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Qualitative studies of some dissipative systems for partial differential equations of evolution-type

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Bayoud Mouhssin

DIRECTEUR DE THÈSE : Khaled ZENNIR M.C.A Qassim University, KSA
CO- DIRECTEUR: Hocine SISSAOUI Prof U.B.M. ANNABA

Devant le jury

PRESIDENT:	Abdelhak DJEBABLA	M.C.A	U.B.M. ANNABA
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EXAMINATEUR :	Abdelaziz MENNOUNI	Prof	U. M.B.B. BATNA 2

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Qualitative studies of some dissipative systems for partial differential equations of evolution-type

Presented By: Bayoud Mouhssin

Under the Supervision of

Dr. Khaled Zennir and Pr. Hocine Sissaoui

Contents

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Abstract

In this thesis, we consider the study of some hyperbolic problems (equations and system of equations) with the presence of a viscoelastic term under some assumptions on initial data and boundary conditions, conditions on damping and source terms. The focuss of the study is on the existence and asymptotic behavior of solutions .

Key Words: Infinite memory, Thermoelastic transmission problem, Polynomial decay, Exponential Stability, Semigroup.
subject classification 2000 : 35L05, 35B40, 93D20, 93C20.

RÉSUMÉ

Dans cette thèse, on considère l'études théorique de quelques problèmes de type hyperbolique (équations et systèmes des équations) à terme viscoélastique sous quelques hypothèses sur les conditions initiale et au bord, des conditions sur les termes de dissipation, termes sources. Nous avons étudié l'existence et le comportement asymptotique de l'énergie des solutions.

Mots Clés: Mémoire infinie, problème de transmission thermoélastique, déccroissance polynôme, stabilité exponentielle, semi groupe.

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Introduction

Motivation

The problem of stabilization and control of partial differential equations play a pivotal role in the current paradigm of fundamental sciences. Evolution equations, i.e. partial differential equations with time t as one of the independent variables, arise not only in many fields of mathematics, but also in other branches of science such as physics, mechanics and material science. For example, Navier-Stokes and Euler equations of fluid mechanics, nonlinear reaction-diffusion equations of heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrodinger equations of quantum mechanics and Cahn-Hilliard equations of material science, to name just a few are special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences. see [44], [32], [34], [52], [35], [36], [39] and [37].
The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.
The problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of

stabilization:

1. Strong stabilization: $E(t) \to 0, \text{as} \;\; t \to \infty.$

2. Uniform stabilization: $E(t) \leq C \;\; exp(-\delta t), \forall t > 0, \;\; (C, \delta > 0).$

3. Polynomial stabilization: $E(t) \leq Ct^{-\delta}, \forall t > 0, \;\; (C, \delta > 0).$

4. Logarithmic stabilization: $E(t) \leq C(ln(t))^{-\delta}, \forall t > 0, \;\; (C, \delta > 0).$

In recent years, an increasing interest has been developed to study the dynamical behavior of several thermoelastic problems so as to describe the thermo-mechanical interactions in elastic materials. In the beginning people mainly considered the dynamical problems of classical thermoelastic systems, the $1-D$ linear model of which is given as follows:

$$\begin{cases} u_{tt} - u_{xx} - b\theta_x = 0, & x \in (0, L), t > 0 \\ \\ \theta_t + \theta_{xx} + bu_{xt} = 0, & x \in (0, L), t > 0 \end{cases} \tag{1}$$

Where $u(x, t)$ denotes the displacement of the rod at time t, and $\theta(x, t)$ is the temperature difference with respect to a fixed reference temperature. In 1960s Dafermos in [6] discussed the existence of solution of the classical thermoelastic system and showed the asymptotic stability of the system under certain condition. Rivera further proved that the solution of this kind of thermoelastic system decays exponentially.

The classical thermoelasticity is mainly modeled based on the Fourier's law, in wish the speed of thermal propagation is infinite. This violates practical conditions, since the whole materials will not fell instantly at a sudden disturbance in some point (see[43]). In order to eliminate this paradox, Lord and Shulman in [42], employed the modified Fourier's law, proposed by Cattaneo (named Cattaneo's law), and developed what now is known as extended thermoelasticity. Based on this non classical thermoelastic theory, many nice results on large time behavior of the thermoelastic systems.

In 1990s, three thermoelastic theories, known as type I, type II and type III, respectively, were proposed by Green and Naghdi[14]. They developed their theories by introducing the thermal displacement τ satisfying the following equation.

$$\tau(., t) = \int_0^t \theta(., s)ds + \tau(., 0) \tag{2}$$

The type I theory is consistent with the classical thermoelasticity, the type II is also named thermoelasticity without dissipation, that is the energy is conservative, these two theories type I and type II are restricted cases of the type III given as follows.

$$\begin{cases} \rho u'' - (au_x - l\theta)_x = 0, \\ \\ c\tau'' + lu'_x - (\beta\theta_x + k\tau_x)_x = 0. \end{cases} \tag{3}$$

When $k = 0$, the above system becomes (1), the so-called type I thermoelasticity (classical one), and when $b = 0$, the following thermoelastic system is obtained named type II

$$\begin{cases} \rho u'' - (au_x - l\theta)_x = 0, \\ \\ c\tau'' + lu'_x - k\tau_{xx} = 0. \end{cases} \tag{4}$$

Based on these three types of thermoelasticity, there has been an extensive literature on the decay rate for thermoelastic systems in recent years. We refer for instance, ([56]) for the exponential decay and polynomial decay of multi-dimensional thermoelasticity of type III by observability estimates; [57] for the exponential decay for thermoelasticity of type II with porous damping based on frequency domain analysis; ([30]-[59]) for the stability analysis of thermoelastic Timoshenko-type systems of type III by energy multiplier method; [30] for the spectral properties of thermoelasticity of type II and for the stability analysis of transmission problem between thermoelasticity and pure elasticity at the interfaces; and [58] for analyticity of solution of thermoelasticities.

From the above results on asymptotic behavior of the systems, we find that for the linear $1 - D$ thermoelastic models of type I and type III, the thermal effects are all always strong enough to stabilize the system exponentially, while the one of type II is a conservative system in which there is no dissipation. Thus an interesting issue is roused that whether or not the system can achieve exponential decay rate when mixing two of them (type I, type II, type III) together, that is, in one part of the domain we have a type of thermoelasticity, but in the other part of the domain, we have another type of thermoelasticity coupling with certain transmission condition at the interface. The dynamical behavior of this kind of transmission problem is difficult to analyze, since coupling exist not only between the therm and elasticity but also at the interface. Liu and Quintanilla in [60], considered the asymptotic behavior of the mixed type II and type III thermoelastic system. They proved that the system is lack of exponential decay rate bat achieves polynomial decay under certain condition. However the sharpness of the polynomial decay rate for this kind of system is still unknown, wish is very tough issue due to the complex couplings.

Conserved and dissipated quantities

The notion of dissipative of number, energy, mass, momentum is a fundamental principle that can be used to derive many partial differential equations.

Any function, especially one with several independent variables, carries a huge amount of information. The questions we want to answer about partial differential equations are often simple however. Complete knowledge of the details of an equation's solution are frequently unavailable, and would be overkill in any event. It is therefore useful to study coarse grained quantities that arise in partial differential equations in order to circumvent a complete analysis of these problems. Notice this philosophy has a long history in science, physicists and chemists like to talk about a system's energy or entropy, which can be understood without any intimate knowledge of the microscopic details.

For some solution of a partial differential equations $u(x,t)$, we can define a coarse-grained quantity as a functional, which is a mapping from u to the real numbers for example,

$$\int_\Omega u dx, \qquad \int_\Omega u_x^2 dx, \qquad \int_\Omega u_{xx}^4 dx,$$

are all examples of functionals. It often happens that functionals represent quantities of physical interest mass, energy, momentum, etc. but such an interpretation is not essential for these objects to be useful.

Suppose E is some functional of $u(x,t)$ of the form

$$E[u] = \int_\Omega f(u, u_x, \dots) dx.$$

So that E depends on t, but not on the variable x which has been integrated out. There are two common properties which depend on the time evolution of E. If $E' = 0$, then E is called conserved. If $E' \leq 0$, then E is called dissipated.

Suppose u solves the wave equation and boundary conditions

$$u'' - u_{xx} = 0, \qquad u(0,t) = 0 = u(L,t).$$

Then the energy functional (essentially the sum of kinetic and potential energy)

$$E(t) = \frac{1}{2}\int_0^L u'^2 + u_x^2 dx,$$

is conserved indeed,

$$E'(t) = \int_0^L u'u'' + u_x u_x' dx = [u_x u']_0^L + \int_0^L u'u'' + u'u_{xx} dx = 0$$

where integration by parts and the boundary condition was used for the second equality. The fact that E remains the same for all t has profound qualitative implications. Any solution which has wave oscillations initially (so that the energy is positive) must continue to have oscillations for all time - they never die out, for example. Conversely if the initial conditions are quiescent, so that $E = 0$, then this must happen forever. Notice we learn these things without ever finding a solution of the equation.

As another example, suppose u solves the diffusion equation

$$u' - u_{xx} = 0, \qquad u(0,t) = 0 = u(L,t).$$

Then the energy functional

$$E(t) = \frac{1}{2}\int_0^L u_x^2 dx,$$

is dissipated, since

$$E'(t) = \int_0^L u_x u_x' dx = -\int_0^L u' u_{xx} dx = -\int_0^L u_{xx}^2 dx < 0$$

where again integration by parts and the boundary condition was used.

We can interpret E as follows. The arc length of $x-$cross sections of u can be approximated for small u_x as

$$\int_0^L \sqrt{1 + u_x^2} dx \equiv \int_0^L 1 + \frac{1}{2} u_x^2 dx.$$

Since $E' \leq 0$, the approximate arc length must also diminish over time. This means the graph of $u(x,.)$ gradually becomes smoother, and oscillations die away. This statement will be made perfectly quantitative by solving the equation outright using separation of variables.

Overview of the dissertation and Target problems

The thesis divided in to four chapters beginning by a general introduction.

The First Chapter

This chapter summarizes some concepts, definitions and results which are mostly relevant to the undergraduate curriculum and are thus assumed as basically known, or have specific roots in rather distant areas and have rather auxiliary character with respect to the purpose of this study. In the next four chapters, we develop our main results for nonlinear evolution problems of hyperbolic type.

The second Chapter

In this chapter, we describes a polynomial decay rate of solution for a transmission problem with $1-D$ mixed type I and type II thermoelastic system with infinite memory acting in the first part. The main contributions here are to show that the infinite memory lets our problem still dissipative, and that the system is not exponentially stable, in spite of the kernel in the memory term is sub-exponential. Also we establish that the t^{-1} is the sharp decay rate. We extend the results in [21]. This is subject of publication in J. Applied Sciences, Vol.2 .

The third Chapter

This chapter we consider a class of degenerate viscoelastic wave equation with density in Kirchhoff-type. Under appropriate conditions on the initial datums, we prove a very general decay rate of solution using the Lyapunov functions. In order to compensate the lack of Poincare's inequality in $\mathbb{R}^n$ we use spaces weighted by density function.This is subject of publication in Int. J. Appl. Comput. Math (2018) (Springer).

The fourth Chapter

This chapter , we study a viscoelastic wave equations of the Kirchhoff type

$$u'' - \phi(x)\left(M(\|\nabla_x u\|_2^2)\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) = au\ln|u|^k$$

defined in any spaces dimension. It is well known that from a class of nonlinearities the logarithmic nonlinearity is distinguished by several interesting physical properties. We use weighted spaces to establish the long-time behavior of solution of (3.1). Furthermore, under convenient hypotheses on g and the initial data, the local-in-time existence of solution is established.This is subject of publictain in J.Appl.Math.Inf.Sci.10,No.61-10(2016).

Chapter 1

Technical tools

The aim of this chapter is to recall the essential notions and results used throughout this work. First, we recall some definitions and results on Sobolev spaces and the spaces $L^p(0, T, X)$ and give the statement of some important theorems in the analysis of problems to be studied and eventually some notations used throughout this study.

1.1 Function Analysis

Normed spaces, Banach spaces and their properties

Let V be linear space.

Definition 1.1.1. *A non-negative, degree-1 homogeneous, sub-additive functional $\|.\|_V : V \longrightarrow \mathbb{R}$ is called a norm if it vanishes only at 0, often we will write briefly $\|.\|$ instead of $\|.\|_V$ if the following properties are satisfying respectively*

$$\begin{cases} \|v\| \geq 0 \\ \|av\| = |a|\|v\| \\ \|u+v\| \leq \|u\| + \|v\| \\ \|v\| = 0 \longrightarrow v = 0. \end{cases}$$

for any $v \in V$ and $a \in \mathbb{R}$.

A linear space equipped with a norm is called a normed linear space. If the last (i.e.$\|v\|_v = 0 \longrightarrow v = 0$) is missing, we call such a functional a semi-norm.

Definition 1.1.2. *A Banach space is a complete normed linear space V. Its dual space V' is the linear space of all continuous linear functional $u : V \longrightarrow \mathbb{R}$.*

Example of Banach spaces

1. $L^\infty[\alpha, \beta]$.
   This is the space of all measurable (complex valued) functions u on $[\alpha, \beta]$ which are essentially bounded, i.e., for every $u \in L^\infty[\alpha, \beta]$ there exists $a > 0$ such that $|u(t)| \leq a$ almost everywhere. Define $\|u\|$ to be the infinitum of such a. (Here also we identify two functions which are equal almost everywhere).

2. V' equipped with the norm $\|.\|_{V'}$ defined by

$$\|u\|_{V'} = sup|u(x)| \text{ for all } \|x\| \leq 1,$$

   is also a Banach space.
   If V is a Banach space such that, for any

$$v \in V, V \longrightarrow \mathbb{R} : u \longrightarrow \|u + v\|^2 - \|u - v\|^2,$$

   is linear, then V is called a Hilbert space. In this case, we define the inner product (also called scalar product) by

$$(u, v) = \frac{1}{4}\|u + v\|^2 - \frac{1}{4}\|u - v\|^2.$$

Definition 1.1.3. *Since u is linear we see that*

$$u : V \longrightarrow V'',$$

is a linear isometric of V onto a closed subspace of V'' , we denote this by

$$V \longrightarrow V''.$$

Let V be a Banach space and $u \in V'$. Denote by

$$\phi_u : V \longrightarrow \mathbb{R}$$

$$x \longmapsto \phi_u(V),$$

when u covers V', we obtain a family of applications to $V \in \mathbb{R}$.

Definition 1.1.4. *The weak topology on V, denoted by $\sigma(V, V')$, is the weakest topology on V for which every $(\phi_u)_{u\in V'}$ is continuous. We will define the third topology on V', the weak star topology, denoted by $\sigma(V', V)$. For all $x \in V$, denote by*

$$\begin{array}{rcl} \phi_x : V' & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \phi_x(u) =< u, x >_{V',V} \end{array}$$

when x cover V, we obtain a family $(\phi_x)_{x\in V}$, of applications to V' in $\mathbb{R}$.

Theorem 1.1.1. *Let V be Banach space. Then, V is reflexive, if and only if,*

$$B_V = \{x \in V : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(V, V')$.

Corollary 1.1.1. *Every weakly y^* convergent sequence in V' must be bounded if V is a Banach space. In particular, every weakly convergent sequence in a reflexive Banach V must be bounded.*

Remark 1.1.1. *The weak convergence does not imply strong convergence in general*

Definition 1.1.5. *Let V be a Banach space and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in V. Then (u_n) converges strongly to u in V if and only if*

$$\lim_{n\longrightarrow\infty} \|u_n - u\|_V = 0$$

and this is denoted by $u_n \longrightarrow u$, or

$$\lim_{n\longrightarrow\infty} u_n = u$$

Example 1.1.1. *We shall now show by an example that weak convergence does not imply strong convergence in general. Consider the sequence* $\sin(n\pi t)$ *in* $L^2(0,1)$ *(real). This sequence converges weakly to zero. Since, by the Riesz theorem, any linear functional is given by the scalar product with a function we have to show that*

$$\int_0^t f(t)\sin(n\pi t)dt \to 0, \quad for \quad each \quad f \in L^2(0,1).$$

But By Bessel's inequality

$$\sum_{n=1}^{\infty}\int_0^1 |f(t)\sin(n\pi t)dt|^2 \leq \int_0^1 |f(t)|^2 dt,$$

so $\int_0^t f(t)\sin(n\pi t)dt \to 0$ *as* $n \to \infty$. *But* $\sin(n\pi t)$ *is not strongly convergent, since*

$$\begin{aligned} \|\sin(n\pi t) - \sin(m\pi t)\|^2 &= \int_0^1 |\sin(n\pi t) - \sin(m\pi t)|^2 dt \\ &= 2 \quad for \quad n \neq m. \end{aligned}$$

Functional spaces

Definition 1.1.6. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and let* m *be a non-negative integer. We define by* $C^m(\Omega)$ *the linear space of continuous functions on* Ω *whose partial derivatives* $D^\alpha u, |\alpha| \leq m$, *exist and continuous, where*

$$D^\alpha u = \frac{\partial^\alpha u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}$$

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *is called a multi-index of dimension* n *and* $|\alpha| = \sum_{i=1}^n \alpha_i$.

Definition 1.1.7. *The support of a continuous function* u *defined on* $\mathbb{R}^n$ *is the closure of the set of point where* $u(x)$ *is nonzero:*

$$suppu = \{\overline{x \in \mathbb{R}^n : u(x) \neq 0}\}$$

The closed and bounded sets in $\mathbb{R}^n$ *are precisely the compact sets, so if* $suppu$ *is bounded, we say* u *has a compact support and denote the set of such functions by* $C^0(\mathbb{R}^n)$. *Similarly,* $C^0(\Omega)$ *denotes the set of continuous functions on* Ω *whose supports are compact subsets of* Ω.

The $L^p(\Omega)$ spaces

Definition 1.1.8. *Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$; by:*

$$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}, \quad u \text{ is measurable and }, \quad \int_\Omega |u(x)|^p dx < \infty\}.$$

With the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}},$$

if $p = \infty$, we have

$$L^\infty(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}, u \text{ is measurable and }, \quad \exists C \in \mathbb{R}_+, |u(x)| \leq C \quad a.e\ in\, \Omega\}.$$

With the norm

$$\|u\|_\infty = inf\{C; |u| \leq C \quad a.e\ in\, \Omega\} \tag{1.1}$$

Theorem 1.1.2. *It is well known that $L^p(\Omega)$ equipped with the norm $\|.\|_p$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.1.2. *In particular, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$< f, g >_{L^2(\Omega)} = \int_\Omega f(x).g(x)dx,$$

is a Hilbert space.

Theorem 1.1.3. *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

Theorem 1.1.4. *if $u \in L^p(\Omega), 1 \leq p < \infty$, then there exist a sequence $(f_n) \subset C_0^\infty(\Omega)$ which converges to f with respect to the norm $\|.\|_p$. This implies that $C_0^\infty(\Omega)$ is dence in $L^p(\Omega)$.*

Sobolev spaces

Modern theory of differential equations is based on spaces of functions whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 1.1.1. *Let Ω be an open domain in $\mathbb{R}^n$, then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$if there exists a function $u \in L^p(\Omega)$ such that*

$$< T, \phi >= \int_{\Omega} u(x)\phi(x)dx, \forall \phi \in D(\Omega),$$

where $1 \leq p \leq \infty$, and it's well-known that u is unique.

Definition 1.1.9. *Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $u \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) \quad = \quad \{u \in L^p(\Omega) \quad where, D^\alpha u \in L^p(\Omega) \quad \forall \alpha \in \mathbb{N}^n\}.$$

Theorem 1.1.5. *$W^{m,p}(\Omega)$ is a Banach space with its usual norm*

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, 1 \leq p < \infty \quad \forall u \in W^{m,p}(\Omega).$$

Notation 1.1.1. *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Space $H^m(\Omega)$:

Definition 1.1.10. *When $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ endowed with the norm*

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} (\|D^\alpha u\|)^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

which renders $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$< u, v >_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u D^\alpha v dx.$$

Theorem 1.1.6. *1- $H^m(\Omega)$ endowed with inner product $< ., . >_{H^m(\Omega)}$ is a Hilbert space.*

2- If $m < m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$, with continuous embedding.

Lemma 1.1.1. *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω and we have*

$$D(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow D'(\Omega)$$

Theorem 1.1.7. *(Gronwell lemma in integral form)*
Let $T > 0$, and let ψ be a function such that,$\psi \in L^1[0,T], \phi \geq 0$, almost everywhere and ψ be a function such that $\phi \in L^1[0,T], \phi \geq 0$, almost everywhere and $\phi\psi \in L^1[0,T], C_1, C_2 \geq 0$. Suppose that

$$\phi(t) \leq C_1 + C_2 \int_0^t \phi(t)\psi(t)ds, for \quad a.e \quad t \in]0,T[,$$

then

$$\phi(t) \leq C_1 \exp\left(C_2 \int_0^t \phi(t)\psi(t)ds\right), for \quad a.e \quad t \in]0,T[.$$

1.2 Green's Formula

Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary, then $\forall u \in H^1(\Omega), \forall v \in H^2(\Omega)$, we have

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} u \nabla v \eta ds \tag{1.2}$$

Where η is the outer unit normal to $\partial\Omega$.

Remark 1.2.1. *If $u \in H_0^1(\Omega)$, the Green's Formula is reduced to*

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \nabla v dx$$

1.3 Useful technical lemmas

Lemma 1.3.1. *For any* $v, g \in C^1\left([0,T], H^1(\mathbb{R}^n)\right)$ *Let* $\alpha \in C^1(\mathbb{R}^n)$ *we have*

$$
\begin{aligned}
& -\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
& = \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g \circ A^{1/2} v\right)(t) \\
& - \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx ds\right] \\
& - \frac{1}{2}\alpha(t)\left(g'^{1/2}v\right)(t) + \frac{1}{2}\alpha(t) g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx ds \\
& - \frac{1}{2}\alpha'(t)\left(g \circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s) ds \int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx ds.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
& = \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2} v'^{1/2} v(s) dx ds \\
& = \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2} v'(t)\left[A^{1/2}v(s) - A^{1/2}v(t)\right] dx ds \\
& + \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2} v'^{1/2} v(t) dx ds.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
& = -\frac{1}{2}\alpha(t)\int_0^t g(t-s)\frac{d}{dt}\int_{\mathbb{R}^n}\left|A^{1/2}v(s) - A^{1/2}v(t)\right|^2 dx ds \\
& + \alpha(t)\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx\right) ds
\end{aligned}
$$

Which implies

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) Av(s) v'(t) ds dx \\
&= -\frac{1}{2}\frac{d}{dt}\left[\alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left|A^{1/2}v(s) - A^{1/2}v(t)\right|^2 dx ds\right] \\
&+ \frac{1}{2}\frac{d}{dt}\left[\alpha(t) \int_0^t g(s) \int_{\mathbb{R}^n} \left|A^{1/2}v(t)\right|^2 dx ds\right] \\
&+ \frac{1}{2}\alpha(t) \int_0^t g'(t-s) \int_{\mathbb{R}^n} \left|A^{1/2}v(s) - A^{1/2}v(t)\right|^2 dx ds \\
&- \frac{1}{2}\alpha(t) g(t) \int\limits_{\mathbb{R}^n} \left|A^{1/2}v(t)\right|^2 dx ds. \\
&+ \frac{1}{2}\alpha'(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left|A^{1/2}v(s) - A^{1/2}v(t)\right|^2 dx ds \\
&- \frac{1}{2}\alpha'(t) \int_0^s g(s) ds \int_{\mathbb{R}^n} \left|A^{1/2}v(t)\right|^2 dx ds.
\end{aligned}$$

□

1.4 Some algebraic and integral inequalities

We give here some important integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Theorem 1.4.1. *Let a and b be strictly positive realities p and q such as,* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $1 < p < \infty$,$1 < q < \infty$ *we have :*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The function f defined by:

$$f(x) = \frac{x^p}{p} - x$$

reached its minimum point $x = 1$ indeed :

$$y' = x^{p-1} \quad et \quad y'' = (p-1)x^{p-2} > 0$$

from where

$$f(ab^{1-q}) \geq f(1)$$

which gives

$$\frac{\left(ab^{1-q}\right)^{p}}{p} - ab^{1-q} \geq \frac{1}{p} - 1 = -\frac{1}{q}$$

so that

$$\frac{a^p}{p} b^{(1-q)p} - ab^{1-q} + \frac{1}{q} \quad \geq 0$$

By dividing the two members by $b^{(1-q)p}$ we obtain :

$$\frac{a^p}{p} - ab^{(1-q)-p+pq} + \frac{b^q}{q} \geq 0$$

which yields

$$\frac{a^p}{p} - ab + \frac{b^q}{q} \geq 0$$

so that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Remark 1.4.1. *A simple case of Young's inequality is the inequality for $p = q = 2$*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

which also gives Young's inequality for all $\delta > 0$

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2$$

Theorem 1.4.2. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$.*
Then for a.e $x \in \mathbb{R}^n$ the function $f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ and we define

$$(f * g) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

In addition
$(f * g) \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

The following is an extension of Theorem 1.4.2.

Theorem 1.4.3. *(Young) Assume* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$ *with* $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$*. Therefore*

$$(f * g) \in L^r(\mathbb{R}^n)$$

and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Theorem 1.4.4. *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *with* $1 \leq p \leq \infty, 1 \leq q \leq \infty$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$*, Then* $fg \in L^1(\Omega)$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

when $p = q = 2$*, we get the Cauchy-Schwartz inequality*

Corollary 1.4.1. *(Holder's inequality general form) Let* $f_1, f_2, ...f_k$ *be k functions such that,* $f_i \in L^{p_i}(\Omega)$*,* $i = \overline{1:k}, 1 \leq p_i \leq \infty$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq 1.$$

Then, the product $f_1, f_2, ...f_k \in L^p(\Omega)$ *and* $\|f_1 f_2 ... f_k\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} ... \|f_k\|_{p_k}$.

Lemma 1.4.1. *(Minkowski inequality) let* $f, g \in L^p(\Omega)$ *For* $1 \leq p \leq \infty$*, we have*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

1.5 Semi-group approach

Definition 1.5.1. *Let* $\{T_t\}_{t \geq 0}$ *be a one-parameter family of linear operators on a Banach space X into itself satisfying the following conditions:*

(1) $T_t T_s = T_{t+s}, T_0 = I$*, I denoting the identity operator on X (Semi -group property).*

(2) $s - \lim\limits_{t \to t_0} T_t x = T_{t_0} x \leq 0$ *and each* $x \in X$ *(strong continuity).*

(3) *there exists a real number* $\beta \geq 0$ *such that* $\|T_t\| \leq e^{\beta t}$ *for* $t \geq 0$.

We call such a family $\{T_t\}$ a semi group of linear operators of normal type on the Banach space X, or simply a semi-group.

Theorem 1.5.1. *Linear operator $\mathcal{A}$ is dissipative if and only if*

$$\|(\lambda I - \mathcal{A})\|_X \geq \lambda \|X\|_X, \forall x \in D(\mathcal{A}), \lambda > 0.$$

Theorem 1.5.2. *Linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ generates a strongly continuous semi group of contractions $(T(t))_{t\geq 0}$ on X if and only if $\mathcal{A}$ is m-dissipative, i.e, it satisfies:*

1- $\Re(\mathcal{A}v, v) \leq 0 \ \forall v \in D(\mathcal{A})$.

2- $\exists \lambda > 0, (\lambda I - \mathcal{A})$ *is surjective.*

Theorem 1.5.3. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semi group of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\zeta : \zeta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\zeta| \to \infty}} \|(i\zeta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Lemma 1.5.1. *A C_0 semi group e^{At} of contraction on Hilbert space satisfies*

$$\|e^{At} U_0\| \leq C t^{\frac{-1}{l}} \|U_0\|_{D(A)} \quad \forall U_0 \in D(A)$$

for some constant $C > 0$, if and only if the following conditions hold:

$$\rho(\mathcal{A}) \supseteq \{i\zeta : \zeta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\zeta| \to \infty}} \zeta^{-l} \|(i\zeta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Chapter 2

Transmission problem with 1-D mixed type in thermoelasticity and infinite memory

2.1 Introduction and position of problem

In this chapter, we consider a transmission problem with $1-D$ mixed type I and type II thermoelastic system and memory term for $t>0$ in the following:

$$\begin{cases} \rho_1 u'' - a_1\Big(u_{xx} - \int_{-\infty}^{t} \mu(t-s)u_{xx}(s)ds\Big) + \beta_1\theta_x = 0, & x \in (-L,0),\\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L,0),\\ \rho_2 v'' - a_2 v_{xx} + \beta_2 q_x = 0, & x \in (0,L),\\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0,L),\\ \\ u(0,t) = v(0,t), \\ \theta(0,t) = q(0,t), \\ w_1(0,t) = w_2(0,t), \\ l\theta_x(0,t) = k w_{2,x}(0,t), \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), \end{cases} \tag{2.1}$$

Where u, v are the displacement of the system at time t in $(-L,0)$ and $(0,L)$ in $(-L,0)$ and $(0,L)$ and θ, q are respectively the temperature difference with respect to a fixed reference temperature, w_1, w_2 are the so-called thermal displacement, which satisfies

$$w_1(.,t) = \int_0^t \theta(.,s)ds + w_1(.,0)$$

and

$$w_2(.,t) = \int_0^t q(.,s)ds + w_2(.,0).$$

The parameters $a_1, a_2, \rho_1, \rho_2, \beta_1, \beta_2, c_1.c_2, k, l$ and $L < \infty$ are assumed to be positive constants. The system (2.1) satisfies the Dirichlet boundary conditions:

$$\begin{cases} u(-L,t) = v(L,t) = 0, & t > 0, \\ \\ w_1(-L,t) = w_2(L,t) = 0, & t > 0, \end{cases} \tag{2.2}$$

And the following initial conditions:

$$\begin{cases} u(.,0) = u^0(x), u'(.,0) = u^1(x), w_1(.,0) = w_1^0(x), \theta(.,0) = \theta^0(x), \\ v(.,0) = v^0(x), v'(.,0) = v^1(x), w_2(.,0) = w_2^0(x), q(.,0) = q^0(x). \end{cases} \tag{2.3}$$

We treat the infinite memory as Dafermos [8], adding a new variable η to the system which corresponds to the relative displacement history. Let us define the auxiliary variable

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in (-L,0) \times \mathbb{R}^+.$$

By differentiation, we have

$$\frac{d}{dt}\eta^t(x,s) = -\frac{d}{ds}\eta^t(x,s) + \frac{d}{dt}u(x,t), \quad (x,s) \in (-L,0) \times \mathbb{R}^+.$$

We can take as initial condition $(t = 0)$

$$\eta^0(x,s) = u^0(x) - u(x,-s), \quad (x,s) \in (-L,0) \times \mathbb{R}^+.$$

Thus, the original memory term can be rewritten as follows

$$\begin{aligned} \int_{-\infty}^t \mu(t-s)u_{xx}(s)ds &= \int_0^\infty \mu(s)u_{xx}(t-s)ds \\ &= \left(\int_0^\infty \mu(t)dt\right) u_{xx} - \int_0^\infty \mu(s)\eta_{xx}^t(s)ds. \end{aligned}$$

The problem (2.1) is transformed into the system

$$\begin{cases} \rho_1 u'' - a_1\Big(\mu_0 u_{xx} + \int_0^\infty \mu(s)\eta^t_{xx}(s)ds\Big) + \beta_1\theta_x = 0, & x \in (-L,0), \\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L,0), \\ \rho_2 v'' - a_2 v_{xx} + \beta_2 q_x = 0, & x \in (0,L), \\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0,L), \\ \frac{d}{dt}\eta^t(x,s) + \frac{d}{ds}\eta^t(x,s) - \frac{d}{dt}u(x,t) = 0, & x \in (-L,0), \\ \\ u(0,t) = v(0,t), \\ \theta(0,t) = q(0,t), \\ w_1(0,t) = w_2(0,t), \\ l\theta_x(0,t) = k w_{2,x}(0,t), \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), \\ \eta^0(x,s) = u^0(x,0) - u^0(x,-s), s > 0, \end{cases} \tag{2.4}$$

Where $\mu_0 = 1 - \int_0^\infty \mu(t)dt$.

The stability of various transmission problems on thermoelasticity have been considered [10], [12], [30], [18], [19] and [20]. Without infinite memory, it is proved in [21] that the energy of system (2.1) cannot achieve exponential decay rate. This paper is devoted to show that our system can achieve polynomial decay rate. That is, our main result here is to show that for these types of materials the dissipation produced by the viscoelastic part is not strong enough to produce an exponential decay of the solution despite that the infinite memory satisfies assumptions (2.7) and (2.8).

2.2 Previous Results on Its Stability

The transmission problem to hyperbolic equations was studied by Dautray and Lions [9], where the existence and regularity of solutions for the linear problem have been proved. In

[18], the authors considered the transmission problem of viscoelastic waves

$$\begin{cases} \rho_1 u'' - \alpha_1 u_{xx} = 0, & x \in (0, L_0), \\ \rho_2 v'' - \alpha_2 v_{xx} + \int_0^t g(t-s) v_{xx}(s) ds = 0, & x \in (L_0, L), \end{cases} \tag{2.5}$$

Satisfying boundary conditions and initial conditions. The authors studied the wave propagations over materials consisting of elastic and viscoelastic components. They showed that the viscoelastic part produce exponential decay of the solution. In [16], the authors investigated a 1D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero. Marzocchi et al [17] studied a multidimensional linear thermoelastic transmission problem. An existence and regularity result has been proved. When the solution is supposed to be spherically symmetric, the authors established an exponential decay result similar to [16]. Next, Rivera and *all* [19], considered a transmission problem in thermoelasticity with memory. As time goes to infinity, they showed the exponential decay of the solution in case of radially symmetric situations. We must mention the pioneer work by Rivera and *all* in [12], where a semilinear transmission problem for a coupling of an elastic and a thermoelastic material is considered. The heat conduction is modeled by Cattaneo's law removing the physical paradox of infinite propagation speed of signals. The damped, totally hyperbolic system is shown to be exponentially stable. In 2009, Mesaoudi and *all* [30] proposed and studied a 1D linear thermoelastic transmission problem, where the heat conduction is described by the theories of Green and Naghdi. By using the energy method, they proved that the thermal effect is strong enough to produce an exponential stability of the solution. The earliest result in this direction was established in [21], where the dynamical behavior of the system is described by

$$\begin{cases} \rho_1 u_1'' - a_1 u_{1,xx} + \beta_1 \theta_{1,x} = 0, & x \in (-1, 0), \\ c_1 \tau_1'' - b\theta_{1,xx} + \beta_1 u_{1,x}' = 0, & x \in (-1, 0), \\ \rho_2 u_2'' - a_2 u_{2,xx} + \beta_2 \theta_{2,x} = 0, & x \in (0, 1), \\ c_2 \tau_2'' - k\tau_{2,xx} + \beta_2 u_{2,x}' = 0, & x \in (0, 1). \end{cases} \tag{2.6}$$

The system consists of two kinds of thermoelastic components, one is of type I, another one is of type II. Under certain transmission conditions, these two components are coupled at

the interface. The authors proved that the system is lack of exponential decay rate and they obtain the sharp polynomial decay rate.
First we recall and make use the following assumptions on the functions μ:
We assume that the function $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$1 - \int_0^\infty \mu(t)dt = \mu_0 > 0, \quad \forall t \in \mathbb{R}^+, \tag{2.7}$$

And that there exists a constants $k_1 > 0$ such that

$$\mu'(t) + k_1\mu(t) \leq 0 \quad \forall t \in \mathbb{R}^+. \tag{2.8}$$

We denote by $\mathcal{A}$ the unbounded operator in an appropriate Hilbert state space
Let

$$\begin{aligned}
&V^k(0,L) = \{h \in H^k(0,L); h(L) = 0\},\\
&V^k(-L,0) = \{h \in H^k(-L,0); h(-L) = 0\},\\
&\mathcal{H} = V^1(-L,0) \times L^2(-L,0) \times L^2(-L,0) \times V^1(0,L) \times L^2(0,L) \times V^1(0,L) \times L^2(0,L).
\end{aligned}$$

equipped, for $(u, u^1, \theta, v, v^1, w_2, q), (\tilde{u}, \tilde{u^1}, \tilde{\theta}, \tilde{v}, \tilde{v^1}, \tilde{w}_2, \tilde{q}) \in \mathcal{H}$, with an inner product

$$\begin{aligned}
&\Big\langle (u, u^1, \theta, v, v^1, w_2, q), (\tilde{u}, \tilde{u^1}, \tilde{\theta}, \tilde{v}, \tilde{v^1}, \tilde{w}_2, \tilde{q}) \Big\rangle_{\mathcal{H}}\\
&= \int_{-L}^0 \left[a_1\Big(\mu_0 u_x + \int_0^t \mu(s)\eta_x^t(s)ds\Big)\overline{\tilde{u}_x} + \rho_1 u^1\overline{\tilde{u^1}} + c_1\theta\overline{\tilde{\theta}}\right]dx\\
&+ \int_0^L \left[a_2 v_x\overline{\tilde{v}_x} + \rho_2 v^1\overline{\tilde{v^1}} + kw_{2,x}\overline{\tilde{w}_{2,x}} + c_2 q_x\overline{\tilde{q}_x}\right]dx.
\end{aligned}$$

With domain

$$\mathcal{D}(\mathcal{A}) = (u, u^1, \theta, v, v^1, w_2, q) \in \mathcal{H} : \begin{cases}
u, \theta \in H^2(-L,0), u^1 \in H^1(-L,0),\\
v \in H^2(0,L), v^1, q \in H^1(0,L), w_2 \in H^2(0,L),\\
\theta(-L) = q(L) = 0, l\theta_x(0) = kw_{2,x}(0)\\
a_1\mu_0 u_x(0) - \beta_1\theta(0) = a_2 v_x(0) - \beta_2 q(0)\\
u(0) = v(0), \theta(0) = q(0),
\end{cases} \tag{2.9}$$

And

$$\mathcal{A}\begin{pmatrix} u \\ u^1 \\ \theta \\ v \\ v^1 \\ w_2 \\ q \end{pmatrix} = \begin{pmatrix} u^1 \\ \rho_1^{-1}\Big(a_1\Big(\mu_0 u_{xx} + \int_0^\infty \mu(s)\eta_{xx}^t(s)ds\Big) - \beta_1\theta_x\Big) \\ c_1^{-1}\Big(-\beta_1 u_x^1 + l\theta_{xx}\Big) \\ v^1 \\ \rho_2^{-1}\Big(a_2 v_{xx} - \beta_2 q_x\Big) \\ q \\ c_2^{-1}\Big(-\beta_2 v_x^1 + k w_{2,xx}\Big) \end{pmatrix} \tag{2.10}$$

For $\mathcal{U} = (u, u^1, \theta, v, v^1, w_2, q)^T$, the problem (2.4) can be reformulated in the abstract from

$$\mathcal{U}' = \mathcal{A}\mathcal{U}, \tag{2.11}$$

Where $\mathcal{U}(0) = (u^0, u^1, \theta^0, v^0, v^1, w_2^0, q^0)^T \in \mathcal{H}$ is given.
We will use necessary and sufficient conditions for C_0-semigroups being exponentially stable in a Hilbert space. This result was obtained by Gearhart [13] and Huang [11]

2.3 Lack of Exponential Stability

Following the techniques in [4], it is easy to check that $(\mathcal{H}, \|.\|_{\mathcal{H}})$ is a Hilbert space. In this section we prove the lack of exponential decay using Theorem 1.5.3, that is we show that there exists a sequence of values h_m such that

$$\|(ih_m I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty. \tag{2.12}$$

It is equivalent to prove that there exist a sequence of data $F_m \in \mathcal{H}$ and a sequence of real numbers $h_m \in \mathbb{R}$, with $\|F_m\|_{\mathcal{H}} \leq 1$ such that

$$\|(ih_m I - \mathcal{A})^{-1}F_m\|_{\mathcal{H}} = \|U_m\|_{\mathcal{H}}^2 \to \infty. \tag{2.13}$$

Theorem 2.3.1. *Assume that the kernel is of the form $\mu(s) = e^{-hs}, s \in \mathbb{R}^+$, with $h > 1$. The semi group $S(t)$ on $\mathcal{H}$ is not exponentially stable.*

Proof. As in [3], we will find a sequence of bounded functions

$$F_m = (f_{1,m}, f_{2,m}, f_{3,m}, f_{4,m}, f_{5,m}, f_{6,m}, f_{7,m}, f_{8,m})^T \in \mathcal{H}, h \in \mathbb{R},$$

for which the corresponding solutions of the resolvent equations is not bounded. This will prove that the resolvent operator is not uniformly bounded. We consider the spectral equation

$$ihU_m - \mathcal{A}U_m = F_m.$$

And show that the corresponding solution U_m is not bounded when F_m is bounded in $\mathcal{H}$. Rewriting the spectral equation in term of its components, we get

$$\begin{cases} ihu - u^1 = f_{1m} \\ ih\rho_1 u^1 - \Big(a_1\Big(\mu_0 u_{xx} + \int_0^\infty \mu(s)\eta^t_{xx}(s)ds\Big) - \beta_1\theta_x\Big) = \rho_1 f_{2m} \\ ihc_1\theta - \Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = c_1 f_{3m} \\ ihv - v^1 = f_{4m} \\ ih\rho_2 v^1 - \Big(a_2 v_{xx} - \beta_2 q_x\Big) = \rho_2 f_{5m} \\ ihw_2 - q = f_{6m} \\ ihc_2 q - \Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = c_2 f_{7m} \\ ih\eta^t - u^1 + \eta^t_s = f_{8m}. \end{cases} \tag{2.14}$$

We prove that there exists a sequence of real numbers h_m so that (2.14) verified. Let us consider $f_{1m} = f_{4m} = f_{6m} = f_{8m} = 0$. We eliminate the terms u^1, v^1. We can choose $f_{2m} = f_{3m} = f_{5m} = f_{7m} = \lambda_m$ and we obtain $u^1 = ihu$, $v^1 = ihv$ and $q = ihw_2$. Then, the system (2.14) takes the form

$$\begin{cases} -h^2 u - \rho_1^{-1}\Big(a_1\Big(\mu_0 u_{xx} + \int_0^\infty \mu(s)\eta^t_{xx}(s)ds\Big) - \beta_1\theta_x\Big) = \lambda_m \\ ih\theta - c_1^{-1}\Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = \lambda_m \\ -h^2 v - \rho_2^{-1}\Big(a_2 v_{xx} - \beta_2 ihw_{2,x}\Big) = \lambda_m \\ -h^2 w_2 - c_2^{-1}\Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = \lambda_m \\ ih\eta^t - ihu + \eta^t_s = 0 \end{cases} \tag{2.15}$$

We look for solutions of the form

$$u = a\lambda_m, v = b\lambda_m, \theta = c\lambda_m, w_2 = d\lambda_m, u^1 = e\lambda_m, v^1 = f\lambda_m, \eta^t(x,s) = \gamma(s)\lambda_m$$

With $a, b, c, d, e, f \in \mathbb{C}$ and $\gamma(s)$ depend on h and will be determined explicitly in what follows. From (2.15), we get a, b, c, d, e and f satisfy

$$\begin{cases} -h^2 a - \rho_1^{-1}\Big(a_1 h_m \Big(\mu_0 a + \int_0^\infty \mu(s)\gamma(s)ds\Big) - \beta_1 ch\Big) = 1, \\ ihc - c_1^{-1}\Big(- \beta_1 e + lh_m c\Big) = 1, \\ -h^2 b - \rho_2^{-1}\Big(a_2 h_m b - \beta_2 ihd\Big) = 1, \\ ihd - c_2^{-1}\Big(- \beta_2 f + kh_m d\Big) = 1, \\ \gamma_s + ih\gamma - iha = 0. \end{cases} \tag{2.16}$$

From $(2.16)_5$ we get

$$\gamma(s) = a - ae^{-ihs}. \tag{2.17}$$

Then, from (2.17) we have

$$\begin{aligned} \int_0^\infty \mu(s)\gamma(s)\, ds &= \int_0^\infty \mu(s)(a - ae^{-ihs})ds \\ &= a\int_0^\infty \mu(s)ds - a\int_0^\infty \mu(s)ae^{-ihs}ds \\ &= a(1-\mu_0) - a\int_0^\infty \mu(s)e^{-ihs}\, ds. \end{aligned} \tag{2.18}$$

Now, we would like to find the parameters constants. To this end, we choose

$$c_1 ih = h_m l, \qquad c_2 ih = kh_m, \tag{2.19}$$

And using the equations $(2.16)_2$ and $(2.16)_4$, we obtain

$$e = \frac{c_1}{\beta_1}, \tag{2.20}$$

$$f = \frac{c_2}{\beta_2}. \tag{2.21}$$

We choose $-h^2\rho_2 = a_2 h_m$. By equations $(2.16)_1$ and $(2.16)_3$, we have

$$c = \frac{1}{(-h^2 - \rho_1^{-1}h_m a_1 \mu_0)}\Big(1 + \rho_1^{-1}h_m a_1 \int_0^\infty \mu(s)\gamma(s)ds - \rho_1^{-1}h_m\beta_1 c\Big),$$

$$d = \frac{\rho_2}{\beta_2 ih}.$$

Since $c_2 l = c_1 k$, recalling from (2.20), (2.21) that

$$\begin{aligned} u^1 + v^1 &= e\lambda_m + f\lambda_m \\ &= \frac{c_1}{\beta_1}\lambda_m + \frac{c_2}{\beta_2}\lambda_m, \end{aligned}$$

We get

$$\|u^1\|_2^2 + \|v^1\|_2^2 = \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big] h_m^2.$$

Therefore we have

$$\begin{aligned} \lim_{m\to\infty} \|U_m\|_{\mathcal{H}}^2 &\geq \lim_{m\to\infty} [\|u^1\|_2^2 + \|v^1\|_2^2] \\ &= \lim_{m\to\infty} \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big] h_m^2 \\ &= +\infty \end{aligned}$$

Which completes the proof. □

2.4 Polynomial Stability

Our main result reads as follows.

Theorem 2.4.1. *Assume that (2.7) and (2.8) hold. Then t^{-1} is the sharp decay rate. Therefore there exists positive constant C such that the solution of our system satisfies*

$$E(t) \leq \frac{C}{t}, \qquad \forall t \in \mathbb{R}^+. \tag{2.22}$$

Proof. We will follow the idea for the proof of the corresponding results in [21]. We would show that

$$\lim_{\zeta\to\infty} \zeta^{-1} \|(i\zeta I - \mathcal{A})^{-1}\| < \infty \tag{2.23}$$

We prove that there exist a sequence

$$V_n = (u_n, u_n^1, \theta_n, v_n, v_n^1, w_{2,n}, q_n) \in \mathcal{D}(\mathcal{A}),$$

with $\|V_n\|_{\mathcal{H}} = 1$, and a sequence $\zeta_n \in \mathbb{R}$ with $\zeta_n \to \infty$ such that

$$\lim_{n\to\infty} \zeta_n \|(i\zeta_n I - \mathcal{A})V_n\|_{\mathcal{H}} = 0$$

or

$$\zeta_n(i\zeta_n u_n - u_n^1) \quad \to 0, \quad in \quad H^1(-L,0), \tag{2.24}$$

$$\zeta_n\Big(i\zeta_n u_n^1 - \rho_1^{-1}\Big(a_1\Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s)\eta_{n,xx}^t(s)ds\Big) - \beta_1\theta_{n,x}\Big)\Big) \quad \to 0, \quad in \quad L^2(-L,0), \tag{2.25}$$

$$\zeta_n\Big(i\zeta_n\theta_n - c_1^{-1}\Big(-\beta_1 u_{n,x}^1 + l\theta_{n,xx}\Big)\Big) \quad \to 0, \quad in \quad L^2(-L,0), \tag{2.26}$$

$$\zeta_n(i\zeta_n v_n - v_n^1) \quad \to 0, \quad in \quad H^1(0,L), \tag{2.27}$$

$$\zeta_n\Big(i\zeta_n v_n^1 - \rho_2^{-1}\Big(a_2 v_{n,xx} - \beta_2 q_{n,x}\Big)\Big) \quad \to 0, \quad in \quad L^2(0,L), \tag{2.28}$$

$$\zeta_n(i\zeta_n w_{2,n} - q_n) \quad \to 0, \quad in \quad H^1(0,L), \tag{2.29}$$

$$\zeta_n\Big(i\zeta_n q_n - c_2^{-1}\Big(-\beta_2 v_{n,x}^1 + kw_{2,n,xx}\Big)\Big) \quad \to 0, \quad in \quad L^2(0,L), \tag{2.30}$$

$$ih\eta^t - u_{1,n}^1 + \eta_s^t = 0 \tag{2.31}$$

Note that

$$Re\langle\zeta_n(i\zeta_n - \mathcal{A})V_n, V_n\rangle_{\mathcal{H}} = \zeta_n\|\sqrt{l} \quad \theta_{n,x}\|_{L^2}^2 \to 0.$$

Then

$$\sqrt{\zeta_n} \quad \theta_{n,x} \to 0, \quad in \quad L^2(-L,0). \tag{2.32}$$

By Poincaré's inequality, we get

$$\sqrt{\zeta_n} \quad \theta_n \to 0, \quad in \quad L^2(-L,0). \tag{2.33}$$

Thanks to the Gagliardo-Nirenberg inequality, we have

$$\|\sqrt{\zeta_n}\quad\theta_n\|_{L^\infty}\leq C_1\sqrt{\|\sqrt{\zeta_n}\quad\theta_{n,x}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n}\quad\theta_n\|_{L^2}}+C_2\|\sqrt{\zeta_n}\quad\theta_n\|_{L^2}. \tag{2.34}$$

Thus,

$$\sqrt{\zeta_n}\quad\theta_n(0)\to 0. \tag{2.35}$$

From (2.24), we have $\beta_1(i\zeta_n)^{-1}u^1_{n,x}$ is bounded in $L^2(-L,0)$. By (2.26) we have the boundedness of $(i\zeta_n)^{-1}\theta_{n,xx}$ in $L^2(-L,0)$.

Using again the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}\|_{L^\infty} &\leq d_1\sqrt{\|(\zeta_n)^{-1}\quad\theta_{n,xx}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n}\quad\theta_{n,x}\|_{L^2}}+d_2\|\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}\|_{L^2}\\ &\to 0.\end{aligned}$$

which gives

$$\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}(-L)\to 0,\qquad \left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}(0)\to 0. \tag{2.36}$$

Multiplying (2.25) by $p(x)u_{n,x}$ in L^2-norm for $p(x)\in C^1[-L,0]$, we get

$$\begin{aligned}&-\zeta_n^2\langle u_n,p(x)u_{n,x}\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\left\langle\mu_0u_{n,xx},p(x)u_{n,x}\right\rangle_{L^2(-L,0)}\\ &-\rho_1^{-1}a_1\left\langle\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds,p(x)u_{n,x}\right\rangle_{L^2(-L,0)}\\ &+\rho_1^{-1}\beta_1\langle\theta_{n,x},p(x)u_{n,x}\rangle_{L^2(-L,0)}\to 0.\end{aligned} \tag{2.37}$$

Integration by parts gives

$$\begin{aligned}&-\zeta_n^2\langle u_n,p(x)u_{n,x}\rangle_{L^2(-L,0)}\\ &=\frac{1}{2}\zeta_n^2p(-L)|u_n(-L)|^2-\zeta_n^2p(0)|u_n(0)|^2+\zeta_n^2\left\langle p_x(x)u_n,u_n\right\rangle_{L^2(-L,0)}.\\ &-\rho_1^{-1}a_1\mu_0\left\langle u_{n,xx},p(x)u_{n,x}\right\rangle_{L^2(-L,0)}\\ &=-\frac{1}{2}\rho_1^{-1}a_1\mu_0p(0)|u_{n,x}(0)|^2+\rho_1^{-1}a_1\mu_0p(-L)|u_{n,x}(-L)|^2+\rho_1^{-1}a_1\mu_0\left\langle p_x(x)u_{n,x},u_{n,x}\right\rangle_{L^2(-L,0)}\end{aligned}$$

And

$$-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)} = -\rho_1^{-1}a_1 p(0)\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds u_{n,x}(0)$$
$$+\rho_1^{-1}a_1 p(-L)\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds u_{n,x}(-L)$$
$$+\rho_1^{-1}a_1\Big\langle p_x(x)\int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)}.$$

Since

$$\rho_1^{-1}\beta_1\langle\theta_{n,x}, p(x)u_{n,x}\rangle_{L^2(-L,0)} \to 0,$$

then by the above integrations, for $p(x) = x \in C^1[-L,0]$, (2.37) takes the form

$$\begin{aligned}
&-\zeta_n^2|u_n(-L)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\mu_0|u_{n,x}(-L)|^2 + \rho_1^{-1}a_1\mu_0\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds u_{n,x}(-L)\\
&+\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0, \qquad (2.38)
\end{aligned}$$

and hence, $u_{n,x}(-L)$ and $\zeta_n u_n(-L)$ are bounded.

Similarly, taking $p(x) = x + L \in C^1[-L,0]$, (2.37) takes the form

$$\begin{aligned}
&-\zeta_n^2|u_n(0)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\mu_0|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_0\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds u_{n,x}(0)\\
&+\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \qquad (2.39)
\end{aligned}$$

Then, we get boundedness of $\zeta_n u_n(0)$ and $u_{n,x}(0)$.

Multiplying (2.26) by $u_{n,x}$ and taking the integration, we get

$$i\zeta_n\Big\langle\theta_n, u_{n,x}\Big\rangle_{L^2(-L,0)} + c_1^{-1}\beta_1\Big\langle u_{1,n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} - c_1^{-1}l\Big\langle\theta_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.$$

By (2.33), after dividing by $i\sqrt{\zeta_n}$, we have, where we have used $\zeta_n > 0$

$$i\zeta_n\Big\langle\theta_n, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0$$

Integrating by parts, we get

$$l(i\sqrt{\zeta_n})^{-1}\Big(\theta_{n,x}(-L)\overline{u_{n,x}(-L)}-\theta_{n,x}(0)\overline{u_{n,x}(0)}\Big)+l\Big\langle\sqrt{\zeta_n}\theta_{n,x},(i\zeta_n)^{-1}u_{n,xx}\Big\rangle_{L^2(-L,0)}$$
$$+\beta_1\sqrt{\zeta_n}\Big\langle u_{1,n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0 \tag{2.40}$$

By (2.36) and the boundedness of $u_{n,x}(-L)$ and $u_{n,x}(0)$, we have

$$l(i\sqrt{\zeta_n})^{-1}\Big(\theta_{n,x}(-L)\overline{u_{n,x}(-L)}-\theta_{n,x}(0)\overline{u_{n,x}(0)}\Big)\to 0$$

Moreover, from (2.25), we obtain that $(i\zeta_n)^{-1}u_{n,xx}$ is bounded in $L^2(-L,0)$. Thus

$$l(\sqrt{\zeta_n}\theta_{n,x},(i\zeta_n)^{-1}u_{n,xx})\to 0$$

Hence by (2.40), we get

$$\sqrt{\sqrt{\zeta_n}}\quad u_{n,x}\to 0,\qquad in\qquad L^2(-L,0). \tag{2.41}$$

Thanks to the Poincaré inequality, we have

$$\sqrt{\sqrt{\zeta_n}}\quad u_n\to 0,\quad in\qquad L^2(-L,0) \tag{2.42}$$

By (2.41), (2.42) and Galiardo-Nirenberg inequality, we get

$$\sqrt{\sqrt{\zeta_n}}\quad u_n(0)\to 0. \tag{2.43}$$

From (2.25) and (2.32), using $\zeta_n>0$, we have

$$i\zeta_n u_{1,n}-\rho_1^{-1}a_1\Big(\mu_0 u_{n,xx}+\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds\Big)\to 0,\qquad in\quad L^2(-L,0), \tag{2.44}$$

Multiplying the above by u_n, we get

$$i\zeta_n\Big\langle u_{1,n},u_n\Big\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\Big\langle\Big(\mu_0 u_{n,xx}+\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds\Big),u_n\Big\rangle_{L^2(-L,0)}\to 0.$$

Integrating by parts, we get

$$-\Big\langle u_{1,n},u_{1,n}\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\mu_0 u_{n,x}(0)\overline{u_n(0)}+\rho_1^{-1}a_1\mu_0 u_{n,x}(-L)\overline{u_n(-L)}-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}$$
$$+\rho_1^{-1}a_1\int_0^\infty\mu(s)\eta^t_{n,x}(0,s)ds\overline{u_n(0)}-\rho_1^{-1}a_1\int_0^\infty\mu(s)\eta^t_{n,x}(-L,s)ds\overline{u_n(-L)}$$
$$+\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0.$$

Since $u_{n,x}(0), u_{n,x}(-L)$ are bounded, by (2.41) and $u_n(-L) \to 0, u_n(0) \to 0$, we have

$$u_{1,n}, \zeta_n u_n \to 0, \quad in \quad L^2(-L,0). \tag{2.45}$$

Multiplying (2.25) by $(x+L)u_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re\Big[-\Big\langle \zeta_n^2 u_{1,n}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\Big\langle\Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds\Big), (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}\Big] \\
&= -\zeta_n^2|u_n(0)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_0|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_0\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds u_{n,x}(0) + \rho_1^{-1}a_1\Big\langle\int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned}$$

Hence, by (2.41) and (2.45), we get

$$\zeta_n u_n(0), u_{n,x}(0) \to 0 \tag{2.46}$$

Now, multiplying (2.25) by $xu_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re\Big[-\Big\langle \zeta_n^2 u_n^1, xu_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\Big\langle\Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds\Big), xu_{n,x}\Big\rangle_{L^2(-L,0)}\Big] \\
&= -\zeta_n^2|u_n(-L)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_0|u_{n,x}(-L)|^2 + \rho_1^{-1}a_1\mu_0\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds u_{n,x}(-L) + \rho_1^{-1}a_1\Big\langle\int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned} \tag{2.47}$$

Then

$$\zeta_n u_n(-L), u_{n,x}(-L) \to 0. \tag{2.48}$$

Taking again (2.25), multiplying by u_n, we have

$$\sqrt{\zeta_n}\Big\langle i\zeta_n u^1 1, n, u_n\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle \theta_{n,x}, u_n\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_0\Big\langle u_{n,xx}, u_n\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, u_n\Big\rangle_{L^2(-L,0)} \to 0. \tag{2.49}$$

By (2.41) and (2.46), we have

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_0\Big\langle u_{n,xx}, u_n\Big\rangle_{L^2(-L,0)}$$

$$= -\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}u_{n,x}(0)\overline{u_n(0)} + \rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}u_{n,x}(-L)\overline{u_n(-L)}$$

$$+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0$$

And

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, u_n\Big\rangle_{L^2(-L,0)}$$

$$= -\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds\overline{u_n(0)}$$

$$+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds\overline{u_n(-L)}$$

$$+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{2.50}$$

Thus, by(2.50) and (2.32), we go to

$$\sqrt{\sqrt{\zeta_n}}u_n^1 \to 0, \quad in \qquad L^2(-L,0). \tag{2.51}$$

Multiplying (2.25) by $(x+L)u_{n,x}$, we have

$$\Big\langle i\sqrt{\zeta_n}\zeta_n u_n^1, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle \theta_{n,x}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_0\Big\langle u_{n,xx}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{2.52}$$

Integrating by parts and using (2.32) and the boundedness of $u_{n,x}$ in $L^2(-L,0)$, we get

$$\begin{aligned}
&-\sqrt{\zeta_n}|u_n^1(0)|^2 + \sqrt{\zeta_n}\Big\langle u_n^1, u_n^1\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}|u_{n,x}(0)|^2\\
&+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds\,u_{n,x}(0)\\
&-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_{-L}^0 \Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned} \tag{2.53}$$

Thus by (2.41) and (2.51), we go to

$$\sqrt{\sqrt{\zeta_n}}u_n^1(0), \sqrt{\sqrt{\zeta_n}}u_{n,x}(0) \to 0 \tag{2.54}$$

Multiplication of (2.44) by $u_{n,x}$ yields

$$\begin{aligned}
&i\zeta_n\Big\langle u_n^1, u_{n,x}\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned} \tag{2.55}$$

Due to (2.46)and (2.48), we get

$$\begin{aligned}
&-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&= \frac{1}{2}(-\rho_1^{-1}a_1\mu_0)|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_0)|u_{n,x}(-L)|^2\\
&-\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds\,u_{n,x}(0) + \rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds\,u_{n,x}(-L)\\
&+\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned} \tag{2.56}$$

Thus, it follows from (2.55) that

$$(i\zeta_n u_n^1, u_{n,x}) \to 0. \tag{2.57}$$

Taking the product of (2.44) with θ_n, yields

$$\begin{aligned} &i\zeta_n \Big\langle u_{1,n}, \theta_n \Big\rangle_{L^2(-L,0)} - \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,xx}, \theta_n \Big\rangle_{L^2(-L,0)} \\ &- \rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta_{n,xx}^t(s) ds, \theta_n \Big\rangle_{L^2(-L,0)} \to 0 \qquad in \quad L^2(-L,0). \end{aligned} \tag{2.58}$$

Due to (2.32),(2.35) and (2.46), we have

$$\begin{aligned} &- \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,xx}, \theta_n \Big\rangle_{L^2(-L,0)} \\ &= -\rho_1^{-1} a_1 \mu_0 u_{n,x}(0) \overline{\theta_n(0)} + \rho_1^{-1} a_1 \mu_0 u_{n,x}(-L) \overline{\theta_n(-L)} \\ &+ \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,x}, \theta_{n,x} \Big\rangle_{L^2(-L,0)} \to 0 \end{aligned} \tag{2.59}$$

And

$$\begin{aligned} &- \rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta_{n,xx}^t(s) ds, \theta_n \Big\rangle_{L^2(-L,0)} \\ &= -\rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta_{n,x}^t(0,s) ds \overline{\theta_n(0)} \\ &+ \rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta_{n,x}^t(-L,s) ds \overline{\theta_n(-L)} \\ &+ \rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta_{n,x}^t(s) ds, \theta_{n,x} \Big\rangle_{L^2(-L,0)} \to 0. \end{aligned} \tag{2.60}$$

Then from (2.58), we obtain

$$i\zeta_n \Big\langle u_n^1, \theta_n \Big\rangle_{L^2(-L,0)} \to 0 \tag{2.61}$$

Multiplying (2.26) by u_n^1, we have

$$\Big\langle i\zeta_n \theta_n, u_n^1 \Big\rangle_{L^2(-L,0)} - c_1^{-1} l \Big\langle \theta_{n,xx}, u_n^1 \Big\rangle_{L^2(-L,0)} + c_1^{-1} \beta_1 \Big\langle u_{n,x}^1, u_n^1 \Big\rangle_{L^2(-L,0)} \to 0. \tag{2.62}$$

By (2.57), (2.61), we have

$$\Big\langle \theta_{n,xx}, u_n^1 \Big\rangle_{L^2(-L,0)} \to 0. \tag{2.63}$$

Integrating by parts

$$\theta_{n,x}(0) \overline{u_n^1(0)} - \theta_{n,x}(-L) \overline{u_n^1(-L)} - \Big\langle \theta_{n,x}, u_{n,x}^1 \Big\rangle_{L^2(-L,0)} \to 0. \tag{2.64}$$

Due to (2.36)and (2.54), we get

$$\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} \to 0. \tag{2.65}$$

From (2.64) we have

$$\left\langle \theta_{n,x}, u_{n,x}^1 \right\rangle_{L^2(-L,0)} \to 0. \tag{2.66}$$

Multiplying (2.26) by $(x+L)\theta_{n,x}$ and integrating, we get

$$\Re\Big[\left\langle i\zeta_n\theta_n, (x+L)\theta_{n,x} \right\rangle_{L^2(-L,0)} - c_1^{-1}\left\langle (l\theta_{n,xx} - \beta_1 u_{n,x}^1), (x+L)\theta_{n,x} \right\rangle_{L^2(-L,0)}\Big] \to 0$$

By (2.32) and (2.33), we obtain

$$\left\langle i\zeta_n\theta_n, (x+L)\theta_{n,x} \right\rangle_{L^2(-L,0)} \to 0. \tag{2.67}$$

Thus by (2.67) and (2.32), we have

$$-c_1^{-1} l\theta_{n,x}(0)\overline{\theta_{n,x}(0)} + 2\Re[c_1^{-1}\beta_1(u_{n,x}^1, (x+L)\theta_{n,x})] \to 0. \tag{2.68}$$

Then, by (2.66), we get

$$\theta_{n,x}(0) \to 0 \tag{2.69}$$

Hence, by (2.56),(2.46),(2.35)and (2.69), we have

$$u_{n,x}(0), u_n(0), \theta_n(0), \theta_{n,x}(0) \to 0. \tag{2.70}$$

Taking the product of (2.30) with $(x-L)w_{2,n,x}$, yields

$$\begin{aligned}\Re\Big[i\zeta_n\left\langle q_n, (x-L)w_{2,n,x} \right\rangle_{L^2(0,L)} + c_2^{-1}\beta_2\left\langle v_{n,x}^1, (x-L)w_{2,n,x} \right\rangle_{L^2(0,L)} \\ - c_2^{-1}k\left\langle w_{2,n,xx}, (x-L)w_{2,n,x} \right\rangle_{L^2(0,L)}\Big] \to 0. \end{aligned} \tag{2.71}$$

Using the transmission conditions in (2.1), we get

$$(q_n, q_n) + c_2^{-1}k(w_{2,n,x}, w_{2,n,x}) - 2\Re\Big[c_2^{-1}\beta_2\left\langle v_{n,x}, (x-L)q_{n,x} \right\rangle_{L^2(0,L)}\Big] \to 0. \tag{2.72}$$

Taking the product of (2.28) with $(x-L)v_{n,x}$, we obtain

$$i\zeta_n\left\langle v_n^1, (x-L)v_{n,x} \right\rangle_{L^2(0,L)} - \rho_2^{-1}a_2\left\langle v_{n,xx}, (x-L)v_{n,x} \right\rangle_{L^2(0,L)} \tag{2.73}$$

$$+ \rho_2^{-1}\beta_2\left\langle q_{n,x}, (x-L)v_{n,x} \right\rangle_{L^2(0,L)} \to 0. \tag{2.74}$$

Integrating (3.2) by parts we have

$$\left\langle v_n^1, v_n^1 \right\rangle_{L^2(0,L)} + \rho_2^{-1} a_2 \left\langle v_{n,x}, v_{n,x} \right\rangle_{L^2(0,L)} \\ +2\Re\left[\rho_2^{-1}\beta_2 \left\langle q_{n,x}, (x-L) q_{n,x} \right\rangle_{L^2(0,L)}\right] \to 0. \tag{2.75}$$

Thus by (2.72) and (2.75), we obtain

$$a_2 \left\langle v_{n,x}, v_{n,x} \right\rangle_{L^2(0,L)} + \left\langle \rho_2 v_n^1, v_n^1 \right\rangle_{L^2(0,L)} + k \left\langle w_{2,n,x}, w_{2,n,x} \right\rangle_{L^2(0,L)} \\ + c_2 \left\langle q_n, q_n \right\rangle_{L^2(0,L)} \to 0. \tag{2.76}$$

Then

$$v_{n,x}, v_n^1, w_{2,n,x}, q_n \to 0, \qquad in \qquad L^2(0,L). \tag{2.77}$$

Thus (2.77) together with (2.33), (2.45) and (2.77), we give

$$V_n = (u_n, u_n^1, \theta_n, v_n, v_n^1, w_{2,n}, q_n)^T \to 0, \tag{2.78}$$

Which contradicts $\|V_n\| = 1$. Therefore, (2.23) holds. □

Chapter 3

Decay of solution for Degenerate Wave Equation of Kirchhof Type in viscoelasticity

3.1 Position of problem and related results

In this chapter, we consider the following degenerate problem

$$\rho(x)\left(|w'|^{p-2}w'\right)' - M(\|\nabla_x w\|_2^2)div[a(x)\nabla_x w] + \int_0^t \mu(t-s)div[a(x)\nabla_x w(s)]ds = 0, \quad (3.1)$$

Where $x \in \mathbb{R}^n, t > 0, p, n \geq 2$, and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$ and the scalar function $\mu(s)$ (so-called relaxation kernel) is assumed to satisfy (A1).

Equation (3.1) is a prototype for PDE of hyperbolic in Kirchhoff type with degenerate memory when it is equipped by the following initial data.

$$w(0,x) = w_0 \in \mathcal{H}(\mathbb{R}^n), \quad w'(0,x) = w_1 \in L^p_\rho(\mathbb{R}^n), \quad (3.2)$$

Where the weighted spaces $\mathcal{H}$ is given in Definition (3.2.1) and the density function satisfies

$$\rho : \mathbb{R}^n \to \mathbb{R}_+, \quad \rho \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \quad (3.3)$$

with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^{s_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s_1 = \frac{2n(q-1)}{qn-2n+2q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathbb{R}_+$ is the space of all nonnegative real numbers.

The self-adjoint non-positive operator of the form

$$Aw = div(a(x)\nabla_x w(t))$$

represent the class of degenerate term considered in Equation (3.1), where $a \in C^1(\mathbb{R}^n)$ may vanish in a subset Ω_0 of $\mathbb{R}^n$.

In this framework, (see [4], [24], [25], [26], [33], [37]), it is well known that, for any initial data $w_0 \in \mathcal{H}(\mathbb{R}^n), w_1 \in L^p_\rho(\mathbb{R}^n)$, the problem (3.1)-(3.2) has a unique solution $w \in C([0,T), \mathcal{H}(\mathbb{R}^n)), w' \in C([0,T), L^p_\rho(\mathbb{R}^n))$ for T small enough, under hypothesis $(A0)-(A3)$.

The energy of w at time t is defined by

$$E(t) = \frac{(p-1)}{p}\|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{1}{2}\Big(m_0 - \int_0^t \mu(s)ds\Big)\|\nabla_x w\|^2_{V_a} + \frac{1}{2}(\mu \circ \nabla_x w) + \frac{m_1}{2(\gamma+1)}\|\nabla_x w\|_2^{2(\gamma+1)} \tag{3.4}$$

And the following energy functional law holds:

$$E'(t) = \frac{1}{2}(\mu' \circ \nabla_x w)(t) - \frac{1}{2}\mu(t)\|\nabla_x w(t)\|^2_{V_a} \quad \text{for all} \quad t \geq 0. \tag{3.5}$$

Which means that, our energy is uniformly bounded and decreasing along the trajectories.

The following notation will be used throughout this paper

$$(\mu \circ \nabla_x w)(t) = \int_0^t \mu(t-\tau)\, \|\nabla_x w(t) - \nabla_x w(\tau)\|^2_{V_a} d\tau, \tag{3.6}$$

For $w(t) \in \mathcal{H}(\mathbb{R}^n), t \geq 0$.

This kind of systems appears in the models of nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [29] in the case $n = 1$ this type of problem describes a small amplitude vibration of an elastic string.

The original equation is:

$$\rho h u_{tt} + \tau u_t = \left(P_0 + \frac{Eh}{2L}\int_0^L |u_x(x,t)|^2 ds\right) u_{xx} + f, \tag{3.7}$$

Where $0 \leq x \leq L$ and $t > 0, u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, P_0 the initial axial tension, τ the resistance modulus, E the Young modulus and f the external force (for example the action of gravity).

In the non-degenerate case where $a \equiv 1$, problem (3.1) was studied in [37], where the author allowed a wider class of relaxation functions and obtained a very general decay results. The motivation of our work is due to some results regarding viscoelastic wave equations of Kirchhoff type in a bounded domain. The wave equation of the form

$$u'' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t \mu(t-s)\Delta_x u(s)ds + h(u') = f(u), \quad x \in \Omega, t > 0 \tag{3.8}$$

is a model to describe the motion of deformable solids as hereditary effect is incorporated. Equation (3.8) was studied by [36] and they proved the existence of weak solution for large datum. Later, for small datum and under some restrictions, global solutions and exponential decay to zero is shown in [34].

The work with weighted spaces was studied by many authors (see in this direction [44], [32], [35] and [39]). For the decay rate in $\mathbb{R}^n$, $M \equiv 1$, we quote essentially the results of [22], [4], [26], [27], [28], [33], [38].

The problem (3.1),(3.2) for the case $q = 2, \rho(x) = 1, M \equiv 1, a \equiv 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive non-increasing function was considered in [33], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$\mu'(t) \leq -H(\mu(t)), t \geq 0, \quad H(0) = 0 \tag{3.9}$$

For a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], r < 1$. This improves the conditions considered recently by Alabau-Boussouira and Cannarsa [22] on the relaxation functions

$$\mu'(t) \leq -\chi(\mu(t)), \quad \chi(0) = \chi'(0) = 0 \tag{3.10}$$

Where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. The main purpose of this work is to obtain a very general decay results for the degenerate problem to improve earlier results by the first author [37].

3.2 Preliminaries, Assumptions and functions spaces

Definition 3.2.1 ([26], [35]). *We define the function spaces of our problem and its norm as follows.*

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n . \right\} \tag{3.11}$$

Note that $\mathcal{H}(\mathbb{R}^n)$ can be embedded continuously in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. The space $L^2_\rho(\mathbb{R}^n)$ we define to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{3.12}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

First we recall and make use the following assumptions on the functions a and μ as follows.

(A0) Let $a \in C^1(\mathbb{R}^n)$ be a positive function such that $a(x) \geq A^2 > 0$ and

$$V_a = \{ w \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} a(x) |\nabla w|^2 dx < \infty \} \tag{3.13}$$

is a Hilbert space endowed with the norm

$$\|w\|^2_{V_a} = \int_{\mathbb{R}^n} a |\nabla w|^2 dx$$

(A1) To guarantee the hyperbolic of the system, we assume that the function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is of class C^1 satisfying

$$m_0 - \overline{\mu} = l > 0, \quad \mu(0) = \mu_0 > 0,$$

Where $\overline{\mu} = \int_0^\infty \mu(t) dt$.

(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$\mu'(t) + H(\mu(t)) \leq 0, t \geq 0, \quad H(0) = 0 \tag{3.14}$$

And H is linear or strictly increasing and strictly convex C^2-function on $(0, r], r < 1$.

Note that

$$||\nabla_x w||_2 \leq \frac{1}{|A|} ||w||_{V_a}$$

for any $w \in V_a$. Really, we have

$$\begin{aligned}
||\nabla_x w||_2^2 &= \int_{\mathbb{R}^n} |\nabla_x w|^2 dx \\
&= \int_{\mathbb{R}^n} \frac{a(x)}{a(x)} |\nabla_x w|^2 dx \\
&\leq \frac{1}{A^2} \int_{\mathbb{R}^n} a(x) |\nabla_x w|^2 dx \\
&= \frac{1}{A^2} ||w||_{V_a}^2,
\end{aligned}$$

Where up on we get the desired estimate. For the reader we shall develop here the next important technical Lemma.

Lemma 3.2.1. *For any $w \in C^1\left(0, T, H^1(\mathbb{R}^n)\right)$ we have*

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^t \mu(t-s) div[a(x)\nabla_x w(s)] w'(t) ds dx = \frac{1}{2}\frac{d}{dt}\left(\mu \circ \nabla_x w\right)(t) \\
&- \frac{1}{2}\frac{d}{dt}\left[\int_0^t \mu(s) \int_{\mathbb{R}^n} a(x) \left|\nabla_x w(t)\right|^2 dx ds\right] \\
&- \frac{1}{2}\left(\mu' \circ \nabla_x w\right)(t) + \frac{1}{2}\mu(t) \int_{\mathbb{R}^n} a(x) \left|\nabla_x w(t)\right|^2 dx.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^t \mu(t-s) div[a(x)\nabla_x w(s)] w'(t) ds dx \\
&= -\int_0^t \mu(t-s) \int_{\mathbb{R}^n} a(x) \nabla_x w(s) \nabla_x w'(t) dx ds \\
&= -\int_0^t \mu(t-s) \int_{\mathbb{R}^n} \nabla_x w'(t) a(x) \left[\nabla_x w(s) - \nabla_x w(t)\right] dx ds \\
&- \int_0^t \mu(t-s) \int_{\mathbb{R}^n} \nabla_x w'(t) a(x) \nabla_x w(t) dx ds.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^t \mu(t-s) div[a(x)\nabla_x w(s)] w'(t) ds dx \\
&= \frac{1}{2} \int_0^t \mu(t-s) \frac{d}{dt} \int_{\mathbb{R}^n} a(x) \left|\nabla_x w(s) - \nabla_x w(t)\right|^2 dx ds \\
&- \frac{1}{2} \int_0^t \mu(s) \left(\frac{d}{dt} \int_{\mathbb{R}^n} a(x) \left|\nabla_x w(t)\right|^2 dx\right) ds
\end{aligned}$$

Which implies

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^t \mu(t-s)div[a(x)\nabla_x w(s)]w'(t)dsdx\\
&=\frac{1}{2}\frac{d}{dt}\left[\int_0^t \mu(t-s)\int_{\mathbb{R}^n} a(x)\left|\nabla_x w(s)-\nabla_x w(t)\right|^2 dxds\right]\\
&-\frac{1}{2}\frac{d}{dt}\left[\int_0^t \mu(s)\int_{\mathbb{R}^n} a(x)\left|\nabla_x w(t)\right|^2 dxds\right]\\
&-\frac{1}{2}\int_0^t \mu'(t-s)\int_{\mathbb{R}^n} a(x)\left|\nabla_x w(s)-\nabla_x w(t)\right|^2 dxds\\
&+\frac{1}{2}\mu(t)\int_{\mathbb{R}^n} a(x)\left|\nabla_x v(t)\right|^2 dx.
\end{aligned}$$

□

Remark 3.2.1. *[33] By* (A1) *we have that μ is a positive function and $m_0-\overline{\mu}>0$. Then $\int_0^\infty \mu(t)dt \geq 0$ and $\int_0^\infty \mu(t)dt \leq m_0$. Therefore $\lim_{t\to\infty}\mu(t)=0$. Because $\mu(t)$ is a positive function on $\mathbb{R}_+$, $\mu(0)>0$ and $\lim_{t\to\infty}\mu(t)=0$, we conclude that there is a $t_1>0$, large enough, so that $\mu(t)$ is a decreasing function on $[t_1,\infty)$ and $\mu(t_1)>0$. Hence, $\mu'(t)\leq 0$ on $[t_1,\infty)$. Since $\mu(t)\to 0$ as $t\to\infty$, we get that $\mu'(t)\to 0$ as $t\to\infty$. We set*

$$H_0(t) := H(D(t))$$

For a C^1-function D such that $D(0)=0$, H_0 is strictly increasing and strictly convex C^2-function on $(0,r]$. If there is a need we enlarge $t_1>0$ so that $\mu(t_1)>0$ and

$$\max\{\mu(s),-\mu'(s)\} < \min\{r, H(r), H_0(r)\} \tag{3.15}$$

For any $s\in[t_1,\infty)$. Note that

$$\begin{aligned}
\int_0^\infty \mu(s)H_0\left(-\mu'(s)\right)ds &= \int_0^{t_1}\mu(s)H_0\left(-\mu'(s)\right)ds+\int_{t_1}^\infty \mu(s)H_0\left(-\mu'(s)\right)ds\\
&\leq \int_0^{t_1}\mu(s)H_0\left(-\mu'(s)\right)ds+\int_{t_1}^\infty \mu(s)H_0(r)ds\\
&= \int_0^{t_1}\mu(s)H_0\left(-\mu'(s)\right)ds+H_0(r)\int_{t_1}^\infty \mu(s)ds\\
&< \infty.
\end{aligned}$$

Since μ is a positive function on $(0, t_1]$ and H is a positive continuous function on $\mathbb{R}_+$, we have that there exist positive constants α_1 and β_1 such that

$$\alpha_1 \leq H\left(\mu(t)\right) \leq \beta_1 \quad for \quad any \quad t \in [0, t_1]$$

Let

$$m_1 = \min_{t\in[0,t_1]} \mu(t), \quad M_1 = \max_{t\in[0,t_1]} \mu(t).$$

Then $m_1 > 0$ and $M_1 > 0$ and

$$-H\left(\mu(t)\right) \leq -\alpha_1 = -\frac{\alpha_1}{M_1} M_1 \leq -\frac{\alpha_1}{M_1} \mu(t)$$

for any $t \in [0, t_1]$. We set

$$k := \frac{\alpha_1}{M_1}.$$

Then $k > 0$ and

$$-H\left(\mu(t)\right) \leq -k\mu(t) \quad for \quad any \quad t \in [0, t_1].$$

Hence,

$$\mu'(t) \leq -H\left(\mu(t)\right) \leq -k\mu(t) \quad for \quad any \quad t \in [0, t_1]. \tag{3.16}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [1], then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0, H_0'(r))$$

And satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r]. \tag{3.17}$$

We are now ready to state and prove our main results.

3.3 Lyapunov techniques and main results

We will start with the following useful Lemma.

Lemma 3.3.1. *Let ρ satisfies (3.3), then for any $w \in \mathcal{H}(\mathbb{R}^n)$ there exists $\alpha > 0$ such that*

$$\|w\|_{L^p_\rho(\mathbb{R}^n)} \leq \alpha^{\frac{1}{p}} \|\rho\|^{\frac{1}{p}}_{L^{s_1}(\mathbb{R}^n)} \|\nabla_x w\|_{L^2(\mathbb{R}^n)} \tag{3.18}$$

with $s_1 = \frac{2n(q-1)}{nq-2n+2q}, 2 \leq q$.

Proof. Let $w \in \mathcal{H}(\mathbb{R}^n)$ be arbitrarily chosen and fixed. Since $\mathcal{H}(\mathbb{R}^n)$ is embedded continuously in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, there exists a positive constant α such that

$$\|w\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq \alpha^{\frac{1}{p}} \|\nabla_x w\|_{L^2(\mathbb{R}^n)}. \tag{3.19}$$

Now we apply Hölder's inequality for

$$s_1 = \frac{2n(q-1)}{nq-2n+2q} \quad \text{and} \quad s_2 = \frac{2n}{(n-2)p}$$

and we get

$$\begin{aligned}
\|w\|^p_{L^p_\rho(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \rho |w|^p dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho^{s_1} dx \right)^{\frac{1}{s_1}} \left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} dx \right)^{\frac{p(n-2)}{2n}} \\
&= \|\rho\|_{L^{s_1}(\mathbb{R}^n)} \|w\|^p_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}.
\end{aligned}$$

Now we apply (3.19) and we get

$$\|w\|^p_{L^p_\rho(\mathbb{R}^n)} \leq \alpha \|\rho\|_{L^{s_1}(\mathbb{R}^n)} \|\nabla_x w\|^p_{L^2(\mathbb{R}^n)},$$

Which completes the proof. □

To construct a Lyapunov functional L, equivalent to E, we introduce the following functionals

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) w |w'|^{p-2} w' dx, \tag{3.20}$$

$$\psi_2(t) = -\int_{\mathbb{R}^n} \rho(x) |w'|^{p-2} w' \int_0^t \mu(t-s)(w(t) - w(s)) ds dx. \tag{3.21}$$

Lemma 3.3.2. *Under the assumptions (A0) - (A2), the functional ψ_1 satisfies, along the solution of (3.1),(3.2), the following inequality*

$$\psi_1'(t) \leq \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|^{2+2\gamma}_{V_a} + (\sigma - l)||w||^2_{V_a} + \frac{1}{4\sigma}(m_0 - l)\,(\mu \circ \nabla_x w)$$

For any $\sigma > 0$.

Proof. We differentiate (3.20) and using (3.1), we get

$$\psi_1'(t) = \int_{\mathbb{R}^n} \rho(x)|w'|^p dx + \int_{\mathbb{R}^n} \rho(x) w \left(|w'|^{p-2}w'\right)' dx$$
$$= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \int_{\mathbb{R}^n} w \left(M\left(\|\nabla_x w\|_2^2\right) \text{div}\,[a(x)\nabla_x w] - \int_0^t \mu(t-s)\text{div}\,[a(x)\nabla_x w(s)]\,ds \right) dx$$
$$= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \int_{\mathbb{R}^n} w\Big(\left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma}\right) \text{div}\,[a(x)\nabla_x w] - \int_0^t \mu(t-s)\text{div}\,[a(x)\nabla_x w(s)]\,ds\Big)dx$$
$$= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma}\right) \int_{\mathbb{R}^n} w\text{div}\,[a(x)\nabla_x w]\,dx$$
$$- \int_{\mathbb{R}^n} w \int_0^t \mu(t-s)\text{div}\,[a(x)\nabla_x w(s)]\,dsdx$$
$$= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} - \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma}\right) \int_{\mathbb{R}^n} a(x)|\nabla_x w|^2 dx$$
$$- \int_{\mathbb{R}^n} w \int_0^t \mu(t-s)\text{div}\,[a(x)\nabla_x w(s)]\,dsdx$$
$$= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} - \left(m_0 + m_1\|\nabla_x\|_2^{2\gamma}\right) \|w\|^2_{V_a}$$
$$+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\nabla_x w(s)dsdx$$
$$= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} - \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma}\right) \|w\|^2_{V_a}$$
$$+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx$$
$$+ \left(\int_{\mathbb{R}^n} a(x)|\nabla_x w|^2 dx\right) \int_0^t \mu(s)ds.$$

We get

$$
\begin{aligned}
\psi_1'(t) &\leq \|w'\|_{L^p_\rho(\mathbb{R}^n)} - \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma}\right)\|w\|_{V_a}^2 \\
&+ \|w\|_{V_a}^2 \int_0^\infty \mu(s)ds + \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx \\
&= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} - \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma}\right)\|w\|_{V_a}^2 + (m_0 - l)\|w\|_{V_a}^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx \\
&= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} - m_1\|\nabla_x w\|_2^{2\gamma}\|w\|_{V_a}^2 - l\|w\|_{V_a}^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx \\
&\leq \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + m_1\|\nabla_x w\|_2^{2\gamma}\|w\|_{V_a}^2 - l\|w\|_{V_a}^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx \\
&\leq \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|_{V_a}^{2+2\gamma} - l\|w\|_{V_a}^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\psi_1'(t) &\leq \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|_{V_a}^{2+2\gamma} - l\|w\|_{V_a}^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x w \int_0^t \mu(t-s)a(x)\left(\nabla_x w(s) - \nabla_x w(t)\right) dsdx.
\end{aligned}
$$

Let $\sigma > 0$ be arbitrarily chosen. Now we apply Young's inequality with σ and we get

$$
\begin{aligned}
\psi_1'(t) &\leq \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|_{V_a}^{2+2\gamma} - l\|w\|_{V_a}^2 + \sigma \int_{\mathbb{R}^n} a(x)|\nabla_x w|^2 dx \\
&+ \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t \mu(t-s)\left(a(x)\right)^{\frac{1}{2}}\left(\nabla_x w(s) - \nabla_x w(t)\right) ds\right)^2 dx \\
&= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|_{V_a}^{2+2\gamma} + (\sigma - l)\|w\|_{V_a}^2 \\
&+ \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t \mu(t-s)\left(a(x)\right)^{\frac{1}{2}}\left(\nabla_x w(s) - \nabla_x w(t)\right) ds\right)^2 dx.
\end{aligned}
$$

Now we apply Inequality the Cauchy-Schwartz we obtain

$$\begin{aligned}
\psi_1'(t) &\leq \|w'\|^p_{L^p_{|rho}(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|^{2+2\gamma}_{V_a} + (\sigma - l)\,\|w\|^2_{V_a} \\
&+ \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t \mu(s)ds\right)\int_0^t \mu(t-s)a(x)|\nabla_x w(s) - \nabla_x w(t)|^2 ds dx \\
&\leq \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|^{2+2\gamma}_{V_a} + (\sigma - l)\,\|w\|^2_{V_a} \\
&+ \frac{1}{4\sigma}\int_0^\infty \mu(s)ds\int_{\mathbb{R}^n}\int_0^t \mu(t-s)a(x)|\nabla_x w(s) - \nabla_x w(t)|^2 ds dx \\
&= \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + \frac{m_1}{A^{2\gamma}}\|w\|^{2+2\gamma}_{V_a} + (\sigma - l)\,\|w\|^2_{V_a} \\
&+ \frac{1}{4\sigma}(m_0 - l)\,(\mu \circ \nabla_x w)\,,
\end{aligned}$$

□

Remark 3.3.1. *By the proof of Lemma 3.3.2 it follows*

$$\int_{\mathbb{R}^n}\nabla_x w\int_0^t \mu(t-s)a(x)(\nabla_x w(s) - \nabla_x w(t))ds dx \leq \sigma\|w'\|^2_{V_a} + \frac{1}{4\sigma}(m_0 - l)\,(\mu \circ \nabla_x w)$$

for every $\sigma > 0$.

Lemma 3.3.3. *Under the assumptions (A0) - (A2), the functional ψ_2 satisfies, along the solution of (3.1),(3.2), the following inequality*

$$\begin{aligned}
\psi_2'(t) \leq &\left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma} - \int_0^t \mu(s)ds\right)\left(\sigma\|w'\|^2_{V_a} + \frac{m_0 - l}{4\sigma}(\mu \circ \nabla_x w)\right) \\
&+ \left(\sigma - \int_0^t \mu(s)ds\right)\|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + (\mu \circ \nabla_x w)\int_0^t \mu(s)ds \\
&+ \frac{\alpha Q^{\frac{p}{2}} C(\sigma)}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(|\mu'| \circ \nabla_x w\right)^{\frac{p}{2}}
\end{aligned}$$

For any $\sigma > 0$ and for some $\alpha > 0$. Here

$$Q = \max_{t\in[0,\infty)}\int_0^t |\mu'(s)|ds, \quad C(\sigma) = \frac{1}{(\sigma p)^{\frac{q}{p}}q}.$$

Proof. By the definition of $\psi_2(t)$ we get

$$
\begin{aligned}
\psi_2'(t) = &- \int_{\mathbb{R}^n} \rho(x) \left(|w'|^{p-2}w'\right)' \int_0^t \mu(t-s)\left(w(t)-w(s)\right) dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t)-w(s)\right) dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|w'|^{p} dx \int_0^t \mu(s) ds \\
= &- \int_{\mathbb{R}^n} \left(M\left(\|\nabla_x w\|_2^2\right) \operatorname{div}\left[a(x)\nabla_x w\right] - \int_0^t \mu(t-s) \operatorname{div}\left[a(x)\nabla_x w(s)\right] ds \right) \\
&\times \left(\int_0^t \mu(t-s)\left(w(t)-w(s)\right) ds \right) dx \\
&- \int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t)-w(s)\right) dsdx - \|w'\|_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds \\
= &- \int_{\mathbb{R}^n} M\left(\|\nabla_x w\|_2^2\right) \operatorname{div}\left[a(x)\nabla_x w\right] \int_0^t \mu(t-s)\left(w(t)-w(s)\right) dsdx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-s) \operatorname{div}\left[a(x)\nabla_x w(s)\right] ds \right) \left(\int_0^t \mu(t-s)\left(w(t)-w(s)\right) ds \right) dx \\
&- \int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t)-w(s)\right) dsdx - \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds \\
= &\int_{\mathbb{R}^n} M\left(\|\nabla_x w\|_2^2\right) a(x)\nabla_x w \int_0^t \mu(t-s)\left(\nabla_x w(t)-\nabla_x w(s)\right) dsdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-s) a(x)\nabla_x w(s) ds \right) \left(\int_0^t \mu(t-s)\left(\nabla_x w(t)-\nabla_x w(s)\right) ds \right) dx \\
&- \int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t)-w(s)\right) dsdx - \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds
\end{aligned}
$$

we get

$$\begin{aligned}
\psi_2'(t) &= \int_{\mathbb{R}^n} \left(m_0 + m_1 \|\nabla_x\|_2^{2\gamma}\right) a(x) \nabla_x w \int_0^t \mu(t-s) \left(\nabla_x w(t) - \nabla_x w(s)\right) ds dx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-s) \left(a(x)\right)^{\frac{1}{2}} \left(\nabla_x w(t) - \nabla_x w(s)\right) ds\right)^2 dx \\
&- \int_{\mathbb{R}^n} \rho(x) |w'|^{p-2} w' \int_0^t \mu'(t-s) \left(w(t) - w(s)\right) ds dx - \|w'\|_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds \\
&- \int_{\mathbb{R}^n} \nabla_x w(t) a(x) \int_0^t \mu(t-s) \left(\nabla_x w(t) - \nabla_x w(s)\right) ds dx \int_0^t \mu(s) ds \\
&= \left(m_0 + m_1 \|\nabla_x w\|_2^{2\gamma} - \int_0^t \mu(s) ds\right) \int_{\mathbb{R}^n} a(x) \nabla_x w(t) \int_0^t \mu(t-s) \left(\nabla_x w(t) - \nabla_x w(s)\right) ds dx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-s) \left(a(x)\right)^{\frac{1}{2}} \left(\nabla_x w(t) - \nabla_x w(s)\right) ds\right)^2 dx \\
&- \int_{\mathbb{R}^n} \rho(x) |w'|^{p-2} w' \int_0^t \mu'(t-s) \left(w(t) - w(s)\right) ds dx - \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds,
\end{aligned}$$

i.e

$$\begin{aligned}
\psi_2'(t) \quad \leq \quad & \left(m_0 + m_1 \|\nabla_x w\|_2^{2\gamma} - \int_0^t \mu(s) ds\right) \int_{\mathbb{R}^n} a(x) \nabla_x w(t) \int_0^t \mu(t-s) \left(\nabla_x w(t) - \nabla_x w(s)\right) ds dx \\
& + \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-s) \left(a(x)\right)^{\frac{1}{2}} \left(\nabla_x w(t) - \nabla_x w(s)\right) ds\right)^2 dx \\
& - \int_{\mathbb{R}^n} \rho(x) |w'|^{p-2} w' \int_0^t \mu'(t-s) \left(w(t) - w(s)\right) ds dx - \|w'\|^p_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds.
\end{aligned} \tag{3.22}$$

Let $\sigma > 0$ be arbitrarily chosen. Then using (3.3.1) we get that

$$\begin{aligned}
\psi_2'(t) \quad \leq \quad & \left(m_0 + m_1 \|\nabla_x w\|_2^{2\gamma} - \int_0^t \mu(s) ds\right) \left(\sigma \|w\|^2_{V_a} + \frac{1}{4\sigma} \left(m_0 - l\right) \left(\mu \circ \nabla_x w\right)\right) \\
& + \int_{\mathbb{R}^n} \left(\int_0^t \mu(t-s) \left(a(x)\right)^{\frac{1}{2}} \left(\nabla_x w(t) - \nabla_x w(s)\right) ds\right)^2 dx \\
& - \|w'\|_{L^p_\rho(\mathbb{R}^n)} \int_0^t \mu(s) ds - \int_{\mathbb{R}^n} \rho(x) |w'|^{p-2} w' \int_0^t \mu'(t-s) \left(w(t) - w(s)\right) ds dx.
\end{aligned} \tag{3.23}$$

We apply Lemma 2.4 for $\theta = \frac{1}{2}$, μ and $(a(x))^{\frac{1}{2}}(\nabla_x w(t) - \nabla_x w(s))$ and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_0^t \mu(t-s)\,(a(x))^{\frac{1}{2}}\left(\nabla_x w(t) - \nabla_x w(s)\right)ds\right)^2 dx \\
&\le \int_{\mathbb{R}^n}\left(\int_0^t \mu(s)ds\right)\left(\int_0^t \mu(t-s)a(x)\left(\nabla_x w(t) - \nabla_x w(s)\right)^2 ds\right)dx \\
&= \left(\int_{\mathbb{R}^n} a(x)\int_0^t \mu(t-s)\left(\nabla_x w(t) - \nabla_x w(s)\right)^2 dsdx\right)\int_0^t \mu(s)ds \\
&= \left(\mu \circ \nabla_x w\right)(t)\int_0^t \mu(s)ds.
\end{aligned}$$

Hence and (3.23) we get

$$\begin{aligned}
\psi_2'(t) \le\ & \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma} - \int_0^t \mu(s)ds\right)\left(\sigma\|w\|_{V_a}^2 + \frac{1}{4\sigma}\left(m_0 - l\right)\left(\mu \circ \nabla_x w\right)(t)\right) \\
& + \left(\mu \circ \nabla_x w\right)(t)\int_0^t \mu(s)ds - \|w'\|_{L^p_\rho(\mathbb{R}^n)}^p \int_0^t \mu(s)ds \qquad (3.24) \\
& - \int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t) - w(s)\right)dsdx.
\end{aligned}$$

Now we consider

$$-\int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t) - w(s)\right)dsdx.$$

Applying Hölder's inequality and Young's inequality with σ we get

$$\begin{aligned}
&-\int_{\mathbb{R}^n} \rho(x)|w'|^{p-2}w' \int_0^t \mu'(t-s)\left(w(t) - w(s)\right)dsdx \\
&\le \int_{\mathbb{R}^n} \rho(x)|w'|^{p-1}\int_0^t |\mu'(t-s)||w(t) - w(s)|dsdx \\
&= \int_{\mathbb{R}^n} (\rho(x))^{\frac{1}{q}}|w'|^{p-1}(\rho(x))^{\frac{1}{p}}\int_0^t |\mu'(t-s)||w(t) - w(s)|dsdx \\
&\le \left(\int_{\mathbb{R}^n} \rho(x)|w'|^p dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t |\mu'(t-s)||w(t) - w(s)|ds\right|^p dx\right)^{\frac{1}{p}} \\
&\le \sigma\left(\int_{\mathbb{R}^n} \rho(x)|w'|^p dx\right) + C(\sigma)\left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t |\mu'(t-s)||w(t) - w(s)|ds\right|^p dx\right)^{\frac{1}{p}} \\
&= \sigma|w'|_{L^p_\rho(\mathbb{R}^n)}^p + C(\sigma)\left\|\int_0^t |\mu'(t-s)||w(t) - w(s)|ds\right\|_{L^p_\rho(\mathbb{R}^n)}^p.
\end{aligned}$$

Hence and (3.24) we get

$$\begin{aligned}\psi_2'(t) \;\leq\; & \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma} - \textstyle\int_0^t \mu(s)ds\right)\left(\sigma\|w\|_{V_a}^2 + \tfrac{1}{4\sigma}\left(m_0 - l\right)\left(\mu\circ\nabla_x w\right)(t)\right) \\ & + \left(\mu\circ\nabla_x w\right)(t)\textstyle\int_0^t \mu(s)ds - \|w'\|_{L^p_\rho(\mathbb{R}^n)}^p \int_0^t \mu(s)ds \\ & + \sigma\|w'\|_{L^p_\rho(\mathbb{R}^n)}^p + C(\sigma)\|\textstyle\int_0^t |\mu'(t-s)||w(t)-w(s)|ds\|_{L^p_\rho(\mathbb{R}^n)}^p.\end{aligned} \tag{3.25}$$

Now we consider

$$\|\int_0^t |\mu'(t-s)||w(t)-w(s)|ds\|_{L^p_\rho(\mathbb{R}^n)}^p.$$

Applying Lemma 3.3.1, we obtain

$$\begin{aligned}&\|\int_0^t |\mu'(t-s)||w(t)-w(s)|ds\|_{L^p_\rho(\mathbb{R}^n)}^p \\ &\leq \alpha\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\|\int_0^t |\mu'(t-s)||\nabla_x w(t)-\nabla_x w(s)|ds\|_{L^2(\mathbb{R}^n)}^p \\ &= \alpha\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\left|\int_0^t |\mu'(t-s)||\nabla_x w(t)-\nabla_x w(s)|ds\right|^2 dx\right)^{\frac{p}{2}} \\ &= \alpha\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\frac{a(x)}{a(x)}\left(\int_0^t \left|\mu'(t-s)\right|\left|\nabla_x w(t)-\nabla_x w(s)\right|ds\right)^2 dx\right)^{\frac{p}{2}} \\ &\leq \frac{\alpha}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\left(\int_0^t \left|\mu'(t-s)\right|\left(a(x)\right)^{\frac{1}{2}}\left|\nabla_x w(t)-\nabla_x w(s)\right|ds\right)^2 dx\right)^{\frac{p}{2}} \\ &\leq \frac{\alpha}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\left(\int_0^t |\mu'(t-s)|ds\right)\left(\int_0^t |\mu'(t-s)|a(x)|\nabla_x w(t)-\nabla_x w(s)|^2 ds\right)dx\right)^{\frac{p}{2}} \\ &\leq \frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} a(x)\int_0^t |\mu'(t-s)||\nabla_x w(t)-\nabla_x w(s)|^2 ds dx\right)^{\frac{1}{2}} \\ &= \frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\left(|\mu'|\circ\nabla_x w\right)(t)\right)^{\frac{p}{2}}.\end{aligned}$$

Hence and (3.25), we get

$$\psi_2'(t) \le \left(m_0 + m_1\|\nabla_x w\|_2^{2\gamma} - \int_0^t \mu(s)ds\right)\left(\sigma\|w'\|_{V_a}^2 + \frac{1}{4\sigma}(m_0 - l)(\mu \circ \nabla_x w)(t)\right)$$
$$+ (\mu \circ \nabla_x w)(t)\int_0^t \mu(s)ds - \|w'\|^p_{L^p_{|rho}(\mathbb{R}^n)}\int_0^t \mu(s)ds$$
$$+ \sigma\|w'\|^p_{L^p_\rho(\mathbb{R}^n)} + C(\sigma)\frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\left(|\mu'| \circ \nabla_x w\right)(t)\right)^{\frac{p}{2}}$$

□

Remark 3.3.2. *By the proof of Lemma 3.3.3 it follows that*

$$\|\int_0^t \mu'(t-s)\left(w(t) - w(s)\right)ds\|^p_{L^p_\rho(\mathbb{R}^n)} \le \frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\left(|\mu'| \circ \nabla_x w\right)(t)\right)^{\frac{p}{2}}. \tag{3.26}$$

We define the Lyapunov function as

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2\psi_2(t) \tag{3.27}$$

for ξ_1, ξ_2 large enough. We need the next Lemma, which means that the Lyapunov function and energy function are equivalent, that is for ξ_1, ξ_2 large enough, we have

$$\beta_1 L(t) \le E(t) \le \beta_2 L(t) \tag{3.28}$$

Holds for some positive constants β_1 and β_2.

Lemma 3.3.4. *For ξ_1, ξ_2 large enough, we have*

$$L(t) \sim E(t). \tag{3.29}$$

Proof. By (3.27) we have

$$|L(t) - \xi_1 E(t)| \le |\psi_1(t)| + \xi_2|\psi_2(t)|$$
$$\le \int_{\mathbb{R}^n}\left|\rho(x)w|w'|^{p-2}w'\right|dx$$
$$+ \xi_2\int_{\mathbb{R}^n}\left|\rho(x)|w'|^{p-2}w'\int_0^t \mu(t-s)(w(t) - w(s))ds\right|dx.$$

Using Hölder's and Young's inequalities, Lemma 3.3.1 and the proof of Lemma 3.3.3, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\rho(x)w|w'|^{p-2}w'\right|dx &\leq \left(\int_{\mathbb{R}^n}\rho(x)|w|^p dx\right)^{1/p}\left(\int_{\mathbb{R}^n}\rho(x)|w'|^p dx\right)^{(p-1)/p}\\ &\leq \frac{1}{p}\left(\int_{\mathbb{R}^n}\rho(x)|w|^p dx\right)+\frac{p-1}{p}\left(\int_{\mathbb{R}^n}\rho(x)|w'|^p dx\right)\\ &= \frac{1}{p}\|w\|^p_{L^p_\rho(\mathbb{R}.)^p}+\frac{p-1}{p}\|w'\|^p_{L^p_\rho(\mathbb{R}^n)}\\ &\leq \frac{\alpha}{p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\|\nabla_x w\|^p_{L^2(\mathbf{R}^n)}+\frac{p-1}{p}\|w'\|^p_{L^p_\rho(\mathbb{R}^n)}\end{aligned}$$

And

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\rho(x)|w'|^{p-2}w'\int_0^t\mu(t-s)\left(w(t)-w(s)\right)ds\right|dx\\ &\leq \sigma\|w'\|^p_{L^p_\rho(\mathbb{R}^n)}+C(\sigma)\frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\left(\mu\circ\nabla_x w\right)(t)\right)^{\frac{p}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}|L(t)-\xi_1E(t)| &\leq \frac{\alpha}{p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\|\nabla_x w\|^p_{L^2(\mathbf{R}^n)}+\frac{p-1}{p}\|w'\|^p_{L^p_\rho(\mathbb{R}^n)}\\ &\quad+\xi_2\left(\sigma\|w'\|^p_{L^p_\rho(\mathbb{R}^n)}+C(\sigma)\frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\left(\mu\circ\nabla_x w\right)(t)\right)^{\frac{p}{2}}\right)\\ &= \frac{\alpha}{p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\|\nabla_x w\|^p_{L^2(\mathbb{R}^n)}\\ &\quad+\left(\frac{p-1}{p}+\sigma\xi_2\right)\|w'\|^p_{L^p_\rho(\mathbb{R}^n)}\\ &\quad+\xi_2C(\sigma)\frac{\alpha Q^{\frac{p}{2}}}{A_2^p}\|\rho\|_{L^{s_1}(\mathbb{R}^n)}\left(\left(\mu\circ\nabla_x w\right)(t)\right)^{\frac{p}{2}}\\ &\leq BE(t).\end{aligned}$$

For some positive constant B. If there is a need we enlarge ξ_1 so that $\xi_1 \geq B$. Therefore

$$-BE(t)\leq L(t)-\xi_1E(t)\leq BE(t)$$

Or

$$\left(\xi_1-B\right)E(t)\leq L(t)\leq\left(\xi_1+B\right)E(t),$$

□

Our main result reads as follows.

Theorem 3.3.1. *Let* $(u_0, u_1) \in \mathcal{H}(\mathbb{R}^n) \times L^q_\rho(\mathbb{R}^n)$ *and suppose that* $(A0) - (A2)$ *hold. Then there exist positive constants* $\beta_0, \beta_1, \beta_2, \beta_3$ *such that the energy of solution given by (3.1),(3.2) satisfies,*

$$E(t) \leq \beta_3 h^{-1}(\beta_1 t + \beta_2), \quad \text{for all} \quad t \geq 0,$$

where

$$h(t) = \int_t^1 \frac{1}{sH_0'(\beta_0 s)} ds \tag{3.30}$$

Proof. Note that

$$L'(t) = \xi_1 E'(t) + \psi_1'(t) + \xi_2 \psi_2'(t).$$

By Lemma 3.3.2 and Lemma 3.3.3 it follows that we can choose $\xi_1 > 1$ and $\xi_2 > 1$ large enough and $\sigma > 0$ small enough so That

$$L'(t) \leq N_0(\mu \circ \nabla_x w) + c_1 m_1 \|\nabla_x w\|^{2(\gamma+1)}_{V_a} - cE(t), \quad \text{for all} \quad t \geq 0, \tag{3.31}$$

for some positive constants c_1 and c, where $N_0 = \dfrac{4\xi_2 c + m_0 - l}{4\sigma}$. Now we set $F(t) = L(t) + cE(t)$. Because $L(t) \sim E(t)$, we have that $F(t) \sim E(t)$. Then by (3.31), if there is a need we enlarge c, we get

$$\begin{aligned} F'(t) &= L'(t) + cE'(t) \\ &\leq -cE(t) + c \int_{\mathbb{R}^n} \int_{t_1}^t \mu(t-s) a(x) |\nabla_x w(t) - \nabla_x w(s)|^2 ds dx, \quad \text{for all} \quad t \geq t_1. \end{aligned} \tag{3.32}$$

By (3.5) and Remark 3.2.1, we have for any $t \geq t_1$

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_0^{t_1} \mu(t-s) a(x) |\nabla_x w(t) - \nabla_x w(s)|^2 ds dx \\ &\leq -\frac{1}{k} \int_{\mathbb{R}^n} \int_0^{t_1} \mu'(t-s) a(x) |\nabla_x w(t) - \nabla_x w(s)|^2 ds dx \\ &\leq -cE'(t). \end{aligned}$$

We define

$$J(t) = \int_{t_1}^t H_0(-\mu'(s))(\mu \circ \nabla_x w)(t) ds. \tag{3.33}$$

Since $\int_0^{+\infty} H_0(-\mu'(s))\mu(s)ds < +\infty$, if there is a need we enlarge t_1, from (3.5) we have

$$\begin{aligned} J(t) &= \int_{t_1}^{t} H_0(-\mu'(s)) \int_{\mathbb{R}^n} \mu(s)a(x)|\nabla_x w(t) - \nabla_x w(t-s)|^2 dxds \\ &\leq 2\int_{t_1}^{t} H_0(-\mu'(s))\mu(s) \int_{\mathbb{R}^n} \left(a(x)|\nabla_x w(t)|^2 + a(x)|\nabla_x w(t-s)|^2\right) dxds \\ &\leq cE(0)\int_{t_1}^{t} H_0(-\mu'(s))\mu(s)ds \\ &< 1. \end{aligned}$$

We define the following functional.

$$\Psi(t) = -\int_{t_1}^{t} \frac{1}{H_0^{-1}(-\mu'(s))}\mu'(s) \int_{\mathbb{R}^n} \mu(s)a(x)|\nabla_x w(t) - \nabla_x w(t-s)|^2 dxds.$$

Using Remark 3.2.1 and (A1), we get that there exists a positive constant k_0 such that

$$\frac{1}{H_0^{-1}(-\mu'(s))} \leq k_0 \quad \text{for} \quad \text{any} \quad s \geq t_1.$$

Hence, for any $t \geq t_1$, we have

$$\begin{aligned} \Psi(t) &\leq -k_0\int_{t_1}^{t} \mu'(s) \int_{\mathbb{R}^n} a(x)|\nabla_x w(t) - \nabla_x w(t-s)|^2 dxds \\ &\leq -2k_0\int_{t_1}^{t} \mu'(s) \int_{\mathbb{R}^n} \left(a(x)|\nabla_x w(t)|^2 + a(x)|\nabla_x w(t-s)|^2\right) dxds \\ &\leq -cE(0)\int_{t_1}^{t} \mu'(s)ds \\ &\leq cE(0)\mu(t_1) \\ &< \min\{r, H(r), H_0(r)\}. \end{aligned}$$

Using the properties of H_0 (strictly convex in $(0,r], H_0(0) = 0$), then for $x \in (0,r], \theta \in [0,1]$, we have

$$H_0(\theta x) \leq \theta H_0(x).$$

Using Remark 3.2.1, (3.34), (3.34) and Jensen's inequality, we go to

$$
\begin{aligned}
\Psi(t) &= \frac{1}{J(t)}\int_{t_1}^{t} J(t)H_0[H_0^{-1}(-\mu'(s))]\frac{1}{H_0^{-1}(-\mu'(s))}\int_{\mathbb{R}^n}\mu(s)a(x)|\nabla_x w(t)-\nabla_x w(t-s)|^2 dxds\\
&\geq \frac{1}{J(t)}\int_{t_1}^{t} H_0[J(t)H_0^{-1}(-\mu'(s))]\frac{1}{H_0^{-1}(-\mu'(s))}\int_{\mathbb{R}^n}\mu(s)a(x)|\nabla_x w(t)-\nabla_x w(t-s)|^2 dxds\\
&\geq H_0\left(\frac{1}{J(t)}\int_{t_1}^{t} J(t)H_0^{-1}(-\mu'(s))\frac{1}{H_0^{-1}(-\mu'(s))}\int_{\mathbb{R}^n}\mu(s)a(x)|\nabla_x w(t)-\nabla_x w(t-s)|^2 dxds\right)\\
&\geq H_0\left(\int_{t_1}^{t}\int_{\mathbb{R}^n}\mu(s)a(x)|\nabla_x w(t)-\nabla_x w(t-s)|^2 dxds\right),
\end{aligned}
$$

Which implies

$$
\int_{t_1}^{t}\int_{\mathbb{R}^n}\mu(s)a(x)|\nabla_x w(t)-\nabla_x w(t-s)|^2 dxds \leq H_0^{-1}(\Psi(t)).
$$

We have

$$
F'(t) \quad\leq\quad -cE(t)+cH_0^{-1}(\Psi(t)), \quad \text{for all} \quad t\geq t_1.
$$

Now we define the functional

$$
F_1(t) = H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)F(t)+cE(t), \quad \beta_0<r, 0<c.
$$

Since $F(t)\sim E(t)$, we have that $F_1(t)\sim E(t)$. Also, using that $E'(t)\leq 0$ for any $t\geq 0$ and $H_0'>0$, $H_0''>$ on $(0,r]$, we get

$$
\begin{aligned}
F_1'(t) &= \beta_0\frac{E'(t)}{E(0)}H_0''\left(\beta_0\frac{E(t)}{E(0)}\right)F(t)+H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)F'(t)+cE'(t)\\
&\leq -cE(t)H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)+cH_0'\left(\beta_0\frac{E(t)}{E(0)}\right)H_0^{-1}(\Psi(t))+cE'(t).
\end{aligned}
$$

Let H_0^* be given as in Remark 3.2.1. Hence, using Young's inequality (3.17) with $A=H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)$, $B=H_0^{-1}(\Psi(t))$, we obtain

$$
\begin{aligned}
F_1'(t) &\leq -cE(t)H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)+cH_0^*\left(H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)\right)+c\Psi(t)+cE'(t)\\
&\leq -cE(t)H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)+c\beta_0\frac{E(t)}{E(0)}H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)-c'E'(t)+cE'(t).
\end{aligned}
$$

We can choose β_0, c, and c' such that

$$F_1'(t) \leq -k\frac{E(t)}{E(0)}H_0'\left(\beta_0\frac{E(t)}{E(0)}\right)$$
$$= -kH_2\left(\frac{E(t)}{E(0)}\right) \quad \text{for} \quad \text{any} \quad t \geq t_1,$$

where $H_2(t) = tH_0'(\beta_0 t)$. Using the strict convexity of H_0 on $(0, r]$, we conclude that H_2' and H_2 are strict positive on $(0, 1]$. Therefore

$$R(t) = \tau\frac{k_1 F_1(t)}{E(0)} \sim E(t), \quad \tau \in (0, 1),$$

For some positive constant k_1, hence

$$R'(t) \leq -\tau k_2 H_2(R(t)) \quad \text{for} \quad \text{any} \quad t \geq t_1$$

and for some positive constant k_2. Then a simple integration and a suitable choice of τ yield

$$R(t) \leq h^{-1}(\beta_1 t + \beta_2), \quad \beta_1, \beta_2 \in (0, +\infty), t \geq t_1.$$

Here $h(t) = \int_t^1 H_2^{-1}(s)ds$. From (3.34), for a positive constant β_3, we have

$$E(t) \leq \beta_3 h^{-1}(\beta_1 t + \beta_2), \quad \beta_1, \beta_2 \in (0, +\infty), t \geq t_1.$$

The fact that h is strictly decreasing function on $(0, 1]$ and due to properties of H_2, we have

$$\lim_{t \to 0} h(t) = +\infty.$$

Then

$$E(t) \leq \beta_3 h^{-1}(\beta_1 t + \beta_2) \quad \text{for all} \quad t \geq 0.$$

□

Chapter 4

Wave equation with Logarithmic nonlinearities in Kirchhoff type

4.1 Introduction

In this chapter, we consider the wave equation with logarithmic nonlinearity (4.1),

$$u'' - \phi(x)\left(M(\|\nabla_x u\|_2^2)\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) = au\ln|u|^k \tag{4.1}$$

where $x \in \mathbb{R}^n, t > 0, n \geq 2, k, a > 0$ and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$ and the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (A1).

It is well known that from a class of nonlinearities, the logarithmic nonlinearity is distinguished by several interesting physical properties. Eqation (4.1) is equipped by the following initial data.

$$u\,(0,x) = u_0(x) \in \mathcal{H}(\mathbb{R}^n), u'\,(0,x) = u_1(x) \in L^2_\rho(\mathbb{R}^n), \tag{4.2}$$

Where the weighted spaces $\mathcal{H}$ is given in Definition 3.2.1 and the density function $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ satisfies

$$\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \tag{4.3}$$

with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.
This kind of systems appears in the models of nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [29] in the case $n = 1$ this type of problem describes a small amplitude vibration of an elastic string. The original equation (see 3.7)

For the decay rate in $\mathbb{R}^n$, we quote essentially the results of [22], [26], [27], [28], [33]. In [27], authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (4.1), (4.2) with $\rho(x) = 1, M \equiv 1, a = 0$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In the case $M \equiv 1, a = 0$, in [26], author looked into a linear Cauchy viscoelastic problem with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [26], was considered in [28], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem (4.1),(4.2) without source, for the case $\rho(x) = 1, M \equiv 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered in [33], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, \quad H(0) = 0 \tag{4.4}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. This improves the conditions considered in [22] on the relaxation functions

$$g'(t) \leq -\chi(g(t)), \quad \chi(0) = \chi'(0) = 0 \tag{4.5}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$.

The goal of the present paper is to establish the existence of solution to the problem (4.1)-(4.2). We obtain also, a fast decay results.

4.2 Material, Assumptions and technical lemmas

For simplicity reason, we take $a = 1$
We recall and make use the following hypothesis on the function g as: (A3) According to results in [33], we have

1. We can deduce that there exists $t_1 > 0$ large enough such that:

   (a) $\forall t \geq t_1$: We have $\lim_{s\to+\infty} g(s) = 0$, which implies that $\lim_{s\to+\infty} -g'(s)$ cannot be positive, so
   $\lim_{s\to+\infty} -g'(s) = 0$. Then $g(t_1) > 0$ and

   $$\max\{g(s), -g'(s)\} < \min\{r, H(r), H_0(r)\}, \tag{4.6}$$

   where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

   $$\int_0^{+\infty} g(s)H_0(-g'(s))ds < +\infty.$$

   (b) $\forall t \in [0, t_1]$: As g is non increasing, $g(0) > 0$ and $g(t_1) > 0$ then $g(t) > 0$ and

   $$g(0) \geq g(t) \geq g(t_1) > 0.$$

   Therefore, since H is a positive continuous function, then

   $$a \leq H(g(t)) \leq b$$

   For some positive constants a and b. Consequently

   $$g'(t) \leq -H(g(t)) \leq -kg(t), \quad k > 0$$

Which gives

$$g'(t) \leq -kg(t), k > 0 \tag{4.7}$$

2. Let H_0^* be the convex conjugate of H_0 in the sense of Young (see[1], then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0, H_0'(r))$$

And satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r]. \tag{4.8}$$

The space $\mathcal{H}(\mathbb{R}^n)$ (see 3.11) as the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the norm

$$\|u\|^2_{\mathcal{H}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla_x u|^2 dx.$$

The following technical Lemmas will play an important role in the sequel.

Lemma 4.2.1. *[23] (Lemma 1.1) For any two functions $g, v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$\begin{aligned} v'(t)\int_0^t g(t-s)v(s)ds = &-\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)|v(t)-v(s)|^2 ds \\ &+\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\right)|v(t)|^2 \\ &+\frac{1}{2}\int_0^t g'(t-s)|v(t)-v(s)|^2 ds \\ &-\frac{1}{2}g(t)|v(t)|^2. \end{aligned}$$

and

$$\begin{aligned} &\left|\int_0^t g(t-s)(v(t)-v(s))ds\right|^2 \\ &\leq \left(\int_0^t |g(s)|^{2(1-\theta)}ds\right)\left(\int_0^t |g(t-s)|^{2\theta}|v(t)-v(s)|^2 ds\right) \end{aligned}$$

Lemma 4.2.2. *Let $u \in \mathcal{H}(\mathbb{R}^n)$ be any function and $c_1, c_2 > 0$ be any numbers. Then*

$$2\int_{\mathbb{R}^n} \rho(x)|u|^2 \ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)dx + n(1+c_1)\|u\|^2_{L^2_\rho} \leq c_2\frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla_x u\|^2_2$$

Definition 4.2.1. *By the weak solution of (4.1) over $[0,T]$ we mean a function*

$$u \in C([0,T], \mathcal{H}(\mathbb{R}^n)) \cap C^1([0,T], L^2_\rho(\mathbb{R}^n)) \cap C^2([0,T], \mathcal{H}^{-1}(\mathbb{R}^n))$$

with $u' \in L^2([0,T], \mathcal{H}(\mathbb{R}^n))$, such that $u(0) = u_0, u'(0) = u_1$ and for all $v \in \mathcal{H}, t \in [0,T]$,

$$\int_{\mathbb{R}^n} \rho(x)u \ln|u|^k v dx = \int_{\mathbb{R}^n} \rho(x)u'' v dx + M(\|\nabla_x u\|^2_2)\int_{\mathbb{R}^n} \nabla_x u \nabla_x v dx - \int_{\mathbb{R}^n}\int_0^t g(t-s)\nabla_x u(s) ds \nabla_x v dx$$

Multiplying the equation (4.1) by $\rho(x)u'$, and integrating by parts over $\mathbb{R}^n$, we have the energy of u at time t is given by

$$E(t) = \frac{1}{2}\Big(\|u'\|^2_{L^2_\rho} + \Big(m_0 - \int_0^t g(s)ds\Big)\|\nabla_x u\|^2_2 + (g \circ \nabla_x u) - \int_{\mathbb{R}^n} \rho(x)u^2 \ln|u|^k dx\Big) + \frac{k}{4}\|u\|^2_{L^2_\rho} + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|^{2(\gamma+1)}_2 \tag{4.9}$$

And the following energy functional law holds

$$E'(t) = \frac{1}{2}(g' \circ \nabla_x u)(t) - \frac{1}{2}g(t)\|\nabla_x u(t)\|^2_2, \forall t \geq 0. \tag{4.10}$$

which means that, our energy is uniformly bounded and decreasing along the trajectories. The following notation will be used throughout this paper

$$(g \circ \nabla_x u)(t) = \int_0^t g(t-\tau)\,\|\nabla_x u(t) - \nabla_x u(\tau)\|^2_2\, d\tau, \tag{4.11}$$

For $u \in \mathcal{H}(\mathbb{R}^n), t \geq 0$.

4.3 Global existence in times

According to logarithmic Sobolev inequality and similar to the proof in ([45], [46], [47], [53], [54]), we have the following result.

Theorem 4.3.1. *(Local existence) Let $u_0(x) \in \mathcal{H}(\mathbb{R}^n), u_1(x) \in L^2_\rho(\mathbb{R}^n)$ be given. Then under hypothesis $(A1), (A2)$ and (4.3), the problem (4.1) has a unique local solution*

$$u \in C([0,T], \mathcal{H}(\mathbb{R}^n)) \cap C^1([0,T], L^2_\rho(\mathbb{R}^n))$$

Now we introduce two functionals

$$\begin{aligned} J(t) = & \frac{1}{2}\Big(\Big(m_0 - \int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g \circ \nabla_x u) \\ & - \int_{\mathbb{R}^n} \rho(x)u^2 \ln|u|^k dx\Big) \\ & + \frac{k}{4}\|u\|^2_{L^2_\rho} + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)} \end{aligned} \tag{4.12}$$

And

$$\begin{aligned} I(t) = & \Big(m_0 - \int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g \circ \nabla_x u) \\ & - \int_{\mathbb{R}^n} \rho(x)u^2 \ln|u|^k dx + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)} \end{aligned} \tag{4.13}$$

Then

$$J(t) = \frac{1}{2}I(t) + \frac{k}{4}\|u\|^2_{L^2_\rho} \tag{4.14}$$

As in ([49]) to establish the corresponding method of potential wells which is related to the logarithmic nonlinear term, we introduce the stable set as follows

$$W = \{u \in \mathcal{H}(\mathbb{R}^n) : I(t) > 0, J(t) < d\} \cup \{0\} \tag{4.15}$$

Remark 4.3.1. *We notice that the mountain pass level d given in (4.15) defined by*

$$d = \inf\{\sup_{u \in \mathcal{H}(\mathbb{R}^n)\setminus\{0\} \mu \geq 0} J(\mu u)\} \tag{4.16}$$

Also, by introducing the so called "Nehari manifold"

$$\mathcal{N} = \{u \in \mathcal{H}(\mathbb{R}^n) \backslash \{0\} : I(t) = 0\}$$

Similar to results in [55], it is readily seen that the potential depth d is also characterized by

$$d = \inf_{u \in \mathcal{N}} J(t). \tag{4.17}$$

This characterization of d shows that

$$dist\,(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|u\|_{\mathcal{H}(\mathbb{R}^n)} \tag{4.18}$$

By the fact that (4.10), we will prove the invariance of the set W. That is if for some $t_0 > 0$ if $u(t_0) \in W$, then $u(t) \in W$, $\forall t \geq t_0$, let us beginning by giving the existence Lemma of the potential depth. (See [47] Lemma 2.4)

Lemma 4.3.1. *d is positive constant.*

Lemma 4.3.2. *Let $u \in \mathcal{H}(\mathbb{R}^n)$ and $\beta = e^{\frac{1}{2}n(1+c_1)}$. if $0 < \|u\|^2_{L^2_\rho} < \beta$, then $I(t) > 0$; if $I(t) = 0, \|u\|^2_2 \neq 0$, then $\|u\|^2_{L^2_\rho} > \beta$.*

Proof. By (A1), (4.13) and Lemma 4.2.2, we have

$$\begin{aligned} I(t) &= \Big(m_0 - \int_0^t g(s)ds\Big) \|\nabla_x u\|_2^2 + (g \circ \nabla_x u) \\ &- \int_{\mathbb{R}^n} \rho(x) u^2 \ln |u|^k dx + \frac{m_1}{2(\gamma+1)} \|\nabla_x u\|_2^{2(\gamma+1)} \\ &\geq l \|\nabla_x u\|_2^2 - k \int_{\mathbb{R}^n} \rho(x) u^2 \Big(\ln \frac{|u|}{\|u\|^2_{L^2_\rho}} + \ln \|u\|^2_{L^2_\rho} \Big) dx \\ &\geq \left(l - \frac{kc_2}{2\pi} \|\rho\|^2_{L^2_\rho} \right) \|\nabla_x u\|_2^2 + \frac{1}{2} kn(1+c_1) \|u\|^2_{L^2_\rho} \\ &- k \|u\|^2_{L^2_\rho} \ln \|u\|^2_{L^2_\rho} \end{aligned}$$

Choosing c_2 such that $l > \frac{kc_2}{2\pi} \|\rho\|^2_{L^2_\rho}$, then

$$I(t) \geq k \Big(\frac{1}{2} n(1+c_1) - \ln \|u\|^2_{L^2_\rho} \Big) \|u\|^2_{L^2_\rho}$$

Therefore, if $0 < \|u\|^2_{L^2_\rho} < \beta$, then $I(t) > 0$; if $I(t) = 0, \|u\|^2_2 \neq 0$, we have $\beta < \|u\|^2_{L^2_\rho}$ then, $\|u\|^2_{L^2_\rho} > \beta$. ∎

Theorem 4.3.2. *(Global Existence) Let $u_0(x) \in \mathcal{H}(\mathbb{R}^n), u_1(x) \in L^2_\rho(\mathbb{R}^n)$ and $0 < E(0) < d, I(0) > 0$. Then, under hypothesis $(A1), (A2)$ and conditions (4.3), the problem (4.1) has a global solution in time.*

Proof. From the definition of energy for solution and by (4.10), we have

$$\frac{1}{2}\|u'\|^2_{L^2_\rho} + J(t) \leq \frac{1}{2}\|u_1\|^2_{L^2_\rho} + J(0), \quad \forall t \in [0, T_{max}) \tag{4.19}$$

Where T_{max} is the maximal existence time of solution of u. Then by the definition of the stable set and using Lemma 4.3.2, we have $u \in W, \quad \forall t \in [0, T_{max})$ □

4.4 Decay estimates

We apply the multiplier techniques to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions. For this purpose, we introduce the functionals

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) u u' dx \tag{4.20}$$

Lemma 4.4.1. *Under the hypothesis (A1) and (A2), the functional ψ_1 satisfies, along the solution of (4.1),(4.2)*

$$\begin{aligned}\psi_1'(t) \leq\ & \|u'\|^2_{L^2_\rho} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} + \frac{(1-l)}{4\sigma}(g \circ \nabla_x u) \\ & + \left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right) + k\|\rho\|^2_{L^2}\Big(\ln \|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\Big)\right] \|\nabla u\|_2^2.\end{aligned}$$

Proof. From (4.20), integrate over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n} \rho(x)|u'|^2 dx + \int_{\mathbb{R}^n} \rho(x) u u'' dx \\
&= \int_{\mathbb{R}^n} \left(\rho(x)|u'|^2 + M(\|\nabla_x u\|_2^2) u \Delta_x u - u \int_0^t g(t-s)\Delta_x u(s,x) ds \right) dx \\
&+ \int_{\mathbb{R}^n} \rho(x) u^2 \ln |u|^k dx \\
&\leq \|u'\|^2_{L^2_\rho(\mathbb{R}^n)} + m_1 \|\nabla_x u\|_2^{2(\gamma+1)} - l\|\nabla_x u\|_2^2 \\
&+ k \int_{\mathbb{R}^n} \rho(x) u^2 \Big(\ln \Big(\frac{|u|}{\|u\|^2_{L^2_\rho}} \Big) + \ln \|u\|^2_{L^2_\rho} \Big) dx \\
&+ \int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(s) - \nabla_x u(t)) ds dx.
\end{aligned}$$

We have by using the Logarithmic Sobolev inequality in Lemma 4.2.2 and generalized version of Poincare's inequality in Lemma3.3.1 Using Young's inequality and Lemma 4.2.1 for $\theta = 1/2$, we obtain

$$\begin{aligned}
\psi_1'(t) &\leq \|u'\|^2_{L^2_\rho} + m_1 \|\nabla_x u\|_2^{2(\gamma+1)} + \Big(\frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l \Big) \|\nabla_x u\|_2^2 \\
&+ k\|u\|^2_{L^2_\rho} \ln \|u\|^2_{L^2_\rho} \\
&+ \sigma \|\nabla_x u\|_2^2 + \frac{1}{4\sigma} \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)|\nabla_x u(s) - \nabla_x u(t)| ds \right)^2 dx \\
&- \frac{1}{2} kn(1+c_1)\|u\|^2_{L^2_\rho} \\
&\leq \|u'\|^2_{L^2_\rho} + m_1 \|\nabla_x u\|_2^{2(\gamma+1)} + \Big(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l \Big) \|\nabla_x u\|_2^2 \\
&+ \frac{(1-l)}{4\sigma}(g \circ \nabla_x u) + k\Big(\ln \|u\|^2_{L^2_\rho} - \frac{1}{2} n(1+c_1) \Big) \|u\|^2_{L^2_\rho}.
\end{aligned}$$

Then

$$\begin{aligned}
\psi_1'(t) &\leq \|u'\|^2_{L^2_\rho} + m_1 \|\nabla_x u\|_2^{2(\gamma+1)} + \frac{(1-l)}{4\sigma}(g \circ \nabla_x u) \\
&+ \left[\Big(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l \Big) + k\|\rho\|^2_{L^2} \Big(\ln \|u\|^2_{L^2_\rho} - \frac{1}{2} n(1+c_1) \Big) \right] \|\nabla u\|_2^2.
\end{aligned}$$

□

The existence of the memory term forces us to make second modification of the associate energy functional. Set

$$\psi_2(t) = -\int_{\mathbb{R}^n} \rho(x)u' \int_0^t g(t-s)(u(t)-u(s))dsdx. \tag{4.21}$$

Lemma 4.4.2. *Under the hypothesis (A1) and (A2), the functional ψ_2 satisfies, along the solution of (4.1),(4.2), for any $\sigma \in (0, m_0)$*

$$\begin{aligned}
\psi_2'(t) \leq & \left[\sigma + k\Big(\sigma\frac{c_2}{2\pi} + \ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\Big)\right]\|\nabla_x u\|_2^2 \\
& + cm_1\|\nabla_x u\|_2^{2(\gamma+1)} + c_\sigma(1 + (k\frac{c_2}{2\pi}+1)\|\rho\|^2_{L^2})(g \circ \nabla_x u) \\
& - c_\sigma\|\rho\|^2_{L^2}(g' \circ \nabla_x u) + \left(\sigma - \int_0^t g(s)ds\right)\|u'\|^2_{L^2_\rho}.
\end{aligned}$$

Proof. Exploiting equation (4.1), (4.21) to get

$$\begin{aligned}
\psi_2'(t) = & -\int_{\mathbb{R}^n} \rho(x)u'' \int_0^t g(t-s)(u(t)-u(s))dsdx \\
& - \int_{\mathbb{R}^n} \rho(x)u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
& - \int_0^t g(s)ds\|u'\|^2_{L^2_\rho} \\
= & \int_{\mathbb{R}^n} M(\|\nabla u\|_2^2)\nabla_x u \int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
& - \int_{\mathbb{R}^n} \rho(x)u\ln|u|^k \int_0^t g(t-s)(u(t)-u(s))dsdx \\
& - \int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla_x u(s,x)ds\right) \times \\
& \left(\int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right)dx \\
& - \int_{\mathbb{R}^n} \rho(x)u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
& - \int_0^t g(s)ds\|u'\|^2_{L^2_\rho}
\end{aligned}$$

By (A1), we have

$$\begin{aligned}
&\psi_2'(t) = \left(m_0 - \int_0^t g(s)ds\right) \times \\
&\int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right)^2 dx \\
&+ cm_1 \|\nabla_x u\|_2^{2(\gamma+1)} \\
&- \int_{\mathbb{R}^n} \rho(x) u \ln |u|^k \int_0^t g(t-s)(u(t) - u(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x) u' \int_0^t g'(t-s)(u(t) - u(s))dsdx \\
&- \int_0^t g(s)ds \|u'\|_{L^2_\rho}^2 + c(g \circ \nabla_x u)(t).
\end{aligned}$$

By Holder's and Young's inequalities and Lemma 3.3.1, we estimate

$$\begin{aligned}
&- \int_{\mathbb{R}^n} \rho(x) u' \int_0^t g'(t-s)(u(t) - u(s))dsdx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'|^2 dx\right)^{1/2} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x) \left| \int_0^t g'(t-s)(u(t) - u(s))ds\right|^2\right)^{1/2} \\
&\leq \sigma \|u'\|_{L^2_\rho}^2 + c_\sigma \left\| \int_0^t -g'(t-s)(u(t) - u(s))ds \right\|_{L^2_\rho}^2 \\
&\leq \sigma \|u'\|_{L^2_\rho}^2 - c_\sigma \|\rho\|_{L^2}^2 (g' \circ \nabla_x u)(t).
\end{aligned}$$

And

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho(x) u' \int_0^t g(t-s)(u(t) - u(s))dsdx \\
&\leq \sigma \|u'\|_{L^2_\rho}^2 + c_\sigma \|\rho\|_{L^2}^2 (g \circ \nabla_x u)(t).
\end{aligned}$$

And by Lemma 3.3.1 and Lemma 4.2.2 and conditions in Lemma 4.3.2, we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} \rho(x) \ln |u|^k u \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leq k \int_{\mathbb{R}^n} \rho(x) \Big(\ln \Big(\frac{|u|}{\|u\|^2_{L^2_\rho}} \Big) + \ln \|u\|^2_{L^2_\rho} \Big) u \times \\
&\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leq k \Big(\ln \|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2} \Big) \|u\|^2_{L^2_\rho} \\
&+ k \frac{c_2}{2\pi} \Big\| u \int_0^t g(t-s)(u(t)-u(s))ds \Big\|^2_{L^2_\rho} \\
&\leq k \Big(\ln \|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2} \Big) \|\rho\|^2_{L^2} \|\nabla_x u\|^2_2 \\
&+ k \frac{c_2}{2\pi} \|\rho\|^2_{L^2} \Big\| \nabla u \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds \Big\|^2_{L^2_\rho} \\
&\leq k \Big(\sigma \frac{c_2}{2\pi} + \ln \|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2} \Big) \|\rho\|^2_{L^2} \|\nabla_x u\|^2_2 \\
&+ c_\sigma k \frac{c_2}{2\pi} \|\rho\|^2_{L^2} (g \circ \nabla_x u).
\end{aligned}
$$

Using Young's and Poincare's inequalities and Lemma 4.2.1 for $\theta = 1/2$, we obtain

$$
\begin{aligned}
\psi_2'(t) \leq &\left[\sigma + k \Big(\sigma \frac{c_2}{2\pi} + \ln \|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2} \Big) \right] \|\nabla_x u\|^2_2 \\
&+ c m_1 \|\nabla_x u\|_2^{2(\gamma+1)} \\
&+ c_\sigma (1 + (k \frac{c_2}{2\pi} + 1) \|\rho\|^2_{L^2}) (g \circ \nabla_x u) - c_\sigma \|\rho\|^2_{L^2} (g' \circ \nabla_x u) \\
&+ \left(\sigma - \int_0^t g(s)ds \right) \|u'\|^2_{L^2_\rho}.
\end{aligned}
$$

□

Now, let us define

$$
L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2 \psi_2(t) \tag{4.22}
$$

For ξ_1, ξ_2 large enough. We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions, that is for ξ_1, ξ_2 large enough, we have

$$
\beta_1 L(t) \leq E(t) \leq \beta_2 L(t)
$$

Holds for two positive constants β_1 and β_2.

Lemma 4.4.3. *For ξ_1, ξ_2 large enough, we have*

$$L(t) \sim E(t).$$

Proof. By (4.22) we have

$$\begin{aligned}
&|L(t) - \xi_1 E(t)| \leq |\psi_1(t)| + \xi_2 |\psi_2(t)| \\
&\leq \int_{\mathbb{R}^n} |\rho(x) u u'| \, dx \\
&+ \xi_2 \int_{\mathbb{R}^n} \left| \rho(x) u' \int_0^t g(t-s)(u(t) - u(s)) ds \right| dx.
\end{aligned}$$

Thanks to Holder and Young's inequalities, we have by using Lemma 3.3.1

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\rho(x) u u'| \, dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x) |u|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \rho(x) |u'|^2 dx \right)^{1/2} \\
&\leq \frac{1}{2} \left(\int_{\mathbb{R}^n} \rho(x) |u|^2 dx \right) + \frac{1}{2} \left(\int_{\mathbb{R}^n} \rho(x) |u'|^2 dx \right) \\
&\leq c \|u'\|^2_{L^2_\rho} + c \|\rho\|^2_{L^s} \|\nabla_x u\|^2_2
\end{aligned}$$

And

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \left(\rho(x)^{\frac{1}{2}} u' \right) \left(\rho(x)^{\frac{1}{2}} \int_0^t g(t-s)(u(t) - u(s)) ds \right) \right| dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x) |u'|^2 dx \right)^{1/2} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x) \left| \int_0^t g(t-s)(u(t) - u(s)) ds \right|^2 dx \right)^{1/2} \\
&\leq \frac{1}{2} \|u'\|^2_{L^2_\rho} + \frac{1}{2} \left\| \int_0^t g(t-s)(u(t) - u(s)) ds \right\|^2_{L^2_\rho} \\
&\leq \frac{1}{2} \|u'\|^2_{L^2_\rho} + \frac{1}{2} \|\rho\|^2_{L^s} (g \circ \nabla_x u).
\end{aligned}$$

Then

$$|L(t) - \xi_1 E(t)| \leq c E(t).$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{4.23}$$

□

Lemma 4.4.4. *For all $t \geq t_1 > 0$, we have*

$$\int_{t_1}^{t} (g \circ \nabla_x u)(s) ds \leq H_0^{-1}\Big(- \int_{t_1}^{t} H_0(-g'(s))g'(s) \times \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds\Big).$$

Where H_0 introduced in (4.6).

Proof. By (4.10) and (A3), we have for all $t \geq t_1$

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_0^{t_1} g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx \\ &\leq -\frac{1}{k} \int_{\mathbb{R}^n} \int_0^{t_1} g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx \\ &\leq -cE'(t). \end{aligned}$$

Now, we define

$$I(t) = \int_{t_1}^{t} H_0(-g'(s))(g \circ \nabla_x u)(t) ds. \tag{4.24}$$

Since $\int_0^{+\infty} H_0(-g'(s))g(s)ds < +\infty$, from (4.10) we have

$$\begin{aligned} I(t) &= \int_{t_1}^{t} H_0(-g'(s)) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\ &\leq 2\int_{t_1}^{t} H_0(-g'(s))g(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\ &\leq cE(0) \int_{t_1}^{t} H_0(-g'(s))g(s)ds < 1. \end{aligned} \tag{4.25}$$

We define again a new functional $\lambda(t)$ related with $I(t)$ as

$$\lambda(t) = -\int_{t_1}^{t} H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds.$$

From (A1)-(A3) and , we get

$$\frac{1}{H_0^{-1}(-g'(s))} \leq k_0.$$

For some positive constant k_0. Then, for all $t \geq t_1$

$$\begin{aligned}
\lambda(t) &\leq -k_0 \int_{t_1}^{t} g'(s) \int_{\mathbb{R}^n} |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&\leq -k_0 \int_{t_1}^{t} g'(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\
&\leq -cE(0) \int_{t_1}^{t} g'(s) ds \\
&\leq cE(0) g(t_1) \\
&< \min\{r, H(r), H_0(r)\}. \qquad (4.26)
\end{aligned}$$

Using the properties of H_0 (strictly convex in $(0, r], H_0(0) = 0$), then for $x \in (0, r], \theta \in [0, 1]$

$$H_0(\theta x) \leq \theta H_0(x).$$

Using hypothesis in (A3), (4.25), (4.26) and Jensen's inequality leads to

$$\begin{aligned}
&\lambda(t) = \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g'(s))] \frac{1}{H_0^{-1}(-g'(s))} \\
&\int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&\geq \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g'(s))] \frac{1}{H_0^{-1}(-g'(s))} \\
&\int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&\geq H_0\Big(\frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(-g'(s)) \frac{1}{H_0^{-1}(-g'(s))} \\
&\int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds\Big) \\
&\geq H_0 \left(\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \right)
\end{aligned}$$

Which implies

$$\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \leq H_0^{-1}(\lambda(t)).$$

□

Our next main result reads as follows.

Theorem 4.4.1. *Let $(u_0, u_1) \in \mathcal{H}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ and suppose that (A1)- (A2) hold. Then there exist positive constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that the energy of solution given by (4.1),(4.2) satisfies,*

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \text{for all} \quad t \geq 0,$$

Where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\alpha_0 s} ds$$

Proof. From (4.10), results of Lemma 4.4.1 and Lemma 4.4.2, we have

$$\begin{aligned} L'(t) &= \xi_1 E'(t) + \psi_1'(t) + \xi_2 \psi_2'(t) \\ &\leq (\frac{1}{2}\xi_1 - c_\sigma \|\rho\|^2_{L^2} \xi_2)(g' \circ \nabla_x u) + M_0 (g \circ \nabla_x u) \\ &- M_1 \|u'\|^2_{L^2_\rho} - M_2 \|\nabla_x u\|_2^2 + (c\xi_2 + 1) m_1 \|\nabla_x u\|_2^{2(\gamma+1)} \end{aligned}$$

Where

$$M_0 = \Big(\xi_2 c_\sigma (1 + (k\frac{c_2}{2\pi} + 1)\|\rho\|^2_{L^2}) + \frac{(1-l)}{4\sigma}\Big) > 0,$$

$$M_1 = \left(\xi_2 \left(\int_0^{t_1} g(s) ds - \sigma\right) - 1\right),$$

$$\begin{aligned} M_2 = \frac{1}{2}\xi_1 g(t_1) - \Big[\Big(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\Big) \\ + k\|\rho\|^2_{L^2}\Big(\ln \|u\|^2_{L^2_\rho} - \frac{1}{2}n(1 + c_1)\Big)\Big] \\ -\xi_2 \left[\sigma + k\Big(\sigma\frac{c_2}{2\pi} + \ln \|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\Big)\right] \end{aligned}$$

And t_1 was introduced in (A3). We choose σ so small that $\xi_1 > 2c_\sigma \|\rho\|^2_{L^2} \xi_2$. Whence σ is fixed, we can choose

$$\xi_2 > \left(\int_0^{t_1} g(s) ds - \sigma\right)^{-1}$$

and ξ_1 large enough so that $M_2 > 0$, which yields

$$L'(t) \leq M_0 (g \circ \nabla_x u) + (c\xi_2 + 1) m_1 \|\nabla_x u\|_2^{2(\gamma+1)} - cE'(t),$$

$$\forall t \geq t_1.$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$.

Then we get for some $c > 2(c\xi_2 + 1)(\gamma + 1)$

$$\begin{aligned} F'(t) &= L'(t) + cE'(t) \\ &\leq -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^{t} g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 ds dx, \quad \text{for all} \quad t \geq t_1. \end{aligned}$$

Using Lemma(4.4.4), we obtain

$$F'(t) \leq -cE(t) + cH_0^{-1}(\lambda(t)), \quad \text{for all} \quad t \geq t_1.$$

Now, we will following the steps in ([33]) and using the fact that $E' \leq 0, 0 < H_0', 0 < H_0''$ on $(0, r]$ to define the functional

$$F_1(t) = H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) F(t) + cE(t), \quad \alpha_0 < r, c > 0.$$

Where $F_1(t) \sim E(t)$ and

$$\begin{aligned} F_1'(t) &= \alpha_0 \frac{E'(t)}{E(0)} H_0''\left(\alpha_0 \frac{E(t)}{E(0)}\right) F(t) \\ &+ H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) F'(t) + cE'(t) \\ &\leq -cE(t) H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) \\ &+c\ H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) H_0^{-1}(\lambda(t)) + cE'(t). \end{aligned}$$

Let H_0^* given in (A3) and using Young's inequality (4.8) with $A = H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right), B = H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned} F_1'(t) &\leq -cE(t) H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + cH_0^*\left(H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right)\right) \\ &+ c\lambda(t) + cE'(t) \\ &\leq -cE(t) H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + c\alpha_0 \frac{E(t)}{E(0)} H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) \\ &- c'E'(t) + cE'(t). \end{aligned}$$

Choosing α_0, c, c', such that for all $t \geq t_1$ we have

$$\begin{aligned} F_1'(t) &\leq -k\frac{E(t)}{E(0)} H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) \\ &= -kH_2\left(\frac{E(t)}{E(0)}\right), \end{aligned}$$

Where $H_2(t) = tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0, r]$, to find that H_2', H_2 are strict positives on $(0, 1]$, then

$$R(t) = \tau \frac{k_1 F_1(t)}{E(0)} \sim E(t), \quad \tau \in (0, 1) \tag{4.27}$$

And

$$R'(t) \leq -\tau k_0 H_2(R(t)), \quad k_0 \in (0, +\infty), t \geq t_1.$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t) \leq H_1^{-1}(\alpha_1 t + \alpha_2), \quad \alpha_1, \alpha_2 \in (0, +\infty), t \geq t_1.$$

Here $H_1(t) = \int_t^1 H_2^{-1}(s)ds$. From (4.27), for a positive constant α_3, we have

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \alpha_1, \alpha_2 \in (0, +\infty), t \geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0, 1]$ and due to properties of H_2, we have

$$\lim_{t \to 0} H_1(t) = +\infty.$$

Then

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \text{for all} \quad t \geq 0.$$

This completes the proof of Theorem 4.4.1. □

4.5 Concluding comments

The coupled systems of wave equations abound in the world. One reason is that nature is full of those physical phenomenos. Another reason is that systems are often used to model a large class of engineering sciences, where propagation and transmission of informations or material are involved.

1- It will be also interesting to consider, derived from (4.1), and study the questions of asymptotic behavior of the related coupled system

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \phi(x)A\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds\right) \\ = au_2 \ln|u_1|^k, \\ \left(|u_2'|^{l-2}u_2'\right)' + \phi(x)A\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds\right) \\ = au_1 \ln|u_2|^k, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (\mathcal{H}(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases}$$

where our weak coupling is given by the logarithmic nonlinearities terms for $a \neq 0, l, n \geq 2$ and A is a linear, selfadjoint operator in $L^2(\mathbb{R}^n)$.

2. Let us remark that, it is similar to study the question of existence and decay of solution of the same problem with the presence of weak-viscoelasticity in the form

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \phi(x)A\left(u_1 + \alpha_1(t)\int_0^t g_1(s)u_1(t-s,x)ds\right) \\ = au_2 \ln|u_1|^k, \\ \left(|u_2'|^{l-2}u_2'\right)' + \phi(x)A\left(u_2 + \alpha_2(t)\int_0^t g_2(s)u_2(t-s,x)ds\right) \\ = au_1 \ln|u_2|^k, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (\mathcal{H}(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases}$$

where we should need additional conditions on α as follows

$$1 - \alpha_i(t)\int_0^t g_i(t)dt \geq k_i > 0, \int_0^\infty g_i(t)dt < +\infty, \alpha_i(t) > 0,$$

$$\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)\xi(t)} = 0 \tag{4.28}$$

where

$$\alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\} \quad \forall t \geq 0.$$

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