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Par
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Dedication

I dedicate this work to the spirit of my parents my mother and my father, may God have mercy on them,

and to make this work in the balance of their good deeds. And to all those who helped us in this work from a far or from close by, especially my family and friends.

Acknowledgement

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Abstract

This thesis deals with the study of a resonance problem generated by a second order nonlinear differential equation with multipoint integral boundary conditions. In fact we propose to establish the existence of the solution of a differential equation whose nonlinear term depends on the first derivative with nonlocal conditions of integral type by means of the concept of the theory of the degree of coincidence of Mawhin.

Keywords: Multipoint boundary value problem, Fredholm operator, Resonance, Mawhin coincidence degree theory, Existence of solution.

Résumé

On s'intéresse dans cette thése à l'étude d'un problème en résonance engendré par une équation différentielle non linéaire du second ordre avec des conditions aux limites de type multipoints et intégrales. Au fait nous nous proposons d'établir l'existence de la solution d'une équation différentielle dont le terme non linéaire dépend de la première dérivée avec des conditions non locales de type intégrale moyennant le concept de la théorie du degré de coincidence de Mawhin.

Mots clés: multipoint problème aux limites, Opérateur de Fredholm, Résonance, Théorie du degré de coincidence de Mawhin, Existence de la solution.

Notation

$\|.\|$ the norm.

$dist$ the distance associated with this norm.

Ω : an open bounded set.

$\overline{\Omega}$ the closure of Ω and $\partial\Omega$ its boundary.

$B(x_0, r)$ the open ball of center x_0 and radius r.

$u'(t)$ the derivative with respect to t.

$\oplus$ direct sum.

$\langle , \rangle$ scalar product.

$\mathbb{R}$ the set of real numbers.

(M, d) metric space.

$d(.,.)$: distance maps.

$C([a.b])$: the space of continuous functions.

max: maximum.

$\overline{B}$: the closed unit ball.

$(E, \|\|)$ Banach space.

A: operator.

$D(L)$: domaine of definition of L

$kerL$: kernel of L.

$\operatorname{Im} L$: image of L.

$dim(F)$: dimension of F.

$codim(F)$: codimension of F.

deg : Brouwer degree.

deg_{LS} : Leray-Schuder degree.

CHAPTER 1

Introduction

Boundary value problems model many phenomena in applied sciences and engineering, that's why their study is an important research area despite it's difficult as long as there is no general method to apply, and then to discuss the qualitative and quantitative properties of solutions for a given boundary value problem, different methods are used such the upper and lower solutions method, Mawhin theory, numerical methods... [25],

The objective of this thesis is the study of the existence of solutions of boundary value problems generated by a class of second order nonlinear differential equations with boundary conditions of integral and multipoint type, by applying the degree of coincidence of Mawhin.

The Mawhin theory permits the use of an approach of topological degree type to problems which can be written as an abstract operator equation of the form $Lx = Nx$, where L is a linear noninvertible operator and N is a nonlinear operator acting on a given Banach space.

In 1972, Mawhin has developed a method to solve this equation in his famous paper "Topological degree and boundary value problems for nonlinear differential equations" [40], he assumed that L is a Fredholm operator of index zero. Hence he has developed a new theory of topological degree known as the degree of coincidence for (L, N), that is also known as Mahwin's coincidence degree theory in honor of him.

A boundary value problem is said to be at resonance if the corresponding linear homogenous problem has nontrivial solution, otherwise it's said to be at resonance.

Many authors studied ordinary boundary value problems at resonance using Mawhin coincidence degree theory, we can cite Feng and Webb [11], Guezane-Lakoud and Frioui [14], Mahin and Ward [42], Infante [26], and the references [4,10,12,15,18-20,30-34,40,42,46].

Let us consider the following second order differential equation:

$$u''(t) = f(t, u(t), u'(t)) + e(t),\ t \in (0, 1), \tag{1.1}$$

jointly with the multipoint boundary conditions of type

$$u(0) = 0,\ u(1) = \alpha u(\xi), \tag{1.2}$$

or

$$u'(0) = 0,\ u(1) = \alpha u(\xi) \tag{1.3}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $e \in L^1(0, 1)$, $\xi \in (0, 1)$, $\alpha \in \mathbb{R}$.

In [24], Gupta, Ntouyas and Tsamatos used the Leray-Schauder continuation theorem to prove the existence of solutions of problem (1.1),(1.2) in the case $\alpha \neq 1$ and for the problem (1.1),(1.3), under the assumption $\alpha < \frac{1}{\xi}$, in both cases the problems are at nonresonance.

Later Feng and Webb in [11], considered the problems (1.1),(1.2) and (1.1),(1.3) in the resonance case that is when $\alpha = 1$ for problem (1.1),(1.2) and $\alpha = \frac{1}{\xi}$ for problem (1.1),(1.3). The authors proved by using Mawhin coincidence degree theory, that these problems have at least one solution.

In [26], Infante and Zima studied the existence of positive solutions for the following boundary value problem at resonance

$$u''(t) = f(t, u(t)),\ t \in (0, 1),$$

$$u'(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Caratheodory's conditions, $\alpha_i > 0$, $i = 1, 2, ..., m-2$, $0 < \xi_1 \leq \xi_2 \leq ... \leq \xi_{m-2} < 1$, $\sum\limits_{i=1}^{m-2} \alpha_i = 1$. Their approach is based on the Leggett-Williams norm-type theorem for coincidences obtained by O'Regan and Zima, see [47].

Recently in [14], by the help of Mawhin coincidence degree, Guezane-Lakoud and Frioui, proved the existence of solutions for a third order multipoint boundary value problem at resonance of the form

$$u'''(t) = f(t, u(t), u'(t)),\ t \in (0, 1),$$

$$u(0) = u''(0) = 0,\ u(1) = \frac{2}{\eta^2} \int_0^{\eta} u(t)dt,$$

where f is Caratheodory's function and $0 < \eta < 1$.

Ordinary differential equations with multipoint boundary conditions occur naturally arise in some applications such in population dynamics model, in semiconductor problems, thermal conduction problems, hydrodynamic problems..., see [7, 8, 27, 50]. For example, if a dynamic system has m degrees of freedom, then we have exactly m states observed at m different times, and consequently we obtain an m-point boundary value problem. The discretization of some boundary value problems for partial differential equations on irregular domains with the line method leads to multipoint boundary value problems.

The aim of this thesis is the study of a resonance boundary value problem generated by a second order nonlinear differential equation with multipoint and integral boundary conditions:

$$u''(t) = f(t, u(t), u'(t)),\ t \in (0, 1),$$

$$u(0) = 0,\ u(1) = \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} u(t)dt$$

where f is a Caratheodory's function, $0 < \eta_k < 1$, $\lambda_k > 0$, $k = 0, ..., m$, $\sum_{k=1}^{m} \lambda_k \eta_k^2 = 2$. We apply the Mawhin coincidence degree to prove the existence of at least one solution in a Banach space that we will define later.

Differential equations with nonlocal conditions, especially integral conditions, plays an important role in both theory and applications. The study of these problems is motivated by various applications, including thermoelasticity, chemical engineering, plasma physics.....

Let us give a brief outline of each chapter of the thesis.

The first chapter presents a review of some fixed point theorems, notably the Banach contraction principle, Leray-Schauder's nonlinear alternative, notion of homotopy, Fredholm's operators of zero index, the concept of the topological degree and its properties is discussed; two degrees are defined: the degree of Brouwer in finite dimension then the degree of Leray-Schauder in infinite dimension. We cite the theorem of Mawhin coincidence degree and its proof.

In the second chapter, we collect interesting results for some classes of second-order boundary value problems at resonance. For this purpose we will summarize the basic results in the literature and present the main ideas of some research on these problems.

The third chapter is devoted to the study of a second order boundary value problem at resonance. In fact we propose to establish the existence of the solution of a differential equation with multipoint conditions of integral type. The proofs are based on Mahwin's theory of coincidence. The results of this chapter are published:

R. Khaldi, M. Kouidri, Solvability of multipoint value problems with integral condition at resonance, International Journal of Analysis and Applications, Vol 16, Number 3 (2018), 306–316.

The thesis is clotured by some interesting references.

CHAPTER 2

Preliminaries

In this chapter we provide the basic notions and results that will be used in the sequel. We will cite some fixed point theorems, such the Banach contraction principle, Leray-Schauder's nonlinear alternative. The notion of the topological degree is also treated in both cases the degree of Brouwer in finite dimension and the degree of Leray-Schauder in infinite dimension, then we will expose their properties. Finally, we give the theorem of Mawhin coincidence degree with the proof.

2.1 Fixed point theorems

The Banach contraction principle, established in 1922 by the Polish mathematician Stefan Banach, is one of the most significant results in analysis and is considered the main source of the metric fixed point theory. The important part of Banach's contraction is to stretch the existence, the uniqueness and the sequence of the successive approximation that converges to a solution of the problem. For more results we refer to [1, 3, 5, 6, 28, 30, 51].

Definition 1 *Let (X, d) be a metric space and let $f : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of f if $x = f(x)$.*

Definition 2 *f is called contraction if there exists a fixed constant $k < 1$ such that*

$$d(f(x), f(y)) \leq kd(x, y), \text{ for all } x, y \in X.$$

Theorem 3 *(Banach Contraction Principle) Let (X, d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.*

Proof. Let x and y be fixed points of f, then $d(x, y) = d(f(x), f(y)) \leq kd(x, y)$. Since $k < 1$, we get $x = y$, that the uniqueness holds.
Now, we will construct explicitly a sequence converging to the fixed point. Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $(x_n)_{n\in\mathbb{N}}$ in X by

$$x_n = f(x_{n-1}) = f^n(x_0), \quad \text{for all } n \geq 1.$$

Since f is a contraction, we get

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq kd(x_{n-1}, x_n), \quad \text{for any } n \geq 1.$$

Thus, we obtain

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \; for \; all \; n \geq 1.$$

Hence, for any $m > n$, we have

$$d(x_n, x_m) \leq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) d(x_0, x_1) \leq \frac{k^n}{1-k} d(x_0, x_1).$$

We deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space X, denote by $x \in X$ its limit. Since f is continuous, we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x).$$

■

Theorem 4 *(Brouwer Fixed Point Theorem) Let B be a closed ball in $\mathbb{R}^n$. Then, any continuous mapping $T : B \to B$ has at least one fixed point.*

Brouwer fixed point theorem is not true in infinite dimensional spaces. The first fixed point theorem in an infinite dimensional Banach space was given by Schauder in 1930.

Theorem 5 *(Leary-schauder Fixed Point Theorem) Let B be the closed unit ball of a Banach E and $f : B \to B$ compact, then f has a fixed point.*

Let X and Y be two normed vector spaces, Ω an open set of X. Let give the definitions of compact map and completely continuous map.

Definition 6 *A continuous mapping $T : \Omega \subset X \to Y$ is called compact if $T(\Omega)$ is relatively compact.*

Lemma 7 *A continuous mapping $T : \Omega \subset X \to Y$ is said to be completely continuous, if the image of any bounded subset B of Ω is relatively compact.*

Theorem 8 *(Ascoli-Arzela) Let $E = C([a, b])$ denotes the space of the continuous functions and $M \subset E$ such that*

1. M is equicontinuous,

2.M is uniformly bounded,

then M is relatively compact in E.

Proposition 9 *Any mapping bounded and of finite rank is completely continuous.*

Remark 10 *Any compact mapping is completely continuous (because for any bounded $B \subset \Omega$ we have $T(B) \subset T(\Omega)$). The converse is true if Ω is bounded.*

Lemma 11 *If $T : X \to Y$ is a linear mapping, with X and Y Banach spaces, for T to be compact it suffices that $T(B(0, 1))$ is precompact. If at least one spaces X or Y is of finite dimension, so T is compact if and only if T is too.*

The principle of continuation is to deform one map into an other simpler one for which we know the existence of a fixed point. This deformation known as homotopy verify certain conditions.

Definition 12 *Let X and Y be two topological spaces. We say that the two continuous applications $f, g : X \to Y$, are homotopic if there exists*

$$H : X \times [0,1] \to X$$

such that

$$H(x,0) = f(x) \quad and \quad H(x,1) = g(x).$$

Example 13 *Let $X = Y = \mathbb{R}^n$, we consider $c : \mathbb{R}^n \to \mathbb{R}^n$ the constant map $c(x) = 0$, and $i : \mathbb{R}^n \to \mathbb{R}^n$ the application $i(x) = x$. Let us show that c and i are homotopes. Let $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that: $H(x,t) = (1-t)c(x) + ti(x)$, we have*

$$H(x,0) = (1-0) \times 0 + 0 \times x = 0$$

and

$$H(x,1) = (1-1) \times 0 + 1 \times x = x$$

then

$$H(x,t) = tx, \quad H(x,0) = c(x), \quad H(x,1) = i(x).$$

Example 14 *Let $X = Y = \mathbb{R}^n - \{0\}$, let $p(x) = \frac{x}{\|x\|}$, and $i(x) = x$. We see that p and i are homotopes by taking:*

$$H : \mathbb{R}^n - \{0\} \times [0,1] \to \mathbb{R}^n - \{0\}$$

such that:

$$H(x,t) = (1-t)i(x) + tp(x),$$

we have

$$\begin{aligned} H(x,0) &= (1-0) \times x + 0 \times x \frac{x}{\|x\|} \\ &= xH(x,1) = (1-1) \times x + 1 \times \frac{x}{\|x\|} = \frac{x}{\|x\|} \end{aligned}$$

then

$$
\begin{aligned}
H(x,t) &= (1-t)x + t\frac{x}{\|x\|}, \\
H(x,0) &= i(x) \;\; and \;\; H(x,1) = p(x).
\end{aligned}
$$

Let (X,d) be a complete metric space, and U be an open subset of X.

Definition 15 *Let $F : U \to X$ and $G : U \to X$ be two contractions; here U denotes the closure of U in X. We say that F and G are homotopic if there exists $H : U \times [0,1] \to X$ with the following properties:*

(a) $H(\cdot,0) = g$ and $H(\cdot,1) = f$;

(b) $x = H(x,t)$ for every $x \in \partial U$ and $t \in [0,1]$ (here ∂U denotes the boundary of U in X);

(c) there exists α, $0 \leq \alpha < 1$, such that $d(H(x,t),H(y,t)) \leq \alpha d(x,y)$ for every $x,y \in \overline{u}$ and $t \in [0,1]$,

(d) there exists $M, M \geq 0$, such that $d(H(x,t),H(x,s)) \leq M|t-s|$ for every $x \in \overline{u}$ and $t,s \in [0,1]$.

Theorem 16 *Let (X,d)be a complete metric space and U an open subset of X. Suppose that $F : \overline{U} \to X$and $G : \overline{U} \to X$ are two homotopic contractive maps and G has a fixed point in U. Then F has a fixed point in U.*

Proof. Consider the set

$$A = \{\lambda \in [0,1] : x = H(x,\lambda) \;\, for\ some\ x \in U\},$$

where H is a homotopy between F and G as described in Definition15. Notice A is nonempty since G has a fixed point, that is, $0 \in A$. We will show that A is both open and closed in $[0,1]$ and hence by connectedness we have that $A = [0,1]$. As a result, F has a fixed point in U. A is closed in $[0,1]$, in fact let

$$\{\lambda_n\}_{n=1}^{\infty} \subseteq A \;\; with \;\; \lambda_n \to \lambda \in [0,1] \; as \; n \to \infty.$$

Since $\lambda_n \in A$ for $n = 1, 2, ...$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$. Also for $n, m \in \{1, 2, ...\}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\ &\leq d(H(x_n, \lambda_n), H(x_{n,}\lambda_m)) + d(H(x_{n,}\lambda m), H(x_m, \lambda_m)) \\ &\leq M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m), \end{aligned}$$

that is,

$$d(x_n, x_m) \leq \left(\frac{M}{1-\alpha}\right) |\lambda_n - \lambda_m|.$$

Since $\{\lambda_n\}$ is a Cauchy sequence then $\{x_n\}$is also a Cauchy sequence, and since X is complete there exists $x \in U$ with $\lim_{n\to\infty} x_n = x$. In addition, $x = H(x, \lambda)$, since

$$\begin{aligned} d(x_n, H(x, \lambda)) &= d(H(x_n, \lambda_n), H(x, \lambda)) \\ &\leq M|\lambda_n - \lambda| + \alpha d(x_n, x). \end{aligned}$$

Thus $\lambda \in A$ and A is closed in $[0, 1]$.

Now, we show that A is open in $[0, 1]$. Let $\lambda_0 \in A$, then there exists $x_0 \in U$ with $x_0 = H(x_0, \alpha_0)$. Fix $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{(1-\alpha)r}{M} \quad where \quad r < dist(x_0, \partial U),$$

and where

$$dist(x_0, \partial U) = \inf\{d(x_0, x) : x \in \partial U\}.$$

Fix $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Then for

$$\begin{aligned} &x \in B(x_0, r) = \{x : d(x, x_0) \leq r\}, d(x_0, H(x, \lambda)) \\ &\leq d(H(x_0, \lambda_0), H(x, \lambda_0)) + d(H(x, \lambda_0), H(x, \lambda)) \\ &\leq \alpha d(x_0, x) + M|\lambda - \lambda_0| \leq \alpha r + (1-\alpha)r = r. \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, $H(\cdot, \lambda) : B(x_0, r) \to B(x_0, r)$. Then we deduce that $H(\ \cdot, \lambda)$ has a fixed point in U. Hence $\lambda \in A$ for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and therefore A is open in $[0, 1]$. ∎

2.2 Topological degree

Let Ω be a bounded open of $\mathbb{R}^N$, $f : \overline{\Omega} \to \mathbb{R}^N$ a continuous function and $b \in \mathbb{R}^N$ such that

$$f(x) = b \tag{2.1}$$

We want to obtain a quantity that give us the number of zeros for the equation (2.1), this quantity should give us the exact number of zeros and should be invariant by small deformations of f. So that we prevent the zeros of f to leave the domain we will impose that $b \notin f(\partial\Omega)$.

2.2.1 Topological degree of Brouwer

Definition 17 *Let Ω be an open bounded $\mathbb{R}^N$, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and b a regular value of f such that $b \notin f(\partial\Omega)$. Then the degree $deg(f, \Omega, b)$ is defined by*

$$deg(f, \Omega, b) = \sum_{x \in f^{-1}(b)} \mathrm{Si}\, gn J_f(x)$$

where $J_f(x)$ is the Jacobi matrix of f in b.

Properties of Brouwer degree

The degree satisfies the following properties:

1) If $deg(f, \Omega, b) = 0$, then the equation $f(x) = b$ has at least one solution in Ω.

2) Normalisation: $deg(id, \Omega, b) = 1$, for all $b \in \Omega$, and $deg(id, \Omega, b) = 0$, for all $b \in \mathbb{R}^N / \overline{\Omega}$, where I is the identity on $\overline{\Omega}$.

3) Additivity $deg(f, \Omega, b) = deg(f, \Omega_1, b) + deg(f, \Omega_2, b)$,if Ω_1, Ω_2 are disjoint in $\Omega = \Omega_1 \cup \Omega_2$,and $b \notin f(\overline{\Omega} / (\Omega_1 \cup \Omega_2))$.

4) Homotopy invariance: If f and g are homotopy equivalent via a homotopy $H(t,.)$ such that $H(t,0) = f$, $H(t,1) = g$, $b \notin H(t, \partial\Omega)$ then $deg(f, \Omega, b) = deg(g, \Omega, b)$.

Proposition 18 *Let Ω be a bounded open set of $\mathbb{R}^N$ and two functions $f, g \in C(\overline{\Omega}, \mathbb{R}^N)$, Assume that $f = g$ on $\partial\Omega$ and that $b \notin f(\partial\Omega)$, So we have*

$$deg(f, \Omega, b) = deg(g, \Omega, b).$$

Proof. Just use homotopy invariance of the topological degree, considering the homotopy $H(x,t) = tf(x) + (1-t)g(x)$. for all $t \in [0,1]$ we have that

$$deg(H(.,t), \Omega, b) = deg(H(.,0), \Omega, b).$$

and the result follows. For $t = 1$

$$deg(f, \Omega, b) = deg(g, \Omega, b)$$

■

Lemma 19 *(Sard) Let Ω be a bounded open and $f \in C^1(\mathbb{R}^N)$ and*

$$S = \{x \in \Omega, J_f(x) = 0\},$$

the set of singular points of f. Then $f(S)$ is of zero measure.

2.2.2 Topological degree of Leray-Schauder

We will now present a degree having the same role as the degree of Brouwer, but in infinite dimension, ie a tool which makes it possible to ensure that an equation of the form $f(x) = y$, where f is continuous of a Banach E in itself, have at least one solution x. The degree of Leray-Schauder, is built on the mapping which differ from the identity by a compact mapping, ie. the degree of Leray-Schauder is defined for applications that are compact perturbations of the identity of the type $I-T$ where T is compact and I.

Lemma 20 *Let Ω an open bounded set of a Banach spaceX. If $T:\overline{\Omega}\to X$ is a compact operator then for any $\varepsilon>0$ there exist E_ε a subspace of finite dimension and a continuous application $T_\varepsilon:\overline{\Omega}\to E$ such $\|T_\varepsilon u-Tu\|<\varepsilon$, for all $u\in\overline{\Omega}$.*

Definition 21 *Let X be a Banach space, Ω a bounded open set of X, $T:\overline{\Omega}\to X$ a compact operator such that $z\notin(I-T)\,\partial\Omega$. We define the topological degree of Leray-Schauder by*

$$deg_{LS}(I-T,\Omega,z)=deg(I-T_\varepsilon,\Omega\cap E_\varepsilon,z).$$

Remark 22 *In the previous definition $deg_{LS}(I-T,\Omega,z)$ depends only on T and Ω.*

The Leray–Schauder degree conserves the basic properties of Brouwer degree.

Theorem 23 *The Leray-Schauder degree has the following properties*

1) Additivity: $deg_{LS}(I-T,\Omega,z)=deg_{LS}(I-T,\Omega_1,z)+deg_{LS}(I-T,\Omega_2,z)$,if Ω_1, Ω_2 are disjoint in $\Omega=\Omega_1\cup\Omega_2$,and $z\notin(I-T)(\partial\Omega_1)\cup(I-T)(\partial\Omega_2)$

2)Existence: If $deg_{LS}(I-T,\Omega,z)\neq 0$, then $z\in(I-T)(\Omega)$.

3) Homotopy invariance: Let $H(t,.)$ such that $H:[0,1]\times\overline{\Omega}$ a compact homotopy such that $z\notin(I-H(t,.))(\partial\Omega)$,then $deg_{LS}(I-H(t,.),\Omega,z)$ is independent of t.

2.3 Mahwin's coincidence degree theory

In 1970, Gaines and Mawhin introduced the theory of the degree of coincidence in the analysis of functional and differential equations. Mawhin has made important contributions since then, and this theory is also known as Mahwin's theory of coincidence. Coincidence theory is considered to be the very powerful technique, especially with regard to questions about the existence of solutions in nonlinear differential equations. Furthermore, many researchers have used it to solve boundary value problems at resonance, see [37, 38, 39, 40, 41, 43, 47].

Let us define the direct sums, projections and topological complement.

Definition 24 *Let E and F be two closed subspaces of a normed vector $\mathbb{R}-spaceX$. We say that E is a topological complement of F if X is the direct sum of F and E (i.e. $X = F \oplus E$).*

Definition 25 *Let X be a vector space. We say that a linear operator $P : X \to X$ is a projection if for all $x \in X$, we have $P(P(x)) = P^2x = P(x)$*

Proposition 26 *Let X be a vector space. A linear operator $P : X \to X$is a projection if and only if $(I - P)$ is a projection. Moreover, if the space X is normed, then P is continuous if and only if $(I - P)$ is continuous.*

Proof. Let P be a projection. So for all $x \in X$

$$\begin{aligned}(I-P)^2(x) &= (I-P)((I-P)(x)) \\ &= I(I-P)(x) - P(I-P)(x) \\ &= I(x-Px) - P(x-Px) \\ &= x - p(x) - p(x) + p^2(x) \\ &= x - 2p(x) + p^2(x) \\ &= x - p(x) = (I-P)(x)\end{aligned}$$

Reciprocally, if $(I - P)$is a projection,$(I - (I - P)) = P$ is too. For the topological framework, as the identity is a continuous mapping and the sum of two continuous mappings is also continuous, then P is continuous if and only if $(I - P)$is. ■

Proposition 27 *If P is a projection in X, then $\ker P = Im(I - P)$ and $ImP = \ker(I - P)$.*

Proof. We prove that $\ker P = Im(I - P)$. If $x \in \ker P \Longrightarrow P(x) = 0$ then

$$(I-P)(x) = x - P(x) = x \Longrightarrow x \in Im(I-P)$$

which implies

$$KerP \subset Im(I-P)$$

Next, if $x \in Im(I-P)$, then

$$\begin{aligned} P((I-P)(x)) &= P(x) - P^2(x) \\ &= P(x) - P(x) = 0 \Longrightarrow (I-P)x \in Ker(P) \end{aligned}$$

hence $Im(I-P) \subset \ker P$ and then

$$KerP = Im(I-P)$$

■

Remark 28 *A topological space X is separated (or Hausdorff) if $\forall x, y \in X : x \neq y$, $\exists Vx$, Vy open as $Vx \cap Vy = \phi$.*

Lemma 29 *The image of any continuous projection in a Hausdorff space is closed. In particular, the images of continuous projections of the Banach spaces are closed.*

Theorem 30 *If P is a continuous projection in a topological vector space of Hausdorff X, then X is the direct sum of ImP and $\ker P$, (ie $X = ImP \oplus \ker P$).*

Definition 31 *If the quotient space X/F is of finite dimension, we say that the subspace closed vector F of X is of finite codimension in X and we write*

$$co\dim(F) = dim(X/F).$$

Proposition 32 *$co\dim(F) = n < \infty$ if, and only if there is a vector subspace closed E of X, as*

$$X = F \oplus E \text{ and } \dim(E) = n$$

2.3.1 Fredholm operators

Definition 33 *Let X and Y be two normed vector spaces; we say that a linear mapping $L : D(L) \subset X \to Y$, is from Fredholm if it satisfies the following conditions*

1. $\ker(L) = L^{-1}(0)$ *is of finite dimension.*

2. $Im(L) = L(D(L))$*is closed and of finite codimension.*

Definition 34 *The index of a Fredholm operator L is the integer*

$$ind(L) = \dim(\ker(L)) - \dim co(Im(L)).$$

Examples.

1. If X and Y are of finite dimensions, then any linear mapping $L : X \to Y$ is from Fredholm with

$$ind(L) = \dim(X) - \dim(Y).$$

If X and Y are Banach spaces and $L : X \to Y$ is a linear mapping bijective, then L is a Fredholm operator of index 0, in fact

$$\dim(\ker(L)) = \dim co(Im(L) = 0.$$

2. The identity is a Fredholm operator of index 0.

Lemma 35 *If L is a Fredholm operator, u is a compact linear application; so $L+u$ is from Fredholm and*

$$ind(L+u) = ind(L).$$

In particular, any perturbation compact identity is a Fredholm index operator 0.

Proposition 36 *If L is a Fredholm operator of zero index, so L is surjective if and only if L is injective.*

2.3.2 Generalized inverse

Let $L : D(L) \subset X \to Y$ be a Fredholm operator of index 0. Let P and Q be two continuous projectors; $P : X \to X$ and $Q : Y \to Y$ such that

$$Im(P) = \ker L \quad \text{and} \quad \ker Q = Im(L).$$

Set

$$X_1 = Im(I-P) = \ker P \quad \text{and} \quad Y_1 = Im(Q),$$

so we can write

$$X = \ker L \oplus X_1, Y = Im(L) \oplus Y_1.$$

Consider an isomorphism,

$$J : \ker L \to Y_1$$

whose existence is ensured by the fact that $\dim \ker L = \dim Y_1 = n$. Note that

$$D(L) = \ker L \oplus (D(L) \cap X_1)$$

and that the restriction of L to $D(L) \cap X_1$ is an isomorphism on $Im(L)$. Denote by Lp this restriction and by $L_p^{-1} : Im(L) \to D(L) \cap X_1$ the inverse of L_p. So the operator

$$J^{-1} \oplus L_p^{-1} : Y = Y_1 \oplus R(L) \to X = \ker L \oplus D(L) \cap X_1,$$

is an isomorphism whose inverse is the operator,

$$L + JP : D(L) \cap Im(I-P) \oplus kerL \to R(L) \oplus Y_1$$

indeed, for every $x \in D(L) \cap Im(I-P) \oplus \ker L,$ we write it in the form $x = (I-P)x + Px$,so

$$(L+JP)((I-P)x + Px) = L(I-P)x + JP(Px) = L(I-P)x + JPx,$$

consequently

$$(J^{-1} \oplus L_p^{-1})(L(I-P)x + JPx) = (I-P)x + Px = x.$$

On the other hand, for all $y \in Y$ we have

$$\left(J^{-1} \oplus L_p^{-1}\right) y = \left(J^{-1} \oplus L_p^{-1}\right)(Qy + (I-Q)y) = J^{-1}Qy + L_p^{-1}(I-Q)y,$$

by setting $K_{P,Q} = L_p^{-1}(I-Q)$,($K_{P,Q}$ is the inverse on the right of L associated with P and Q respectively), then we get $(L+JP)^{-1} = J^{-1}Q + K_{P,Q}$.

2.3.3 Perturbations of a Fredholm operator of zero index L-compact

To solve the equation $Lx = y$, we can write $x = Px + (I-P)x$ and $y = Qy + (I-Q)y$ and by substitution of x and y in the previous equation, we obtain

$$L(Px + (I-P)x) = Qy + (I-Q)y,$$

and since $Qy = 0$ and $LPx = 0$ (because $y \in Im(L)$ and $Px \in \ker L$), then

$$L(I-P)x = (I-Q)y,$$

which leads to

$$x - Px = Lp^{-1}(I-Q)y$$

and thus

$$x = Px + J^{-1}Qy + L_p^{-1}(I-Q)y.$$

Now consider the equation $Lx = Nx$, where $N : G \subset X \to Y$ is an operator (usually nonlinear) according to the above result; this last equation with $x \in D(L) \cap G$ is equivalent to

$$\begin{aligned} x &= Px + J^{-1}QNx + K_{P,Q}P, \\ QNx &= Mx, \end{aligned}$$

which is a fixed point problem.

Definition 37 *Let X and Y be two Banach spaces and and $L : D(L) \subset X \to Y$ a Fredholm operator of index 0. Let Ω an open bounded set of X such that $D(L) \cap \Omega \neq 0$.*

1) The map $N : X \to Y$ is $L-$ compact on $\overline{\Omega}$ if only if the operator $QN\left(\overline{\Omega}\right)$ is bounded and $K_{P,Q}N\Omega : \overline{\Omega} \to X$ is compact.

2) The degree of coincidence of L and N on is defined by

$$deg[(L,N),\Omega] = deg_{LS}(I - M, \Omega, 0)$$

where $M = P + J^{-1}QN + K_{P,Q}N$.

2.3.4 Mawhin's theorem

Theorem 38 *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied.*

Theorem 39 *(i) $Lx \neq \lambda Nx$, for every $(x, \lambda) \in [(D(L) \backslash KerL) \cap \partial\Omega] \times (0,1)$.*

(ii) $Nx \notin ImL$, for every $x \in KerL \cap \partial\Omega$.

(iii) $\deg(JQN_{|\ker L}, \ker L \cap \partial\Omega, 0) \neq 0$, where $J :\ ImQ \to \ker L$ is a linear isomorphism, $Q : Y \to Y$ is a projection as above with $ImL = \ker Q$. Then, the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Proof. For $\lambda \in [0,1]$, consider the family of problems

$$x \in D(L) \cap \overline{\Omega},\ Lx = \lambda Nx + (1-\lambda)QNx \tag{2.2}$$

Let $M : [0,1] \times \overline{\Omega} \to Y$ be a homotopy defined by

$$M(\lambda, x) = Px + J^{-1}QNx + \lambda K_{P,Q}Nx$$

The problem (2.2) is equivalent to a fixed point problem

$$\begin{aligned} x &= Px + J^{-1}Q(\lambda N + (1-\lambda)QN)x + K_{P,Q}(\lambda N + (1-\lambda)QN)x \\ &= Px + \lambda J^{-1}QNx + (1-\lambda)J^{-1}QNx + \lambda K_{P,Q}Nx + (1-\lambda)K_{P,Q}QNx \\ &= M(\lambda, x). \end{aligned}$$

So this last equation is equivalent to a fixed point problem

$$x = M(\lambda, x), \quad x \in \overline{\Omega}, \tag{2.3}$$

If there exists an $x \in \partial\Omega$ such that $Lx = Nx$, then the proof is completed. Now suppose that

$$Lx \neq Nx \text{ for all } x \in D(L) \cap \Omega \tag{2.4}$$

and on the other hand

$$Lx \neq \lambda Nx + (1-\lambda)QNx \tag{2.5}$$

for all $(\lambda, x) \in]0,1[\times(D(L) \cap \Omega)$. If

$$Lx = Nx + (1-\lambda)QNx$$

for all $(\lambda, x) \in]0,1[\times(D(L) \cap \Omega)$, we obtain by application of Q to both members of the previous equality

$$QNx = 0, \quad Lx = \lambda Nx$$

The first of these equalities and the condition (ii) imply that $x \notin KerL \cap \partial\Omega$ i.e $x \in (D(L)\backslash KerL) \cap \partial\Omega$ and therefore the second equality contradicts (i). By using other times (ii), it follows that

$$Lx \neq QNx, \quad \text{for every} \quad x \in D(L) \cap \partial\Omega. \tag{2.6}$$

using (2.4), (2.5) and (2.6), we deduce that

$$x \neq M(\lambda, x) \text{ for all } (\lambda, x) \in [0,1] \times \partial\Omega \tag{2.7}$$

Since N is L-compact then $M(\lambda, x)$ is compact because. Using the homotopy invariance property of the Leray-Schauder degree, we obtain

$$deg_{LS}(I - M(0,.), \Omega, 0) = deg_{LS}(I - M(1,.), \Omega, 0) \tag{2.8}$$

On the other hand we have

$$deg_{LS}(I - M(0,\lambda), \Omega, 0) = deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) \tag{2.9}$$

Since the image of $P + J^{-1}QN$ is contained in $Ker(L)$, then using the property of reduction of the Leray-Schauder degree and the fact that $P\mid_{KerL} = I\mid_{KerL}$, (since $Ker(L) = Im(P) = Ker(I - P)$), we obtain

$$\begin{aligned} deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) &= deg(I - (P + J^{-1}QN), \Omega \cap KerL, 0) \\ &= deg(J^{-1}QN, \Omega \cap KerL, 0) \end{aligned} \tag{2.10}$$

Thanks to (2.8), (2.9) and (2.10), it follows that $deg_{LS}(I - M(1,.), \Omega, 0) \neq 0$, and so the existence property of the Leray-Schauder degree implies the existence of an $x \in \Omega$ such as $x = M(1, x)$ i.e $x \in D(L) \cap \Omega$, $Lx = Nx$. ■

CHAPTER 3

On some boundary value problems at resonance

3.1 Introduction

In this chapter we collect some interesting results for various classes of second boundary value problems at resonance. More precisely, we will summarize basic results in the literature related, and present the main ideas of some researches in resonant boundary value problems. we omit the corresponding proofs.

3.2 A Second order m-point boundary value problem at resonance

Gupta in [25], used Mawhin coincidence degree theory, to investigate the existence of solutions for the following two boundary value problem at resonance,

$$(P1)\begin{cases} u''(t) = f(t, u(t), u'(t)) + e\,(t)\,,\ t \in (0,1), \\ u(0) = 0,\ u'(1) = u'(\xi) \end{cases}$$

$$(P2)\begin{cases} u''(t) = f(t,u(t),u'(t)) + e\,(t)\,,\ t \in (0,1), \\ u(0) = 0,\ u'(1) = \sum\limits_{i=1}^{m-2} \alpha_i u'(\xi_i) \end{cases}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying Caratheodory's conditions, $0 < \xi < 1$, $\alpha_i > 0$, $i = 1, 2, ..., m-2$, $0 < \xi_1 \leq \xi_2 \leq ... \leq \xi_{m-2} < 1$, $\sum\limits_{i=1}^{m-2} \alpha_i = 1$, $e\,(t) \in L^1(0,1)$.

Denote by X and Y the Banach spaces $C^1[0,1]$ and $L^1[0,1]$ respectively, with their usual norms. Let Y_2 be the subspace of Y defined as

$$Y_2 = \{y(t) \in Y, y(t) = At, \quad t \in [0,1], A \in \mathbb{R}\}$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. Define the canonical projection operators

$$\begin{aligned} Q &: Y \to Y_2 \\ Qu &= \frac{2t}{\sum_{i=1}^{m-2} \alpha_i \left(1 - \xi_i^2\right)} \left[\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{1} u\,(s)\,ds \right] \end{aligned}$$

$$\begin{aligned} P &: X \to X \\ Px &= x\,(t) - \frac{2t}{\sum_{i=1}^{m-2} \alpha_i \left(1 - \xi_i^2\right)} \left[\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{1} u\,(s)\,ds \right] \end{aligned}$$

Define the operator

$$\begin{aligned} Lu &= u'', \\ D(L) &= \left\{ u \in W^{2,1}\,(0,1)\,, u(0) = 0,\ u'(1) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i) \right\}. \end{aligned}$$

Define $N : X \to Y$, $Nu = f(t,u(t),u'(t)) + e\,(t)$. Then the boundary value problem (P2) can be written as $Lu = Nu$.

Under the condition $\sum\limits_{i=1}^{m-2} \alpha_i = 1$, the problem (P1) is at resonance, in this case the operator L is not invertible and then the Leray-Schauder continuation theory cannot be used. The existence results are proved by means of the coincidence degree theory of Mawhin under a growth condition on the nonlinear term f.

Theorem 40 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t) \in L^1(0,1)$ such that*

$$|f(t, x_1, x_2)| < p(t)\,|x_1| + q(t)\,|x_2| + r(t)$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in \mathbb{R}^2$. Assume that for every $x \in X$

$$(Qx)(t).(QNx)(t) \geq 0, \;\; for\; t \in [0,1],$$

then for $e \in Y_1$, i.e $e \in L^1(0,1)$ and $\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 e\,(s)\,ds = 0$, the boundary value problem (P2) has at least one solution in $C^1[0,1]$ provided

$$\|p\|_1 + \|q\|_1 < 1.$$

Remark 41 *Theorem 40 remains valid if the condition*

$$(Qx)(t).(QNx)(t) \geq 0, \;\; for\; t \in [0,1],$$

is replaced by

$$(Qx)(t).(QNx)(t) \leq 0, \;\; for\; t \in [0,1],$$

or by

$$f(t, x_1, x_2)x_1 > 0, for\ almost\ all\ t \in (\xi_1, 1) \;\; and\ all\;\; (x_1, x_2) \in \mathbb{R}^2.$$

The existence of solution for problem (P1) is given in the following theorem.

Theorem 42 *Let f be a function as in Theorem 40, then for $e\,(t) \in L^2(0,1)$ and $\int_\eta^1 e\,(s)\,ds = 0$, the boundary value problem (P1) has at least one solution in $C^1[0,1]$*

provided

$$\|p\|_1 + \|q\|_1 < 1.$$

3.3 Existence results for a multipoint boundary value problem at resonance

In [48], Przeradzki and Stanczy, applied coincidence degree theory of Mawhin, to prove the existence results for the following boundary value problem at resonance,

$$(P3)\begin{cases} u''(t) = f(t, u(t), u'(t)),\ t \in (0,1), \\ u(0) = 0,\ u(1) = \sum\limits_{i=1}^{m} \alpha_i u(\xi_i) \end{cases}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, $\alpha_i > 0$, $i = 1, 2, ..., m$, $0 < \xi_1 \leq \xi_2 \leq ... \leq \xi_m < 1$, $\sum\limits_{i=1}^{m-2} \alpha_i \xi_i = 1$.

Denote X and Y the Banach spaces $X = C^1[0,1]$ and $Y = L^1[0,1]$ with their usual norms. Define the operator

$$\begin{aligned} Lu &= u'', \\ D(L) &= \left\{ u \in C^2[0,1], u(0) = 0,\ u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i) \right\}. \end{aligned}$$

Under the condition $\sum\limits_{i=1}^{m-2} \alpha_i \xi_i = 1$, the problem (P1) is at resonance, and the operator L is not invertible. The authors proved that $L : D(L) \subset X \to Y$ is a Fredholm operator of index 0, and $KerL = \{at, a \in \mathbb{R}\}$. Define $N : X \to Z$, $Nu = f(t, u(t), u'(t))$, then the boundary value problem (P2) can be written as $Lu = Nu$. The linear projection Q is

$$Qy = t\left(\int_0^{\xi_i} (1-r)\, y(r)\, dr - \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - r)\, y(r)\, dr \right),\ y \in Y$$

For problem (P3), some conditions are imposed on the nonlinear term f to ensure the existence of solution.

Theorem 43 *Suppose that there exist two continuous functions c, $d : \mathbb{R} \to \mathbb{R}$, such that*

$$|f(t,x,y)| \leq c(x) + d(x) y^2,$$

for any $t \in [0,1]$ and $x, y \in \mathbb{R}$ and there exists a positive number a_0 such that

$$af(t, at, sgn(a) y) \geq 0$$

for any $t \in [0,1]$ and $|a| \geq y \geq a_0$. Then the multipoint boundary value problem (P3) has at least one solution in $C^2[0,1]$.

In the second part, the authors investigated the multidimensional case, that is $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, and proved by Mawhin coincidence degree theory the following results

Theorem 44 *Assume that f is continuous and the conditions*

$$|f_i(t,x,y)| \leq b_i(x) + c_i(x) y_i^2 + d_i(x, y_1, ...y_{i-1})$$

are satisfied for $i = 1,...n$, $t \in [0,1]$, $x, y \in \mathbb{R}^n$, where b_i, c_i, d_i are continuous nonnegative functions and there exists a positive number a_0 such that

$$a_i f_i(t, at, y) \geq 0, \ \ i = 1, ...n$$

for any $t \in [0,1]$, where $y = (y_1, ..., y_n)$, $a = (a_1, ..., a_n)$ is such that $|a_i| = \max_j |a_j|$ and $|a_i| \geq sgn(a_i) y_i \geq a_0$. Then the multipoint boundary value problem (P3) has at least one solution.

Remark 45 *The assumption imposed on the nonlinear term f is weaker, since the authors allow quadratic growth of the derivative of f and no growth condition is imposed with respect to the function f.*

3.4 On multipoint boundary value problem at resonance

In [34], Liu and Yu considered the following second-order ordinary differential equation

$$u''(t) = f(t, u(t), u'(t)) + e\,(t)\,,\ t \in (0,1), \tag{3.1}$$

subject to one of the following boundary value conditions:

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u(1) = \beta u\,(\eta)\,, \tag{3.2}$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = \beta u'\,(\eta)\,, \tag{3.3}$$

$$u'(0) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i), \quad u(1) = \beta u\,(\eta)\,, \tag{3.4}$$

$$u'(0) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i), \quad u'(1) = \beta u'\,(\eta)\,, \tag{3.5}$$

$\alpha_i, \beta \in \mathbb{R}$, $i = 1, 2, ..., m-2$, $0 < \xi_1 \leq \xi_2 \leq ... \leq \xi_{m-2} < 1$, $0 < \eta < 1$.

Denote X, Y be the Banach spaces $X = C^1[0,1]$ and $Y = L^1[0,1]$ with their usual norms. Define the operator

$$\begin{aligned} Lu &= u'', \\ D(L) &= \left\{ u \in W^{2,1}(0,1)\,, u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u(1) = \beta u\,(\eta) \right\}. \end{aligned}$$

Define $N : X \to Z$, $Nu = f(t, u(t), u'(t)) + e\,(t)\,$, Then the boundary value problem (3.1)-(3.2) can be written as $Lu = Nu$.

The authors proved that if $\beta = 1$, $\sum_{i=1}^{m-2} \alpha_i = 1, \sum_{i=1}^{m-2} \alpha_i \xi_i = 0$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 \neq 0$, then the problem (3.1)-(3.2) is at resonance, and the operator L is not invertible, moreover the operator $L : D(L) \subset X \rightarrow Y$ is a Fredholm operator of index 0 and

$$KerL = \{x \in D(L), x = d \in \mathbb{R}\},$$

$$\operatorname{Im} L = \left\{ y \in Y, \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s y(r)\, drds = 0 \right\}.$$

The linear projections Q and P are

$$\begin{aligned} Qy &= \frac{2}{\sum_{i=1}^{m-2} \alpha_i \xi_i^2} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s y(r)\, drds, \quad y \in Y \\ Px &= x(0), \quad x \in X \end{aligned}$$

the generalized inverse of L, is $K_{P,Q} : \operatorname{Im} L \rightarrow D(L) \cap KerP$

$$K_P y = \frac{t}{1-\eta} \int_\eta^1 \int_0^s y(r)\, drds + \int_0^t \int_0^s y(r)\, drds$$

The authors studied the existence of solution for boundary value problems (3.1)-(3.2), (3.1)-(3.3), (3.1)-(3.4) and (3.1)-(3.5) at resonance cases, and established some existence theorems under nonlinear growth restriction on f, by the help of the coincidence degree theory of Mawhin

Theorem 46 *Let $f : [0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function. Assume that*

(A1) There exist functions a, b, c, r in $L^1(0,1)$ and a constant $\theta \in [0,1)$ such that for all $x, y \in \mathbb{R}$, $t \in [0,1]$, either

$$|f(t,x,y)| < a(t)\,|x| + b(t)\,|y| + c(t)\,|y|^\theta + d(t)$$

or else

$$|f(t,x,y)| < a(t)\,|x| + b(t)\,|y| + c(t)\,|x|^\theta + d(t).$$

(A2) There exists constant $M > 0$ such that, for $x \in D(L)$, if $|x(t)| > M$ for all

$t \in [0,1]$, *then*

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \left[\int_0^s f\left(r, x\left(r\right), x'\left(r\right)\right) + e\left(r\right) dr \right] ds \neq 0$$

(A3) There exists constant $M^* > 0$ *such that for any* $d \in \mathbb{R}$*, if* $|d| > M^*$*, then either*

$$d. \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \left[\int_0^s f\left(r, d, 0\right) + e\left(r\right) dr \right] ds < 0,$$

or else

$$d. \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \left[\int_0^s f\left(r, d, 0\right) + e\left(r\right) dr \right] ds > 0.$$

Then, for every $e\left(t\right) \in L^1(0,1), \beta = 1, \sum_{i=1}^{m-2} \alpha_i = 1, \sum_{i=1}^{m-2} \alpha_i \xi_i = 0, \sum_{i=1}^{m-2} \alpha_i \xi_i^2 \neq 0$, *the boundary value problem (3.1)-(3.2) has at least one solution in* $C^1[0,1]$ *provided*

$$\|a\|_1 + \|b\|_1 < \frac{1}{3}.$$

Remark 47 *The authors imposed some assumptions on the nonlinear term* f *to prove the existence of solution for problems (3.1)-(3.3), (3.1)-(3.4) and (3.1)-(3.5), we refer to the paper [34] for further existence results.*

CHAPTER 4

Nonlocal Boundary Value Problems at Resonance

In this chapter, we study a boundary value problem at resonance with a multi integral boundary conditions. By constructing suitable operators, we establish an existence theorem upon the coincidence degree theory of Mawhin. The results of this chapter are published:

R. Khaldi, M. Kouidri, Solvability of multipoint value problems with integral condition at resonance, International Journal of Analysis and Applications, V16, Number 3 (2018), 306–316.

4.1 Introduction

Boundary value problem involves ordinary differential equation with non local condition appears in physical science and applied mathematics. Moreover the theory of boundary value problems with integral condition is found in different areas like applied mathematics and applied physics for example plasma physics, heat conduction, themo-elasticity, underground water flew. In recent years, the boundary value problem at resonance for ordinary differential equations have been extensively studied and many results have been obtained, we refer to [4,10,12,15,18-20,30-34,40,42,46] and the references therein. Moreover, lots of works on multipoint boundary value problems have appeared, for examples, see [10, 20, 22, 24, 26, 29, 33, 35, 45, 48].

The goal of this chapter is to provide sufficient conditions that ensure the existence of solutions for the following multipoint boundary value problem:

$$x''(t) = f(t, x(t), x'), t \in (0,1) \tag{4.1}$$

$$x(0) = 0, x(1) = \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} x(t)dt, \eta_k \in (0,1), \sum_{k=1}^{m} \lambda_k \eta_k^2 = 2 \tag{4.2}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is caratheodary function, and $\eta_k \in (0,1)$.

In the present work, if $\sum\limits_{k=1}^{m} \lambda_k \eta_k^2 = 2$, then, BVP (4.1)-(4.2) is at resonance, since equation

$$x''(t) = 0, t \in (0,1)$$

with boundary condition (4.2) has nontrivial solutions $x = ct$, $c \in \mathbb{R}$, $t \in [0,1]$.

We recall the Mawhin theorem of coincidence degree:

Theorem 48 *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied.*
(i) $Lx \neq \lambda Nx$, for every $(x, \lambda) \in [(D(L) \backslash KerL) \cap \partial\Omega] \times (0,1)$.
(ii) $Nx \notin ImL$, for every $x \in KerL \cap \partial\Omega$.
(iii) $\deg(JQN_{|\ker L}, \ker L \cap \partial\Omega, 0) \neq 0$, where $J : ImQ \to \ker L$ is a linear isomorphism, $Q : Y \to Y$ is a projection as above with $ImL = \ker Q$. Then, the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

In the following, we shall use the classical spaces $X = C[0,1]$ and $Y = L^1[0,1]$ equipped respectively withe the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, $\|x\|_\infty = \max\limits_{t\in[0,1]} |x(t)|$, and $\|y\|_1 = \int_0^1 |y(t)|\, dt$.

We will use the space $AC^2[a,b] = \{x \in C^1[a,b], x' \in AC[a,b]\}$,where $AC[a,b]$is the space of absolutely continuous functions on $[a,b]$.

4.2 Existence of Solutions

Define the operator $L : D(L) \subset X \to Y$ by $Lx = x''$, where $X = C^1[0,1]$, $Y = L^1[0,1]$,

$$\begin{aligned} D(L) &= \{x \in W^{2,1}(0,1) : x(0) = 0, \\ x(1) &= \sum_{k=1}^{m} \lambda_k \int\limits_0^{\eta_k} x(t)dt, \eta_k \in (0,1), \sum_{k=1}^{m} \lambda_k \eta_k^2 = 2\}. \end{aligned}$$

Let $N : X \to Y$ be the operator

$$Nx = f(t, x(t), x'(t)), t \in (0,1).$$

Then, the boundary value problem (4.1),(4.2) can be written as $Lx = Nx$.

We need the following Lemma.

Lemma 49 *(i)* $\ker L = \{x \in D(L) : x = ct, c \in \mathbb{R}, t \in [0,1]\}$,

(ii) $\operatorname{Im} L = \{y \in Y : \int_0^1 (1-s)y(s)ds - \frac{1}{2}\sum\limits_{k=1}^{m} \lambda_k \int_0^{\eta} (\eta_k - s)^2 y(s)ds = 0\}$,

(iii) $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero, and the linear continuous projector operator $Q : Y \to Y$ can be defined as $Qy = k.(Ry).t$ such that $k_0 = 6\left(\sum\limits_{i=1}^{m} \lambda_i \eta_i^4\right)^{-1}$.

$$Ry = \int\limits_0^1 (1-s)y(s)ds - \frac{1}{2}\sum_{i=1}^{m} \lambda_i \int\limits_0^{\eta_i} (\eta_i - s)^2 y(s)ds$$

(iv) The linear operator $Kp : \operatorname{Im} L \to D(L) \cap \ker P$ can be written as

$$K_p y = \int\limits_0^t (t-s)y(s)ds$$

(v) for all $y \in ImL$, we have

$$\|K_p y\| < \|y\|_1 . \tag{4.3}$$

Proof. (i) For $\forall x \in \ker L$, we have $x'' = 0$. Then, we obtain

$$x(t) = a + bt$$

where $a, b \in \mathbb{R}$. From $x(0) = 0$, we have $a = 0$. Again, from

$$x(1) = \sum_{i=1}^{m} \lambda_i \int_0^{\eta i} x(t)dt,$$

one has

$$\ker L = \{x \in D(L) : x = ct, c \in \mathbb{R}, t \in [0, 1]\}$$

(ii) to prove that

$$ImL = \{y \in Y : \int_0^1 (1-s)y(s)ds - \frac{1}{2}\sum_{i=1}^{m} \lambda_i \int_0^{\eta_i} (\eta_i - s)^2 y(s)ds = 0\} \tag{4.4}$$

we show that, the linear equation

$$x'' = y \tag{4.5}$$

has a solution $x(t)$ satisfied

$$x(0) = 0, x(1) = \sum_{i=1}^{m} \lambda_i \int_0^{\eta_i} x(t)dt, \eta_i \in (0, 1), \sum_{i=1}^{m} \lambda_i \eta_i^2 = 2$$

if and only if

$$\int_0^1 (1-s)y(s)ds - \frac{1}{2}\sum_{i=1}^{m} \lambda_i \int_0^{\eta_i} (\eta_i - s)^2 y(s)ds = 0 \tag{4.6}$$

In fact, by integrating equation (4.5) and taking into account that $x(0) = 0$, we get

$$x(t) = x(0) + x'(0)t + \int_0^t (t-s)y(s)ds$$

again from

$$x(1) = \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} x(t)dt,$$

we obtain

$$\begin{aligned}
x(1) = x'(0) + \int_0^1 (1-s)y(s)ds &= \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} \left[x'(0)t + \int_0^t (t-s)y(s)ds \right] dt \\
&= \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} \left[x'(0)t + \int_0^t (t-s)y(s)ds \right] dt \\
&= \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} x'(0)tdt + \int_0^{\eta_k} \int_0^t (t-s)y(s)dsdt \right] \\
&= \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} x'(0)tdt \right] + \left[\int_0^{\eta_k} \int_0^t (t-s)y(s)dsdt \right] \\
&= \sum_{k=1}^{m} \lambda_k \left[x'(0) \int_0^{\eta_k} tdt \right] + \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} \int_0^t (t-s)y(s)dsdt \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{m} \lambda_k x'(0) \left[\frac{1}{2} t^2 \right]_0^{\eta_k} + \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} \int_0^t (t-s)y(s)dsdt \right] \\
&= \frac{1}{2} x'(0) \left(\sum_{k=1}^{m} \lambda_k \eta_k^2 \right) + \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} \int_0^t (t-s)y(s)dsdt \right]
\end{aligned}$$

$$= \frac{1}{2}(2)\,x'(0) + \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} \int_0^{t} (t-s)y(s)dsdt \right]$$
$$= x'(0) + \sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} \int_0^{t} (t-s)y(s)dsdt \right]$$

by change of order of integration, it yields

$$x(1) = x'(0) + \frac{1}{2}\sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \right]$$

which implies

$$\int_0^1 (1-s)y(s)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \right] = 0$$

(iii) For $y \in Y$, we take the projector Qy as

$$Qy = k_0. \left(\int_0^1 (1-s)y(s)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \right] \right) .t$$

Let $y_1 = y - Qy$, we obtain

$$\int_0^1 (1-s)y_1(s)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)y_1(s)ds \right]$$

$$\begin{aligned} &= \int_0^1 (1-s)\,(y - Qy)\,(s)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)\,(y - Qy)\,ds \right] \\ &= 0 \end{aligned}$$

then, $y_1 \in ImL$. Hence, $Y = ImL + R$, since $ImL \cap R = \{0\}$, we have $Y = ImL \oplus R$, thus, $\dim\ker L = \dim R = co\dim L = 1$. Hence, L is a Fredholm operator of index zero.

(iv) Taking $P : X \to X$ as follows,

$$Px = x'(0)t$$

then, the generalized inverse $Kp : ImL \to D(L) \cap KerP$ of L can be written as

$$Kpy = \int_0^1 (1-s)y(s)ds$$

In fact, for $y \in ImL$, we have

$$(LKp)y(t) = [(Kpy)(t)]'' = y(t)$$

and, for $x \in D(L) \cap KerP$, we know

$$(KpL)x(t) = (Kp)\, x''(t) = \int_0^t sx''(s)ds = x(t)$$

This shows that $K_P = (L_{|D(L)\cap\ker P})^{-1}$.

(v) $\|K_p y\| < \|y\|_1$, for all $y \in ImL$. In fact from the definition of K_P, we have

$$\|K_P y\|_\infty \le \int_0^1 (1-s)\,|y(s)|\,ds \le \int_0^1 |y(s)|\,ds = \|y\|_1$$

then

$$\|K_p y\| < \|y\|_1\,.$$

■

Now, we give the result on the existence of a solution for the problem (4.1)-(4.2).

Theorem 50 *Assume that*

(H1) There exist nonnegative functions $\alpha, \beta, \gamma \in L^1[0,1]$, *such that, for all* $(x, y) \in \mathbb{R}^2$, $t \in [0,1]$,

satisfying the following inequalities:

$$|f(t,x,y)| \leq \alpha(t)\,|x| + \beta(t)\,|y| + \gamma(t). \tag{4.7}$$

(H2) There exists a constant $M > 0$, *such that, for* $x \in D(L)$, *if* $|x'(t)| > M$, *for all* $t \in [0,1]$, *then,*

$$\int_0^1 (1-s)y(s)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \neq 0. \tag{4.8}$$

(H3) There exists a constant $M^* > 0$, *such that, for* $c \in \mathbb{R}$, *if* $|c| > M^*$, *then, either*

$$c \times k_0 \times \left[\int_0^1 (1-s)f(s,x(s),c)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),c)ds \right] < 0, \tag{4.9}$$

or else

$$c \times k_0 \times \left[\int_0^1 (1-s)f(s,x(s),c)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),c)ds \right] > 0, \tag{4.10}$$

then BVP (4.1)-(4.2) has at least one solution in $C^1[0,1]$, *provided* $\|\alpha\| + \|\beta\| \leq \frac{1}{2}$.

Next, in order to prove Theorem 50, we need the following Lemma.

Lemma 51 *Suppose that* Ω *is an open bounded subset of* X *such that* $D(L) \cap \Omega \neq \emptyset$. *Then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. Suppose that $\Omega \subset X$ is a bounded set. Without loss of generality, we may assume that $\Omega = B(0,r)$, then for any $x \in \overline{\Omega}, \|x\| \leq r$, For $x \in \overline{\Omega}$, and by condition (4.7), we obtain

$$\begin{aligned}
|QNx| &\le k_0 \int_0^1 |f(s,x(s),x'(s))|\,ds + \frac{k_0}{2}\sum_{k=1}^{m}\lambda_k \int_0^{\eta_k} |f(s,x(s),x'(s))|\,ds \\
&\le k_0 \int_0^1 |\alpha(s)|\,|x(s)| + |\beta(s)|\,|y(s)| + |\gamma(s)|\,ds \\
&\quad + \frac{k_0}{2}\sum_{k=1}^{m}\lambda_k \int_0^{\eta_k} |\alpha(s)|\,|x(s)| + |\beta(s)|\,|y(s)| + |\gamma(s)|\,ds \\
&\le (k_0 + \frac{k_0}{2}\sum_{k=1}^{m}\lambda_k)\left[r\left(\|\alpha\|_1 + \|\beta\|_1\right) + \|\gamma\|_1\right]
\end{aligned}$$

thus,

$$\|QNx\| \le (k_0 + \frac{k_0}{2}\sum_{k=1}^{m}\lambda_k)\left[r\,\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1\right], \tag{4.11}$$

which implies that $QN\left(\overline{\Omega}\right)$is bounded. Next, we show that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is compact. For $x \in \overline{\Omega}$ by condition (4.7) we have

$$\|Nx\|_1 = \int_0^1 |f(t,x(s),x\prime(s))|\,ds \le r\left(\|\alpha\|_1 + \|\beta\|_1\right) + \|\gamma\|_1, \tag{4.12}$$

On the other hand, from the definition of K_P and together with (4.3),(4.11) and (4.12) one gets

$$\begin{aligned}
\|K_P(I-Q)Nx\| &\le \|(I-Q)Nx\| \le \|Nx\|_1 + \|QNx\|_1 \\
&\le (1 + k_0 + \frac{k_0}{2}\sum_{k=1}^{m}\lambda_k)\left[r\,\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1\right],
\end{aligned}$$

It follows that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is uniformly bounded.

Let us prove that $K_P(I-Q)N\left(\overline{\Omega}\right)$is equicontinuous. For any $x \in \left(\overline{\Omega}\right)$, and any

$t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$\begin{aligned}(K_P(I-Q)Nx)(t_1) - (K_P(I-Q)Nx)(t_2) &= \\ \left|\int_0^{t_1}(t_1 - s)(I-Q)Nx(s)ds - \int_0^{t_2}(t_2 - s)(I-Q)Nx(s)ds\right| &\leq \\ \int_0^{t_1}(t_2 - t_1)(I-Q)Nx(s)ds - \int_0^{t_2}(t_2 - s)(I-Q)Nx(s)ds &\to 0\end{aligned}$$

as $t_1 \to t_2$. So $(K_P(I-Q)Nx)(\overline{\Omega})$ is equicontinuous. so, the Ascoli-Arzela theorem ensure that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. The proof is completed. ■

Now we give the proof of Theorem 50

Proof. of Theorem 50. We need to construct the set Ω satisfying all the conditions in Theorem 48, which is separated into the following four steps.

Step 1. Let

$$\Omega_1 = \{x \in D(L) \backslash \ker L : Lx = \lambda Nx, \text{ for some } \lambda \in [0,1]\},$$

then, Ω_1 is bounded. Suppose that $x \in \Omega_1$, and $Lx = \lambda Nx$, thus, $\lambda \neq 0$, $QNx = 0$, so it yields

$$\int_0^1 (1-s)f(s, x(s), x'(s))ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s))ds = 0,$$

thus, from hypothesis (H2), there exists $t_0 \in [0,1]$, such that $|x'(t_0)| \leq M$. In view of

$$x(0) = x(t_0) - \int_0^{t_0} x'(t)dt,$$

then,

$$\|Px\| = |x'(0)| < M + \|x''\|_1 = M + \|Nx\|_1 . \tag{4.13}$$

Again for $x \in \Omega_1$, $x \in D(L) \backslash \ker L$, then $(I-P)x \in D(L) \cap KerP$, $Px = 0$, thus from Lemma 49, we know

$$\|(I-P)x\| = \|KpL(I-P)x\| \leq \|L(I-P)x\|_1 = \|x\|_1 \leq \|Nx\|_1 . \tag{4.14}$$

From (4.13) and (4.14), we have

$$\|x\| \leq \|Px\| + \|(I-P)x\| = \|x\prime(0)\| + \|(I-P)x\| \leq 2\|Nx\| + M. \tag{4.15}$$

If (4.7) holds, then from (4.15), we obtain

$$\|x\| \leq 2\left[\|\alpha\|_1 \|x\|_\infty + \|\beta\|_1 \|x'\|_\infty + \|\gamma\|_1 + \frac{M}{2}\right]. \tag{4.16}$$

From $\|x\|_\infty \leq \|x\|$, (4.16) we have

$$\|x\|_\infty \leq \frac{2}{1-2\|\alpha\|_1}\left[\|\beta\|_1 \|x'\|_\infty + \|\gamma\|_1 + \frac{M}{2}\right]. \tag{4.17}$$

Taking into account $\|x'\|_\infty \leq \|x\|$, (4.16) and (4.17) it yields

$$\|x'\|_\infty \left[1 - \frac{2\|\beta\|_1}{1-2\|\alpha\|_1}\right] \leq \frac{2}{1-2\|\alpha\|_1}\left[\|\gamma\|_1 + \frac{M}{2}\right],$$

therefore,

$$\|x'\|_\infty \left[\frac{1-2\|\alpha\|_1 - 2\|\beta\|_1}{1-2\|\alpha\|_1}\right] \leq \frac{1}{1-2\|\alpha\|_1}\left[2\|\gamma\|_1 + M\right],$$

thus

$$\|x'\|_\infty \leq \frac{2\left[\|\gamma\|_1 + \frac{M}{2}\right]}{1-2\|\alpha\|_1 - 2\|\beta\|_1}, \tag{4.18}$$

from (4.18) there exists $M_1 = \frac{2\left[\|\gamma\|_1 + \frac{M}{2}\right]}{1-2\|\alpha\|_1-2\|\beta\|_1} > 0$, such that

$$\|x'\|_\infty \leq M_1, \tag{4.19}$$

hence, from (4.17), there exist $M_2 > 0$, such that

$$\|x\|_\infty \leq M_2, \tag{4.20}$$

so,$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \leq \max\{M_1, M_2\}$, that implies that Ω_1 is bounded.

Step 2. The set $\Omega_2 = \{x \in \ker L : Nx \in ImL\}$ is bounded. In fact, let $x \in \Omega_2$, then,

$$x \in \ker L = \{x \in D(L) : x = ct, c \in \mathbb{R}, t \in [0,1]\}$$

and $QNx = 0$, thus,

$$\int_0^1 (1-s)f(s,x(s),x'(s))ds - \frac{1}{2}\sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),x'(s))ds = 0.$$

From (H2), there exists $t_0 \in [0,1]$, such that $|x'(t_0)| \leq M$ ie $|b| \leq M$. Then, $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} = |b| \leq M$, so the set Ω_2 is bounded.

Step 3. If the first part of (H1) holds, that is, there exists $M^* > 0$, such that for any $c \in \mathbb{R}$, if $|c| > M^*$, then,

$$c \times k_0 \times \left[\int_0^1 (1-s)f(s,x(s),c)ds - \frac{1}{2}\sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),c)ds\right] < 0. \tag{4.21}$$

Indeed, let

$$\Omega_3 = \{x \in \ker L : -\lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\},$$

here, $J : ImQ \to \ker L$ is the linear isomorphism given by $J(c) = ct, \forall c \in \mathbb{R}, t \in [0,1]$. Then, Ω_3 is bounded. Since, for $x = c_0 t$, then, for $t \in [0,1]$, we obtain

$$\lambda c_0 = (1-\lambda)k_0 \times \left[\int_0^1 (1-s)f(s,x(s),c)ds - \frac{1}{2}\sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),c)ds\right] < 0,$$

if $\lambda = 1$, then $c_0 = 0$. otherwise, if $|c_0| > M^*$, then in view of (17) one has

$$\lambda c_0^2 = (1-\lambda)k_0 \times \left[\int_0^1 (1-s)f(s,x(s),c)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),c)ds \right] < 0,$$

which contradicts $\lambda c_0^2 \geq 0$. Thus, $\Omega_3 \subset \{x \in KerL : \|x\| \leq M^*\}$ is bounded.

Step 4. If the second part of (H3) holds, that is, there exists $M^* > 0$, such that, for any $c \in \mathbb{R}$, if $|c| > M^*$, then,

$$c(1-\lambda)k_0 \times \left[\int_0^1 (1-s)f(s,x(s),c)ds - \frac{1}{2}\sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s,x(s),c)ds \right] > 0.$$

Let

$$\Omega_3 = \{x \in \ker L : -\lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\},$$

here, J as in Step 3. Similar to the argument in Step 3, we can verify Ω_3 is bounded.

Now, we shall prove that all the conditions of Theorem 48 are satisfied. Let Ω be a bounded open subset of X, such that $\cup_{i=1}^{i=3}\overline{\Omega} \subset \Omega$. By the Ascoli-Arzela theorem, we can show that $Kp(I-Q)N : \overline{\Omega} \to Y$is compact, thus, N is L-compact on $\overline{\Omega}$. Then, by the above argument, we have

(i) $Lx \neq \lambda$Nx, for every $(x,\lambda) \in [(D(L)\setminus \ker L) \cap \partial\Omega]]x(0,1)$,

(ii) $Nx \notin ImL$, for every $x \in \ker L \cap \partial\Omega$,

(iii) If the first part of (H3) holds, then, let

$$H(x,\lambda) = -\lambda x + (1-\lambda)JQNx.$$

According to the above argument, we know $H(x,\lambda) \neq 0$, for $x \in \ker L \cap \partial\Omega$, by the homotopy property of degree, we get

$$\begin{aligned}\deg(JQN_{|\ker L}, \Omega\cap\ker L, 0) &= \deg(H(.,0), \Omega\cap\ker L, 0)\\ &= \deg(H(.,1), \Omega\cap\ker L, 0)\\ &= \deg(-I, \Omega\cap\ker L, 0).\end{aligned}$$

According to definition of degree on a space which is isomorphic to $\mathbb{R}$, and

$$\Omega\cap KerL = \{ct : |c| < d\}.$$

We have

$$\begin{aligned}\deg(-I, \Omega\cap\ker L, 0) &= \deg(-J^{-1}IJ, J^{-1}(\Omega\cap\ker L), J^{-l}\{0\})\\ &= \deg(-I, (-d, d), 0) = -1 \neq 0.\end{aligned}$$

If the second part of condition (3) of Theorem 48 holds, let

$$H(x, \lambda) = \lambda x + (1-\lambda)JQNx.$$

Similar to the above argument, we have

$$\begin{aligned}\deg(JQN_{|\ker L}, \Omega\cap\ker L, 0) &= \deg(H(.,0), \Omega\cap\ker L, 0) =\\ \deg(H(.,1), \Omega\cap\ker L, 0) &= \deg(I, \Omega\cap\ker L, 0) = 1.\end{aligned}$$

Then, we obtain

$$\deg(JQN_{|\ker L}, \Omega\cap\ker L, 0) \neq 0.$$

Then by, Theorem 48, $Lx = Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$, so that the boundary value problem (4.1)(-(4.2) has at least one solution in $C^1[0,1]$. The proof is completed. ■

4.3 An illustrative Example

In this section we give an example to illustrate the usefulness of our main results. Consider the multipoint boundary value problem

$$(P)\begin{cases} x''(t) = f(t, x(t), x'(t)),\ t \in (0,1) \\ x(0) = 0, x(1) = 4\int\limits_0^{\frac{1}{2}} x(t)dt + \frac{16}{9}\int\limits_0^{\frac{2}{3}} x(t)dt \end{cases}$$

with

$$f(t,x,y) = \frac{t}{4}x + \left(\frac{1-t^2}{4}\right)y + t.$$

Since

$$\sum_{k=1}^{2} \lambda_k \eta_k^2 = 4(\frac{1}{2})^2 + \frac{16}{9}(\frac{3}{4})^2 = 2,$$

the problem (P) is at resonance. We have

$$|f(t,x,y)| \le \alpha(t)\,|x| + \beta(t)\,|y| + \gamma(t),$$

where the functions

$$\alpha(t) = \frac{t}{4}, \beta(t) = \left(\frac{1-t^2}{4}\right), \gamma(t) = t,$$

are nonnegative and belong to $L^1[0,1]$, so, hypothesis (H1) of Theorem 50 is satisfied.

We claim that condition (H2) of Theorem 50 is satisfied, indeed, for $M = 1.8214 >$

0 and $x \in D(L)$, $x(t) = ct$, if $|x'(t)| > M$, for all $t \in [0,1]$, then,

$$\begin{aligned}
&\int_0^1 (1-s) f(s, x(s), x'(s)) ds \\
&- \frac{1}{2} \sum_{k=1}^{2} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s)) ds \\
&= \int_0^1 (1-s) \left(\frac{c}{4} + s\right) ds - \\
&\frac{1}{2}\left(4 \int_0^{\frac{1}{2}} (\eta_k - s)^2 \left(\frac{c}{4} + s\right) ds + \frac{16}{9} \int_0^{\frac{3}{4}} (\eta_k - s)^2 \left(\frac{c}{4} + s\right) ds\right) \\
&= \frac{7}{96} c + \frac{17}{128} \neq 0.
\end{aligned}$$

Now, for $M^* = 2 > 0$ and any $x(t) = ct \in kerL$ with $|c| > M$, we have

$$\begin{aligned}
&c\left[\int_0^1 (1-s) f(s, x(s), x'(s)) ds - \frac{1}{2} \sum_{k=1}^{2} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s)) ds\right] \\
&= \frac{7}{96} c^2 + \frac{17}{128} c > 0,
\end{aligned}$$

consequently, condition (H3) of Theorem 50 is satisfied. Finally, a simple calculus gives

$$\|\alpha\|_1 + \|\beta\|_1 = \frac{1}{8} + \frac{1}{6} \leq \frac{1}{2}.$$

We conclude from Theorem 50 that the problem (P) has at least one solution in $C^1[0,1]$.

Conclusion. In this thesis, we investigated the existence of a solution of a boundary value problem at resonance generated by a second order differential equation and multipoint integral conditions. by Mawhin's coincidence theorem, and under some conditions on the nonlinear term, we proved the existence of solution, then we gave an example to illustrat the main results. This study can be extended to higher order boundary value problem.

Since differential equations of fractional order have recently been shown to be valuable tools for modeling many phenomena in various fields of science and engineering, this study can be done for multipoint fractional boundary value problems at resonance. Articles on this subject can be found in the literature, for examples, see [52, 53, 54, 55, 56, 57, 58, 59]..

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ملخص

نهتم في هذه الأطروحة بدراسة مسالة الرنين الناتجة عن معادلة تفاضلية غير خطية مع شروط حدية متعددة النقط و التكامل. ولهذا لإثبات وجود حل للمعادلة التفاضلية التي طرفها غير الخطي يعتمد على المشتقة الأولى مع شروط حدودية غير محلية من نوع التكامل نقترح نظرية درجة مصادفة ما موهين.

الكلمات المفتاحية:

مسالة متعددة النقاط، مؤثرات فريد هلم، الرنين، نظرية درجة مصادفة ماوهين، وجود الحل.

Résumé

On s'intéresse dans cette thèse à l'étude d'un problème en résonance engendré par une équation différentielle non linéaire du second ordre avec des conditions aux limites de type multi-points et intégrales. Au fait nous nous proposons d'établir l'existence de la solution d.une équation différentielle dont le terme non linéaire dépend de la première dérivée avec des conditions non locales de type intégrale moyennant le concept de la théorie du degré de coïncidence de Mawhin.

Mots clés: Multi-point problème aux limites, Opérateur de Fredholm, Résonance, Théorie du degré de coïncidence de Mawhin, Existence de la solution.

Abstract

This thesis deals with the study of a resonance problem generated by a second order nonlinear differential equation with multi-point integral boundary conditions. In fact we propose to establish the existence of the solution of a differential equation whose nonlinear term depends on the first derivative with non-local conditions of integral type by means of the concept of the theory of the degree of coincidence of Mawhin.

Keywords: Multi-point boundary value problem, Fredholm operator, Resonance, Mawhin coincidence degree theory, Existence of solution.