem.

2.2 Existence and uniqueness results for IVP with a single base point

First, let us give the definition of the solution of the problem (2.1)-(2.3) .

Definition 2.2.1 *A function $u \in PC\,(J)$ is said to be a solution of the problem (2.1)-(2.3) if u satisfies the differential equation*

$$\left({}^{C}D_{0^+}^{\alpha}u\right)(t) = f\,(t, u)\,,\ \ t \in J' = (0, 1] \setminus \{t_1, \ldots, t_p\}$$

and the conditions

$$\begin{aligned} u\,(0) &= u_0 \\ \Delta u\,(t_j) &= h_j\left(t_j^-,\ u(t_j^-)\right), j = 1, 2...p, \end{aligned}$$

Remark 2.2.1 *Note that*

$$u\,(t) = u_0 - \frac{1}{\Gamma\,(\alpha)}\int_0^a (a-s)^{\alpha-1} f\,(s)\,ds + \frac{1}{\Gamma\,(\alpha)}\int_0^t (t-s)^{\alpha-1} f\,(s)\,ds$$

solves the Cauchy problems

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) = f(t), t \in J \\ u(0) = u_0 - \frac{1}{\Gamma(\alpha)}\int\limits_0^a (a-s)^{\alpha-1} f(s)\, ds \end{cases}$$

where $u_0 \in \mathbb{R}$, *and* $a \in \mathbb{R}_+^*$.

We get the following result immediately

Lemma 2.2.1 *Let* $g : J \to \mathbb{R}$ *be continuous. A function* $u \in C(J)$ *is a solution of the fractional integral equation*

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)}\int\limits_0^a (a-s)^{\alpha-1} g(s)\, ds + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} g(s)\, ds$$

if and only if u *is a solution of the following fractional Cauchy problems*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) = g(t), & t \in J, \\ u(a) = u_0, & a > 0. \end{cases}$$

As a consequence of Lemma 2.2.1 we have the following result which is useful in the sequel.

Lemma 2.2.2 *Let* g *be continuous on* J. *A function* u *is a solution of the fractional integral equation*

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} g(s)\, ds, t \in (0, t_1], \\ u_0 + \sum\limits_{j=1}^{k} h_j\left(u\left(t_j^-\right)\right) + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} g(s)\, ds, t \in (t_k, t_{k+1}], k = 1, ..., p. \end{cases} \tag{2.7}$$

if and only if u is a solution of the following impulsive problem

$$\begin{cases} \left({}^{C}D_{0^+}^{\alpha}u\right)(t) = g(t), \ t \in J' = J \setminus \{t_1, ..., t_p\}; \ J = [0,1], \\ \Delta u(t_j) = h_j\left(u(t_j^-)\right), \ j = 1, 2...p, \\ u(0) = u_0. \end{cases} \tag{2.8}$$

Proof Suppose that u satisfies (2.8) and let $t \in (0, t_1]$, thus

$$\left({}^{C}D_{0^+}^{\alpha}u\right)(t) = g(t), \ t \in (0, t_1], \ u(0) = u_0. \tag{2.9}$$

Applying the fractional integral $I_{0^+}^{\alpha}$ to the equation (2.9), it yields

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds.$$

If $t \in (t_1, t_2]$, then

$$\left({}^{C}D_{0^+}^{\alpha}u\right)(t) = g(t), \ t \in (t_1, t_2] \text{ with } u\left(t_1^+\right) = u\left(t_1^-\right) + h_1\left(u(t_1^-)\right),$$

Lemma 2.2.1 implies

$$\begin{aligned} u(t) &= u\left(t_1^+\right) - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} g(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds \\ &= u\left(t_1^-\right) + h_1\left(u\left(t_1^-\right)\right) - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} g(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds \\ &= u_0 + h_1\left(u\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again by Lemma 2.2.1 we deduce

$$u(t) = u\left(t_k^+\right) - \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} g(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} g(s)\, ds$$
$$= u_0 + \sum_{j=1}^{k} h_j\left(u\left(t_j^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} g(s)\, ds.$$

consequently u satisfies (2.7).

Now, let us show the converse. If $t \in (0, t_1]$ then $u(0) = u_0$, since ${}^C D_{0+}^{\alpha}$ is the left inverse of I_{0+}^{α} we get (2.9). If $t \in (t_k, t_{k+1}], k = 1, ..., p$, then using the fact that the Caputo derivative of a constant is equal to zero, we obtain $\left({}^C D_{0+}^{\alpha} u\right)(t) = g(t)$, $t \in (t_k, t_{k+1}]$ and $u\left(t_k^+\right) = u\left(t_k^-\right) + h_k\left(u\left(t_k^-\right)\right)$. ■

2.2.1 Uniqueness of solution

We make the following assumptions.

$H1)$ $f \in C(J \times \mathbb{R}, \mathbb{R})$.

$H2)$ There exists a nonnegative function $k_1 \in C(J, \mathbb{R}_+)$, such that

$$|f(t, x) - f(t, y)| \leq k_1(t)\,|x - y|, \; t \in [0, 1], \; x, \; y \in \mathbb{R}.$$

$H3)$ $h_j \in C(\mathbb{R}, \mathbb{R})$, $j = 1, ..., p$.

$H4)$ There exists a positive constant $k_2 > 0$, such that

$$|h_j(x) - h_j(y)| \leq k_2\,|x - y|, \text{ for all } x, \; y \in \mathbb{R}, \; j = 1, ..., p$$

The following theorem gives an uniqueness result.

Theorem 2.2.1 *Assume that $(H1) - (H4)$ hold. If*

$$pk_2 + \frac{L}{\Gamma(\alpha + 1)} < 1, \quad L = \|k_1\|_{C(J, \mathbb{R}_+)}.$$

then the problem (2.1)-(2.3) has a unique solution.

Proof Taking Lemma 2.2.2 into account, we transform the problem (2.1)-(2.3) to a fixed point problem. In fact, define the operator $F_1 : PC(J) \to PC(J)$ as

$$\left(F_1 u\right)(t) = u_0 + \sum_{j=1}^{k} h_j\left(u\left(t_j^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s)\right) ds, \tag{2.10}$$

Obviously F_1 is well defined due to $(H1)$.

We shall prove that F_1 is a contraction. Indeed, let $u, v \in PC(J)$ and $t \in (t_k, t_{k+1}]$, $j = 1, ...p$, we have

$$\begin{aligned} \left|\left(F_1 u\right)(t) - \left(F_1 v\right)(t)\right| &\leq \sum_{j=1}^{k} \left|h_j\left(u\left(t_j^-\right)\right) - h_j\left(v\left(t_j^-\right)\right)\right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f\left(s, u(s)\right) - f\left(s, v(s)\right)\right| ds \\ &\leq \left(pk_2 + \frac{L}{\Gamma(\alpha+1)}\right) \|u - v\|_{PC(J)}. \end{aligned}$$

By Banach's fixed point theorem, we deduce that F_1 has a unique fixed point that is a solution of the problem (2.1)-(2.3). ■

2.2.2 Existence of solutions

Let us introduce the following conditions:

$(H'2)$ There exists a constant $a_1 > 0$ such that

$$|f(t, u)| \leq a_1(1 + |u|), \text{ for each } t \in J \text{ and all } u \in \mathbb{R}.$$

$(H'4)$ There exists a constant $a_2 > 0$ such that

$$|h_j(u)| \leq a_2(1 + |u|), \text{ for all } u \in \mathbb{R} \text{ and } j = 1, ..., p.$$

Now we are ready to give the existence results.

Theorem 2.2.2 *Assume that* $(H1), (H3), (H'2)$ *and* $(H'4)$ *hold. If*

$$a_2 p + \frac{a_1}{\Gamma(\alpha+1)} < 1,$$

then the problem (2.1)-(2.3) *has at least one solution in* $PC(J)$.

Proof For convenience, the proof will be done in several steps.

Claim 1. F_1 is continuous. In fact, let (u_n) be a sequence such that $u_n \to u$ in $PC(J)$. Then for each $t \in (t_k, t_{k+1}], k = 1, ...p$, we have

$$\begin{aligned}
|(F_1 u_n)(t) - (F_1 u)(t)| &\leq \sum_{j=1}^{k} \left| h_j\left(u_n\left(t_j^-\right)\right) - h_j\left(u\left(t_j^-\right)\right) \right| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u_n(s)) - f(s, u(s))| \, ds \\
&\leq p \left\| h_j(u_n(.)) - h_j(u(.)) \right\|_{PC(J)} \\
&+ \frac{1}{\Gamma(\alpha+1)} \left\| f(., u_n(.)) - f(., u(.)) \right\|_{PC(J)}.
\end{aligned}$$

In view of $(H1)$ and $(H3)$, it yields

$$\|(F_1 u_n)(.) - (F_1 u)(.)\|_{PC(J)} \to 0 \text{ as } n \to \infty.$$

Claim 2. F_1 maps bounded sets into bounded sets in $PC(J)$. Indeed, it suffices to show that for $\eta > 0$, there exists a $l > 0$ such that for each $u \in B_\eta = \left\{ u \in PC(J), \|u\|_{PC(J)} \leq \eta \right\}$, we have $\|F_1 u\| \leq l$. For each $t \in (t_k, t_{k+1}], k = 1, ...p$, we have

$$\begin{aligned}
|(F_1 u)(t)| &\leq |u_0| + \sum_{j=1}^{k} \left| h_j\left(u\left(t_j^-\right)\right) \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s))| \, ds \\
&\leq |u_0| + a_2 p \left(1 + \|u\|_{PC(J)}\right) + \frac{a_1}{\Gamma(\alpha+1)} \left(1 + \|u\|_{PC(J)}\right) \\
&\leq |u_0| + a_2 p (1+\eta) + \frac{a_1}{\Gamma(\alpha+1)} (1+\eta),
\end{aligned}$$

consequently

$$\|F_1 u\| \leq |u_0| + a_2 p\,(1+\eta) + \frac{a_1}{\Gamma(\alpha+1)}\,(1+\eta) = l.$$

Claim 3. F_1 maps bounded sets into equicontinuous sets of $PC(J)$. Let $t_k \leq t_1 < t_2 < t_{k+1}, k = 1, ..., p$, $u \in B_\eta$. Thanks to $(H'2)$ and $(H'4)$ we obtain

$$\begin{aligned} |(F_1 u)(t_2) - (F_1 u)(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right) |f(s, u(s))|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, u(s))|\, ds \\ &\leq \frac{a_1 \left(1 + \|u\|_{PC(J)}\right)}{\Gamma(\alpha+1)} \left(2(t_2 - t_1)^\alpha + (t_1)^\alpha - (t_2)^\alpha\right) \\ &\leq \frac{a_1(1+\eta)}{\Gamma(\alpha+1)} \left(2(t_2 - t_1)^\alpha + (t_1)^\alpha - (t_2)^\alpha\right). \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero, hence F_1 is equicontinuous on the intervals $(t_k, t_{k+1}]$, $k = 1, ..., p$.

We conclude by Arzela-Ascoli Theorem 1.1.1 that F_1 is completely continuous.

Claim 4. Let $\mathcal{B}$ be the closed convex set in $PC(J)$,

$$\mathcal{B} = \left\{u \in PC(J), \|u\|_{PC(J)} \leq \mathcal{R}\right\}$$

such that

$$\mathcal{R} > \frac{(|u_0| + a_2 p)\,\Gamma(\alpha+1) + a_1}{\Gamma(\alpha+1) - (a_2 p \Gamma(\alpha+1) + a_1)}.$$

We shall prove that $F_1(\mathcal{B}) \subset \mathcal{B}$. Indeed for each $(t_k, t_{k+1}]\,, k = 1, ..., p$, we have

$$|(F_1 u)(t)| \leq |u_0| + \sum_{j=1}^{k} \left|h_j\left(u\left(t_j^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s))|\, ds$$

$$\le |u_0| + a_2 p \left(1 + \|u\|_{PC(J)}\right) + \frac{a_1 \left(1 + \|u\|_{PC(J)}\right)}{\Gamma(\alpha+1)}$$
$$\le |u_0| + a_2 p + \frac{a_1}{\Gamma(\alpha+1)} + \left(a_2 p + \frac{a_1}{\Gamma(\alpha+1)}\right) \|u\|_{PC(J)} < \mathcal{R}.$$

By Schauder's fixed point theorem, we deduce that F_1 has a fixed point and then the problem (2.1)-(2.3) has at least one solution. ■

2.3 Existence and uniqueness results for IVP with multi-base points

Let us give the definition of the solution of the problem (2.4)-(2.6) .

Definition 2.3.1 *A function $u \in PC(J)$ is said to be a solution of the problem (2.4)-(2.6) if u satisfies the equation*

$$\left({}^{C}D^{\alpha}_{t_j^+} u\right)(t) = f(t,u) \;\; on \; J_j = (t_j, t_{j+1}]$$

and the conditions

$$\begin{aligned} \Delta u(t_j) &= h_j\left(u(t_j^-)\right), j = 1, 2...p, \\ u(0) &= u_0. \end{aligned}$$

A direct result is the following

Lemma 2.3.1 *Let g be continuous on J. A function u is a solution of the frac-*

tional integral equation

$$u(t)=\begin{cases} u_0+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} g(s)\,ds, t\in(0,t_1], \\ u_0+\sum\limits_{j=1}^{k} h_j\left(u\left(t_j^-\right)\right)+\sum\limits_{j=1}^{k}\frac{1}{\Gamma(\alpha)}\int\limits_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1} g(s)\,ds \\ +\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t}(t-s)^{\alpha-1} g(s)\,ds, t\in(t_k,t_{k+1}], k=1,...,p. \end{cases} \tag{2.11}$$

if and only if u is a solution of the following impulsive problem

$$\begin{cases} \left({}^C D^{\alpha}_{t_j^+}u\right)(t)=g(t),\ t\in(t_j,t_{j+1}], j=0,1...p, \\ \Delta u(t_j)=h_j\left(u(t_j^-)\right),\ j=1,2...p, \\ u(0)=u_0. \end{cases} \tag{2.12}$$

Proof Assume that u satisfies (2.12) and let $t\in(0,t_1]$, then

$$\left({}^C D^{\alpha}_{0^+}u\right)(t)=g(t),\ t\in(0,t_1] \text{ with } u(0)=u_0 \tag{2.13}$$

applying the fractional integral $I^{\alpha}_{0^+}$ to (2.13), we get

$$u(t)=u_0+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} g(s)\,ds.$$

If $t\in(t_1,t_2]$, then

$$\left({}^C D^{\alpha}_{t_1^+}u\right)(t)=g(t),\ t\in(t_1,t_2] \text{ with } u\left(t_1^+\right)=u\left(t_1^-\right)+h_1\left(u(t_1^-)\right), \tag{2.14}$$

let us apply the fractional integral $I_{t_1^+}^{\alpha}$ to the equation (2.14), and taking into account conditions (2.6) it yields

$$
\begin{aligned}
u(t) &= u\left(t_1^+\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} g(s)\, ds \\
&= u\left(t_1^-\right) + h_1\left(u\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} g(s)\, ds \\
&= u_0 + h_1\left(u\left(t_1^-\right)\right) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} g(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} g(s)\, ds.
\end{aligned}
$$

If $t \in (t_k, t_{k+1}]$ then conditions (2.6) imply

$$
\begin{aligned}
u(t) &= u\left(t_k^+\right) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g(s)\, ds \\
&= u_0 + \sum_{j=1}^{k} h_j\left(u\left(t_j^-\right)\right) + \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} g(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)\, ds.
\end{aligned}
$$

Conversely, assume that u satisfies (2.11). If $t \in (0, t_1]$ then $u(0) = u_0$. Taking into account that ${}^cD_{0^+}^{\alpha}$ is the left inverse of $I_{0^+}^{\alpha}$ we obtain (2.12). Now let $t \in (t_k, t_{k+1}], k = 1, ..., p$ then $\left({}^cD_{t_k^+}^{\alpha} u\right)(t) = g(t)\,,\ t \in (t_k, t_{k+1}]$ and $u\left(t_k^+\right) = u\left(t_k^-\right) + h_k\left(u\left(t_k^-\right)\right)$, since the Caputo derivative of a constant is equal to zero. ■

2.3.1 Uniqueness of solution

Theorem 2.3.1 *Assume that $(H1) - (H4)$ holds. If*

$$
pk_2 + \frac{(p+1)\,L}{\Gamma(\alpha+1)} < 1,
$$

then the problem (2.4)-(2.6) *has a unique solution.*

Proof We transform the problem (2.4)-(2.6) into a fixed point problem as follows. Consider the operator $F_2 : PC(J) \to PC(J)$ defined by

$$(F_2u)(t) = u_0 + \sum_{j=1}^{k} h_j\left(u\left(t_j^-\right)\right) + \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} f(s, u(s))\, ds \quad (2.15)$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, u(s))\, ds$$

In view of $(H1)$, F_2 is well defined.

F_2 is a contraction mapping, in fact for arbitrary $u, v \in PC(J)$ and each $t \in (t_k, t_{k+1}]$, $j = 1, ...p$, we have

$$\begin{aligned} |(F_2u)(t) - (F_2v)(t)| &\leq \sum_{j=1}^{k} \left|h_j\left(u\left(t_j^-\right)\right) - h_j\left(v\left(t_j^-\right)\right)\right| \\ &+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))|\, ds \\ &\leq \left(pk_2 + \frac{p\,\|k_1\|_{C([0,\,1],\,\mathbb{R}_+)}}{\Gamma(\alpha+1)} + \frac{\|k_1\|_{C([0,\,1],\,\mathbb{R}_+)}}{\Gamma(\alpha+1)}\right) \|u - v\|_{PC(J)}\,. \end{aligned}$$

Thanks to the Banach's fixed point theorem, we deduce that F_2 has a unique fixed point which is a the unique solution for the problem (2.4)-(2.6). ■

2.3.2 Existence of solution

Theorem 2.3.2 *Assume that* $(H1), (H3), (H'2)$ *and* $(H'4)$ *hold. If*

$$a_2 p + \frac{a_1(p+1)}{\Gamma(\alpha+1)} < 1,$$

then the problem (2.4)-(2.6) *has at least one solution in* $PC(J)$.

Proof Let F_2 be defined as in (2.15). The proof will be done in several steps.

Claim 1. F_2 is continuous. Let (u_n) be a sequence such that $u_n \to u$ in $PC(J)$. Then for each $t \in (t_k, t_{k+1}]\,, k = 1, ...p$, we have

$$\begin{aligned}
|(F_2 u_n)(t) - (F_2 u)(t)| &\leq \sum_{j=1}^{k} \left|h_j\left(u_n\left(t_j^-\right)\right) - h_j\left(u\left(t_j^-\right)\right)\right| \\
&+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} \left|f(s, u_n(s)) - f(s, u(s))\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \left|f(s, u_n(s)) - f(s, u(s))\right| ds \\
&\leq p \left\|h_j(u_n(.)) - h_j(u(.))\right\|_{PC(J)} \\
&+ \frac{p+1}{\Gamma(\alpha+1)} \left\|f(., u_n(.)) - f(., u(.))\right\|_{PC(J)}.
\end{aligned}$$

Thanks to $(H1)$ and $(H3)$, we get

$$\|(F_2 u_n)(.) - (F_2 u)(.)\|_{_{PC(J)}} \to 0 \text{ as } n \to \infty.$$

Claim 2. F_2 maps bounded sets into bounded sets in $PC(J)$. In fact, let us prove that for any $\eta > 0$, there exists a $l > 0$ such that for each $u \in B_\eta = \left\{u \in PC(J), \|u\|_{PC(J)} \leq \eta\right\}$, we have $\|F_2 u\| \leq l$. For each $t \in (t_k, t_{k+1}]\,, k = 1, ...p$, we obtain

$$\begin{aligned}
|(F_2u)(t)| &\leq |u_0| + \sum_{j=1}^{k} \left|h_j\left(u\left(t_j^-\right)\right)\right| \\
&+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |f(s, u(s))| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s, u(s))| \, ds \\
&\leq |u_0| + a_2 p \left(1 + \|u\|_{PC(J)}\right) + \frac{a_1(p+1)}{\Gamma(\alpha+1)} \|u\|_{PC(J)} \\
&\leq |u_0| + a_2 p (1+\eta) + \frac{a_1(p+1)}{\Gamma(\alpha+1)} (1+\eta),
\end{aligned}$$

hence

$$\|F_2u\| \leq |u_0| + a_1 p (1+\eta) + \frac{a_2(p+1)}{\Gamma(\alpha+1)} (1+\eta) = l.$$

Claim 3. F_2 maps bounded sets into equicontinuous sets of $PC(J)$. For $t_k \leq t_1 < t_2 < t_{k+1}, k = 1, ..., p$ and $u \in B_\eta$ we obtain by the help of conditions $[H'2]$ and $[H'4]$

$$\begin{aligned}
|(F_2u)(t_2) - (F_2u)(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_1} \left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right) |f(s, u(s))| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, u(s))| \, ds \\
&\leq \frac{a_1\left(1 + \|u\|_{PC(J)}\right)}{\Gamma(\alpha+1)} \left(2(t_2 - t_1)^\alpha + (t_1 - t_k)^\alpha - (t_2 - t_k)^\alpha\right) \\
&\leq \frac{a_1(1+\eta)}{\Gamma(\alpha+1)} \left(2(t_2 - t_1)^\alpha + (t_1 - t_k)^\alpha - (t_2 - t_k)^\alpha\right),
\end{aligned}$$

when $t_1 \to t_2$, the right hand side of the above inequality tends to zero, therefore F_2 is equicontinuous on the intervals $(t_k, t_{k+1}], k = 1, ..., p$.

From the above steps and Arzela-Ascoli Theorem 1.1.1, we deduce that F_2 is completely continuous.

Claim 4. We shall prove that $F_2(\Lambda) \subset \Lambda$, where

$$\Lambda = \left\{ u \in PC(J), \|u\|_{PC(J)} \leq R \right\}$$

such that

$$R > \frac{(|u_0| + a_2 p)\,\Gamma(\alpha+1) + a_1(p+1)}{\Gamma(\alpha+1) - (a_2 p \Gamma(\alpha+1) + a_1(p+1))}.$$

Indeed for each $(t_k, t_{k+1}], k = 1, ..., p$, we get

$$\begin{aligned} |(F_2 u)(t)| &\leq |u_0| + \sum_{j=1}^{k} \left| h_j\left(u\left(t_j^-\right)\right) \right| \\ &+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |f(s, u(s))|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |f(s, u(s))|\, ds \end{aligned}$$

$$\begin{aligned} &\leq |u_0| + a_2 p \left(1 + \|u\|_{PC(J)}\right) \\ &+ \frac{a_1(p+1)\left(1 + \|u\|_{PC(J)}\right)}{\Gamma(\alpha+1)} \\ &\leq |u_0| + a_2 p + \frac{a_1(p+1)}{\Gamma(\alpha+1)} \\ &+ \left(a_2 p + \frac{a_1(p+1)}{\Gamma(\alpha+1)}\right) \|u\|_{PC(J)} < R. \end{aligned}$$

We deduce by Schauder's fixed point theorem that F_2 has a fixed point and consequently the problem (2.4)-(2.6) has at least one solution. ∎

2.4 Examples

2.4.1 Example 1

Consider in problems (2.1)-(2.3) and (2.4)-(2.6), $\alpha = \frac{1}{2}$, $p = 1$ and the functions f and h_1 given as

$$\begin{aligned} f(t, u(t)) &= \frac{e^{-t}|u(t)|}{2(1+e^t)(1+|u(t)|)}, u, v \in \mathbb{R}, t \in J\backslash\{t_1\} = (0,1]\backslash\left\{\frac{1}{2}\right\} \\ h_1\left(u\left(t_1^-\right)\right) &= \frac{\left|u\left(t_1^-\right)\right|}{20\left(1+\left|u\left(t_1^-\right)\right|\right)}, u \in \mathbb{R} \\ u_0 &= 0. \end{aligned}$$

We shall check that conditions $(H1)$-$(H4)$ are satisfied. In fact

$$\begin{aligned} |f(t,u) - f(t,v)| &\leq \frac{e^{-t}}{2(1+e^t)}|u - v| \\ &= k_1(t)|u - v|, \text{ for all } u, v \in \mathbb{R}, \text{ and } t \in J, \end{aligned}$$

and

$$|h_1(u) - h_1(v)| \leq \frac{1}{20}|u - v| = k_2|u - v|, \text{ for all } u, v \in \mathbb{R}.$$

By computations, we obtain

$$\begin{aligned} L = \frac{1}{4},\ k_2 &= \frac{1}{20} \\ pk_2 + \frac{pL}{\Gamma(\alpha+1)} &= 0.332\,09 < 1 \\ pk_2 + \frac{(p+1)L}{\Gamma(\alpha+1)} &= 0.614\,19 < 1 \end{aligned}$$

Then all assumptions of Theorems 2.2.1 and 2.3.1 are satisfied. Hence we conclude the uniqueness of solution for problems (2.1)-(2.3) and (2.4)-(2.6) respectively.

2.4.2 Example 2

Consider the problems (2.1)-(2.3) and (2.4)-(2.6) with $\alpha = \frac{1}{2}$, $p = 2$ and

$$f\left(t,u\left(t\right)\right) = \frac{e^{-t}\left(1+\left|u\left(t\right)\right|\right)}{2\left(1+e^{t}\right)\left(2+\left|u\left(t\right)\right|\right)}, t \in J\setminus\left\{t_1,t_2\right\} = (0,1]\setminus\left\{\frac{1}{3},\frac{2}{3}\right\},$$

$$u\left(0\right) = u_0 = 0$$

$$h_1\left(u\left(t_1^-\right)\right) = \frac{1+\left|u\left(t_1^-\right)\right|}{10\left(5+\left|u\left(t_1^-\right)\right|\right)},\ h_2\left(u\left(t_2^-\right)\right) = \frac{1+\left|u\left(t_2^-\right)\right|}{20\left(3+\left|u\left(t_2^-\right)\right|\right)}.$$

We claim that $(H1)$,$(H3)$, $(H'2)$ and $(H'4)$ hold. Indeed,

$$\left|f\left(t,u\right)\right| \leq \frac{e^{-t}}{2\left(1+e^{t}\right)}\left(1+\left|u\right|\right) \leq \frac{1}{4}\left(1+\left|u\right|\right) = a_1\left(1+\left|u\right|\right), u \in \mathbb{R}, t \in J,$$

and

$$\begin{aligned}\left|h_1\left(u\right)\right| &\leq \frac{1}{20}\left(1+\left|u\right|\right) = \frac{1}{20}\left(1+\left|u\right|\right) = a_1\left(1+\left|u\right|\right), u \in \mathbb{R},\\ \left|h_2\left(u\right)\right| &\leq \frac{1}{20}\left(1+\left|u\right|\right) = \frac{1}{20}\left(1+\left|u\right|\right) = a_1\left(1+\left|u\right|\right), u \in \mathbb{R}.\end{aligned}$$

Moreover

$$\begin{aligned}a_2p + \frac{a_1}{\Gamma\left(\alpha+1\right)} &= 0.382\,09 < 1,\\ a_2p + \frac{a_1\left(p+1\right)}{\Gamma\left(\alpha+1\right)} &= 0.946\,28 < 1.\end{aligned}$$

$$\frac{\left(\left|u_0\right| + a_2p\right)\Gamma\left(\alpha+1\right) + a_1}{\Gamma\left(\alpha+1\right) - \left(a_2p\Gamma\left(\alpha+1\right) + a_1\right)} = 0.618\,37 < 0,7 = \mathcal{R},$$

$$\frac{\left(\left|u_0\right| + a_2p\right)\Gamma\left(\alpha+1\right) + a_1\left(p+1\right)}{\Gamma\left(\alpha+1\right) - \left(a_2p\Gamma\left(\alpha+1\right) + a_1\left(p+1\right)\right)} = 17.617 < 17,7 = R.$$

Since all assumptions of Theorems 2.2.2 and 2.3.2 are satisfied, then there exists at least one solution to problems (2.1)-(2.3) and (2.4)-(2.6) in $\mathcal{B}$ and Λ respectively.

CHAPTER 3

Impulsive Mixed Fractional Differential Equations with Delay

3.1 Introduction and motivation

Impulsive differential equations describe processes that endure a sudden change in their state at certain times.

Processes with such a characteristic, are modeled by impulsive differential equations and occur naturally in different fields of science such in physics, economics, population dynamics...

This chapter concerns the existence of solutions for an impulsive boundary value problem with delay and involving multi-base points right and left Caputo derivatives:

$$ {}^{C}D^{\alpha}_{t^{-}_{j+1}}\left({}^{C}D^{\beta}_{t^{+}_{j}}u(t)\right) = f\left(t,u_t\right),\ 0<t<1,\ t\neq t_j;\ j=0,1...p, \tag{3.1}$$

$$ u\left(t\right)=\varphi(t),\ t\in[-r,0] \tag{3.2}$$

$$ u'\left(0\right)=0 \tag{3.3}$$

$$ \left({}^{C}D^{\beta}_{t^{+}_{j}}u\right)_{|_{t=t^{-}_{j+1}}} = g_j\left(t^{-}_{j+1},u\left(t^{-}_{j+1}\right)-u\left(t^{+}_{j}\right)\right),\ j=0,1,...,p \tag{3.4}$$

$$ \Delta u\left(t_j\right)=h_j\left(t^{-}_{j},u(t^{-}_{j})\right),\ j=1,2...p \tag{3.5}$$

$$ \Delta u'\left(t_j\right)=\tilde{h}_j\left(t^{-}_{j},u'(t^{-}_{j})\right),\ j=1,2...p \tag{3.6}$$

where $0<\alpha<1$, $1<\beta<2$, such that $\alpha+\beta>2$, ${}^{C}D^{\beta}_{t^{+}_{j}}$, ${}^{C}D^{\alpha}_{t^{-}_{j+1}}$ are respectively the left and the right Caputo fractional derivatives, $j=0,\cdots,p$, u is the unknown function and the history of state is $u_t\left(\theta\right)=u\left(t+\theta\right)$, for $\theta\in[-r,0]$, $f:[0,1]\times D\to\mathbb{R}$, where

$$\begin{aligned} D &= \{u:[-r,1]\to\mathbb{R},\ u \text{ is continuous every where except for a finite number} \\ &\quad \text{of point } \theta \text{ at which } u\left(\theta\right) \text{ and the right limit } u\left(\theta^{-}\right) \text{ exist} \\ &\quad \text{and } u\left(\theta^{-}\right)=u\left(\theta\right)\} \end{aligned}$$

and $f\left(t,u_t\right)$ is measurable on $[0,1]$ with respect to the t for any $u_t\in D$. The functions h_j, $\tilde{h}_j$, $g_j:[0,1]\times\mathbb{R}\to\mathbb{R}$, for $j=1,...,p$ are given. The initial function $\varphi:\mathbb{R}\to\mathbb{R}$, satisfies $\varphi\left(0\right)=0$. The impulsive moments t_k are such, $0=t_0<$

$t_1 < \cdots < t_p < t_{p+1} = 1$, $\Delta u(t_j) = u\left(t_j^+\right) - u\left(t_j^-\right)$, $u\left(t_j^+\right) = \lim\limits_{h\to 0^+} u(t_j + h)$, and $u\left(t_j^-\right) = \lim\limits_{h\to 0^-} u(t_j + h)$ are the right and the left limits of $u(t)$ at the point $t = t_j$, $j = 1, \ldots, p$ respectively. $\Delta u'(t_j) = u'\left(t_j^+\right) - u'\left(t_j^-\right)$, $u'\left(t_j^+\right) = \lim\limits_{h\to 0^+} u'(t_j + h)$, and $u'\left(t_j^-\right) = \lim\limits_{h\to 0^-} u'(t_j + h)$ are the right and the left limits of $u'(t)$ to the point $t = t_j$, $j = 1, \ldots, p$ respectively, and $\left({}^C D^{\beta}_{t_j^+} u\right)_{|t=t_{j+1}^-} = \lim\limits_{t\to t_j^-} \left({}^C D^{\beta}_{t_j^+} u\right)(t)$.

Since impulsive fractional differential equations have several applications in various fields of research, they have been investigated in numerous articles in the literature, see [1, 3, 5, 8, 9, 17, 18, 21, 22, 39, 46, 64, 71, 72, 76, 77, 81, 86]., where questions of existence, uniqueness and stability of solutions are addressed. In addition, when these processes deal with hereditary phenomena or delay the argumentation that can lead to undesirable performance in the system, it is necessary to analyze the effect of delay on the dynamic behaviors of the impulsive fractional differential equations. For more results on the impulsive fractional differential equations with delay, we refer to [1, 3, 8, 18].

Recently, fractional differential equations involving the left and right fractional derivatives have been considered in [6, 11, 15, 16, 20, 30, 35, 36, 52]. The left and right fractional derivatives may arise naturally as in some physical situations, where the state of the process depends on all its past states and on the results of its future development, for more details see [6, 16].

In [71], the existence of at least one solution or infinitely many solutions for the following impulsive fractional problem is investigated by means of variational method and critical point theory:

$$\begin{cases} {}^c D^{\alpha}_{T^-}\left({}^c D^{\alpha}_{0^+} x(t)\right) = f(t,\ x),\ 0 \le t \le T,\ t \ne t_j, \\ \Delta\left({}^c D^{\alpha}_{T^-}\left({}^c D^{\alpha}_{0^+} x\right)\right)\left(t_j^-\right) = I_j\left(x\left(t_j^-\right)\right),\ j = 1, 2, ..., n, \\ x(0) = x(T) = 0, \end{cases}$$

where $\alpha \in \left(\frac{1}{2},\ 1\right]$, $0 = t_0 < t_1 < ... < t_n < t_{n+1} = T$.

In [18], an initial value problem for fractional differential equations with infinite delay is considered:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = f(t,\ x_t),\ 0 \leq t \leq b,\ 0 < \alpha < 1, \\ x(t) = \varphi(t),\ t \in (-\infty,\ 0]. \end{cases}$$

The existence of solutions is established by the help of Banach fixed point theorem and the nonlinear alternative of Leray-Schauder.

This chapter is organized as follows. In the next Section, we transform the problem (3.1) − (3.6) to an equivalent integral equation then we give some properties of the Green's function. In Section 3, we present the main result which is the existence of at least one solution, for this end, we rewritte the posed problem as a sum of a compact operator and a contraction, then we apply Krasnoselskii's fixed point theorem to conclude the existence of a nontrivial solution. Two illustrative examples are given in Section 4.

3.2 Equivalent integral equation

Let us define the functional space where the problem (3.1) − (3.6) will be solved. Denote by E the Banach space

$$E = PC([-r, 1], \mathbb{R}) \cap PC^1([0, 1], \mathbb{R})$$

with the norm

$$\|u\|_E = \max_{t\in[-r,1]} |u(t)|.$$

Definition 3.2.1 *A function $u \in E$ is said to be a solution for problem (3.1)-(3.6) if it satisfies the differential equation (3.1) and the conditions (3.2)-(3.6).*

First, we establish an equivalence between the problem (3.1)-(3.6) and some integral equations.

Lemma 3.2.1 *The boundary value problem (3.1) − (3.6) is equivalent to the following integral equation:*

$$u(t) = \varphi(t),\ t \in [-r, 0],$$

if $t \in [0, t_1)$ *then*

$$u(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left(\int_0^\tau (t-s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^{t_1} \left(\int_0^t (t-s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau) d\tau$$
$$+ \frac{t^\beta}{\Gamma(\beta+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right),$$

if $t \in [t_1, t_2)$, *then*

$$u(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_1}^t \left(\int_{t_1}^\tau (t-s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^{t_2} \left(\int_{t_1}^t (t-s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{t_1} \left(\int_{t_0}^\tau (t_1-s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau) d\tau$$
$$+ h_1\left(t_1^-, u(t_1^-)\right) + \frac{t_1^\beta}{\Gamma(\beta+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right)$$
$$+ \frac{(t-t_1)^\beta}{\Gamma(\beta+1)} g_1\left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right)$$
$$+ (t-t_1)\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta-1)} \int_0^{t_1} \left(\int_0^\tau (t_1-s)^{\beta-2} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau) d\tau \right.$$
$$\left. + \tilde{h}_1\left(u(t_1^-)\right) + \frac{t_1^{\beta-1}}{\Gamma(\beta)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \right),$$

If $t \in (t_j, t_{j+1}]$, $j = 2, ..., p$, *then*

$$
\begin{aligned}
u(t) = &\int_{t_j}^{t_{j+1}} G_j(t,\tau) f(\tau, u_\tau)\, d\tau \\
&+ \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} F_k(\tau) f(\tau, u_\tau)\, d\tau + h_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\
&\left. + \frac{(t_k - t_{k-1})^\beta}{\Gamma(\beta+1)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right) \\
&+ \sum_{k=1}^{j-1} (t_j - t_k) \left(\int_{t_{k-1}}^{t_k} H_k(\tau) f(\tau, u_\tau)\, d\tau + \tilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\
&\left. + \frac{(t_k - t_{k-1})^{\beta-1}}{\Gamma(\beta)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right) \\
&+ (t - t_j) \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} H_k(\tau) f(\tau, u_\tau)\, d\tau + \tilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\
&\left. + \frac{(t_k - t_{k-1})^{\beta-1}}{\Gamma(\beta)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right)
\end{aligned}
$$

where

$$
G_j(t,\tau) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \begin{cases} \int_{t_j}^{\tau} (t-s)^{\beta-1} (\tau - s)^{\alpha-1}\, ds, & t_j \le \tau < t \le t_{j+1}, \\ \int_{t_j}^{t} (t-s)^{\beta-1} (\tau - s)^{\alpha-1}\, ds, & t_j \le t < \tau \le t_{j+1}. \end{cases}
$$

for $\forall j = 0, \cdots, p$, *and*

$$
F_j(\tau) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_{j-1}}^{\tau} (t_j - s)^{\beta-1} (\tau - s)^{\alpha-1}\, ds, \;\; t_{j-1} \le \tau \le t_j,
$$

$$H_j(\tau) = \frac{1}{\Gamma(\alpha)\Gamma(\beta-1)} \int_{t_{j-1}}^{\tau} (t_j - s)^{\beta-2} (\tau - s)^{\alpha-1} ds, \ t_{j-1} \leq \tau \leq t_j.$$

where, $j = 1, \cdots, p$.

Proof It is obvious that $u(t) = \varphi(t), t \in [-r, 0]$ since it is the history condition. Let $t \in [0, t_1]$, then equation (3.1) becomes

$$^{C}D^{\alpha}_{t_1^-}\left({}^{C}D^{\beta}_{0^+} u(t)\right) = f(t, u_t), \ 0 < t < t_1, \tag{3.7}$$

then by applying the fractional integral $I^{\alpha}_{t_1^-}$ to the equation (3.7), we obtain

$$^{C}D^{\beta}_{0^+} u(t) = I^{\alpha}_{t_1^-} f(t, u_t) + c_0, \tag{3.8}$$

condition (3.4) implies

$$c_0 = g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right),$$

substituting c_0 in (3.8), we get

$$^{C}D^{\beta}_{0^+} u(t) = I^{\alpha}_{t_1^-} f(t, u_t) + g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right). \tag{3.9}$$

Now, applying the fractional integral $I^{\beta}_{0^+}$ to the equation (3.9), it yields

$$u(t) = I^{\beta}_{0^+} I^{\alpha}_{t_1^-} f(t, u_t) + I^{\beta}_{0^+}\left(g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right)\right) + c_1 + c_2 t. \tag{3.10}$$

thanks to (3.2) and (3.3), we obtain

$$c_1 = \varphi(0) = 0, \ c_2 = u'(0) = 0,$$

that we substitute in (3.10) to get

$$\begin{aligned} u(t) &= I_{0^+}^{\beta} I_{t_1^-}^{\alpha} f(t,u_t) + \frac{t^{\beta}}{\Gamma(\beta+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left((t-s)^{\beta-1} \int_s^{t_1} (\tau-s)^{\alpha-1} f(\tau,u_\tau)\, d\tau \right) ds \\ &+ \frac{t^{\beta}}{\Gamma(\beta+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right). \end{aligned}$$

By using the Fubini theorem, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left(\int_0^{\tau} (t-s)^{\beta-1} (\tau-s)^{\alpha-1}\, ds \right) f(\tau,u_\tau)\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^{t_1} \left(\int_0^{t} (t-s)^{\beta-1} (\tau-s)^{\alpha-1}\, ds \right) f(\tau,u_\tau)\, d\tau \\ &+ \frac{t^{\beta}}{\Gamma(\beta+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right). \end{aligned}$$

Now, let $t \in [t_1, t_2]$. Applying the fractional integral $I_{t_2^-}^{\alpha}$ to the equation (3.1), then the condition (3.4) implies

$$^{C}D_{t_1^+}^{\beta} u(t) = I_{t_2^-}^{\alpha} f(t,u_t) + g_1\left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right). \tag{3.11}$$

We get by applying the fractional integral $I_{t_1^-}^{\alpha}$ to the equation (3.11), the following

$$u(t) = I_{t_1^+}^{\beta} I_{t_2^-}^{\alpha} f(t,u_t) + \frac{(t-t_1)^{\beta}}{\Gamma(\beta+1)} g_1\left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right) + b_1 + b_2(t-t_2). \tag{3.12}$$

In view of conditions (3.5) and (3.6), we obtain

$$\begin{aligned} b_1 = u\left(t_1^{+}\right) &= h_1\left(t_1^{-}, u(t_1^{-})\right) + u\left(t_1^{-}\right) \\ &= h_1\left(t_1^{-}, u(t_1^{-})\right) + \frac{t_1^{\beta}}{\Gamma(\beta+1)} g_0\left(t_1^{-}, u\left(t_1^{-}\right) - u\left(0^{+}\right)\right) \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{t_1} \left(\int_0^{\tau} (t_1 - s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau)\, d\tau \end{aligned}$$

and

$$\begin{aligned} b_2 = u'\left(t_1^{+}\right) &= \tilde{h}_1\left(u(t_1^{-})\right) + u'\left(t_1^{-}\right) \\ &= \tilde{h}_1\left(u(t_1^{-})\right) + \frac{t_1^{\beta-1}}{\Gamma(\beta)} g_0\left(u\left(t_1^{-}\right) - u\left(0^{+}\right)\right) \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta-1)} \int_0^{t_1} \left(\int_0^{\tau} (t_1 - s)^{\beta-2} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau)\, d\tau, \end{aligned}$$

substituting b_1 and b_2 in (3.12), and then using Fubini theorem, it yields

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_1}^{t} \left(\int_{t_1}^{\tau} (t - s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau)\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t}^{t_2} \left(\int_{t_1}^{t} (t - s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau)\, d\tau \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{t_1} \left(\int_0^{\tau} (t_1 - s)^{\beta-1} (\tau - s)^{\alpha-1} ds \right) f(\tau, u_\tau)\, d\tau \\ &+ h_1\left(t_1^{-}, u(t_1^{-})\right) + \frac{t_1^{\beta}}{\Gamma(\beta+1)} g_0\left(t_1^{-}, u\left(t_1^{-}\right) - u\left(0^{+}\right)\right) \end{aligned}$$

$$+\frac{(t-t_1)^{\beta}}{\Gamma(\beta+1)}g_1\left(t_2^-,u\left(t_2^-\right)-u\left(t_1^+\right)\right)$$
$$+(t-t_1)\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta-1)}\int_0^{t_1}\left(\int_0^{\tau}(t_1-s)^{\beta-2}(\tau-s)^{\alpha-1}ds\right)f(\tau,u_\tau)\,d\tau\right.$$
$$\left.+\tilde{h}_1\left(u(t_1^-)\right)+\frac{t_1^{\beta-1}}{\Gamma(\beta)}g_0\left(t_1^-,u\left(t_1^-\right)-u\left(0^+\right)\right)\right).$$

By proceeding in a similar manner for $t\in[t_p,1]$, we get

$$u(t)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_p}^{t}\left(\int_{t_p}^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\right)f(\tau,u_\tau)\,d\tau$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{t}^{1}\left(\int_{t_p}^{t}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\right)f(\tau,u_\tau)\,d\tau$$
$$+\sum_{k=1}^{p}\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{\tau}(t_k-s)^{\beta-1}(\tau-s)^{\alpha-1}ds\right)f(\tau,u_\tau)\,d\tau+h_k\left(t_k^-,u\left(t_k^-\right)\right)\right.$$

$$\left.+\frac{(t_k-t_{k-1})^{\beta}}{\Gamma(\beta+1)}g_{k-1}\left(t_k^-,u\left(t_k^-\right)-u\left(t_{k-1}^+\right)\right)\right)$$
$$+\sum_{k=1}^{p-1}(t_p-t_k)\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta-1)}\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{\tau}(t_k-s)^{\beta-2}(\tau-s)^{\alpha-1}ds\right)f(\tau,u_\tau)\,d\tau\right.$$

$$\left.+\tilde{h}_k\left(t_k^-,u\left(t_k^-\right)\right)+\frac{(t_k-t_{k-1})^{\beta-1}}{\Gamma(\beta)}g_k\left(t_k^-,u\left(t_k^-\right)-u\left(t_{k-1}^+\right)\right)\right)$$
$$+(t-t_p)\sum_{k=1}^{p}\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta-1)}\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{\tau}(t_k-s)^{\beta-2}(\tau-s)^{\alpha-1}ds\right)f(\tau,u_\tau)\,d\tau\right.$$
$$\left.+\tilde{h}_k\left(t_k^-,u\left(t_k^-\right)\right)+\frac{(t_k-t_{k-1})^{\beta-1}}{\Gamma(\beta)}g_k\left(t_k^-,u\left(t_k^-\right)-u\left(t_{k-1}^+\right)\right)\right)$$

Conversely, supposing that u satisfies the integral equations in Lemma 3.2.1, then by direct computations, we prove that u satisfies the problem (3.1) − (3.6). ■

The properties of the functions G_j, F_j, and H_j, for $j = 0, \cdots, p$ are given in the following lemma.

Lemma 3.2.2 *The functions G_j, F_j, and H_j, are nonnegative and satisfy the following estimates for all $t, \tau \in [t_j, t_{j+1}]$, $j = 0, \cdots, p$:*
1)

$$G_j(t,\tau) \leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)},$$

2)

$$\begin{aligned} F_j(\tau) &\leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}, \\ H_j(\tau) &\leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}. \end{aligned}$$

Proof From the expressions of G_j, F_j, and H_j, for $j = 0, \cdots, p$ we see that are nonnegative functions. Let $t_j \leq \tau < t \leq t_{j+1}$, $j = 0, \cdots, p$, then

$$\begin{aligned} G_j(t,\tau) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_j}^{\tau} (t-s)^{\beta-1} (\tau-s)^{\alpha-1} ds \\ &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_j}^{\tau} (\tau-s)^{\alpha-1} ds = \frac{(\tau-t_j)^{\alpha}}{\alpha\Gamma(\alpha)\Gamma(\beta)} \\ &\leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}. \end{aligned}$$

For $t_j \leq \tau < t \leq t_{j+1}$, $j = 0, \cdots, p$, we have

$$
\begin{aligned}
G_j(t,\tau) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_j}^{t} (t-s)^{\beta-1} (\tau - s)^{\alpha-1} ds \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_j}^{t} (t-s)^{\alpha+\beta-2} ds = \frac{(t-t_j)^{\alpha+\beta-1}}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} \\
&\leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)},
\end{aligned}
$$

thus,

$$
G_j(t,\tau) \leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}, \; t_j \leq \tau, t \leq t_{j+1}, j = 0, \cdots, p,
$$

In the same way, we prove that for all $\tau \in [t_{j-1}, t_j]$, $j = 1, \cdots, p$ we have

$$
F_j(\tau) \leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)},
$$

and

$$
H_j(\tau) \leq \frac{1}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}.
$$

■

3.3 Existence of solutions

In this section, we shall prove the existence of solution for problem (3.1)-(3.6) by means of Krasnoselski fixed point Theorem 1.2.1. Define the operators A and B on E by

$$
Au(t) = \begin{cases} 0, \; t \in [-r, 0] \\ A_j u(t), \; t \in [t_j, t_{j+1}], \; j = 0, \cdots, p. \end{cases}
$$

$$
Bu(t) = \begin{cases} \varphi(t), \; t \in [-r, 0] \\ B_j u(t), \; t \in [t_j, t_{j+1}], \; j = 0, \cdots, p. \end{cases}
$$

where

$$A_j u(t) = \int_{t_j}^{t_{j+1}} G_j(t,\tau) f(\tau, u_\tau)\, d\tau,\ t \in [t_j, t_{j+1}],\ j = 0, \cdots, p$$

and

$$B_j u(t) = \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} F_k(\tau) f(\tau, u_\tau)\, d\tau + h_k\left(t_k^-, u\left(t_k^-\right)\right) \right.$$
$$\left. + \frac{(t_k - t_{k-1})^\beta}{\Gamma(\beta+1)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right)$$
$$+ \sum_{k=1}^{j-1} (t_j - t_k) \left(\int_{t_{k-1}}^{t_k} H_k(\tau) f(\tau, u_\tau)\, d\tau + \tilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right.$$

$$\left. + \frac{(t_k - t_{k-1})^{\beta-1}}{\Gamma(\beta)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right)$$
$$+ (t - t_j) \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} H_k(\tau) f(\tau, u_\tau)\, d\tau + \tilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right.$$
$$\left. + \frac{(t_k - t_{k-1})^{\beta-1}}{\Gamma(\beta)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right),$$

for $t \in [t_j, t_{j+1}]$, $j = 0, \cdots, p$.

Then u is a solution for problem (3.1) $-$ (3.6) if and only if

$$Au(t) + Bu(t) = u(t),\ t \in [-r, 1]$$

that is u is a fixed point for the operator $A + B$.

We will use the following assumptions:

H_1) *The function $f(.,0)$ is continuous and not identically null on $[0,1]$, and there exists a nonnegative function $k \in L_1([0,1], \mathbb{R}_+)$, such that*

$$|f(t,x) - f(t,y)| \leq k(t)|x-y|,\ t \in [0,1],\ x,\ y \in \mathbb{R};$$

and

$$\|k\|_{L_1([0,1],\mathbb{R}_+)} < \frac{(\alpha+\beta-2)\,\Gamma(\alpha)\,\Gamma(\beta)}{24p}. \tag{3.13}$$

H_2) *The functions* $h_j(.,0) = 0$, *for all* $j = 1,\cdots,p$, *and there exist nonnegative functions* $a_j \in C([0,1],\mathbb{R}_+)$, $j = 1,\cdots,p$, *such that*

$$|h_j(t,x) - h_j(t,y)| \leq a_j(t)\,|x-y|\,,\ 0 \leq t \leq 1,\ x,\ y \in \mathbb{R},\ j = 1,\cdots,p,$$

and

$$a = \max_{j=1,\cdots,p}\left(\|a_j\|_{C([0,1],\mathbb{R}_+)}\right) < \frac{1}{8p}. \tag{3.14}$$

H_3) $\tilde{h}_j(.,0) = 0$, *for all* $j = 1,\cdots,p$, *and there exist nonnegative functions* $b_j \in C([0,1],\ \mathbb{R}_+)$, $j = 1,\cdots,p$, *such that*

$$\left|\tilde{h}_j(t,x) - \tilde{h}_j(t,y)\right| \leq b_j(t)\,|x-y|\,,\ 0 \leq t \leq 1\ ,x,\ y \in \mathbb{R},\ j = 1,\cdots,p,$$

$$b = \max_{j=1,\cdots,p}\left(\|b_j\|_{C([0,1],\mathbb{R}_+)}\right) < \frac{1}{16p}. \tag{3.15}$$

H_4) *There exist nonnegative functions* $c_j \in C([0,1],\mathbb{R}_+)$, $j = 0,\cdots,p$, *such that*

$$|g_j(t,x) - g_j(t,y)| \leq c_j(t)\,|x-y|\,,\ 0 \leq t \leq 1,\ x,\ y \in \mathbb{R},\ j = 0,\cdots,p,$$

$$c = \max_{j=0,\cdots,p}\left(\|c_j\|_{C([0,1],\mathbb{R}_+)}\right) < \frac{\Gamma(\beta)}{48p}. \tag{3.16}$$

Let

$$M = \{u \in E, \|u\| \leq R\}\,.$$

where R is chosen such that

$$R \geq \max\left(\frac{24pL}{(\alpha+\beta-2)\,\Gamma(\alpha)\,\Gamma(\beta) - 24p\,\|k\|_{L_1}}, \frac{24pd}{\Gamma(\beta)}\right) \tag{3.17}$$

where

$$L = \max_{t\in[0,1]}|f(t,0)|\,, \qquad d = \max_{j=0,\cdots,p}\left(\max_{t\in[0,1]}|g_j(t,0)|\right).$$

Clearly, M is a nonempty, bounded and convex subset of E.

Theorem 3.3.1 *Under assumptions* $(H_1)-(H_4)$*, the problem* (3.1)-(3.6) *has at least one nontrivial solution in* M*.*

Proof We shall prove that all assumptions of krasnoselskii's fixed point Theorem are satisfied, for this the proof will be done in some steps.
Claim 1. The mapping A is continuous on M. Let $(u_n)_n$ be a convergent sequence in M, that is $u_n \to u$ in M. By Lemma 3.2.2 and assumption (H_1) we deduce for $t \in [t_j, t_{j+1}]$, $j = 0, \cdots, p$

$$\begin{aligned}
|Au_n(t) - Au(t)| &= |A_j u_n(t) - A_j u(t)| \\
&\leq \int_{t_j}^{t_{j+1}} G_j(t,\tau) |f(\tau, u_{n\tau}) - f(\tau, u_\tau)| \, d\tau \\
&\leq \frac{\|k\|_{L_1([0,1],\mathbb{R}_+)}}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} \|u_n - u\|.
\end{aligned}$$

Claim 2. (Au) is uniformly bounded on M. In fact, by assumption (H_1) it yields for $u \in M$ and $t \in [t_j, t_{j+1}]$, $j = 0, \cdots, p$

$$|Au(t)| = |A_j u(t)| \leq \int_{t_j}^{t_{j+1}} G_j(t,\tau) |f(\tau, u_\tau)| \, d\tau \tag{3.18}$$

$$\begin{aligned}
&\leq \int_{t_j}^{t_{j+1}} G_j(t,\tau) |f(\tau, u_\tau) - f(\tau, 0)| \, d\tau + \int_{t_j}^{t_{j+1}} G_j(t,\tau) |f(\tau, 0)| \, d\tau \\
&\leq \frac{\|u\| . \|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} \leq \frac{R\|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}.
\end{aligned}$$

Claim 3. (Au) is equicontinuous on M. We have, for $u \in M$, and $t_j < \tau_1 < \tau_2 < t_{j+1}$, for $j = 0, \cdots, p$

$$|Au(\tau_2) - Au(\tau_1)| = |A_j u(\tau_2) - A_j u(\tau_1)|$$

$$\leq \int_{t_j}^{\tau_1} |G_j(\tau_2,\tau) - G_j(\tau_1,\tau)| \, |f(\tau,u_\tau)| \, d\tau$$
$$+ \int_{\tau_1}^{\tau_2} |G_j(\tau_2,\tau) - G_j(\tau_1,\tau)| \, |f(\tau,u_\tau)| \, d\tau$$
$$+ \int_{\tau_2}^{t_{j+1}} |G_j(\tau_2,\tau) - G_j(\tau_1,\tau)| \, |f(\tau,u_\tau)| \, d\tau$$

$$\leq \left(\frac{R \|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{\Gamma(\alpha)\Gamma(\beta)} \right)$$
$$\times \left(\int_{t_j}^{\tau_1} \left(\int_{t_j}^{\tau} \left((\tau_2 - s)^{\beta-1} - (\tau_1 - s)^{\beta-1} \right) (\tau - s)^{\alpha-1} \, ds \right) d\tau \right.$$
$$+ \int_{\tau_1}^{t_{j+1}} \left(\int_{t_j}^{\tau_1} \left((\tau_2 - s)^{\beta-1} - (\tau_1 - s)^{\beta-1} \right) (\tau - s)^{\alpha-1} \, ds \right) d\tau$$
$$+ \int_{\tau_1}^{\tau_2} \left(\int_{\tau_1}^{\tau} (\tau_2 - s)^{\beta-1} (\tau - s)^{\alpha-1} \, ds \right) d\tau$$
$$\left. + \int_{\tau_1}^{t_{j+1}} \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\beta-1} (\tau - s)^{\alpha-1} \, ds \right) d\tau \right)$$
$$\leq 3(\tau_2 - \tau_2) \left(\frac{R \|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{\Gamma(\alpha+1)\Gamma(\beta+1)} \right),$$

hence, $|Au(\tau_2) - Au(\tau_1)|$ tends to zero as $\tau_1 \to \tau_2$. Consequently (Au) is equicontinuous for $t \neq t_j$, $j = 1, \ldots, p+1$. We shall prove the equicontinuity of (Au) at $t = t_j$, for this, we firstly prove the equicontinuity of (Au) at $t = t_j^-$. Let us fix

$\delta_1 > 0$ such that $\{t_k,\ k \neq j\} \cap [t_j - \delta_1,\ t_j + \delta_1] = \emptyset$, for $0 < h < \delta_1$, we have that

$$\begin{aligned}
|Au(t_j) - Au(t_j - h)| &= |A_{j-1}u(t_j) - A_{j-1}u(t_j - h)| \\
&\leq \left(2(t_j - t_{j-1})^{\beta} - 2(t_j - t_{j-1} - h)^{\beta} - h^{\beta}\right) \\
&\times \left(\frac{R\|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right),
\end{aligned}$$

as $h \to 0$, then the right-hand side tends to zero. Similarly, we prove the equicontinuity of (Au) at $t = t_j^+$. Fixing $\delta_2 > 0$ such that $\{t_k,\ k \neq j\} \cap [t_j - \delta_2,\ t_j + \delta_2] = \emptyset$, then for $0 < h < \delta_2$, it yields

$$\begin{aligned}
|Au(t_j + h) - Au(t_j)| &= |A_j u(t_j + h) - A_j u(t_j)| \\
&\leq h^{\beta}\left(\frac{R\|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{\Gamma(\alpha)\Gamma(\beta)}\right), \text{ as } h \to 0.
\end{aligned}$$

We conclude by Arzela-Ascoli Theorem 1.1.1 that A is completely continuous on M.

Claim 4. The mapping B is contraction on M. Taking assumptions $(H_1) - (H_4)$ into account, then we get for $u,\ v \in M$ and $t \in [t_j,\ t_{j+1}]$, $j = 1, \cdots, p$,

$$\begin{aligned}
|Bu(t) - Bv(t)| &= |B_j u(t) - B_j v(t)| \\
&\leq \left[\frac{3p\|k\|_{L_1([0,1],\mathbb{R}_+)}}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} + pa\right. \\
&\left. +2pb + \frac{3pc}{\Gamma(\beta)}\right]\|u - v\| \\
&\leq \frac{\|u - v\|}{2},
\end{aligned}$$

thus B is contraction on M.

Claim 5. $(Au + Bv) \in M$ for all $u, v \in M$. Indeed, in view of (3.17) and (3.18), it yields

$$|Au(t)| \leq \frac{R\|k\|_{L_1([0,1],\mathbb{R}_+)} + L}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} \leq \frac{R}{24}.$$

Taking (3.14),(3.15) and (3.16) into account, we obtain for $v \in M$

$$\begin{aligned}|Bv(t)| \leq & \frac{Rp}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} \\ & \times \left(3\|k\|_{L_1} + a(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta) + 2b\Gamma(\beta) + 6c\right) \\ & + 3p\frac{L + d(\alpha+\beta-2)\Gamma(\alpha)}{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)} \\ & \leq \frac{5R}{8},\end{aligned}$$

hence

$$|Au(t) + Bv(t)| \leq |Au(t)| + |Bv(t)| \leq \frac{2R}{3} < R,\ u,\ v \in M.$$

Finally, we conclude by Krasnoselskii fixed point Theorem 1.2.1, that $A+B$ has a fixed point $u \in M$ and then the problem (3.1)-(3.6) has at least one nontrivial solution in M. ■

3.4 Examples

3.4.1 Example 1

Consider the problem (3.1)-(3.6), with $p = 1$, $t_1 = \frac{1}{2}$, $\alpha = 0.75$, $\beta = 1.75$, and

$$\begin{aligned}{}^{C}D^{\alpha}_{\frac{1}{2}^-}\left({}^{C}D^{\beta}_{0^+}u(t)\right) &= f(t, u_t),\ 0 \leq t < \frac{1}{2}, \\ {}^{C}D^{\alpha}_{1^-}\left({}^{C}D^{\beta}_{\frac{1}{2}^+}u(t)\right) &= f(t, u_t),\ \frac{1}{2} \leq t < 1, \\ u(t) = \varphi(t),\ t \in [-r, 0],\ & u'(0) = 0, \\ {}^{C}D^{\beta}_{0^+}(u(t))|_{t=\frac{1}{2}^-} &= {}^{C}D^{\beta}_{\frac{1}{2}^+}(u(t))|_{t=1^-} = 0,\end{aligned}$$

$$\begin{aligned}\Delta u\left(\frac{1}{2}\right) &= h_1\left(\frac{1}{2}^-, u\left(\frac{1}{2}\right)\right), \\ \Delta u'\left(\frac{1}{2}\right) &= \tilde{h}_1\left(\frac{1}{2}^-, u\left(\frac{1}{2}\right)\right),\end{aligned}$$

$$f(t,x)=\frac{2\sin t^2}{45}\left(x-\frac{t}{2(1+x^2)}\right),\ t\in[0,1],\ x\in\mathbb{R},$$

$$h_1(t,x)=x\frac{\cos t^2}{8},\ t\in[0,1],\ x\in\mathbb{R},\ \tilde{h}_1(t,x)=x\frac{\sin t^2}{16},\ t\in[0,1],\ x\in\mathbb{R}.$$

Let us check the assumptions of Theorem 3.3.1. We have

$$f(t,0)=\frac{t\sin t^2}{45}\ \text{nonidentically null on}\ [0,1],$$

$$|f(t,x)-f(t,y)|=\frac{2\sin t^2}{15}|x-y|=k(t)|x-y|,t\in[0,1],x,y\in\mathbb{R},$$

$$\begin{aligned}\|k\|_{L_1}&=\int_0^1\frac{\sin t^2}{15}dt=0.020685\\&<0.023463=\frac{(\alpha+\beta-2)\Gamma(\alpha)\Gamma(\beta)}{24}.\end{aligned}$$

Then hypothesis (H_1) is satisfied. Moreover we have

$$\begin{aligned}h_1(t,0)&=0,\\|h_1(t,x)-h_1(t,y)|&=\frac{\cos t^2}{8}|x-y|,\ t\in[0,1],\ x,\ y\in\mathbb{R},\\a_1(t)&=\frac{\cos t^2}{8},\ a=\frac{\cos 1}{8}=0.067538<\frac{1}{8}=0.125,\end{aligned}$$

and

$$\begin{aligned}\tilde{h}_1(t,0)&=0,\\\left|\tilde{h}_1(t,x)-\tilde{h}_1(t,y)\right|&=\frac{\sin t^2}{16}|x-y|,\ t\in[0,\ 1],\ x,\ y\in\mathbb{R},\\b&=\frac{\sin 1}{16}=0.052592<\frac{1}{16}=0.0625.\end{aligned}$$

Thus hypotheses (H_2) and (H_3) hold. By calculation, we get

$$c=\max_{j=0,1}\|c_j\|_{L_1}=0<\frac{\Gamma(1.75)}{24},$$

$$L=\sup\left\{\left|f\left(t,\ 0\right)\right|,\ 0\leq t\leq 1\right\}=\frac{2\sin 1}{45}=0.037399,$$
$$d=0,$$

and
$$\max\left(\frac{24pL}{\left(\alpha+\beta-2\right)\Gamma\left(\alpha\right)\Gamma\left(\beta\right)-24p\left\|k\right\|_{L_1}},\frac{24pd}{\Gamma\left(\alpha\right)}\right)=6.7311,$$

hence, hypothesis (H_4) is satisfied. Finally, we conclude by Theorem 3.3.1, that the problem (3.1)-(3.6) has at least one nontrivial solution u satisfying $\|u\|\leq R$, with $R>6.7311$.

3.4.2 Example 2

Let us consider the impulsive boundary value problem with delay (3.1)-(3.6), with $p=1$, $t_1=\frac{1}{2}$, $\alpha=\beta=1.5$,

$$\begin{aligned}
{}^{C}D^{\alpha}_{\frac{1}{2}^-}\left({}^{C}D^{\beta}_{0^+}u\left(t\right)\right)&=f\left(t,u_t\right),\ 0\leq t<\frac{1}{2},\\
{}^{C}D^{\alpha}_{1^-}\left({}^{C}D^{\beta}_{\frac{1}{2}^+}u\left(t\right)\right)&=f\left(t,u_t\right),\ \frac{1}{2}\leq t<1,\\
u\left(t\right)=\varphi\left(t\right),\ t\in\left[-r,0\right],\ u'\left(0\right)&=0,\\
{}^{C}D^{\beta}_{0^+}\left(u\left(t\right)\right)|_{t=\frac{1}{2}^-}={}^{C}D^{\beta}_{\frac{1}{2}^+}\left(u\left(t\right)\right)|_{t=1^-}&=0,
\end{aligned}$$

$$\begin{aligned}
\Delta u\left(\frac{1}{2}\right)&=h_1\left(\frac{1}{2}^-,u\left(\frac{1}{2}^-\right)\right)=\frac{tx}{15},t\in\left[0,1\right],\ x\in\mathbb{R},\\
\Delta u'\left(\frac{1}{2}\right)&=\tilde{h}_1\left(\frac{1}{2}^-,u\left(\frac{1}{2}^-\right)\right)=\frac{tx}{18},t\in\left[0,1\right],\ x\in\mathbb{R}.
\end{aligned}$$

$$f\left(t,x\right)=\frac{\left(1+x\right)e^{-t}}{\left(25+e^t\right)},\ t\in\left[0,1\right],\ x\in\mathbb{R}.$$

All assumptions of Theorem 3.3.1 are satisfied. In fact, we have

$$f\left(t,0\right)=\frac{e^{-t}}{\left(25+e^t\right)}\ \text{nonidentically null on}\ \left[0,1\right],$$

$$\begin{aligned}\left|f\left(t,x\right)-f\left(t,y\right)\right| &= \frac{e^{-t}}{\left(25+e^{t}\right)}\left|x-y\right| \\ &= k\left(t\right)\left|x-y\right|, t\in\left[0,1\right], x,y\in\mathbb{R},\end{aligned}$$

$$\begin{aligned}\left\|k\right\|_{L_1} = \int_0^1 \frac{e^{-t}}{\left(25+e^{t}\right)}dt = 0.02378\,7 \\ < 0.03272\,5 = \frac{\left(\alpha+\beta-2\right)\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{24}.\end{aligned}$$

$$\begin{aligned}h_1\left(t,0\right) &= 0, \\ \left|h_1\left(t,x\right)-h_1\left(t,y\right)\right| &= \frac{t}{10}\left|x-y\right|,\ t\in\left[0,1\right],\ x,\ y\in\mathbb{R}, \\ a_1\left(t\right) &= \frac{t}{10},\ t\in\left[0,\ 1\right],\ a=\frac{1}{15}=0.06667<\frac{1}{8}=0.125.\end{aligned}$$

$$\begin{aligned}\left|\tilde{h}_1\left(t,x\right)-\tilde{h}_1\left(t,y\right)\right| &= \frac{t}{18}\left|x-y\right|,\ t\in\left[0,1\right],\ x,\ y\in\mathbb{R}, \\ \tilde{h}_1\left(t,0\right) &= 0, b_1\left(t\right)=\frac{t}{18},\ t\in\left[0,1\right], \\ b &= \frac{1}{18}=0.05555\,6<\frac{1}{16}=0.0625.\end{aligned}$$

Thus assumptions (H_1)-(H_4) hold. Furthermore, by computation it yields

$$c=\max_{j=0,1}\left\|c_j\right\|_{L_1}=0<\frac{\Gamma\left(1.5\right)}{24},$$

$$\begin{aligned}L &= \sup\left\{\left|f\left(t,0\right)\right|,\ 0\leq t\leq 1\right\}=\frac{1}{26}=0.03846\,2, \\ d &= 0\end{aligned}$$

$$\max\left(\frac{24pl}{\left(\alpha+\beta-2\right)\Gamma\left(\alpha\right)\Gamma\left(\beta\right)-24p\left\|k\right\|_{L_1}},\ \frac{24pd}{\Gamma\left(\alpha\right)}\right)=4.303\,2.$$

Thanks to Theorem 3.3.1, we deduce that the problem (3.1)-(3.6) has at least one nontrivial solution u satisfying $\|u\| \leq R$ where $R > 4.3032$.

CHAPTER 4

Positive Solutions for Impulsive Mixed Fractional Differential Equations with p-Laplacian Operator

4.1 Introduction and motivation

This chapter is devoted to the study of the existence, uniqueness and positivity of solutions for the following nonlinear impulsive boundary value problem involving mixed type fractional derivatives and the p-Laplacian operator

$$ {}^{C}D_{1^-}^{\beta}\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(t\right)\right) = f(t,u(t)),\ t\in J=\left[0,1\right],\ t\neq t_i,\ i=1,\ldots,m, \tag{4.1}$$

$$ \Delta\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(t_i\right)\right) = I_i\left(u\left(t_i\right)\right),\ i=1,\ldots,m, \tag{4.2}$$

$$ \Delta\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(t_i\right)\right)\right)' = J_i\left(u\left(t_i\right)\right),\ i=1,\ldots,m, \tag{4.3}$$

$$ a\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(0\right)\right) - b\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(1\right)\right) = 0,\ b>a>0, \tag{4.4}$$

$$ a\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(0\right)\right)\right)' - b\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\left(1\right)\right)\right)' = 0,\ b>a>0, \tag{4.5}$$

$$ \Delta u\left(t_i\right) = \tilde{I}_i\left(u\left(t_i\right)\right),\ i=1,\ldots,m, \tag{4.6}$$

$$ \Delta u\left(t_i\right) = \tilde{J}_i\left(u\left(t_i\right)\right),\ i=1,\ldots,m, \tag{4.7}$$

$$ \gamma u\left(0\right) - \eta u\left(1\right) = 0,\ \gamma>\eta>0, \tag{4.8}$$

$$ \gamma u'\left(0\right) - \eta u'\left(1\right) = 0,\ \gamma>\eta>0. \tag{4.9}$$

where $1<\alpha$, $\beta<2$, $\gamma>\eta>0$, $b>a>0$, $p\geq 1$. We make the following notations for $i=1,\ldots,m$

$$\begin{aligned}
\Delta\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\left(t_i\right) &= \left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)\left(t_i^+\right) - \left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)\left(t_i^-\right),\\
\Delta\left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)'\left(t_i\right) &= \left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)'\left(t_i^+\right) - \left(\phi_p\left({}^{C}D_{0^+}^{\alpha}u\right)\right)'\left(t_i^-\right),\\
\Delta u\left(t_i\right) &= u\left(t_i^+\right) - u\left(t_i^-\right),\\
\Delta u'\left(t_i\right) &= u'\left(t_i^+\right) - u'\left(t_i^-\right),
\end{aligned}$$

here

$$\phi_p\left({}^C D_{0^+}^{\alpha} u\left(t_i^+\right)\right) = \lim_{t\to t_i^+} \phi_p\left({}^C D_{0^+}^{\alpha} u\right)(t),$$
$$\phi_p\left({}^C D_{0^+}^{\alpha} u\left(t_i^-\right)\right) = \lim_{t\to t_i^-} \phi_p\left({}^C D_{0^+}^{\alpha} u\right)(t),$$

$$\left(\phi_p\left({}^C D_{0^+}^{\alpha} u\right)\right)'\left(t_i^+\right) = \lim_{t\to t_i^+} \left(\phi_p\left({}^C D_{0^+}^{\alpha} u\right)\right)'(t),$$
$$\left(\phi_p\left({}^C D_{0^+}^{\alpha} u\left(t_i^-\right)\right)\right)' = \lim_{t\to t_i^-} \left(\phi_p\left({}^C D_{0^+}^{\alpha} u\right)\right)'(t),$$

$$u\left(t_i^+\right) = \lim_{t\to t_i^+} u(t),\ u\left(t_i^-\right) = \lim_{t\to t_i^-} u(t),$$
$$u'\left(t_i^+\right) = \lim_{t\to t_i^+} u'(t),\ u'\left(t_i^-\right) = \lim_{t\to t_i^-} u'(t).$$

Denote ${}^C D_{0^+}^{\beta}$ and ${}^C D_{1^-}^{\alpha}$ the left and the right Caputo fractional derivatives respectively, u is the unknown function, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$, the functions J_i, $\tilde{I}_i$, $\tilde{J}_i : \mathbb{R}^+ \to \mathbb{R}^+$, $I_i : \mathbb{R}^+ \to \mathbb{R}^-$, $i = 0, ..., m$ are given, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$.

The existence and uniqueness of solution are obtained by means of Banach contraction principle and Schauder fixed point theorem while the existence of positive solutions is established by the help of a fixed point theorem in cones.

Fractional differential equations involving the p-Laplacian operator have several applications in many fields such as in turbulent filtration in porous media, blood circulation problems, rheology, viscoelasticity modeling,....

The p-Laplacian differential equation was introduced, to model a mechanical problem that is the turbulent flow in a porous medium, by Leibenson [57] where he studied the following equation

$$\phi_p\left(u'(t)\right)' = f\left(t, u(t)\right) = 0,\ 0 < t < 1.$$

Since then, much attention has been paid to this type of differential equations but few papers has been done on the existence of positive solutions to fractional boundary value problems involving the p-Laplacian operator, see [24, 44, 64, 85, 91, 94]. Let us cite in particular a few articles in the literature

Using critical point theory and variational methods, the authors in [94], obtained the existence of multiple solutions for the following p-Laplacian impulsive fractional boundary value problems

$$\begin{cases} {}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u(t)\right)+|u(t)|^{p-2}u(t)=f(t,u(t)), & 0<t<T,\ t\neq t_j,\\ \Delta\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)(t_j)=I_j(u(t_j)), & j=1,\dots,m,\\ u(0)=u(T)=0. \end{cases}$$

Where $0<\alpha<1$, $p>1$, ${}_tD^{\alpha}_T$ denotes the right Riemann-Liouville derivative and ${}_0^CD^{\alpha}_t$ denotes the left Caputo derivative, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, and

$$\Delta\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)(t_j)=\Delta\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)\left(t_j^+\right)-\Delta\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)\left(t_j^-\right),$$

where

$$\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)\left(t_j^+\right)=\lim_{t\to t_i^+}\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)(t),$$
$$\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)\left(t_j^-\right)=\lim_{t\to t_i^-}\left({}_tD^{\alpha}_{T^-}\phi_p\left({}_0^CD^{\alpha}_t u\right)\right)(t).$$

In [24], the author established the existence and multiplicity of positive solutions for a boundary value problem of fractional differential equation with p-Laplacian operator

$$\begin{cases} D^{\beta}_{0^+}\phi_p\left(D^{\alpha}_{0^+}u(t)\right)=f(t,u(t)) & 0<t<1\\ u(0)=0,u(1)+\sigma D^{\gamma}_{0^+}u(1)=0, & D^{\alpha}_{0^+}u(0)=0, \end{cases}$$

where $D^{\alpha}_{0^+}$, $D^{\beta}_{0^+}$and $D^{\gamma}_{0^+}$ are the standard Riemann-Liouville derivatives with $1<\alpha<2$, $0<\beta,\gamma\leq 1$, $0\leq\alpha-\gamma-1$, the constant σ is a positive number. Thanks to the fixed point theorem in cones the main results are proved.

This chapter is structured as follows. In Section 2, we prove the equivalence between the problem (4.1)-(4.9) and an integral equation then we give some properties of the Green's function. In Section 3, we prove the existence of a unique solution by using Banach's contration principle. The existence of at least one solution is proved by the help of Schauder fixed point theorem and the existence of positive solutions is discussed by means of a fixed point theorems in the cone. Some illustrative examples are given in Section 4.

4.2 Equivalent integral equation

Define what we mean by a solution for the problem (4.1)-(4.9).

Definition 4.2.1 *A function $u \in PC(J, \mathbb{R}_+)$ is said to be a solution to the boundary value problem (4.1)-(4.9) if u satisfies (4.1) and conditions(4.2)-(4.9).*

The corresponding linear differential equation

$$ {}^{C}D^{\beta}_{1^-}\phi_p\left({}^{C}D^{\alpha}_{0^+}u(t)\right) = h(t), \ t \in J = [0,1], \ t \neq t_i, \ i = 1, \ldots, m, \tag{4.10} $$

Now we are ready to give the expression of the auxiliary problem (4.10)-(4.2)-(4.9).

Lemma 4.2.1 *The solutions $u \in PC(J, \mathbb{R}^+)$ of the impulsive fractional boundary value problem (4.10)-(4.2)-(4.9) is given by the integral equation*

$$ \begin{aligned} u(t) = &\int_0^1 K_1(t,s)\left(\int_0^1 G_1(s,\tau)h(\tau)\,d\tau + \sum_{i=1}^{m} G_2(s,t_i)\left(-I_i(u(t_i))\right)\right. \\ &\left. + \sum_{i=1}^{m} G_3(s,t_i) J_i(u(t_i))\right)^{q-1} ds \\ &+ \sum_{i=1}^{m} K_2(t,t_i)\tilde{I}_i(u(t_i)) + \sum_{i=1}^{m} K_3(t,t_i)\tilde{J}_i(u(t_i)) \end{aligned} \tag{4.11} $$

where

$$ K_1(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta^2(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)} + \frac{t\eta(1-s)^{\alpha-2}}{(\gamma-\eta)\Gamma(\alpha-1)}, \\ 0 < s \leq t < 1, \\ \frac{\eta(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta^2(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)} + \frac{t\eta(1-s)^{\alpha-2}}{(\gamma-\eta)\Gamma(\alpha-1)}, \ 0 < s \leq t < 1. \end{cases} \tag{4.12} $$

$$G_1(t,s)=\begin{cases}\frac{(s-t)^{\beta-1}}{\Gamma(\beta)}+\frac{as^{\beta-1}}{(b-a)\Gamma(\beta)}+\frac{a^2s^{\beta-2}}{(b-a)^2\Gamma(\beta-1)}+\frac{(1-t)as^{\beta-2}}{(b-a)\Gamma(\beta-1)},\\ 0<t\leq s<1,\\ \frac{as^{\beta-1}}{(b-a)\Gamma(\beta)}+\frac{a^2s^{\beta-2}}{(b-a)^2\Gamma(\beta-1)}+\frac{(1-t)as^{\beta-2}}{(b-a)\Gamma(\beta-1)},\ 0<s\leq t<1.\end{cases} \tag{4.13}$$

$$K_2(t,t_i)=\begin{cases}\frac{\gamma}{\gamma-\eta},0<t\leq t_i<1,\\ \frac{\eta}{\gamma-\eta},0<t_i\leq t<1\end{cases} \tag{4.14}$$

$$G_2(t,t_i)=\begin{cases}\frac{a}{b-a},\ 0<t\leq t_i<1\\ \frac{b}{b-a},\ 0<t_i\leq t<1\end{cases} \tag{4.15}$$

$$K_3(t,t_i)=\begin{cases}\frac{\eta\gamma}{(\gamma-\eta)^2}+\frac{\gamma(t-t_i)}{(\gamma-\eta)},\ 0<t\leq t_i<1\\ \frac{\eta\gamma}{(\gamma-\eta)^2}+\frac{\eta(t-t_i)}{(\gamma-\eta)},\ 0<t_i\leq t<1\end{cases} \tag{4.16}$$

$$G_3(t,t_i)=\begin{cases}\frac{a}{b-a}(1-t_i)+\frac{a}{b-a}(1-t)+\frac{ab}{(b-a)^2},\ 0<t\leq t_i<1,\\ \frac{b}{b-a}(1-t_i)+\frac{b}{b-a}(1-t)+\frac{ab}{(b-a)^2},\ 0<t_i\leq t<1,\end{cases} \tag{4.17}$$

for all $i=1,\ldots,m$.

Proof Set $x(t)=\phi_p\left({}^CD_{0^+}^{\alpha}u(t)\right)$, then equation (4.10) becomes

$$^CD_{1^-}^{\beta}x(t)=h(t),\ t\in J,\ t\neq t_i,\ i=1,\ldots,m$$

By Lemma 1.4.1 we get for $k=0,\ldots,m$,

$$x(t)=\frac{1}{\Gamma(\beta)}\int_t^1(s-t)^{\beta-1}h(s)\,ds+c_k+d_k(1-t),\ t\in[t_k,t_{k+1}] \tag{4.18}$$

Differentiating (4.18) we obtain

$$x'(t)=\frac{-1}{\Gamma(\beta-1)}\int_t^1(s-t)^{\beta-2}h(s)\,ds-d_k,\ t\in(t_k,t_{k+1}],\ k=0,\ldots,m \tag{4.19}$$

Using conditions (4.4) and (4.5), it yields

$$\frac{a}{\Gamma(\beta)}\int_0^1 s^{\beta-1}h(s)\,ds+ac_0+ad_0-bc_m=0, \tag{4.20}$$

$$-\frac{a}{\Gamma(\beta-1)}\int_0^1 s^{\beta-2}h(s)\,ds-ad_0+bd_m=0. \tag{4.21}$$

Applying the impulsive condition (4.2)-(4.3) we get

$$c_k-c_{k-1}=I_k(u(t_k))-(1-t_k)J_k(u(t_k)),\ \ k=1,\dots,m \tag{4.22}$$

$$d_{k-1}-d_k=J_k(u(t_k)),\ \ k=1,\dots,m, \tag{4.23}$$

Moreover using (4.21) and (4.23), we obtain

$$d_0=\frac{a}{(b-a)\Gamma(\beta-1)}\int_0^1 s^{\beta-2}h(s)\,ds+\frac{b}{(b-a)}\sum_{i=1}^m J_i(u(t_i)), \tag{4.24}$$

$$d_m=\frac{a}{(b-a)\Gamma(\beta-1)}\int_0^1 s^{\beta-2}h(s)\,ds+\frac{b}{(b-a)}\sum_{i=1}^m J_i(u(t_i)), \tag{4.25}$$

$$\begin{aligned} d_k &= d_0-\sum_{i=1}^k J_i(u(t_i)) \qquad (4.26)\\ &= \frac{a}{(b-a)\Gamma(\beta-1)}\int_0^1 s^{\beta-2}h(s)\,ds+\frac{b}{(b-a)}\sum_{i=1}^m J_i(u(t_i))-\sum_{i=1}^k J_i(u(t_i)), \end{aligned}$$

taking into account (4.20), (4.22) and (4.24), we get

$$\begin{aligned} c_0 &= \frac{a}{(b-a)\Gamma(\beta)}\int_0^1 s^{\beta-1}h(s)\,ds+\frac{a^2}{(b-a)^2\Gamma(\beta-1)}\int_0^1 s^{\beta-2}h(s)\,ds \qquad (4.27)\\ &+\frac{ab}{(b-a)^2}\sum_{i=1}^m J_i(u(t_i))-\frac{b}{(b-a)}\sum_{i=1}^m I_i(u(t_i))+\frac{b}{(b-a)}\sum_{i=1}^m (1-t_i)J_i(u(t_i)), \end{aligned}$$

$$
\begin{aligned}
c_k &= c_0 + \sum_{i=1}^{k} \left(I_i\left(u\left(t_i\right)\right) - \left(1-t_i\right) J_i\left(u\left(t_i\right)\right)\right) \qquad (4.28)\\
&= \frac{a}{(b-a)\Gamma(\beta)} \int_0^1 s^{\beta-1} h(s)\, ds + \frac{a^2}{(b-a)^2 \Gamma(\beta-1)} \int_0^1 s^{\beta-2} h(s)\, ds \\
&+ \frac{ab}{(b-a)^2} \sum_{i=1}^{m} J_i\left(u\left(t_i\right)\right) - \frac{b}{(b-a)} \sum_{i=1}^{m} I_i\left(u\left(t_i\right)\right) \\
&+ \frac{b}{(b-a)} \sum_{i=1}^{m} \left(1-t_i\right) J_i\left(u\left(t_i\right)\right) \\
&+ \sum_{i=1}^{k} I_i\left(u\left(t_i\right)\right) - \sum_{i=1}^{k} \left(1-t_i\right) J_i\left(u\left(t_i\right)\right)
\end{aligned}
$$

Substituting (4.24) and (4.27) in (4.18) for $t \in J_0 = [0, t_1]$, we obtain

$$
\begin{aligned}
x(t) &= \frac{1}{\Gamma(\beta)} \int_t^1 (s-t)^{\beta-1} h(s)\, ds + \frac{a}{(b-a)\Gamma(\beta)} \int_0^1 s^{\beta-1} h(s)\, ds \\
&+ \frac{a^2}{(b-a)^2 \Gamma(\beta-1)} \int_0^1 s^{\beta-2} h(s)\, ds \\
&+ \frac{a}{(b-a)\Gamma(\beta-1)} \int_0^1 (1-t)\, s^{\beta-2} h(s)\, ds \\
&+ \sum_{i=1}^{m} \frac{b}{(b-a)} \left(-I_i\left(u\left(t_i\right)\right)\right) \\
&+ \sum_{i=1}^{m} \left(\frac{ab}{(b-a)^2} + \frac{b}{(b-a)}\left(1-t_i\right) + \frac{b}{(b-a)}(1-t) \right) J_i\left(u\left(t_i\right)\right) \\
&= \int_t^1 G_1(t,s)\, h(s)\, ds + \sum_{i=1}^{m} G_2\left(t,t_i\right)\left(-I_i\left(u\left(t_i\right)\right)\right) + \sum_{i=1}^{m} G_3\left(t,t_i\right) J_i\left(u\left(t_i\right)\right)
\end{aligned}
$$

where the functions $G_1(t,s)$, $G_2(t,t_i)$ and $G_1(t,t_i)$, for $i = 1, \ldots, m$ are defined in (4.13), (4.15) and (4.17).

Similarly, substituting (4.26) and (4.28) in (4.18) for $t \in [t_k, t_{k-1}]$, $k = 1, \ldots, m$, it yields

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\beta)} \int_t^1 (s-t)^{\beta-1} h(s)\, ds + \frac{a}{(b-a)\Gamma(\beta)} \int_0^1 s^{\beta-1} h(s)\, ds \\
&+ \frac{a^2}{(b-a)^2 \Gamma(\beta-1)} \int_0^1 s^{\beta-2} h(s)\, ds + \frac{a}{(b-a)\Gamma(\beta-1)} \int_0^1 (1-t) s^{\beta-2} h(s)\, ds \\
&+ \sum_{i=1}^{k} \frac{a}{(b-a)} \left(-I_i(u(t_i))\right) + \sum_{i=k+1}^{m} \frac{b}{(b-a)} \left(-I_i(u(t_i))\right) \\
&+ \sum_{i=1}^{k} \left(\frac{ab}{(b-a)^2} + \frac{a}{(b-a)}(1-t_i) + \frac{a}{(b-a)}(1-t) \right) J_i(u(t_i)), \\
&+ \sum_{i=k+1}^{m} \left(\frac{ab}{(b-a)^2} + \frac{b}{(b-a)}(1-t_i) + \frac{b}{(b-a)}(1-t) \right) J_i(u(t_i)),
\end{aligned}$$

hence

$$x(t) = \int_t^1 G_1(t,s) h(s)\, ds + \sum_{i=1}^{m} G_2(t,t_i)\left(-I_i(u(t_i))\right) + \sum_{i=1}^{m} G_3(t,t_i) J_i(u(t_i)).$$

Now, since

$$x(t) = \phi_p\left({}^C D_{0^+}^{\alpha} u(t)\right), \ t \in (t_k, \ t_{k-1}], \ k = 0, \ldots, m,$$

then, the problem (4.10), (4.2)-(4.9) can be rewritten as

$$\begin{cases}
{}^C D_{0^+}^{\alpha} u(t) = \phi_q(x(t)), \ t \in J = [0,1], \ t \neq t_i, i = 1, \ldots, m, \\
\Delta u(t_i) = \tilde{I}_i(u(t_i)), \ i = 1, \ldots, m, \\
\Delta u'(t_i) = \tilde{J}_i(u(t_i)), \ i = 1, \ldots, m, \\
\gamma u(0) - \eta u(1) = 0, \ \gamma > \eta > 0, \\
\gamma u'(0) - \eta u'(1) = 0, \ \gamma > \eta > 0.
\end{cases}$$

Setting $\varphi(t) = \phi_q(x(t))$, then similarly to the above arguments, we conclude the required result. ■

The properties of the functions K_1, K_2, K_3, G_1, G_2 and G_3 are given in the next lemma.

Lemma 4.2.2 *The functions K_1, K_2, K_3, G_1, G_2 and G_3 are positive and satisfy the following properties for t, $t_i \in J$, $i = 1, \ldots, m$, $s \in (0,1)$:*

$$\frac{\eta}{\gamma} M_K(s) \leq K_1(t,s) \leq M_K(s),$$

$$\frac{a}{b} M_G(s) \leq G_1(t,s) \leq M_G(s),$$

$$\frac{\eta}{\gamma-\eta} \leq K_2(t,t_i) \leq \frac{\gamma}{\gamma-\eta}, \ ,$$

$$\frac{a}{b-a} \leq G_2(t,t_i) \leq \frac{b}{b-a},$$

$$\frac{\eta^2}{(\gamma-\eta)^2} \leq K_3(t,t_i) \leq \frac{\gamma^2}{(\gamma-\eta)^2},$$

$$\frac{a^2}{(b-a)^2} \leq G_3(t,t_i) \leq \frac{2b^2}{(b-a)^2},$$

where

$$M_K(s) = \frac{\gamma(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta\gamma(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)},$$
$$M_G(s) = \frac{bs^{\beta-1}}{(b-a)\Gamma(\beta)} + \frac{abs^{\beta-2}}{(b-a)^2\Gamma(\beta-1)}.$$

Proof The positivity of the functions K_1, K_2, K_3, G_1, G_2 and G_3 is direct from their expressions. Moreover K_1 and G_1 are increasing with respect to $t \in J$ for all $s \in (0,1)$. To simplify we use the following notations

$$K_1^{(1)}(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta^2(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)} + \frac{t\eta(1-s)^{\alpha-2}}{(\gamma-\eta)\Gamma(\alpha-1)}, \quad 0 < s \leq t < 1$$

$$K_1^{(2)}(t,s) = \frac{\eta(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta^2(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)} + \frac{t\eta(1-s)^{\alpha-2}}{(\gamma-\eta)\Gamma(\alpha-1)}, \quad 0 < t \leq s < 1.$$

$$G_1^{(1)}(t,s) = \tfrac{(s-t)^{\beta-1}}{\Gamma(\beta)} + \tfrac{as^{\beta-1}}{(b-a)\Gamma(\beta)} + \tfrac{a^2s^{\beta-2}}{(b-a)^2\Gamma(\beta-1)} + \tfrac{(1-t)as^{\beta-2}}{(b-a)\Gamma(\beta-1)},\quad 0<t\le s<1,$$

$$G_1^{(2)}(t,s) = \tfrac{as^{\beta-1}}{(b-a)\Gamma(\beta)} + \tfrac{a^2s^{\beta-2}}{(b-a)^2\Gamma(\beta-1)} + \tfrac{(1-t)as^{\beta-2}}{(b-a)\Gamma(\beta-1)},\quad 0<s\le t<1.$$

By computation we get

$$\begin{aligned}\max_{t\in J}K_1(t,s) &= \max\left(\max_{t\in(1,s)}K_1^{(1)}(t,s),\max_{t\in(0,s)}K_1^{(2)}(t,s)\right)\\ &= \max\left(K_1^{(1)}(1,s),K_1^{(2)}(s,s)\right)\\ &= K_1^{(1)}(1,s) = \frac{\gamma(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta\gamma(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)}\\ &= M_K(s),\end{aligned}$$

$$\begin{aligned}\min_{t\in J}K_1(t,s) &= \min\left(\min_{t\in(1,s)}K_1^{(1)}(t,s),\min_{t\in(0,s)}K_1^{(2)}(t,s)\right)\\ &= \min\left(K_1^{(1)}(s,s),K_1^{(2)}(0,s)\right)\\ &= K_1^{(2)}(0,s) = \frac{\eta(1-s)^{\alpha-1}}{(\gamma-\eta)\Gamma(\alpha)} + \frac{\eta^2(1-s)^{\alpha-2}}{(\gamma-\eta)^2\Gamma(\alpha-1)}\\ &= \frac{\eta}{\gamma}M_K(s),\end{aligned}$$

$$\begin{aligned}\max_{t\in J}G_1(t,s) &= \max\left(\max_{t\in(0,s)}G_1^{(1)}(t,s),\max_{t\in(s1,)}G_1^{(2)}(t,s)\right)\\ &= \max\left(G_1^{(1)}(0,s),G_1^{(2)}(s,s)\right)\\ &= G_1^{(1)}(0,s) = \frac{bs^{\beta-1}}{(b-a)\Gamma(\beta)} + \frac{abs^{\beta-2}}{(b-a)^2\Gamma(\beta-1)}\\ &= M_G(s),\end{aligned}$$

$$\begin{aligned}
\min_{t\in J} G_1(t,s) &= \min\left(\min_{t\in(0,s)} G_1^{(1)}(t,s), \min_{t\in(s,1)} G_1^{(2)}(t,s)\right) \\
&= \min\left(G_1^{(1)}(s,s), G_1^{(2)}(1,s)\right) \\
&= G_1^{(2)}(1,s) = \frac{as^{\beta-1}}{(b-a)\Gamma(\beta)} + \frac{a^2 s^{\beta-2}}{(b-a)^2\Gamma(\beta-1)} \\
&= \frac{a}{b} M_G(s).
\end{aligned}$$

Further, we easily obtain for $t,\ t_i \in J$ and $i = 1,\ldots,m$

$$\begin{aligned}
\frac{\eta}{\gamma-\eta} \leq K_2(t,t_i) \leq \frac{\gamma}{\gamma-\eta}, \\
\frac{a}{b-a} \leq G_2(t,t_i) \leq \frac{b}{b-a},
\end{aligned}$$

$$\begin{aligned}
\frac{\eta^2}{(\gamma-\eta)^2} \leq K_3(t,t_i) \leq \frac{\gamma^2}{(\gamma-\eta)^2}, \\
\frac{a^2}{(b-a)^2} \leq G_3(t,t_i) \leq \frac{2b^2}{(b-a)^2}.
\end{aligned}$$

■

4.3 Uniqueness, existence and existence of positive solutions

Let us define the functional space needed in this study. Denote by E the Banach space

$$\begin{aligned}
E &= PC\left(J,\mathbb{R}^+\right) \\
&= \left\{u : J \to \mathbb{R}^+,\ u \in C\left((t_k,t_{k+1}],\ \mathbb{R}^+\right),\ \text{for } k = 0,\cdots,p \right. \\
&\quad \left. \text{and there exist } u\left(t_k^+\right) \text{ and } u\left(t_k^-\right) \text{ with } u\left(t_k^-\right) = u(t_k)\right\},
\end{aligned}$$

according to the norm

$$\|u\| = \max_{t\in J} |u(t)|.$$

Define the positive cone

$$\begin{aligned} \Omega_r &= \{u \in E, \|u\| \le r\}, r > 0 \\ C &= \{u \in E,\ u(t) \ge \delta \|u\|,\ t \in J\}, \\ 0 &< \delta = \frac{\eta^2}{\gamma^2}\left(\frac{a^2}{2b^2}\right)^{q-1} < 1. \end{aligned}$$

We will use the notations

$$E_1 = \int_0^1 K_1^{(1)}(1,s)\,ds = \frac{\gamma^2 + (\alpha-1)\gamma\eta}{(\gamma-\eta)^2\,\Gamma(\alpha+1)},$$

$$E_2 = \int_0^1 G_1^{(1)}(0,s)\,ds = \frac{b^2 + (\beta-1)ab}{(b-a)^2\,\Gamma(\beta+1)}.$$

Define the operator $A : E \to E$ by

$$\begin{aligned} Au(t) = &\int_0^1 K_1(t,s)\left(\int_0^1 G_1(s,\tau) f(\tau,u(\tau))\,d\tau \right. \\ &\left. + \sum_{i=1}^m G_2(s,t_i)(-I_i(u(t_i))) + \sum_{i=1}^m G_3(s,t_i) J_i(u(t_i))\right)^{q-1} ds \\ &+ \sum_{i=1}^m K_2(t,t_i)\,\tilde{I}_i(u(t_i)) + \sum_{i=1}^m K_3(t,t_i)\,\widetilde{J}_i(u(t_i)) \end{aligned} \tag{4.29}$$

To prove the existence of solutions for problem (4.1)-(4.9) it suffices to prove that the operator A has a fixed point i.e. $Au = u$, hence the fixed point of the operator A coincides with the solution of problem (4.1)-(4.9).

4.3.1 Uniqueness of a solution

Let us introduce the following hypotheses.

$H1)$ $f \in C\left(J \times \mathbb{R}^{+}, \mathbb{R}^{+}\right).$
$H2)$ $I_i \in C\left(\mathbb{R}^{+}, \mathbb{R}^{-}\right),\ J_i,\ \tilde{I}_i,\ \tilde{J}_i \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right).$
$H3)$ There exist a constant $k_1 > 0$ such that

$$\left|f(t,x) - f(t,y)\right| \leq k_1 \left|x - y\right|, \text{ for each } t \in [0,1] \text{ and all } x, y \in \mathbb{R}^{+}.$$

$H4)$ There exist a constants $k_2, c_2 > 0$ such that

$$\left|I_k(x) - I_k(y)\right| \leq k_2 \left|x - y\right|, \text{ for all } x, y \in \mathbb{R}^{+}, k = 1, 2, ..., m.$$

$H5)$ There exists a constant $k_3 > 0$ such that

$$\left|J_k(x) - J_k(y)\right| \leq k_3 \left|x - y\right|, \text{ for all } x, y \in \mathbb{R}^{+}, k = 1, 2, ..., m.$$

$H6)$ There exists a constant $k_4 > 0$ such that

$$\left|\tilde{I}_k(x) - \tilde{I}_k(y)\right| \leq k_4 \left|x - y\right|, \text{ for all } x, y \in \mathbb{R}^{+}, k = 1, 2, ..., m.$$

$H7)$ There exists a constant $k_5 > 0$ such that

$$\left|\tilde{J}_k(x) - \tilde{J}_k(y)\right| \leq k_5 \left|x - y\right|, \text{ for all } x, y \in \mathbb{R}^{+}, k = 1, 2, ..., m.$$

In the next theorem we give an uniqueness result in the case $1 < p < 2$.

Theorem 4.3.1 *Suppose that $1 < p < 2$, the hypotheses $(H1)-(H7)$ are satisfied, and further suppose that the following hypotheses hold:*
$H8)$ *There exists a nonnegative function $w \in C[0,1]$, such that*

$$\left|f(t,x)\right| \leq w(t), \textit{ for all } t \in [0,1] \textit{ and } x \in \mathbb{R}^{+}.$$

$H9)$ *There exists a constants $c_2 > 0$ such that*

$$\sum_{k=1}^{m} \left(-I_k\left(u\left(t_k\right)\right)\right) \leq c_2.$$

$H10$) There exists a constant $c_3 > 0$ such that

$$\sum_{k=1}^{m} J_k\left(u\left(t_k\right)\right) \leq c_3.$$

$H11$) If

$$r = (q-1)\, M^{q-2} E_1 \left(E_2 k_1 + \frac{bm}{b-a} k_2 + \frac{2b^2 m}{(b-a)^2} k_3 \right) + \left(\frac{\eta m}{\gamma - \eta} k_4 + \frac{\eta^2 m}{(\gamma-\eta)^2} k_5 \right) < 1,$$

where

$$M = c_1 E_2 + \frac{bc_2}{b-a} + \frac{2b^2 c_3}{(b-a)^2}$$

and

$$c_1 = \max_{t \in [0,1]} w\left(t\right)$$

Then the problem (4.1)-(4.9) has a unique solution in E.

Proof We will show that A is a contraction mapping. Taking hypotheses $(H8)-(H10)$ and the fact that $1 < p < 2$, then $q > 2$, into account, it yields

$$\begin{aligned} &\int_0^1 G_1\left(s,\tau\right) f\left(\tau, u\left(\tau\right)\right) d\tau + \sum_{i=1}^{m} G_2\left(s,t_i\right)\left(-I_i\left(u\left(t_i\right)\right)\right) + \sum_{i=1}^{m} G_3\left(s,t_i\right) J_i\left(u\left(t_i\right)\right) \\ &\leq \int_0^1 G_1\left(s,\tau\right) w\left(\tau\right) d\tau + \frac{b}{b-a}\sum_{i=1}^{m}\left(-I_i\left(u\left(t_i\right)\right)\right) + \frac{2b^2}{(b-a)^2}\sum_{i=1}^{m} J_i\left(u\left(t_i\right)\right) \\ &\leq c_1 E_1 + \frac{bc_2}{b-a} + \frac{2b^2 c_3}{(b-a)^2} = M \end{aligned}$$

Applying the first statement of Lemma 1.1.4 and Lemma 4.2.2, we get for all

$u, v \in E$ and $t \in [t_k, t_{k+1}]$, $k = 0, 1, ..., m$,

$$\left| \left(\int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau + \sum_{i=1}^m G_2(s,t_i)(-I_i(u(t_i))) + \sum_{i=1}^m G_3(s,t_i) J_i(u(t_i)) \right)^{q-1} \right.$$
$$\left. - \left(\int_0^1 G_1(s,\tau) f(\tau, v(\tau)) d\tau + \sum_{i=1}^m G_2(s,t_i)(-I_i(v(t_i))) + \sum_{i=1}^m G_3(s,t_i) J_i(v(t_i)) \right)^{q-1} \right|$$
$$+ \sum_{i=1}^m K_2(t,t_i) \left| \tilde{I}_i(u(t_i)) - \tilde{I}_i(v(t_i)) \right| + \sum_{i=1}^m K_3(t,t_i) \left| \tilde{J}_i(u(t_i)) - \tilde{J}_i(v(t_i)) \right|$$
$$\leq (q-1) M^{q-2} \left(\int_0^1 G_1(s,\tau) |f(\tau,u(\tau)) - f(\tau,v(\tau))| d\tau \right.$$
$$\left. + \sum_{i=1}^m G_2(s,t_i) |I_i(u(t_i)) - I_i(v(t_i))| + \sum_{i=1}^m G_3(s,t_i) |J_i(u(t_i)) - J_i(v(t_i))| \right)$$
$$+ \sum_{i=1}^m K_2(t,t_i) \left| \tilde{I}_i(u(t_i)) - \tilde{I}_i(v(t_i)) \right| + \sum_{i=1}^m K_3(t,t_i) \left| \tilde{J}_i(u(t_i)) - \tilde{J}_i(v(t_i)) \right|$$

Thanks to hypotheses $(H3)$-$(H7)$ we get

$$\begin{aligned} |Au(t) - Av(t)| &\leq \left[(q-1) M^{q-2} E_1 \left(E_2 k_1 + \frac{bm}{b-a} k_2 + \frac{2b^2 m}{(b-a)^2} k_3 \right) \right. \\ &\quad \left. + \left(\frac{\eta m}{\gamma - \eta} k_4 + \frac{\eta^2 m}{(\gamma-\eta)^2} k_5 \right) \right] \|u - v\| \\ &= r \|u - v\|. \end{aligned}$$

We have from condition $(H11)$ that $r < 1$, then by Banach's fixed point theorem, we conclude the existence of a unique solution in E for the problem (4.1)-(4.9). ■

In the following theorem we shall give an uniqueness result in the case $p \geq 2$.

Theorem 4.3.2 *Assume that $p \geq 2$, the assumptions $(H1) - (H7)$ are satisfied and further suppose that the following assumptions hold:*

H12) There exists a constant $\lambda > 0$ *such that*

$$|f(t,x)| \geq \lambda, \text{ for all } t \in [0,1] \text{ and } x \in \mathbb{R}^+$$

H13)

$$r_1 = (q-1) M_1^{q-2} E_1 \left(E_2 k_1 + \frac{bm}{b-a} k_2 + \frac{2b^2 m}{(b-a)^2} k_3 \right) + \left(\frac{\eta m}{\gamma - \eta} k_4 + \frac{\eta^2 m}{(\gamma-\eta)^2} k_5 \right) < 1,$$

where $M_1 = \frac{a}{b} \lambda E_2$.

Then the problem (4.1)-(4.9) has a unique solution in E.

Proof Let us show that A is contraction mapping. According to hypothesis $(H12)$ and the fact that $p \geq 2$ then $1 < q \leq 2$, we obtain

$$\begin{aligned} & \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau + \sum_{i=1}^m G_2(s,t_i)(-I_i(u(t_i))) + \sum_{i=1}^m G_3(s,t_i) J_i(u(t_i)) \\ & \geq \int_0^1 G_1(s,\tau) \lambda d\tau \geq \frac{a}{b} \lambda E_2 = M_1 > 0 \end{aligned}$$

By hypotheses $(H3)$-$(H7)$, $(H13)$ and from Lemmas 1.1.5 and 4.2.2, we obtain for all $u, v \in E$, and $t \in [t_k, t_{k+1}]$, $k = 0, 1, ..., m$,

$$\begin{aligned} |Au(t) - Av(t)| & \leq \Bigg[(q-1) M_1^{q-2} E_1 \left(E_2 k_1 + \frac{bm}{b-a} k_2 + \frac{2b^2 m}{(b-a)^2} k_3 \right) \\ & \quad + \left(\frac{\eta m}{\gamma-\eta} k_4 + \frac{\eta^2 m}{(\gamma-\eta)^2} k_5 \right) \Bigg] \|u - v\| \\ & = r_1 \|u - v\| \end{aligned}$$

Finally Banach's fixed point theorem guarantees the existence of a unique solution in E for the problem (4.1)-(4.9). ■

4.3.2 Existence of solutions

Lemma 4.3.1 *Assume that hypotheses $(H1)$ and $(H2)$ hold, then the operator$A$ is completely continuous and $A(C) \subset C$.*

Proof The proof will be done in some steps.
Claim 1: The operator A maps bounded sets into bounded sets. Indeed let B be an arbitrary bounded set in E. Then there exists $M > 0$ such that $\|u\| < M$, for all $u \in B$. In view of hypotheses $(H1) - (H2)$, we set for $t \in (t_k, t_{k+1}]$, $i = 1, \dots, m$, $u \in B$:

$$\begin{aligned} L &= \max_{(t,x)\in J\times[0,M]} f(t,x) < \infty, \\ l_1 &= \max_{i=\overline{1,m}} \left(\max_{x\in[0,M]} \left(-I_i(x)\right)\right) < \infty, \\ l_2 &= \max_{i=\overline{1,m}} \left(\max_{x\in[0,M]} J_i(x)\right) < \infty, \\ \tilde{l}_1 &= \max_{i=\overline{1,m}} \left(\max_{x\in[0,M]} \tilde{I}_i(x)\right) < \infty, \\ \tilde{l}_2 &= \max_{i=\overline{1,m}} \left(\max_{x\in[0,M]} \tilde{J}_i(x)\right) < \infty. \end{aligned} \tag{4.30}$$

Obviously the constants L, l_i, $\tilde{l}_i$, for $i = 1, 2$ are positive. Let $t \in J_k$, $i = 1, \dots, m$, $u \in B$, then

$$\begin{aligned} 0 \le (Au)(t) &\le \left(\int_0^1 M_K(s)\, ds \right) \left(L \int_0^1 M_G(\tau)\, d\tau + \frac{bml_1}{b-a} + \frac{2b^2 ml_2}{(b-a)^2} \right)^{q-1} \\ &+ \frac{\eta m\, \tilde{l}_1}{\gamma - \eta} + \frac{\eta^2 m \tilde{l}_2}{(\gamma-\eta)^2} \\ &\le E_1 \left(LE_2 + \frac{bml_1}{b-a} + \frac{2b^2 ml_2}{(b-a)^2} \right)^{q-1} + \frac{\eta m \tilde{l}_1}{\gamma-\eta} + \frac{\eta^2 m \tilde{l}_2}{(\gamma-\eta)^2} = C_1 \end{aligned}$$

hence $\|Au\| \le C_1$.
Claim 2: A is equicontinuous on B. In fact, fix $s \in J$ then for $u \in B$ and any τ_1,

$\tau_2 \in (t_k, t_{k+1}]$, $\tau_1 < \tau_2$, it yields

$$\begin{aligned}|Au(\tau_2) - Au(\tau_1)| = \Bigg| &\int_0^1 (K_1(\tau_2, s) - K_1(\tau_1, s)) \left(\int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau \right. \\ &\left. + \sum_{i=1}^m G_2(s, t_i)(-I_i(u(t_i))) + \sum_{i=1}^m G_3(s, t_i) J_i(u(t_i)) \right)^{q-1} ds \\ &+ \sum_{i=1}^m (K_2(\tau_2, t_i) - K_2(\tau_1, t_i)) \tilde{I}_i(u(t_i)) \\ &+ \sum_{i=1}^m (K_3(\tau_2, t_i) - K_3(\tau_1, t_i)) \tilde{J}_i(u(t_i)) \Bigg|\end{aligned}$$

$$\begin{aligned}&\leq \left(LE_2 + \frac{bml_1}{b-a} + \frac{2b^2ml_2}{(b-a)^2} \right)^{q-1} \\ &\times \left(\int_0^{\tau_1} \left(K_1^{(1)}(\tau_2, s) - K_1^{(1)}(\tau_1, s) \right) ds + \int_{\tau_1}^{\tau_2} \left(K_1^{(1)}(\tau_2, s) - K_1^{(2)}(\tau_1, s) \right) ds \right. \\ &\left. + \int_{\tau_2}^1 \left(K_1^{(2)}(\tau_2, s) - K_1^{(2)}(\tau_1, s) \right) ds \right) + \tilde{l}_1 \sum_{i=1}^m |K_2(\tau_2, t_i) - K_2(\tau_1, t_i)| \\ &+ \tilde{l}_2 \sum_{i=1}^m |K_3(\tau_2, t_i) - K_3(\tau_1, t_i)|\end{aligned}$$

$$\begin{aligned}&\leq \left(LE_2 + \frac{bml_1}{b-a} + \frac{2b^2ml_2}{(b-a)^2} \right)^{q-1} \left(\frac{\eta}{(\gamma-\eta)} (\tau_2 - \tau_1) + \frac{\tau_2^\alpha - \tau_1^\alpha}{\Gamma(\alpha+1)} \right) \\ &+ \frac{\tilde{l}_2 m(\eta+\gamma)}{(\gamma-\eta)} (\tau_2 - \tau_1) \to 0 \text{ as } \tau_1 \to \tau_2.\end{aligned}$$

Thus (Au) is equicontinuous on all the subinterval $(t_k, t_{k+1}]$, $k = 0, \ldots, m$. By PC type Ascoli-Arzela Theorem 1.1.1, we conclude that A is compact.

Claim 3: The operator A is continuous. Let $(u_n)_n$ be a convergent sequence in E

i.e. $u_n \to u_0 \in E$, then

$$
\begin{aligned}
&0 \leq f\left(t, u_n(t)\right) \leq L,\ t \in J, \\
&0 \leq \left(-I_k\left(u_n\left(t_k\right)\right)\right) \leq l_1,\ t \in J_k,\ k = 1, \ldots, m, \\
&0 \leq J_k\left(u_n\left(t_k\right)\right) \leq l_2,\ t \in J_k,\ k = 1, \ldots, m,
\end{aligned}
$$

where L, l_i, for $i = 1, 2$ are defined in (4.30). By computation we get

$$
\begin{aligned}
&\int_0^1 G_1(s,\tau) f(\tau, u(\tau))\, d\tau + \sum_{i=1}^{m} G_2(s,t_i)\left(-I_i\left(u\left(t_i\right)\right)\right) \\
&+ \sum_{i=1}^{m} G_3(s,t_i) J_i\left(u\left(t_i\right)\right) \\
&\leq LE_2 + \frac{bml_1}{b-a} + \frac{2b^2 ml_2}{(b-a)^2} = c
\end{aligned}
$$

Now, since the functions f, I_k, J_k, $\tilde{I}_k$, $\tilde{J}_k$, $k = 1, \ldots, m$, are uniformly continuous for and $\|u_n - u_0\| \to 0$, then there exists $n_0 \geq 1$ such that for $n \geq n_0$, the following estimate hold

$$
\begin{aligned}
\left|f\left(t, u_n(t)\right) - f\left(t, u_0(t)\right)\right| &< \varepsilon, \\
\left|I_k\left(u_n\left(t_k\right)\right) - I_k\left(u_0\left(t_k\right)\right)\right| &< \varepsilon, \\
\left|\tilde{I}_k\left(u_n\left(t_k\right)\right) - \tilde{I}_k\left(u_0\left(t_k\right)\right)\right| &< \varepsilon, \\
\left|J_k\left(u_n\left(t_k\right)\right) - J_k\left(u_0\left(t_k\right)\right)\right| &< \varepsilon, \\
\left|\tilde{J}_k\left(u_n\left(t_k\right)\right) - \tilde{J}_k\left(u_0\left(t_k\right)\right)\right| &< \varepsilon.
\end{aligned}
$$

We have two cases to explore.

Case 1. If $1 < q \leq 2$, then from the second statement of Lemma 1.1.4, we have

$$\left| \left(\int_0^1 G_1(s,\tau) f(\tau, u_n(\tau))\, d\tau + \sum_{i=1}^{m} G_2(s,t_i)(-I_i(u_n(t_i))) + \sum_{i=1}^{m} G_3(s,t_i) J_i(u_n(t_i)) \right)^{q-1} - \left(\int_0^1 G_1(s,\tau) f(\tau, u_0(\tau))\, d\tau + \sum_{i=1}^{m} G_2(s,t_i)(-I_i(u_0(t_i))) + \sum_{i=1}^{m} G_3(s,t_i) J_i(u_0(t_i)) \right)^{q-1} \right|$$

$$\leq \left(\int_0^1 G_1(s,\tau) \left| f(\tau, u_n(\tau)) - f(\tau, u_0(\tau)) \right| d\tau + \sum_{i=1}^{m} G_2(s,t_i) \left| I_i(u_n(t_i)) - I_i(u_0(t_i)) \right| + \sum_{i=1}^{m} G_3(s,t_i) \left| J_i(u_n(t_i)) - J_i(u_0(t_i)) \right| \right)^{q-1}$$
$$\leq \varepsilon^{q-1} \left(E_2 + \frac{bm}{b-a} + \frac{2b^2 m}{(b-a)^2} \right)^{q-1},$$

that implies

$$\|Au_n - Au_0\| \leq \varepsilon^{q-1} E_1 \left(E_2 + \frac{bm}{b-a} + \frac{2b^2 m}{(b-a)^2} \right)^{q-1} + \left(\frac{\eta m}{\gamma - \eta} + \frac{\eta^2 m}{(\gamma - \eta)^2} \right) \varepsilon,$$

thus A is continuous on E.

Case 2. If $q > 2$, then the first statement of Lemma 1.1.4 implies

$$\left| \left(\int_0^1 G_1(s,\tau) f(\tau, u_n(\tau)) \, d\tau + \sum_{i=1}^m G_2(s,t_i)(-I_i(u_n(t_i))) + \sum_{i=1}^m G_3(s,t_i) J_i(u_n(t_i)) \right)^{q-1} \right.$$
$$\left. - \left(\int_0^1 G_1(s,\tau) f(\tau, u_0(\tau)) \, d\tau + \sum_{i=1}^m G_2(s,t_i)(-I_i(u_0(t_i))) + \sum_{i=1}^m G_3(s,t_i) J_i(u_0(t_i)) \right)^{q-1} \right|$$
$$\leq (q-1) c^{q-2} \left(\int_0^1 G_1(s,\tau) \left| f(\tau, u_n(\tau)) - f(\tau, u_0(\tau)) \right| d\tau \right.$$
$$\left. + \sum_{i=1}^m G_2(s,t_i) \left| I_i(u_n(t_i)) - I_i(u_0(t_i)) \right| + \sum_{i=1}^m G_3(s,t_i) \left| J_i(u_n(t_i)) - J_i(u_0(t_i)) \right| \right)$$
$$\leq (q-1) c^{q-2} \left(E_2 + \frac{bm}{b-a} + \frac{2b^2 m}{(b-a)^2} \right) \varepsilon,$$

consequently,

$$\|Au_n - Au_0\| \leq (q-1) c^{q-2} E_1 \left(E_2 + \frac{bm}{b-a} + \frac{2b^2 m}{(b-a)^2} \right) \varepsilon$$
$$+ \left(\frac{\eta m}{\gamma - \eta} + \frac{\eta^2 m}{(\gamma-\eta)^2} \right) \varepsilon,$$

thus A is continuous operator on E.

From the above analysis, we deduce that the operator A is completely continuous on E.

Finally, let us show that $A(C) \subset C$. In fact, in view of Lemmas 4.2.1 and 4.2.2 we obtain for $t \in (t_k, t_{k+1}]$ and $u \in C$, $k = 0, \ldots, m$,

$$(Au)(t) \geq \left(\frac{\eta}{\gamma} \int_0^1 M_K(s) \, ds \right) \left(\frac{a}{b} \int_0^1 M_G(\tau) f(\tau, u(\tau)) \, d\tau + \sum_{i=1}^m \frac{a}{b-a} (-I_i(u(t_i))) \right.$$
$$\left. + \sum_{i=1}^m \frac{a^2}{(b-a)^2} J_i(u(t_i)) \right)^{q-1} + \sum_{i=1}^m \frac{\eta}{\gamma-\eta} \tilde{I}_i(u(t_i)) + \sum_{i=1}^m \frac{\eta^2}{(\gamma-\eta)^2} \tilde{J}_i(u(t_i)).$$

Moreover, we have

$$(Au)(t) \leq \left(\int_0^1 M_K(s)\,ds\right)\left(\int_0^1 M_G(\tau) f(\tau, u(\tau))\,d\tau + \sum_{i=1}^m \frac{b}{b-a}(-I_i(u(t_i)))\right.$$
$$\left. + \sum_{i=1}^m \frac{2b^2}{(b-a)^2} J_i(u(t_i))\right)^{q-1} + \sum_{i=1}^m \frac{\gamma}{\gamma-\eta}\tilde{I}_i(u(t_i)) + \sum_{i=1}^m \frac{\gamma^2}{(\gamma-\eta)^2}\tilde{J}_i(u(t_i)),$$

then

$$\begin{aligned}
\delta\|Au\| &= \frac{\eta^2}{\gamma^2}\left(\frac{a^2}{2b^2}\right)^{q-1} \max_{t\in J_k} Au(t) \\
&\leq \left(\frac{\eta}{\gamma}\int_0^1 \frac{\eta}{\gamma} M_K(s)\,ds\right)\left(\frac{a}{2b}\int_0^1 \frac{a}{b} M_G(\tau) f(\tau, u(\tau))\,d\tau\right. \\
&\left. + \frac{a}{2b}\sum_{i=1}^m \frac{a}{b-a}(-I_i(u(t_i))) + \sum_{i=1}^m \frac{a^2}{(b-a)^2} J_i(u(t_i))\right)^{q-1} \\
&+ \frac{\eta}{\gamma}\left(\frac{a^2}{2b^2}\right)^{q-1} \sum_{i=1}^m \frac{\eta}{\gamma-\eta}\tilde{I}_i(u(t_i)) + \left(\frac{a^2}{2b^2}\right)^{q-1} \sum_{i=1}^m \frac{\eta^2}{(\gamma-\eta)^2}\tilde{J}_i(u(t_i)) \\
&\leq Au(t),
\end{aligned}$$

hence $(Au)(t) \geq \delta\|Au\|$, for $u \in C$, that is $A(C) \subset C$. ■

Now, we give an existence result for the problem (4.1)-(4.9), the proof is based on Schauder's fixed point theorem, for this end, we make the following hypotheses.

$(H14)$ There exist $p_1 \in C(J, \mathbb{R}^+)$ and two positive constants q_1 and δ_1, $0 < \delta_1 < \frac{1}{p-1}$ such that

$$|f(t,u)| \leq p_1(t) + q_1 u^{\delta_1}, \text{ for each } t \in J \text{ and all } u \in \mathbb{R}^+,$$

$(H15)$ There exist two positive constants $p_2, q_2 > 0$ such that

$$|I_k(u)| \leq p_2 + q_2 u^{\delta_1}, \text{ for all } u \in \mathbb{R}^+,$$

$(H16)$ There exist two positive constants $p_3, q_3 > 0$, such that

$$|J_k(u)| \leq p_3 + q_3 u^{\delta_1}, \text{ for all } u \in \mathbb{R}^+,$$

$(H17)$ There exist positive constants $p_4, q_4 > 0$, and $\delta_2 \in (0,1]$ such that

$$\left|\tilde{I}_k(u)\right| \leq p_4 + q_4 u^{\delta_2}, \text{ for all } u \in \mathbb{R}^+,$$

$(H18)$ There exist $p_2, q_2 > 0$, such that

$$\left|\tilde{J}_k(u)\right| \leq p_5 + q_5 u^{\delta_2}, \text{ for all } u \in \mathbb{R}^+,$$

Theorem 4.3.3 *Assume that $(H1)$-$(H2)$ and $(H14)$-$(H18)$ hold. Then the problem (4.1)-(4.9) has at least one solution in E.*

Proof We introduce the following notations

$$\begin{aligned}
A_1 &= 3\max\left\{2^{q-2}, 1\right\} E_1 \left(E_2 p_1^* + \frac{mb}{b-a} p_2 + \frac{2mb^2}{(b-a)^2} p_3\right)^{q-1} \\
&\quad + 3\left(\frac{m\gamma}{\gamma-\eta} p_4 + \frac{m\gamma^2}{(\gamma-\eta)^2} p_5\right) \\
A_2 &= 3\left(E_2 q_1 + \frac{mb}{b-a} q_2 + \frac{2mb^2}{(b-a)^2} q_3\right)^{q-1} \\
A_3 &= 3\left(\frac{m\gamma}{\gamma-\eta} q_4 + \frac{m\gamma^2}{(\gamma-\eta)^2} q_5\right) \\
P_1^* &= \max_{t\in J} P_1(t)
\end{aligned}$$

Let

$$r > \max\left\{A_1, (A_2)^{\frac{1}{1-\delta_1(p-1)}}, (A_3)^{\frac{1}{1-\delta_2}}\right\},$$

and set $\Omega_r = \{u \in E, \|u\| \leq r\}$. We shall show that $A(\Omega_r) \subset \Omega_r$. Taking into account Lemmas 1.1.3 and 4.2.2 and hypotheses $(H1)$-$(H2)$ and $(H14)$-$(H18)$, we get for all $u \in \Omega_r$, $t \in (t_k, t_{k+1}]$, $k = 1, ..., m$,

$$\left| \int_0^1 K_1(s,\tau) \left(\int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau + \sum_{i=1}^m G_2(s,t_i)(-I_i(u(t_i))) \right. \right. \tag{4.31}$$

$$\left. \left. + \sum_{i=1}^m G_3(s,t_i) J_i(u(t_i)) \right)^{q-1} ds \right|$$

$$\leq \int_0^1 K_1(s,\tau) \left(\int_0^1 G_1(s,\tau) |f(\tau, u(\tau))| d\tau + \sum_{i=1}^m G_2(s,t_i) |I_i(u(t_i))| \right.$$

$$\left. + \sum_{i=1}^m G_3(s,t_i) |J_i(u(t_i))| \right)^{q-1} ds$$

$$\leq \int_0^1 K_1(s,\tau) \left(\int_0^1 G_1(s,\tau) \left(p_1(t) + q_1 |u|^{\delta_1}\right) d\tau + \sum_{i=1}^m G_2(s,t_i) \left(p_2 + q_2 |u|^{\delta_1}\right) \right.$$

$$\left. + \sum_{i=1}^m G_3(s,t_i) \left(p_3 + q_3 |u|^{\delta_1}\right) \right)^{q-1} ds$$

$$\leq E_1 \left[E_2 \left(p_1^* + q_1 r^{\delta_1}\right) + \frac{mb}{b-a} \left(p_2 + q_2 r^{\delta 1}\right) + \frac{2mb^2}{(b-a)^2} \left(p_3 + q_3 r^{\delta_1}\right) \right]^{q-1}$$

$$\leq E_1 \left(\left(E_2 p_1^* + \frac{mb}{b-a} p_2 + \frac{2mb^2}{(b-a)^2} p_3 \right) \right.$$

$$\left. + \left(E_2 q_1 + \frac{mb}{b-a} q_2 + \frac{2mb^2}{(b-a)^2} q_3 \right) r^{\delta_1} \right)^{q-1}$$

$$\leq \max\left\{2^{q-2}, 1\right\} E_1 \left[\left(E_2 p_1^* + \frac{mb}{b-a} p_2 + \frac{2mb^2}{(b-a)^2} p_3 \right)^{q-1} \right.$$

$$\left. + \left(E_2 q_1 + \frac{mb}{b-a} q_2 + \frac{2mb^2}{(b-a)^2} q_3 \right)^{q-1} r^{\delta_1 (q-1)} \right].$$

From (4.31) and hypotheses $(H1)$-$(H2)$, $(H10)$-$(H11)$ and Lemma 4.2.2, it yields

$$\begin{aligned}|Au(t)| \leq\ & \max\left\{2^{q-2},1\right\} E_1\left[\left(E_2 p_1^* + \frac{mb}{b-a}p_2 + \frac{2mb^2}{(b-a)^2}p_3\right)^{q-1}\right. \\ & \left. + \left(E_2 q_1 + \frac{mb}{b-a}q_2 + \frac{2mb^2}{(b-a)^2}q_3\right)^{q-1} r^{\delta_1(q-1)}\right] \\ & + \left(\frac{m\gamma}{\gamma-\eta}p_4 + \frac{m\gamma^2}{(\gamma-\eta)^2}p_5\right) + \left(\frac{m\gamma}{\gamma-\eta}q_4 + \frac{m\gamma^2}{(\gamma-\eta)^2}q_5\right) r^{\delta_2} < r,\end{aligned}$$

so $A(\Omega_r) \subset \Omega_r$. Since the operators A is completely continuous by Lemma 4.3.1, then Schauder's fixed point theorem, guarantees that A has at least one fixed point which is a solution of the problem (4.1)-(4.9). ■

4.3.3 Existence of positive solutions

Let us introduce the following hypotheses.

$(H19)$ $\limsup_{x\to+\infty}\left(\max_{t\in J}\frac{f(t,x)}{x^{p-1}}\right) < l$;

$H20)$ $\liminf_{x\to 0^+}\left(\min_{t\in J}\frac{f(t,x)}{x^{p-1}}\right) > \lambda$.

where

$$\begin{aligned} l &= \frac{\left(\max\left\{2^{\gamma-1},1\right\}E_1\right)^{\frac{1}{1-q}}}{E_2}, \\ \lambda &= \frac{b\gamma^{p-1}E_1^{1-p}}{a(\delta\eta)^{p-1}2^p E_2}.\end{aligned}$$

Set

$$N_1 = \max\left\{\frac{b}{(b-a)}\sum_{i=1}^{m}\left(-I_i(u(t_i))\right), \frac{2b^2}{(b-a)^2}\sum_{i=1}^{m}J_i(u(t_i))\right\},$$

$$N_2 = \max\left\{\frac{\gamma}{(\gamma-\eta)}\sum_{i=1}^{m}\tilde{I}_i(u(t_i)), \frac{\gamma^2}{(\gamma-\eta)^2}\sum_{i=1}^{m}\tilde{J}_i(u(t_i))\right\}.$$

Now, we are ready to give an existence result for the positive solutions. The proof is based on Lemma 1.2.1.

Theorem 4.3.4 *Assume that* $(H1)-(H2)$ *and* $(H19)-(H20)$ *hold. Then problem* (4.1)-(4.9) *has at least one positive solution.*

Proof Let $A : C \to C$, then from Lemma 4.3.1 we have A is completely continuous. Firstly, taking hypothesis $(H18)$ into account, we can choose $\varepsilon_0 \in (0, l)$, such that

$$\lim_{x\to+\infty} \sup \left(\max_{t\in J} \frac{f(t,x)}{x^{p-1}} \right) < l - \varepsilon_0,$$

thus, there exists $R_0 > 0$, such that the inequality

$$f(t,x) < (l - \varepsilon_0) x^{p-1}, \ t \in J, x \geq R_0$$

is satisfied. Let $M = \max_{(t,x)\in J\times[0,R_0]} f(t,x)$, then

$$f(t,x) \leq (l - \varepsilon_0) x^{p-1} + M, \ \forall x \in \mathbb{R}^+, \ t \in J. \tag{4.32}$$

Since $(l - \varepsilon_0)^{q-1} < l^{q-1}$, we can choose $k > 0$, such that $(l - \varepsilon_0)^{q-1} < l^{q-1} - k$. Set

$$D_1 = \max\left\{2^{q-2}, \ 1\right\} E_1 E_2{}^{q-1}, \ F = kD_1$$
$$G = \max\left\{2^{q-2}, \ 1\right\} E_1 (E_2 M + 2N_1)^{q-1} + 2N_2.$$

Let $R > \frac{G}{F}$ set $H_R = \{u \in C, \ \|u\| < R\}$. We shall prove that the relation

$$Au \neq \mu u, \ \forall u \in \partial H_R, \ \mu \geq 1, \tag{4.33}$$

holds. Indeed, Reasoning by contradiction, then there exists $u_0 \in \partial H_R$, and $\mu_0 \geq 1$, with $Au_0 = \mu_0 u_0$. from (4.32), it yields

$$\begin{aligned} f(t, u_0(t)) &\leq (l - \varepsilon_0) u_0^{p-1} + M \\ &\leq (l - \varepsilon_0) R^{p-1} + M, \end{aligned}$$

thus

$$0 \leq (Au_0)(t) \leq \left(\int_0^1 M_K(s)\, ds \right)$$
$$\times \left(\int_0^1 M_G(\tau) \left((l-\varepsilon_0) R^{p-1} + M \right) d\tau + 2N_1 \right)^{q-1} + 2N_2$$
$$\leq E_1 \left(E_2 \left((l-\varepsilon_0) R^{p-1} + M \right) + 2N_1 \right)^{q-1} + 2N_2,$$

remembering that $(p-1)(q-1) = 1$, then we get by Lemma 1.1.3,

$$\left(E_2 \left((l-\varepsilon_0) R^{p-1} + M \right) + 2N_1 \right)^{q-1} \leq$$

$$\max\left\{2^{q-2},\ 1\right\} \left(E_2^{q-1} (l-\varepsilon_0)^{q-1} R + (E_2 M + 2N_1)^{q-1} \right)$$
$$\leq \max\left\{2^{q-2}, 1\right\} \left(E_2^{q-1} \left(l^{q-1} - k \right) R + (E_2 M + 2N_1)^{q-1} \right),$$

hence

$$(Au_0)(t) \leq \max\left\{2^{q-2}, 1\right\}$$
$$\times \left(E_1 E_2^{q-1} \left(l^{q-1} - k \right) R + E_1 (E_2 M + 2N_1)^{q-1} \right) + 2N_2$$
$$\leq \max\left\{2^{q-2}, 1\right\} E_1 E_2^{q-1} \left(l^{q-1} - k \right) R$$
$$+ \max\left\{2^{q-2}, 1\right\} E_1 (E_2 M + 2N_1)^{q-1} + 2N_2$$
$$\leq \left(D_1 l^{q-1} - F \right) R + G,$$

thus

$$u_0(t) \leq \mu_0 u_0(t) = (Au_0)(t) \leq \left(D_1 l^{q-1} - F \right) R + G, t \in J. \tag{4.34}$$

Since $D_1 l^{q-1} = 1$, then (4.34) implies

$$R = \|u_0\| \leq (1-F) R + G$$

and so $R \leq \frac{G}{F}$, that contradicts the choice of R and consequently condition (4.33) is satisfied.

Now, in view of hypothesis $(H19)$, we can choose $\varepsilon_0 > 0$, such that

$$\lim_{x\to 0^+}\inf\left(\min_{t\in J}\frac{f(t,x)}{x^{p-1}}\right) > \lambda+\varepsilon_0,$$

then there exists $r_1 > 0$, such that

$$f(t,x) > (\lambda+\varepsilon_0)\,x^{p-1},\ t\in J,\ x\in[0,r_1].$$

Choose r such $0 < r < \min\left(R, r_1,\ 2^p\left(\left(E_1\frac{\eta}{\gamma}\right)^{p-1}\frac{a^2}{b^2}N_1\right)^{q-1}, \frac{4\eta^2}{\gamma^2}N_2\right)$, and set

$$H_r = \{u\in P,\ \|u\| < r\}$$

Now, we will show that
i) $\underset{u\in\partial H_r}{Inf}\ \|Au\| > 0.$
ii) $Au \neq \mu u$, $\forall u\in\partial H_r$, $\mu\in(0,1]$.
Let $u\in\partial H_r$, we have $\delta\|u\| \leq u(t) \leq \|u\|$, where $\delta = \frac{\eta^2}{\gamma^2}\left(\frac{a^2}{2b^2}\right)^{q-1}$, then

$$\begin{aligned} f(t,u(t)) &\geq (\lambda+\varepsilon_0)\,u(t)^{p-1} \\ &\geq (\lambda+\varepsilon_0)\,(\delta\|u\|)^{p-1} \\ &\geq (\lambda+\varepsilon_0)\,(\delta r)^{p-1},\ t\in J \end{aligned}$$

hence

$$\begin{aligned} (Au)(t) &\geq \int_0^1 K_1(t,s)\left(\int_0^1 G_1(s,\tau)(\lambda+\varepsilon_0)(\delta r)^{p-1}d\tau + \frac{a^2}{b^2}N_1\right)^{q-1} ds + \frac{2\eta^2}{\gamma^2}N_2 \\ &\geq \frac{\eta}{\gamma}E_1\left(\frac{a}{b}E_2(\lambda+\varepsilon_0)(\delta r)^{p-1} + \frac{a^2}{b^2}N_1\right)^{q-1} + \frac{2\eta^2}{\gamma^2}N_2, \end{aligned}$$

thus

$$\|Au\| \geq \left(\left(\frac{\eta}{\gamma}\delta E_1\right)^{p-1}\frac{a}{b}E_2(\lambda+\varepsilon_0)\,r^{p-1} + \left(\frac{\eta}{\gamma}E_1\right)^{p-1}\frac{a^2}{b^2}N_1\right)^{q-1} + \frac{2\eta^2}{\gamma^2}N_2 = C > 0, \tag{4.35}$$

consequently, the first statement is satisfied. Now suppose that there exists a $u_0 \in \partial H_r$, and $\mu_0 \in (0,1]$, such that $\mu_0 u_0 = Au_0$. By similar argument used to get (4.35) and using the definition of λ, we get

$$\|Au_0\| \geq \left(\left(\left(\frac{\eta}{\gamma}\delta E_1\right)^{p-1}\frac{a}{b}E_2\left(\lambda+\varepsilon_0\right)r^{p-1}+\frac{1}{2^p}2^p\left(\left(\frac{\eta}{\gamma}E_1\right)^{p-1}\frac{a^2}{b^2}N_1\right)^{(q-1)(p-1)}\right)^{q-1} + \frac{2\eta^2}{\gamma^2}N_2 > \left(\frac{r^{p-1}}{2^p}+\frac{r^{p-1}}{2^p}\right)^{q-1}+\frac{r}{2}=r.$$

Since $\mu_0 \in (0,1]$, then

$$r = \|u_0\| \geq \mu_0 \|u_0\| = \|Au_0\| > r,$$

which is impossible, consequently the statement (ii) holds. Then from Lemma 1.2.1, the operator A has a fixed point $u \in C \cap \left(\overline{H_R}\diagdown H_r\right)$. Therefore, u is a positive solution to the problem (4.1)-(4.9). ■

4.4 Examples

4.4.1 Example 1

Consider the fractional boundary value problem (4.1)-(4.9) with $\gamma = b = 2$, $\eta = a = 1$, $\alpha = \beta = \frac{3}{2}$, $p = \frac{3}{2}$, $q = 3$,

$$f(t,x) = \frac{t^2}{100} + \frac{e^{-t}x}{(99+e^t)(x+1)}, t \in J = [0,1], \ t \neq \frac{1}{2}, \ x \in \mathbb{R}^+,$$

$$\begin{aligned} I_1(x) &= -\frac{1}{16}e^{-\frac{1}{2}x}, \ x \in \mathbb{R}^+, \\ J_1(x) &= \frac{1}{64}e^{-\frac{1}{2}x}, \ x \in \mathbb{R}^+, \\ \tilde{I}_1(x) &= \frac{1}{50}\frac{x}{1+x}, \ x \in \mathbb{R}^+, \\ \tilde{J}_1(x) &= \frac{1}{50}\frac{x}{1+x}, \ x \in \mathbb{R}^+. \end{aligned}$$

Then all assumptions of Theorem 4.3.1 are satisfied. In fact, we have

$$\begin{aligned}
|f(t,x) - f(t,y)| &\leq \frac{e^{-t}}{(99+e^{t})}\left|\frac{x}{(x+1)} - \frac{y}{y+1}\right| \\
&\leq \frac{1}{100}\left|\frac{x-y}{(x+1)(y+1)}\right| \\
&\leq \frac{1}{100}|x-y| = k_1|x-y|, t \in [0,1], \ x, y \in \mathbb{R}^{+}
\end{aligned}$$

Further

$$\begin{aligned}
f(t,x) &\leq \frac{t^2}{100} + \frac{e^{-t}}{(99+e^{t})} \\
&\leq \frac{t^2}{100} + \frac{1}{100}, t \in [0,1], \ x \in \mathbb{R}^{+},
\end{aligned}$$

so there exists a nonnegative function $w \in C[0,1]$, such that

$$w(t) = \frac{t^2}{100} + \frac{1}{100}, t \in [0,1],$$

$$c_1 = \max_{t\in[0,1]} w(t) = \frac{1}{50}.$$

By computation, we get

$$\begin{aligned}
|I_1(x) - I_1(y)| &\leq \frac{1}{16}\left|e^{-\frac{1}{2}x} - e^{-\frac{1}{2}y}\right| \\
&\leq \frac{1}{16}|x-y| = k_2|x-y|, x \in \mathbb{R}^{+}, \\
-I_1(x) &\leq \frac{1}{16} = c_2
\end{aligned}$$

$$\begin{aligned}
|J_1(x) - J_1(y)| &\leq \frac{1}{64}|x-y| = k_3|x-y|, x \in \mathbb{R}^{+} \\
J_1(x) &\leq \frac{1}{64} = c_3
\end{aligned}$$

$$\begin{aligned}\left|\tilde{I}_1(x)-\tilde{I}_1(y)\right| &\leq \frac{1}{50}\left|\frac{x}{1+x}-\frac{y}{1+y}\right| \\ &\leq \frac{1}{50}|x-y| = k_4|x-y|, x\in\mathbb{R}^+\end{aligned}$$

$$\left|\tilde{J}_1(x)-\tilde{J}_1(y)\right| \leq \frac{1}{50}|x-y| = k_5|x-y|, x\in\mathbb{R}^+.$$

Then

$$E_1 = \frac{\gamma^2+(\alpha-1)\gamma\eta}{(\gamma-\eta)^2\Gamma(\alpha+1)} = 3.7613,$$

$$E_2 = \frac{b^2+(\beta-1)ba}{(b-a)^2\Gamma(\beta+1)} = 3.7613,$$

$$M = c_1E_2+\frac{bc_2}{b-a}+\frac{2b^2c_3}{(b-a)^2} = 0.325\,23$$

$$\begin{aligned} r &= (q-1)M^{q-2}E_1\left(E_2k_1+\frac{bm}{b-a}k_2+\frac{2b^2m}{(b-a)^2}k_3\right)+\left(\frac{\eta m}{\gamma-\eta}k_4+\frac{\eta^2m}{(\gamma-\eta)^2}k_5\right) \\ &= 0.743\,67<1\end{aligned}$$

We conclude by Theorem 4.3.1, that the problem (4.1)-(4.9) has a unique solution in Ω_r.

4.4.2 Example 2

Consider the fractional boundary value problem (4.1)-(4.9) with $\gamma=b=2$, $\eta=a=1$, $\alpha=\beta=\frac{3}{2}$, $p=3$, $q=\frac{3}{2}$,

$$f(t,x) = \frac{80}{(1+t)}+\frac{x}{(1+x)},\ t\in J=[0,1],\ t\neq\frac{1}{2},\ x\in\mathbb{R}^+,$$

$$\begin{aligned} I_1(x) &= -\frac{x}{100},\ x \in \mathbb{R}^+, \\ J_1(x) &= \frac{x}{200},\ x \in \mathbb{R}^+, \\ \tilde{I}_1(x) &= \frac{1}{60}\frac{x}{1+x},\ x \in \mathbb{R}^+, \\ \tilde{J}_1(x) &= \frac{1}{80}\frac{x}{1+x},\ x \in \mathbb{R}^+. \end{aligned}$$

Let us check that all assumptions of Theorem 4.3.2 hold. We have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left| \frac{x}{(x+1)} - \frac{y}{y+1} \right| \\ &\leq |x-y| = k_1 |x-y|,\ t \in [0,1],\ x, y \in \mathbb{R}^+, \end{aligned}$$

$$f(t,x) \geq \frac{80}{(1+t)} \geq 40 = \lambda,$$

$$E_1 = E_2 = 3.7613,$$

$$M_1 = \frac{a}{b}\lambda E_2 = \frac{1}{2} \times 3.7613 \times 40 = 75.226,$$

$$\begin{aligned} r &= (q-1) M_1^{\frac{3}{2}-2} E_1 \left(E_2 k_1 + \frac{bm}{b-a} k_2 + \frac{2b^2 m}{(b-a)^2} k_3 \right) + \left(\frac{\eta m}{\gamma - \eta} k_4 + \frac{\eta^2 m}{(\gamma-\eta)^2} k_5 \right) \\ &= 0.857\,75 < 1. \end{aligned}$$

From Theorem 4.3.2, we deduce that the problem (4.1)-(4.9) has unique solution in Ω_r.

4.4.3 Example 3

Consider the fractional boundary value problem (4.1)-(4.9) with $\gamma = b = 2$, $\eta = a = 1$, $\alpha = \beta = \frac{3}{2}$, $p = 3$, $q = \frac{3}{2}$,

$$f(t,x) = \frac{400t + x^{\frac{1}{3}}}{10(e^x + x)},\ t \in J = [0,1],\ t \neq \frac{1}{2},\ x \in \mathbb{R}^+,$$

$$
\begin{aligned}
I_1(x) &= -20 - \frac{1}{10}x^{\frac{1}{3}}, \; x \in \mathbb{R}^+, \\
J_1(x) &= 5 + \frac{1}{80}x^{\frac{1}{3}}, \; x \in \mathbb{R}^+, \\
\tilde{I}_1(x) &= \frac{1}{50} + \frac{1}{50}\frac{x^{\frac{1}{2}}}{1+x}, \; x \in \mathbb{R}^+, \\
\tilde{J}_1(x) &= \frac{1}{200} + \frac{1}{200}\frac{x^{\frac{1}{2}}}{1+x}, \; x \in \mathbb{R}^+.
\end{aligned}
$$

It is clear that all assumptions of Theorem 4.3.3 hold. Indeed,

$$
\begin{aligned}
|f(t,u)| &\leq 40t + \frac{1}{10}x^{\frac{1}{3}}, \text{ for each } t \in J \text{ and all } x \in \mathbb{R}^+, \\
p_1(t) &= 40t, \quad p_1^* = 40
\end{aligned}
$$

$$
|I_1(u)| \leq 20 + \frac{x^{\frac{1}{3}}}{20}, \text{ for all } x \in \mathbb{R}^+,
$$

$$
|J_1(u)| \leq 5 + \frac{x^{\frac{1}{3}}}{80}, \text{ for all } x \in \mathbb{R}^+,
$$

$$
\left|\tilde{I}_1(u)\right| \leq \frac{1}{50} + \frac{x^{\frac{1}{2}}}{50}, \text{ for all } x \in \mathbb{R}^+,
$$

$$
\left|\tilde{J}_1(u)\right| \leq \frac{1}{200} + \frac{x^{\frac{1}{2}}}{200}, \text{ for all } x \in \mathbb{R}^+,
$$

By calculation it yields

$$
p_2 = 20, p_3 = 5, p_4 = \frac{1}{50}, p_5 = \frac{1}{200}, E_1 = E_2 = 3.7613,
$$

$$
\begin{aligned}
A_1 = 3\max\left\{2^{q-2}, 1\right\} E_1 \left(E_2 p_1^* + \frac{mb}{b-a}p_2 + \frac{2mb^2}{(b-a)^2}p_3\right)^{q-1} \\
+ 3\left(\frac{m\gamma}{\gamma-\eta}p_4 + \frac{m\gamma^2}{(\gamma-\eta)^2}p_5\right) = 0.923\,31,
\end{aligned}
$$

$$
A_2 = 3\left(E_2 q_1 + \frac{mb}{b-a}q_2 + \frac{2mb^2}{(b-a)^2}q_3\right)^{q-1} = 0.197\,62,
$$

$$A_3 = 3\left(\frac{m\gamma}{\gamma-\eta}q_4 + \frac{m\gamma^2}{(\gamma-\eta)^2}q_5\right) = 0.18.$$

Consequently, the problem (4.1)-(4.9) has at least one solution in Ω_r, where

$$r = 1 > \max\left\{A_1, (A_2)^{\frac{1}{1-\delta_1(p-1)}}, (A_3)^{\frac{1}{1-\delta_2}}\right\} = 0.923\,31.$$

4.4.4 Example 4

Let us choose for problem (4.1)-(4.9), $\gamma = b = 2$, $\eta = a = 1$, $\alpha = \beta = \frac{3}{2}$, $p = 3$, $q = \frac{3}{2}$,

$$f(t,x) = \frac{x}{(1+t)(1+x)},\ t \in J = [0,1],\ t \neq \frac{1}{2},\ x \in \mathbb{R}^+$$

$$\begin{aligned}
I_1(x) &= -\frac{1}{8}\left(1+\frac{1}{1+x}\right),\ x \in \mathbb{R}^+,\\
J_1(x) &= \frac{1}{32}\left(1+\frac{1}{1+x}\right),\ x \in \mathbb{R}^+,\\
\tilde{I}_1(x) &= \frac{1}{8}\left(1+\frac{1}{1+x}\right),\ x \in \mathbb{R}^+,\\
\tilde{J}_1(x) &= \frac{1}{16}\left(1+\frac{1}{1+x}\right),\ x \in \mathbb{R}^+,
\end{aligned}$$

By computation we get

$$\begin{aligned}
E_1 &= \frac{\gamma^2 + (\alpha-1)\gamma\eta}{(\gamma-\eta)^2\Gamma(\alpha+1)} = \frac{5}{\Gamma\left(\frac{5}{2}\right)} = 3.7613,\\
E_2 &= \frac{b^2 + (\beta-1)ba}{(b-a)^2\Gamma(\beta+1)} = \frac{5}{\Gamma\left(\frac{5}{2}\right)} = 3.7613,\\
\delta &= \frac{1}{8\sqrt{2}},
\end{aligned}$$

$$l = \frac{\left(\max\left\{2^{q-2}, 1\right\} E_1\right)^{\frac{1}{1-q}}}{E_2} = \frac{\left(\Gamma\left(\frac{5}{2}\right)\right)^3}{5^3} = 1.879\,3 \times 10^{-2},$$
$$\mu = \frac{b\left(\frac{\eta}{\gamma} E_1 \delta\right)^{1-p}}{a 2^3 E_2} = \frac{2\Gamma\left(\frac{5}{2}\right) 8^2}{5^3} = 1.3612.$$

On the other hand we have

$$\lim_{x\to+\infty}\sup\left(\max_{t\in J}\frac{f(t,x)}{x^{p-1}}\right) = \lim_{x\to+\infty}\sup\left(\max_{t\in J}\left(\frac{1}{(1+t)}\frac{1}{x(1+x)}\right)\right) = 0 < l,$$

$$\lim_{x\to0^+}\inf\left(\min_{t\in J}\frac{f(t,x)}{x^{p-1}}\right) = \lim_{x\to0^+}\inf\left(\min_{t\in J}\left(\frac{1}{(1+t)}\frac{1}{x(1+x)}\right)\right) = +\infty > \mu.$$

It follows from Theorem 4.3.4 that the problem (4.1)-(4.9) has at least one positive solution.

Conclusion

The aim of this thesis is to discuss, in Banach spaces, the existence of solutions for some classes of impulsive fractional differential equations with delay, involving the right and left Caputo derivatives and the p-Laplacian operator. The results are obtained using some fixed point theorems such the Banach's contraction principle, Schauder's fixed point theorem and a fixed point theorem in cones. The presence of impulsive moments with left and right fractional derivatives in the problem makes it more complicated and interesting.

To conclude this thesis,we outline a possible further development on this subject.

First, we can consider the case of the infinite interval to obtain more specific results similar to those of the impulsive that we have studied in the finite interval [0,1]. Then another interesting branch would be questions about the stability of solutions. In addition, the study of the existence of solutions to problems by the method of lower and upper solutions can be very interesting.

Second, we can also extend our results by considering nonlinear differential equations with more general forms for the nonlinear term f. Similar problems with different types of fractional derivatives will be investigated in future works.

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