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THÈSE

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Etude d'existence, unicité et stabilité de quelques problèmes d'évolution à retard

Option
Mathématiques Appliquées

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DEDICATION

I dedicate the fruit of this modest work: To my family, my friends and all who taught me.

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ABSTRACT

In this thesis, we study some time-delay evolution systems with the presence of different mechanisms of dissipation. We begin this thesis by presenting a brief summary about dynamical systems theory where we will introduce a historical overview of its origins. Then, we recall some reminders on functional spaces, and semigroups theory.

The monograph is composed of two parts, the first one is divided to three chapters, the first chapter is devoted to study a thermo-viscoelastic system of Timoshenko-type with nonlinear damping and a distributed delay acting on transverse displacement. We use the energy method and some properties of convex functions to prove a general decay estimate. In chapter 2, we concern with a one-dimensional Timoshenko system of thermoelasticity of type III with infinite memory damped by weakly nonlinear feedbacks. We obtain a general stability estimates using the multiplier method without assuming equal or nonequal speeds of propagation of waves. The third chapter is devoted to study a thermo-viscoelastic Bresse system with second sound and delay terms. We obtain results regardless of the speeds of wave propagation and the stable number which is introduced in some works before. The second part is divided to two chapters, we study non-uniform flexible structures systems, first one with second sound and a distributed delay term, and the second in thermoelasticity with micro-temperature effect. We prove the well-posed of each system as well its stability results under suitable assumptions.

Keywords: Evolution systems, Semigroups theory, Lyapunov functional, Exponential stability, Viscoelasticity, Thermelasticity, Delay.

مــلخص

ندرس في هذه الأطروحة بعض أنظمة التطور ذات التأخير الزمني مع وجود آليات مختلفة مـــــــن التبديد. نستهـــــــل هذه الأطروحـــة بتقديم ملخص موجز لنظرية الأنظمة الديناميكية، حيث سنقدم لمحة تاريخية عن نشأتها. بعـــد ذلـــــك، نقدم تذكير حـــــول بعض الفضاءات التابعية ونظريـــة **شبه الزمر**.

دراستنا سوف تكـون من جزئـــــين، الجزء الأول يضم ثلاث فصول، الأول مخصـــــص لدراســـــة نظـام حراري لزج مرن من نوع **تيموشينكو** مع التـخـميــــد غير الخطــــي والتأخيـر الموزع الــذي يعمــل على الإزاحة العرضية. نستخــدم طريـقة الطـــاقة وبعض خصــــائص الدوال المحدبة لإثبــــات الاضمحلال العام للنـظام.

في الفصل الثاني ، سنهتم بنظام **تيموشينكو** أحادي البعد للمــــرونة الحرارية من النوع الثـــــالث مع ذاكـرة مخمــد بواسطة دالة غـير خطية. في ظل شروط مناسبة ، نثبت وجود الحل و وحدانيته باستـخدام نظرية شبه الزمر، وكذلك تقديرات الاستقرار العـام باستخدام طريقة المضاعف دون افتراض سرعات متساوية أو غير متساوية لانتشـــــار الموجات.

الفصل الثالث مخصص لدراسة نظام **براس** . النتائج المحصل عليها هنا غير متعلقة بسرعات انتشار الموجــــــــة و الثــــــابت الذي تم ذكره في بعض الأعمال من قبــل. أما الجزء الثـاني فـيضم الفصـلين 4 و 5 ، سنـدرس الهياكل المرنة غيـر المنتظمة ، الأولى مع الصـوت الثاني و التـأخير الموزع، والثانية في المرونة الحــرارية مـع تأثير درجة الحـرارة عبر الجزيئات الدقيقة. لقد أثبتنا وجود الحل و وحدانيته بالنسبة لكل نظــــام بالإضافة إلى استقراره في ظــــل فرضـــــيات مناسبة.

الكلمات المفتاحية: أنظمة التطور، نظرية شبه الزمر، دالة **ليابونوف**، الإستقرار الأســـي، اللـــــزوجة، المـــرونة الحرارية ، التأخير.

RÉSUMÉ

Dans cette thèse, nous étudions certains systèmes d'évolution à retard de différents mécanismes de dissipation. Nous commençons cette thèse en présentant un bref résumé sur la théorie des systèmes dynamiques où nous présenterons un aperçu historique de ses origines. Ensuite, nous rappelons quelques notions sur certains espaces fonctionnels et la théorie des semi-groupes.

La monographie est composée de deux parties, la première est divisée en trois chapitres, le premier chapitre est consacré à l'étude d'un système thermo-viscoélastique de type Timoshenko à amortissement non linéaire et à retard distribué agissant sur le déplacement transversal. Nous utilisons la méthode de l'énergie et certaines propriétés des fonctions convexes pour prouver une estimation de décroissance générale. Dans le chapitre 2, nous nous intéressons à un système de Timoshenko unidimensionnel de thermoélasticité de type III à mémoire infinie amorti par des rétroactions faiblement non linéaires. On obtient une estimation générale de la stabilité en utilisant la méthode du multiplicateur sans supposer des vitesses égales ou non de propagation des ondes. Le troisième chapitre est consacré à l'étude d'un système de Bresse thermo-viscoélastique avec deuxième son à retard. Nous obtenons des résultats quelles que soient les vitesses de propagation des ondes et le nombre stable qui est introduit dans certains travaux précédents. La deuxième partie est divisée en deux chapitres, nous étudions des systèmes des structures flexibles non uniformes, la première avec deuxième son à retard distribué et la seconde en thermélasticité avec l'effet de micro-température. Nous prouvons le bien-posé de chaque système ainsi que ses stabilités sous des hypothèses appropriées.

Mots-clés: Systèmes d'évolution, Théorie des semigroupes, Fonctionnelle de Lyapunov, Stabilité exponentielle, Viscoélasticité, Thermoélasticité, Retard.

CONTENTS

CHAPTER 1
INTRODUCTION

This thesis is devoted to a study of stability of some time-delay evolution systems, a subject that has long been vigorously pursued by such authors as much diversely represented as in mathematics, science, engineering, and economics. It is comforting, that the subject is more enduring than transient, and indeed has sustained a surprising degree of vitality. In particular, for the last decade or so, it has received visible research attention and the advance has been notable. It appears warranted to assert, due to this period of creative work, that the subject has undergone a significant leap conceptually and on practical measures, both its nature and scope have been dramatically advanced and broadened.

What defines a time-delay system is the feature that the system's future evolution depends not only on its present state, but also on a period of its history. This particular cause-effect relationship can be most succinctly captured, and indeed has been traditionally so modeled by differential-difference equations, or more generally, by functional differential equations. While in practice many dynamical systems may be satisfactorily described by ordinary differential equations alone, for which the system's future evolution depends solely on its current state, there are times when delay effect cannot be neglected, or it will be more beneficial for it to be accounted for. In a word, we reckon that many will agree time delay is by no means a matter of rarity, in fact, it is more prevalent than uncommon, numerous results mentioned in this thesis and elsewhere serve to solidify this standpoint. It is thus unsurprising, due to their omnipresence, and for their intrinsic scientific interest and practical implication, that time-delay systems have been studied long and well. It has for decades been an active area of scientific research in mathematics, biology, ecology, economics, and in engineering, under such terms as hereditary systems, systems with aftereffect, or systems with time-lag (system with time-delay terms), and more generally as a subclass of functional differential equations and infinite dimensional systems. The field of time-delay systems as a whole has its beginning dated back to the eighteenth century, and it received substantial attention in the early twentieth century in works devoted to the modeling of biological, ecological, as well as engineering systems.

A differential equation with delay describing a dynamical system belongs to the class of retarded functional differential equations (also sometimes called retarded differential-difference equations). One can also consider other classes of delay differential equations (DDE), namely neutral DDEs and advanced DDEs. If the evolution of a DDE depends on the past rates of changes in addition to its present and past values, then the corresponding DDE is referred to as a neutral DDE. An advanced type DDE is the one in which the evolution depends on its present and future values [88]. For example, consider the simple case of a linear scalar first order equation

$$a_0 x_t(t) + a_1 x_t(t-\tau) + b_0 x(t) + b_1 x(t-\tau) = f(t),$$

where a_0, a_1, b_0 and b_1 are arbitrary constants and $f(t)$ is a forcing function. The above equation is said to be a DDE of retarded type if $a_0 \neq 0$ and $a_1 = 0$, it is said to be of neutral type if $a_0 \neq 0$ and $a_1 \neq 0$, and of advanced type if $a_0 = 0$ and $a_1 \neq 0$.

In particular, the evolution of a dynamical variable corresponding to a retarded DDE depends not only on its present value, $x(t)$, but also on its values at earlier times, $x(t')$, $t' \in (-\tau, 0)$, where $\tau > 0$ is the delay time. As a consequence, a time-dependent solution of a system of DDEs is not uniquely determined by its initial state at a given moment alone. Instead, the solution profile (initial function) on an interval of length equal to the maximal delay prior to the time $t = 0$ has to be prescribed. That is, we need to define a set of infinite (but continuous) number of initial conditions for $-\tau < t < 0$ and hence DDEs are effectively infinite-dimensional systems, even if we have only a single scalar delay differential equation.

The most common type of infinite-dimensional dynamical systems involve the evolution of functions in time. For instance, if we want to study the evolution of chemical concentrations in time and space, we can phrase the problem as the change in time of the spatial distribution of chemicals. This distribution can be represented by a function of the spatial variables, that is, $C = C(r)$. This is also one of the reasons for increasing interest of physics community for DDEs as they provide a natural link with space extended systems by means of the two variable representation of the time $t = \varsigma + \theta\tau$, where $\varsigma \in (0, \tau)$ is the continuous space variable, and $\theta \in \mathbb{N}$ is a discrete temporal variable.

Generally, stabilization of DDE systems aims to attenuate vibrations by feedback, so it consists in guaranteeing the decrease of the energy of the solutions towards 0 more or less quickly by a dissipation mechanism. More precisely, the stabilization problem in which we are interested comes down to determining the asymptotic behavior of the energy that we note $\mathcal{E}(t)$, to study its limit in order to determine if this limit is zero or not, and if this limit is zero, to give an estimate of its rate of decrease towards zero.

Description and objective of the thesis

We present in this thesis a study on the existence, uniqueness and stability of some elastic, thermoelastic, viscoelastic and delayed evolution problems, such as the Timoshenko systems, Bresse systems and systems of flexible structures.

In recent decades, much work on local existence, global existence, and asymptotic behavior of solutions to some initial condition and boundary problems as well as Cauchy problems in one-dimensional and multidimensional thermoelasticity have been done.

In this regard, the objective of this thesis is to study the behavior of solutions of certain evolution systems where dissipation is introduced by the presence of a thermoelastic, micro-temperature, nonlinear or viscoelastic term, and the retardation by a constant or distributed delay term. Several results concerning the decrease of solutions in non-classical thermoelasticity have been proved. In this study, we generalize and improve various previous results.

In the following, we will give a brief analysis of the content of the thesis which is divided into six chapters.

The first one consists of a theoretical support for the study, we will find the different tools on which our study will be based. We begin this chapter by presenting a brief summary of dynamical systems theory where we will introduce a historical overview of its origins. Then, we recall some reminders on Hilbert spaces, the L^p spaces, Sobolev spaces and semigroups theory.

The second chapter is devoted to study a thermo-viscoelastic system of Timoshenko-type with nonlinear damping and a distributed delay acting on transverse displacement

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t(x,t-s)\,ds = 0 \\ \rho_2\psi_{tt} - \beta\psi_{xx} + K(\varphi_x+\psi) + \int_0^t g(t-s)\left(a(x)\psi_x(s)\right)_x ds + \mu_3(t)b(x)f(\psi_t) + \gamma\theta_x = 0 \\ \rho_3\theta_t + kq_x + \gamma\psi_{tx} = 0 \\ \rho_4 q_t + \delta q + k\theta_x = 0. \end{cases}$$

The heat flux of the system is governed by Cattaneo's law. We use the energy method and some properties of convex functions to prove, regardless of the speeds of wave propagation, general decay estimate from which the exponential, logarithmic and polynomial types of decay are only special cases. (see [49]).

In third chapter, we consider a one-dimensional Timoshenko system of thermoelasticity of type III with infinite memory damped by weakly nonlinear feedbacks

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x+\psi\right)_x + \beta\theta_{tx} + \alpha(t)\,f(\varphi_t) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x+\psi\right) + \int_0^\infty g(s)\,\psi_{xx}(t-s)\,ds - \beta\theta_t = 0 \\ \rho_3\theta_{tt} - \delta\theta_{xx} - k\theta_{xxt} + \gamma\varphi_{tx} + \gamma\psi_t + \alpha(t)\,f(\theta_t) = 0. \end{cases}$$

Under suitable conditions, we establish the well-posedness of the problem using semi-groups theory, and a general stability estimates using the multiplier method with no growth assumption on f at the origin and without assuming equal or nonequal speeds of propagation of waves which is mentioned in numerous works (e.g. [14, 29, 35, 48, 68]). Our results show that the damping effect leads to general decay rate for the energy function and also remove the necessity of the assumption on equal speeds which has been imposed in the prior literature (see [51]).

The fourth chapter is devoted to study a thermo-viscoelastic Bresse system with second sound and delay terms,

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t-\tau_1\right) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \delta\int_0^t g\left(t-s\right)\psi_{xx}\left(x,s\right)ds + \gamma\theta_x = 0 \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \lambda_1\omega_t + \lambda_2\omega_t\left(x, t-\tau_2\right) = 0 \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0 \\ \alpha q_t + \beta q + \theta_x = 0, \end{cases}$$

where the heat flux is given by Cattaneo's law. Regardless of the speeds of wave propagation and the stable number, which is introduced in [59, 64], we prove an exponential stability result using energy method under suitable assumptions on the weights of the delays and the frictional damping (see [50]).

In the fifth chapter, we study a non-uniform delayed flexible structure damped by a non-linear dissipation term,

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + f(u_t) + \int_{\tau_1}^{\tau_2}\mu(s)u_t\left(t-s\right)ds = 0 \\ \theta_t + \eta u_{tx} + kq_x = 0 \\ \tau q_t + \beta q + k\theta_x = 0, \end{cases}$$

where the heat flux is given by Cattaneo's law. We prove the well-posed of the system using semi-group theory and general stability using multiplier method under suitable assumptions on the weights of the delay, heating effect and material damping and regardless of growth assumption on the nonlinear damping term f at the origin. (see [52]).

The final chapter is devoted to study a non-uniform flexible structure with micro-temperature effect

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + dw_x + \eta\theta_x = 0 \\ c\theta_t - k\theta_{xx} + \eta u_{tx} + k_1 w_x = 0 \\ \tau w_t - k_3 w_{xx} + k_2 w + k_1\theta_x + du_{tx} = 0. \end{cases}$$

We prove the well-posed of the problem using semi-group theory, as well as an exponential stability using the multiplier method without any restriction or relation on the coefficients of the system (see [53]).

Methodology

In this work, to ensure the well-posed of our problems, we use the theory of semi-groups to establish the existence and uniqueness of the solutions. In semigroups theory, the Hille-Yosida theorem is a powerful and fundamental tool relating the energy dissipation properties of an unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ to the existence, uniqueness and regularity of the solutions of a stationary differential equation (Cauchy problem)

$$\begin{cases} \Phi'(t) = \mathcal{A}(t)\Phi(t), \quad t > 0 \\ \Phi(0) = \Phi_0, \end{cases}$$

For the stability results, we use the multiplier method based on the construction of a Lyapunov function $\mathcal{L}$ equivalent to the energy $\mathcal{E}$ of the solution. We denote by $\mathcal{L} \sim \mathcal{E}$ the equivalence

$$c_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_2\mathcal{E}(t), \quad \forall t \geq 0, \tag{1.1}$$

for two positive constants c_1 and c_2. For example, to establish exponential stability, it suffices to show that

$$\mathcal{L}'(t) \leq -c\mathcal{L}(t), \quad \forall t > 0, \tag{1.2}$$

for some $c > 0$. A simple integration of (1.2) over $[0, t]$ with (1.1) leads to the desired result of exponential stability. It is worth noting that Lyapunov's theorems are only sufficient conditions for the stability and the difficulty here is to find the adequate Lyapunov function.

Publications

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CHAPTER 2

GENERALITIES

In this chapter, we shall introduce a brief summary of dynamical systems theory and then state some necessary materials needed in the proof of our results, such as basic results which concerning the Hilbert spaces, the L^p space, Sobolev spaces, some theorems on these last and existence and uniqueness theorem. The knowledge of all these notions and results are important for our study.

2.1 About dynamical systems theory

The qualitative analysis of dynamical systems introduced by H. Poincaré at the end of the nineteenth century [82] gave birth to the fruitful field on dynamical systems theory, with all the profound implications and applications we have nowadays including, among others, systems and control theory. Before Poincaré's differential equations were mostly viewed as equations to be solved, similarly to as algebraic equations. Poincaré had the bright idea to try to study differential equations in a qualitative way, which essentially means that finding solutions is not the objective anymore, but instead, we focus on establishing certain properties of the solutions. This point of view is particularly relevant since many differential equations do not admit closed-form solutions and can only be solved numerically. In the same vein of Poincaré's ideas, A.M. Lyapunov developed the theory of stability of dynamical systems during his Ph.D. thesis [67], which was supervised by P. Chebyshev. Stability is a fundamental property of dynamical systems having deep consequences in sciences and engineering. Stability essentially means that solutions of a dynamical system starting close to an equilibrium point (which is a resting point of the system), remain close to this equilibrium point. A typical example is the pendulum example. Pendulums with rigid rod admit two equilibrium points, one is when the rod is vertical and the mass down, the other is when the mass is up. Consider the first equilibrium point and assume that there is no friction. A small push from this resting position will result in sustained oscillations of bounded amplitude around it. This equilibrium point is therefore stable. An equilibrium point is, moreover, said to be asymptotically stable if it is stable and the trajectories starting nearby to it converge back to it. Taking again the pendulum example and adding friction to the problem will result in damped oscillations around the equilibrium point. Eventually, the pendulum will stop oscillating and will return to its resting position. This equilibrium point is therefore asymptotically stable. Opposed to stable equilibrium points, unstable ones are resting positions from which arbitrarily small perturbations will be amplified, pushing then the dynamical system away from them. For instance, the second equilibrium point of the pendulum with friction, i.e., the one with the mass up, is unstable since when slightly pushed from its equilibrium position, it does not return there. Instead, it converges to the asymptotically stable equilibrium point. A fundamental and appealing feature of Lyapunov's results is that, in the same spirit as Poincaré's ideas, the properties of the trajectories in a neighborhood of an equilibrium point can be assessed without even computing the solutions of the dynamical system. This can be actually performed using potential functions, now referred to as Lyapunov functions. These functions form the cornerstones of the powerful Stability theory also called Lyapunov's theory of stability or even Lyapunov theory.

This theory has been broadly accepted by systems and control theorists as a fundamental starting point for dealing with the analysis and control of dynamical systems. Whenever control systems are concerned, stability is one of the most important properties a control system should possess. Ensuring asymptotic stability of the closed-loop system is an efficient way for assessing that the controlled process behaves in the desired way, for instance, converges to a desired equilibrium point. Another striking point is the versatility of the approach which has been adapted, since then, to an immense variety of systems such as time-varying systems, discrete-time systems, hybrid systems, and infinite-dimensional systems. The dynamical systems we are interested in this thesis do not escape this rule, and Lyapunov theory will be shown to be an adequate tool for dealing with time-delay and linear parameter-varying systems. Whereas time-delay systems can be approached as a pure mathematical problem arising from a scientific field such as biology, ecology or physics, parameter-varying systems essentially come up from engineering problems such as filtering and control. In this regard, the field of linear parameter-varying time-delay systems is mostly of engineering interest only. We refer the interested readers to the books [22, 42, 45, 60, 62] for details discussion on the theory and to [91] for different examples of time-delay problems.

2.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. We will suffice to mention its definition.

Definition 2.1 [24] A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.

2.3 Functional Spaces

The $L^p(\Omega)$ spaces:

Definition 2.2 [24] Let $1 \leq p \leq \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

Notation 2.1 *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, denote by*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$\begin{aligned} L^\infty(\Omega) &= \{ f : \Omega \to \mathbb{R} : f \textit{ is measurable and there exists a constant } C \\ &\quad \textit{such that}, |f(x)| \leq C \textit{ a.e in } \Omega \}. \end{aligned}$$

Also, we denote by

$$\|f\|_\infty = \inf \{ C, \ |f(x)| \leq C \textit{ a.e in } \Omega \}.$$

Notation 2.2 *Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark 2.1 In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx,$$

is a Hilbert space.

The Sobolev spaces $\mathbf{W}^{m,p}(\Omega)$:

Definition 2.3 Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as

$$W^{m,p}(\Omega) = \left\{ f \in L^p(\Omega),\ \text{such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that } |\alpha| = \sum_{j=1}^{n} \alpha_j \leq m,\ \text{where } \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} \right\}.$$

Theorem 2.1 *([28]) $W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p},\ 1 \leq p \leq \infty,\ \textit{for all } f \in L^p(\Omega).$$

Definition 2.4 When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,p}(\Omega) = H_0^m(\Omega)$ for $p \in [0, \infty[$ supplied with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \left(\|\partial^\alpha f\|_{L^2} \right)^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

The next result provides a basic characterization of functions in $W_0^{1,p}(\Omega)$.

Theorem 2.2 *[24] Let $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.*

Remark 2.2

1. Theorem 2.2 explains the central role played by the space $W_0^{1,p}(\Omega)$. Differential equations (or partial differential equations) are often coupled with boundary conditions, i.e., the value of u is prescribed on $\partial\Omega$.

2. We have the following characterization of $H_0^m(\Omega)$

$$H_0^m(\Omega) = \{u \in H^m(\Omega),\ u = u' = ... = u^{(m-1)} = 0 \text{ on } \partial\Omega\}$$

It is essential to notice the distinction between

$$H_0^2(\Omega) = \{u \in H^2(\Omega),\ u = u' = 0 \text{ on } \partial\Omega\},$$

and

$$H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega),\ u = 0 \text{ on } \partial\Omega\}.$$

2.4 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Lemma 2.1 *([24], Hölder's Inequality) Let* $1 \leq p \leq \infty$*, assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *then,* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |fg|\, dx \leq \|f\|_p \|g\|_q . \tag{2.1}$$

The next result is an important prototype of a Sobolev inequality (also called a Sobolev embedding).

Lemma 2.2 *([24]) There exists a constant* C *(depending only on* $|I| \leq \infty$*) such that*

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}, \ \forall\ u \in W^{1,p}(I), \forall\ 1 \leq p \leq \infty. \tag{2.2}$$

Lemma 2.3 *[24] (Poincaré's inequality) Suppose* I *is a bounded interval. Then there exists a constant* C *(depending on* $|I| < \infty$*) such that*

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^P(I)}, \ \textit{for all } u \in W_0^{1,p}(I). \tag{2.3}$$

Lemma 2.4 *[74] (Poincaré type Scheeffer's inequality): Let* $h \in H_0^1(0, L)$*. Then it holds*

$$\int_0^L |h|^2\, dx \leq\ l \int_0^L |h_x|^2\, dx, \quad l = \frac{L^2}{\pi^2}. \tag{2.4}$$

2.5 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 2.5 *([24], Cauchy-Schwarz Inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\| . \tag{2.5}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Lemma 2.6 *[24](Young's Inequality) For all* $a, b \in \mathbb{R}^+$*, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{2.6}$$

where ϵ *is any positive constant.*

Now, Let us denote by h^* the conjugate function in the sense of Young of a convex function h (see [13], p. 64), i.e.,

$$h^*(p) = \sup_{t\in\mathbb{R}_+}(pt - h(t)).$$

Assume that $h'' > 0$, then for $p \geq 0$ a given number, h^* is the Legendre transform of h (see Liu and Zuazua [66]), which is given by

$$h^*(p) := p\,[h']^{-1}(p) - h([h']^{-1}(p)), \tag{2.7}$$

and which satisfies the following inequality

Lemma 2.7 *[81](Young's Inequality for the convex functions) Let h a convex function, h^* its conjugate in the sense of Young, we have*

$$px \leq h(x) + h^*(p) \quad \forall p, x \geq 0. \tag{2.8}$$

Remark 2.3 The relation (2.7) and the fact that $h(0) = 0$ and $(h')^{-1}$, h are increasing functions yield

$$h^*(p) \leq p\,[h']^{-1}(p) \quad \forall p \geq 0. \tag{2.9}$$

2.6 Existence and uniqueness theorem

Lax-Milgram Lemma

The existence and uniqueness of a solution to the weak formulation of the problem can be proved using the Lax-Milgram Lemma. This states that the weak formulation admits a unique solution.

Lemma 2.8 *[24] (Lax-Milgram lemma). Let $a(\cdot,\cdot)$ be a bilinear form on a Hilbert space $\mathcal{H}$ equipped with norm $\|\cdot\|_{\mathcal{H}}$ and the following properties:*

i) $a(\cdot,\cdot)$ is continuous, that is

$$\exists \gamma_1 > 0 \text{ such that } |a(w,v)| \leq \gamma_1 \|w\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \ \forall w, v \in \mathcal{H},$$

ii) $a(\cdot,\cdot)$ coercive (or $\mathcal{H}$-elliptic), that is

$$\exists \alpha > 0 \text{ such that } |a(v,v)| \leq \alpha \|v\|_{\mathcal{H}}^2, \ \forall v \in \mathcal{H},$$

iii) L is a linear mapping on $\mathcal{H}$ (thus L is continuous), that is

$$\exists \gamma_2 > 0 \text{ such that } |L(w)| \leq \gamma_2 \|w\|_{\mathcal{H}}, \ \forall w \in \mathcal{H},$$

Then there exists a unique $u \in \mathcal{H}$ such that

$$a(w,u) = L(w), \ \forall w \in \mathcal{H}.$$

C_0-Semigroup of bounded linear operators

Throughout this section $\mathcal{H}$ denotes a Hilbert space.

Definition 2.5 [81] Let X be a Banach space. A one parameter family $(S(t))_{t\geq 0}$ of bounded linear operators defined from X into X is a strongly continuous semigroup of bounded linear operators on X if:

- $S(0) = I$ (I identity operator on X).
- $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$.
- $S(t)x \to x$, as $t \to 0$, $\forall x \in X$.

Such a semigroup is called a C_0-semigroup.

Definition 2.6 [81] The infinitesimal generator $\mathcal{A}$ of the semigroup $(S(t))_{t\geq 0}$ is defined by:

$$D(\mathcal{A}) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

and

$$\mathcal{A}x = \lim_{t\to 0} \frac{S(t)x - x}{t}, \; x \in D(\mathcal{A}).$$

Definition 2.7 [24] An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is said to be monotone if it satisfies

$$(\mathcal{A}u, u) \geq 0, \; \forall u \in D(\mathcal{A}),$$

It is called maximal monotone if, in addition

$$R(\mathcal{I} + \mathcal{A}) = \mathcal{H}, \text{ i.e.,}$$

$$\forall f \in \mathcal{H}, \exists u \in D(\mathcal{A}) \text{ such that } u + Au = f.$$

Proposition 2.1 *[24] Let A be a maximal monotone operator. Then $D(\mathcal{A})$ is dense in $\mathcal{H}$.*

Generally speaking, the first step in dealing with the study of the well-posedness of the solution is to rewrite our evolution system of partial differential equations as a Cauchy problem on some appropriate Hilbert space $\mathcal{H}$ called the energy space

$$\begin{cases} u' + \mathcal{A}(t)u = 0, \\ u(0) = u_0, \end{cases}$$

where $\mathcal{A}(t)$ is an unbounded operator on $\mathcal{H}$. Then we prove that $\mathcal{A}(t)$ is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$ in order to deduce the existence of a solution in a certain Hilbert space. The solution is hence of the form $u(t) = S(t)u_0$. We mention here Hille–Yosida Theorem: Lumer-Phillips form (see [24]) which is applied to justify the existence and uniqueness of solutions of some partial differential equations.

Theorem 2.3 *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C([0,\infty[, D(\mathcal{A})) \cap C^1([0,\infty[, \mathcal{H})$$

satisfying

$$\begin{cases} u' + \mathcal{A}(t)u = 0 \quad on\ [0,\infty[\\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \leq |u_0|, \ \forall t \geq 0 \ and \ |\frac{du}{dt}(t)| = |Au(t)| \leq |Au_0|, \, \forall t > 0.$$

Remark 2.4

1. The main interest of Theorem 2.3 lies in the fact that we reduce the study of an "evolution problem" to the study of the "stationary equation" $u' + Au = f$.

2. The space $D(A)$ is equipped with the graph norm $|u| + |Au|$ or with the equivalent Hilbert norm $\sqrt{|u|^2 + |Au|^2}$.

3. We refer the interested readers to [61, 90] and references therein for details discussion on existence and uniqueness of local or global solutions of nonlinear evolution equations.

Part I

Timoshenko–Bresse Systems

CHAPTER 3

GENERAL DECAY FOR A VISCOELASTIC DAMPED TIMOSHENKO SYSTEM OF SECOND SOUND WITH DISTRIBUTED DELAY

3.1 Introduction

We consider the following Timoshenko-type system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t\,(x, t-s)\,ds = 0, \\ \rho_2\psi_{tt} - \beta\psi_{xx} + K(\varphi_x + \psi) + \int_0^t g\,(t-s)\,(a(x)\psi_x(s))_x\,ds + \mu_3(t)b(x)f(\psi_t) + \gamma\theta_x = 0, \\ \rho_3\theta_t + kq_x + \gamma\psi_{tx} = 0, \\ \rho_4 q_t + \delta q + k\theta_x = 0, \end{cases} \tag{3.1}$$

where $t \in (0, +\infty)$ denotes the time variable and $x \in (0, 1)$ is the space variable. Here φ, ψ, θ and q are respectively the transverse displacement of the beam, the rotation angle, the difference temperature and the heat flux. a, b, f, μ_3 and g are specific functions, $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0$ are initial data. The coefficients, $\rho_1, \rho_2, \rho_3, \rho_4, \gamma, \delta, k, \beta, \mu_1$ and K are positive constants, $\mu_2 : [\tau_1; \tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1, τ_2 two real numbers satisfying $0 \leq \tau_1 < \tau_2$.
We consider the following initial and boundary conditions:

$$\begin{array}{l} \varphi\,(.,0) = \varphi_0\,(x)\,,\;\; \varphi_t\,(.,0) = \varphi_1\,(x)\,,\;\; \theta\,(.,0) = \theta_0\,(x) \quad \text{in } \;(0,1)\,, \\ \psi\,(.,0) = \psi_0\,(x)\,,\;\; \psi_t\,(.,0) = \psi_1\,(x)\,,\;\; q\,(.,0) = q_0\,(x) \quad \text{in } \;(0,1)\,, \\ \varphi\,(0,t) = \varphi\,(1,t) = \psi\,(0,t) = \psi\,(1,t) = \theta\,(0,t) = \theta\,(1,t) = 0 \text{ in } \;(0,\infty)\,, \\ \varphi_t\,(x,-t) = f_0(x,t) \quad \text{in } \;(0,1)\times(0,\tau_2)\,, \end{array} \tag{3.2}$$

where f_0 is the history function.

It is well-known that a great number of processes of the applied sciences (like treatments of physical, biological, chemical, economic, and thermal phenomena), can be modeled by means of delay differential equations, these last are differential equations involving not only the function and their derivatives at present state t but also the function and\or their derivatives at some past times. Then this makes the control of PDEs with time delay effects becomes an active area of research. In recent years, the issue of existence and stability of evolution problems with delay has attracted a great deal of attention [9, 19, 21, 35, 36, 79, 92]. Kafini et al. [57] concerned with the following Timoshenko system of thermoelasticity of type III with distributive delay:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_{tx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - k\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\,\theta_{txx}(x, t-s)ds + \gamma\psi_{tx} = 0, \end{cases} \tag{3.3}$$

where $\tau_1 < \tau_2$ are non-negative constants. They proved an exponential decay in the case of equal wave speeds and a polynomial decay result in the case of nonequal wave speeds with smooth initial data. Also, Kafini et al. [58] considered the following Timoshenko system of thermoelasticity of type III with delay:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x + \psi)_x + \mu_1\phi_t(x,t) + \mu_2\phi_t(x, t-\tau) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) + \beta\theta_{tx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0. \end{cases} \tag{3.4}$$

They established the well-posedness and the stability of the system for the cases of equal and nonequal speeds of wave propagation, they showed that the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay, and in the opposite case it decays polynomially. Very recently, Hao and Wang [48] considered the following Timoshenko-type system with distributed delay and past history:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \beta\theta_{tx} = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \int_0^{\infty} g(s)\,\psi_{xx}(x, t-s)ds - \beta\theta_t + f(\psi) = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - l\theta_{txx} + \gamma\varphi_{tx} + \gamma\psi_t - \int_{\tau_1}^{\tau_2} \mu(\varsigma)\,\theta_{txx}(x, t-\varsigma)d\varsigma = 0. \end{cases} \tag{3.5}$$

They proved the well-posedness and the stability of the system for the cases of equal and nonequal speeds of wave propagation. Their results show that the damping effect is strong enough to uniformly stabilize the system even in the existence of time delay under suitable conditions. Chen et al. [29] studied the following thermo-viscoelastic system of Timoshenko of type III with frictional damping and delay terms:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} + \int_0^t g(t-s)\,\theta_{xx}ds + \mu_1\theta_t(x,t) + \mu_2\theta_t(x, t-\tau) = 0. \end{cases} \tag{3.6}$$

Under a hypothesis between the weights of the frictional damping and the delay, they proved the global existence of solutions by using the Faedo–Galerkin approximations, and established a general energy decay result from which the polynomial and exponential types of decay are only special cases.

In the absence of delay ($\mu_2 = 0$), viscoelastic term ($g = 0$) and nonlinear damping ($f = 0$), Messaoudi et al. [72] considered (3.1) for both linear and nonlinear case and proved that the system is exponentially stable without any restriction on the coefficients. Apalara [9] and Ouchenane [79] extended this last to a Timoshenko system with delay term of the form $\int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t\,(x, t-s)\,ds$ and $\mu_2\varphi_t(x, t-\tau)$ respectively. Under suitable assumptions on the weights of the delay and the frictional damping, both authors established the well-posedness result and proved that the system is also exponentially stable regardless of the speeds of wave propagation. Fareh and Messaoudi [35] extended the result obtained by Apalara [9] to a thermoelastic system of type III, they proved the well-posedness and exponential stability results in the presence and the absence of an extra frictional damping under some conditions.

Motivated by the works mentioned above, we investigate system (3.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$) and nonlinear damping ($f \neq 0$), we can establish a general energy decay regardless also of the speeds of wave propagation. We prove our result by using the energy method together with some properties of convex functions. These arguments of convexity were introduced by Lasiecka and Tataru [63] and used by Liu and Zuazua [66] and others.

3.2 Preliminaries

In this section, we present some materials needed in the proof of our results. We also state, without proof, a local existence result for problem (3.1). The proof can be established by using Faedo–Galerkin method [29]. Throughout this thesis, c represents a generic positive constant and is different in various occurrences.

As in [78], Taking the following new variable

$$z(x, \rho, s, t) = \varphi_t\,(x, t-\rho s)\,, \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty)\,.$$

Then we obtain

$$\begin{cases} s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = \varphi_t(x,t). \end{cases}$$

Consequently, problem (3.1)-(3.2) is equivalent to

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)ds = 0, \\ \rho_2\psi_{tt} - \beta\psi_{xx} + K(\varphi_x+\psi) + \int_0^t g(t-s)\left(a(x)\psi_x(s)\right)_x ds + \mu_3(t)b(x)f(\psi_t) + \gamma\theta_x = 0, \\ \rho_3\theta_t + kq_x + \gamma\psi_{tx} = 0, \\ \rho_4 q_t + \delta q + k\theta_x = 0, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \end{cases} \tag{3.7}$$

where $(x,\rho,s,t) \in (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty)$, with the following initial and boundary conditions:

$$\begin{array}{l} \varphi(.,0) = \varphi_0(x),\ \varphi_t(.,0) = \varphi_1(x),\ \theta(.,0) = \theta_0(x) \quad \text{in } (0,1), \\ \psi(.,0) = \psi_0(x),\ \psi_t(.,0) = \psi_1(x),\ q(.,0) = q_0(x) \quad \text{in } (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0 \text{ in } (0,\infty), \\ z(x,\rho,s,0) = f_0(x,\rho s) \quad \text{in } (0,1)\times(0,1)\times(0,\tau_2). \end{array} \tag{3.8}$$

We shall use the following assumptions:

Assumption 2.1. $\mu_2 : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds < \mu_1 \tag{3.9}$$

Assumption 2.2. The functions a and b will be supposed continuous, non-negatives and satisfy

$$\begin{array}{l} a \in C^1([0,1]), \\ a = 0 \text{ or } a(0) + a(1) > 0, \\ \inf_{x\in[0,1]} \{a(x) + b(x)\} > 0. \end{array}$$

Assumption 2.3. $f : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function such that there exist positive constants k_1, k_2 and l and a convex, continuous and increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying: $h(0) = 0$ and $h'' = 0$ on $[0,l]$ or ($h'(0) = 0$ and $h'' > 0$ on $(0,l]$) such that

$$\begin{array}{ll} h\left(s^2 + f^2(s)\right) \le f(s)\, s & \text{for } |s| \le l, \\ k_1 s^2 \le f(s)\, s \le k_2 s^2 & \text{for } |s| > l. \end{array} \tag{3.10}$$

Assumption 2.4. $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0,\ \int_0^\infty g(s)\, ds = g_1,\ \beta - \|a\|_\infty \int_0^\infty g(s)\, ds > 0. \tag{3.11}$$

Assumption 2.5. There exists a non-increasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(s) \le -\eta(s)\, g(s), \text{ for } s \ge 0.$$

3.2. Preliminaries

Assumption 2.6. $\mu_3 : \mathbb{R}_+ \rightarrow]0, +\infty[$ is a non-increasing C^1-function satisfying

$$\int_0^\infty \mu_3(s)\,ds = +\infty.$$

Remark 3.1 1. *From* ***assumption*** 2.3 *we easily infer that* $f(s)s \geq 0$ *for* $s \in \mathbb{R}$.

2. ***Assumption*** 2.6 *implies that* μ_3 *is bounded.*

3. *According to our knowledge,* ***Assumption*** 2.3 *was first introduced by Lasiecka and Tataru [63].*

4. *Since* g *is positive and* $g(0) > 0$ *then for any* $t_0 > 0$ *we have*

$$\int_0^t g(s)\,ds \;\geq \int_0^{t_0} g(s)\,ds = g_0 > 0, \quad \forall t \geq t_0. \tag{3.12}$$

Using the fact that $a(0) > 0$ and a is continuous, then there exists $\varepsilon > 0$ such that $\inf_{x\in[0,\varepsilon]} a(x) \geq \varepsilon$. Let us denote

$$d = \min\left\{\varepsilon, \inf_{x\in[0,1]}\{a(x) + b(x)\}\right\} > 0,$$

and let $\alpha \in C^1([0,1])$ be such that $0 \leq \alpha \leq a$ and

$$\begin{cases} \alpha(x) = 0 & \text{if } a(x) \leq \frac{d}{4}, \\ \alpha(x) = a(x) & \text{if } a(x) \geq \frac{d}{2}. \end{cases}$$

Lemma 3.1 *[26] The function* α *is not identically zero and satisfies*

$$\inf_{x\in[0,1]}\{\alpha(x) + b(x)\} \geq \frac{d}{2}.$$

Now, for $\epsilon_0 > 0$ we define the functions J and K by:

$$J(t) := \begin{cases} t & \text{if } h'' = 0 \text{ on } [0,l], \\ th'(\epsilon_0 t) & \text{if } h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0,l]. \end{cases} \tag{3.13}$$

$$K(t) = \int_t^1 \frac{1}{J(s)}\,ds. \tag{3.14}$$

To facilitate our calculations we introduce the following notations

$$(\phi * \psi)(t) := \int_0^t \phi(t-\tau)\,\psi(\tau)\,d\tau,$$

$$(\phi \diamond \psi)(t) := \int_0^t \phi(t-\tau)\,(\psi(t) - \psi(\tau))^2\,dxd\tau,$$

$$(g \vartriangle v)(t) := \int_0^1 \alpha(x)\int_0^t g(s)\,(v(t) - v(s))\,dsdx, \ \forall v \in L^2(0,1),$$

$$(g \circ \nu)(t) := \int_0^1 a(x)\int_0^t g(t-s)\,(\nu(t) - \nu(s))^2 dsdx, \ \forall \nu \in L^2(0,1).$$

Lemma 3.2 *[75] For any function $\phi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,1)$, we have*

$$\begin{aligned}(\phi * \psi)(t)\,\psi_t(t) &= -\frac{1}{2}\phi(t)\,|\psi(t)|^2 + \frac{1}{2}(\phi' \diamond \psi)(t)\\ &\quad -\frac{1}{2}\frac{d}{dt}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)\,d\tau\right)|\psi(t)|^2\right\}.\end{aligned}$$

Lemma 3.3 *[46] There exists a positive constant c such that*

$$(g \bigtriangleup v)^2 \leq cg \circ v_x, \;\; \forall v \in H_0^1(0,1).$$

The energy functional associated to (3.7)-(3.8), is

$$\begin{aligned}E(t,\varphi,\psi,\theta,q,z) &= \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \left(\beta - a(x)\int_0^t g(s)\,ds\right)\psi_x^2\right\}dx\\ &\quad +\frac{1}{2}\int_0^1 \left\{K(\varphi_x+\psi)^2 + \rho_3\theta^2 + \rho_4 q^2\right\}dx + \frac{1}{2}(g \circ \psi_x)\\ &\quad +\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx, \qquad (3.15)\end{aligned}$$

we denote $E(t) = E(t,\varphi,\psi,\theta,q,z)$ and $E(0) = E(0,\varphi_0,\psi_0,\theta_0,q_0,f_0)$ for simplicity of notations.

For state a local existence result, we introduce the vector function $\Phi = (\varphi,u,\psi,v,\theta,q,z)^T$, where $u = \varphi_t$ and $v = \psi_t$, using the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms for define the space $\mathcal{H}$ as follows

$$\mathcal{H} := \left[H_0^1(0,1) \times L^2(0,1)\right]^2 \times \left[L^2(0,1)\right]^2 \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).$$

Proposition 3.1 *Let $\Phi_0 = (\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,q_0,f_0)^T \in \mathcal{H}$ be given. Assume that* assumption 2.1 - assumption 2.4 *are satisfied, Problem (3.7)-(3.8) possesses then a unique global (weak) solution satisfying*

$$\Phi = (\varphi,u,\psi,v,\theta,q,z)^T \in C(\mathbb{R}_+;\mathcal{H}).$$

3.3 Technical Lemmas

In this section we establish several lemmas needed for the proof of our main result.

Lemma 3.4 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.7)-(3.8), then the energy E is decreasing function and satisfies, for all $t \geq 0$ and $\eta_0 > 0$,*

$$\begin{aligned}E'(t) &= -\delta\int_0^1 q^2dx - \frac{1}{2}g(t)\int_0^1 a(x)\,\psi_x^2dx - \int_0^1 b(x)\,\psi_t f(\psi_t)\,dx\\ &\quad +\frac{1}{2}(g' \circ \psi_x) - \mu_1\int_0^1 \varphi_t^2dx - \int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)\,ds\,dx\\ &\leq -\delta\int_0^1 q^2dx - \mu_3(t)\int_0^1 b(x)\,\psi_t f(\psi_t)\,dx + \frac{1}{2}(g' \circ \psi_x) - \eta_0\int_0^1 \varphi_t^2dx \leq 0, \qquad (3.16)\end{aligned}$$

Proof. Multiplying $(3.7)_1$,$(3.7)_2$,$(3.7)_3$,$(3.7)_4$,$(3.7)_5$ by φ_t, ψ_t, θ, q and $|\mu_2(s)|\, z$ respectively, and integrating over $(0,1)$, using integration by parts, Lemma (3.3), Young's and Cauchy–Schwarz inequalities we get (3.16). ■

Lemma 3.5 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.7)-(3.8). Then the functional*

$$\begin{aligned} F_1(t) \;:\; &= -\rho_2 \int_0^1 \alpha(x)\psi_t \int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx \\ &+\frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)\, q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx, \end{aligned}$$

satisfies

$$\begin{aligned} F_1'(t) \;\leq\; & -\left(\rho_2\int_0^t g(s)\,ds-\varepsilon_1\left(\rho_2^2+\int_0^t g(s)\,ds\right)\right)\int_0^1 \alpha(x)\psi_t^2 dx \\ &+\varepsilon_1' K^2\int_0^1(\varphi_x+\psi)^2 dx + c\varepsilon_1\int_0^1 b(x) f^2(\psi_t)dx \\ &+\varepsilon_1'\left(2\beta^2+1\right)\int_0^1 \psi_x^2 dx+\left(c\varepsilon_1+\frac{1}{\varepsilon_1}\int_0^t g(s)\,ds\right)\int_0^1 q^2 dx \\ &+c\left(\varepsilon_1'+\frac{1}{\varepsilon_1'}\right) g\circ\psi_x + c\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right) g\circ\psi_x-\frac{c}{\varepsilon_1} g'\circ\psi_x, \end{aligned} \tag{3.17}$$

for any $\varepsilon_1,\ \varepsilon_1'>0.$

Proof. For simplicity we write

$$F_1(t) := I_1(t)+I_2(t).$$

where

$$\begin{aligned} I_1(t) &= -\rho_2\int_0^1 \alpha(x)\psi_t\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx, \\ I_2(t) &= \frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)\, q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx. \end{aligned}$$

Differentiating I_1 gives

$$\begin{aligned} I_1'(t) \;=\; & -\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx \\ & -\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx \\ & -\int_0^1 \rho_2\alpha(x)\psi_t^2\int_0^t g(s)\,dsdx. \end{aligned} \tag{3.18}$$

Using $(3.7)_2$, we get

$$\begin{aligned}
&-\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
=&\int_0^1 \beta\alpha(x)\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\\
&+\int_0^t K\alpha(x)(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&-\int_0^1 \alpha(x)a(x)\left(\int_0^t g(t-s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx\\
&+\mu_3(t)\int_0^1 b(x)f(\psi_t)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\
&+\int_0^1 \alpha(x)\gamma\theta_x\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\
&+\int_0^1 \alpha'(x)\left(\beta\psi_x-a(x)\int_0^t g(s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx.
\end{aligned}\tag{3.19}$$

Next, by using Lemma 3.3, we have for any $\varepsilon_1>0$

$$\begin{aligned}
&-\int_0^1 \rho_2\alpha(x)\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\
\leq\;&\varepsilon_1\rho_2^2\int_0^1 \alpha(x)\psi_t^2dx-\frac{c}{\varepsilon_1}g'\circ\psi_x.
\end{aligned}\tag{3.20}$$

Also,

$$\begin{aligned}
I_2'(t)=\;&\frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)q_t\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&+\frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)q\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\
&+\frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)q\psi_t\int_0^t g(s)\,ds.
\end{aligned}$$

Using $(3.7)_4$, which gives

$$\begin{aligned}
I_2'(t)=\;&-\frac{\gamma\delta}{\kappa}\int_0^1 \alpha(x)q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&-\int_0^1 \alpha(x)\gamma\theta_x\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx\\
&+\frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)q\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\\
&+\frac{\gamma\rho_4}{\kappa}\left(\int_0^t g(s)\,ds\right)\int_0^1 \alpha(x)q\psi_tdx.
\end{aligned}\tag{3.21}$$

Similarly to (3.20), we treat the terms in the right-hand side of (3.19) as follows

$$\begin{aligned} & \int_0^1 \beta\alpha(x)\psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx \\ \leq\ & \varepsilon_1'\beta^2 \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_1'} g\circ\psi_x. \end{aligned} \tag{3.22}$$

Also,

$$\begin{aligned} & \int_0^t K\alpha(x)(\varphi_x+\psi) \int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx \\ \leq\ & \varepsilon_1' K^2 \int_0^1 (\varphi_x+\psi)^2 dx + \frac{c}{\varepsilon_1'} g\circ\psi_x. \end{aligned} \tag{3.23}$$

By the same method used in [46], we have these estimates

$$\begin{aligned} & -\int_0^1 \alpha(x)a(x)\left(\int_0^t g(s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx \\ \leq\ & \varepsilon_1' \int_0^1 \psi_x^2 dx + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g\circ\psi_x, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} & \mu_3(t)\int_0^1 b(x)f(\psi_t)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx \\ \leq\ & \varepsilon_1 c \int_0^1 b(x)f^2(\psi_t)dx + c\left(\varepsilon_1 + \frac{1}{\varepsilon_1}\right) g\circ\psi_x. \end{aligned} \tag{3.25}$$

Finally,

$$\begin{aligned} & \int_0^1 \alpha'(x)\left(\beta\psi_x - a(x)\int_0^t g(s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\right)dx \\ \leq\ & \varepsilon_1'\beta^2 \int_0^1 \psi_x^2 dx + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g\circ\psi_x. \end{aligned} \tag{3.26}$$

As in (3.20), we find easily that

$$\begin{aligned} & \frac{\gamma\rho_4}{\kappa}\int_0^1 \alpha(x)q\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx \\ \leq\ & \varepsilon_1 \int_0^1 q^2 dx - \frac{c}{\varepsilon_1} g'\circ\psi_x. \end{aligned} \tag{3.27}$$

Also, we estimate the first term in the right-hand side of (3.21) as follows

$$\begin{aligned} & -\frac{\gamma\delta}{\kappa}\int_0^1 \alpha(x)q\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx \\ \leq\ & \left(\frac{\gamma\delta}{\kappa}\right)^2 \varepsilon_1 \int_0^1 q^2 dx + \frac{c}{\varepsilon_1} g\circ\psi_x, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} & \frac{\gamma\rho_4}{\kappa}\left(\int_0^t g(s)\,ds\right)\int_0^1 \alpha(x)\,q\psi_t dx \\ \leq & \left(\int_0^t g(s)\,ds\right)\frac{1}{\varepsilon_1}\int_0^1 q^2 dx + \left(\int_0^t g(s)\,ds\right)c\varepsilon_1\int_0^1 \psi_t^2 dx. \end{aligned} \tag{3.29}$$

By combining the estimates (3.18)−(3.29), we complete the proof. ■

As in [48], we introduce the multiplier w which is the solution of

$$-w_{xx} = \psi_x, \qquad w(0) = w(1) = 0. \tag{3.30}$$

Lemma 3.6 *The solution of (3.30) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx, \tag{3.31}$$

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx. \tag{3.32}$$

Lemma 3.7 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.7)-(3.8). Then the functional*

$$F_2(t) := \rho_2\int_0^1 \psi_t\psi dx + \rho_1\int_0^1 \varphi_t w dx - \frac{\gamma\rho_4}{\kappa}\int_0^1 \psi q dx. \tag{3.33}$$

satisfies

$$\begin{aligned} F_2'(t) \leq & -\left(\beta + \frac{c\varepsilon_2\mu_1}{2} - \frac{\delta\gamma\varepsilon_2}{2\kappa}\right)\int_0^1 \psi_x^2 dx + \left(\frac{\rho_1}{2\varepsilon_2} + \frac{\mu_1}{2\varepsilon_2}\right)\int_0^1 \varphi_t^2 dx \\ & + \left(\rho_2 + \frac{\gamma\rho_4\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right)\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_2} g\circ\psi_x \\ & + \left(\frac{\gamma\rho_4}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\int_0^1 q^2 dx + \frac{c}{2\varepsilon_2}\int_0^1 b(x)\,f^2(\psi_t)\,dx \\ & + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\,ds\,dx, \end{aligned} \tag{3.34}$$

for any $\varepsilon_2 > 0$.

Proof. By taking the derivative of F_2, we get

$$\begin{aligned} F_2'(t) = & \underbrace{\int_0^1 \left(\rho_2\psi_{tt}\psi + \rho_2\psi_t^2\right)dx}_{:=J_1} + \underbrace{\int_0^1 \left(\rho_1\varphi_{tt}w + \rho_1\varphi_t w_t\right)dx}_{:=J_2} \\ & \underbrace{-\frac{\gamma\rho_4}{\kappa}\int_0^1 \left(\psi_t q + \psi q_t\right)dx.}_{:=J_3} \end{aligned} \tag{3.35}$$

Next, using $(3.7)_1$ and $(3.7)_4$, we obtain

$$\begin{aligned} J_2 + J_3 &= -K\int_0^1 \varphi\psi_x dx + K\int_0^1 w_x^2 dx + \rho_1\int_0^1 \varphi_t w_t dx \\ &\quad -\frac{\gamma\rho_4}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx + \gamma\int_0^1 \psi\theta_x dx \\ &\quad -\mu_1\int_0^1 \varphi_t w dx - \int_0^1 w\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx. \end{aligned} \tag{3.36}$$

Using $(3.7)_2$, we get also

$$\begin{aligned} J_1 &= -\beta\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x\int_0^t g(t-s)\, a(x)\, \psi_x(s)\, ds dx \\ &\quad -K\int_0^1 \psi^2 dx - K\int_0^1 \varphi_x\psi dx - \mu_3(t)\int_0^1 b(x)\,\psi f(\psi_t)\, dx - \int_0^1 \gamma\psi\theta_x dx. \end{aligned} \tag{3.37}$$

From (3.36), (3.37) and by using Lemma 3.6, we deduce

$$\begin{aligned} F_2'(t) &\leq -\mu_1\int_0^1 \varphi_t w dx + \rho_1\int_0^1 \varphi_t w_t dx - \frac{\gamma\rho_4}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx \\ &\quad -\beta\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx - \mu_3(t)\int_0^1 b(x)\,\psi f(\psi_t)\, dx \\ &\quad +\int_0^1 a(x)\,\psi_x\int_0^t g(t-s)\,\psi_x(s)\, ds dx - \int_0^1 w\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx. \end{aligned}$$

By exploiting the inequality

$$|ab| \leq \frac{\nu}{2}a^2 + \frac{1}{2\nu}b^2, \qquad a, b \in \mathbb{R},\ \nu > 0,$$

we easily find, for any $\varepsilon_2 > 0$,

$$\begin{aligned} F_2'(t) &\leq -\beta\int_0^1 \psi_x^2 dx + \frac{\mu}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w^2\right) + \frac{\rho_1}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w_t^2\right) dx \\ &\quad +\frac{\gamma\rho_4}{2\kappa}\int_0^1 \left(\varepsilon_2\psi_t^2 + \frac{1}{\varepsilon_2}q^2\right) dx + \frac{\delta\gamma}{2\kappa}\int_0^1 \left(\varepsilon_2\psi^2 + \frac{1}{\varepsilon_2}q^2\right) dx \\ &\quad -\int_0^1 w\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx + \rho_2\int_0^1 \psi_t^2 dx - \mu_3(t)\int_0^1 b(x)\,\psi f(\psi_t)\, dx \\ &\quad +\int_0^1 a(x)\,\psi_x\int_0^t g(t-s)\,\psi_x(s)\, ds dx. \end{aligned} \tag{3.38}$$

Now, we estimate the last three terms in the right-hand side of (3.38). by Young's, Cauchy-Schwartz and Poincaré inequalities, we arrive at

$$\left|\mu_3(t)\int_0^1 b(x)\,\psi f(\psi_t)\, dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{c}{2\varepsilon_2}\int_0^1 b(x)\, f^2(\psi_t)\, dx. \tag{3.39}$$

$$\left|\int_0^1 a(x)\psi_x\int_0^t g(t-s)\psi_x(s)\,dsdx\right| \le \varepsilon_2 c\int_0^1 \psi_x^2dx + \frac{c}{\varepsilon_2}g\circ\psi_x, \tag{3.40}$$

$$\begin{aligned} &-\int_0^1 w\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx \\ \le\ & \frac{c\varepsilon_2\mu_1}{2}\int_0^1 \psi_x^2dx + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)dsdx. \end{aligned} \tag{3.41}$$

Then, plugging (3.39) and (3.41) into (3.38) and using (3.32) completes the proof. ■

Lemma 3.8 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.7)-(3.8). Then the functional*

$$F_3(t) := \rho_1\int_0^1 \varphi_t\left(-\varphi+\int_0^x \psi(y)dy\right)dx. \tag{3.42}$$

satisfies

$$\begin{aligned} F_3'(t) \ \le\ & -\frac{K}{2}\int_0^1(\varphi_x+\psi)^2dx + \varepsilon_3\int_0^1 \psi_t^2dx + c(1+\frac{1}{\varepsilon_3})\int_0^1 \varphi_t^2dx \\ & +\frac{\mu_1}{K}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,s,t)dsdx, \end{aligned} \tag{3.43}$$

for any $\varepsilon_3 > 0$.

Proof. Taking the derivative of (3.42), exploiting $(3.7)_1$ and integrating by parts, we obtain

$$\begin{aligned} F_3'(t) \ =\ & \rho_1\int_0^1 \varphi_t\left(\int_0^x \psi_t(y)dy\right)dx - \int_0^1\left(\varphi+\int_0^x\psi(y)dy\right)\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx \\ & -K\int_0^1(\varphi_x+\psi)^2dx + \rho_1\int_0^1\varphi_t^2dx - \mu_1\int_0^1\varphi_t\left(\varphi+\int_0^x\psi(y)dy\right)dx. \end{aligned} \tag{3.44}$$

Using Young's, Poincaré and Cauchy-Schwarz inequalities and the fact that

$$-\mu_1\int_0^1\varphi_t\left(\varphi+\int_0^x\psi(y)dy\right)dx \le \frac{K}{4}\int_0^1(\varphi_x+\psi)^2dx + c\int_0^1\varphi_t^2dx,$$

then lead to Estimate (3.43). ■

Lemma 3.9 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (3.7)-(3.8). Then the functional*

$$F_4(t) := -\rho_3\rho_4\int_0^1 q\left(\int_0^x\theta(y)dy\right)dx, \tag{3.45}$$

satisfies

$$\begin{aligned} F_4'(t) \ \le\ & \left(-\rho_3\kappa+\frac{\varepsilon_4\rho_3\delta c}{2}\right)\int_0^1\theta^2dx + \frac{\varepsilon_4\rho_4\gamma}{2}\int_0^1\psi_t^2dx \\ & +\left(\rho_4\kappa+\frac{\rho_3\delta}{2\varepsilon_4}+\frac{\rho_4\gamma}{2\varepsilon_4}\right)\int_0^1 q^2dx, \end{aligned} \tag{3.46}$$

for any $\varepsilon_4 > 0$.

Proof. Taking the derivative of (3.45) and using $(3.7)_3$, $(3.7)_4$, integration by parts and Young's inequality, we obtain (3.46). ■

Lemma 3.10 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (3.7)-(3.8). Then, for $\eta_1 > 0$, the functional*

$$F_5(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \tag{3.47}$$

satisfies

$$\begin{aligned} F_5'(t) &\le -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \\ &\quad -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \mu_1 \int_0^1 \varphi_t^2 dx. \end{aligned} \tag{3.48}$$

Proof. Differentiating (3.47) and using $(3.7)_5$, the fact that $z(x, 0, s, t) = \varphi_t$ and $e^{-s} \le e^{-s\rho} \le 1$ we get for all $\rho \in [0, 1]$

$$\begin{aligned} F_5'(t) &\le \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 \varphi_t^2 dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Since $s \mapsto -e^{-s}$ is an increasing function, we have $-e^{-s} \le -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$. Finally, setting $\eta_1 = -e^{-\tau_2}$ and recalling (3.9), we obtain (3.48). ■

3.4 Proof of the stability result

In this section we prove our stability result. First, we define a Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(t) := NE(t) + N_1 F_1(t) + F_2(t) + F_3(t) + F_4(t) + N_2 F_5(t) \tag{3.49}$$

where N_1, N_2 and N are positive real numbers to be chosen appropriately later.

Lemma 3.11 *For N sufficiently large we have*

$$\mathcal{L}(t) \sim E(t). \tag{3.50}$$

Proof. Let $\mathcal{L}(t) := N_1 F_1(t) + F_2(t) + F_3(t) + F_4(t) + N_2 F_5(t)$,
we get

$$|\mathcal{L}(t)| \le \rho_2 N_1 \int_0^1 \left| \alpha(x) \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s)) ds \right| dx$$

$$
\begin{aligned}
&+N_1\frac{\gamma\rho_4}{\kappa}\int_0^1\left|\alpha(x)\,q\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)ds\right|dx\\
&+\rho_2\int_0^1|\psi_t\psi|\,dx+\rho_1\int_0^1|\varphi_t w|\,dx+\frac{\gamma\rho_4}{\kappa}\int_0^1|\psi q|\,dx\\
&+\rho_1\int_0^1\left|\varphi_t\left(\varphi+\int_0^x\psi(y)dy\right)\right|dx+\rho_3\rho_4\int_0^1\left|q\left(\int_0^x\theta(y)dy\right)\right|dx\\
&+N_2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|e^{-s\rho}\mu_2(s)\right|z^2(x,\rho,s,t)dsd\rho dx \qquad (3.51)
\end{aligned}
$$

By the same techniques used in the proof of Lemma (3.5), we easily estimate the first and the second term in the right-hand side of (3.51). Exploiting Young's, Poincaré and Cauchy-Schwarz inequalities, (3.15) and the fact that $e^{-s\rho}\leq 1$ for all $\rho\in[0,1]$, we obtain

$$
\begin{aligned}
|\mathcal{L}(t)| &\leq c\int_0^1\left[\varphi_t^2+\psi_t^2+\psi_x^2+(\varphi_x+\psi)^2+\theta^2+q^2\right]dx\\
&\quad +cg\circ\psi_x+c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2(s)\right|z^2(x,\rho,s,t)dsd\rho dx\\
&\leq cE(t).
\end{aligned}
$$

Thus, $|\mathcal{L}(t)-NE(t)|\leq cE(t)$, which yields

$$
(N-c)\,E(t)\leq\mathcal{L}(t)\leq(N+c)\,E(t).
$$

Choosing N large enough, then there exist two positive constants β_1 and β_2 such that

$$
\beta_1E(t)\leq\mathcal{L}(t)\leq\beta_2E(t).
$$

This completes the proof. ■

Theorem 3.1 *Let* $(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,q_0,f_0)^T\in\mathcal{H}$ *be given. Assume that* assumption 2.1 - assumption 2.6 *are satisfied, then there exist* c_1,$c_2>0$ *for which the (weak) solution of problem (3.7)-(3.8) satisfies*

$$
E(t)\leq c_1K^{-1}\left(c_2\int_0^t(\mu_3\eta)(s)\,ds\right),\ \forall t\geq 0, \qquad (3.52)
$$

where $\eta=1$ *if* $a=0$.

1. *The last result is satisfied regardless of the speeds of wave propagation.*
2. *Since* $\lim_{t\to 0^+}K(t)=+\infty$, *then for* $\int_0^{+\infty}(\mu_3\eta)(s)\,ds=+\infty$ *we have the strong stability of (3.7)-(3.8), that is*

$$
\lim_{t\to+\infty}E(t)=0.
$$

3.4. Proof of the stability result

Proof. By combining (3.16), (3.17), (3.34), (3.43) and (3.46), using (3.11) and (3.12), we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -N_1 \left\{\rho_2 g_0 - \varepsilon_1 \left(\rho_2^2 + g_1\right)\right\} \int_0^1 \left(\alpha\left(x\right) + b\left(x\right)\right) \psi_t^2 dx \\
& + \left\{\rho_2 + \frac{\gamma\rho_4\varepsilon_2}{2\kappa} + \frac{\rho_4\gamma\varepsilon_4 + \rho_1\varepsilon_2}{2} + \varepsilon_3\right\} \int_0^1 \psi_t^2 dx - N\mu_3(t) \int_0^1 b\left(x\right) \psi_t f\left(\psi_t\right) dx \\
& + \left\{\frac{\mu_1 + \rho_1}{2\varepsilon_2} + c\left(1 + \frac{1}{\varepsilon_3}\right) + N_2\mu_1 - N\eta_0\right\} \int_0^1 \varphi_t^2 dx \\
& + \left\{N_1 c\varepsilon_1 + \frac{c}{2\varepsilon_2}\right\} \int_0^1 b(x) f^2(\psi_t) dx + N_1 \left\{\rho_2 g_0 - \varepsilon_1 \left(\rho_2^2 + g_1\right)\right\} \int_0^1 b\left(x\right) \psi_t^2 dx \\
& + \left\{N_1 \varepsilon_1' \left(2\beta^2 + 1\right) - \left(\beta + \frac{c\varepsilon_2\mu_1}{2} + \frac{c\delta\gamma\varepsilon_2}{2\kappa}\right)\right\} \int_0^1 \psi_x^2 dx \\
& + \left\{cN_1\left(\varepsilon_1 + \frac{1}{\varepsilon_1}\right) + cN_1\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) + \frac{c}{\varepsilon_2}\right\} g \circ \psi_x + \left(\frac{N}{2} - \frac{cN_1}{\varepsilon_1}\right) g' \circ \psi_x \\
& + \left\{N_1\left(c\varepsilon_1 + \frac{g_1}{\varepsilon_1}\right) + \frac{\delta\gamma + \gamma\rho_4}{2\kappa\varepsilon_2} + \frac{\rho_4\gamma + \rho_3\delta}{2\varepsilon_4} + \rho_4\kappa - \delta N\right\} \int_0^1 q^2 dx \\
& + \left\{\frac{1}{2\varepsilon_2} + \frac{\mu_1}{K} - N_2\eta_1\right\} \int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| z^2(x,1,s,t) ds dx \\
& + \left\{N_1\varepsilon_1' K^2 - \frac{K}{2}\right\} \int_0^1 (\varphi_x + \psi)^2 dx + \left\{-\rho_3\kappa + \frac{\varepsilon_4\rho_3\delta c}{2}\right\} \int_0^1 \theta^2 dx \\
& - N_2\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z^2(x,\rho,s,t) ds d\rho dx. \qquad (3.53)
\end{aligned}$$

for all $t \geq t_0$.

Now, we have to choose our constants very carefully. First, let us take $\varepsilon_2 = 1$, ε_1 and ε_4 small enough such that

$$\varepsilon_1 < \frac{\rho_2 g_0}{\rho_2^2 + g_1}, \ \varepsilon_4 < \frac{2\kappa}{\delta c}.$$

Next, taking $\varepsilon_3 = \left(\tau_0\gamma\varepsilon_4 + \rho_1\right)/2$, using Lemma 3.1, and choosing N_1, N_2 large enough such that

$$N_1\left(\rho_2 g_0 - \varepsilon_1\left(\rho_2^2 + g_1\right)\right) > \left(\rho_1 + \rho_2 + \rho_4\gamma\varepsilon_4 + \frac{\gamma\rho_4}{2k}\right)\frac{2}{d}, \ N_2 > \frac{1}{2\eta_1} + \frac{\mu_1}{K\eta_1},$$

then, we can select ε_1' so small such that

$$\varepsilon_1' < \min\left\{\frac{1}{2N_1 K}, \ \left(\beta + \frac{c\mu_1}{2} + \frac{c\gamma\delta}{2k}\right) / \ N_1\left(2\beta^2 + 1\right)\right\}.$$

Finally, we choose N large enough (even larger so that (3.50) remains valid) and

$$\frac{\mu_1 + \rho_1}{2} + c\left(1 + \frac{1}{\varepsilon_3}\right) + N_2\mu_1 - N\eta_0 < 0, \ \ \frac{cN_1}{\varepsilon_1} - \frac{N}{2} < 0,$$

$$N_1\left(c\varepsilon_1+\frac{g_1}{\varepsilon_1}\right)+\frac{\delta\gamma+\gamma\rho_4}{2\kappa}+\frac{\rho_4\gamma+\rho_3\delta}{2\varepsilon_4}+\rho_4\kappa-\delta N<0.$$

Therefore, (3.53) becomes

$$\begin{aligned}\mathcal{L}'(t) \leq & -c\int_0^1\left[\varphi_t^2+(\alpha(x)+b(x))\psi_t^2+\psi_x^2+(\varphi_x+\psi)^2+\theta^2+q^2\right]dx\\ & -c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|z^2(x,\rho,s,t)dsd\rho dx+cg\circ\psi_x\\ & +c\int_0^1 b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx.\end{aligned}$$

By using lemma (3.1) and estimate (3.15) then lead to

$$\mathcal{L}'(t)\leq -cE(t)+cg\circ\psi_x+c\int_0^1 b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx,\ \forall t\geq t_0. \tag{3.54}$$

Let us define the following sets

$$\Sigma_\psi=\{x\in(0,1):|\psi_t(x,t)|>l\}\ \text{ and }\ \Theta_\psi=(0,1)\backslash\Sigma_\psi.$$

We work now for estimate the last term in the right-hand side of (3.54) First, note that

$$\begin{aligned}\int_0^1 b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx = & \int_{\Sigma_\psi} b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx\\ & +\int_{\Theta_\psi} b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx\end{aligned}$$

Using ***assumption*** 2.3 and (3.16), we easily show that

$$\begin{aligned}\mu_3(t)\int_{\Sigma_\psi} b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx \leq & \left(k_1^{-1}+k_2\right)\int_{\Sigma_\psi}\mu_3(t)b(x)\psi_t f(\psi_t)dx\\ \leq & \left(k_1^{-1}+k_2\right)\int_0^1\mu_3(t)b(x)\psi_t f(\psi_t)dx\\ \leq & -cE'(t).\end{aligned} \tag{3.55}$$

If $h''=0$ on $[0,l]$: This implies that there exist $k_1',k_2'>0$ such that $k_1's^2\leq f(s)s\leq k_2's^2$ for all $s\in\mathbb{R}_+$, and then (3.55) is also satisfied for $|\psi_t(x,t)|\leq l$, then on all $(0,1)$. From (3.54), (3.55) and the fact that $\mu_3'\leq 0$, we arrive at

$$(\mu_3(t)\mathcal{L}(t)+cE(t))'\leq -c\mu_3(t)J(E(t))+cg\circ\psi_x,\quad \forall t\geq t_0, \tag{3.56}$$

where J is defined in (3.13).

If $h'(0)=0$ and $h''>0$ on $(0,l]$: Since h is convex and increasing, h^{-1} is concave and increasing, by using ***assumption*** 2.3, the reversed Jensen's inequality for concave function (see [87], p. 61), and (3.16), we obtain,

$$\begin{aligned}
\mu_3(t)\int_{\Theta_\psi} b(x)\left(\psi_t^2+f^2(\psi_t)\right)dx &\leq \mu_3(t)\int_{\Theta_\psi} b(x)\,h^{-1}(\psi_t f(\psi_t))\,dx \\
&\leq \mu_3(t)\int_{\Theta_\psi} h^{-1}(b(x)\,\psi_t f(\psi_t))\,dx \\
&\leq \mu_3(t)\,|\Theta_\psi|\,h^{-1}\left(\int_{\Theta_\psi}\frac{1}{|\Theta_\psi|}b(x)\,\psi_t f(\psi_t)\,dx\right) \\
&\leq c\mu_3(t)h^{-1}\left(\int_{\Theta_\psi} b(x)\,\psi_t f(\psi_t)\,dx\right) \\
&\leq c\mu_3(t)h^{-1}\left(\int_0^1 b(x)\,\psi_t f(\psi_t)\right)dx \\
&\leq c\mu_3(t)h^{-1}(-cE'(t)) . \qquad (3.57)
\end{aligned}$$

Therefore, from (3.54), (3.55) and (3.57), we find that

$$\mu_3(t)\mathcal{L}'(t) \leq -c\mu_3(t)E(t) + c\mu_3(t)h^{-1}(-cE'(t)) - cE'(t) + cg\circ\psi_x, \quad \forall t\geq t_0.$$

By using Young's inequality (2.8) and the fact that

$$h^*(p) \leq p[h']^{-1}(p),\ \ E'\leq 0,\ \ h''>0 \text{ and } \mu_3'\leq 0,$$

we obtain for $\varepsilon_0>0$ small enough and $c_0>0$ large enough,

$$\begin{aligned}
& [h'(\varepsilon_0 E(t))\,[\mu_3(t)\mathcal{L}(t)+cE(t)]+c_0E(t)]' \\
={}& \varepsilon_0 E'(t)\,h''(\varepsilon_0 E(t))\,[\mu_3(t)\mathcal{L}(t)+cE(t)] \\
& +h'(\varepsilon_0 E(t))\,[\mu_3(t)\mathcal{L}'(t)+\mu_3'(t)\mathcal{L}(t)+cE'(t)]+c_0E'(t) \\
\leq{}& -c\mu_3(t)h'(\varepsilon_0 E(t))\,E(t)+c\mu_3(t)h'(\varepsilon_0 E(t))\,h^{-1}(-cE'(t)) \\
& +c_0E'(t)+ch'(\varepsilon_0 E(t))\,g\circ\psi_x \\
\leq{}& -c\mu_3(t)h'(\varepsilon_0 E(t))\,E(t)+c\mu_3(t)h^*(h'(\varepsilon_0 E(t)))-cE'(t) \\
& +c_0E'(t)+ch'(\varepsilon_0 E(0))\,g\circ\psi_x \\
\leq{}& -c\mu_3(t)h'(\varepsilon_0 E(t))\,E(t)+c\varepsilon_0\mu_3(t)h'(\varepsilon_0 E(t))\,E(t)+cg\circ\psi_x \\
\leq{}& -c\mu_3(t)h'(\varepsilon_0 E(t))\,E(t)+cg\circ\psi_x = -c\mu_3(t)J(E(t))+cg\circ\psi_x. \qquad (3.58)
\end{aligned}$$

Now, let us define the following functional:

$$\mathcal{F}(t)=\begin{cases}\mu_3(t)\mathcal{L}(t)+cE(t) & \text{if } h''=0 \text{ on } [0,l]\,, \\ h'(\varepsilon_0 E(t))\,[\mu_3(t)\mathcal{L}(t)+cE(t)]+c_0E(t) & \text{if } h'(0)=0 \text{ and } h''>0 \text{ on } (0,l]\,.\end{cases}$$

Using (3.50), we have

$$\mathcal{F}\sim E,$$

and exploiting (3.56) and (3.58), we easily deduce that

$$\mathcal{F}'(t) \leq -c\mu_3(t) J(E(t)) + cg \circ \psi_x, \quad \forall t \geq t_0.$$

By using (3.16) and ***assumption*** 2.5, we obtain

$$\begin{aligned}
(\eta(t)\mathcal{F}(t))' &= \eta'(t)\mathcal{F}(t) + \eta(t)\mathcal{F}'(t) \\
&\leq -c\mu_3(t)\eta(t) J(E(t)) + c\eta(t) g \circ \psi_x \\
&\leq -c\mu_3(t)\eta(t) J(E(t)) + c(\eta g) \circ \psi_x \\
&\leq -c\mu_3(t)\eta(t) J(E(t)) - cg' \circ \psi_x \\
&\leq -c\mu_3(t)\eta(t) J(E(t)) - cE'(t).
\end{aligned}$$

Next, let

$$\mathcal{R}(t) = \varepsilon(\eta(t)\mathcal{F}(t) + cE(t)),$$

where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\eta(t)\mathcal{F}(t) + cE(t) \leq \frac{1}{\bar{\varepsilon}} E(t), \quad \forall t \geq 0.$$

We also have

$$\mathcal{R} \sim E, \tag{3.59}$$

and for $t \geq t_0$

$$\mathcal{R}'(t) \leq -c\varepsilon\mu_3(t)\eta(t) J(\mathcal{R}(t)). \tag{3.60}$$

Noting that $K' = -1/J$ (see (3.14)), we get from (3.60)

$$\mathcal{R}'(t) K'(\mathcal{R}(t)) \geq c\varepsilon\mu_3(t)\eta(t), \quad \forall t \geq t_0,$$

A simple integration over (t_0, t) then yields

$$K(\mathcal{R}(t)) \geq K(\mathcal{R}(t_0)) + c\varepsilon \int_0^t \mu_3(s)\eta(s)\, ds - c\varepsilon \int_0^{t_0} \mu_3(s)\eta(s)\, ds,$$

On the other hand, since $\lim_{t \to 0^+} K(t) = +\infty$ and

$$0 \leq \mathcal{R}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(0),$$

we obtain for ε small enough

$$K(\mathcal{R}(t_0)) - c\varepsilon \int_0^{t_0} \mu_3(s)\eta(s)\, ds > 0.$$

Then, thanks to the fact that K^{-1} is decreasing , we infer

$$\begin{aligned}
\mathcal{R}(t) &\leq K^{-1}\left(K(\mathcal{R}(t_0)) + c\varepsilon \int_0^t \mu_3(s)\eta(s)\, ds - c\varepsilon \int_0^{t_0} \mu_3(s)\eta(s)\, ds\right) \\
&\leq K^{-1}\left(c\varepsilon \int_0^t (\mu_3\eta)(s)\, ds\right).
\end{aligned}$$

From this end inequality and (3.59) we get easily (3.52). Then the proof is completed. ■

3.4. Proof of the stability result

CHAPTER 4

GLOBAL EXISTENCE AND GENERAL DECAY OF A WEAKLY NON-LINEAR DAMPED TIMOSHENKO SYSTEM OF THERMOELASTICITY OF TYPE III WITH INFINITE MEMORY

4.1 Introduction

Consider the following weakly nonlinear damped Timoshenko-type system for thermoelasticity of type III with infinite memory:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \beta\theta_{tx} + \alpha\left(t\right) f\left(\varphi_t\right) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \int_0^\infty g\left(s\right)\psi_{xx}\left(t-s\right) ds - \beta\theta_t = 0 \\ \rho_3\theta_{tt} - \delta\theta_{xx} - k\theta_{xxt} + \gamma\varphi_{tx} + \gamma\psi_t + \alpha\left(t\right) f\left(\theta_t\right) = 0, \end{cases} \tag{4.1}$$

where $t \in (0, +\infty)$ denotes the time variable and $x \in (0,1)$ is the space variable. Here φ, ψ and θ are respectively the transverse displacement of the beam, the rotation angle and the difference temperature. α, f, and g are specific functions satisfying some conditions to be determined later. $\alpha\left(t\right) f\left(\varphi_t\right)$ and $\alpha\left(t\right) f\left(\theta_t\right)$ are the weak nonlinear dissipative terms, the infinite integral depending on g represents the infinite memory term. $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0$, θ_1 are initial data. The coefficients, $\rho_1, \rho_2, \rho_3, \gamma, \delta, \beta, b$ and k are positive constants.

With the following initial and boundary conditions:

$$\begin{aligned} &\varphi\left(x,0\right) = \varphi_0\left(x\right),\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right),\ \theta\left(x,0\right) = \theta_0\left(x\right),\ \text{in } (0,1), \\ &\psi\left(x,0\right) = \psi_0\left(x\right), \psi_t\left(x,0\right) = \psi_1\left(x\right),\ \theta_t\left(x,0\right) = \theta_1\left(x\right) \text{ in } (0,1), \\ &\varphi\left(0,t\right) = \varphi\left(1,t\right) = \psi\left(0,t\right) = \psi\left(1,t\right) = \theta\left(0,t\right) = \theta\left(1,t\right) = 0 \text{ in } (0,\infty). \end{aligned} \tag{4.2}$$

There is a large number of publications concerning the stabilization of Timoshenko systems with dissipative mechanisms of several types, such as viscoelastic or memory type, feedback and control forces (e.g. [26, 35, 40, 48, 57, 58, 71]). In the context of asymptotic stabilization with nonlinear feedback damping, first results are given in [1] where the author studies the asymptotic behavior of the system governing the non-linear vibrations of a Timoshenko beam,

$$\begin{cases} u_{tt} - \alpha\beta\left(u_x - v\right)_x - \gamma\left\|u_x\right\|^2 u_{xx} + g\left(u_t\right) = 0, \\ \frac{1}{\alpha}v_{tt} - v_{xx} - \alpha\beta\left(u_x - v\right) + g\left(v_t\right) = 0, \end{cases} \tag{4.3}$$

where $g : R \to R$ is a C^1-class, non-increasing function with $g(0) = 0$ and satisfying

$$\begin{aligned} c_1\left|x\right|^r \leq g\left(x\right) \leq c_2\left|x\right|^{1/r} \quad &\text{for} \ \ |x| \leq 1, \\ c_3\left|x\right|^k \leq g\left(x\right) \leq c_4\left|x\right|^s \quad &\text{for} \ \ |x| > 1. \end{aligned}$$

Alabau-Boussouira [4] considered the asymptotic behavior for the following Timoshenko system with a single nonlinear damping:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - bv_{xx} + k\left(\varphi_x + \psi\right) + \alpha\left(\psi_t\right) = 0. \end{cases} \tag{4.4}$$

He established a general semi-explicit formula for the decay rate of the energy at infinity in the case of the same speed of propagation in the two equations of the system $\left(\text{i.e. } \frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. Park and Kang [80] concerned with the decay property of the solutions for Timoshenko beam with a weak non-linear dissipation

$$\begin{cases} u_{tt} - \left(u_x + v\right)_x + \sigma\left(t\right)g\left(u_t\right) = 0, \\ v_{tt} - v_{xx} + \left(u_x + v\right) + \sigma\left(t\right)g\left(v_t\right) = 0. \end{cases} \tag{4.5}$$

Without assuming equal speeds of propagation of waves, Cavalcanti et al. [25] considered the Timoshenko model for vibrating beams under effect of two nonlinear and localized frictional damping mechanisms

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \alpha_1\left(x\right)g_1\left(\varphi_t\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \alpha_2\left(x\right)g_2\left(\psi_t\right) = 0. \end{cases} \tag{4.6}$$

They proved that the damping placed on an arbitrarily small support, unquantitized at the origin, leads to uniform decay rates (asymptotic in time) for the energy function. Feng and Yang [37] studied the nonlinear Timoshenko system with a time delay term in the internal feedback

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = h\left(x\right), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \mu_1\psi_t + \mu_2\psi_t(x, t-\tau) + f\left(\psi\right) = g\left(x\right). \end{cases} \tag{4.7}$$

Under some suitable assumptions on the weights of feedback, the authors established the existence of a global attractor with finite fractal dimension for the case of equal speed wave propagation, as well as the existence of exponential attractors.

For Timoshenko systems in classical thermoelasticity of type III, Djebabla and Tatar [33] considered the system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta_x = 0, \\ \rho_3\theta_{tt} - l\theta_{xx} + \gamma\psi_{ttx} + \beta\int_0^t g\left(t-s\right)\theta_{xx}\left(x,s\right)ds = 0, \\ \varphi\left(x,0\right) = \varphi_0\left(x\right),\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right),\ \psi\left(x,0\right) = \psi_0\left(x\right),\ \psi_t\left(x,0\right) = \psi_1\left(x\right),\ \text{in }\ (0,1), \\ \varphi\left(0,t\right) = \varphi\left(1,t\right) = \psi\left(0,t\right) = \psi\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right)\ \text{ in }\ (0,\infty). \end{cases} \tag{4.8}$$

and proved, under suitable conditions on its coefficients, and for g decaying exponentially, that the energy function also decays exponentially. Messaoudi and Fareh [69, 70] discussed a similar system of porous-thermoelasticity with viscoelastic damping term in the second equation of the form $\int_0^t g\left(s\right)\psi_{xx}\left(x,t-s\right)ds$. They established some general decay results for the solutions in the case of equal wave speeds $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$ as well as for different speeds of wave propagation $\left(\frac{k}{\rho_1} \neq \frac{b}{\rho_2}\right)$. Kafini [56] improved the result of Djebabla and Tatar [33] with more general relaxation functions. He proved, under the same conditions on the coefficients, a general decay result, from which the usual exponential and polynomial decays are only special cases. We refer the interested readers to [3, 12, 14, 15, 17] and references therein for details discussion on problems with weak or strong non-linear dissipation.

Motivated by in works of Cavalcanti et al., Kafini and Messaoudi and Fareh mentioned above, we investigate (4.1) under suitable conditions and establish the well-posedness of the problem using semigroups theory, as well as the energy decay of solution which depends on α, f and g by using the multiplier method with some properties of convex functions. Our purpose in the present manuscript is to obtain general decay rate estimates of the energy for the thermoelastic Timoshenko system with infinite memory subjected to a weakly nonlinear damping placed in first and third equations, but without any restriction or relation on the coefficients. We prove our result then regardless of equal speeds of propagation of waves and no growth assumption on f at the origin.

4.2 Preliminaries and well-posedness result

In this section, we present some materials needed in the proof of our results. We also state, with proof, an existence and uniqueness result for problem (4.1). The proof is established by using semi-group method.

First, to facilitate our calculations we introduce the following notation

$$\left(g\circ\nu\right)\left(t\right) := \int_0^1\int_0^\infty g\left(s\right)\left(\nu\left(x,t\right) - \nu\left(x,t-s\right)\right)^2 ds dx,\ \forall\nu\in L^2(0,1).$$

It is easy to obtain the following inequalities, we omit their proofs.

Lemma 4.1 *[48] The following inequalities hold,*

$$\begin{aligned}
\int_0^1 \left(\int_0^\infty g(s)\psi_x(t-s)ds\right)^2 dx &\leq 2g_0(g\circ\psi_x)(t)+2g_0\int_0^1 \psi_x^2 dx,\\
\int_0^1 \left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\right)^2 dx &\leq g_0(g\circ\psi_x)(t),\\
\int_0^1 \left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\right)^2 dx &\leq d_1(g\circ\psi_x)(t),\\
\int_0^1 \left(\int_0^\infty g'(s)(\psi(t)-\psi(t-s))ds\right)^2 dx &\leq -d_2(g'\circ\psi_x)(t),\\
\int_0^1 \left(\int_0^\infty g'(s)(\psi_x(t)-\psi_x(t-s))ds\right)^2 dx &\leq -g(0)(g'\circ\psi_x)(t),
\end{aligned}$$

where d_1 and d_2 are positive constants.

The energy functional associated to (4.1)-(4.2), is

$$\begin{aligned}
E(t) \quad &:= E\left(t,\varphi,\psi,\theta\right)\\
&= \frac{\gamma}{2}\int_0^1 \left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+k(\varphi_x+\psi)^2+\left(b-\int_0^\infty g(s)ds\right)\psi_x^2\right\}dx\\
&\quad +\frac{\gamma}{2}(g\circ\psi_x)(t)+\frac{\beta}{2}\int_0^1\left\{\rho_3\theta_t^2+\delta\theta_x^2\right\}dx, \qquad (4.9)
\end{aligned}$$

we denote $E(t)=E\left(t,\varphi,\psi,\theta\right)$ and $E(0)=E\left(0,\varphi_0,\psi_0,\theta_0\right)$ for simplicity of notations. Then the energy E is decreasing function and satisfies, for all $t\geq 0$,

$$\begin{aligned}
E'(t) &= -k\beta\int_0^1 \theta_{tx}^2 dx-\alpha(t)\left\{\gamma\int_0^1 \varphi_t f\left(\varphi_t\right)dx+\beta\int_0^1 \theta_t f\left(\theta_t\right)dx\right\}\\
&\quad +\frac{\gamma}{2}\left(g'\circ\psi_x\right)(t)\leq 0.
\end{aligned}$$

To obtain precise decay rates of $E(t)$ as $t\to+\infty$, we consider the following assumptions:

$\mathbf{A}_1$. $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous and non-decreasing function such that there exist positive constants k_1 and l and a convex, continuous and increasing function $h:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1\left(\mathbb{R}_+\right)\cap C^2\left(]0,+\infty[\right)$ satisfying: $h\left(0\right)=0$ and

$$h''=0 \text{ on } [0,l], \qquad (4.10)$$

or

$$h'\left(0\right)=0 \text{ and } h''>0 \text{ on } (0,l], \qquad (4.11)$$

such that

$$h\left(f^{2}\left(s\right)\right) \leq f\left(s\right)s \quad \text{for } |s| \leq l,$$
$$|f\left(s\right)| \leq k_{1}\left|s\right| \quad \text{for } |s| \geq l.$$

$\mathbf{A}_2$. $g:\mathbb{R}_{+}\to\mathbb{R}_{+}$ is a differentiable function such that

$$g\left(0\right)>0,\ \int_{0}^{\infty}g\left(s\right)ds=g_{0},\ \ell=b-g_{0}>0.$$

$\mathbf{A}_3$. There exists a non-increasing differentiable function $\eta:\mathbb{R}_{+}\to\mathbb{R}_{+}$ satisfying

$$g'\left(s\right)\leq-\eta\left(s\right)g\left(s\right),\ \text{for } s\geq0.$$

$\mathbf{A}_4$. $\alpha:\mathbb{R}_{+}\to]0,+\infty[$ is a non-increasing C^1-function satisfying

$$\int_{0}^{\infty}\alpha\left(s\right)ds=+\infty.$$

Next, we take the following notation

$$\eta^{t}(x,s)=\psi(x,t)-\psi(x,t-s),\quad t\in\mathbb{R}_{+},\ (x,s)\in(0,1)\times\mathbb{R}_{+},\tag{4.12}$$

let

$$\eta_{0}(x,s):=\eta^{0}(x,s)=\psi_{0}(x,0)-\psi_{0}(x,s),\quad (x,s)\in(0,1)\times\mathbb{R}_{+}.$$

η^t is the relative history of ψ, we have

$$\begin{aligned}\eta_{t}^{t}+\eta_{s}^{t}-\psi_{t} &= 0,\quad (x,t,s)\in(0,1)\times\mathbb{R}_{+}\times\mathbb{R}_{+},\\ \eta^{t}(0,s) &= \eta^{t}(1,s)=0,\quad (t,s)\in\mathbb{R}_{+}\times\mathbb{R}_{+},\\ \eta^{t}(x,0) &= 0,\quad (x,t)\in(0,1)\times\mathbb{R}_{+}.\end{aligned}\tag{4.13}$$

Then the second equation of (4.1) becomes

$$\rho_{2}\psi_{tt}-b\psi_{xx}+k(\varphi_{x}+\psi)+g_{0}\psi_{xx}(x,t)-\int_{0}^{\infty}g(s)\eta_{xx}^{t}(x,s)ds-\beta\theta_{t}=0.$$

For state an existence result, we set $u=\varphi_t, v=\psi_t$ and $\omega=\theta_t$ and introduce the vector function $\Phi=(\varphi,u,\psi,v,\theta,\omega,\eta^t)^T$. Using the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms for define the space H as follows

$$\mathcal{H}:=\left[H_{0}^{1}(0,1)\times L^{2}(0,1)\right]^{3}\times L_{g},$$

where

$$L_{g}=\left\{w:\mathbb{R}_{+}\to H_{0}^{1}(0,1),\ \int_{0}^{1}\int_{0}^{\infty}g(s)w_{x}^{2}\,ds\,dx<\infty\right\}.$$

Then problem (4.1) becomes the following problem for an abstract first-order evolutionary equation,

$$\begin{aligned} \frac{d}{dt}\Phi + \mathcal{A}\Phi &= B(\Phi), \\ \Phi(0) &= \Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, \eta_0)^T, \end{aligned} \tag{4.14}$$

where $\Phi = (\varphi, u, \psi, v, \theta, \omega, \eta^t)$ and the linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -u \\ -\frac{k}{\rho_1}(\varphi_x + \psi)_x + \frac{\beta}{\rho_1}\omega_x \\ -v \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{g_0}{\rho_2}\psi_{xx} - \frac{1}{\rho_2}\int_0^\infty g(s)\eta^t_{xx}(x,t,s)ds - \frac{\beta}{\rho_2}\omega \\ -\omega \\ -\frac{\delta}{\rho_3}\theta_{xx} - \frac{k}{\rho_3}\omega_{xx} + \frac{\gamma}{\rho_3}u_x + \frac{\gamma}{\rho_3}v \\ \eta^t_s - v \end{pmatrix}, \tag{4.15}$$

$$B(\Phi) = \begin{pmatrix} 0 \\ -\frac{\alpha(t)}{\rho_1}f(\varphi_t) \\ 0 \\ 0 \\ 0 \\ -\frac{\alpha(t)}{\rho_3}f(\theta_t) \\ 0 \end{pmatrix}. \tag{4.16}$$

For any $\Phi = (\varphi, u, \psi, v, \theta, \omega, \eta^t)^T \in H$, $\widetilde{\Phi} = (\widetilde{\varphi}, \widetilde{u}, \widetilde{\psi}, \widetilde{v}, \widetilde{\theta}, \widetilde{\omega}, \widetilde{\eta^t})^T \in H$, we equip H with the inner product defined by

$$\begin{aligned} \langle \Phi, \widetilde{\Phi} \rangle_{\mathcal{H}} = \gamma \int_0^1 (\rho_1 u\widetilde{u} + k(\varphi_x + \psi)(\widetilde{\varphi_x} + \widetilde{\psi}) + \rho_2 v\widetilde{v} - g_0\psi_x\widetilde{\psi_x} + b\psi_x\widetilde{\psi_x})dx \\ + \beta \int_0^1 (\delta\theta_x\widetilde{\theta_x} + \rho_3\omega\widetilde{\omega})dx + \gamma < \eta^t, \widetilde{\eta^t} >_{L_g}, \end{aligned}$$

where

$$\langle w_1, w_2 \rangle_{L_g} = \int_0^1 \int_0^\infty g(s) w_{1x}(s) w_{2x}(s)\, ds\, dx.$$

The domain of A is

$$D(\mathcal{A}) = \left\{ \Phi \in \mathcal{H} : \varphi, \psi, \theta \in H^2(0,1) \cap H^1_0(0,1), u, v, \omega \in H^1_0(0,1), \eta^t \in L_g \right\},$$

which is dense in H.

Proposition 4.1 *Assume $\Phi_0 \in \mathcal{H}$ and A_1 - A_4 hold. Then, there exists a unique solution $\Phi \in C(\mathbb{R}_+, \mathcal{H})$ of problem (4.1)-(4.2). Moreover, if $\Phi_0 \in D(A)$ then*

$$\Phi \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. We use the semigroups approach. We prove that A is a maximal monotone operator. First, we prove that A is monotone, for any $\Phi \in D(A)$, we have

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = \beta k \int_0^1 \omega_x^2 dx + \gamma \int_0^1 \int_0^\infty g(s)\eta_{sx}^t \eta_x^t(x,t,s) ds dx. \tag{4.17}$$

Since $g : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, then we get

$$\begin{aligned} \int_0^\infty g(s)\eta_{sx}^t \eta_x^t(x,t,s) ds &= \frac{1}{2}\int_0^\infty g(s)\frac{d}{ds}\left(\eta_x^t\right)^2(x,t,s) ds \\ &= -\frac{1}{2}\int_0^\infty g'(s)\left(\eta_x^t\right)^2(x,t,s) ds. \end{aligned}$$

The last term in the left-hand side of (4.17) gives

$$\gamma \int_0^1 \int_0^\infty g(s)\eta_{sx}^t \eta_x^t(x,t,s) ds dx = -\frac{\gamma}{2} g' \circ \psi_x.$$

Consequently,

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = \beta k \int_0^1 \omega_x^2 dx - \frac{\gamma}{2} g' \circ \psi_x \geq 0.$$

Thus, A is monotone. Next, we prove that the operator $I + A$ is surjective. Given $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in H$, we prove that there exists a unique $\Phi \in D(A)$ such that

$$(I + \mathcal{A})\Phi = \mathcal{G}. \tag{4.18}$$

That is,

$$\begin{cases} \varphi - u = g_1 & \in H_0^1(0,1), \\ \rho_1 u - k(\varphi_x + \psi)_x + \beta\omega_x = \rho_1 g_2 & \in L^2(0,1), \\ \psi - v = g_3 & \in H_0^1(0,1), \\ \rho_2 v - b\psi_{xx} + k(\varphi_x + \psi) - \beta\omega - \int_0^\infty g(s)\eta_{xx}^t(x,t,s) ds & \\ +\psi_{xx}\int_0^\infty g(s) ds = \rho_2 g_4 & \in L^2(0,1), \\ \theta - \omega = g_5 & \in H_0^1(0,1), \\ \rho_3\omega - \delta\theta_{xx} - k\omega_{xx} + \gamma u_x + \gamma v = \rho_3 g_6 & \in L^2(0,1), \\ \eta^t + \eta_s^t - v = g_7 & \in L_g. \end{cases} \tag{4.19}$$

Using last line in the above equation, we obtain

$$\eta^t = e^{-s}\int_0^s e^{\tau}(v + g_7(\tau)) d\tau. \tag{4.20}$$

Inserting $u = \varphi - g_1$, $v = \psi - g_3$, $\omega = \theta - g_5$ and (4.20) in $(4.19)_2$, $(4.19)_4$ and $(4.19)_6$, we obtain

$$\begin{array}{ll} \rho_1\varphi - k(\varphi_x + \psi)_x + \beta\theta_x = h_1 & \in L^2(0,1), \\ \rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi) - \beta\theta - \int_0^\infty g(s)e^{-s}\int_0^s \psi_{xx}e^{\tau}d\tau ds & \\ + \int_0^\infty g(s)ds\psi_{xx} = h_2 & \in L^2(0,1), \\ \rho_3\theta - \delta\theta_{xx} - k\theta_{xx} + \gamma\varphi_x + \gamma\psi = h_3 & \in L^2(0,1), \end{array} \tag{4.21}$$

here

$$\begin{aligned} h_1 &= g_2\rho_1 + g_1\rho_1 + \beta g_{5x}, \\ h_2 &= \rho_2 g_4 + \rho_2 g_3 - \beta g_5 + \int_0^\infty g(s)e^{-s}\int_0^s (g_7 - g_3)_{xx}e^{\tau}d\tau ds, \\ h_3 &= \rho_3 g_5 - k g_{5xx} + \gamma g_{1x} + \gamma g_3 + g_6\rho_3. \end{aligned}$$

Considering the following variational formulation

$$K((\varphi,\psi,\theta),\ (\varphi_1,\psi_1,\theta_1)) = F(\varphi_1,\psi_1,\theta_1), \tag{4.22}$$

where $K : [H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)]^2 \to R$ is the bilinear form defined by

$$\begin{aligned} & K((\varphi,\psi,\theta),(\varphi_1,\psi_1,\theta_1)) \\ = \ & \gamma\rho_1\int_0^1 \varphi\varphi_1 dx + k\gamma\int_0^1(\varphi_x+\psi)(\varphi_{1x}+\psi_1)dx + \gamma\beta\int_0^1\theta_x\varphi_1 dx \\ & +\gamma\rho_2\int_0^1\psi\psi_1 dx + b\gamma\int_0^1\psi_x\psi_{1x}dx - \gamma\int_0^\infty g(s)ds\int_0^1\psi_x\psi_{1x}dx \end{aligned}$$

$$\begin{aligned} & + \gamma\int_0^1\psi_{1x}\int_0^\infty g(s)e^{-s}\int_0^s\psi_x e^{\tau}d\tau ds dx - \beta\gamma\int_0^1\theta\psi_1\,dx \\ & + \rho_3\beta\int_0^1\theta\theta_1 dx + \beta\delta\int_0^1\theta_x\theta_{1x}dx + k\beta\int_0^1\theta_x\theta_{1x}dx + \gamma\beta\int_0^1\varphi_x\theta_1 dx \\ & + \gamma\beta\int_0^1\psi\theta_1 dx, \end{aligned}$$

and $F : H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1) \to R$ is the linear functional

$$F[(\varphi_1,\psi_1,\theta_1)] = \gamma\int_0^1 h_1\varphi_1 dx + \gamma\int_0^1 h_2\psi_1 dx + \beta\int_0^1 h_3\theta_1 dx.$$

Now, for $V = [H_0^1(0,1)]^3$ equipped with the norm

$$\|\varphi,\psi,\theta\|_V^2 = \|\varphi\|_2^2 + \|\theta\|_2^2 + \|\varphi_x+\psi\|_2^2 + \|\psi_x\|_2^2 + \|\theta_x\|_2^2,$$

where $\|.\|_2$ is the usual norm, using integration by parts we have

$$\begin{aligned}
&K((\varphi,\psi,\theta),(\varphi,\psi,\theta))\\
&=\gamma\rho_1\int_0^1\varphi^2dx+k\gamma\int_0^1(\varphi_x+\psi)^2dx+\gamma\rho_2\int_0^1\psi^2dx\\
&\quad+(b-\int_0^\infty g(s)ds)\gamma\int_0^1\psi_x^2dx+\gamma\int_0^1\psi_x^2dx\int_0^\infty g(s)\int_0^s e^\tau d\tau e^{-s}ds\\
&\quad+\rho_3\beta\int_0^1\theta^2dx+\beta\delta\int_0^1\theta_x^2dx+k\beta\int_0^1\theta_x^2dx,\\
&\geq\lambda\|\varphi,\psi,\theta\|_V^2,
\end{aligned}$$

for some $\lambda>0$. Thus, K is coercive.

On the other hand, using Hölder's (2.1) and Poincaré's inequalities (2.3), we obtain

$$|K((\varphi,\psi,\theta),(\varphi_1,\psi_1,\theta_1))|\leq c\|\varphi,\psi,\theta\|_V\|\varphi_1,\psi_1,\theta_1\|_V.$$

Similarly

$$|F(\varphi_1,\psi_1,\theta_1)|\leq c\|\varphi_1,\psi_1,\theta_1\|_V.$$

Consequently, by the Lax-Milgram Lemma, system (4.21) has a unique solution

$$(\varphi,\psi,\theta)\in\left[H_0^1(0,1)\right]^3,$$

satisfying

$$K((\varphi,\psi,\theta),(\varphi_1,\psi_1,\theta_1))=F(\varphi_1,\psi_1,\theta_1),\quad(\varphi_1,\psi_1,\theta_1)\in V.$$

The substitution of φ, ψ and θ into $(4.19)_1$, $(4.19)_3$ and $(4.19)_5$ yields

$$(u,v,\omega)\in\left[H_0^1(0,1)\right]^3.$$

Similarly, inserting v in (4.20) and bearing in mind $(4.19)_7$, we obtain $\eta^t\in L_g$. Moreover, if we take $(\varphi_1,\theta_1)\equiv(0,0)\in[H_0^1(0,1)]^2$ in (4.22), we obtain

$$\begin{aligned}
&k\int_0^1(\varphi_x+\psi)\psi_1dx+\rho_2\int_0^1\psi\psi_1dx+b\int_0^1\psi_x\psi_{1x}dx-\beta\int_0^1\theta\psi_1dx\\
&-\int_0^\infty g(s)ds\int_0^1\psi_x\psi_{1x}dx+\int_0^\infty g(s)(1-e^{-s})ds\int_0^1\psi_x\psi_{1x}dx\\
&=\int_0^1 h_2\psi_1dx.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&b\int_0^1\psi_x\psi_{1x}dx-\int_0^\infty g(s)ds\int_0^1\psi_x\psi_{1x}dx+\int_0^\infty g(s)(1-e^{-s})ds\int_0^1\psi_x\psi_{1x}dx\\
&=\int_0^1[-k(\varphi_x+\psi)-\rho_2\psi+\beta\theta+h_2]\psi_1dx,\quad\psi_1\in H_0^1(0,1).
\end{aligned}$$

By noting that $-k(\varphi_x+\psi)-\rho_2\psi+\beta\theta+h_2 \in L^2(0,1)$, we obtain $\psi \in H^2(0,1)\cap H_0^1(0,1)$. Consequently using integration by parts we get

$$\int_0^1 [-b\psi_{xx}+\int_0^\infty g(s)ds\psi_{xx}dx - \int_0^\infty g(s)(1-e^{-s})ds\psi_{xx} + k(\varphi_x+\psi)+\rho_2\psi-\beta\theta-h_2]\psi_1 dx = 0, \quad \psi_1 \in H_0^1(0,1).$$

Therefore,

$$-b\psi_{xx}+\int_0^\infty g(s)ds\psi_{xx}dx - \int_0^\infty g(s)(1-e^{-s})ds\psi_{xx}+k(\varphi_x+\psi)+\rho_2\psi-\beta\theta = h_2.$$

This gives $(4.21)_2$. Similarly, if we take $(\varphi_1,\psi_1)\equiv(0,0)\in[H_0^1(0,1)]^2$ in (4.22), we can show that

$$\theta \in H^2(0,1)\cap H_0^1(0,1),$$

and $(4.21)_3$ are satisfied.

If we take $(\psi_1,\theta_1)\equiv(0,0)\in[H_0^1(0,1)]^2$ in (4.22), we can show that

$$\varphi \in H^2(0,1)\cap H_0^1(0,1),$$

and $(4.21)_1$ are satisfied.

Finally, from (4.20) we can get $\eta^t \in L_g$. Hence, there exists a unique $\Phi \in D(A)$ such that (4.18) is satisfied. Therefore, A is a maximal monotone operator, then $D(A)$ is dense in H (see Proposition 7.1 in [24]).

Now, we prove that the operator B defined in (4.14) is locally Lipschitz in H. Let $\Phi = (\varphi,u,\psi,v,\theta,\omega,\eta^t)^T$ and $\widetilde{\Phi} = (\widetilde{\varphi},\widetilde{u},\widetilde{\psi},\widetilde{v},\widetilde{\theta},\widetilde{\omega},\widetilde{\eta}^t)^T$, since Lipschitz continuous function, then we have

$$\begin{aligned}
\|\mathcal{B}(\Phi)-\mathcal{B}(\widetilde{\Phi})\|_{\mathcal{H}} &\le \frac{\alpha(t)}{\rho_1}\left(\|f(\varphi_t)-f(\widetilde{\varphi}_t)\|_{L^2}+\|f(\theta_t)-f(\widetilde{\theta}_t)\|_{L^2}\right)\\
&\le c\left(\|\varphi_t-\widetilde{\varphi}_t\|_{L^2}+\|\theta_t-\widetilde{\theta}_t\|_{L^2}\right)\\
&\le c\left(\|u-\widetilde{u}\|_{L^2}+\|\omega-\widetilde{\omega}\|_{L^2}\right)\\
&\le c\|\Phi-\widetilde{\Phi}\|_{\mathcal{H}}.
\end{aligned}$$

Then the operator $\mathcal{B}$ is locally Lipschitz in $\mathcal{H}$. Consequently, $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{H}$. Hence, the result of Proposition 4.1 follows (see [61, 83]) and the references therein. ■

4.3 Technical Lemmas and stability result

In this section, we start with establish several lemmas needed for our work then proof our main result.

Let us first prove that the energy function E is decreasing, we have

Lemma 4.2 *Let (φ,ψ,θ) be the solution of (4.1)-(4.2), then the energy E is decreasing function and satisfies, for all $t\geq 0$,*

$$E'(t) = -k\beta\int_0^1 \theta_{tx}^2 dx - \alpha(t)\left\{\gamma\int_0^1 \varphi_t f(\varphi_t)\,dx + \beta\int_0^1 \theta_t f(\theta_t)\,dx\right\} + \frac{\gamma}{2}(g'\circ\psi_x)(t) \leq 0. \tag{4.23}$$

Proof. Multiplying $(4.1)_1$,$(4.1)_2$ and $(4.1)_3$ by $\gamma\varphi_t$, $\gamma\psi_t$ and $\beta\theta_t$ respectively, and integrating over $(0,1)$, using integration by parts, we get

$$\begin{aligned}&\frac{\gamma}{2}\frac{d}{dt}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2+k(\varphi_x+\psi)^2+b\psi_x^2\right)dx\\ &+\frac{\beta}{2}\frac{d}{dt}\int_0^1(\rho_3\theta_t^2+\delta\theta_x^2)dx+\gamma\int_0^1\psi_t\int_0^\infty g(s)\,\psi_{xx}(t-s)\,dsdx\\ =\;& -k\beta\int_0^1\theta_{tx}^2dx-\alpha(t)\left[\gamma\int_0^1\varphi_t f(\varphi_t)\,dx+\beta\int_0^1\theta_t f(\theta_t)\,dx\right].\end{aligned} \tag{4.24}$$

Using Lemma (4.1) and Young's inequality, the last term in the left-hand side of (4.24) becomes

$$\begin{aligned}&\int_0^1\psi_t\int_0^\infty g(s)\,\psi_{xx}(t-s)\,dsdx\\ =\;&\int_0^1\psi_{xt}\int_0^\infty g(s)\left(\psi_x(t)-\psi_x(t-s)\right)dsdx-\left(\int_0^\infty g(s)\,ds\right)\int_0^1\psi_{xt}\psi_x dx\\ =\;&\frac{1}{2}\frac{d}{dt}\left[(g\circ\psi_x)(t)-\left(\int_0^\infty g(s)\,ds\right)\int_0^1\psi_x^2dx\right]-\frac{1}{2}(g'\circ\psi_x)(t).\end{aligned} \tag{4.25}$$

A combination of (4.24) and (4.25) gives

$$\begin{aligned}E'(t) =\;& -k\beta\int_0^1\theta_{tx}^2dx+\frac{\gamma}{2}(g'\circ\psi_x)(t)\\ &-\alpha(t)\left[\gamma\int_0^1\varphi_t f(\varphi_t)\,dx+\beta\int_0^1\theta_t f(\theta_t)\,dx\right]\leq 0.\end{aligned}$$

■

Lemma 4.3 *The functional*

$$I_1(t) := -\rho_1\int_0^1\varphi_t\varphi\,dx-\rho_2\int_0^1\psi_t\psi\,dx \tag{4.26}$$

satisfies

$$\begin{aligned}I_1'(t)\leq\;& -\rho_1\int_0^1\varphi_t^2dx-\rho_2\int_0^1\psi_t^2dx+(k+2c+\beta\varepsilon_1)\int_0^1(\varphi_x+\psi)^2dx\\ &+\frac{\beta}{\varepsilon_1}\int_0^1\theta_t^2dx+c(2+\varepsilon_1)\int_0^1\psi_x^2dx+c(g\circ\psi_x)(t)+c\int_0^1 f^2(\varphi_t),\end{aligned} \tag{4.27}$$

for any $\varepsilon_1>0$.

Proof. By computations, using (4.1), we obtain

$$I_1'(t) = -\rho_1 \int_0^1 \varphi_t^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx - \beta \int_0^1 \theta_t \varphi_x \, dx - \rho_2 \int_0^1 \psi_t^2 dx$$
$$+ b \int_0^1 \psi_x^2 dx - \beta \int_0^1 \theta_t \psi \, dx - \int_0^1 \psi_x \int_0^\infty g(s)\psi_x(x, t-s) \, ds \, dx$$
$$+ \alpha(t) \int_0^1 \varphi f(\varphi_t) \, dx.$$

By using Young's inequality and Poincaré's inequality, we obtain for $\varepsilon_1 > 0$,

$$I_1'(t) \le -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\beta\varepsilon_1}{2} \int_0^1 \varphi_x^2 dx$$
$$+ \frac{\beta}{\varepsilon_1} \int_0^1 \theta_t^2 dx + (2b + \frac{\beta\varepsilon_1}{2}) \int_0^1 \psi_x^2 dx$$
$$+ \frac{1}{4b} \int_0^1 \left(\int_0^\infty g(s)\psi_x(x, t-s) ds \right)^2 dx + \alpha(t) \int_0^1 \varphi f(\varphi_t) \, dx. \tag{4.28}$$

By using Lemma (4.1) and Poincaré's inequality, we have

$$\int_0^1 \varphi_x^2 dx \le 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi^2 dx \le 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx, \tag{4.29}$$

$$\alpha(t) \int_0^1 \varphi f(\varphi_t) \, dx \le \frac{1}{2} \int_0^1 \varphi^2 dx + c \int_0^1 f^2(\varphi_t) \, dx$$
$$\le c \int_0^1 \psi_x^2 dx + c \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 f^2(\varphi_t) \, dx, \tag{4.30}$$

$$\int_0^1 \left(\int_0^\infty g(s)\psi_x(x, t-s) ds \right)^2 dx \le c \left(g \circ \psi_x(t) + \int_0^1 \psi_x^2 dx \right). \tag{4.31}$$

The substitution of (4.29), (4.30) and (4.31) into (4.28) gives (6.36). ■

Lemma 4.4 *The functional*

$$I_2(t) := -\rho_2 \int_0^1 \psi_t \int_0^\infty g(s)(\psi(t) - \psi(t-s)) \, ds \, dx$$

satisfies

$$I_2'(t) \le \varepsilon_2 c \int_0^1 \psi_x^2 dx - (\rho_2 g_0 - \varepsilon_2' \rho_2) \int_0^1 \psi_t^2 dx + c(\varepsilon_2 + \frac{1}{\varepsilon_2})(g \circ \psi_x)(t)$$
$$+ \varepsilon_2 k \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_2 \beta^2 \int_0^1 \theta_t^2 dx - \frac{c}{4\varepsilon_2'}(g' \circ \psi_x)(t), \tag{4.32}$$

for any $\varepsilon_2, \varepsilon_2' > 0$.

Proof. First, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^{\infty} g(s)(\psi(t)-\psi(t-s))ds\right)\\
&=-\frac{\partial}{\partial t}\left(\int_t^{-\infty} g(t-s)(\psi(t)-\psi(s))ds\right)\\
&=\frac{\partial}{\partial t}\left(\int_{-\infty}^{t} g(t-s)(\psi(t)-\psi(s))ds\right)\\
&=\int_{-\infty}^{t} g'(t-s)(\psi(t)-\psi(s))ds+\int_{-\infty}^{t} g(t-s)\psi_t(t)ds\\
&=g_0\psi_t(t)+\int_0^{\infty} g'(s)(\psi(t)-\psi(t-s))ds.
\end{aligned}$$

Then, by differentiating $I_2(t)$ and using (4.1), we find

$$\begin{aligned}
I_2'(t)=&\ b\int_0^1 \psi_x \int_0^{\infty} g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx - g_0\rho_2\int_0^1 \psi_t^2 dx\\
&-\rho_2\int_0^1 \psi_t \int_0^{\infty} g'(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&+k\int_0^1 (\varphi_x+\psi)\int_0^{\infty} g(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&-\beta\int_0^1 \theta_t\int_0^{\infty} g(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
&-\int_0^1\int_0^{\infty} g(s)\psi_x(t-s)ds\int_0^{\infty} g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx. \qquad (4.33)
\end{aligned}$$

By using Young's and Poincaré's inequalities,

$$\begin{aligned}
&b\int_0^1 \psi_x\int_0^{\infty} g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx\\
\leq&\ \varepsilon_2 b\int_0^1 \psi_x^2 dx+\frac{b}{4\varepsilon_2}\int_0^1\left(\int_0^{\infty} g(s)(\psi_x(t)-\psi_x(t-s))ds\right)^2 dx\\
\leq&\ \varepsilon_2 b\int_0^1 \psi_x^2 dx+\frac{bg_0}{4\varepsilon_2}(g\circ\psi_x)(t), \qquad (4.34)\\
&-\rho_2\int_0^1 \psi_t\int_0^{\infty} g'(s)(\psi(t)-\psi(t-s))\,ds\,dx\\
\leq&\ \varepsilon_2'\int_0^1 \rho_2\psi_t^2 dx+\frac{\rho_2}{4\varepsilon_2'}\int_0^1\left(\int_0^{\infty} g'(s)(\psi(t)-\psi(t-s))ds\right)^2 dx\\
\leq&\ \varepsilon_2'\int_0^1 \rho_2\psi_t^2 dx-\frac{\rho_2 d_2}{4\varepsilon_2'}(g'\circ\psi_x)(t), \qquad (4.35)
\end{aligned}$$

$$
\begin{aligned}
& k\int_0^1(\varphi_x+\psi)\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
\leq\ & \varepsilon_2 k\int_0^1(\varphi_x+\psi)^2dx+\frac{k}{4\varepsilon_2}\int_0^1\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\right)^2dx \\
\leq\ & \varepsilon_2 k\int_0^1(\varphi_x+\psi)^2dx+\frac{kd_1}{4\varepsilon_2}(g\circ\psi_x)(t),
\end{aligned} \tag{4.36}
$$

$$
\begin{aligned}
& \beta\int_0^1\theta_t\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\,dx \\
\leq\ & \varepsilon_2\beta^2\int_0^1\theta_t^2dx+\frac{1}{4\varepsilon_2}\int_0^1\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\right)^2dx \\
\leq\ & \varepsilon_2\beta^2\int_0^1\theta_t^2dx+\frac{d_1}{4\varepsilon_2}(g\circ\psi_x)(t),
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
& \int_0^1\int_0^\infty g(s)\psi_x(t-s)ds\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx \\
& \leq \varepsilon_2\int_0^1\left(\int_0^\infty g(s)\psi_x(t-s)ds\right)^2dx \\
& +\frac{1}{4\varepsilon_2}\int_0^1\left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\right)^2dx \\
& \leq\left(2\varepsilon_2+\frac{1}{4\varepsilon_2}\right)g_0(g\circ\psi_x)+2\varepsilon_2 g_0\int_0^1\psi_x^2dx.
\end{aligned} \tag{4.38}
$$

By substituting (4.34)-(4.38) into (4.33), we obtain (4.32). ■

Lemma 4.5 *The functional*

$$
I_3(t):=\frac{k}{2}\int_0^1\theta_x^2dx+\rho_3\int_0^1\theta_t\theta\,dx+\gamma\int_0^1\varphi_x\theta\,dx
$$

satisfies

$$
\begin{aligned}
I_3'(t)\leq & \left(c\varepsilon_3-\frac{3\delta}{4}\right)\int_0^1\theta_x^2dx+\left(\rho_3+\frac{\gamma^2}{2\varepsilon_3}\right)\int_0^1\theta_t^2dx+\varepsilon_3\int_0^1\psi_x^2dx \\
& +\frac{\gamma^2}{\delta}\int_0^1\psi_t^2dx+\varepsilon_3\int_0^1(\varphi_x+\psi)^2dx+\frac{c}{4\varepsilon_3}\int_0^1 f^2\left(\theta_t\right)\,dx,
\end{aligned} \tag{4.39}
$$

for any $\varepsilon_3>0$.

Proof. By differentiating $I_3(t)$ and using (4.1), we obtain

$$
\begin{aligned}
I_3'(t)= & \rho_3\int_0^1\theta_t^2dx+\gamma\int_0^1\varphi_x\theta_tdx-\delta\int_0^1\theta_x^2dx-\gamma\int_0^1\psi_t\theta\,dx \\
& -\alpha\left(t\right)\int_0^1 f\left(\theta_t\right)\theta\,dx.
\end{aligned}
$$

By using Young's and Poincaré's inequalities, we obtain for any $\varepsilon_3 > 0$

$$\begin{aligned}
\gamma \int_0^1 \varphi_x \theta_t dx &\leq \varepsilon_3 \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_3 \int_0^1 \psi_x^2 dx + \frac{\gamma^2}{2\varepsilon_3} \int_0^1 \theta_t^2 dx, \\
\gamma \int_0^1 \psi_t \theta \, dx &\leq \frac{\delta}{4} \int_0^1 \theta_x^2 dx + \frac{\gamma^2}{\delta} \int_0^1 \psi_t^2 dx, \ \left(\text{for } \epsilon = \frac{\delta}{4}\right), \\
-\alpha(t) \int_0^1 \theta f(\theta_t) \, dx &\leq \varepsilon_3 \int_0^1 \theta^2 dx + \frac{c}{4\varepsilon_3} \int_0^1 f^2(\theta_t) \, dx \\
&\leq c\varepsilon_3 \int_0^1 \theta_x^2 dx + \frac{c}{4\varepsilon_3} \int_0^1 f^2(\theta_t) \, dx.
\end{aligned}$$

Using these last inequalities completes the proof. ■

Lemma 4.6 *The functional*

$$I_4(t) := 2\rho_1 \int_0^1 x \varphi_x \varphi_t \, dx \tag{4.40}$$

satisfies

$$\begin{aligned}
I_4'(t) \leq -\rho_1 \int_0^1 \varphi_t^2 dx - (k - c\varepsilon_4) \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\beta^2}{2\varepsilon_4} \int_0^1 \theta_{tx}^2 dx \\
+ \left(c\varepsilon_4 + \frac{1}{\varepsilon_4}\right) \int_0^1 \psi_x^2 dx + \frac{c}{2\varepsilon_4} \int_0^1 f^2(\varphi_t) \, dx,
\end{aligned} \tag{4.41}$$

for any $\varepsilon_4 > 0$.

Proof. by differentiating $I_4(t)$ and using (4.1), we find that

$$\begin{aligned}
I_4'(t) &= 2 \int_0^1 [k(\varphi_x + \psi)_x - \beta \theta_{tx} - \alpha(t) f(\varphi_t)] (x\varphi_x) \, dx + 2\rho_1 \int_0^1 x \frac{d}{dt} \varphi_t^2 dx \\
&= 2k \int_0^1 (\varphi_x + \psi)_x x \varphi_x dx - 2\beta \int_0^1 \theta_{tx} x \varphi_x dx - 2\alpha(t) \int_0^1 f(\varphi_t) x \varphi_x dx \\
&\quad -2\rho_1 \int_0^1 \varphi_t^2 dx
\end{aligned}$$

$$\begin{aligned}
&= k \int_0^1 x \left[(\varphi_x + \psi)^2\right]_x dx - 2k \int_0^1 x(\varphi_x + \psi)\psi dx - 2\beta \int_0^1 \theta_{tx} x \varphi_x dx \\
&\quad -2\alpha(t) \int_0^1 f(\varphi_t) x \varphi_x dx - 2\rho_1 \int_0^1 \varphi_t^2 dx \\
&= -k \int_0^1 (\varphi_x + \psi)^2 dx + 2k \int_0^1 x(\varphi_x + \psi)\psi_x dx + 2k \int_0^1 (\varphi_x + \psi)\psi dx \\
&\quad -2\beta \int_0^1 \theta_{tx} x \varphi_x dx - 2\alpha(t) \int_0^1 f(\varphi_t) x \varphi_x dx - 2\rho_1 \int_0^1 \varphi_t^2 dx,
\end{aligned}$$

then

$$\begin{aligned} 2k\int_0^1 x(\varphi_x+\psi)\psi_x dx &\leq k^2\varepsilon_4\int_0^1(\varphi_x+\psi)^2+\frac{1}{\varepsilon_4}\int_0^1\psi_x^2dx,\\ -2\beta\int_0^1\theta_{tx}x\varphi_x dx &\leq 2\varepsilon_4\int_0^1\varphi_x^2+\frac{\beta^2}{2\varepsilon_4}\int_0^1\theta_{tx}^2dx,\\ &\leq 4\varepsilon_4\int_0^1(\varphi_x+\psi)^2+c\varepsilon_4\int_0^1\psi_x^2dx+\frac{\beta^2}{2\varepsilon_4}\int_0^1\theta_{tx}^2dx, \end{aligned}$$

and

$$-2\alpha(t)\int_0^1 f(\varphi_t)\,x\varphi_x dx\leq 2\varepsilon_4\int_0^1\varphi_x^2+\frac{c}{2\varepsilon_4}\int_0^1 f^2(\varphi_t)dx.$$

The substitution of these last inequalities gives (4.41). ■

As in [37], we introduce the multiplier ω which is the solution of

$$-w_{xx}=\psi_x,\quad w(0)=w(1)=0. \tag{4.42}$$

Lemma 4.7 *The solution of (4.42) satisfies*

$$\begin{aligned} \int_0^1 w_x^2dx\leq\int_0^1\psi^2dx\leq\int_0^1\psi_x^2dx,\\ \int_0^1 w_t^2dx\leq\int_0^1 w_{xt}^2dx\leq\int_0^1\psi_t^2dx. \end{aligned}$$

Lemma 4.8 *The functional*

$$I_5(t):=\int_0^1(\rho_1\varphi_t w+\rho_2\psi_t\psi)dx,$$

satisfies

$$\begin{aligned} I_5'(t)\leq\left(c\varepsilon_5'-\frac{\ell}{2}\right)\int_0^1\psi_x^2(t)dx+\frac{3\beta^2}{\ell}\int_0^1\theta_t^2(t)dx+\varepsilon_5\int_0^1\varphi_t^2dx\\ +(\rho_2+\frac{\rho_1^2}{4\varepsilon_5})\int_0^1\psi_t^2dx+\frac{3g_0}{2\ell}(g\circ\psi_x)(t)+\frac{c}{4\varepsilon_5'}\int_0^1 f^2(\varphi_t)dx. \end{aligned}\tag{4.43}$$

for any $\varepsilon_5,\varepsilon_5'>0$.

Proof. By a simple differentiation of $I_5(t)$ and using (4.1), we get

$$\begin{aligned} I_5'(t)=\beta\int_0^1\theta_t w_x\,dx+k\int_0^1 w_x^2dx+\rho_1\int_0^1\varphi_t w_t dx+\beta\int_0^1\theta_t\psi\,dx\\ -b\int_0^1\psi_x^2dx-k\int_0^1\psi^2dx+\rho_2\int_0^1\psi_t^2dx-\alpha(t)\int_0^1 f(\varphi_t)w\,dx\\ +\int_0^1\int_0^\infty g(s)\psi_x(x,t-s)ds\psi_x(t)dx, \end{aligned}$$

where we have used integration by parts, (4.42) and the boundary conditions (4.2). By using Young's, Poincaré's inequalities, Lemma 4.1 and Lemma 4.7, we have

$$\beta \int_0^1 \theta_t \omega_x \, dx \leq \frac{\ell}{6} \int_0^1 \psi_x^2 \, dx + \frac{3\beta^2}{2\ell} \int_0^1 \theta_t^2 dx,$$
$$\rho_1 \int_0^1 \varphi_t \omega_t dx \leq \varepsilon_5 \int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_5} \int_0^1 \psi_t^2 dx,$$
$$\beta \int_0^1 \theta_t \psi \, dx \leq \frac{\ell}{6} \int_0^1 \psi_x^2 \, dx + \frac{3\beta^2}{2\ell} \int_0^1 \theta_t^2 dx,$$
$$-\alpha(t) \int_0^1 f(\varphi_t) w \, dx \leq c\varepsilon_5' \int_0^1 \psi_x^2(t) dx + \frac{c}{4\varepsilon_5'} \int_0^1 f^2(\varphi_t) dx,$$

and

$$\begin{aligned}
&\int_0^1 \int_0^\infty g(s)\psi_x(x,t-s) ds \psi_x(t) dx \\
&= \int_0^1 \int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t) + \psi_x(t)) ds \psi_x(t) dx \\
&= \int_0^1 \int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t)) ds \psi_x(t) dx + \int_0^\infty g(s) ds \int_0^1 \psi_x^2(t) dx \\
&\leq \frac{\ell}{6} \int_0^1 \psi_x^2(t) dx + \frac{3}{2\ell} \int_0^1 \left(\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t)) ds \right)^2 dx \\
&\quad + \int_0^\infty g(s) ds \int_0^1 \psi_x^2(t) dx \\
&\leq \frac{\ell}{6} \int_0^1 \psi_x^2(t) dx + \frac{3g_0}{2\ell} (g \circ \psi_x)(t) + g_0 \int_0^1 \psi_x^2(t) dx.
\end{aligned}$$

By using these last inequalities, we get (4.43). ■

Now, we prove our stability result. First, we define a Lyapunov functional L by

$$\mathcal{L}(t) := NE(t) + I_1(t) + N_1 I_2(t) + I_3(t) + N_2 I_4(t) + N_3 I_5(t), \tag{4.44}$$

where N_1, N_2, N_3 and N are positive real numbers to be chosen appropriately later. One sees that, for N sufficiently large we have

$$\mathcal{L}(t) \sim E(t). \tag{4.45}$$

Theorem 4.1 *Let $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)^T \in H$ be given. Assume that A_1- A_4 are satisfied, then there exist c_1,$c_2 > 0$ for which the solution of problem (4.1)-(4.2) satisfies*

$$E(t) \leq c_1 H_1^{-1} \left(c_2 \int_0^t (\alpha\eta)(s) \, ds \right), \ \forall t \geq 0, \tag{4.46}$$

where the functions H_1 and H_2 are defined by:

for $\epsilon_0 > 0$

$$H_2(t) := \begin{cases} t & \text{if } h'' = 0 \text{ on } [0,l], \\ th'(\epsilon_0 t) & \text{if } h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0,l], \end{cases} \tag{4.47}$$

and

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds. \tag{4.48}$$

Proof. By combining (4.23), (4.27), (4.32), (4.39), (4.41) and (4.43), and letting $\varepsilon_1 = 1$ and $\varepsilon_2 = \frac{1}{N_1}$, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq\; & \left\{N_1\rho_2(\varepsilon_2' - g_0) - \rho_2 + \left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_5}\right)N_3 + \frac{\gamma^2}{\delta}\right\}\int_0^1 \psi_t^2 dx \\
& + \{-\rho_1(N_2+1) + \varepsilon_5 N_3\}\int_0^1 \varphi_t^2 dx + \left\{c\varepsilon_3 - \frac{3\delta}{4}\right\}\int_0^1 \theta_x^2 dx \\
& + \left\{N_3\left(c\varepsilon_5' - \frac{\ell}{2}\right) + \varepsilon_3 + 3c + N_2\left(c\varepsilon_4 + \frac{1}{\varepsilon_4}\right) + c\right\}\int_0^1 \psi_x^2 dx \\
& + \{N_2(c\varepsilon_4 - k) + 2c + \beta + 2k + \varepsilon_3\}\int_0^1 (\varphi_x + \psi)^2 dx \\
& + \left\{c\left(\beta + \rho_3 + \frac{\gamma^2}{2\varepsilon_3} + \beta^2\left(N_2\frac{1}{2\varepsilon_4} + \frac{3}{\ell}N_3 + 1\right)\right) - Nk\beta\right\}\int_0^1 \theta_{xt}^2 dx \\
& + \left\{c\left(N_1^2 + 2\right) + \frac{3g_0}{2\ell}N_3\right\} g \circ \psi_x(t) - \left\{N_1\frac{c}{4\varepsilon_2'} - N\frac{\gamma}{2}\right\} g' \circ \psi_x(t) \\
& + \frac{c}{4\varepsilon_4}\int_0^1 f^2(\theta_t)dx + c\left\{\frac{N_2}{2\varepsilon_4} + \frac{N_3}{4\varepsilon_5'} + 1\right\}\int_0^1 f^2(\varphi_t)dx \\
& - N\alpha(t)\left\{\gamma\int_0^1 \varphi_t f(\varphi_t)\,dx + \beta\int_0^1 \theta_t f(\theta_t)\,dx\right\},
\end{aligned} \tag{4.49}$$

for all $t \geq t_0$.

Now, we have to choose our constants very carefully. First, let us take ε_3, ε_4, ε_2' and ε_5' small enough such that

$$c\varepsilon_3 - \frac{3\delta}{4} < 0,\; c\varepsilon_4 - k < 0,\; \varepsilon_2' - g_0 < 0,\; c\varepsilon_5' - \frac{\ell}{2} < 0.$$

After that, we pick N_2 large enough so that

$$N_2(c\varepsilon_4 - k) + 2c + \beta + 2k + \varepsilon_3 < 0,$$

then, we choose N_3 large enough such that

$$N_3\left(c\varepsilon_5' - \frac{\ell}{2}\right) + \varepsilon_3 + 3c + N_2\left(c\varepsilon_4 + \frac{1}{\varepsilon_4}\right) + c < 0.$$

Now, we select ε_5 sufficiently small so that

$$-\rho_1 (N_2 + 1) + \varepsilon_5 N_3 < 0,$$

next, we take N_1 large enough such that

$$N_1 \rho_2 (\varepsilon_2' - g_0) - \rho_2 + \left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_5}\right) N_3 + \frac{\gamma^2}{\delta} < 0.$$

Finally, we choose N large enough (even larger so that (4.45) remains valid) and

$$c\left(\beta + \rho_3 + \frac{\gamma^2}{2\varepsilon_3} + \beta^2 \left(N_2 \frac{1}{2\varepsilon_4} + \frac{3}{\ell} N_3 + 1\right)\right) - Nk\beta < 0,$$

$$N_1 \tfrac{c}{4\varepsilon_2'} - N\tfrac{\gamma}{2} < 0.$$

Therefore, (4.49) becomes

$$\begin{aligned} \mathcal{L}'(t) \leq & -c \int_0^1 \left[\varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta_x^2 + \theta_t^2\right] dx + cg \circ \psi_x(t) + cg' \circ \psi_x(t) \\ & + c\int_0^1 f^2(\theta_t) dx + c\int_0^1 f^2(\varphi_t) dx - c\int_0^1 (\varphi_t f(\varphi_t) + \theta_t f(\theta_t))\, dx. \end{aligned}$$

By using the estimate (4.9) then lead to

$$\mathcal{L}'(t) \leq -cE(t) + cg \circ \psi_x(t) + c\int_0^1 \left(f^2(\theta_t) + f^2(\varphi_t)\right) dx, \ \forall t \geq t_0. \tag{4.50}$$

Let us define the following sets

$$\begin{aligned} \Sigma_{\varphi_+} &= \{x \in (0,1) : |\varphi_t(x,t)| > l\}, \ \Sigma_{\varphi_-} = (0,1) \setminus \Sigma_{\varphi_+}, \\ \Sigma_{\theta_+} &= \{x \in (0,1) : |\theta_t(x,t)| > l\}, \ \Sigma_{\theta_-} = (0,1) \setminus \Sigma_{\theta_+}. \end{aligned}$$

We work now for estimate the last term in the right-hand side of (4.50). First, note that

$$\begin{aligned} \int_0^1 \left(f^2(\varphi_t) + f^2(\theta_t)\right) dx &= \int_{\Sigma_{\varphi_+}} f^2(\varphi_t)\, dx + \int_{\Sigma_{\theta_+}} f^2(\theta_t)\, dx \\ &+ \int_{\Sigma_{\varphi_-}} f^2(\varphi_t)\, dx + \int_{\Sigma_{\theta_-}} f^2(\theta_t)\, dx. \end{aligned}$$

Using A_1 and (4.23), we easily show that

$$\begin{aligned} \alpha(t)\left[\int_{\Sigma_{\varphi_+}} f^2(\varphi_t)\, dx + \int_{\Sigma_{\theta_+}} f^2(\theta_t)\, dx\right] &\leq k_1 \int_{\Sigma_{\varphi_+}} \alpha(t)\varphi_t f(\varphi_t)\, dx + k_1' \int_{\Sigma_{\theta_+}} \alpha(t)\theta_t f(\theta_t)\, dx \\ &\leq k_1 \int_0^1 \alpha(t)\varphi_t f(\varphi_t)\, dx + k_1' \int_0^1 \alpha(t)\theta_t f(\theta_t)\, dx \\ &\leq -(\lambda_1 + \lambda_2) E'(t) \\ &\leq -cE'(t), \end{aligned} \tag{4.51}$$

for $\lambda_1, \lambda_2 > 0$.

If $h'' = 0$ on $[0, l]$: This implies that there exist $k_2 > 0$ such that $|f(s)| \leq k_1' |s|$ for all $s \in \mathbb{R}_+$, and then (4.51) is also satisfied for $|\varphi_t(x,t)| \leq l$ and $|\theta_t(x,t)| \leq l$, then on all $(0, 1)$. From (4.50), (4.51) and the fact that $\alpha' \leq 0$, we arrive at

$$(\alpha(t)\mathcal{L}(t) + cE(t))' \leq -c\alpha(t)H_2(E(t)) + cg \circ \psi_x(t), \quad \forall t \geq t_0, \tag{4.52}$$

where H_2 is defined in (4.47).

If $h'(0) = 0$ and $h'' > 0$ on $(0, l]$: Since h is convex and increasing, h^{-1} is concave and increasing, by using A_1, the reversed Jensen's inequality for concave function (see [87], p. 61), and (4.23), we obtain,

$$\begin{aligned}
\alpha(t)\left[\int_{\Sigma_{\varphi_-}} f^2(\varphi_t)\,dx + \int_{\Sigma_{\theta-}} f^2(\theta_t)\,dx\right] &\leq \alpha(t)\int_{\Sigma_{\varphi_-}} h^{-1}(\varphi_t f(\varphi_t))\,dx \\
&\quad + \alpha(t)\int_{\Sigma_{\theta_-}} h^{-1}(\theta_t f(\theta_t))\,dx \\
&\leq C\alpha(t)h^{-1}\left(\int_{\Sigma_{\varphi_-}} \varphi_t f(\varphi_t)\,dx\right) \\
&\quad + C\alpha(t)h^{-1}\left(\int_{\Sigma_{\theta_-}} \theta_t f(\theta_t)\,dx\right) \\
&\leq C\alpha(t)h^{-1}\left(\int_0^1 \varphi_t f(\varphi_t)\,dx\right) \\
&\quad + C\alpha(t)h^{-1}\left(\int_0^1 \theta_t f(\theta_t)\,dx\right) \\
&\leq C\alpha(t)\left\{h^{-1}(-c_1'E'(t)) + h^{-1}(-c_2'E'(t))\right\} \\
&\leq 2C\alpha(t)h^{-1}(-cE'(t)) \\
&\leq c\alpha(t)h^{-1}(-cE'(t)),
\end{aligned}$$

where $c = \max\{c_1', c_2'\}$.

Therefore, from (4.50), (6.45), (6.47) and the fact that $\alpha' \leq 0$, we find that

$$(\alpha(t)\mathcal{L}(t) + cE(t))' \leq c\alpha(t)h^{-1}(-cE'(t)) - c\alpha(t)E(t) + cg \circ \psi_x(t), \quad \forall t \geq t_0.$$

By using Young's inequality (2.8) and the fact that

$$h^*(p) \leq p[h']^{-1}(p), \;\; E' \leq 0, \;\; h'' > 0,$$

we obtain for $\varepsilon_0 > 0$ small enough and $c_0 > 0$ large enough,

$$\begin{aligned} & \left[h'\left(\varepsilon_0 E\left(t\right)\right)\left[\alpha(t)\mathcal{L}(t)+cE\left(t\right)\right]+c_0 E\left(t\right)\right]' \\ = \ & \varepsilon_0 E'\left(t\right) h''\left(\varepsilon_0 E\left(t\right)\right)\left[\alpha(t)\mathcal{L}(t)+cE\left(t\right)\right]+c_0 E'\left(t\right) \\ & +h'\left(\varepsilon_0 E\left(t\right)\right)\left[\alpha(t)\mathcal{L}'(t)+\alpha'(t)\mathcal{L}(t)+cE'\left(t\right)\right] \\ \leq \ & -c\alpha(t)h'\left(\varepsilon_0 E\left(t\right)\right)E\left(t\right)+c\alpha(t)h'\left(\varepsilon_0 E\left(t\right)\right)h^{-1}(-cE'\left(t\right)) \\ & +c_0 E'\left(t\right)+ch'\left(\varepsilon_0 E\left(t\right)\right)g\circ\psi_x(t) \\ \leq \ & -c\alpha(t)h'\left(\varepsilon_0 E\left(t\right)\right)E\left(t\right)+c\alpha(t)h^*(h'\left(\varepsilon_0 E\left(t\right)\right))-cE'\left(t\right) \\ & +c_0 E'\left(t\right)+ch'\left(\varepsilon_0 E\left(0\right)\right)g\circ\psi_x(t) \\ \leq \ & -c\alpha(t)h'\left(\varepsilon_0 E\left(t\right)\right)E\left(t\right)+c\varepsilon_0\alpha(t)h'\left(\varepsilon_0 E\left(t\right)\right)E\left(t\right)+cg\circ\psi_x(t) \\ \leq \ & -c\alpha(t)h'\left(\varepsilon_0 E\left(t\right)\right)E\left(t\right)+cg\circ\psi_x(t)=-c\alpha(t)H_2\left(E\left(t\right)\right)+cg\circ\psi_x(t). \end{aligned} \tag{4.53}$$

Now, let us define the following functional:

$$\mathcal{F}\left(t\right)=\begin{cases}\alpha(t)\mathcal{L}(t)+cE\left(t\right) & \text{if (4.10) holds,}\\ h'\left(\varepsilon_0 E\left(t\right)\right)\left[\alpha(t)\mathcal{L}(t)+cE\left(t\right)\right]+c_0 E\left(t\right) & \text{if (4.11) holds.}\end{cases}$$

Using (4.45), we have

$$\mathcal{F}\sim E,$$

and exploiting (4.52) and (4.53), we easily deduce that

$$\mathcal{F}'\left(t\right)\leq -c\alpha(t)H_2\left(E\left(t\right)\right)+cg\circ\psi_x(t),\quad \forall t\geq t_0.$$

By using (4.23) and A_3, we obtain

$$\begin{aligned} \left(\eta\left(t\right)\mathcal{F}\left(t\right)\right)' &= \eta'\left(t\right)\mathcal{F}\left(t\right)+\eta\left(t\right)\mathcal{F}'\left(t\right) \\ &\leq -c\alpha(t)\eta\left(t\right)H_2\left(E\left(t\right)\right)+c\eta\left(t\right)g\circ\psi_x(t) \\ &\leq -c\alpha(t)\eta\left(t\right)H_2\left(E\left(t\right)\right)+c\left(\eta g\right)\circ\psi_x(t) \\ &\leq -c\alpha(t)\eta\left(t\right)H_2\left(E\left(t\right)\right)-cg'\circ\psi_x(t) \\ &\leq -c\alpha(t)\eta\left(t\right)H_2\left(E\left(t\right)\right)-cE'\left(t\right). \end{aligned}$$

Next, let

$$\mathcal{R}\left(t\right)=\varepsilon\left(\eta\left(t\right)\mathcal{F}\left(t\right)+cE\left(t\right)\right),$$

where $0<\varepsilon<\bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\eta\left(t\right)\mathcal{F}\left(t\right)+cE\left(t\right)\leq\frac{1}{\bar{\varepsilon}}E\left(t\right),\quad \forall t\geq 0.$$

We also have

$$\mathcal{R}\sim E, \tag{4.54}$$

and for $t \geq t_0$

$$\mathcal{R}'(t) \leq -c\varepsilon\alpha(t)\eta(t) H_2(\mathcal{R}(t)). \tag{4.55}$$

Noting that $H_1' = -1/H_2$ (see (4.48)), we get from (4.55)

$$\mathcal{R}'(t) H_1'(\mathcal{R}(t)) \geq c\varepsilon\alpha(t)\eta(t), \quad \forall t \geq t_0,$$

A simple integration over (t_0, t) then yields

$$H_1(\mathcal{R}(t)) \geq H_1(\mathcal{R}(t_0)) + c\varepsilon \int_0^t \alpha(t)\eta(s)\, ds - c\varepsilon \int_0^{t_0} \alpha(t)\eta(s)\, ds.$$

On the other hand, since $\lim_{t\to 0^+} H_1(t) = +\infty$ and

$$0 \leq \mathcal{R}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(0),$$

we obtain for ε small enough

$$H_1(\mathcal{R}(t_0)) - c\varepsilon \int_0^{t_0} \alpha(t)\eta(s)\, ds > 0.$$

Then, thanks to the fact that H_1^{-1} is decreasing , we infer

$$\begin{aligned} \mathcal{R}(t) &\leq H_1^{-1}\left(H_1(\mathcal{R}(t_0)) + c\varepsilon \int_0^t \alpha(t)\eta(s)\, ds - c\varepsilon \int_0^{t_0} \alpha(t)\eta(s)\, ds\right) \\ &\leq H_1^{-1}\left(c\varepsilon \int_0^t (\alpha\eta)(s)\, ds\right). \end{aligned}$$

From this end inequality and (4.54) we get easily (4.46). Then the proof is completed. ∎

Remark 4.1 We can obtain the same result by using a similar techniques even in the absence of the second nonlinear term $f(\theta_t)$.

CHAPTER 5

EXPONENTIAL DECAY FOR A THERMO-VISCOELASTIC BRESSE SYSTEM WITH SECOND SOUND AND DELAY TERMS

5.1 Introduction

In the present chapter, we consider the following thermo-viscoelastic Bresse system with second sound and delay terms

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t-\tau_1\right) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \delta\int_0^t g\left(t-s\right)\psi_{xx}\left(x, s\right)ds + \gamma\theta_x = 0 \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \lambda_1\omega_t + \lambda_2\omega_t\left(x, t-\tau_2\right) = 0 \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0 \\ \alpha q_t + \beta q + \theta_x = 0, \end{cases} \tag{5.1}$$

with the initial data and boundary conditions

$$\begin{aligned} &\varphi\left(x,0\right) = \varphi_0\left(x\right),\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right),\ \psi\left(x,0\right) = \psi_0\left(x\right),\ \psi_t\left(x,0\right) = \psi_1\left(x\right), \\ &\omega\left(x,0\right) = \omega_0\left(x\right), \omega_t\left(x,0\right) = \omega_1\left(x\right), \theta\left(x,0\right) = \theta_0\left(x\right),\ \theta_t\left(x,0\right) = \theta_1\left(x\right), \\ &q\left(x,0\right) = q_0\left(x\right),\ q_t\left(x,0\right) = q_1\left(x\right). \end{aligned}$$

$$\begin{aligned} &\varphi\left(0,t\right) = \psi_x\left(0,t\right) = \omega_x\left(0,t\right) = \theta\left(0,t\right) = \omega\left(L,t\right) = \psi\left(L,t\right) = \varphi_x\left(L,t\right) \\ &= q\left(L,t\right) = 0,\ t \in (0,+\infty). \\ &\varphi_t\left(x, t-\tau_1\right) = f_0\left(x, t-\tau_1\right),\ \ (x,t) \in (0,L)\times(0,\tau_1), \\ &\omega_t\left(x, t-\tau_2\right) = \widetilde{f}_0\left(x, t-\tau_2\right),\ \ (x,t) \in (0,L)\times(0,\tau_2). \end{aligned} \tag{5.2}$$

where $(x,t) \in (0,L)\times\mathbb{R}_+$, $\rho_1, \rho_2, \rho_3, \alpha, \beta, k, k_0, l, b, \delta, \gamma, \mu_1, \lambda_1$ are positive constants, μ_2 and λ_2 are real numbers, $\tau_1, \tau_2 > 0$ represent the time delays, θ is the difference temperature, q is the heat flux and g is a positive function satisfying some conditions to be determined later.

Originally, the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as (see [23]):

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1 \\ \rho_2\psi_{tt} = M_x - Q + F_2 \\ \rho_1\omega_{tt} = N_x - lQ + F_3, \end{cases} \tag{5.3}$$

where in our work

$$\begin{aligned} M &= b\psi_x - \delta\int_0^t g(t-s)\,\psi_x(.,s)\,ds,\ \ N = k_0(\omega_x - l\varphi),\ \ Q = k(\varphi_x + \psi + l\omega), \\ F_1 &= -\mu_1\varphi_t - \mu_2\varphi_t(.,t-\tau_1),\ \ F_2 = 0,\ \text{ and } F_3 = -\lambda_1\omega_t - \lambda_2\omega_t(.,t-\tau_2). \end{aligned}$$

N, Q and M denote the axial force, the shear force and the bending moment. By ω, φ, and ψ, we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1 = \rho A$, $\rho_2 = \rho l$, $b = EI$, $k_0 = EA$, $k = k_0 GA$ and $l = R^{-1}$. For material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, k for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denote external forces. The Bresse system (5.3), is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $(l = 0)$.

The issue of existence and stability of Bresse system has attracted a great deal of attention in the last decades (e.g. [5, 7, 16, 23, 38, 39]). In the absence of viscoelastic damping $(g = 0)$, frictional damping $\mu_1 = \lambda_1 = 0$ and delay terms $\mu_2 = \lambda_2 = 0$, Keddi et al. [59] studied the following one-dimensional thermoelastic Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma\theta_x = 0 \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0 \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0 \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{5.4}$$

where the heat conduction is given by Cattaneo's law effective in the shear angle displacement. They established the well-posedness of the system and proved, under a condition on the parameters ζ, k and k_0, which is

$$\zeta := \left(1 - \frac{\tau k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\tau\gamma^2}{b} = 0 \text{ and } k = k_0,$$

that the system was exponentially stable depending on the stable number of the system, and showed that in general, the system was polynomially stable if $\zeta \neq 0$ and $k = k_0$. Li et al. [64] extended this last result to the following Bresse system with delay

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu\varphi_t\left(x, t-\tau_0\right) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_x = 0 \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = 0 \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0 \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{5.5}$$

They proved that the system is well-posed by using the semigroups method, and under a similar condition on the precedent parameters, that is

$$\zeta := \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\gamma^2\rho_1}{bk\rho_3} = 0 \text{ and } k = k_0,$$

they showed that the dissipation induced by the heat is strong enough to exponentially stabilize the system in the presence of a "small" delay when the stable number is zero.

Motivated by the works mentioned above, we investigate system (5.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$), the frictional damping ($\lambda_1, \mu_1 \neq 0$) and the second delay term ($\lambda_2 \neq 0$), we can establish an exponential decay result regardless of the stable number ζ. We prove our result by using the energy method together with some hypotheses on the weights of the delays and the frictional damping as well the relaxation function g.

5.2 Preliminaries

In this section, we present some materials needed in the proof of our results. We also state, without proof, a local existence result for problem (5.1). The proof can be established by using Faedo–Galerkin method [29].

We shall use the following assumptions:

($\mathbf{A}_1$) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad b - \delta\int_0^\infty g(s)ds = b - \delta g_1 = l > 0, \tag{5.6}$$

($\mathbf{A}_2$) There exists a non-increasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\eta(t)g(t), \qquad t \geq 0 \quad \text{and} \quad \int_0^\infty \eta(t)dt = +\infty. \tag{5.7}$$

We introduce the new variable as in [78]

$$z_1(x,\rho,t) = \varphi_t(x, t-\tau_1\rho), \qquad x \in (0,L), \rho \in (0,1), t > 0, \tag{5.8}$$

$$z_2(x,\rho,t) = \omega_t(x, t-\tau_2\rho), \qquad x \in (0,L), \rho \in (0,1), t > 0. \tag{5.9}$$

Then, we have

$$\begin{aligned} \tau z_{1t}(x,\rho,t)+z_{1\rho}(x,\rho,t) &= 0, \qquad x\in(0,L),\rho\in(0,1),t>0,\\ \tau z_{2t}(x,\rho,t)+z_{2\rho}(x,\rho,t) &= 0, \qquad x\in(0,L),\rho\in(0,1),t>0. \end{aligned}$$

Hence, problem (5.1)-(5.2) is equivalent to the following system, where $(x,\rho,t)\in(0,L)\times(0,1)\times(0,+\infty)$

$$\begin{cases} \rho_1\varphi_{tt}-k\left(\varphi_x+\psi+l\omega\right)_x-k_0l\left(\omega_x-l\varphi\right)+\mu_1\varphi_t+\mu_2 z_1(x,1,t)=0\\ \tau_1 z_{1_t}(x,\rho,t)+z_{1_\rho}(x,\rho,t)=0\\ \rho_2\psi_{tt}-b\psi_{xx}+k\left(\varphi_x+\psi+l\omega\right)+\delta\int_0^t g\left(t-s\right)\psi_{xx}\left(x,s\right)ds+\gamma\theta_x=0\\ \rho_1\omega_{tt}-k_0\left(\omega_x-l\varphi\right)_x+kl\left(\varphi_x+\psi+l\omega\right)+\lambda_1\omega_t+\lambda_2 z_2(x,1,t)=0\\ \tau_2 z_{2_t}(x,\rho,t)+z_{2_\rho}(x,\rho,t)=0\\ \rho_3\theta_t+q_x+\gamma\psi_{tx}=0\\ \alpha q+\beta q+\theta_x=0. \end{cases} \tag{5.10}$$

With the following initial data and boundary conditions

$$\begin{cases} \varphi\left(x,0\right)=\varphi_0\left(x\right),\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right), & x\in(0,L),\\ \psi\left(x,0\right)=\psi_0\left(x\right),\psi_t\left(x,0\right)=\psi_1\left(x\right), & x\in(0,L),\\ \omega\left(x,0\right)=\omega_0\left(x\right),\ \omega_t\left(x,0\right)=\omega_1\left(x\right), & x\in(0,L),\\ q\left(x,0\right)=q_0\left(x\right),\ q_t\left(x,0\right)=q_1\left(x\right), & x\in(0,L),\\ \theta\left(x,0\right)=\theta_0\left(x\right),\ \theta_t\left(x,0\right)=\theta_1\left(x\right), & x\in(0,L),\\ z_1(x,\rho,0)=f_0(x,-\rho\tau_1),\ z_2(x,\rho,0)=\widetilde{f}_0(x,-\rho\tau_2) & (x,\rho)\in(0,L)\times(0,1)\\ z_1(x,0,t)=\varphi_t(x,t),\quad z_2(x,0,t)=\omega_t(x,t) & (x,t)\in(0,L)\times(0,+\infty),\\ \varphi\left(0,t\right)=\psi_x\left(0,t\right)=\omega_x\left(0,t\right)=\theta\left(0,t\right)=0, & t\in(0,+\infty),\\ \omega\left(L,t\right)=\psi\left(L,t\right)=\varphi_x\left(L,t\right)=q\left(L,t\right)=0, & t\in(0,+\infty). \end{cases} \tag{5.11}$$

Along this section, we use the following notations

$$(f\diamond v)(t)=\int_0^t f(t-s)\left(v(t)-v(s)\right)ds,\quad \forall v\in L^2(0,L),$$

$$(f\circ v)(t)=\int_0^t f(t-s)(v(s)-v(t))^2ds.$$

The energy functional associated to (5.10)-(5.11), is

$$\begin{aligned} \mathcal{E}(t) \;=\; & \frac{1}{2}\int_0^L\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+\rho_3\theta^2+\alpha q^2+\left(b-\delta\int_0^t g(s)ds\right)\psi_x^2\right\}dx\\ &+\frac{1}{2}\int_0^L\left\{\xi_1\int_0^1 z_1^2(x,\rho,t)d\rho+\xi_2\int_0^1 z_2^2(x,\rho,t)d\rho+k\left(\varphi_x+\psi+l\omega\right)^2\right\}dx\\ &+\frac{1}{2}\int_0^L\left\{k_0\left(\omega_x-l\varphi\right)^2+\delta(g\circ\psi_x)\right\}dx \end{aligned} \tag{5.12}$$

5.2. Preliminaries

we denote $\mathcal{E}(t) = \mathcal{E}(t, \varphi, \psi, \omega, \theta, q, z_1, z_2)$ and $\mathcal{E}(0) = \mathcal{E}\left(0, \varphi_0, \psi_0, \omega_0, \theta_0, q_0, f_0, \widetilde{f}_0\right)$ for simplicity of notations.

For state a local existence result, we introduce the vector function:
$\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z_1, z_2)^T$, where $u = \varphi_t$, $v = \psi_t$ and $w = \omega_t$, using the standard Lebesgue space $L^2(0, L)$ and the Sobolev space $H_0^1(0, L)$ with their usual scalar products and norms for define the space $\mathcal{H}$ as follows

$$\mathcal{H} := H_*^1(0, L) \times L^2(0, L) \times \left[\widetilde{H_*^1}(0, L) \times L^2(0, L)\right]^2 \times \left[L^2(0, L)\right]^2 \times \left[L^2((0, L) \times (0, 1))\right]^2,$$

where

$$\begin{aligned} H_*^1(0, L) &= \{f \in H^1(0, L),\ f(0) = 0\}, \\ \widetilde{H_*^1}(0, L) &= \{f \in H^1(0, L),\ f(L) = 0\}, \\ H_*^2(0, L) &= H^2(0, L) \cap H_*^1(0, L), \\ \widetilde{H_*^2}(0, L) &= H^2(0, L) \cap \widetilde{H_*^1}(0, L). \end{aligned}$$

Proposition 5.1 *Let $\Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_0, q_0, f_0, \widetilde{f}_0)^T \in \mathcal{H}$ be given. Assume that $(A_1), (A_2)$, $\mu_1 > |\mu_2|$ and $\lambda_1 > |\lambda_2|$ are satisfied. Then Problem (5.10)-(5.11) possesses a unique global (weak) solution satisfying*

$$\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z_1, z_2)^T \in C\left(\mathbb{R}_+; \mathcal{H}\right).$$

5.3 Exponential stability result

In this section, we state and prove our exponential decay result for the energy of the solution of system (5.1)-(5.2), using the Lyapunov functional which is equivalent to the energy functional. To achieve our goal, we need the following technical lemmas.

The two inequalities in the following lemma are introduced in [33] and [48] respectively.

Lemma 5.1 *For any function $g \in C([0, +\infty), \mathbb{R}_+)$ and any $v \in L^2(0, L)$ we have*

$$[g \diamond v(t)]^2\, dx \le \left(\int_0^t g(s)ds\right) g \circ v(t), \quad \forall t \ge 0, \tag{5.13}$$

$$\int_0^L \left(\int_0^t g(t-s) v_x(s) ds\right)^2 dx \le 2g_1 \int_0^L g \circ v_x dx + 2g_1 \int_0^L v_x^2 dx. \tag{5.14}$$

Lemma 5.2 *[38] There exists a positive constant c such that the following inequality holds for every $(\varphi, \psi, \omega) \in \left[H_0^1(0, L)\right]^3$*

$$\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx \le c \int_0^L \left[b\psi_x^2 + k(\varphi_x + \psi_x + \omega_x)^2 + k_0(\omega_x - l\varphi)^2\right] dx. \tag{5.15}$$

Lemma 5.3 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11). Then the energy functional satisfies, for some n_0, $n_0^{'} > 0$,*

$$\begin{aligned}\mathcal{E}^{'}(t) \;\leq\; & -\beta\int_0^L q^2dx + \frac{\delta}{2}\int_0^L (g^{'}\circ\psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2dx \\ & -n_0\left(\int_0^L \varphi_t^2dx + \int_0^L z_1^2(x,1,t)dx\right) - n_0^{'}\left(\int_0^L \omega_t^2dx + \int_0^L z_2^2(x,1,t)dx\right) \leq 0\end{aligned}$$

where

$$\tau_1|\mu_2| < \xi_1 < \tau_1(2\mu_1 - |\mu_2|) \text{ and } \tau_2|\lambda_2| < \xi_2 < \tau_2(2\lambda_1 - |\lambda_2|). \quad (5.16)$$

Proof. *Multiplying Equation $(5.10)_1$ by φ_t , $(5.10)_3$ by ψ_t, $(5.10)_4$ by ω_t, $(5.10)_6$ by θ_t and $(5.10)_7$ by q, then integrating over $(0,L)$. Next, multiplying $(5.10)_2$ by $(\xi_1/\tau_1)z_1$ and $(5.10)_5$ by $(\xi_2/\tau_2)z_2$ and integrating over $(0,L)\times(0,1)$ with respect to ρ and x, we get*

$$\begin{aligned}& \frac{1}{2}\frac{d}{dt}\int_0^L\left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + \rho_3\theta^2 + b\psi_x^2\right\}dx \qquad (5.17)\\ & +\frac{1}{2}\frac{d}{dt}\int_0^L\left\{k(\varphi_x+\psi+l\omega)^2 + k_0(\omega_x - l\varphi)^2 + \alpha q^2\right\}dx \\ = & -\mu_1\int_0^L\varphi_t^2 - \lambda_1\int_0^L\omega_t^2 - \mu_2\int_0^L z_1(x,1,t)\varphi_tdx - \beta\int_0^L q^2dx \\ & -\lambda_2\int_0^L z_2(x,1,t)\omega_tdx - \delta\int_0^L\psi_t\int_0^t g(t-s)\psi_{xx}(s)dsdx,\end{aligned}$$

and

$$\begin{aligned}\frac{\xi_1}{\tau_1}\int_0^L\int_0^1 z_1z_{1\rho}(x,\rho,t)d\rho dx &= \frac{\xi_1}{\tau_1}\int_0^L\int_0^1\frac{d}{2d\rho}z_1^2(x,\rho,t)d\rho dx \\ &= \frac{\xi_1}{2\tau_1}\int_0^L[z_1^2(x,1,t) - z_1^2(x,0,t)]dx \\ &= \frac{\xi_1}{2\tau_1}\int_0^L z_1^2(x,1,t)dx - \frac{\xi_1}{2\tau_1}\int_0^L\varphi_t^2dx,\end{aligned}$$

$$\begin{aligned}\frac{\xi_2}{\tau_2}\int_0^L\int_0^1 z_2z_{2\rho}(x,\rho,t)d\rho dx &= \frac{\xi_2}{\tau_2}\int_0^L\int_0^1\frac{d}{2d\rho}z_2^2(x,\rho,t)d\rho dx \\ &= \frac{\xi_2}{2\tau_2}\int_0^L[z_2^2(x,1,t) - z_2^2(x,0,t)]dx \\ &= \frac{\xi_2}{2\tau_2}\int_0^L z_2^2(x,1,t)dx - \frac{\xi_2}{2\tau_2}\int_0^L\omega_t^2dx.\end{aligned}$$

Now, we estimate the last term on the left-hand side of (5.17).

$$\begin{aligned}\delta\int_0^L\psi_t(t)\int_0^t g(t-s)\psi_{xx}(s)dsdx &= \frac{\delta}{2}\frac{d}{dt}\int_0^L(g\circ\psi_x)dx + \frac{\delta}{2}g(t)\int_0^L\psi_x^2(t)dx \\ & -\frac{\delta}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_0^1\psi_x^2(t)dx\right) - \frac{\delta}{2}\int_0^L(g^{'}\circ\psi_x)dx.\end{aligned}$$

We have also

$$\begin{aligned} -\mu_2 \int_0^L z_1(x,1,t)\varphi_t dx &\leq \frac{|\mu_2|}{2}\left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx\right), \\ -\lambda_2 \int_0^L z_2(x,1,t)\omega_t dx &\leq \frac{|\lambda_2|}{2}\left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx\right) \end{aligned}$$

So, we conclude

$$\begin{aligned} \mathcal{E}'(t) \leq\ & \frac{\delta}{2}\int_0^L (g' \circ \psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2 dx - \left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\int_0^L \varphi_t^2 dx \\ & - \left(\lambda_1 - \frac{\xi_2}{2\tau_2} - \frac{|\lambda_2|}{2}\right)\int_0^L \omega_t^2 dx - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\int_0^L z_1^2(x,1,t)dx \\ & - \left(\frac{\xi_2}{2\tau_2} - \frac{|\lambda_2|}{2}\right)\int_0^L z_2^2(x,1,t)dx. \end{aligned}$$

Using (5.16), we have, for some n_0, $n_0' > 0$,

$$\begin{aligned} \mathcal{E}'(t) \leq\ & \frac{\delta}{2}\int_0^L (g' \circ \psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2 dx - n_0\left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx\right) \\ & -n_0'\left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx\right) \leq 0. \end{aligned}$$

■

Lemma 5.4 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be a solution of (5.10)-(5.11). Then the functional*

$$\mathcal{I}_1(t) = -\rho_2 \int_0^L \psi_t \left(\int_0^t g(t-s)(\psi(t) - \psi(s)ds\right)dx \tag{5.18}$$

satisfies for any $\delta' > 0$

$$\begin{aligned} \mathcal{I}_1'(t) \leq\ & -\rho_2\left(g_0 - \delta'\right)\int_0^L \psi_t^2 dx + \left(b^2 + \delta^2 g_1^2 - 2b\delta g_0\right)\delta' \int_0^L \psi_x^2 dx \\ & + k\delta' \int_0^L (\varphi_x + \psi + l\omega)^2 dx - \frac{\rho_2 g(0)}{4\delta'}\int_0^L (g' \circ \psi_x) dx \\ & + C\left(\delta'\right)\int_0^L g \circ \psi_x dx + \frac{1}{2}\int_0^L \theta^2 dx. \end{aligned} \tag{5.19}$$

Proof. *Taking the derivative of $\mathcal{I}_1$, using the third equation in (5.10), we obtain*

$$\begin{aligned} \mathcal{I}_1'(t) =\ & -\rho_2 \int_0^L \psi_t (g' \diamond \psi) dx - \rho_2 \left(\int_0^t g(s) ds\right)\int_0^L \psi_t^2 dx \\ & + \left(b - \delta \int_0^t g(s) ds\right)\int_0^L (g \diamond \psi_x)\psi_x dx + k\int_0^L (\varphi_x + \psi + lw)(g \diamond \psi) dx \\ & + \delta \int_0^L (g \diamond \psi_x)^2 dx - \int_0^L \theta (g \diamond \psi_x) dx \end{aligned} \tag{5.20}$$

By using Young's inequality, and (5.13), we get, for any $\delta^{'} > 0$

$$\delta \int_0^L (g \diamond \psi_x)^2 \, dx \leq \delta g_1 \int_0^L (g \circ \psi_x) \, dx \tag{5.21}$$

$$-\int_0^L \psi_t \left(g' \diamond \psi\right) dx \leq \delta^{'} \int_0^L \psi_t^2 dx - \frac{\rho_2 g\left(0\right)}{4\delta^{'}} \int_0^L \left(g' \circ \psi_x\right) dx \tag{5.22}$$

$$k \int_0^L (\varphi_x + \psi + lw) \left(g \diamond \psi\right) dx \ \leq k\delta^{'} \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \frac{g_1 k}{4\delta^{'}} \int_0^L (g \circ \psi_x) \, dx \tag{5.23}$$

$$\begin{aligned} \left(b - \delta \int_0^t g\left(s\right) ds\right) \int_0^L \left(g \diamond \psi_x\right) \psi_x dx \ &\leq \ \left(b^2 + \delta^2 g_1^2 - 2b\delta g_0\right) \delta^{'} \int_0^L \psi_x^2 dx \\ &+ \frac{g_1}{4\delta^{'}} \int_0^L \left(g \circ \psi_x\right) dx \end{aligned} \tag{5.24}$$

$$-\int_0^L \theta \left(g \diamond \psi_x\right) dx \leq \frac{1}{2} \int_0^L \theta^2 dx + \frac{g_1}{2} \int_0^L \left(g \circ \psi_x\right) dx \tag{5.25}$$

Combining (5.20)-(5.25), the result follows. ■

Lemma 5.5 *Let* $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ *be the solution of (5.10)-(5.11), then for* $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, *the functional*

$$\mathcal{I}_2(t) = -\frac{\rho_2 \rho_3}{\gamma} \int_0^L \theta \int_0^x \psi_t(y) dy dx \tag{5.26}$$

satisfies the estimate

$$\begin{aligned} \mathcal{I}_2^{'}(t) \ &\leq \ -\frac{\rho_2}{\gamma} \int_0^L \psi_t^2 dx + \epsilon_1 \int_0^L (\varphi_x + \psi + l\omega)^2 \, dx + c\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + 1\right) \int_0^L \theta^2 dx \\ &+ (\epsilon_2 + 2g_1\epsilon_3) \int_0^L \psi_x^2 dx + c \int_0^L q^2 dx + 2g_1 \epsilon_3 \int_0^L g \circ \psi_x dx. \end{aligned} \tag{5.27}$$

Proof. A simple differentiation of $\mathcal{I}_2$, then exploiting the third and sixth equations in (5.10), leads to

$$\begin{aligned} \mathcal{I}_2^{'}(t) \ &= \ -\rho_2 \int_0^L \psi_t^2 dx + \rho_3 \int_0^L \theta^2 dx - \frac{\rho_2}{\gamma} \int_0^L q\psi_t dx - \frac{b\rho_3}{\gamma} \int_0^L \theta \psi_x dx \\ &- \frac{k\rho_3}{\gamma} \int_0^L (\varphi_x + \psi + l\omega) \int_0^x \theta(y) dy dx + \frac{\delta \rho_3}{\gamma} \int_0^L \theta \int_0^t g(t-s) \psi_x ds dx \end{aligned}$$

Estimate (5.27) follows by using Cauchy–Schwarz and Young's inequalities. ■

Lemma 5.6 *Let* $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ *be the solution of (5.10)-(5.11), then for* $\epsilon_4 > 0$, *the functional*

$$\mathcal{I}_3(t) = \alpha \rho_3 \int_0^L \theta \int_0^x q(y) dy dx \tag{5.28}$$

satisfies the estimate

$$\mathcal{I}_3'(t) \leq -\frac{\rho_3}{2}\int_0^L \theta^2 dx + \delta' \int_0^L \psi_t^2 dx + c\left(1+\frac{1}{4\delta'}\right)\int_0^L q^2 dx. \tag{5.29}$$

Proof. A simple differentiation of $\mathcal{I}_3$, then exploiting the last two equations in (5.10), leads to

$$\mathcal{I}_3'(t) = -\rho_3 \int_0^L \theta^2 dx + \alpha \int_0^L q^2 dx + \alpha\gamma \int_0^L q\psi_t dx - \beta\rho_3 \int_0^L \theta \int_0^x q(y)dydx$$

Estimate (5.29) follows by using Cauchy–Schwarz and Young's inequalities. ■

Lemma 5.7 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11), then for $\delta' > 0$, the functional*

$$\mathcal{I}_4(t) = \rho_1 \int_0^L \varphi_t \left(\varphi + \int_0^x \psi(y)dy\right) dx \tag{5.30}$$

satisfies the estimate

$$\begin{aligned} \mathcal{I}_4'(t) \quad &\leq \quad -\frac{k}{2}\int_0^L (\varphi_x+\psi+l\omega)^2 dx - \frac{lk_0}{2}\int_0^L (\omega_x - l\varphi)^2 dx + \delta' c \int_0^L \psi_t^2 dx \\ &\quad + \left(c+\frac{1}{4\delta'}\right)\int_0^L \varphi_t^2 dx + c\int_0^L z_1^2(x,1,t)dx. \end{aligned} \tag{5.31}$$

Proof. A simple differentiation of $\mathcal{I}_4$, then exploiting the first equation in (5.10), leads to

$$\begin{aligned} \mathcal{I}_4'(t) \quad &= \quad \rho_1 \int_0^L \varphi_t \int_0^x \psi_t(y)dydx - \mu_2 \int_0^L \left(\varphi + \int_0^x \psi(y)dy\right) z_1(x,1,t)dx \\ &\quad -k\int_0^L (\varphi_x+\psi+l\omega)^2 dx + \rho_1 \int_0^L \varphi_t^2 dx - lk_0 \int_0^L (\omega_x - l\varphi)^2 dx \\ &\quad -\mu_1 \int_0^L \varphi_t \left(\varphi + \int_0^x \psi(y)dy\right) dx. \end{aligned}$$

Using Cauchy–Schwarz, Poincaré's and Young's inequalities gives (5.31). ■

Lemma 5.8 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11), then for $\delta', \epsilon_4 > 0$, the functional*

$$\mathcal{I}_5(t) = \rho_2 \int_0^L \psi\psi_t dx \tag{5.32}$$

satisfies the estimate

$$\begin{aligned} \mathcal{I}_5'(t) \quad &\leq \quad \left(-\frac{b}{2}+\frac{\delta^2}{4\delta'}+\frac{\gamma^2}{\epsilon_4}+2g_1\delta'\right)\int_0^L \psi_x^2 dx + 2g_1\delta' \int_0^L (g\circ\psi_x)\,dx \\ &\quad +\rho_2 \int_0^L \psi_t^2 dx + \frac{k^2}{b}\int_0^L (\varphi_x+\psi+l\omega)^2 dx + \epsilon_4 \int_0^L \theta^2 dx. \end{aligned} \tag{5.33}$$

Proof. A simple differentiation of $\mathcal{I}_5$, then exploiting the first equation in (5.10), leads to

$$\begin{aligned}\mathcal{I}_5^{'}(t) &= -\frac{b}{2}\int_0^L \psi_x^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \gamma \int_0^L \theta \psi_x dx \\ &\quad -k\int_0^L (\varphi_x + \psi + l\omega)\psi dx + \delta \int_0^L \psi_x \int_0^t g(t-s)\psi_x ds dx.\end{aligned}$$

Using (5.13), (5.14), Cauchy–Schwarz, Poincaré's and Young's inequalities gives (5.33). ■

Lemma 5.9 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11) and for $k = k_0$ and $\delta^{'} > 0$, the functional*

$$\mathcal{I}_6(t) = -\rho_1 \int_0^L \varphi_t (\omega_x - l\varphi)\, dx - \rho_1 \int_0^L \omega_t (\varphi_x + \psi + l\omega)\, dx \tag{5.34}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_6^{'}(t) &\leq \left(2\delta^{'} - k_0 l\right) \int_0^L (\omega_x - l\varphi)^2 dx + \left(\rho_1 l + \frac{\mu_1^2}{4\delta^{'}}\right) \int_0^L \varphi_t^2 dx \\ &\quad + \left(kl + 2\delta^{'}\right)\int_0^L (\varphi_x + \psi + l\omega)^2 dx + \left(\frac{\rho_1^2}{4\delta^{'}} + \frac{\lambda_1^2}{4\delta^{'}} - \rho_1 l\right)\int_0^L \omega_t^2 dx \\ &\quad + \delta^{'}\int_0^L \psi_t^2 dx + \frac{\mu_2^2}{4\delta^{'}}\int_0^L z_1^2(x,1,t)dx + \frac{\lambda_2^2}{4\delta^{'}}\int_0^L z_2^2(x,1,t)dx.\end{aligned} \tag{5.35}$$

Proof. A simple differentiation of $\mathcal{I}_6$, using the first and fourth equations in (5.10), leads to

$$\begin{aligned}\mathcal{I}_6^{'}(t) &= -k_0 l \int_0^L (\omega_x - l\varphi)^2 dx + \rho_1 l \int_0^L \varphi_t^2 dx + kl \int_0^L (\varphi_x + \psi + l\omega)^2 dx \\ &\quad -\rho_1 l \int_0^L \omega_t^2 dx - \rho_1 \int_0^L \omega_t \psi_t dx + \mu_1 \int_0^L \varphi_t (\omega_x - l\varphi)\, dx \\ &\quad + \lambda_1 \int_0^L \omega_t (\varphi_x + \psi + l\omega)\, dx + \mu_2 \int_0^L z_1(x,1,t)(\omega_x - l\varphi)\, dx \\ &\quad + \lambda_2 \int_0^L z_2(x,1,t)(\varphi_x + \psi + l\omega)\, dx\end{aligned}$$

Using Young's inequality for the last five terms in the right-hand side gives (5.35) under the condition $k = k_0$. ■

Lemma 5.10 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be a solution of (5.10)-(5.11). Then the functional*

$$\mathcal{I}_7(t) = -\rho_1 \int_0^L (\varphi\varphi_t + \omega\omega_t)\, dx - \frac{\mu_1}{2}\int_0^L \varphi^2 dx - \frac{\lambda_1}{2}\int_0^L \omega^2 dx$$

satisfies, for $c>0$, the estimate

$$\begin{aligned} \mathcal{I}_7'(t) &\leq -\rho_1 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \omega_t^2 dx + c \int_0^L (\varphi_x + \psi + l\omega)^2 dx \qquad (5.36) \\ &+c \int_0^L (\omega_x - l\varphi)^2 dx + c \int_0^L \psi_x^2 dx + \frac{\mu_2^2}{2} \int_0^L z_1^2(x,1,t)dx \\ &+\frac{\lambda_2^2}{2} \int_0^L z_2^2(x,1,t)dx \qquad (5.37) \end{aligned}$$

Proof. Taking the derivative of $\mathcal{I}_7$, by using equations in (5.10), we get

$$\begin{aligned} \mathcal{I}_7'(t) &= -\rho_1 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \omega_t^2 dx + k \int_0^L (\varphi_x + \psi + l\omega)^2 dx \qquad (5.38) \\ &+k_0 \int_0^L (\omega_x - l\varphi)^2 dx - k \int_0^L (\varphi_x + \psi + lw)\, \psi dx \\ &+\mu_2 \int_0^L \varphi z_1(x,1,t)dx + \lambda_2 \int_0^L \omega z_2(x,1,t)dx, \qquad (5.39) \end{aligned}$$

according to (5.15), we have the following relation where c is a positive constant

$$\int_0^L \left[\varphi_x^2 + \psi_x^2 + \omega_x^2\right] dx \leq c \int_0^L \left[(\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx. \qquad (5.40)$$

We obtain the result by using (5.40) and Young's inequality. ■

Lemma 5.11 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11). Then the functional $\mathcal{I}_8$ defined by*

$$\mathcal{I}_8(t) = \tau_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1^2(x,\rho,t) d\rho dx \qquad (5.41)$$

satisfies

$$\mathcal{I}_8'(t) \leq -2I_8(t) - C_1 \int_0^L z_1^2(x,1,t)dx + \int_0^L \varphi_t^2 dx \qquad (5.42)$$

Proof. *By differentiating $\mathcal{I}_8$, then by using $(5.10)_2$ and $(5.10)_5$, and integrating by parts, we get*

$$\begin{aligned} \mathcal{I}_8'(t) &= -2 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1 z_{1_\rho}(x,\rho,t) d\rho dx \\ &= -2\tau_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1^2(x,\rho,t) d\rho dx - \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-2\tau_1 \rho} z_1^2(x,\rho,t)\right) d\rho dx \\ &= -2I_8(t) - \int_0^L e^{-2\tau_1} z_1^2(x,1,t)dx + \int_0^L \varphi_t^2 dx. \\ &= -2I_8(t) - C_1 \int_0^L z_1^2(x,1,t)dx + \int_0^L \varphi_t^2 dx. \end{aligned}$$

for $C_1 > 0$. ■

5.3. Exponential stability result

Lemma 5.12 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11). Then the functional $\mathcal{I}_8$ defined by*

$$\mathcal{I}_9(t) = \tau_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2^2(x, \rho, t) d\rho dx \tag{5.43}$$

satisfies

$$\mathcal{I}_9'(t) \leq -2I_9(t) - C_2 \int_0^L z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx \tag{5.44}$$

Proof. *By differentiating $\mathcal{I}_8$, then by using $(5.10)_2$ and $(5.10)_5$, and integrating by parts, we get*

$$\begin{aligned} \mathcal{I}_9'(t) &= -2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2 z_{2_\rho}(x, \rho, t) d\rho dx \\ &= -2\tau_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2^2(x, \rho, t) d\rho dx - \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-2\tau_2 \rho} z_2^2(x, \rho, t) \right) d\rho dx \\ &= -2I_9(t) - \int_0^L e^{-2\tau_2} z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx. \\ &= -2I_9(t) - C_2 \int_0^L z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx. \end{aligned}$$

for $C_2 > 0$. ■

Now, we are ready to state and prove the main result of this section. First, we define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = N\mathcal{E}(t) + \sum_{i=1}^{9} N_i \mathcal{I}_i(t) \tag{5.45}$$

satisfies, for N_i, $i = 1, 2, ..., 9$ are positive constants to be properly chosen later, with sufficiently large N, one can easily prove that

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t), \qquad \forall t \geq 0 \tag{5.46}$$

where α_1 and α_2 are positive constants.

Theorem 5.1 *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (5.10)-(5.11) and assume that $(A_1), (A_2)$, $k = k_0$, $\mu_1 > |\mu_2|$ and $\lambda_1 > |\lambda_2|$ hold. Then, the energy functional (5.12) satisfies,*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 \int_{t_0}^t \eta(s) ds}, \qquad \forall t \geq 0$$

where c_1 and c_2 are positive constants.

Proof. *From the estimates of the previous lemmas we have*

$$\begin{aligned}
\mathcal{L}^{'}(t) \leq\ & \{-n_0N + cN_4 + l\rho_1 N_6 - \rho_1 N_7 + N_8\} \int_0^L \varphi_t^2 dx \\
& + \left\{-\rho_2 g_0 N_1 - \frac{\rho_2}{\gamma} N_2 + \rho_2 N_5\right\} \int_0^L \psi_t^2 dx \\
& + \left\{-n_0^{'} N - l\rho_1 N_6 - \rho_1 N_7 + N_9\right\} \int_0^L \omega_t^2 dx + \{-\beta N + cN_2 + cN_3\} \int_0^L q^2 dx \\
& + \left\{-N\frac{\delta}{2} g(t) + (\epsilon_2 + 2g_1\epsilon_3) N_2 + \left(\frac{-b}{2} + \frac{\gamma^2}{\epsilon_4}\right) N_5 + cN_7\right\} \int_0^L \psi_x^2 dx \\
& + \left\{\frac{N_1}{2} + cN_2\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + 1\right) - \frac{\rho_3}{2} N_3 + \epsilon_4 N_5\right\} \int_0^L \theta^2 dx \\
& + \left\{-\frac{lk_0}{2} N_4 - lk_0 N_6 + cN_7\right\} \int_0^L (\omega_x - l\varphi)^2 dx \\
& + \left\{\epsilon_1 N_2 - \frac{k}{2} N_4 + \frac{k^2}{b} N_5 + lkN_6 + cN_7\right\} \int_0^L (\varphi_x + \psi + l\omega)^2 dx \\
& + \left\{-n_0 N + cN_4 + \frac{\mu_2^2}{2} N_7 - C_1 N_8\right\} \int_0^L z_1^2(x,1,t) dx \\
& + \left\{-n_0^{'} N + \frac{\lambda_2^2}{2} N_7 - C_1^{'} N_9\right\} \int_0^L z_2^2(x,1,t) dx \\
& + \{-mN_8\} \int_0^L z_1^2(x,\rho,t) dx + \{-mN_9\} \int_0^L z_2^2(x,\rho,t) dx \\
& + \left\{c(\delta^{'} N_1 + 2g_1\epsilon_3 N_2)\right\} \int_0^L (g \circ \psi_x) dx + N\frac{\delta}{2} \int_0^L \left(g^{'} \circ \psi_x\right) dx \\
& + \delta^{'} \int_0^L \left[(N_3 + \rho_2 N_1 + cN_4 + N_6) \psi_t^2 + (2N_6 + kN_1)(\varphi_x + \psi + l\omega)^2\right. \\
& \left. + 2N_6 (\omega_x - l\varphi)^2 + \left[\left(b^2 + \delta^2 - 2\delta b g_0\right) N_1 + 2g_1 N_5\right] \psi_x^2 + 2g_1 N_5 (g \circ \psi_x)\right] dx \\
& + \frac{1}{\delta^{'}} \int_0^L \left[\frac{\rho_2 g(0)}{4} N_1 (g^{'} \circ \psi_x) + \frac{N_3}{4} cq^2 + \frac{\delta^2}{4} N_5 \psi_x^2 + \left(\frac{\rho_1^2}{4} + \frac{\lambda_1^2}{4}\right) N_6 \omega_t^2\right] dx \\
& + \frac{1}{\delta^{'}} \int_0^L \left[\left(\frac{\mu_1^2}{4} N_6 + \frac{N_4}{4}\right) \varphi_t^2 + \frac{\mu_2^2}{4} N_6 z_1^2(x,1,t) + \frac{\lambda_2^2}{4} N_6 z_2^2(x,1,t)\right] dx \\
& + \{-mN_8\} \left(\int_0^L z_1^2(x,\rho,t) dx + \int_0^L z_2^2(x,\rho,t) dx\right)
\end{aligned}$$

$$\begin{aligned}
& + \left\{\left(c(\delta^{'}) + 2g_1\right) N_1\right\} \int_0^L (g \circ \psi_x) dx + N\frac{\delta}{2} \int_0^L \left(g^{'} \circ \psi_x\right) dx \\
& + \delta^{'} C_1(N_1, N_3, N_4) E(t) - \frac{1}{\delta^{'}} C_2(N_1, N_3, N_4) E^{'}(t).
\end{aligned}$$

5.3. Exponential stability result

By taking $\epsilon_2 = \epsilon_3 = \epsilon_4 = N_5 = N_6 = N_7 = 1$, $N_1 = N_2$ and $N_8 = N_9$, we arrive at

$$\begin{aligned}
\mathcal{L}^{'}(t) \leq\; & \{-n_0 N + cN_4 + l\rho_1 - \rho_1 + N_8\} \int_0^L \varphi_t^2 dx + \left\{ \left(-\rho_2 g_0 - \frac{\rho_2}{\gamma}\right) N_1 + \rho_2 \right\} \int_0^L \psi_t^2 dx \\
& + \left\{ -n_0^{'} N - l\rho_1 - \rho_1 + N_8 \right\} \int_0^L \omega_t^2 dx + \{-\beta N + cN_2 + cN_3\} \int_0^L q^2 dx \\
& + \left\{ -N\frac{\delta}{2} g(t) + (1+2g_1) N_1 + \frac{-b}{2} + \gamma^2 + c \right\} \int_0^L \psi_x^2 dx \\
& + \left\{ \left(\frac{1}{2} + c\left(\frac{1}{\epsilon_1} + 3\right)\right) N_1 - \frac{\rho_3}{2} N_3 + 1 \right\} \int_0^L \theta^2 dx \\
& + \left\{ -\frac{lk_0}{2} N_4 - lk_0 + c \right\} \int_0^L (\omega_x - l\varphi)^2 dx \\
& + \left\{ \epsilon_1 N_1 - \frac{k}{2} N_4 + \frac{k^2}{b} + lk + c \right\} \int_0^L (\varphi_x + \psi + l\omega)^2 dx \\
& + \left\{ -n_0 N + cN_4 + \frac{\mu_2^2}{2} - C_1 N_8 \right\} \int_0^L z_1^2(x,1,t) dx \\
& + \left\{ -n_0^{'} N + \frac{\lambda_2^2}{2} - C_1^{'} N_8 \right\} \int_0^L z_2^2(x,1,t) dx \qquad (5.47)
\end{aligned}$$

Let us choose N_4 large enough such that

$$-\frac{lk_0}{2} N_4 - lk_0 + c < 0,$$

Picking N_4 and choose N_1 large enough so that

$$\left(-\rho_2 g_0 - \frac{\rho_2}{\gamma}\right) N_1 + \rho_2 < 0,$$

choose ϵ_1 small enough so that

$$\epsilon_1 N_1 - \frac{k}{2} N_4 + \frac{k^2}{b} + lk + c < 0.$$

Next, we select N_3 large enough such that

$$\left(\frac{1}{2} + c\left(\frac{1}{\epsilon_1} + 3\right)\right) N_1 - \frac{\rho_3}{2} N_3 + 1 < 0,$$

Finally, we choose N sufficiently large to satisfy

$$-n_0 N + cN_4 + N_8 + \rho_1 (l-1) < 0, \quad -n_0^{'} N - C_1^{'} N_8 + \tfrac{\lambda_2^2}{2} < 0.$$

$$-n_0^{'} N + N_8 - \rho_1 (l+1) < 0, \quad -n_0 N + cN_4 - C_1 N_8 + \tfrac{\mu_2^2}{2} < 0,$$

$$-\beta N + cN_1 + cN_3 < 0, \quad -N\tfrac{\delta}{2} g(t) + (1+2g_1) N_1 + \tfrac{-b}{2} + \gamma^2 + c < 0.$$

Therefor, (5.47) *takes the form*

$$\mathcal{L}^{'}(t) \leq -\left[C_0 - C_1\left(N_1, N_3, N_4\right)\delta^{'}\right]\mathcal{E}(t) - \frac{C_2\left(N_1, N_3, N_4\right)}{\delta^{'}}\mathcal{E}^{'}(t) + C_3\int_0^L (g\circ\psi_x)dx$$

for some positive constants C_0, C_1, C_2, C_3*. At this point, we take* $\delta^{'} < \frac{C_0}{C_1}$*, then for some* $m_0 > 0$*, we obtain*

$$\mathcal{L}^{'}(t) \leq -m_0\mathcal{E}(t) + C_3\int_0^L (g\circ\psi_x)dx - \frac{C_2}{\delta^{'}}\mathcal{E}^{'}(t). \tag{5.48}$$

Multiplying (5.48) *by* $\eta(t)$ *gives*

$$\eta(t)\mathcal{L}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) + C_3\eta(t)\int_0^L (g\circ\psi_x)dx - \frac{C_2}{\delta^{'}}\eta(t)\mathcal{E}^{'}(t). \tag{5.49}$$

The second term can be estimated, using (A_2)*, as follows*

$$\begin{aligned} C_3\eta(t)\int_0^L (g\circ\psi_x)dx &= C_3\eta(t)\int_0^L\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)^2 dsdx \\ &\leq -\frac{2C_3}{\beta}\mathcal{E}^{'}(t), \end{aligned}$$

so for some $C_4 > 0$*,* (5.49) *becomes as follows*

$$\eta(t)\mathcal{L}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) - C_4^{'}\mathcal{E}(t) - \frac{C_2}{\delta^{'}}\eta(t)\mathcal{E}^{'}(t). \tag{5.50}$$

We have

$$\mathcal{F}(t) = \eta(t)\left(\mathcal{L}(t) + \frac{C_2}{\delta^{'}}\mathcal{E}(t)\right) \sim \mathcal{E}(t)$$

Therefore, using (5.50) *and the fact that* $\eta^{'}(t) \leq 0$*, we arrive at,*

$$\mathcal{F}^{'}(t) = \eta^{'}(t)\left(\mathcal{L}(t) + \frac{C_2}{\delta^{'}}\mathcal{E}(t)\right) + \eta(t)\left(\mathcal{L}^{'}(t) + \frac{C_2}{\delta^{'}}\mathcal{E}^{'}(t)\right) \leq \eta(t)\left(\mathcal{L}^{'}(t) + \frac{C_2}{\delta^{'}}\mathcal{E}^{'}(t)\right).$$

So

$$\mathcal{F}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) - C_4^{'}\mathcal{E}(t).$$

Now, we set

$$\mathcal{G}(t) = \mathcal{F}(t) + C_4\mathcal{E}(t) \sim \mathcal{E}(t),$$

gives

$$\mathcal{G}^{'}(t) = \mathcal{F}^{'}(t) + C_4\mathcal{E}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t). \tag{5.51}$$

A simple integration of (5.51) *over* (t_0, t) *leads to*

$$\mathcal{G}(t) \leq \mathcal{G}(t_0)e^{-m_0\int_{t_0}^t \eta(s)ds}. \tag{5.52}$$

Recalling (5.46) *and the estimate* (5.52) *completes the proof.* ■

5.3. Exponential stability result

Part II

Flexible Structures Systems

CHAPTER 6

GLOBAL EXISTENCE AND GENERAL DECAY FOR A DELAYED FLEXIBLE STRUCTURE WITH SECOND SOUND SUBJECTED TO WEAKLY NONLINEAR DAMPING

6.1 Introduction

In this chapter, we aim to study the following inhomogeneous delayed flexible structure system of second sound with weakly nonlinear damping

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + f(u_t) + \int_{\tau_1}^{\tau_2} \mu(s)u_t\,(t-s)\,ds = 0 \\ \theta_t + \eta u_{tx} + kq_x = 0 \\ \tau q_t + \beta q + k\theta_x = 0, \end{cases} \tag{6.1}$$

where $u(x,t)$ is the displacement of a particle at position $x \in (0,L)$ and time $t > 0$. $\eta > 0$ is the coupling constant, that accounts for the heating effect, and $\beta, k > 0$. θ is the temperature of the body, $q = q(x,t)$ is the heat flux and the parameter $\tau > 0$ is the relaxation time describing the time lag in the response for the temperature. $s > 0$ is a real number represents the time delay. $m(x)$, $\delta(x)$ and $p(x)$ are responsible for the non-uniform structure of the body, and, respectively, denote mass per unit length of structure, coefficient of internal material damping and a positive function related to the stress acting on the body at a point x, and for τ_1, τ_2 two real numbers satisfying $0 \leq \tau_1 < \tau_2$, $\mu : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function. f is specific function satisfying some conditions to be determined later. Here, $f(u_t)$ is the nonlinear dissipative term. The model of heat condition, originally due to Cattaneo, is of hyperbolic type.

We consider the following initial and boundary conditions:

$$\begin{aligned}
&u\left(.,0\right)=u_0\left(x\right), \; u_t\left(.,0\right)=u_1\left(x\right), \; \theta\left(.,0\right)=\theta_0\left(x\right), q\left(.,0\right)=q_0\left(x\right), \\
&u\left(0,t\right)=u\left(L,t\right)=\theta\left(0,t\right)=\theta\left(L,t\right)=0, \; \forall t\geq 0, \quad \forall x\in[0,L], \\
&u_t\left(x,-t\right)=g_0(x,t), \quad 0<t\leq\tau_2,
\end{aligned} \tag{6.2}$$

where g_0 is the history function.

The issue of existence and stability of flexible structure system has attracted a great deal of attention in the last years (e.g. [26, 43]). S. Misra et al. [73] considered the vibrations of a cantilever structure modeled by the standard linear flexible model of viscoelasticity coupled to an expectedly dissipative effect through heat conduction

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x-k\theta_x=f \\ \theta_t-\theta_{xx}-ku_{tx}=0. \end{cases}$$

The distributed force $f:(0,L)\times\mathbb{R}\to\mathbb{R}$ is the uncertain disturbance appearing in the model which is assumed to be continuously differentiable for all $t\geq 0$. By using semi-groups theory and multiplier technique, they established the well-posedness and an exponential stability of the system when the disturbing force is insignificant. In the absence of both delay and nonlinear damping terms, Alves et al. [8] concerned with the system

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\eta\theta_x=0 \\ \theta_t+kq_x+\eta u_{tx}=0 \\ \tau q_t+\beta q+k\theta_x=0. \end{cases} \tag{6.3}$$

They established the well-posedness of the system and proved its stability exponential and polynomial under suitable boundary conditions.

The original motivation of this type of problem was first introduced by Datko et al. [32] in 1986 when they showed that the presence of the delay may not only destabilize a system which is asymptotically stable in the absence of the delay but may also lead to ill-posedness (see also [78] and [84]). On the other hand, it has been established that voluntary introduction of delay can benefit the control (see [2]). We refer the interested readers to [9, 10, 18, 20, 41, 42] for details discussion on the subject. In the context of asymptotic stabilization with nonlinear feedback damping, first results are given in [1] (in 2002) where the author studies the asymptotic behaviour of the system governing the nonlinear vibrations of a Timoshenko beam,

$$\begin{cases} u_{tt}-\alpha\beta\left(u_x-v\right)_x-\gamma\left\|u_x\right\|^2u_{xx}+g\left(u_t\right)=0 \\ \frac{1}{\alpha}v_{tt}-v_{xx}-\alpha\beta\left(u_x-v\right)+g\left(v_t\right)=0, \end{cases} \tag{6.4}$$

where $g:R\to R$ is a C^1-class, non-increasing function with $g(0)=0$ and satisfying

$$\begin{aligned} &c_1\left|x\right|^r\leq g\left(x\right)\leq c_2\left|x\right|^{1/r} \quad \text{for } \left|x\right|\leq 1, \\ &c_3\left|x\right|^k\leq g\left(x\right)\leq c_4\left|x\right|^s \quad \text{for } \left|x\right|>1. \end{aligned}$$

Muñoz Rivera and Racke [76] treated a system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma\left(\varphi_x, \psi\right)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta_x = 0 \\ \rho_3\theta_t - K\theta_{xx} + \gamma\psi_{xt} = 0. \end{cases} \tag{6.5}$$

where φ, ψ and θ are functions of (x,t) model transverse displacement of the beam, the rotation angle of the filament and the difference temperature, respectively. Under appropriate conditions of σ, ρ_i, b, K and γ, they proved several exponential decay results for the linearized system and nonexponential stability result for the case of different wave speeds of propagation. Also, Muñoz Rivera and Racke [77] considered the following nonlinear Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - \sigma\left(\varphi_x, \psi\right)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + d\psi_t = 0. \end{cases} \tag{6.6}$$

with homogeneous boundary conditions and proved that the system is exponentially stable if and only if $\rho_1/k = \rho_2/b$ and a polynomial stability otherwise. Alabau-Boussouira [4] extended these last results of Muñoz Rivera and Racke to the case of nonlinear feedback $\alpha\left(\psi_t\right)$, instead of $d\psi$. He considered the following nonlinear Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \alpha\left(\psi_t\right) = 0. \end{cases} \tag{6.7}$$

where α is a globally Lipchitz function satisfying some growth conditions at the origin, and established a general semi-explicit formula for the decay rate of the energy at infinity in the case of the same speed of propagation in the two equations of the system $\left(\text{i.e. } \frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$.

Our purpose here is to obtain a general decay rate estimates of the energy, for this end we consider (6.3) with an internal distributed delay term subjected to non-linear damping in the first equation, under a suitable assumption on the weights of the delay, heating effect, material damping and the function f, we establish a well-posed result of the system using semigroups theory and a general stability using the multiplier method with some properties of convex functions and no growth assumption on f at the origin.

6.2 Well-posedness of the problem

In this section, we present some assumptions and give the existence and uniqueness result of system (6.1)-(6.2) using the semigroups theory. Taking the following new variable

$$z(x, \rho, s, t) = u_t\left(x, t - \rho s\right), \text{ in } (0, L) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Then we obtain

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = u_t(x,t). \end{cases}$$

Consequently, problem (6.1)-(6.2) is equivalent to

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + f(u_t) + \int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s)ds = 0 \\ \theta_t + kq_x + \eta u_{tx} = 0 \\ \tau q_t + \beta q + k\theta_x = 0 \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \end{cases} \quad 11m \qquad (6.8)$$

where $(x,\rho,s,t) \in (0,L)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty)$, with the following initial and boundary conditions:

$$\begin{array}{l} u(.,0) = \varphi_0(x), \;\; u_t(.,0) = \varphi_1(x), \;\; \theta(.,0) = \theta_0(x), q(.,0) = q_0(x), \\ u(0,t) = u(L,t) = \theta(0,t) = \theta(L,t) = 0, \;\; \forall t > 0, \;\; \forall x \in [0,L]. \\ z(x,\rho,s,0) = g_0(x,\rho s) \;\; \text{in} \;\; (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{array} \qquad (6.9)$$

We shall use the following assumptions:

(H1) $\mu : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds < \eta. \qquad (6.10)$$

(H2) The functions $m(x)$, $\delta(x)$ and $p(x)$ will be supposed such that:

$$m, \delta, p \in W^{1,\infty}(0,L), m(x), p(x) > 0, 2\delta(x) > l\eta, \forall x \in [0,L], \;\; l = L^2/\pi^2. \qquad (6.11)$$

(H3) $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous and non-decreasing function such that there exist positive constants k_1 and λ and a convex, continuous and increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying: $h(0) = 0$ and

$$h'' = 0 \text{ on } [0,\lambda], \qquad (6.12)$$

or

$$h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0,\lambda], \qquad (6.13)$$

such that

$$\begin{array}{ll} h\left(f^2(s)\right) \leq f(s)\, s & \text{for } \;|s| \leq \lambda, \\ |f(s)| \leq k_1 |s| & \text{for } \;|s| \geq \lambda. \end{array}$$

The aim of this section is to prove that system (6.8) is well-posed. From Equation $(6.8)_3$ and the boundary conditions (6.9), we have that

$$\frac{d}{dt}\int_0^L q(x,t)\, dx + \frac{\beta}{\tau}\int_0^L q(x,t)\, dx = 0.$$

So, if we set

$$\widetilde{q}(x,t) = q(x,t) - \frac{1}{L}\left(\int_0^L q_0(x)\,dx\right)\exp\left(-\frac{\beta t}{\tau}\right),$$

then, $(u, u_t, \theta, \widetilde{q})$ satisfies Equation (6.1), and

$$\int_0^L \widetilde{q}(x,t)\,dx = 0, \ \forall t \geq 0.$$

Therefore, the use of Poincaré's inequality for $\widetilde{q}$ is justified. In the sequel, we shall work with $\widetilde{q}$ but we write q for simplicity.

Let us introducing the vector function $U = (u, v, \theta, q, z)^T$, where $v = u_t$, using the standard Lebesgue space $L^2(0,L)$ and the Sobolev space $H_0^1(0,L)$ with their usual scalar products and norms for define the spaces:

$$\mathcal{H} := H_0^1(0,L) \times \left[L^2(0,L)\right]^2 \times L_*^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),$$

and

$$H_*^1(0,L) = H^1(0,L) \cap L_*^2(0,L),$$

where

$$L_*^2(0,L) = \left\{ w \in L^2(0,L) : \int_0^L w(s)\,ds = 0 \right\}.$$

We equip $\mathcal{H}$ with the inner product

$$\begin{aligned}(U,\widetilde{U})_{\mathcal{H}} &= \int_0^L p(x)\,u_x\widetilde{u}_x dx + \int_0^L m(x)\,v\widetilde{v}dx + \int_0^L \theta\widetilde{\theta}dx + \tau\int_0^L q\widetilde{q}dx \\ &\quad + \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z(x,\rho,s)\widetilde{z}(x,\rho,s)ds d\rho dx.\end{aligned}$$

Next, the system (6.8)-(6.9) can be reduced to the following abstract Cauchy problem:

$$\begin{cases} U'(t) + (\mathcal{A}+\mathcal{B})\,U(t) = 0, & t > 0 \\ U(0) = U_0 = (u_0, u_1, \theta_0, q_0, g_0)^T, \end{cases} \tag{6.14}$$

where the operators A and B are defined by: $A : D(A) \to \mathcal{H}$

$$\mathcal{A}U = \begin{pmatrix} -v \\ \frac{1}{m(x)}\left(-(p(x)u_x + 2\delta(x)v_x - \eta\theta)_x + \int_{\tau_1}^{\tau_2}\mu(s)z(1,s)ds + v\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\right) \\ kq_x + \eta v_x \\ \frac{1}{\tau}(k\theta_x + \beta q) \\ \frac{1}{s}z_\rho(x,\rho,s) \end{pmatrix},$$

and $B : D(\mathcal{B}) = \mathcal{H} \to \mathcal{H}$

$$\mathcal{B}U = \frac{1}{m(x)} \begin{pmatrix} 0 \\ -v \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds + f(v) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H} \mid u \in H^2(0,L) \cap H_0^1(0,L), v, \theta \in H_0^1(0,L), q \in H_*^1(0,L) \\ z, z_\rho \in L^2((0,L) \times (0,1) \times (\tau_1, \tau_2)),\ z(x,0,s) = v \end{array} \right\}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Before state an existence and uniqueness result, we refer the reader to [61] (from page 90), [81] and the references therein, for more details discussion about solutions of (6.14), then we have

Proposition 6.1 *Let $U_0 \in \mathcal{H}$ be given. Assume that (H1)−(H3) are satisfied, Problem (6.14) possesses then a unique solution satisfying $U \in C(\mathbb{R}^+; \mathcal{H})$. If $U_0 \in D(\mathcal{A})$, then $U \in C^1(\mathbb{R}^+; \mathcal{H}) \cap C(\mathbb{R}^+; D(\mathcal{A}))$.*

Proof. We use the semigroups approach to prove that A is a maximal monotone operator and that B is a Lipschitz continuous operator. In what follows, we prove that A is monotone. For any $U \in D(A)$, we have

$$\begin{aligned} (\mathcal{A}U, U)_{\mathcal{H}} &= -\int_0^L p(x)\, v_x u_x dx - \int_0^L [p(x)\, u_x]_x\, v dx + \int_0^L \eta\theta_x v dx \\ &\quad -2\int_0^L [\delta(x)\, v_x]_x\, v dx + \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\ &\quad +k\int_0^L \theta q_x dx + \int_0^L \eta\theta v_x dx + k\int_0^L \theta_x q dx + \int_0^L \beta q^2 dx \\ &\quad +\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z z_\rho ds d\rho dx + \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^L v^2 dx. \end{aligned}$$

Integration by parts and using the fact that

$$\begin{aligned} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z z_\rho ds d\rho dx &= \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 |\mu(s)| \frac{\partial}{\partial \rho} z^2 d\rho ds dx \\ &= \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx \\ &\quad -\frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^L v^2 dx, \end{aligned}$$

we get

$$
\begin{aligned}
(\mathcal{A}U,U)_{\mathcal{H}} &= 2\int_0^L \delta(x)\, v_x^2 dx + \int_0^L \beta q^2 dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^L v^2 dx \\
&\quad + \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds dx \\
&\quad + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s) ds dx. \qquad (6.15)
\end{aligned}
$$

By using Young's inequality, the fourth term on the right-hand side of Equation (6.15) gives

$$
\begin{aligned}
& -\int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds dx \\
\leq\ & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right) \int_0^L v^2 dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s) ds dx, \qquad (6.16)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds dx \\
\geq\ & -\frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right) \int_0^L v^2 dx - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s) ds dx,
\end{aligned}
$$

from this last, the Equation (6.15) yields

$$
(\mathcal{A}U,U)_{\mathcal{H}} \geq 2\int_0^L \delta(x)\, v_x^2 dx + \int_0^L \beta q^2 dx \geq 0.
$$

Hence, $\mathcal{A}$ is monotone. Next, we prove that the operator $I + A$ is surjective.

Given $G = (g_1, g_2, g_3, g_4, g_5)^T \in \mathcal{H}$, we prove that there exists $U \in D(\mathcal{A})$ satisfying

$$
(\mathcal{I} + \mathcal{A})\, U = G, \qquad (6.17)
$$

which gives

$$
\begin{aligned}
& -v + u = g_1, \\
& -(p(x)u_x + 2\delta(x)v_x - \eta\theta)_x + \int_{\tau_1}^{\tau_2} \mu(s) z(1,s) ds + \left(\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds + m(x)\right) v = m(x)\, g_2, \\
& kq_x + \eta v_x + \theta = g_3, \\
& k\theta_x + (\beta + \tau)\, q = \tau g_4, \\
& z_\rho + sz = sg_5.
\end{aligned}
\qquad (6.18)
$$

Suppose that u, q are given with the appropriate regularity. Then, Equations $(6.18)_1$ and $(6.18)_4$ yield

$$
v = u - g_1 \in H_0^1(0,L), \qquad (6.19)
$$

$$\theta_x = \frac{\tau}{k} g_4 - \frac{\beta+\tau}{k} q \in L_*^2(0,L), \tag{6.20}$$

by using (6.20), we get

$$\theta = \frac{\tau}{k}\int_0^x g_4(y)dy - \frac{\beta+\tau}{k}\int_0^x q(y)dy,$$

and then

$$\theta\,(0,t) = \theta\,(L,t) = 0.$$

The last equation in Equation (6.18) together with Equation (6.19) and the fact $z(x,0) = v(x) = u - g_1(x)$ yield

$$z(x,\rho,s) = ue^{-s\rho} - e^{-s\rho} g_1 + se^{-s\rho}\int_0^\rho e^{s\upsilon} g_5(x,\upsilon,s)d\upsilon. \tag{6.21}$$

From equation (6.19)-(6.21), we can easily show that u and q satisfy

$$\begin{cases} -(p(x)u_x + 2\delta(x)v_x)_x - \frac{\eta(\beta+\tau)}{k} q + u\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds + \gamma u(x) = f_1, \\ -k^2 q_x + (\beta+\tau)\int_0^x q(y)dy - k\eta u_x = f_2, \\ -v_x + u_x = f_3, \end{cases} \tag{6.22}$$

where

$$\begin{aligned} &\gamma = m\,(x) + \textstyle\int_{\tau_1}^{\tau_2} |\mu(s)|\,ds, \\ &z_0(x,s) = e^{-s} g_1(x) - se^{-s}\textstyle\int_0^1 e^{s\upsilon} g_5(x,\upsilon,s)d\upsilon, \\ &f_1 = \gamma g_1(x) + m\,(x)\,g_2(x) - \tfrac{\eta\tau}{k} g_4(x) + \textstyle\int_{\tau_1}^{\tau_2}\mu(s)z_0(x,s)ds \in L^2(0,L), \\ &f_2 = -k\eta g_1(x) + \tau\textstyle\int_0^x g_4(y)dy - kg_3 \in L^2(0,L), \\ &f_3 = g_{1x}(x) \in L^2(0,L). \end{aligned}$$

The variational formulation corresponding to Equation (6.22) takes the form

$$B((u,q),(\widetilde{u},\widetilde{q})) = F(\widetilde{u},\widetilde{q}), \tag{6.23}$$

where $B : \left[H_0^1(0,L)\times L_*^2(0,L)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B((u,q),(\widetilde{u},\widetilde{q})) \;=\; &\int_0^L (p(x)+2\delta(x))\,u_x\widetilde{u}_x dx - \frac{\eta\,(\beta+\tau)}{k}\int_0^L q\widetilde{u}dx \\ &+ (\beta+\tau)\int_0^L q\widetilde{q}dx + \gamma\int_0^L u\widetilde{u}dx \\ &+ \frac{(\beta+\tau)^2}{k^2}\int_0^L\left(\int_0^x q(y)dy\int_0^x \widetilde{q}(y)dy\right)dx \\ &+ \frac{\eta\,(\beta+\tau)}{k}\int_0^L u\widetilde{q}dx + \int_0^L u\widetilde{u}\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}dsdx, \end{aligned}$$

and $F : H_0^1(0,L) \times L_*^2(0,L) \to \mathbb{R}$ is the linear functional defined by

$$\begin{aligned} F(\widetilde{u},\widetilde{q}) &= \int_0^L f_1\widetilde{u}dx + \frac{(\beta+\tau)}{k^2}\int_0^L f_2\left(\int_0^x \widetilde{q}(y)dy\right)dx \\ &\quad + \int_0^L 2f_3\delta(x)\widetilde{u}_x dx. \end{aligned}$$

For $V = H_0^1(0,L) \times L_*^2(0,L)$ equipped with the norm

$$\|(u,q)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|q\|_2^2,$$

where $\|.\|_2$ is the usual norm.

One can easily see that B and F are bounded. Also, we get

$$\begin{aligned} B((u,q),(u,q)) &= \int_0^L (p(x)+2\delta(x))\,u_x^2dx + (\beta+\tau)\int_0^L q^2dx \\ &\quad +\gamma\int_0^L u^2dx + \frac{\beta+\tau}{k^2}\int_0^L\left(\int_0^x q(y)dy\right)^2dx \\ &\quad + \int_0^L u^2dx\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds \\ &\geq c\,\|(u,q)\|_V^2. \end{aligned}$$

Then, B is coercive. Consequently, by the Lax–Milgram lemma, system (6.22) has a unique solution

$$u \in H_0^1(0,L), \quad q \in L_*^2(0,L).$$

Moreover, if $\widetilde{q} \equiv 0 \in L_*^2(0,L)$, then Equation (6.23) reduces to

$$\begin{aligned} &-\int_0^L (p(x)u_x + 2\delta(x)u_x)_x\,\widetilde{u}dx + \gamma\int_0^L u\widetilde{u}dx - \frac{\eta(\beta+\tau)}{k}\int_0^L q\widetilde{u}dx \\ &+\int_0^L u\widetilde{u}\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}dsdx \\ =&\int_0^L f_1\widetilde{u}dx - \int_0^L (2f_3\delta(x))_x\,\widetilde{u}dx, \quad \forall\widetilde{u}\in H_0^1(0,L)\,. \end{aligned}$$

That is

$$-\left(p(x)u_x + 2\delta(x)u_x\right)_x + \gamma u - \frac{\eta(\beta+\tau)}{k}q + u\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds = f_1 - (2f_3\delta(x))_x\,,$$

then, we have

$$(p(x)u_x + 2\delta(x)u_x)_x = \gamma u + u\int_{\tau_1}^{\tau_2}\mu(s)e^{-s}ds - \frac{\eta(\beta+\tau)}{k}q - f_1 + (2f_3\delta(x))_x \in L^2(0,L).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$u \in H_0^1(0, L) \cap H^2(0, L).$$

Similarly, if $\widetilde{u} \equiv 0 \in H_0^1(0, L)$, we obtain

$$q \in H_*^1(0, L).$$

Moreover, from (6.19) and (6.20) we deduce that

$$v, \theta \in H_0^1(0, L).$$

Hence, there exists a unique $U \in D(A)$ such that Equation (6.17) is satisfied. Consequently, A is a maximal monotone operator. Then, $D(A)$ is dense in H (see Proposition 7.1 in [24]).

On the other hand, we show that operator B is Lipschitz continuous. In fact, if $U = (u, v, \theta, q, z)^T$ and $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{\theta}, \widetilde{q}, \widetilde{z})^T$ belong to H, we have

$$\left\|\mathcal{B}U - \mathcal{B}\widetilde{U}\right\|_{\mathcal{H}} \leq c \left\|\widetilde{v} - v\right\|_{L^2} + \left\|f(v) - f(\widetilde{v})\right\|_{L^2}. \tag{6.24}$$

Using the embedding of $H^1(0, L)$ into $L^\infty(0, L)$ (see [24] Theorem 8.8, p. 212) and (H3), one sees that

$$c \left\|\widetilde{v} - v\right\|_{L^2} \leq c \left\|\widetilde{v} - v\right\|_{L^\infty(0,L)} \leq c \left\|U - \widetilde{U}\right\|_{\mathcal{H}}. \tag{6.25}$$

$$\left\|f(v) - f(\widetilde{v})\right\|_{L^2} \leq c \left\|v - \widetilde{v}\right\|_{L^2} \leq c\|U - \widetilde{U}\|_{\mathcal{H}} \tag{6.26}$$

Combining (6.24), (6.25) and (6.26), we infer that B is Lipschitz continuous in H (see [18]). Consequently, $A + B$ is the infinitesimal generator of a linear contraction C_0-semigroup on H. Hence, the result of Proposition 6.1 follows (see [61], [83]) and the references therein. ■

To state our decay result, we introduce the energy functional associated to (6.8)-(6.9), namely,

$$\begin{aligned} \mathcal{E}\left(t, \varphi, \psi, \theta, q, z\right) &= \frac{1}{2}\int_0^L \left\{p(x)u_x^2 + m(x)u_t^2 + \theta^2 + \tau q^2\right\} dx \\ &\quad + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x, \rho, s, t) ds d\rho dx, \end{aligned} \tag{6.27}$$

we denote $E(t) = E\left(t, \varphi, \psi, \theta, q, z\right)$ and $E(0) = E\left(0, \varphi_0, \psi_0, \theta_0, q_0, g_0\right)$ for simplicity of notations.

6.3 General decay result

In this section, we introduce some lemmas allow us to achieve our goal, which is the proof of the stability result. The first one will benefit us in the next chapter too.

Lemma 6.1 *[8] Let (u, u_t, θ, q) be the solution to system (6.1)-(6.2), with an initial datum in $D(A)$. Then, for any $t > 0$, there exists a sequence of real numbers (depending on t), denoted by $\xi_i \in [0, L](i = 1, ..., 6)$, such that:*

$$\begin{aligned}
\int_0^L p(x)\, u_x^2 dx &= p(\xi_1) \int_0^L u_x^2 dx, \quad \int_0^L m(x)\, u_t^2 dx = m(\xi_2) \int_0^L u_t^2 dx, \\
\int_0^L m(x)\, u^2 dx &= m(\xi_3) \int_0^L u^2 dx, \quad \int_0^L \delta(x)\, u^2 dx = \delta(\xi_4) \int_0^L u^2 dx, \\
\int_0^L \delta(x)\, u_x^2 dx &= \delta(\xi_5) \int_0^L u_x^2 dx, \quad \int_0^L \delta(x)\, u_{xt}^2 dx = \delta(\xi_6) \int_0^L u_{xt}^2 dx.
\end{aligned}$$

Lemma 6.2 *Let (u, v, θ, q, z) be the solution of (6.8)-(6.9), then the energy E is non-increasing function and satisfies, for all $t \geq 0$,*

$$\begin{aligned}
\mathcal{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - \beta \int_0^L q^2 dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s, t) ds dx \\
&\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x, 1, s, t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx \\
&\quad -\int_0^L f(u_t) u_t dx \\
&\leq -\beta \int_0^L q^2 dx - c\int_0^L u_{xt}^2 dx - \int_0^L f(u_t) u_t dx \leq 0. \qquad (6.28)
\end{aligned}$$

where $c > 0$ is constant.

Proof. Multiplying the equations in $(6.8)_1$,$(6.8)_2$, and $(6.8)_3$ by u_t,θ and q, respectively, and integrate over $(0, L)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \left\{p(x)u_x^2 + m(x)u_t^2 + \theta^2 + \tau q^2\right\} dx \\
&= -\beta \int_0^L q^2 dx - 2\int_0^L \delta(x)\, u_{xt}^2 dx \\
&\quad -\int_0^L f(u_t) u_t dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s, t) ds dx. \qquad (6.29)
\end{aligned}$$

Multiplying the last equation in (6.8) by $|\mu(s)|\, z$, integrating the product over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, and recall that $z(x, 0, s, t) = u_t$, yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\, |\mu(s)|\, z^2(x, \rho, s, t) ds d\rho dx \\
&= -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x, 1, s, t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx. \qquad (6.30)
\end{aligned}$$

Now, a combination of (6.29) and (6.30) gives

$$\begin{aligned}
\mathcal{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - \beta\int_0^L q^2 dx - \int_0^L f(u_t)u_t dx \\
&\quad - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\
&\quad - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx \\
&\quad + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx. \qquad (6.31)
\end{aligned}$$

Meanwhile, using Young and Cauchy–Schwarz inequalities, we have

$$\begin{aligned}
& -\int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx \\
\leq\ & \underbrace{\frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds}_{<\eta} \int_0^L u_t^2 dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx. \qquad (6.32)
\end{aligned}$$

Substitution of (6.32) into (6.31), using (6.10), Lemma 6.1 and (2.4) gives

$$\begin{aligned}
\mathcal{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - \beta\int_0^L q^2 dx - \int_0^L f(u_t)u_t dx \\
&\quad - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds dx + \frac{1}{2}\int_0^L u_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx \\
&\quad - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx \\
&\leq -\beta\int_0^L q^2 dx - 2\int_0^L \delta(x)\, u_{xt}^2 dx + \eta\int_0^L u_t^2 dx - \int_0^L f(u_t)u_t dx \\
&\leq -\beta\int_0^L q^2 dx - 2\delta(\xi_6)\int_0^L u_{xt}^2 dx + l\eta\int_0^L u_{xt}^2 dx - \int_0^L f(u_t)u_t dx \\
&\leq -\beta\int_0^L q^2 dx - (2\delta(\xi_6) - l\eta)\int_0^L u_{xt}^2 dx - \int_0^L f(u_t)u_t dx \\
&\leq -\beta\int_0^L q^2 dx - c\int_0^L u_{xt}^2 dx - \int_0^L f(u_t)u_t dx \leq 0,
\end{aligned}$$

which concludes the proof. ∎

Lemma 6.3 *The functional*

$$I_1(t) := \tau\int_0^L \theta\left(\int_0^x q(t,y)dy\right)dx, \qquad (6.33)$$

satisfies

$$I_1'(t) \leq -(\kappa-\beta\varepsilon_1)\int_0^L \theta^2 dx + \varepsilon_2\tau\eta\int_0^L u_t^2 dx + \left(\tau+\frac{\tau\eta}{\varepsilon_2}+l\frac{\beta}{\varepsilon_1}\right)\int_0^L q^2 dx, \tag{6.34}$$

for $\epsilon_1, \epsilon_2 > 0$

Proof. Taking the derivative of (6.33) and using $(6.8)_2$, $(6.8)_3$, (2.4), integration by parts and Young's inequality, we obtain (6.34). ■

Lemma 6.4 *Then the functional*

$$I_2(t) := \int_0^L \left(\delta(x)u_x^2 + m(x)u_t u\right) dx, \tag{6.35}$$

satisfies

$$\begin{aligned} I_2'(t) \leq & -(p(\xi_1)-(\eta+c+\eta l)\epsilon_3)\int_0^L u_x^2 dx + m(\xi_2)\int_0^L u_t^2 dx \\ & + \frac{\eta}{\epsilon_3}\int_0^L \theta^2 dx + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx \\ & + \frac{1}{4\epsilon_3}\int_0^L f^2(u_t)\, dx, \end{aligned} \tag{6.36}$$

for any $\varepsilon_3 > 0$.

Proof. Differentiating Equation (6.35) with respect to t and using Equations $(6.8)_1$ and (6.9), we get

$$\begin{aligned} I_2'(t) = & -\int_0^L p(x)u_x^2 dx + \int_0^L m(x)u_t^2 dx - \eta\int_0^L \theta_x u dx \\ & -\int_0^L f(u_t)u dx - \int_0^L u\int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s,t)dsdx. \end{aligned}$$

Using Young's inequality, we have for $\epsilon_3 > 0$

$$-\eta\int_0^L \theta_x u dx = \eta\int_0^L u_x\theta dx \leq \eta\epsilon_3\int_0^L u_x^2 dx + \frac{\eta}{\epsilon_3}\int_0^L \theta^2 dx,$$

from Young's inequality, (6.10) and (2.4), we find

$$\begin{aligned} & -\int_0^L u\int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s,t)dsdx \\ \leq & \; \epsilon_3\underbrace{\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds}_{<\eta}\int_0^L u^2 dx + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx \\ \leq & \; l\eta\epsilon_3\int_0^L u_x^2 dx + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx, \end{aligned}$$

and

$$\begin{aligned} -\int_0^L uf(u_t)\,dx &\leq \epsilon_3 \int_0^L u^2 dx + \frac{1}{4\epsilon_3}\int_0^L f^2(u_t)\,dx \\ &\leq c\epsilon_3 \int_0^L u_x^2 dx + \frac{1}{4\epsilon_3}\int_0^L f^2(u_t)\,dx, \end{aligned}$$

application of Lemma 6.1 and the recent inequalities completes the proof. ■

Lemma 6.5 *The functional*

$$I_3(t) = \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu(s)| z^2(x,\rho,s,t) ds d\rho dx, \tag{6.37}$$

satisfies

$$\begin{aligned} I_3'(t) &\leq -\eta_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,s,t) ds d\rho dx \\ &\quad -\eta_1 \int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t) ds dx + \eta \int_0^L u_t^2 dx. \end{aligned} \tag{6.38}$$

for $\eta_1 > 0$.

Proof. Differentiating (6.37) and using the last equation in (6.8), we obtain

$$\begin{aligned} I_3'(t) &= -2\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu(s)|\, z(x,\rho,s,t) z_\rho(x,\rho,s,t) ds d\rho dx \\ &= -\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| \frac{\partial}{\partial\rho}[e^{-s\rho} z^2(x,\rho,s,t)] ds d\rho dx \\ &\quad -\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu(s)|\, z^2(x,\rho,s,t) ds d\rho dx \\ &= -\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| \,[e^{-s} z^2(x,1,s,t) - z^2(x,0,s,t)] ds dx \\ &\quad -\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu(s)|\, z^2(x,\rho,s,t) ds d\rho dx, \end{aligned}$$

using the fact that $z(x,0,s,t) = u_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, we get for $\rho \in [0,1]$

$$\begin{aligned} I_3'(t) &\leq \int_0^L \int_{\tau_1}^{\tau_2} e^{-s} |\mu(s)|\, z^2(x,1,s,t) ds dx + \underbrace{\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds}_{<\eta} \int_0^L u_t^2 dx \\ &\quad -\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s} |\mu(s)|\, z^2(x,\rho,s,t) ds d\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$. Finally, setting $\eta_1 = -e^{-\tau_2}$ and recalling (6.10), we obtain (6.38). ■

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

6.3. General decay result

Lemma 6.6 *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE(t) + N_1 I_1(t) + I_2(t) + N_2 I_3(t). \tag{6.39}$$

where N_1 and N_2 are positive real numbers to be chosen appropriately later, satisfies

$$c_1' \mathcal{E}(t) \leq \mathcal{L}(t) \leq c_2' \mathcal{E}(t), \quad \forall t \geq 0. \tag{6.40}$$

where c_1' and c_2' are positive constants.

Proof. Let

$$\mathfrak{L}(t) := N_1 I_1(t) + I_2(t) + N_2 I_3(t).$$

then, exploiting Young's, Poincaré's, Cauchy-Schwarz inequalities, (6.27), and the fact that $e^{-s\rho} \leq 1$, we obtain

$$\begin{aligned}
|\mathfrak{L}(t)| &\leq N_1 \tau \int_0^L \left| \theta \left(\int_0^x q(t,y)dy \right) \right| dx + \int_0^L \delta(x) u_x^2 dx + \int_0^L m(x) \left| u_t u \right| dx \\
&\quad + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left| e^{-s\rho} \mu(s) \right| z^2(x,\rho,s,t) ds d\rho dx \\
&\leq \int_0^L \delta(x) u_x^2 + \frac{1}{2} \int_0^L m(x) u^2 dx + \frac{1}{2} \int_0^L m(x) u_t^2 dx \\
&\quad + N_1 \tau l \int_0^L |\theta q| \, dx + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left| \mu(s) \right| z^2(x,\rho,s,t) ds d\rho dx \\
&\leq \frac{1}{2} \int_0^L m(x) u_t^2 dx + \frac{\|\delta(x)\|_\infty}{\lambda} \int_0^L p(x) u_x^2 + \frac{N_1 \tau l}{2} \int_0^L \theta^2 dx \\
&\quad + \frac{l \|m(x)\|_\infty}{2\lambda} \int_0^L p(x) u_x^2 + \frac{N_1 \tau l}{2} \int_0^L q^2 dx \\
&\quad + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \left| \mu(s) \right| z^2(x,\rho,s,t) ds d\rho dx \\
&\leq cE(t),
\end{aligned}$$

where $\lambda = \inf_{x \in [0,L]} \{p(x)\}$, and $c > 0$. Consequently,

$$|\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t),$$

which yields

$$(N-c)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+c)\mathcal{E}(t).$$

Choosing N large enough, we obtain estimate (6.40). ■

Now, we are ready to state and prove the main result of this section.

Theorem 6.1 *Let (u, v, θ, q, z) be the solution of (6.8)-(6.9), assume that (H1)-(H3) are satisfied, then there exist c_1,$c_2 > 0$ for which the energy E satisfies, for all $t \geq 0$,*

$$\mathcal{E}(t) \leq c_1 H_1^{-1}(c_2 t), \ \forall t \geq 0, \tag{6.41}$$

where the functions H_1 and H_2 are defined by:

for $\epsilon_0 > 0$

$$H_2(t) := \begin{cases} t & \text{if } h'' = 0 \text{ on } [0, \lambda], \\ th'(\epsilon_0 t) & \text{if } h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0, \lambda], \end{cases} \tag{6.42}$$

and

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds. \tag{6.43}$$

Proof. We differentiate (6.39), and recall (2.4), (6.28), (6.36), (6.34), and (6.38), to obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & N\left(-\beta\int_0^L q^2 dx - c\int_0^L u_{xt}^2 dx - \int_0^L f(u_t)u_t dx\right) \\
& -(p(\xi_1) - (\eta + c + \eta l)\epsilon_3)\int_0^L u_x^2 dx + m(\xi_2)\int_0^L u_t^2 dx \\
& + \frac{1}{4\epsilon_3}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,s,t) ds dx + \frac{\eta}{\epsilon_3}\int_0^L \theta^2 dx \\
& + N_1\left(-(\kappa - \beta\epsilon_1)\int_0^L \theta^2 dx + \epsilon_2\tau\eta\int_0^L u_t^2 dx\right) \\
& + N_1\left(\tau + \frac{\tau\eta}{\epsilon_2} + l\frac{\beta}{\varepsilon_1}\right)\int_0^L q^2 dx + \frac{1}{4\epsilon_3}\int_0^L f^2(u_t) dx \\
& - N_2\eta_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)| z^2(x,\rho,s,t) ds d\rho dx \\
& - N_2\eta_1\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,s,t) ds dx + N_2\eta\int_0^L u_t^2 dx \\
\leq\ & -\left\{\frac{Nc}{l} - N_2\eta - m(\xi_2) - N_1\epsilon_2\tau\eta\right\}\int_0^L u_t^2 dx \\
& -\{p(\xi_1) - (\eta + c + \eta l)\epsilon_3\}\int_0^L u_x^2 dx + \frac{1}{4\epsilon_3}\int_0^L f^2(u_t) dx \\
& -\left\{N\beta - N_1\left(\tau + \frac{\tau\eta}{\epsilon_2} + \frac{\beta}{\varepsilon_1}l\right)\right\}\int_0^L q^2 dx - N\int_0^L f(u_t)u_t dx \\
& -\left\{N_1(\kappa - \beta\epsilon_1) - \frac{\eta}{\epsilon_3}\right\}\int_0^L \theta^2 dx \\
& -\left\{\eta_1 N_2 - \frac{1}{4\epsilon_3}\right\}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,s,t) ds dx \\
& -\eta_1 N_2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx.
\end{aligned}$$

At this point, we take $\epsilon_2 = 1$, then we choose ϵ_1 and ϵ_3 small enough such that

$$\beta\epsilon_1 - k < 0, \quad \epsilon_3 < \frac{p(\xi_1)}{\eta\,(l+1)+c},$$

then we choose N_1 and N_2 large enough so that

$$N_1\,(\kappa-\epsilon_1\beta) - \frac{\eta}{\epsilon_3} > 0, \quad \eta_1 N_2 - \frac{1}{4\epsilon_3} > 0.$$

Once N_1 and N_2 are fixed, we then choose N large enough so that

$$\frac{Nc}{l} - N_2\eta - m(\xi_2) - N_1\tau\eta > 0,$$

$$N\beta - N_1\left(\tau + \tau\eta + l\frac{\beta}{\varepsilon_1}\right) > 0.$$

Thus, using (6.27), we arrive at

$$\mathcal{L}'(t) \leq -cE(t) + c\int_0^L f^2(u_t)\,dx, \quad \forall t > 0. \tag{6.44}$$

Let us define the following sets

$$\Sigma_+ = \{x \in (0,L) : |u_t\,(x,t)| > \lambda\}\,, \; \Sigma_- = (0,L)\setminus\Sigma_+,$$

We work now for estimate the last term in the right-hand side of (6.44). First, note that

$$\int_0^L f^2\,(u_t)\,dx = \int_{\Sigma_+} f^2\,(u_t)\,dx + \int_{\Sigma_-} f^2\,(u_t)\,dx.$$

Using A_1 and (6.28), we easily show that

$$\begin{aligned}\int_{\Sigma_+} f^2\,(u_t)\,dx &\leq k_1\int_{\Sigma_+} u_t f\,(u_t)\,dx\\ &\leq k_1\int_0^L u_t f\,(u_t)\,dx\\ &\leq -cE'\,(t)\,.\end{aligned} \tag{6.45}$$

If $h'' = 0$ on $[0,\lambda]$: This implies that there exist $k_1 > 0$ such that $|f\,(s)| \leq k_1\,|s|$ for all $s \in \mathbb{R}_+$, and then (6.45) is also satisfied for $|u_t\,(x,t)| \leq \lambda$, then on all $(0,L)$. From (6.44), (6.45), we arrive at

$$(\mathcal{L}(t) + cE\,(t))' \leq -cH_2\,(E\,(t))\,, \quad \forall t \geq t_0, \tag{6.46}$$

where H_2 is defined in (6.42).

If $h'(0) = 0$ and $h'' > 0$ on $(0,\lambda]$: Since h is convex and increasing, h^{-1} is concave and increasing, by using (H3), the reversed Jensen's inequality for concave function (see [87]), and (6.28), we obtain,

$$\begin{aligned}\int_{\Sigma_-} f^2(u_t)\,dx &\leq \int_{\Sigma_-} h^{-1}(u_t f(u_t))\,dx \\ &\leq ch^{-1}\left(\int_{\Sigma_-} u_t f(u_t)\,dx\right) \\ &\leq ch^{-1}\left(\int_0^L u_t f(u_t)\,dx\right) \\ &\leq ch^{-1}(-cE'(t)). \end{aligned} \tag{6.47}$$

Therefore, from (6.44), (6.45) and (6.47), we find that

$$(\mathcal{L}(t) + cE(t))' \leq ch^{-1}(-cE'(t)) - cE(t), \quad \forall t \geq t_0.$$

By using Young's inequality (2.8) and the fact that

$$h^*(p) \leq p[h']^{-1}(p), \;\; E' \leq 0 \text{ and } h'' > 0,$$

we obtain for $\varepsilon_0 > 0$ small enough and $c_0 > 0$ large enough,

$$\begin{aligned} & [h'(\varepsilon_0 E(t))[\mathcal{L}(t) + cE(t)] + c_0 E(t)]' \\ = \;& \varepsilon_0 E'(t) h''(\varepsilon_0 E(t))[\mathcal{L}(t) + cE(t)] + c_0 E'(t) \\ & + h'(\varepsilon_0 E(t))[\mathcal{L}'(t) + cE'(t)] \\ \leq \;& -ch'(\varepsilon_0 E(t)) E(t) + c.h'(\varepsilon_0 E(t)) h^{-1}(-cE'(t)) \\ & + c_0 E'(t) \\ \leq \;& -ch'(\varepsilon_0 E(t)) E(t) + ch^*(h'(\varepsilon_0 E(t))) - cE'(t) \\ & + c_0 E'(t) \\ \leq \;& -ch'(\varepsilon_0 E(t)) E(t) + c\varepsilon_0 h'(\varepsilon_0 E(t)) E(t) \\ \leq \;& -ch'(\varepsilon_0 E(t)) E(t) = -cH_2(E(t)). \end{aligned} \tag{6.48}$$

Now, let us define the following functional:

$$\mathcal{F}(t) = \begin{cases} \mathcal{L}(t) + cE(t) & \text{if (6.12) holds,} \\ h'(\varepsilon_0 E(t))[\mathcal{L}(t) + cE(t)] + c_0 E(t) & \text{if (6.13) holds.} \end{cases}$$

Using (6.40), we have

$$\mathcal{F} \sim E,$$

and exploiting (6.46) and (6.48), we easily deduce that

$$\mathcal{F}'(t) \leq -cH_2(E(t)), \quad \forall t \geq t_0.$$

6.3. General decay result

Next, let

$$\mathcal{R}(t) = \varepsilon \mathcal{F}(t),$$

where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\mathcal{F}(t) \leq \frac{1}{\bar{\varepsilon}} E(t), \quad \forall t \geq 0.$$

We also have

$$\mathcal{R} \sim E, \tag{6.49}$$

and for $t \geq t_0$

$$\begin{aligned} \mathcal{R}'(t) &\leq -c\varepsilon H_2(\mathcal{E}(t)) \leq -c\varepsilon H_2(\bar{\varepsilon}\mathcal{F}(t)) & (6.50) \\ &\leq -c\varepsilon H_2(\varepsilon\mathcal{F}(t)) = -c\varepsilon H_2(\mathcal{R}(t)). & (6.51) \end{aligned}$$

Noting that $H_1' = -1/H_2$ (see (6.43)), we get from (6.50)

$$\mathcal{R}'(t) H_1'(\mathcal{R}(t)) \geq c\varepsilon, \quad \forall t \geq t_0.$$

A simple integration over (t_0, t) then yields

$$H_1(\mathcal{R}(t)) \geq H_1(\mathcal{R}(t_0)) + c\varepsilon t - c\varepsilon t_0.$$

On the other hand, since $\lim_{t \to 0^+} H_1(t) = +\infty$ and

$$0 \leq \mathcal{R}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}} E(0),$$

we obtain for ε small enough

$$H_1(\mathcal{R}(t_0)) - c\varepsilon t_0 > 0.$$

Then, thanks to the fact that H_1^{-1} is decreasing , we infer that

$$\begin{aligned} \mathcal{R}(t) &\leq H_1^{-1}(H_1(\mathcal{R}(t_0)) + c\varepsilon t - c\varepsilon t_0) \\ &\leq H_1^{-1}(c\varepsilon t). \end{aligned}$$

From this end inequality and (6.49) we get easily (6.41). Then the proof is completed. ■

CHAPTER 7

ON THE EXPONENTIAL STABILITY OF A FLEXIBLE STRUCTURE IN THERMO-ELASTICITY WITH MICRO-TEMPERATURE EFFECTS

7.1 Introduction

We shall study the following inhomogeneous flexible structure system with micro-temperature effect:

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + dw_x + \eta\theta_x = 0, \\ c\theta_t - k\theta_{xx} + \eta u_{tx} + k_1 w_x = 0, \\ \tau w_t - k_3 w_{xx} + k_2 w + k_1\theta_x + du_{tx} = 0, \end{cases} \tag{7.1}$$

where $u(x,t)$ is the displacement of a particle at position $x \in (0,L)$ and time $t > 0$, θ and w are the temperature of the body and the micro-temperature vector respectively. $\eta > 0$ is the coupling constant, that accounts for the heating effect, and $k, k_1, k_2, k_3, c, d, \tau > 0$. $m(x)$, $\delta(x)$ and $p(x)$ are responsible for the non-uniform structure of the body, and, respectively, denote mass per unit length of structure, coefficient of internal material damping and a positive function related to the stress acting on the body at a point x. We consider the following initial and boundary conditions:

$$\begin{array}{l} u\,(.,0) = u_0\,(x)\,,\ u_t\,(.,0) = u_1\,(x)\,,\ \theta\,(.,0) = \theta_0\,(x)\,,\ w\,(.,0) = w_0\,(x)\,, \quad \forall x \in [0,L] \\ u\,(0,t) = u\,(L,t) = \theta\,(0,t) = \theta\,(L,t) = w_x\,(0,t) = w_x\,(L,t) = 0,\ \forall t \geq 0. \end{array} \tag{7.2}$$

In the presence of second sound, Alves et al. [8] concerned with the system

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x = 0, \\ \theta_t + kq_x + \eta u_{tx} = 0, \\ \tau q_t + \beta q + k\theta_x = 0, \end{cases} \tag{7.3}$$

They established the well-posedness of the system and proved its stability exponential and polynomial under suitable boundary conditions. Li et al. [65] considered (7.3) with a delay term of the form $\mu u_t(x, t - \tau_0)$ in its first equation, they proved that the system is exponential decay under a "small" condition on time delay. For more details discussion on the flexible structure systems see [6, 43] and the references therein.

Historically, the linear theory of thermo-elasticity with micro-temperatures for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess micro-temperatures was constructed by Ieşan and Quintanilla [54, 55]. The work is motivated by increasing use of materials which possess thermal variation at a microstructure level. The same authors proved an existence theorem and established the continuous dependence of solutions of the initial data and body loads. We note that the concept of micro-temperature was just used in the theory of thermodynamics for elastic materials with microstructure. In addition to micro-deformations of the string, the micro-elements of the continuum possess micro-temperatures which represent the variation of the temperature within a micro-volume. Originally, Grot [44] was the first to take into consideration the inner structure of a body in order to develop a thermodynamic theory for thermo-elastic materials where micro-elements, in addition to classic micro-deformations, possess micro-temperatures. While, the fundamental solution of the equations of the theory of thermo-elasticity with micro-temperatures was constructed by Svanadze [89]. Riha [85, 86] developed a further study concerning heat conduction in thermo-elastic materials with inner structure. It is shown that the experimental data for the silicone rubber containing spherical aluminum particles and for human blood are conform closely to the predicted theoretical model of thermo-elasticity with micro-temperatures. We refer the interested readers to [11, 27, 30, 31, 34, 47] for details discussion on the theory.

Motivated by works mentioned above, we investigate (7.1)-(7.2) under suitable condition and establish the well-posedness of the problem using semi-group theory, as well as the stability result of the solution using the multiplier method. Our purpose here is to obtain an exponential decay rate estimates of the energy function of (7.1) without any restriction or relation on the coefficients of the system.

7.2 Existence and uniqueness of solution

In this section, we present some assumptions and give the existence and uniqueness result of system (7.1)-(7.2) using the semigroups theory. Throughout this section, c' represents a generic positive constant and is different in various occurrences.

The aim of this section is to prove that system (7.1)-(7.2) is well-posed. From Equation

$(7.1)_3$ and the boundary conditions (7.2), we have

$$\frac{d}{dt}\int_0^L w\,(x,t)\,dx+\frac{k_2}{\tau}\int_0^L w\,(x,t)\,dx=0,\ \forall t\geq o,$$

thus

$$\int_0^L w\,(x,t)\,dx=\left(\int_0^L w_0dx\right)\exp\left(\frac{-t}{\tau}k_2\right),\ \forall t\geq 0,$$

So, if we set

$$\widetilde{w}\,(x,t)=w\,(x,t)-\frac{1}{L}\left(\int_0^L w_0dx\right)\exp\left(\frac{-t}{\tau}k_2\right),\ t\geq 0,\ x\in[0,L],$$

then, $(u,u_t,\theta,\widetilde{w})$ satisfies Equation (7.1), and

$$\int_0^L \widetilde{w}\,(x,t)\,dx=0,$$

for all $t\geq 0$. In the sequel, we shall work with $\widetilde{w}$ but we write w for simplicity.

The energy functional associated to (7.1)-(7.2), namely,

$$\mathcal{E}\,(t,u,u_t,\theta,w)=\frac{1}{2}\int_0^L\left\{p(x)u_x^2+m(x)u_t^2+c\theta^2+\tau w^2\right\}dx, \tag{7.4}$$

we denote $E(t)=E\,(t,u,u_t,\theta,w)$ and $E(0)=E\,(0,u_0,u_1,\theta_0,w_0)$ for simplicity of notations. Then the energy E is decreasing function and satisfies, for all $t\geq 0$,

$$\begin{aligned}
\mathcal{E}'(t) &= -2\int_0^L\delta\,(x)\,u_{xt}^2dx-k_2\int_0^L w^2dx-k_3\int_0^L w_x^2dx-k\int_0^L\theta_x^2dx\\
&\leq -c'\int_0^L u_t^2dx-k_2\int_0^L w^2dx-k_3\int_0^L w_x^2dx-k\int_0^L\theta_x^2dx \qquad (7.5)\\
&\leq 0. \qquad (7.6)
\end{aligned}$$

To obtain precise decay rates of $E(t)$ as $t\to+\infty$, we assume that

$$m,\ \delta,\ p\in W^{1,\infty}(0,L),\ m(x),\ p(x),\ \delta(x)>0,\quad \forall x\in[0,L]. \tag{7.7}$$

Let us introducing the vector function $U=(u,v,\theta,w)^T$, where $v=u_t$, using the standard Lebesgue space $L^2(0,L)$ and the Sobolev space $H_0^1(0,L)$ with their usual scalar products and norms for define the spaces:

$$\mathcal{H}:=H_0^1(0,L)\times\left[L^2(0,L)\right]^2\times L_*^2(0,L),$$

and

$$H_*^2(0,L)=\left\{w\in H^2(0,L):w_x\,(L)=w_x\,(0)=0\right\},$$

where

$$L_*^2(0,L) = \left\{ w \in L^2(0,L) : \int_0^L w\,(s)\,ds = 0 \right\}.$$

We equip $\mathcal{H}$ with the inner product

$$(U,\widetilde{U})_{\mathcal{H}} = \int_0^L p\,(x)\,u_x \widetilde{u}_x dx + \int_0^L m\,(x)\,v\widetilde{v} dx + c\int_0^L \theta\widetilde{\theta} dx + \tau \int_0^L w\widetilde{w} dx.$$

Next, the system (7.1)-(7.2) can be reduced to the following abstract Cauchy problem:

$$\begin{cases} U'(t) + \mathcal{A}U(t) = 0, & t > 0 \\ U(0) = U_0 = (u_0, u_1, \theta_0, w_0)^T, \end{cases} \tag{7.8}$$

where the operator $A : D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{1}{m(x)}(p(x)u_x + 2\delta(x)v_x - \eta\theta - dw)_x \\ \frac{1}{c}\left(-k\theta_{xx} + \eta u_{tx} + k_1 w_x\right) \\ \frac{1}{\tau}(-k_3 w_{xx} + k_2 w + k_1\theta_x + du_{tx}) \end{pmatrix}.$$

The domain of A is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H} \mid u \in H^2(0,L) \cap H_0^1(0,L),\ v \in H_0^1(0,L),\ \theta \in H^2(0,L), \\ w \in L_*^2(0,L) \cap H_*^2(0,L) \end{array} \right\},$$

which is dense in $\mathcal{H}$.

Proposition 7.1 *Let $U_0 \in \mathcal{H}$ be given. Problem (7.8) possesses then a unique solution satisfying $U \in C(\mathbb{R}^+;\mathcal{H})$. If $U_0 \in D(\mathcal{A})$, then $U \in C^1(\mathbb{R}^+;\mathcal{H}) \cap C(\mathbb{R}^+; D(\mathcal{A}))$.*

Proof. For any $U \in D(A)$, we have

$$(\mathcal{A}U,U)_{\mathcal{H}} = 2\int_0^L \delta\,(x)\,v_x^2 dx + k\int_0^L \theta_x^2 dx + k_2\int_0^L w^2 dx + k_3\int_0^L w_x^2 dx \geq 0.$$

Hence, A is monotone. Next, we prove that the operator $I + A$ is surjective.

Given $G = (g_1, g_2, g_3, g_4)^T \in H$, we prove that there exists $U \in D(A)$ satisfying

$$(\mathcal{I} + \mathcal{A})\,U = G, \tag{7.9}$$

which gives

$$\begin{array}{l} -v + u = g_1 \in H_0^1(0,L), \\ -(p(x)u_x + 2\delta(x)v_x - \eta\theta - dw)_x + m\,(x)\,v = m\,(x)\,g_2 \in L^2(0,L), \\ -k\theta_{xx} + \eta v_x + k_1 w_x + c\theta = cg_3 \in L^2(0,L), \\ -k_3 w_{xx} + k_2 w + k_1\theta_x + dv_x + \tau w = \tau g_4 \in L_*^2(0,L). \end{array} \tag{7.10}$$

Inserting $u = v + g_1$, in $(7.10)_2$, $(7.10)_3$ and $(7.10)_4$, we obtain

$$\begin{cases} -(p(x)u_x + 2\delta(x)u_x - \eta\theta - dw)_x + m(x)\,u = m(x)\,(g_1+g_2) - 2\delta(x)g_{1xx} = f_1 \in L^2(0,L), \\ -k\theta_{xx} + \eta u_x + k_1 w_x + c\theta = cg_3 + \eta g_{1x} = f_2 \in L^2(0,L), \\ -k_3 w_{xx} + k_2 w + k_1\theta_x + du_x + \tau w = \tau g_4 + dg_{1x} = f_3 \in L^2_*(0,L). \end{cases} \tag{7.11}$$

The variational formulation corresponding to Equation (7.11) takes the form

$$B((v,\theta,w),(\widetilde{v},\widetilde{\theta},\widetilde{w})) = F(\widetilde{v},\widetilde{\theta},\widetilde{w}), \tag{7.12}$$

where $B : \left[H_0^1(0,L)\times L^2(0,L)\times L^2_*(0,L)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B((v,\theta,w),(\widetilde{v},\widetilde{\theta},\widetilde{w})) &= \int_0^L \left[(p(x)+2\delta(x))\,u_x - \eta\theta - dw\right]\widetilde{u}_x dx + \int_0^L m(x)\,u\widetilde{u}dx \\ &\quad + k\int_0^L \theta_x\widetilde{\theta}_x dx - \eta\int_0^L u\widetilde{\theta}_x dx - k_1\int_0^L w\widetilde{\theta}_x dx + c\int_0^L \theta\widetilde{\theta}dx \\ &\quad + k_3\int_0^L w_x\widetilde{w}_x dx + (k_2+\tau)\int_0^L w\widetilde{w}dx + k_1\int_0^L \theta_x\widetilde{w}dx \\ &\quad - d\int_0^L u\widetilde{w}_x dx, \end{aligned}$$

and $F : H_0^1(0,L)\times L^2(0,L)\times L^2_*(0,L) \to \mathbb{R}$ is the linear functional defined by

$$F(\widetilde{v},\widetilde{\theta},\widetilde{w}) = \int_0^L f_1\widetilde{u}dx + \int_0^L f_2\widetilde{\theta}dx + \int_0^L f_3\widetilde{w}dx.$$

For $V = H_0^1(0,L)\times L^2(0,L)\times L^2_*(0,L)$ equipped with the norm

$$\|(v,\theta,w)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|w\|_2^2 + \|\theta_x\|_2^2,$$

where $\|.\|_2$ is the usual norm.

One can easily see that B and F are bounded. Also, we get

$$\begin{aligned} B((u,\theta,w),(u,\theta,w)) &= \int_0^L (p(x)+2\delta(x))\,u_x^2 dx + \int_0^L m(x)\,u^2 dx + k\int_0^L \theta_x^2 dx \\ &\quad c\int_0^L \theta^2 dx + k_3\int_0^L w_x^2 dx + k_2\int_0^L w^2 dx \\ &\geq c\,\|(v,\theta,w)\|_V^2 . \end{aligned}$$

Then, B is coercive. Consequently, by the Lax–Milgram lemma, system (7.11) has a unique solution

$$u \in H_0^1(0,L),\ \ \theta \in L^2(0,L),\ \ w \in L^2_*(0,L).$$

From $(7.10)_1$, we infer that

$$v \in H_0^1(0,L).$$

Moreover, if $\left(\widetilde{\theta}, \widetilde{w}\right) \equiv (0,0) \in L^2(0,L) \times L^2_*(0,L)$, then Equation (7.12) reduces to

$$-\int_0^L \left[(p(x)+2\delta(x))\,u_x - \eta\theta - dw\right]_x \widetilde{u}dx + \int_0^L m\,(x)\,u\widetilde{u} = \int_0^L f_1\widetilde{u}dx,$$

That is

$$-\left[(p(x)+2\delta(x))\,u_x\right]_x = \eta\theta_x + dw_x - m\,(x)\,u + f_1 \in L^2(0,L).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$u \in H_0^1(0,L) \cap H^2(0,L).$$

Similarly, if $\left(\widetilde{u}, \widetilde{\theta}\right) \equiv (0,0) \in H_0^1(0,L) \times L^2(0,L)$, then Equation (7.12) reduces to

$$k_3 \int_0^L w_x\widetilde{w}_x dx + (k_2+\tau)\int_0^L w\widetilde{w}dx + k_1\int_0^L \theta_x\widetilde{w}dx - d\int_0^L u\widetilde{w}_x dx = \int_0^L f_3\widetilde{w}dx. \tag{7.13}$$

That is

$$k_3 w_{xx} = (k_2+\tau)\,w + k_1\theta_x + du_x - f_3 \in L^2(0,L), \tag{7.14}$$

then, it follows that $\int_0^L wdx = 0$, and we get

$$w \in L_*^2(0,L) \cap H^2(0,L).$$

Moreover, (7.13) is also true for any $\phi \in C^1([0;L])$ included in $L_*^2(0,L)$. Hence, we have

$$k_3 \int_0^L w_x\phi_x dx + (k_2+\tau)\int_0^L w\phi dx + k_1\int_0^L \theta_x\phi dx - d\int_0^L u\phi_x dx = \int_0^L f_3\phi dx,$$

for all $\phi \in C^1([0;L])$. Thus, using integration by parts and bearing in mind (7.14), we obtain

$$w_x\,(L)\,\phi\,(L) - w_x\,(0)\,\phi\,(0) = 0,\ \forall\phi \in C^1([0;L]).$$

Therefore, $w_x\,(L) = w_x\,(0) = 0$, consequently, we have

$$w \in L_*^2(0,L) \cap H_*^2(0,L).$$

Now, if $(\widetilde{u}, \widetilde{w}) \equiv (0,0) \in H_0^1(0,L) \times L_*^2(0,L)$, then Equation (7.12) reduces to

$$k\int_0^L \theta_x\widetilde{\theta}_x dx - \eta\int_0^L u\widetilde{\theta}_x dx - k_1\int_0^L w\widetilde{\theta}_x dx + c\int_0^L \theta\widetilde{\theta}dx = \int_0^L f_2\widetilde{\theta}dx,$$

that is

$$-k\theta_{xx} = f_2 - \eta u_x - k_1 w_x - c\theta \in L^2(0, L),$$

then, we get

$$\theta \in H^2(0, L).$$

Hence, there exists a unique $U \in D(A)$ such that Equation (7.9) is satisfied. Consequently, A is a maximal monotone operator. Then, $D(A)$ is dense in $\mathcal{H}$ (see Proposition 7.1 in [24]) and the result of Proposition (7.1) follows from Lumer-Phillips theorem. ■

7.3 Stability result

In this section, we introduce some lemmas allow us to achieve our goal, which is the proof of the stability result.

Lemma 7.1 *Let (u, v, θ, w) be the solution of (7.1)-(7.2), then the energy E is non-increasing function and satisfies, for all $t \geq 0$,*

$$\begin{aligned} \mathcal{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - k_2 \int_0^L w^2 dx - k_3 \int_0^L w_x^2 dx - k \int_0^L \theta_x^2 dx \\ &\leq -c' \int_0^L u_t^2 dx - k_2 \int_0^L w^2 dx - k_3 \int_0^L w_x^2 dx - k \int_0^L \theta_x^2 dx \leq 0. \end{aligned} \tag{7.15}$$

where $c' = 2\delta(\xi_6)/l$.

Proof. Multiplying the equations in $(7.1)_1$,$(7.1)_2$, and $(7.1)_3$ by u_t, θ and w, respectively, integrate over $(0, L)$ and using (2.4), we obtain (7.15). ■

Lemma 7.2 *The functional*

$$I_1(t) := \int_0^L \left(\delta(x) u_x^2 + m(x) u_t u\right) dx, \tag{7.16}$$

satisfies

$$\begin{aligned} I_1'(t) \leq\ & -\left(p(\xi_1) - (\eta + d)\,\epsilon_1\right) \int_0^L u_x^2 + m(\xi_2) \int_0^L u_t^2 + \frac{\eta}{4\epsilon_1} \int_0^L \theta^2 \\ & + \frac{d}{4\epsilon_1} \int_0^L w^2 dx, \end{aligned} \tag{7.17}$$

for any $\varepsilon_1 > 0$.

Proof. Differentiating Equation (7.16) with respect to t and using Equations $(7.1)_1$, we get

$$I_1'(t) = -\int_0^L p(x) u_x^2 + \int_0^L m(x) u_t^2 - \eta \int_0^L \theta_x u dx - d \int_0^L u w_x dx.$$

Using Young's inequality, we have for $\epsilon_1 > 0$

$$\begin{aligned} -\eta \int_0^L \theta_x u dx &= \eta \int_0^L u_x \theta dx \leq \eta\epsilon_1 \int_0^L u_x^2 dx + \frac{\eta}{4\epsilon_1} \int_0^L \theta^2 dx, \\ -d \int_0^L w_x u dx &= d \int_0^L u_x w dx \leq d\epsilon_1 \int_0^L u_x^2 dx + \frac{d}{4\epsilon_1} \int_0^L w^2 dx, \end{aligned}$$

application of Lemma 6.1 and the last two inequality completes the proof. ■

Lemma 7.3 *The functional*

$$I_2(t) := \tau c \int_0^L \theta \left(\int_0^x w(y) dy \right) dx, \tag{7.18}$$

satisfies

$$\begin{aligned} I_2'(t) &\leq (-k_1 c + 3\varepsilon_2) \int_0^L \theta^2 dx + \frac{1}{2\varepsilon_2} \int_0^L u_t^2 dx + \frac{1}{4\varepsilon_2} \int_0^L \theta_x^2 dx \\ &\quad + (k_1\tau + 2\varepsilon_2 c' + c') \int_0^L w^2 dx + \frac{1}{4\varepsilon_2} \int_0^L w_x^2 dx, \end{aligned} \tag{7.19}$$

for any $\varepsilon_2 > 0$.

Proof. Taking the derivative of (7.18) and using $(7.1)_2$ and $(7.1)_3$ we find

$$\begin{aligned} I_2'(t) &= \tau \left(k \int_0^L \theta_{xx} \left(\int_0^x w(y) dy \right) dx - \eta \int_0^L u_{tx} \left(\int_0^x w(y) dy \right) dx \right. \\ &\quad \left. - k_1 \int_0^L w_x \left(\int_0^x w(y) dy \right) dx \right) \\ &\quad + c \left(k_3 \int_0^L \theta \left(\int_0^x w_{yy}(y) dy \right) dx - k_2 \int_0^L \theta \left(\int_0^x w(y) dy \right) dx \right. \\ &\quad \left. - k_1 \int_0^L \theta \left(\int_0^x \theta_y(y) dy \right) dx - d \int_0^L \theta \left(\int_0^x u_{ty}(y) dy \right) dx \right). \end{aligned}$$

Integration by parts and the fact that $\int_0^L w(x) dx = 0$, give us

$$\begin{aligned} I_2'(t) &= \tau \left(-k \int_0^L \theta_x w dx + \eta \int_0^L u_t w dx + k_1 \int_0^L w^2 dx \right) \\ &\quad + c \left(k_3 \int_0^L \theta w_x dx - k_2 \int_0^L \theta \left(\int_0^x w(y) dy \right) dx \right. \\ &\quad \left. - k_1 \int_0^L \theta^2 dx - d \int_0^L \theta u_t dx \right), \end{aligned} \tag{7.20}$$

7.3. Stability result

using Young's inequality, we get also

$$
\begin{aligned}
&-k\int_0^L \theta_x w dx \le \tfrac{1}{4\varepsilon_2}\int_0^L \theta_x^2 dx + c'\varepsilon_2\int_0^L w^2 dx \\
&\eta\int_0^L u_t w dx \le \tfrac{1}{4\varepsilon_2}\int_0^L u_t^2 dx + c'\varepsilon_2\int_0^L w^2 dx \\
&k_3\int_0^L \theta w_x dx \le \tfrac{1}{4\varepsilon_2}\int_0^L w_x^2 dx + \varepsilon_2\int_0^L \theta^2 dx \qquad (7.21)\\
&-k_2\int_0^L \theta\left(\int_0^x w(y)dy\right)dx \le \varepsilon_2\int_0^L \theta^2 dx + c'\int_0^L w^2 dx \\
&-d\int_0^L \theta u_t dx \le \tfrac{1}{4\varepsilon_2}\int_0^L u_t^2 dx + \varepsilon_2\int_0^L \theta^2 dx.
\end{aligned}
$$

From (7.20) and (7.21) we infer (7.19). ■

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional.

Lemma 7.4 *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE(t) + I_1(t) + N_1 I_2(t). \qquad (7.22)$$

where N and N_1 are positive real numbers to be chosen appropriately later, satisfies

$$c_1'\mathcal{E}(t) \le \mathcal{L}(t) \le c_2'\mathcal{E}(t), \quad \forall t \ge 0. \qquad (7.23)$$

where c_1' and c_2' are positive constants.

Proof. Let

$$\mathfrak{L}(t) := I_1(t) + N_1 I_2(t).$$

then, exploiting Young's inequality, (2.4) and (7.4), we obtain

$$
\begin{aligned}
|\mathfrak{L}(t)| &\le N_1 c\tau \int_0^L \left|\theta(t,x)\left(\int_0^x w(t,y)dy\right)\right| dx + \int_0^L \left(\delta(x)u_x^2 + m(x)|u_t u|\right) dx \\
&\le \int_0^L \delta(x)u_x^2 + \frac{1}{2}\int_0^L m(x)u^2 dx + N_1\tau cl\int_0^L |\theta(t,x)w(t,y)|\,dx \\
&\quad + \frac{1}{2}\int_0^L m(x)u_t^2 dx \\
&\le \frac{1}{2}\int_0^L m(x)u_t^2 dx + \frac{\|\delta(x)\|_\infty}{\lambda}\int_0^L p(x)u_x^2 + \frac{l\,\|m(x)\|_\infty}{2\lambda}\int_0^L p(x)u_x^2 \\
&\quad + \frac{N_1\tau cl}{2}\int_0^L \theta^2 dx + \frac{N_1\tau cl}{2}\int_0^L w^2(t,y)dx \\
&\le c'E(t),
\end{aligned}
$$

where $\lambda = \inf_{x\in[0,L]} \{p(x)\}$, and $c' > 0$. Consequently,

$$|\mathcal{L}(t) - N\mathcal{E}(t)| \leq c'\mathcal{E}(t),$$

which yields

$$(N - c')\,\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N + c')\,\mathcal{E}(t).$$

Choosing N large enough, we obtain estimate (7.23). ■

Now, we are ready to state and prove the main result of this section.

Theorem 7.1 *Let (u, v, θ, w) be the solution of (7.1)-(7.2), then the energy E satisfies, for all $t \geq 0$,*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 t},$$

where c_1 and c_2 are positive constants.

Proof. We differentiate (7.22), and recall (7.15), (7.17), and (7.19), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq\; & N\left(-c'\int_0^L u_t^2dx - k_2\int_0^L w^2dx - k_3\int_0^L w_x^2dx - k\int_0^L \theta_x^2dx\right) \\
& -\left(p(\xi_1) - (\eta + d)\,\epsilon_1\right)\int_0^L u_x^2 + m(\xi_2)\int_0^L u_t^2 + \frac{\eta}{4\epsilon_1}\int_0^L \theta^2 + \frac{d}{4\epsilon_1}\int_0^L w^2dx \\
& +N_1\left((-k_1c + 3\varepsilon_2)\int_0^L \theta^2dx + \frac{1}{2\varepsilon_2}\int_0^L u_t^2dx + \frac{1}{4\varepsilon_2}\int_0^L \theta_x^2dx\right. \\
& \left.+\left(k_1\tau + 2\varepsilon_2c' + c'\right)\int_0^L w^2dx + \frac{1}{4\varepsilon_2}\int_0^L w_x^2dx\right) \\
\leq\; & \left\{-Nc' + \frac{N_1}{2\varepsilon_2} + m(\xi_2)\right\}\int_0^L u_t^2dx + \left\{-p(\xi_1) + (\eta + d)\,\epsilon_1\right\}\int_0^L u_x^2dx \\
& +\left\{-Nk_2 + N_1\left(k_1\tau + 2\varepsilon_2c' + c'\right) + \frac{d}{4\epsilon_1}\right\}\int_0^L w^2dx \\
& +\left\{N_1\left(-k_1c + 3\varepsilon_2\right) + \frac{\eta}{4\epsilon_1}\right\}\int_0^L \theta^2dx + \left\{-Nk + \frac{N_1}{4\varepsilon_2}\right\}\int_0^L \theta_x^2dx \\
& +\left\{-Nk_3 + \frac{N_1}{4\varepsilon_2}\right\}\int_0^L w_x^2dx
\end{aligned}$$

At this point, we choose ϵ_1 and ϵ_2 small enough such that

$$-p(\xi_1) + (\eta + d)\,\epsilon_1 < 0, \quad -k_1c + 3\varepsilon_2 < 0,$$

then we choose N_1 large enough so that

$$N_1\left(-k_1c + 3\varepsilon_2\right) + \frac{\eta}{4\epsilon_1} < 0.$$

Once N_1 is fixed, we then choose N large enough so that

$$\begin{array}{l} -Nc' + \frac{N_1}{2\varepsilon_2} + m(\xi_2) < 0 \\ -Nk_2 + N_1\left(k_1\tau + 2\varepsilon_2 c' + c'\right) + \frac{d}{4\epsilon_1} < 0, \\ -Nk + \frac{N_1}{4\varepsilon_2} < 0, \\ -Nk_3 + \frac{N_1}{4\varepsilon_2} < 0. \end{array}$$

Thus, using (7.4), we arrive at

$$\mathcal{L}'(t) \leq -c\mathcal{E}(t), \quad \forall t > 0. \tag{7.24}$$

A combination of (7.23) and (7.24) gives

$$\mathcal{L}'(t) \leq -c_2\mathcal{L}(t), \quad \forall t > 0. \tag{7.25}$$

where $c_2 = c/c_2'$, a simple integration of (7.25) over $(0, t)$ yields

$$c_1'\mathcal{E}(t) \leq \mathcal{L}(t) \leq \mathcal{L}(0)e^{-c_2 t}\ , \ \ \forall t > 0.$$

Taking $c_1 = L(0)/c_1'$ which completes the proof. ■

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