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Well-Posedness of a Third Order Partial Differential Equation

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In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds, and prayers and peace be upon Mohamed His servant and messenger. I take pleasure in dedicating this thesis to my family. A special feeling of gratitude to my loving parents and my grandmother 'Mabibia' for their endless support and encouragement during not only this PhD but all my endeavors, My sisters Karima, Dounia, Rofia, Choubeila. Brothers Rassim, Lotfi and Kais have supported me throughout the process and never left my side and are very special. I will always appreciate all what they have done.

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

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Abstract

In this thesis, local and nonlocal boundary value problems for third order partial differential equations in arbitrary Hilbert space with a self-adjoint positive definite operator are studied. It is well-known that local and nonlocal boundary value problems for third order partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Applying all these analytical methods, exact solutions of several problems in the case when the differential equation has constant coefficients are presented. Operator approach permit us to study local and nonlocal boundary value problems for third order partial differential equations in arbitrary Hilbert space with a self-adjoint positive definite operator. Theorems on stability estimates for the solution of these boundary value problems are established. In practice, stability estimates for the solution of several problems for third order partial differential equations are obtained. The difference schemes for the numerical solution of one-dimensional third order partial differential equations are presented. Numerical results are given.

keywords: Well-posedness, Boundary value problems, Third order partial differential equation, Self-adjoint positive definite operator, Hilbert space.

ملخص

في هذه الاطروحة نهتم بدراسة اشكاليات المعادلات التفاضلية الجزئية من الدرجة الثالثة بقيم حدية محلية واخرى غير محلية في فضاء مجرد هيلبرت مع مؤثر معرف موجب قرينا لنفسه .

من المعروف ان اشكاليات المعادلات التفاضلية الجزئية من الدرجة الثالثة ذات القيم الحدية المحلية والغير محلية يمكن حلها تحليليا باستعمال سلاسل فوريي و تحويل لا بلاس وأيضا تحويل فوريي.

سنقدم الحلول الدقيقة لعدة اشكاليات لما تكون المعادلات التفاضلية ذات عوامل ثابتة بتطبيق هذه الطرق التحليلية.
نظرية المؤثرات تسمح لنا بدراسة اشكاليات المعادلات التفاضلية الجزئية من الدرجة الثالثة بقيم حدية محلية واخرى غير محلية في فضاء مجرد هيلبرت مع مؤثر معرف موجب قرينا لنفسه.
ننشأ نظرية الاستقرار لحلول هاته الاشكالات ذات القيم الحدية.
في التطبيقات نقدم نظرية الاستقرار لحلول عدة اشكاليات ذات القيم الحدية.
نعرض مخططات الفروق للحلول العددية للمعادلات التفاضلية الجزئية من الدرجة الثالثة ذات البعد الاول وفي الاخير نقدم النتائج العددية.

الكلمات المفتاحية: فضاء هلبرت، المعادلات التفاضلية الجزئية من الدرجة الثالثة ، المؤثر المعرف الموجب القرين لنفسه.

Résumé

Plusieurs méthodes sont apparus pour l'étude des problèmes aux limites du troisième ordre, ils peuvent être résolues analytiquement par des méthodes classiques à savoir les séries de Fourier, la transformée de Laplace ainsi que la transformée de Fourier. Des résultats sont présentés dans le cas où l'équation différentielle est à coefficients constants.
Dans le cas général, il est difficile de trouver la solution de ce genre de problèmes, néanmoins, l'approche par les opérateurs donne un autre moyen pour l'étude.
Dans ce travail de thèse, on s'intéresse à l'étude de deux problèmes aux limites locale et non locale engendrés par des équations différentielles aux dérivés partielles de troisième ordre dans un espace de Hilbert dont l'opérateur est défini positif et auto-adjoint. Des théorèmes de stabilité des solutions sont ainsi établis.
Dans l'application, les estimations de stabilité ainsi que les schémas de différence de la solution de plusieurs problèmes aux limites de troisième ordre sont obtenues.

Mots clés : Problème bien posé, Problème aux limites, Equations aux dérivés partielles d'ordre trois, Opérateur défini positif auto-adjoint, Espace de Hilbert.

Table of contents

Chapter 1

Introduction

Nonlocal boundary value problems (BVP) for partial differential equations have been a major research area in many branches of science and engineering particularly in applied mathematics when it is impossible to determine the boundary values of the unknown function. In the last century, interest towards to the subject of local and nonlocal boundary value problems for partial differential equations with time and space variables has been substantial and growing tendency because of science and industry. For these reasons, we have worked on these issues in this thesis.

Local and nonlocal boundary value problems for third order ordinary differential equations and system of ordinary differential equations have been considered in the field of science and engineering such as modern physics, mathematical biology, chemical diffusions and fluid mechanics. Additionally, this type of boundary value problems has been studied widely in the literature (for instance, see [60], [84], [103], [104]).

The authors (MR Grossinho, FM Minhos, and AI Santos) [59] studied existence and location result for the boundary value problem for third-order nonlinear ordinary differential equation. In the papers [60], [62], [63], [105], nonlocal boundary value problem for third-order nonlinear ordinary differential equations was considered. Existence of the problems were established by using the Leggett-Williams fixed point theorem [63] and Leray Schauder nonlinear alternative [60]. Additionally, some sufficient conditions for the existence of the problem in Banach spaces were obtained by using fixed point index theory [105]. Moreover, the authors (Z.Liu and S.Kang) [81], (A.Palamides and A.Veloni) [86], (P.Palamides and K.Palamides) [87], (S.Smirnov) [99], [100], investigated local boundary value problem for third-order nonlinear ordinary differential equations. Existence of the problems was established by using the Krasnoselskii's fixed-point theorem of cone (A.Palamides and A.Veloni) [86],(P.Palamides and K.Palamides) [87], and (Z.Liu, S.Kang, and J.Sheok) [81] also established existence of the problem. Similarly, the author (S.Smirnov) [99], [100] also established existence of the problem. Finally, local boundary value problem for system of third-order nonlinear ordinary differential equation was studied in the paper [94]. The multiplicity and existence of the problem was also established by using the Krasnoselskii's fixed-point theorem of cone.

In [17], the problem for a third order ordinary differential equation

$$\begin{cases} \frac{d^3y(t)}{dt^3}+c(t)\frac{d^2y(t)}{dt^2}+b(t)\frac{dy(t)}{dt}+a(t)y(t)=f(t), 0<t<T, \\ y(0)=y(T), y'(0)=y'(T), y''(0)=y''(T) \end{cases} \tag{1.1}$$

is investigated. Here, it is assumed that $a(t)$, $b(t)$, $c(t)$, and $f(t)$ to be such that problem (1.1) has a unique smooth solution on $[0,T]$. Three-step difference schemes are generated for the numerical solutions of problem (1.1). The construction of the three-step difference schemes is based on Taylor's decomposition on four points. The results are shown to be well applicable for the numerical solution of nonlocal boundary-value problems by an example involving periodically time-varying parameters.

Local and nonlocal boundary value problems for third order partial differential equations have been studied widely in the literature (for instance, see [6], [14], [52], [80]).

In the paper [6], the local boundary value problem

$$\begin{cases} \frac{\partial^3 u(x)}{\partial x^3}-\frac{\partial^2 u(y)}{\partial y^2}=f(x,y),\ 0<x<p, 0<y<l, \\ u_y(x,0)=\varphi_1(x),\ u_y(x,l)=\varphi_2(x),\ p>0,\ l>0, \\ u(0,y)=\psi_1(y),\ u(p,y)=\psi_2(y),\ u_x(p,y)=\psi_3(y) \end{cases}$$

for third-order partial differential equations in a rectangular domain was studied. The investigation of authors of this paper is based on the fundamental solutions of corresponding non-homogeneous equation the green function of analyzed problem.

In [14], the boundary value problem for a third order partial differential equation

$$\begin{cases} \frac{d^3u(t)}{dt^3}-Au(t)=f(t), 0<t<1, \\ u(0)=\varphi, u_t(0)=\psi, u_{tt}(1)=\xi \end{cases} \tag{1.2}$$

in a Hilbert space H with a self-adjoint positive definite operator A was investigated. The main theorem on the stability estimates for the solution of problem (1.2), in a Hilbert space was established. Two applications of the main theorem were given. Theorems on the stability estimates for the solutions of these applications were obtained. A first and high order of accuracy difference schemes were constructed for the approximate solution of (1.2). Numerical results were given.

In [33], the nonlocal boundary value problem for a third order partial differential equation

$$\begin{cases} \dfrac{d^3u(t)}{dt^3} + Au(t) = f(t),\ 0 < t < 1, \\ \\ u(0) = u(1) + \varphi, u'(0) = u'(1) + \psi,\ u''(0) = \xi \end{cases} \tag{1.3}$$

in a Hilbert space H with a self-adjoint positive definite operator A was investigated. The main theorem on the stability estimates for the solution of problem (1.3) in a Hilbert space was established. Two applications of the main theorem were given. Theorems on the stability estimates for the solutions of these applications were obtained. A first and high order of accuracy difference schemes were constructed for the approximate solution of (1.3). Numerical results were given.

In this thesis, we study local boundary value problem

$$\begin{cases} \dfrac{d^3u(t)}{dt^3} + Au(t) = f(t), \quad 0 < t < 1, \\ \\ u(0) = \varphi, \quad u(1) = \psi, \quad u'(1) = \xi \end{cases} \tag{1.4}$$

and nonlocal boundary value problem

$$\begin{cases} \dfrac{d^3u(t)}{dt^3} + A\dfrac{du(t)}{dt} = f(t), \quad 0 < t < 1, \\ \\ u(0) = \gamma u(\lambda) + \varphi, \quad u'(0) = \alpha u'(\lambda) + \psi, |\gamma| < 1, \\ u''(0) = \beta u''(\lambda) + \xi, \quad |1 + \beta\alpha| > |\alpha + \beta|, 0 < \lambda \leq 1 \end{cases} \tag{1.5}$$

for third order partial differential equations in a Hilbert space H with a self-adjoint positive definite operator A.

A function $u(t)$ is a solution of problem (1.4) if the following conditions are satisfied:

i) $u(t)$ is three times continuously differentiable on the interval $(0,1)$ and continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in (0,1)$, and function $Au(t)$ is continuous on the segment $[0,1]$.

iii) $u(t)$ satisfies the equation and boundary conditions (1.4).

A function $u(t)$ is a solution of problem (1.5) if the following conditions are satisfied:

i) $u(t)$ is three times continuously differentiable on the interval $(0,1)$ and twice continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $\frac{du(t)}{dt}$ belongs to $D(A)$ for all $t \in (0,1)$, and function $A\frac{du(t)}{dt}$ is continuous on the segment $[0,1]$.

iii) $u(t)$ satisfies the equation and nonlocal boundary conditions (1.5).

It is known that local and nonlocal boundary value problems for third order partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

First, we consider Fourier series method for solution of local and nonlocal boundary value problems for third order partial differential equations.

Example 1.0.1 *Obtain the Fourier series solution of the boundary value problem*

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = e^{-t}\sin x, 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \sin x, u(1,x) = e^{-1}\sin x, u_t(1,x) = -e^{-1}\sin x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, 0 \le t \le 1 \end{cases} \tag{1.6}$$

for a third-order partial differential equation.

Solution. In order to solve the problem, first we consider the Sturm-Liouville problem

$$X''(x) = \lambda X(x), 0 < x < \pi, X(0) = 0, X(\pi) = 0$$

generated by the space operator of problem (1.6). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda = -k^2, X_k(x) = \sin kx, k = 1, 2, \ldots.$$

Therefore, we will seek solution $u(t,x)$ using by the Fourier series

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t) \sin kx.$$

Here $A_k(t), k = 1, 2, ...$ are unknown functions. Putting $u(t,x)$ into the equation and using given boundary conditions, we obtain

$$\frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = \sum_{k=1}^{\infty} \left[A_k^{'''}(t) + (k^2+1)A_k(t) \right] \sin kx = e^{-t} \sin x,$$

and

$$u(0,x) = \sum_{k=1}^{\infty} A_k(0) \sin kx = \sin x,$$

$$u(1,x) = \sum_{k=1}^{\infty} A_k(1) \sin kx = e^{-1} \sin x,$$

$$u_t(1,x) = \sum_{k=1}^{\infty} A_k'(1) \sin kx = -e^{-1} \sin x.$$

So, we get

$$\sum_{k=1}^{\infty} \left[A_k^{'''}(t) + (k^2+1)A_k(t) \right] \sin kx = e^{-t} \sin x,$$

$$\sum_{k=1}^{\infty} A_k(0) \sin kx = \sin x,$$

$$\sum_{k=1}^{\infty} A_k(1) \sin kx = e^{-1} \sin x,$$

$$\sum_{k=1}^{\infty} A_k'(1) \sin kx = -e^{-1} \sin x.$$

Equating coefficients of $\sin kx, k \geq 1$ to zero, we get

$$A_1'''(t) + 2A_1(t) = e^{-t}, A_k'''(t) + (k^2+1)A_k(t) = 0, k \geqslant 2, 0 < t < 1,$$

$$\begin{aligned} A_1(0) &= 1, A_1(1) = e^{-1}, A_1'(1) = -e^{-1}, \\ A_k(0) &= 0, A_k(1) = 0, A_k'(0) = 0, k \geqslant 2. \end{aligned}$$

First, we obtain $A_1(t)$. It is clear that $A_1(t)$ be solution of the following boundary value problem

$$\begin{cases} A_1'''(t) + 2A_1(t) = e^{-t}, 0 < t < 1, \\ \\ A_1(0) = 1, A_1(1) = e^{-1}, A_1'(0) = -e^{-1} \end{cases}$$

for a third order ordinary differential equation. We will seek the general solution of this equation by the formula

$$A_1(t) = A_c(t) + A_p(t),$$

where $A_c(t)$ is the solution of homogeneous equation

$$A_1'''(t) + 2A_1(t) = 0$$

and $A_p(t)$ is the particular solution of non-homogeneous equation

$$A_1'''(t) + 2A_1(t) = e^{-t}.$$

For obtaining $A_c(t)$, we will consider the auxiliary equation

$$m^3 + 2 = 0.$$

We have three roots:

$$m_1 = -\sqrt[3]{2}, m_2 = \frac{\sqrt[3]{2}}{2} + i\frac{\sqrt{3}}{\sqrt[3]{2}}, m_3 = \frac{\sqrt[3]{2}}{2} - i\frac{\sqrt{3}}{\sqrt[3]{2}}.$$

Therefore,

$$A_c(t) = C_1 e^{-\sqrt[3]{2}t} + e^{\frac{\sqrt[3]{2}t}{2}}\left[C_2 \cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right) + C_3 \sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)\right].$$

For obtain $A_p(t)$, we assume that

$$A_p(t) = Be^{-t}.$$

Putting into the equation, we get

$$-Be^{-t} - 2Be^{-t} = -3e^{-t}.$$

From that it follows $B = 1$ and $A_p(t) = e^{-t}$. Therefore, the general solution is

$$A_1(t) = C_1 e^{-\sqrt[3]{2}t} + e^{\frac{\sqrt[3]{2}t}{2}}\left[C_2 \cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right) + C_3 \sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)\right] + e^{-t}.$$

Then,

$$A_1'(t) = -C_1\sqrt[3]{2}e^{-\sqrt[3]{2}t} + C_2 e^{\frac{\sqrt[3]{2}t}{2}}\left[\frac{\sqrt[3]{2}}{2}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right) - \frac{\sqrt{3}}{\sqrt[3]{2}}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)\right]$$

$$+C_3 e^{\frac{\sqrt[3]{2}t}{2}}\left[\frac{\sqrt[3]{2}}{2}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)+\frac{\sqrt{3}}{\sqrt[3]{2}}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)\right]-e^{-t}.$$

Using given boundary conditions $A_1(0)=1, A_1(1)=e^{-1}$, and $A_1'(1)=-e^{-1}$, we get the following system of equations

$$\begin{cases} C_1+C_2+1=1, \\ C_1 e^{-\sqrt[3]{2}}+e^{\frac{\sqrt[3]{2}}{2}}\left[C_2\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)+C_3\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)\right]+e^{-1}=e^{-1}, \\ -C_1\sqrt[3]{2}e^{-\sqrt[3]{2}}+C_2 e^{\frac{\sqrt[3]{2}}{2}}\left[\frac{\sqrt[3]{2}}{2}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)-\frac{\sqrt{3}}{\sqrt[3]{2}}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)\right] \\ +C_3 e^{\frac{\sqrt[3]{2}}{2}}\left[\frac{\sqrt[3]{2}}{2}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)+\frac{\sqrt{3}}{\sqrt[3]{2}}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)\right]-e^{-1}=-e^{-1} \end{cases}$$

or

$$\begin{cases} C_1+C_2=0, \\ C_1 e^{-\sqrt[3]{2}}+e^{\frac{\sqrt[3]{2}}{2}}\left[C_2\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)+C_3\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)\right]=0, \\ -C_1\sqrt[3]{2}e^{-\sqrt[3]{2}}+C_2 e^{\frac{\sqrt[3]{2}}{2}}\left[\frac{\sqrt[3]{2}}{2}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)-\frac{\sqrt{3}}{\sqrt[3]{2}}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)\right] \\ +C_3 e^{\frac{\sqrt[3]{2}}{2}}\left[\frac{\sqrt[3]{2}}{2}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)+\frac{\sqrt{3}}{\sqrt[3]{2}}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}\right)\right]=0. \end{cases}$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ e^{-\alpha} & \frac{\alpha}{2}\cos(\beta) & \frac{\alpha}{2}\sin(\beta) \\ -\alpha e^{-\alpha} & e^{\frac{\alpha}{2}}\left(\frac{\alpha}{2}\cos(\beta)-\beta\sin(\beta)\right) & e^{\frac{\alpha}{2}}\left(\frac{\alpha}{2}\sin(\beta)+\beta\cos(\beta)\right) \end{vmatrix}$$

$$=\frac{1}{2}\alpha\beta e^{\frac{1}{2}\alpha}-\beta e^{-\alpha}e^{\frac{1}{2}\alpha}\cos\beta-\frac{1}{2}\alpha e^{-\alpha}e^{\frac{1}{2}\alpha}\sin\beta-\frac{1}{2}\alpha^2 e^{-\alpha}\sin\beta\neq 0,$$

where $\alpha=\sqrt[3]{2}, \beta=\frac{\sqrt{3}}{\sqrt[3]{2}}$. From that it follows that

$$C_1=C_2=C_3=0.$$

Therefore, $A_1(t)=e^{-t}$.

Second, we obtain $A_k(t), k \neq 1$. It is clear that $A_k(t)$ be solution of the following boundary value problem

$$\begin{cases} A_k'''(t) + (k^2+1)A_k(t) = 0, 0 < t < 1, \\ \\ A_k(0) = 0, A_k(1) = 0, A_k'(0) = 0 \end{cases}$$

for a third order ordinary differential equation. For obtaining $A_k(t)$, we will consider the auxiliary equation

$$m^3 + k^2 + 1 = 0.$$

We have three roots:

$$m_1 = -\sqrt[3]{1+k^2}, m_{2,3} = \frac{\sqrt[3]{1+k^2} \mp i\sqrt{3}\sqrt[3]{1+k^2}}{2}.$$

Therefore, we will seek the general solution of this equation by the formula

$$\begin{aligned} A_k(t) &= C_1 e^{-t\sqrt[3]{1+k^2}} + e^{\frac{\sqrt[3]{1+k^2}t}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+k^2}t \right) \right. \\ &\quad \left. + C_3 \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+k^2}t \right) \right]. \end{aligned}$$

Taking the derivative, we get

$$\begin{aligned} A_k'(t) = \tfrac{1}{2}\sqrt[3]{k^2+1} \Big(&-2C_1 e^{-t\sqrt[3]{k^2+1}} + C_2 e^{\frac{1}{2}t\sqrt[3]{k^2+1}} \left(\cos\tfrac{1}{2}\sqrt{3}t\sqrt[3]{k^2+1} - \sqrt{3}\sin\tfrac{1}{2}\sqrt{3}t\sqrt[3]{k^2+1} \right) \\ &+ C_3 e^{\frac{1}{2}t\sqrt[3]{k^2+1}} \left(\sqrt{3}e^{\frac{1}{2}t\sqrt[3]{k^2+1}} \cos\tfrac{1}{2}\sqrt{3}t\sqrt[3]{k^2+1} + \sin\tfrac{1}{2}\sqrt{3}t\sqrt[3]{k^2+1} \right) \Big). \end{aligned}$$

Using these formulas, given boundary conditions, we can write the following system of equations

$$\begin{cases} C_1 + C_2 = 0, \\ \\ C_1 e^{-\sqrt[3]{1+k^2}} + C_2 \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+k^2} \right) + C_3 \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+k^2} \right) = 0, \\ \\ \frac{1}{2}\sqrt[3]{k^2+1}\left(-2C_1 + C_2 + C_3\sqrt{3} \right) = 0. \end{cases}$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ e^{-\sqrt[3]{1+k^2}} & \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+k^2}\right) & \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+k^2}\right) \\ -\sqrt[3]{k^2+1} & \frac{1}{2}\sqrt[3]{k^2+1} & \frac{\sqrt{3}}{2}\sqrt[3]{k^2+1} \end{vmatrix}$$

$$= \frac{1}{2}\sqrt{3}\left(\cos\frac{1}{2}\sqrt{3}\sqrt[3]{k^2+1}\right)\sqrt[3]{k^2+1} - \frac{1}{2}\sqrt{3}e^{-\sqrt[3]{k^2+1}}\sqrt[3]{k^2+1}$$

$$-\frac{3}{2}\left(\sin\frac{1}{2}\sqrt{3}\sqrt[3]{k^2+1}\right)\sqrt[3]{k^2+1} \neq 0.$$

From that it follows

$$C_1 = C_2 = C_3 = 0.$$

Then, $A_k(t) = 0$ and the exact solution of the problem (1.6) is

$$u(t,x) = e^{-t}\sin x.$$

Note that using similar procedure one can obtain the solution of the following boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \ \ x = (x_1,...,x_n) \in \overline{\Omega}, \ 0 < t < T, \\ \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), u_t(T,x) = \xi(x), \quad x \in \overline{\Omega}, \\ \\ u(t,x) = 0, 0 \leq t \leq T, x \in S \end{cases} \tag{1.7}$$

for the multidimensional a third order partial differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \xi(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary S,

$$\overline{\Omega} = \Omega \cup S.$$

However Fourier series method described in solving (1.7) can be used only in the case when (1.7) has constant coefficients.

Example 1.0.2 *Obtain the Fourier series solution of the nonlocal boundary value problem*

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^3 u(t,x)}{\partial t \partial x^2} + \dfrac{\partial u(t,x)}{\partial t} = 4t\cos x, 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \cos x, u_t(0,x) = \frac{1}{2}u_t(1,x) - \cos x, 0 \le x \le \pi, \\ u_{tt}(0,x) = \frac{1}{2}u_{tt}(1,x) + \cos x, 0 \le x \le \pi, \\ u_x(t,0) = u_x(t,\pi) = 0, 0 \le t \le 1 \end{cases} \tag{1.8}$$

for a third-order partial differential equation.

Solution. In order to solve the problem, first we consider the Sturm-Liouville problem

$$X''(x) = \lambda X(x), 0 < x < \pi, X'(0) = 0, X'(\pi) = 0$$

generated by the space operator of problem (1.8). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda = -k^2, X_k(x) = \cos kx, k = 0, 1, ...$$

Therefore, we will seek solution $u(t,x)$ using by the Fourier series

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos kx.$$

Here $A_k(t), k = 0, 1, ...$ are unknown functions. Putting $u(t,x)$ into the equation and using given boundary conditions, we obtain

$$\frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^3 u(t,x)}{\partial t \partial x^2} + \frac{\partial u(t,x)}{\partial t} = \sum_{k=0}^{\infty} \left[A_k'''(t) + (k^2+1)A_k'(t) \right] \cos kx = 4t\cos x,$$

and

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0) \cos kx = \cos x,$$

$$u_t(0,x) - \frac{1}{2}u_t(1,x) = \sum_{k=0}^{\infty} A_k'(0) \cos x - \frac{1}{2} \sum_{k=0}^{\infty} A_k'(1) \cos kx = -\cos x$$

$$u_{tt}(0,x)-\frac{1}{2}u_{tt}(1,x)=\sum_{k=0}^{\infty}A_k''(0)\cos kx-\frac{1}{2}\sum_{k=0}^{\infty}A_k''(1)\cos kx=\cos x.$$

So, we get

$$\sum_{k=0}^{\infty}\left[A_k'''(t)+(k^2+1)A_k'(t)\right]\cos kx=4t\cos x,$$

$$\sum_{k=0}^{\infty}A_k(0)\cos kx=\cos x,$$

$$\sum_{k=0}^{\infty}\left(A_k'(0)-\frac{1}{2}A_k'(1)\right)\cos kx=-\cos x,$$

$$\sum_{k=0}^{\infty}\left(A_k''(0)-\frac{1}{2}A_k''(1)\right)\cos kx=\cos x.$$

Equating coefficients of $\cos kx, k\geq 0$ to zero, we get

$$A_1'''(t)+2A_1'(t)=4t, A_k'''(t)+(k^2+1)A_k'(t)=0, k\neq 1, 0<t<1,$$

$$\begin{aligned}A_1(0) &= 1, A_1'(0)-\frac{1}{2}A_1'(1)=-1, A_1''(0)-\frac{1}{2}A_1''(1)=1,\\ A_k(0) &= 0, A_k'(0)-\frac{1}{2}A_k'(1)=0, A_k''(0)-\frac{1}{2}A_k''(1)=0, k\neq 1.\end{aligned}$$

First, we obtain $A_1(t)$. It is clear that $A_1(t)$ is solution of the following nonlocal boundary value problem

$$\begin{cases} A_1'''(t)+2A_1'(t)=4t, 0<t<1,\\ A_1(0)=1, A_1'(0)-\frac{1}{2}A_1'(1)=-1, A_1''(0)-\frac{1}{2}A_1''(1)=1\end{cases}$$

for a third order ordinary differential equation. We will seek the general solution of this equation by the formula

$$A_1(t)=A_c(t)+A_p(t),$$

where $A_c(t)$ is the solution of homogeneous equation

$$A_1'''(t)+2A_1'(t)=0$$

and $A_p(t)$ is the particular solution of nonhomogeneous equation

$$A_1'''(t) + 2A_1'(t) = 4t.$$

For obtaining $A_c(t)$, we will consider the auxiliary equation

$$m^3 + 2m = 0.$$

We have three roots: $m_1 = 0, m_{2,3} = \pm i\sqrt{2}$, Therefore,

$$A_c(t) = C_1 + C_2 \cos\left(\sqrt{2}t\right) + C_3 \sin\left(\sqrt{2}t\right).$$

For obtaining $A_p(t)$, we assume that

$$A_p(t) = t\,(at + b).$$

Putting into the equation, we get

$$4at + 2b = 4t.$$

Equating coefficients $t^m, m = 0, 1$ to zero, we get $a = 1, b = 0$ and $A_p(t) = t^2$. Therefore, the general solution is

$$A_1(t) = C_1 + C_2 \cos\left(\sqrt{2}t\right) + C_3 \sin\left(\sqrt{2}t\right) + t^2.$$

Then,

$$A_1'(t) = -\sqrt{2}C_2 \sin\sqrt{2}t + \sqrt{2}C_3 \cos\sqrt{2}t + 2t$$
$$A_1''(t) = -2C_2 \cos\sqrt{2}t - 2C_3 \sin\sqrt{2}t + 2$$

Using nonlocal boundary conditions $A_1(0) = 1, A_1'(0) - \frac{1}{2}A_1'(1) = -1$, and $A_1''(0) - \frac{1}{2}A_1''(1) = 1$, we get the following system of equations

$$\begin{cases} C_1 + C_2 = 1, \\ C_2 \cos\sqrt{2} + C_3\left(\sqrt{2} + \sin\sqrt{2}\right) = 0, \\ C_2\left(-2 + \cos\sqrt{2}\right) + C_3 \sin\sqrt{2} = 0. \end{cases}$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & \cos\sqrt{2} & \left(\sqrt{2}+\sin\sqrt{2}\right) \\ 0 & \left(-2+\cos\sqrt{2}\right) & \sin\sqrt{2} \end{vmatrix}$$

$$= 2\sin\sqrt{2} - \sqrt{2}\cos\sqrt{2} + 2\sqrt{2} \neq 0.$$

From that it follows

$$C_1 = 1, C_2 = C_3 = 0.$$

Therefore,

$$A_1(t) = 1 + t^2.$$

Second, we obtain $A_k(t)$ for all $k \neq 1$. It is clear that $A_k(t)$ is solution of the following nonlocal boundary value problem

$$\begin{cases} A_k'''(t) + (k^2+1)A_k'(t) = 0, 0 < t < 1, \\ A_k(0) = 0, A_k'(0) = \frac{1}{2}A_k'(1), A_k''(0) = \frac{1}{2}A_k''(1). \end{cases}$$

The auxiliary equation is

$$m^3 + (k^2+1)m = 0.$$

We have three roots: $m_1 = 0, m_{2,3} = \pm i\sqrt{k^2+1}$, Therefore,

$$A_k(t) = C_1 \cos\sqrt{k^2+1}t + C_2 \sin\sqrt{k^2+1}t + C_3.$$

Taking derivatives, we obtain

$$A_k'(t) = \sqrt{k^2+1}\left(-C_1 \sin t\sqrt{k^2+1} + C_2 \cos t\sqrt{k^2+1}\right),$$

$$A_k''(t) = \left(k^2+1\right)\left(-C_1 \cos t\sqrt{k^2+1} - C_2 \sin t\sqrt{k^2+1}\right).$$

Using given boundary conditions, the following system of equations are obtained

$$\begin{cases} C_1 + C_3 = 0, \\ \sqrt{k^2+1}\left(C_2 - \frac{1}{2}\left(-C_1 \sin\sqrt{k^2+1} + C_2 \cos\sqrt{k^2+1}\right)\right) = 0, \\ \left(k^2+1\right)\left(-C_1 \cos\sqrt{k^2+1} - C_2 \sin\sqrt{k^2+1}\right) = 0. \end{cases}$$

We have that

$$\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{2}\sin\sqrt{k^2+1} & 1 - \frac{1}{2}\cos\sqrt{k^2+1} & 0 \\ \cos\sqrt{k^2+1} & \sin\sqrt{k^2+1} & 0 \end{vmatrix}$$

$$= \frac{1}{2} - \cos\sqrt{k^2+1} \neq 0.$$

From that it follows

$$C_1 = C_2 = C_3 = 0.$$

Therefore,

$$A_k(t) = 0.$$

Then, the exact solution of the problem (1.8) is

$$u(t,x) = \left(t^2+1\right)\cos x.$$

Note that using similar procedure one can obtain the solution of the following nonlocal-boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^3 u(t,x)}{\partial x_r^2 \partial t} = f(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega},\ 0 < t < T, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), \quad |\gamma| < 1, 0 < \lambda \le T, x \in \overline{\Omega}, \\ u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), x \in \overline{\Omega}, \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), \\ |1 + \alpha\beta| \ge |\alpha| + |\beta|, 0 < \lambda \le T, x \in \overline{\Omega}, \\ u(t,x) = 0, 0 \le t \le T, \quad x \in S \end{cases} \tag{1.9}$$

for the multidimensional a third order partial differential equation. Here $\alpha_r > \alpha > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \xi(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions.

However Fourier series method described in solving (1.9) can be used only in the case when (1.9) has constant coefficients.

Second, we consider the Laplace transform method for solution of local and nonlocal boundary value problems for third order partial differential equations.

Example 1.0.3 *Obtain the solution of the boundary value problem*

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = -e^{-t}e^{-x}, 0 < t < 1, 0 < x < \infty, \\ u(0,x) = e^{-x}, u(1,x) = e^{-1}e^{-x}, u_t(1,x) = -e^{-1}e^{-x}, 0 \le x < \infty, \\ u(t,0) = e^{-t}, u_x(t,0) = -e^{-t}, 0 \le t \le 1 \end{cases}$$

for a third-order partial differential equation by the Laplace transform method.

Solution. Here and in future, we will denote

$$L\{u(t,x)\} = u(t,s).$$

Using formula

$$L\left\{e^{-x}\right\}=\frac{1}{1+s} \tag{1.10}$$

and taking the Laplace transform of both sides of the differential equation, we can write

$$L\{u_{ttt}(t,x)\}-L\{u_{xx}(t,x)\}+L\{u(t,x)\}=-L\left\{e^{-(t+x)}\right\},0<t<1,$$

$$L\{u(0,x)\}=\frac{1}{1+s},L\{u(1,x)\}=e^{-1}\frac{1}{1+s},L\{u_t(1,x)\}=-e^{-1}\frac{1}{1+s}.$$

Applying definition of Laplace transform and conditions $u(t,0)=e^{-t},u_x(t,0)=-e^{-t}$, we can write

$$u_{ttt}(t,s)+(1+s^2)u(t,s)=\left(s-1-\frac{1}{1+s}\right)e^{-t},0<t<1,$$

$$u(0,s)=\frac{1}{1+s},u(1,s)=e^{-1}\frac{1}{1+s},u_t(1,s)=-e^{-1}\frac{1}{1+s}.$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following boundary value problem

$$\begin{cases} u_{ttt}(t,s)+(1+s^2)u(t,s)=\frac{s^2-2}{1+s}e^{-t},0<t<1, \\ u(0,s)=\frac{1}{s+1},u(1,s)=e^{-1}\frac{1}{s+1},u_t(1,s)=-e^{-1}\frac{1}{s+1} \end{cases}$$

for a third order ordinary differential equation. We will seek the general solution of this equation by the formula

$$u(t,s)=u_c(t,s)+u_p(t,s),$$

where $u_c(t,s)$ is the solution of homogeneous equation

$$u_{ttt}(t,s)+(1+s^2)u(t,s)=0$$

and $u_p(t,s)$ is the particular solution of nonhomogeneous equation

$$u_{ttt}(t,s)+(1+s^2)u(t,s)=\frac{s^2-2}{1+s}e^{-t}.$$

For obtain $u_c(t,s)$, we will consider the auxiliary equation

$$m^3+1+s^2=0.$$

We have three roots:

$$m_1 = -\sqrt[3]{1+s^2}, m_{2,3} = \frac{\sqrt[3]{1+s^2} \overset{+}{-} i\sqrt{3}\sqrt[3]{1+s^2}}{2}.$$

Therefore, we will seek the general solution of this equation by the formula

$$\begin{aligned} u_c(t,s) &= C_1 e^{-t\sqrt[3]{1+s^2}} + e^{\frac{\sqrt[3]{1+s^2}t}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right) \right. \\ &\quad \left. + C_3 \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right)\right]. \end{aligned}$$

We seek the particular solution $u_p(t,s)$ of the nonhomogeneous equation by the following formula

$$u_p(t,s) = A(s)\, e^{-t}.$$

Putting into the equation, we get

$$-A(s)e^{-t} + \left(1+s^2\right) A(s) e^{-t} = (s - 1 - \frac{1}{1+s})e^{-t}.$$

Then

$$A(s) = \frac{1}{1+s}.$$

Therefore, the general solution of this equation is

$$\begin{aligned} u(t,s) = C_1 e^{-t\sqrt[3]{1+s^2}} + e^{\frac{\sqrt[3]{1+s^2}t}{2}} &\left[C_2 \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right) \right. \\ &\left. + C_3 \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right)\right] + \frac{e^{-t}}{1+s}. \end{aligned}$$

Taking the derivative, we can write

$$u_t(t,s) = -C_1 \sqrt[3]{1+s^2} e^{-t\sqrt[3]{1+s^2}}$$

$$+C_2 e^{\frac{\sqrt[3]{1+s^2}t}{2}} \frac{\sqrt[3]{1+s^2}}{2} \left[\cos\left(\tfrac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right) - \sqrt{3}\sin\left(\tfrac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right)\right]$$

$$+C_3 e^{\frac{\sqrt[3]{1+s^2}t}{2}} \frac{\sqrt[3]{1+s^2}}{2} \left[\sqrt{3}\cos\left(\tfrac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right) + \sin\left(\tfrac{\sqrt{3}}{2}\sqrt[3]{1+s^2}t\right)\right] - \tfrac{e^{-t}}{1+s}.$$

Using boundary conditions $u(0,s)=\frac{1}{s+1}, u(1,s)=e^{-1}\frac{1}{s+1}, u_t(1,s)=-e^{-1}\frac{1}{s+1}$,we obtain the system of equations

$$\begin{cases} C_1+C_2=0, \\ C_1e^{-\sqrt[3]{1+s^2}}+e^{\frac{\sqrt[3]{1+s^2}}{2}}C_2\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)+e^{\frac{\sqrt[3]{1+s^2}}{2}}C_3\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)=0, \\ -C_1\sqrt[3]{1+s^2}e^{-\sqrt[3]{1+s^2}}+C_2e^{\frac{\sqrt[3]{1+s^2}}{2}}\frac{\sqrt[3]{1+s^2}}{2}\left[\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)\right] \\ +C_3e^{\frac{\sqrt[3]{1+s^2}}{2}}\frac{\sqrt[3]{1+s^2}}{2}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)+\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)\right]=0. \end{cases}$$

We denote that

$$\alpha=\sqrt[3]{1+s^2}.$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ e^{-\alpha} & e^{\frac{\alpha}{2}}\cos\left(\frac{\sqrt{3}}{2}\alpha\right) & e^{\frac{\alpha}{2}}\sin\left(\frac{\sqrt{3}}{2}\alpha\right) \\ -\alpha e^{-\alpha} & e^{\frac{\alpha}{2}}\frac{\alpha}{2}\left(\cos\left(\frac{\sqrt{3}}{2}\alpha\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\alpha\right)\right) & e^{\frac{\alpha}{2}}\frac{\alpha}{2}\left(\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\alpha\right)+\sin\left(\frac{\sqrt{3}}{2}\alpha\right)\right) \end{vmatrix}$$

$$=\frac{1}{2}\sqrt{3}\alpha e^{\alpha}-\frac{1}{2}\sqrt{3}\alpha e^{-\alpha}e^{\frac{1}{2}\alpha}\cos\frac{1}{2}\sqrt{3}\alpha-\frac{3}{2}\alpha e^{-\alpha}e^{\frac{1}{2}\alpha}\sin\frac{1}{2}\sqrt{3}\alpha\neq 0.$$

Therefore,

$$C_1=C_2=C_3=0$$

and

$$u(t,s)=\frac{e^{-t}}{1+s}.$$

Taking the inverse Laplace transform, we can obtain

$$u(t,x)=e^{-(t+x)}.$$

Note that using similar procedure one can obtain the solution of the following boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} a_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \ \ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ 0 < t < T, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), u_t(T,x) = \xi(x), \quad x \in \overline{\Omega}^+, \\ u(t,x) = \alpha(t,x), \quad u_{x_r}(t,x) = \beta(t,x), 1 \le r \le n, 0 \le t \le T, x \in S^+ \end{cases} \tag{1.11}$$

for the multidimensional a third order partial differential equation. Assume that $a_r > a > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}^+\right), \varphi(x), \psi(x), \xi(x), \left(x \in \overline{\Omega}^+\right), \alpha(t,x), \beta(t,x) (t \in [0,T], x \in S^+)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+,

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However Laplace transform method described in solving (1.11) can be used only in the case when (1.11) has constant coefficients.

Example 1.0.4 *Obtain the solution of the nonlocal boundary value problem*

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^3 u(t,x)}{\partial t \partial x^2} + \dfrac{\partial u(t,x)}{\partial t} = 6e^{-x}, 0 < t < 1, 0 < x < \infty, \\ u(0,x) = e^{-x}, u_t(0,x) = \frac{1}{2} u_t(1,x) - \frac{3}{2} e^{-x}, 0 \le x < \infty, \\ u_{tt}(0,x) = \frac{1}{2} u_{tt}(1,x) - 3e^{-x}, 0 \le x < \infty, \\ u(t,0) = t^3 + 1, u_x(t,0) = -\left(t^3 + 1\right), 0 \le t \le 1 \end{cases} \tag{1.12}$$

for a third-order partial differential equation by the Laplace transform method.

Solution. Using formula (1.10) and taking the Laplace transform of both sides of the differential equation, we can write

$$L\{u_{ttt}(t,x)\} - L\{u_{txx}(t,x)\} + L\{u_t(t,x)\} = 6L\left\{e^{-x}\right\}, 0 < t < 1,$$

$$\begin{aligned} L\{u(0,x)\} &= \frac{1}{1+s}, L\{u_t(0,x)\} = \frac{1}{2}L\{u_t(1,x)\} - \frac{3}{2}\frac{1}{1+s}, \\ L\{u_{tt}(0,x)\} &= \frac{1}{2}L\{u_{tt}(1,x)\} - \frac{3}{1+s}. \end{aligned}$$

Applying definition of Laplace transform and conditions $u(t,0) = t^3+1, u_x(t,0) = -\left(t^3+1\right)$, we can write

$$u_{ttt}(t,s) + (1-s^2)u_t(t,s) = \frac{6}{1+s} + (1-s)\,3t^2, 0 < t < 1,$$

$$u(0,s) = \frac{1}{1+s}, u_t(0,s) - \frac{1}{2}u_t(1,x) = -\frac{3}{2}\frac{1}{1+s},$$

$$u_{tt}(0,x) - \frac{1}{2}u_{tt}(1,s) = -\frac{3}{1+s}.$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following nonlocal boundary value problem

$$\begin{cases} u_{ttt}(t,s) + (1-s^2)u_t(t,s) = \frac{6}{1+s} + (1-s)\,3t^2, 0 < t < 1, \\ u(0,s) = \dfrac{1}{1+s}, u_t(0,s) - \frac{1}{2}u_t(1,x) = -\frac{3}{2}\dfrac{1}{1+s}, \\ u_{tt}(0,x) - \frac{1}{2}u_{tt}(1,s) = -\dfrac{3}{1+s} \end{cases}$$

for a third order ordinary differential equation. We will seek the general solution of this equation by the formula

$$u(t,s) = u_c(t,s) + u_p(t,s),$$

where $u_c(t,s)$ is the solution of homogeneous equation

$$u_{ttt}(t,s) + (1-s^2)u_t(t,s) = 0$$

and $u_p(t,s)$ is the particular solution of nonhomogeneous equation

$$u_{ttt}(t,s) + (1-s^2)u_t(t,s) = \left(\frac{6}{1+s} + (1-s)\,3t^2\right).$$

For obtaining $u_c(t,s)$, we will consider the auxiliary equation

$$m^3 + \left(1 - s^2\right) m = 0.$$

We have three roots:

$$m_1 = 0, m_{2,3} = \pm i\sqrt{1 - s^2}.$$

Therefore, we will seek the general solution of this equation by the formula

$$\begin{aligned} u_c(t,s) &= C_1 + C_2 \cos\left(\sqrt{1-s^2}t\right) \\ &\quad + C_3 \sin\left(\sqrt{1-s^2}t\right). \end{aligned}$$

We seek the particular solution $u_p(t,s)$ of the nonhomogeneous equation by the following formula

$$u_p(t,s) = t\left(at^2 + bt + c\right).$$

Putting into the equation, we get

$$6a + \left(1 - s^2\right)\left[3at^2 + 2bt + c\right] = 3(1-s)t^2 + \frac{6}{1+s}.$$

Equating coefficients $t^m, m = 0,1,2$ to zero, we get $a = \frac{1}{1+s}, b = 0, c = 0.$Then

$$u_p(t,s) = \frac{t^3}{1+s}.$$

Therefore, the general solution of this equation is

$$\begin{aligned} u(t,s) = C_1 + \Big[&C_2 \cos\left(\sqrt{1-s^2}t\right) \\ &+ C_3 \sin\left(\sqrt{1-s^2}t\right)\Big] + \frac{t^3}{1+s}. \end{aligned}$$

Taking the derivatives, we can write

$$u_t(t,s) = C_2\left[-\sqrt{1+s^2}\sin\left(\sqrt{1+s^2}t\right)\right] + C_3\left[\sqrt{1+s^2}\cos\left(\sqrt{1+s^2}t\right)\right] + \frac{3t^2}{1+s},$$

$$u_{tt}(t,s) = C_2\left[-\left(1+s^2\right)\left(\cos\sqrt{1+s^2}t\right)\right] - C_3\left[\left(1+s^2\right)\sin\left(\sqrt{1+s^2}t\right)\right] + \frac{6t}{1+s}.$$

Using nonlocal boundary conditions $u(0,s)=\dfrac{1}{1+s}, u_t(0,s)-\frac{1}{2}u_t(1,x)=-\frac{3}{2}\dfrac{1}{1+s}, u_{tt}(0,x)-\frac{1}{2}u_{tt}(1,s)=-3\dfrac{1}{1+s}$,we obtain the system of equations

$$\begin{cases} C_1+C_2=\dfrac{1}{1+s}, \\ C_3\sqrt{1+s^2}\left(1-\frac{1}{2}\cos\left(\sqrt{1+s^2}\right)\right)+C_2\frac{1}{2}\sqrt{1+s^2}\sin\left(\sqrt{1+s^2}\right)=0, \\ C_2\left(1+s^2\right)\left(\frac{1}{2}\left(\cos\sqrt{1+s^2}\right)-1\right)+\frac{1}{2}C_3\left(1+s^2\right)\sin\left(\sqrt{1+s^2}\right)=0. \end{cases}$$

We denote that

$$\alpha=\sqrt{1+s^2}.$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2}\alpha\sin(\alpha) & \alpha\left(1-\frac{1}{2}\cos(\alpha)\right) \\ 0 & \alpha^2\left(\frac{1}{2}(\cos\alpha)-1\right) & \frac{1}{2}\alpha^2\sin(\alpha) \end{vmatrix}$$

$$=\frac{5}{4}\alpha^3-\alpha^3\cos\alpha\neq 0.$$

Therefore,

$$C_1=\frac{1}{1+s}, C_2=C_3=0$$

and

$$u(t,s)=\frac{1}{1+s}+\frac{t^3}{1+s}.$$

Taking the inverse Laplace transform, we can obtain

$$u(t,x)=\left(t^3+1\right)e^{-x}.$$

Note that using similar procedure one can obtain the solution of the following nonlocal boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} a_r \dfrac{\partial^3 u(t,x)}{\partial t \partial x_r^2} = f(t,x), \ \ x=(x_1,...,x_n) \in \overline{\Omega}^+, \ 0<t<T, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), u_t(0,x) = \alpha u_t(\lambda,x) \psi(x), |\gamma|<1, 0<\lambda \le T, \\ u_{tt}(0,x) = \beta u_{tt}(0,x) + \xi(x), |1+\alpha\beta| > |\alpha|+|\beta|, 0<\lambda \le T \quad x \in \overline{\Omega}^+, \\ u(t,x) = \chi(t,x), \quad u_{x_r}(t,x) = \omega(t,x), 1 \le r \le n, 0 \le t \le T, x \in S^+ \end{cases} \tag{1.13}$$

for the multidimensional a third order partial differential equation. Assume that $a_r > a > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}^+\right), \varphi(x), \psi(x), \xi(x), \left(x \in \overline{\Omega}^+\right), \chi(t,x), \omega(t,x) \, (t \in [0,T], x \in S^+)$ are given smooth functions. However Laplace transform method described in solving (1.13) can be used only in the case when (1.13) has constant coefficients.

Third, we consider Fourier transform method for solution of local and nonlocal boundary value problems for third order partial differential equations.

Example 1.0.5 *Obtain the Fourier transform solution of the following boundary value problem*

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^3 u(t,x)}{\partial x^2 \partial t} + u(t,x) = \left(4x^2-2\right) e^{-\left(t+x^2\right)}, 0<t<1, -\infty<x<\infty, \\ u(0,x) = e^{-x^2}, u(1,x) = e^{-1}e^{-x^2}, u_t(1,x) = -e^{-1}e^{-x^2}, -\infty<x<\infty \end{cases}$$

for a third order partial differential equation.

Solution. Here and in future, we will denote

$$F\{u(t,x)\} = u(t,s). \tag{1.14}$$

Taking the Fourier transform of both sides of the differential equation, we can write

$$F\{u_{ttt}(t,x)\} - F\{u_{xx}(t,x)\} + F\{u(t,x)\} = F\left\{\left(-4x^2+2\right)e^{-\left(t+x^2\right)}\right\}, 0<t<1,$$

$$F\{u(0,x)\} = F\left\{e^{-x^2}\right\}, F\{u(1,x)\} = e^{-1}F\left\{e^{-x^2}\right\}, F\{u_t(1,x)\} = -e^{-1}F\left\{e^{-x^2}\right\}.$$

Applying definition of Fourier transform and conditions $u(0,x)=e^{-x^2}, u(1,x)=e^{-1}e^{-x^2}, u_t(1,x)=-e^{-1}e^{-x^2}$, we can write

$$u_{ttt}(t,s)+(1+s^2)u(t,s)=s^2e^{-t}F\left\{e^{-x^2}\right\}, 0<t<1,$$

$$u(0,s)=F\left\{e^{-x^2}\right\}, u(1,s)=e^{-1}F\left\{e^{-x^2}\right\}, u_t(1,s)=-e^{-1}F\left\{e^{-x^2}\right\}.$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following boundary value problem

$$\begin{cases} u_{ttt}(t,s)+(1+s^2)u(t,s)=s^2e^{-t}F\left\{e^{-x^2}\right\}, 0<t<1, \\ u(0,s)=F\left\{e^{-x^2}\right\}, u(1,s)=e^{-1}F\left\{e^{-x^2}\right\}, u_t(1,s)=-e^{-1}F\left\{e^{-x^2}\right\} \end{cases}$$

for a third order ordinary differential equation. We will seek the general solution of this equation by the formula

$$u(t,s)=u_c(t,s)+u_p(t,s),$$

where $u_c(t,s)$ is the solution of homogeneous equation

$$u_{ttt}(t,s)+(1+s^2)u(t,s)=0$$

and $u_p(t,s)$ is the particular solution of nonhomogeneous equation

$$u_{ttt}(t,s)+(1+s^2)u(t,s)=s^2e^{-t}F\left\{e^{-x^2}\right\}.$$

For obtaining $u_c(t,s)$, we will consider the auxiliary equation

$$m^3+\left(1+s^2\right)=0.$$

We have three roots:

$$m_1=-\sqrt[3]{1+s^2}, m_{2,3}=\frac{\sqrt[3]{1+s^2}\pm i\sqrt{3}\sqrt[3]{1+s^2}}{2}.$$

Therefore, we will seek the general solution of this equation by the formula

$$\begin{aligned} u_c(t,s) &= C_1 e^{-t\sqrt[3]{1+s^2}} + e^{\frac{\sqrt[3]{1+s^2}t}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) \right. \\ & \left. + C_3 \sin\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) \right]. \end{aligned}$$

We seek the particular solution $u_p(t,s)$ of the nonhomogeneous equation by the following formula

$$u_p(t,s) = A(s)\, e^{-t}.$$

Putting into the equation, we get

$$-A(s)\, e^{-t} + A(s)\left(1+s^2\right) e^{-t} = e^{-t} s^2 F\left\{ e^{-x^2} \right\}.$$

From that it follows

$$A(s) = F\left\{ e^{-x^2} \right\}.$$

Then

$$u_p(t,s) = e^{-t} F\left\{ e^{-x^2} \right\}.$$

Therefore, the general solution of this equation is

$$\begin{aligned} u(t,s) &= C_1 e^{-t\sqrt[3]{1+s^2}} + e^{\frac{\sqrt[3]{1+s^2}t}{2}} C_2 \cos\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) \\ &+ e^{\frac{\sqrt[3]{1+s^2}t}{2}} C_3 \sin\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) + e^{-t} F\left\{ e^{-x^2} \right\}. \end{aligned}$$

Taking the derivative, we can write

$$\begin{aligned} u_t(t,s) &= -C_1 \sqrt[3]{1+s^2} e^{-t\sqrt[3]{1+s^2}} \\ &+ C_2 e^{\frac{\sqrt[3]{1+s^2}t}{2}} \frac{\sqrt[3]{1+s^2}}{2} \left[\cos\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) \right] \\ &+ C_3 e^{\frac{\sqrt[3]{1+s^2}t}{2}} \frac{\sqrt[3]{1+s^2}}{2} \left[\sqrt{3} \cos\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) + \sin\left(\frac{\sqrt{3}}{2} \sqrt[3]{1+s^2}t \right) \right] \\ &- e^{-t} F\left\{ e^{-x^2} \right\}. \end{aligned}$$

Using boundary conditions $u(0,s)=F\left\{e^{-x^2}\right\}, u(1,s)=e^{-1}F\left\{e^{-x^2}\right\}, u_t(1,s)=-e^{-1}F\left\{e^{-x^2}\right\}$, we obtain the system of equations

$$\begin{cases} C_1+C_2=0, \\ C_1e^{-\sqrt[3]{1+s^2}}+e^{\frac{\sqrt[3]{1+s^2}}{2}}C_2\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)+e^{\frac{\sqrt[3]{1+s^2}}{2}}C_3\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)=0, \\ -C_1\sqrt[3]{1+s^2}e^{-\sqrt[3]{1+s^2}}+C_2e^{\frac{\sqrt[3]{1+s^2}}{2}}\frac{\sqrt[3]{1+s^2}}{2}\left[\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)\right] \\ +C_3e^{\frac{\sqrt[3]{1+s^2}}{2}}\frac{\sqrt[3]{1+s^2}}{2}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)+\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1+s^2}\right)\right]=0. \end{cases}$$

We denote that

$$\alpha=\frac{\sqrt[3]{1+s^2}}{2}.$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ e^{-2\alpha} & e^{\alpha}\left(\cos\left(\sqrt{3}\alpha\right)\right) & e^{\alpha}\sin\left(\sqrt{3}\alpha\right) \\ -2\alpha e^{-\alpha} & e^{\alpha}\alpha\left(\cos\left(\sqrt{3}\alpha\right)-\sqrt{3}\sin\left(\sqrt{3}\alpha\right)\right) & e^{\alpha}\alpha\left(\sqrt{3}\cos\left(\sqrt{3}\alpha\right)+\sin\left(\sqrt{3}\alpha\right)\right) \end{vmatrix}$$

$$\sqrt{3}\alpha e^{2\alpha}-\alpha e^{-\alpha}\left(\left(\sin\sqrt{3}\alpha\right)-\sqrt{3}\left(\cos\sqrt{3}\alpha\right)\right)-2\alpha\left(\sin\sqrt{3}\alpha\right)\neq 0.$$

Therefore,

$$C_1=C_2=C_3=0.$$

and

$$u(t,s)=e^{-t}F\left\{e^{-x^2}\right\}.$$

Taking the inverse Fourier transform, we can obtain

$$u(t,x)=e^{-t}e^{-x^2}.$$

Note that using the same manner one obtain the solution of the following boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{|r|=2m} \alpha_r \dfrac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} ... \partial x_n^{r_n}} = f(t,x), \\ \\ 0 < t < T, x, r \in \mathbb{R}^n, |r| = r_1 + ... + r_n, \\ \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), u_t(T,x) = \xi(x), x \in \mathbb{R}^n \end{cases}$$

for a third order in t and $2m-th$ order in space variables multidimensional partial differential equation. Assume that $\alpha_r \geq \alpha \geq 0$ and $f(t,x), (t \in [0,T], x \in \mathbb{R}^n), \varphi(x), \psi(x), \xi(x), (x \in \mathbb{R}^n)$ are given smooth functions.

Example 1.0.6 *Obtain the Fourier transform solution of the following nonlocal boundary value problem*

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^3 u(t,x)}{\partial t \partial x^2} + \dfrac{\partial u(t,x)}{\partial t} = \left[6 + 3t^2\left(-4x^2+3\right)\right] e^{-x^2}, 0 < t < 1, -\infty < x < \infty, \\ \\ u(0,x) = e^{-x^2}, u_t(0,x) = \frac{1}{2} u_t(1,x) - \frac{3}{2} e^{-x^2}, -\infty < x < \infty, \\ \\ u_{tt}(0,x) = \frac{1}{2} u_{tt}(1,x) - 3e^{-x^2}, -\infty < x < \infty \end{cases}$$

for a third order ordinary differential equation.

Solution. Using formula (1.14) and taking the Fourier transform of both sides of the differential equation, we can write

$$F\{u_{ttt}(t,x)\} - F\{u_{txx}(t,x)\} + F\{u_t(t,x)\} = F\left\{\left[6 + 3t^2\left(-4x^2+3\right)\right] e^{-x^2}\right\}, 0 < t < 1,$$

$$\begin{aligned} F\{u(0,x)\} &= F\left\{e^{-x^2}\right\}, F\{u_t(0,x)\} = F\left\{\frac{1}{2} u_t(1,x) - \frac{3}{2} e^{-x^2}\right\}, \\ F\{u_{tt}(0,x)\} &= F\left\{\frac{1}{2} u_{tt}(1,x) - 3e^{-x^2}\right\}. \end{aligned}$$

Applying definition of Fourier transform and conditions $u(0,x) = e^{-x^2}, u_t(0,x) = \frac{1}{2} u_t(1,x) - \frac{3}{2} e^{-x^2}, u_{tt}(0,x) = \frac{1}{2} u_{tt}(1,x) - 3e^{-x^2}$, we can write

$$u_{ttt}(t,s)+(1+s^2)u_t(t,s)=\left(6+3t^2\left(s^2+1\right)\right)F\left\{e^{-x^2}\right\},0<t<1,$$

$$u(0,s)=F\left\{e^{-x^2}\right\},u_t(0,s)=\frac{1}{2}u_t(1,s)-\frac{3}{2}F\left\{e^{-x^2}\right\},u_{tt}(0,s)=\frac{1}{2}u_{tt}(1,s)-3F\left\{e^{-x^2}\right\}.$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following nonlocal boundary value problem

$$\begin{cases} u_{ttt}(t,s)+(1+s^2)u_t(t,s)=\left(6+3t^2\left(s^2+1\right)\right)F\left\{e^{-x^2}\right\},0<t<1, \\ u(0,s)=F\left\{e^{-x^2}\right\},u_t(0,s)=\frac{1}{2}u_t(1,s)-\frac{3}{2}F\left\{e^{-x^2}\right\}, \\ u_{tt}(0,s)=\frac{1}{2}u_{tt}(1,s)-3F\left\{e^{-x^2}\right\} \end{cases}$$

for a third order ordinary differential equation. We will seek the general solution of this equation by the formula

$$u(t,s)=u_c(t,s)+u_p(t,s),$$

where $u_c(t,s)$ is the solution of homogeneous equation

$$u_{ttt}(t,s)+(1+s^2)u_t(t,s)=0$$

and $u_p(t,s)$ is the particular solution of nonhomogeneous equation

$$u_{ttt}(t,s)+(1+s^2)u_t(t,s)=\left(6+3t^2\left(s^2+1\right)\right)F\left\{e^{-x^2}\right\}.$$

For obtaining $u_c(t,s)$, we will consider the auxiliary equation

$$m^3+\left(1+s^2\right)m=0.$$

We have three roots:

$$m_1=0,m_{2,3}=\pm i\sqrt{s^2+1}.$$

Therefore, we will seek the general solution of this equation by the formula

$$u_c(t,s)=C_1+C_2\cos\left(\sqrt{1+s^2}t\right)+C_3\sin\left(\sqrt{1+s^2}t\right).$$

We seek the particular solution $u_p(t,s)$ of the nonhomogeneous equation by the following formula

$$u_p(t,s) = t\left(at^2+bt+c\right).$$

Putting into the equation, we get

$$3a\left(1+s^2\right)t^2+2b\left(1+s^2\right)t+\left(1+s^2\right)c+2a = \left(6+3t^2\left(s^2+1\right)\right)F\left\{e^{-x^2}\right\}.$$

Equating coefficients $t^m, m=0,1,2$ to zero, we get

$$a = F\left\{e^{-x^2}\right\}, b=0, c=0.$$

Then

$$u_p(t,s) = t^3F\left\{e^{-x^2}\right\}.$$

Therefore, the general solution of this equation is

$$u(t,s) = C_1+C_2\cos\left(\sqrt{1+s^2}t\right)$$
$$+C_3\sin\left(\sqrt{1+s^2}t\right)+t^3F\left\{e^{-x^2}\right\}.$$

Taking the derivatives, we can write

$$u_t(t,s) = -\sqrt{1+s^2}C_2\sin\left(\sqrt{1+s^2}t\right)$$

$$+\sqrt{1+s^2}C_3\cos\left(\sqrt{1+s^2}t\right)\left(3t^2\right)F\left\{e^{-x^2}\right\},$$

$$\begin{aligned} u_{tt}(t,s) &= -\left(1+s^2\right)C_2\cos\left(\sqrt{1+s^2}t\right) \\ &\quad -\left(1+s^2\right)C_3\sin\left(\sqrt{1+s^2}t\right)+6tF\left\{e^{-x^2}\right\}. \end{aligned}$$

Using boundary conditions $u(0,s) = F\left\{e^{-x^2}\right\}, u_t(0,s) = \frac{1}{2}u_t(1,s)-\frac{3}{2}F\left\{e^{-x^2}\right\}, u_{tt}(0,s) = \frac{1}{2}u_{tt}(1,s)-3F\left\{e^{-x^2}\right\}$,we obtain the system of equations

$$\begin{cases} C_1 + C_2 = F\left\{e^{-x^2}\right\}, \\ \frac{1}{2}\left(\sqrt{1+s^2}C_2 \sin\sqrt{1+s^2}\right) + \sqrt{1+s^2}C_3\left(1 - \frac{1}{2}\cos\sqrt{1+s^2}\right) = 0, \\ \left(1+s^2\right)C_2\left(-1 + \frac{1}{2}\cos\sqrt{1+s^2}\right) + \frac{1}{2}\left(1+s^2\right)C_3 \sin\sqrt{1+s^2} = 0. \end{cases}$$

We denote that

$$\alpha = \sqrt{1+s^2}.$$

We have that

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2}(\alpha\sin\alpha) & \alpha\left(1 - \frac{1}{2}\cos\alpha\right) \\ 0 & \alpha^2\left(-1 + \frac{1}{2}\cos\alpha\right) & \frac{1}{2}\alpha^2\sin\alpha \end{vmatrix}$$

$$\frac{5}{4}\alpha^3 - \alpha^3\cos\alpha \neq 0.$$

Therefore,

$$C_1 = F\left\{e^{-x^2}\right\}, C_2 = C_3 = 0$$

and

$$u(t,s) = \left(t^3 + 1\right)F\left\{e^{-x^2}\right\}.$$

Taking the inverse Fourier transform, we can obtain

$$u(t,x) = \left(t^3 + 1\right)e^{-x^2}.$$

Note that using similar procedure one can obtain the solution of the following nonlocal boundary value problem

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{|r|=2m} \alpha_r \frac{\partial^{|r|+1} u(t,x)}{\partial t \partial x_1^{r_1} \ldots \partial x_n^{r_n}} = f(t,x),\ 0 < t < T, x, r \in \mathbb{R}^n, |r| = r_1 + \ldots + r_n, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), |\gamma| < 1, 0 < \lambda \leq T \quad x \in \mathbb{R}^n, \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), |1 + \alpha\beta| > |\alpha| + |\beta|, x \in \mathbb{R}^n \end{cases} \tag{1.15}$$

for a third order in t and $2m+1-th$ order in space variables multidimensional partial differential equation. Assume that $\alpha_r \geq \alpha \geq 0$ and $f(t,x), (t \in [0,T], x \in \mathbb{R}^n)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $(x \in \mathbb{R}^n)$ are given smooth functions.

However, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant coefficients. It is well-known that the most general method for solving partial differential equation with dependent in t and in the space variables is operator method.

Now, let us briefly describe the contents of the various chapters of the thesis. It consists of four chapters.

First chapter is the introduction.

Second chapter the boundary value problem (1.4) for a third order partial differential equation is investigated. The main theorem on the stability estimates for the solution of problem (1.4) in a Hilbert space is established. Three applications of the main theorem are given. Theorems on the stability estimates for the solutions of these applications are obtained. A first and high order of accuracy difference schemes are constructed for the approximate solution of (1.4). Numerical results are given.

Third chapter consists the nonlocal boundary value problem (1.5) for a third order partial differential equation. The main theorem on the stability estimates for the solution of problem (1.5) in a Hilbert space is established. Three applications of the main theorem are given. Theorems on the stability estimates for the solutions of these applications are obtained. A first and high order of accuracy difference schemes are constructed for the approximate solution of (1.5). Numerical results are given.

Fourth chapter contains conclusions.

Chapter 2

Stability of a BVP for the Third Order Partial Differential Equation

2.1 Introduction

In chapter 2 we consider the boundary value problem for the third order partial differential equation

$$\begin{cases} \dfrac{d^3u(t)}{dt^3} + Au(t) = f(t), \quad 0 < t < 1, \\ \\ u(0) = \varphi, \quad u(1) = \psi, \quad u'(1) = \xi \end{cases} \tag{2.1}$$

in a Hilbert space H with a self-adjoint positive definite operator $A \geq \delta I$,where $\delta > 0$.

We are interested in studying the stability of solutions of problem (2.1). A function $u(t)$ is a solution of problem (2.1) if the following conditions are satisfied:

(i) $u(t)$ is thrice continuously differentiable on the interval $(0,1)$ and continuously differentiable on the segment $[0,1]$. The derivatives at the end points of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$, and function $Au(t)$ is continuous on the segment $[0,1]$.

(iii) $u(t)$ satisfies the equation and boundary conditions (2.1).

Applying operator approach, stability estimates for solution of boundary value problem (2.1) are obtained. In practice, boundary value problems for a third order in t partial differential equations are studied. Theorems on the stability estimates for the solutions of these problems are obtained. A first and high order of accuracy difference schemes are constructed for the approximate solution of the one dimensional partial differential equations. Numerical results are given.

The outline of Chapter 2 is as follows. The first section is introduction. In the section 2 main theorem on stability of problem (2.1) is established. Section 3 establishes the stability estimates for the solution of three problems for partial differential equations of third order in t. Section 4 numerical analysis.

2.2 Main theorem on stability

Let us prove some lemmas needed in the sequel.

Lemma 2.2.1 *[53], For $t \geq 0$ the following estimate holds*

$$\left\| \exp\left\{ \pm itA^{\frac{1}{3}} \right\} \right\|_{H\to H} \leq 1. \tag{2.2}$$

Proof. Applying the spectral representation of unit self-adjoint positive definite operator A, we get

$$\left\| \exp\left\{ \pm itA^{\frac{1}{3}} \right\} \right\|_{H\to H} \leq \sup_{\delta\leq\lambda<\infty} \left| \exp\left\{ \pm it\lambda^{\frac{1}{3}} \right\} \right| = 1.$$

■

Lemma 2.2.2 *Assume that $\delta > \left(\frac{1}{3}\ln 4\right)^3$. Then, the operator Δ defined by the following formula*

$$\Delta = \frac{1}{3}\left\{ I - \left(ae^{-(1+a)B} + \bar{a}e^{-(1+\bar{a})B} \right) \right\}$$

has a bounded inverse $T = \Delta^{-1}$ and the following estimate holds

$$\|T\|_{H\to H} \leq \frac{3}{1-2e^{-(3/2)\delta^{\frac{1}{3}}}}. \tag{2.3}$$

Here $a = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \bar{a} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $B = A^{\frac{1}{3}}$.

Proof. Using estimate (2.2) and triangle inequality, we can write

$$\left\| ae^{-(1+a)B} + \bar{a}e^{-(1+\bar{a})B} \right\|_{H\to H}$$

$$\leq \left\| ae^{-(1+a)B} \right\|_{H\to H} + \left\| \bar{a}e^{-(1+\bar{a})B} \right\|_{H\to H}$$

$$\leq |a| \left\| e^{-(1+a)B} \right\|_{H\to H} + |\bar{a}| \left\| e^{-(1+\bar{a})B} \right\|_{H\to H}$$

$$\leq \left\| e^{-(3/2)B} \right\|_{H\to H} \left[\left\| e^{-\left(i\sqrt{3}/2\right)B} \right\|_{H\to H} + \left\| e^{\left(i\sqrt{3}/2\right)B} \right\|_{H\to H} \right]$$

$$\leq 2e^{-(3/2)\delta^{1/3}}.$$

Therefore,

$$\|\Delta\|_{H\to H} = \left\| \frac{1}{3} \left\{ I - \left(ae^{-(1+a)B} + \bar{a}e^{-(1+\bar{a})B} \right) \right\} \right\|_{H\to H}$$

$$\geq \frac{1}{3}\left[1 - \left\| ae^{-(1+a)B} + \bar{a}e^{-(1+\bar{a})B} \right\|_{H\to H} \right] \geq \frac{1}{3}\left[1 - 2e^{-(3/2)\delta^{\frac{1}{3}}} \right] > 0.$$

From that it follows estimate (2.3). Lemma 2.2.2 proved. ∎

Lemma 2.2.3 *Suppose that* $\delta > \left(\frac{1}{3}\ln 4\right)^3$, $\varphi \in D(A)$, $\psi \in D(A)$, $\xi \in D\left(A^{\frac{2}{3}}\right)$ *and* $f(t)$ *is continuously differentiable on* $[0,1]$. *Then there is unique solution of problem (2.1) and the following formula holds*

$$u(t) = e^{-Bt}u(0) + \frac{1}{1+a}B^{-1}\left(e^{-(1-t)B} - e^{-(a+t)B}\right)\left(u'(1) + Bu(1)\right)$$

$$+\frac{1}{a-\bar{a}}B^{-2}\left\{ \frac{1}{1+a}\left(e^{-(1-t)aB} - e^{-(a+t)B}\right) - \frac{1}{1+\bar{a}}\left(e^{-(1-t)\bar{a}B} - e^{-(\bar{a}+t)B}\right) \right\}$$

$$\times \left(u''(1) + \bar{a}Bu'(1) - aB^2u(1)\right) - \frac{1}{a-\bar{a}}B^{-2}$$

$$\times \int_0^t \left[\frac{1}{1+a}\left(e^{-(t-s)B} - e^{-(t+sa)B}\right) - \frac{1}{1+\bar{a}}\left(e^{-(t-s)B} - e^{-(t+s\bar{a})B}\right) \right] f(s)ds, \tag{2.4}$$

where

$$u''(1) = T\left\{ B^2e^{-B}u(0) + \frac{1}{1+a}B\left(I - e^{-(a+1)B}\right)\left(u'(1) + Bu(1)\right) \right.$$

$$+\frac{1}{a-\bar{a}}\left\{ \frac{1}{1+a}\left(a^2I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(\bar{a}^2I - e^{-(\bar{a}+1)B}\right) \right\}\left(\bar{a}Bu'(1) - aB^2u(1)\right)$$

$$-\frac{1}{a-\bar{a}}B^{-1}\left[e^{-(1+a)B} - e^{-(1+\bar{a})B} - \frac{1}{1+a}\left(e^{-(a+1)B} - I\right) + \frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B} - I\right) \right] f(1)$$

$$-\frac{1}{a-\bar{a}}B^{-2}\left[\frac{1}{1+a}\left(I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(I-e^{-(\bar{a}+1)B}\right)\right]f'(1)$$

$$-\frac{1}{a-\bar{a}}\int_0^1\left[\frac{1}{1+a}\left(e^{-(1-s)B}-e^{-(sa+1)B}\right)-\frac{1}{1+\bar{a}}\left(e^{-(1-s)B}-e^{-(s\bar{a}+1)B}\right)\right]f(s)ds\Bigg\}. \tag{2.5}$$

Proof. Obviously, we can write

$$\left(\frac{d}{dt}-\bar{a}B\right)\left(\frac{d}{dt}-aB\right)\left(\frac{d}{dt}+B\right)u(t)=f(t)$$

for all $u(t)\in D(A)$. Therefore, problem (2.1) can be written as the equivalent boundary value problem

$$\begin{cases}\dfrac{du(t)}{dt}+Bu(t)=v(t),u(0)=\varphi,u(1)=\psi,\\[2ex] \dfrac{dv(t)}{dt}-aBv(t)=w(t),u'(1)=\xi,\\[2ex] \dfrac{dw(t)}{dt}-\bar{a}Bw(t)=f(t),\ \ 0<t<1\end{cases} \tag{2.6}$$

for the system of first order differential equations. Integrating these equations, we can write

$$\begin{cases}w(t)=e^{-(1-t)\bar{a}B}w(1)-\int\limits_t^1 e^{-(s-t)\bar{a}B}f(s)\,ds,\\[2ex] v(t)=e^{-(1-t)aB}v(1)-\int\limits_t^1 e^{-(p-t)aB}w(p)dp,\\[2ex] u(t)=e^{-Bt}u(0)+\int\limits_0^t e^{-(t-p)B}v(p)dp.\end{cases} \tag{2.7}$$

Applying equations (2.6), we get

$$v(1)=u'(1)+Bu(1),$$
$$w(1)=v'(1)-aBv(1)=u''(1)+\bar{a}Bu'(1)-aB^2u(1).$$

Then, we have that

$$w(t)=e^{-(1-t)\bar{a}B}w(1)-\int_t^1 e^{-(s-t)\bar{a}B}f(s)\,ds$$

$$= e^{-(1-t)\bar{a}B}\left[u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right]-\int_t^1 e^{-(s-t)\bar{a}B}f(s)\,ds. \tag{2.8}$$

Using formulas (2.7), (2.8), we get

$$v(t)=e^{-(1-t)aB}v(1)-\int_t^1 e^{-(p-t)aB}w(p)dp$$

$$=e^{-(1-t)aB}v(1)-\int_t^1 e^{-(p-t)aB}\left(e^{-(1-s)\bar{a}B}\left[u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right]\right.$$

$$\left.-\int_p^1 e^{-(s-p)\bar{a}B}f(s)\,ds\right)dp=e^{-(1-t)aB}\left(u'(1)+Bu(1)\right)$$

$$-\int_t^1 e^{-(p-t)aB}e^{-(1-p)\bar{a}B}dp\left[u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right]$$

$$+\int_t^1\int_p^1 e^{-(p-t)aB}e^{-(1-s)\bar{a}B}f(s)\,dsdp.$$

Applying formulas

$$\int_t^1 e^{-(p-t)aB}e^{-(1-p)\bar{a}B}dp=-\frac{1}{a-\bar{a}}B^{-1}\left(e^{-(1-t)aB}-e^{-(1-t)\bar{a}B}\right),$$

$$\int_t^1\int_p^1 e^{-(p-t)aB}e^{-(1-p)\bar{a}B}f(s)\,dsdp=\int_t^1\int_t^s e^{-(p-t)aB}e^{-(1-p)\bar{a}B}dpf(s)\,ds$$

$$=-\frac{1}{a-\bar{a}}B^{-1}\int_t^1\left(e^{-(s-t)aB}-e^{-(s-t)\bar{a}B}\right)f(s)\,ds,$$

we obtain

$$v(t)=\frac{1}{a-\bar{a}}B^{-1}\left(e^{-(1-t)aB}-e^{-(1-t)\bar{a}B}\right)\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right) \tag{2.9}$$

$$+e^{-(1-t)aB}\left(u'(1)+Bu(1)\right)-\frac{1}{a-\bar{a}}B^{-1}\int_t^1\left(e^{-(s-t)aB}-e^{-(s-t)\bar{a}B}\right)f(s)\,ds.$$

Applying formulas (2.9) and (2.7), we get

$$u(t)=e^{-Bt}u(0)+\int_0^t e^{-(t-p)B}v(p)dp$$

$$=e^{-Bt}u(0)+\int_0^t e^{-(t-p)B}\left[\frac{1}{a-\bar{a}}B^{-1}\left(e^{-(1-p)aB}-e^{-(1-p)\bar{a}B}\right)\right.$$

$$\times\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right)+e^{-(1-p)aB}\left(u'(1)+Bu(1)\right)$$

$$\left.-\frac{1}{a-\bar{a}}B^{-1}\int_p^1\left(e^{-(s-p)aB}-e^{-(s-p)\bar{a}B}\right)f(s)\,ds\right]dp$$

$$=e^{-Bt}u(0)+\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right)+e^{-(1-p)aB}\left(u'(1)+Bu(1)\right)$$

$$\times\frac{1}{a-\bar{a}}B^{-1}\int_0^t e^{-(t-p)B}\left(e^{-(1-p)aB}-e^{-(1-p)\bar{a}B}\right)dp$$

$$-\frac{1}{a-\bar{a}}B^{-1}\int_0^t\int_p^1 e^{-(t-p)B}\left(e^{-(s-p)aB}-e^{-(s-p)\bar{a}B}\right)f(s)\,dsdp.$$

Making the change of the order of double integral, we can write

$$\int_0^t\int_p^1 e^{-(t-p)B}\left(e^{-(s-p)aB}-e^{-(s-p)\bar{a}B}\right)f(s)\,dsdp$$

$$=\int_0^t\int_0^s e^{-(t-p)B}\left(e^{-(s-p)aB}-e^{-(s-p)\bar{a}B}\right)dpf(s)\,ds$$

$$=B^{-1}\int_0^t\left[\frac{1}{1+a}\left(e^{-(t-s)B}-e^{-(t+sa)B}\right)-\frac{1}{1+\bar{a}}\left(e^{-(t-s)B}-e^{-(t+s\bar{a})B}\right)\right]f(s)ds. \quad (2.10)$$

Applying formulas (2.10) and

$$\int_0^t e^{-(t-p)B}\left(e^{-(1-p)aB}-e^{-(1-p)\bar{a}B}\right)dp$$

$$=\frac{B^{-1}}{1+a}\left(e^{-(1-t)aB}-e^{-(t+a)B}\right)-\frac{B^{-1}}{1+\bar{a}}\left(e^{-(1-t)\bar{a}B}-e^{-(t+\bar{a})B}\right),$$

we obtain formula (2.4). Taking the second order derivative and putting $t=1$, we get the following operator equation with respect to $u''(1)$.

$$u''(1)=B^2e^{-B}u(0)+\frac{1}{1+a}B\left(I-e^{-(a+1)B}\right)\left(u'(1)+Bu(1)\right)$$

$$+\frac{1}{a-\bar{a}}\left\{\frac{1}{1+a}\left(a^2I-e^{-(a+1)B}\right)-\frac{1}{1+\overline{a}}\left(\overline{a}^2I-e^{-(\bar{a}+1)B}\right)\right\}$$

$$\times\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right)-\frac{1}{a-\bar{a}}B^{-1}$$

$$\times\left[e^{-(1+a)B}-e^{-(1+\bar{a})B}-\frac{1}{1+a}\left(e^{-(a+1)B}-I\right)+\frac{1}{1+\overline{a}}\left(e^{-(\bar{a}+1)B}-I\right)\right]f(1)$$

$$-\frac{1}{a-\bar{a}}B^{-2}\left[\frac{1}{1+a}\left(I-e^{-(a+1)B}\right)-\frac{1}{1+\overline{a}}\left(I-e^{-(\bar{a}+1)B}\right)\right]f'(1)$$

$$-\frac{1}{a-\bar{a}}\int_0^1\left[\frac{1}{1+a}\left(e^{-(1-s)B}-e^{-(sa+1)B}\right)-\frac{1}{1+\overline{a}}\left(e^{-(1-s)B}-e^{-(s\overline{a}+1)B}\right)\right]f(s)ds. \quad (2.11)$$

Since

$$\Delta=\frac{1}{a-\bar{a}}\left\{\frac{1}{1+a}\left(a^2I-e^{-(a+1)B}\right)-\frac{1}{1+\overline{a}}\left(\overline{a}^2I-e^{-(\bar{a}+1)B}\right)\right\}$$

$$=-\frac{1}{3}\left\{I-\left(ae^{-(1+a)B}+\bar{a}e^{-(1+\bar{a})B}\right)\right\}$$

has a bounded inverse $T=\Delta^{-1}$, using lemma 2.2.2, we get formula (2.5). Lemma 2.2.3. is proved. ■

Now, we will establish the following main theorem.

Theorem 2.2.4 *Assume that* $\delta>\left(\frac{1}{3}\ln 4\right)^3$, $\varphi\in D(A)$, $\psi\in D(A)$, $\xi\in D\left(A^{2/3}\right)$ *and* $f(t)$ *is continuously differentiable on* $[0,1]$. *Then there is a unique solution of problem (2.1) and* *the following inequalities hold*

$$\max_{0\le t\le 1}\|u(t)\|_H$$

$$\le M\left\{\|\varphi\|_H+\|\xi\|_H+\left\|f'(1)\right\|_H+\|\psi\|_H+\max_{0\le t\le 1}\|f(t)\|_H\right\}, \tag{2.12}$$

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\|Au(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H+\|A\psi\|_H+\left\|A^{2/3}\xi\right\|_H+\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\}, \tag{2.13}$$

where M does not depend on $f(t)$, φ, ψ, ξ.

Proof. First, we estimate $\|u(t)\|_H$ for $t\in[0,1]$. Applying (2.4), (2.2) and triangle inequality, we get

$$\|u(t)\|_H\le\left\|e^{-Bt}u(0)+\frac{1}{1+a}B^{-1}\left(e^{-(1-t)B}-e^{-(a+t)B}\right)\left(u'(1)+Bu(1)\right)\right\|_H$$

$$+\left\|\frac{1}{a-\bar a}B^{-2}\left\{\frac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)-\frac{1}{1+\bar a}\left(e^{-(1-t)\bar aB}-e^{-(\bar a+t)B}\right)\right\}\right.$$

$$\times\left(u''(1)+\bar aBu'(1)-aB^2u(1)\right)\Big\|_H+\left\|-\frac{1}{a-\bar a}B^{-2}\right.$$

$$\times\int_0^t\left[\frac{1}{1+a}\left(e^{-(t-s)B}-e^{-(t+sa)B}\right)-\frac{1}{1+\bar a}\left(e^{-(t-s)B}-e^{-(t+s\bar a)B}\right)\right]f(s)ds\Bigg\|_H$$

$$\le\left\|e^{-Bt}\varphi+\frac{1}{1+a}\left(e^{-(1-t)B}-e^{-(a+t)B}\right)\left(B^{-1}\xi+\psi\right)\right\|_H$$

$$+\frac{1}{|a-\bar a|}\left\|B^{-2}\left\{\frac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)-\frac{1}{1+\bar a}\left(e^{-(1-t)\bar aB}-e^{-(\bar a+t)B}\right)\right\}\right.$$

$$\times\left(u''(1)+\bar aBu'(1)-aB^2u(1)\right)\Big\|_H+\frac{1}{|a-\bar a|}$$

$$\times\int_0^t\left\|\frac{1}{1+a}e^{-(t-s)B}-e^{-(sa+t)B}+\frac{1}{1+\bar a}e^{-(t-s)B}-e^{-(s\bar a+t)B}\right\|_{H\to H}\left\|B^{-2}f(s)\right\|_H ds$$

$$\leq \left\|e^{-Bt}\right\|_{H\to H}\|\varphi\|_H + \frac{1}{|1+a|}\left\|e^{-(1-t)B} - e^{-(a+t)B}\right\|_{H\to H}$$

$$\times \left\|B^{-1}\xi + \psi\right\|_H + \frac{1}{|a-\bar{a}|}$$

$$\times \left\{\frac{1}{|1+a|}\left\|e^{-(1-t)aB} - e^{-(a+t)B}\right\|_{H\to H} + \frac{1}{|1+\bar{a}|}\left\|e^{-(1-t)\bar{a}B} - e^{-(\bar{a}+t)B}\right\|_{H\to H}\right\}$$

$$\times \left(\left\|B^{-2}u''(1)\right\|_H + |\bar{a}|\left\|B^{-1}\xi\right\|_H + |a|\,\|\varphi\|_H\right)$$

$$+\frac{1}{|a-\bar{a}|}\int_0^t \left[\left\|\frac{1}{1+a}\left(e^{-(t-s)B} - e^{-(sa+t)B}\right)\right\|_{H\to H}\right.$$

$$\left. + \left\|\frac{1}{1+\bar{a}}\left(e^{-(t-s)B} - e^{-(s\bar{a}+t)B}\right)\right\|_{H\to H}\right]\left\|B^{-2}f(s)\right\|_H ds$$

$$\leq M\left\{\|\varphi\|_H + \|\xi\|_H + \|\psi\|_H + \max_{0\leq t\leq 1}\|f(t)\|_H + \left\|B^{-2}u''(1)\right\|_H\right\} \tag{2.14}$$

for any $t \in [0,1]$. Applying formula (2.5), we get

$$B^{-2}u''(1) = B^{-2}T\left\{B^2e^{-B}u(0) + \frac{1}{1+a}B\left(I - e^{-(a+1)B}\right)\left(u'(1) + Bu(1)\right)\right.$$

$$+\frac{1}{a-\bar{a}}B^{-2}\left\{\frac{1}{1+a}\left(a^2I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(\bar{a}^2I - e^{-(\bar{a}+1)B}\right)\right\}$$

$$\times\left(\bar{a}Bu'(1) - aB^2u(1)\right) - \frac{1}{a-\bar{a}}B^{-2}B^{-1}$$

$$\times\left[e^{-(1+a)B} - e^{-(1+\bar{a})B} - \frac{1}{1+a}\left(e^{-(a+1)B} - I\right)\right.$$

$$\left. + \frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B} - I\right)\right] f(1)$$

$$-\frac{1}{a-\bar{a}}B^{-2}B^{-2}\left[\frac{1}{1+a}\left(I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(I - e^{-(\bar{a}+1)B}\right)\right] f'(1)$$

$$-\frac{1}{a-\bar{a}}B^{-2}\int_0^1\left[\left[\frac{1}{1+a}\left(e^{-(1-s)B} - e^{-(sa+1)B}\right)\right]\right.$$

$$\left.\left. - \frac{1}{1+\bar{a}}\left(e^{-(1-s)B} - e^{-(s\bar{a}+1)B}\right)\right] f(s)ds\right\},$$

$$= T\left\{e^{-B}\varphi + \frac{1}{1+a}\left(I - e^{-(a+1)B}\right)\left(B^{-1}\xi + \psi\right)\right.$$

$$+\frac{1}{a-\bar{a}}B^{-2}\left\{\frac{1}{1+a}\left(a^2I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(\bar{a}^2I - e^{-(\bar{a}+1)B}\right)\right\}$$

$$\times\left(\bar{a}B\xi - aB^2\psi\right) + \frac{1}{a-\bar{a}}B^{-3}$$

$$\times\left[e^{-(1+a)B} - e^{-(1+\bar{a})B} - \frac{1}{1+a}\left(e^{-(a+1)B} - I\right)\right.$$

$$\left.+\frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B} - I\right)\right] f(1)$$

$$+\frac{1}{a-\bar{a}}B^{-4}\left[\frac{1}{1+a}\left(I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(I - e^{-(\bar{a}+1)B}\right)\right] f'(1)$$

$$+\frac{1}{a-\bar{a}}B^{-2}\int_0^1\left[\frac{1}{1+a}\left(e^{-(1-s)B} - e^{-(sa+1)B}\right)\right.$$

$$\left.\left.-\frac{1}{1+\bar{a}}\left(e^{-(1-s)B} - e^{-(s\bar{a}+1)B}\right)\right] f(s)ds\right\}.$$

Applying the triangle inequality and estimate (2.3), we get

$$\left\|B^{-2}u''(1)\right\|_H \le \|T\|_{H\to H}\left\{\left\|e^{-B}\right\|_{H\to H}\|\varphi\|_H + \frac{1}{|1+a|}\right.$$

$$\times\left\|I - e^{-(a+1)B}\right\|_{H\to H}\left\|B^{-1}\xi + \psi\right\|_H$$

$$+\frac{1}{|a-\bar{a}|}\left\{\frac{1}{|1+a|}\left\|a^2I - e^{-(a+1)B}\right\|_{H\to H} + \frac{1}{|1+\bar{a}|}\left\|\bar{a}^2I - e^{-(\bar{a}+1)B}\right\|_{H\to H}\right\}$$

$$\times\left(|\bar{a}|\left\|B^{-1}\xi\right\|_H + |a|\,\|\varphi\|_H\right) + \frac{1}{|a-\bar{a}|}$$

$$\times\int_0^1\left[\frac{1}{|1+a|}\left\|e^{-(1-s)B} - e^{-(sa+1)B}\right\|_{H\to H}\right.$$

$$\left.+\frac{1}{|1+\bar{a}|}\left\|e^{-(1-s)B} - e^{-(s\bar{a}+1)B}\right\|_{H\to H}\right]\left\|B^{-2}f(s)\right\|_H ds$$

$$+\frac{1}{|a-\bar{a}|}\left[\left\|e^{-(a+1)B} - e^{-(1+\bar{a})B}\right\|_{H\to H} + \frac{1}{|1+a|}\left\|I - e^{-(a+1)B}\right\|_{H\to H}\right.$$

$$+\frac{1}{|1+\bar{a}|}\left\|I-e^{-(\bar{a}+1)B}\right\|_{H\to H}\left\|B^{-3}f(1)\right\|_H\Big]+\frac{1}{|a-\bar{a}|}$$

$$\times\left\{\frac{1}{|1+a|}\left\|I-e^{-(a+1)B}\right\|_{H\to H}+\frac{1}{|1+\bar{a}|}\left\|I-e^{-(\bar{a}+1)B}\right\|_{H\to H}\right\}\left\|B^{-4}f'(1)\right\|_H\Big\}$$

$$\le M\left\{\|\varphi\|_H+\|\xi\|_H+\|\psi\|_H+\left\|f'(1)\right\|_H+\max_{0\le t\le 1}\|f(t)\|_H\right\}. \tag{2.15}$$

From estimates (2.14) and (2.15) it follows estimate (2.12).

Second, we estimate $\|Au(t)\|_H$ for $t\in[0,1]$. Applying formulas (2.5) and

$$\int_0^t e^{-(t-s)B}f(s)ds=B^{-1}f(t)-B^{-1}e^{-tB}f(t)$$

$$-B^{-1}\int_0^t e^{-(t-s)B}f'(s)ds,$$

$$\int_0^t e^{-(t+sa)B}f(s)ds=-\frac{1}{a}B^{-1}e^{-t(a+1)B}f(t)$$

$$+\frac{1}{a}B^{-1}e^{-tB}f(t)+\frac{1}{a}B^{-1}\int_0^t e^{-(t+sa)B}f'(s)ds,$$

$$\int_0^t e^{-(t+s\bar{a})B}f(s)ds=-\frac{1}{\bar{a}}B^{-1}e^{-t(\bar{a}+1)B}f(t)$$

$$+\frac{1}{\bar{a}}B^{-1}e^{-tB}f(t)+\frac{1}{\bar{a}}B^{-1}\int_0^t e^{-(t+s\bar{a})B}f'(s)ds,$$

we get

$$u(t)=e^{-Bt}u(0)+\frac{1}{1+a}B^{-1}\left(e^{-(1-t)B}-e^{-(a+t)B}\right)\left(u'(1)+Bu(1)\right)$$

$$+\frac{1}{a-\bar{a}}B^{-2}\left\{\frac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)\right.$$

$$\left.-\frac{1}{1+\bar{a}}\left(e^{-(1-t)\bar{a}B}-e^{-(\bar{a}+t)B}\right)\right\}$$

$$\times\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right)-\frac{1}{a-\bar{a}}B^{-3}$$

$$\times\left[\left\{\frac{1}{1+a}\left(I+\bar{a}e^{-(1+a)tB}\right)-\frac{1}{1+\bar{a}}\left(I+ae^{-(1+\bar{a})tB}\right)\right\}f(t)\right.$$

$$-\left[\frac{1+\bar{a}}{1+a}-\frac{1+a}{1+\bar{a}}\right]e^{-tB}f(0)$$

$$-\int\limits_0^t\left\{\frac{1}{1+a}\left(e^{-(t-s)B}+\bar{a}e^{-(t+sa)B}\right)\right.$$

$$\left.\left.-\frac{1}{1+\bar{a}}\left(e^{-(t-s)B}+ae^{-(t+s\bar{a})B}\right)\right\}f'(s)ds\right]. \tag{2.16}$$

In the similary manner, applying formula (2.16), we get

$$Au(t)=e^{-Bt}Au(0)+\frac{1}{1+a}AB^{-1}\left(e^{-(1-t)B}-e^{-(a+t)B}\right)$$

$$\times\left(u'(1)+Bu(1)\right)$$

$$+\frac{1}{a-\bar{a}}AB^{-2}\left\{\frac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)\right.$$

$$\left.-\frac{1}{1+\bar{a}}\left(e^{-(1-t)\bar{a}B}-e^{-(\bar{a}+t)B}\right)\right\}$$

$$\times\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right)-\frac{1}{a-\bar{a}}AB^{-3}$$

$$\times\left[\left[\frac{1}{1+a}\left(I+\bar{a}e^{-(1+a)tB}\right)-\frac{1}{1+\bar{a}}\left(I+ae^{-(1+\bar{a})tB}\right)\right]f(t)\right.$$

$$-\left[\frac{1+\bar{a}}{1+a}-\frac{1+a}{1+\bar{a}}\right]e^{-tB}f(0)$$

$$-\int\limits_0^t\left\{\frac{1}{1+a}\left(e^{-(t-s)B}+\bar{a}e^{-(t+sa)B}\right)\right.$$

$$-\frac{1}{1+\overline{a}}\left(e^{-(t-s)B}+ae^{-(t+s\overline{a})B}\right)\Big\}f'(s)ds\Big].$$

$$=e^{-Bt}Au(0)+\frac{1}{1+a}\left(e^{-(1-t)B}-e^{-(a+t)B}\right)\left(B^2u'(1)+Au(1)\right)$$

$$+\frac{1}{a-\bar{a}}B\left(\frac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)\right.$$

$$\left.-\frac{1}{1+\overline{a}}\left(e^{-(1-t)\bar{a}B}-e^{-(\bar{a}+t)B}\right)\right)$$

$$\times\left(u''(1)+\bar{a}Bu'(1)-aB^2u(1)\right)-\frac{1}{a-\bar{a}}$$

$$\times\left[\left[\frac{1}{1+a}\left(I+\overline{a}e^{-(1+a)tB}\right)-\frac{1}{1+\overline{a}}\left(I+ae^{-(1+\overline{a})tB}\right)\right]f(t)\right.$$

$$-\left[\frac{1+\overline{a}}{1+a}-\frac{1+a}{1+\overline{a}}\right]e^{-tB}f(0)$$

$$-\int_0^t\left[\frac{1}{1+a}\left(e^{-(t-s)B}+\overline{a}e^{-(t+sa)B}\right)\right.$$

$$\left.\left.-\frac{1}{1+\overline{a}}\left(e^{-(t-s)B}+ae^{-(t+s\overline{a})B}\right)\right]f'(s)ds\right].$$

Applying the triangle inequality and estimate (2.2), we get

$$\|Au(t)\|_H\leq\left\|e^{-Bt}\right\|_{H\to H}\|Au(0)\|_H+\frac{1}{|1+a|}$$

$$\times\left\|e^{-(1-t)B}-e^{-(a+t)B}\right\|_{H\to H}\left(\left\|B^2u'(1)\right\|_H+\|Au(1)\|_H\right)$$

$$+\left\|\frac{1}{a-\bar{a}}AB^{-2}\left\{\frac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)-\frac{1}{1+\overline{a}}\left(e^{-(1-t)\bar{a}B}-e^{-(\bar{a}+t)B}\right)\right\}\right\|_{H\to H}$$

$$\times\left(\left\|Bu''(1)\right\|_H+|\bar{a}|\left\|B^2u'(1)\right\|_H+|a|\,\|Au(1)\|_H\right)+\frac{1}{|a-\bar{a}|}$$

$$\times\left[\left[\frac{1}{|1+a|}\left\|I+\bar{a}e^{-(1+a)tB}\right\|_{H\to H}+\frac{1}{|1+\bar{a}|}\left\|I+ae^{-(1+\bar{a})tB}\right\|_{H\to H}\right]\right.$$

$$\times\|f(t)\|_H+\left|\frac{1+\bar{a}}{1+a}-\frac{1+a}{1+\bar{a}}\right|\left\|e^{-tB}\right\|_{H\to H}\|f(0)\|_H$$

$$+\int_0^t\left[\frac{1}{|1+a|}\left\|e^{-(t-s)B}+\bar{a}e^{-(t+sa)B}\right\|_{H\to H}\right.$$

$$\left.\left.+\frac{1}{|1+\bar{a}|}\left\|e^{-(t-s)B}+ae^{-(t+s\bar{a})B}\right\|_{H\to H}\left\|f'(s)\right\|_H ds\right]\right],$$

$$\leq M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\|A\psi\|_H\right.$$

$$\left.+\|f(0)\|_H+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_H+\left\|Bu''(1)\right\|_H\right\} \tag{2.17}$$

for any $t\in[0,1]$. Applying formula (2.5) we get

$$Bu''(1)=T\left\{B^3e^{-B}u(0)+\frac{1}{1+a}B^2\left(I-e^{-(a+1)B}\right)\right.$$

$$\times\left(u'(1)+Bu(1)\right)+\frac{1}{a-\bar{a}}$$

$$\times\left\{\frac{1}{1+a}\left(a^2I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(\bar{a}^2I-e^{-(\bar{a}+1)B}\right)\right\}$$

$$\times\left(\bar{a}B^2u'(1)-aB^3u(1)\right)-\frac{1}{a-\bar{a}}$$

$$\times\left[e^{-(1+a)B}-e^{-(1+\bar{a})B}-\frac{1}{1+a}\left(e^{-(a+1)B}-I\right)+\frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B}-I\right)\right]f(1)$$

$$-\frac{1}{a-\bar{a}}B^{-1}\left[\frac{1}{1+a}\left(I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(I-e^{-(\bar{a}+1)B}\right)\right]f'(1)$$

$$\left.-\frac{1}{a-\bar{a}}B\int_0^1\left[\frac{1}{1+a}\left(e^{-(1-s)B}-e^{-(sa+1)B}\right)-\frac{1}{1+\bar{a}}\left(e^{-(1-s)B}-e^{-(s\bar{a}+1)B}\right)\right]f(s)ds\right\}.$$

Applying formulas

$$\int_0^1 \left(e^{-(1-s)B} - e^{-(sa+1)B}\right) f(s)ds = B^{-1}\left[\left(I + \frac{1}{a}e^{-(a+1)B}\right) f(1) - \left(I + \frac{1}{a}\right) e^{-B} f(0)\right]$$

$$-B^{-1}\int_0^1 \left(e^{-(1-s)B} + \frac{1}{a}e^{-(sa+1)B}\right) f'(s)ds,$$

$$\int_0^1 \left(e^{-(1-s)B} - e^{-(s\bar{a}+1)B}\right) f(s)ds = B^{-1}\left[\left(I + \frac{1}{\bar{a}}e^{-(\bar{a}+1)B}\right) f(1) - \left(I + \frac{1}{\bar{a}}\right) e^{-B} f(0)\right]$$

$$-B^{-1}\int_0^1 \left(e^{-(1-s)B} + \frac{1}{\bar{a}}e^{-(s\bar{a}+1)B}\right) f'(s)ds,$$

we get

$$Bu''(1) = T\left\{B^3 e^{-B}\varphi + \frac{1}{1+a}B^2\left(I - e^{-(a+1)B}\right)(\xi + B\psi)\right.$$

$$+\frac{1}{a-\bar{a}}\left\{\frac{1}{1+a}\left(a^2 I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(\bar{a}^2 I - e^{-(\bar{a}+1)B}\right)\right\}\left(\bar{a}B^2\xi - aB^3\psi\right)$$

$$-\frac{1}{a-\bar{a}}\left[e^{-(1+a)B} - e^{-(1+\bar{a})B} - \frac{1}{1+a}\left(e^{-(a+1)B} - I\right)\right.$$

$$\left.+\frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B} - I\right)\right] f(1) - \frac{1}{a-\bar{a}}B^{-1}$$

$$\times\left[\frac{1}{1+a}\left(I - e^{-(a+1)B}\right) - \frac{1}{1+\bar{a}}\left(I - e^{-(\bar{a}+1)B}\right)\right] f'(1)$$

$$-\frac{1}{a-\bar{a}}B\left(\frac{1}{1+a}B^{-1}\left[\left(I + \frac{1}{a}e^{-(a+1)B}\right) f(1) - \left(I + \frac{1}{a}\right) e^{-B} f(0)\right]\right.$$

$$+\frac{1}{1+a}B^{-1}\int_0^1 \left(e^{-(1-s)B} + \frac{1}{a}e^{-(sa+1)B}\right) f'(s)ds - \frac{1}{1+\bar{a}}B^{-1}$$

$$\times\left[\left(I + \frac{1}{\bar{a}}e^{-(\bar{a}+1)B}\right) f(1) - \left(I + \frac{1}{\bar{a}}\right) e^{-B} f(0)\right]$$

$$
-\frac{1}{1+\bar{a}}B^{-1}\int_{0}^{1}\left(e^{-(1-s)B}+\frac{1}{\bar{a}}e^{-(s\bar{a}+1)B}\right)f'(s)ds\Bigg)\Bigg\}.
$$

Applying the triangle inequality and estimate (2.3), we get

$$
\begin{aligned}
\left\|Bu''(1)\right\|_H \le \|T\|_{H\to H}\Bigg\{\left\|e^{-B}\right\|_{H\to H}\left\|B^3\varphi\right\|_H+\frac{1}{|1+a|}\\
\times\left\|\left(I-e^{-(a+1)B}\right)\right\|_{H\to H}\left\|\left(B^2\xi+B^3\psi\right)\right\|_H+\frac{1}{|a-\bar{a}|}\\
\times\left\|\frac{1}{1+a}\left(a^2I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(\bar{a}^2I-e^{-(\bar{a}+1)B}\right)\right\|_{H\to H}\\
\times\left\|\left(\bar{a}B^2\xi-aB^3\psi\right)\right\|_H+\frac{1}{|a-\bar{a}|}\|f(1)\|_H\\
\times\left\|\left[e^{-(1+a)B}-e^{-(1+\bar{a})B}-\frac{1}{1+a}\left(e^{-(a+1)B}-I\right)+\frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B}-I\right)\right]\right\|_{H\to H}\\
+\left\|\frac{1}{a-\bar{a}}B^{-1}\left[\frac{1}{1+a}\left(I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(I-e^{-(\bar{a}+1)aB}\right)\right]f'(1)\right\|_H\\
+\frac{1}{|a-\bar{a}|}\left(\frac{1}{|1+a|}\left\|\left[\left(I+\frac{1}{a}e^{-(a+1)B}\right)f(1)-\left(I+\frac{1}{a}\right)e^{-B}f(0)\right]\right\|_H\right.\\
\left.+\frac{1}{|1+a|}\int_0^1\left\|\left(e^{-(1-s)B}+\frac{1}{a}e^{-(sa+1)B}\right)f'(s)\right\|_H ds\right)\\
+\frac{1}{|a-\bar{a}|}\left(\frac{1}{|1+\bar{a}|}\left\|\left(I+\frac{1}{\bar{a}}e^{-(\bar{a}+1)B}\right)f(1)-\left(I+\frac{1}{\bar{a}}\right)e^{-B}f(0)\right\|_H\right.\\
\left.+\frac{1}{|1+\bar{a}|}\int_0^1\left\|\left(e^{-(1-s)B}+\frac{1}{\bar{a}}e^{-(s\bar{a}+1)B}\right)f'(s)\right\|_H ds\right)\Bigg\|_H\Bigg\}\\
\le M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\|A\psi\|_H\right.\\
\left.+\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\}. \qquad (2.18)
\end{aligned}
$$

Applying estimates (2.17) and (2.18), we get

$$\max_{0\le t\le 1}\|Au(t)\|_H \le M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\|A\psi\|_H\right.$$

$$\left.+\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\}.$$

From that and equation (2.1) and triangle inequality it follows that

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H \le \max_{0\le t\le 1}\|Au(t)\|_H+\max_{0\le t\le 1}\|f(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\|A\psi\|_H+\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\}.$$

Theorem 2.2.4 is proved. ■

Moreover, we have the following theorem on stability.

Theorem 2.2.5 *Assume that* $\delta>\left(\frac{1}{3}\ln 4\right)^3$, $\varphi\in D(A)$, $\psi\in D(A)$, $\xi\in D\left(A^{2/3}\right)$ *and* $f(t)$ *is continuous on* $[0,1]$ *and there exists* $f'(1)$ *and* $f(t)\in D\left(A^{1/3}\right)$. *Then there is a unique solution of problem (2.1) and the following inequalities hold*

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\|Au(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H+\|A\psi\|_H+\left\|A^{2/3}\xi\right\|_H+\left\|f'(1)\right\|_H+\max_{0\le t\le 1}\left\|A^{1/3}f(t)\right\|_H\right\}, \tag{2.19}$$

where M *does not depend on* $f(t)$, φ, ψ, ξ.

Proof. We will estimate $\|Au(t)\|_H$ for $t\in[0,1]$. Applying formula (2.4), we get

$$Au(t)=e^{-Bt}A\varphi+\tfrac{1}{1+a}\left(e^{-(1-t)B}-e^{-(a+t)B}\right)\left(B^2\xi+A\psi\right)$$

$$+\tfrac{1}{a-\bar a}\left\{\tfrac{1}{1+a}\left(e^{-(1-t)aB}-e^{-(a+t)B}\right)-\tfrac{1}{1+\bar a}\left(e^{-(1-t)\bar aB}-e^{-(\bar a+t)B}\right)\right\}$$

$$\times\left(Bu''(1)+\bar aB^2\xi-aA\psi\right)-\tfrac{1}{a-\bar a}$$

$$\times\int_0^t\left[\tfrac{1}{1+a}\left(e^{-(t-s)B}-e^{-(t+sa)B}\right)-\tfrac{1}{1+\bar a}\left(e^{-(t-s)B}-e^{-(t+s\bar a)B}\right)\right]Bf(s)ds.$$

Applying the triangle inequality and estimate (2.2), we get

$$\|Au(t)\|_H \leq \left\|e^{-Bt}\right\|_{H\to H} \|A\varphi\|_H + \left|\frac{1}{1+a}\right| \left\|e^{-(1-t)B} - e^{-(a+t)B}\right\|_{H\to H}$$

$$\times \left(\left\|B^2\xi\right\|_H + \|A\psi\|_H\right)$$

$$+\frac{1}{a-\bar{a}}AB^{-2}\left\{\frac{1}{1+a}\left(e^{-(1-t)aB} - e^{-(a+t)B}\right) - \frac{1}{1+\bar{a}}\left(e^{-(1-t)\bar{a}B} - e^{-(\bar{a}+t)B}\right)\right\}$$

$$+\left|\frac{1}{a-\bar{a}}\right|\left\{\frac{1}{1+a}\left(e^{-(1-t)aB} - e^{-(a+t)B}\right) - \frac{1}{1+\bar{a}}\left(e^{-(1-t)\bar{a}B} - e^{-(\bar{a}+t)B}\right)\right\}$$

$$\times \left(\left\|Bu''(1)\right\|_H + |\bar{a}|\left\|B^2\xi\right\|_H + |a|\,\|A\psi\|_H\right)$$

$$+\left|\frac{1}{a-\bar{a}}\right|\int_0^t \left[\left|\frac{1}{1+a}\right| \left\|e^{-(t-s)B} + \bar{a}e^{-(t+sa)B}\right\|_{H\to H}\right.$$

$$\left.+\left|\frac{1}{1+\bar{a}}\right| \left\|e^{-(t-s)B} + ae^{-(t+s\bar{a})B}\right\|_{H\to H}\right] \|Bf(s)\|_H\, ds\Bigg]$$

$$\leq M\left[\|A\varphi\|_H + \left\|B^2\xi\right\|_H + \|A\psi\|_H + \max_{0\leq t\leq 1}\|Bf(t)\|_H + \left\|Bu''(1)\right\|_H\right] \tag{2.20}$$

for any $t \in [0,1]$. Applying formula (2.5), we get

$$Bu''(1) = T\left\{B^3e^{-B}\varphi + \frac{1}{1+a}B^2\left(I - e^{-(a+1)B}\right)\right.$$

$$\times (\xi + B\psi)$$

$$+\frac{1}{a-\bar{a}}\left\{\frac{1}{1+a}\left(a^2I - e^{-(a+1)B}\right)\right.$$

$$\left.-\frac{1}{1+\bar{a}}\left(\bar{a}^2I - e^{-(\bar{a}+1)B}\right)\right\}\left(\bar{a}B^2\xi - aB^3\psi\right)$$

$$-\frac{1}{a-\bar{a}}\left[e^{-(1+a)B} - e^{-(1+\bar{a})B} - \frac{1}{1+a}\right.$$

$$\times\left(e^{-(a+1)B}-I\right)+\frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B}-I\right)\Big]f(1)-\frac{1}{a-\bar{a}}B^{-1}$$

$$\times\left[\frac{1}{1+a}\left(I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(I-e^{-(\bar{a}+1)aB}\right)\right]f'(1)$$

$$-\frac{1}{a-\bar{a}}B\int_0^1\left[\frac{1}{1+a}\left(e^{-(1-s)B}-e^{-(sa+1)B}\right)\right.$$

$$\left.-\frac{1}{1+\bar{a}}\left(e^{-(1-s)B}-e^{-(s\bar{a}+1)B}\right)\right]f(s)ds\Big\}.$$

Applying the triangle inequality and estimate (2.3), we get

$$\left\|Bu''(1)\right\|_H\leq\|T\|_{H\to H}\left\{\left\|B^3e^{-B}\varphi+\frac{1}{1+a}B^2\left(I-e^{-(a+1)B}\right)\right\|_H\right.$$

$$\times\|(\xi+B\psi)\|_H$$

$$+\frac{1}{|a-\bar{a}|}\left\{\left\|\frac{1}{1+a}\left(a^2I-e^{-(a+1)B}\right)\right.\right.$$

$$\left.\left.-\frac{1}{1+\bar{a}}\left(\bar{a}^2I-e^{-(\bar{a}+1)B}\right)\right\|_{H\to H}\right\}\left\|\left(\bar{a}B^2\xi-aB^3\psi\right)\right\|_H$$

$$+\frac{1}{|a-\bar{a}|}\left[\left\|e^{-(1+a)B}-e^{-(1+\bar{a})B}-\frac{1}{1+a}\right.\right.$$

$$\left.\times\left(e^{-(a+1)B}-I\right)+\frac{1}{1+\bar{a}}\left(e^{-(\bar{a}+1)B}-I\right)\Big]f(1)\right\|_H+\left\|\frac{1}{a-\bar{a}}B^{-1}\right.$$

$$\left.\times\left[\frac{1}{1+a}\left(I-e^{-(a+1)B}\right)-\frac{1}{1+\bar{a}}\left(I-e^{-(\bar{a}+1)aB}\right)\right]f'(1)\right\|_H$$

$$\frac{1}{|a-\bar{a}|}\left\|B\int_0^1\left[\frac{1}{1+a}\left(e^{-(1-s)B}-e^{-(sa+1)B}\right)\right.\right.$$

$$\left.\left.-\frac{1}{1+\bar{a}}\left(e^{-(1-s)B}-e^{-(s\bar{a}+1)B}\right)\right]f(s)ds\right\|_H\Big\}.$$

$$\leq M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\left\|f'(1)\right\|_H+\|A\psi\|_H+\max_{0\leq t\leq 1}\|Bf(t)\|_H\right\}. \qquad (2.21)$$

Applying estimates (2.20) and (2.21), we get

$$\max_{0\le t\le 1}\|Au(t)\|_H \le M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\|A\psi\|_H+\left\|f'(1)\right\|_H+\max_{0\le t\le 1}\|Bf(t)\|_H\right\}.$$

From that and equation (2.1) and triangle inequality it follows that

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H \le \max_{0\le t\le 1}\|Au(t)\|_H+\max_{0\le t\le 1}\|f(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H+\left\|B^2\xi\right\|_H+\|A\psi\|_H+\left\|f'(1)\right\|_H+\max_{0\le t\le 1}\|Bf(t)\|_H\right\}.$$

Theorem 2.2.5 is proved. ■

From Theorem 2.2.4 and Theorem 2.2.5 it follows the following theorem on stability.

Theorem 2.2.6 *Assume that $\delta > \left(\frac{1}{3}\ln 4\right)^3$, $\varphi\in D(A)$, $\psi\in D(A)$, $\xi\in D\left(A^{2/3}\right)$ and $f(t)$ is continuously differentiable on $[0,1]$ and $f(t)\in D\left(A^{1/3}\right)$. Then there is a unique solution of problem (2.1) and the following inequalities hold*

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\|Au(t)\|_H \le M\left\{\|A\varphi\|_H+\|A\psi\|_H+\left\|A^{2/3}\xi\right\|_H\right.$$

$$\left.+\min\left\{\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H,\left\|f'(1)\right\|_H+\max_{0\le t\le 1}\left\|A^{1/3}f(t)\right\|_H\right\}\right\},$$

where M does not depend on $f(t)$, φ, ψ, ξ.

2.3 Applications

In this section we will consider three applications of the main Theorem 2.2.4. First, for the application of the Theorem 2.2.4 we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3}-(a(x)u_x(t,x))_x+\delta u(t,x)=f(t,x),\ \ 0<t,x<1,\\ u(0,x)=\varphi(x),\quad u(1,x)=\psi(x),\quad u_t(1,x)=\xi(x),\ \ 0\le x\le 1,\\ u(t,0)=u(t,1),\quad u_x(t,0)=u_x(t,1),\quad 0\le t\le 1 \end{cases} \tag{2.22}$$

Problem (2.22) has a unique smooth solution $u(t,x)$ for smooth $a(x) \geq a > 0$, $x \in (0,1)$, $\delta > \left(\frac{1}{3}\ln 4\right)^3$, $a(1) = a(0)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t \in (0,1), x \in (0,1))$ functions. This allows us to reduce problem (2.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (2.22). Let us give a number of corollaries of abstract Theorem 2.2.4.

Theorem 2.3.1 *For the solution of the problem (2.22), the stability inequalities*

$$\max_{0\leq t\leq 1} \|u(t,.)\|_{L_2[0,1]}$$

$$\leq M\left[\max_{0\leq t\leq 1} \|f(t,.)\|_{L_2[0,1]} + \|f_t(1,.)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]} + \|\xi\|_{L_2[0,1]}\right], \quad (2.23)$$

$$\max_{0\leq t\leq 1} \|u(t,.)\|_{W_2^2[0,1]} + \max_{0\leq t\leq 1} \left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2[0,1]}$$

$$\leq M\left[\max_{0\leq t\leq 1} \|f_t(t,.)\|_{L_2[0,1]} + \|f(0,.)\|_{L_2[0,1]} + \|\varphi\|_{W_2^2[0,1]} + \|\psi\|_{W_2^2[0,1]} + \|\xi\|_{W_2^2[0,1]}\right] \quad (2.24)$$

hold where M does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (2.22) can be written in abstract form

$$\begin{aligned} &\frac{d^3u(t)}{dt^3} + Au(t) = f(t), \quad 0 \leq t \leq 1, \\ &u(0) = \varphi, \quad u(1) = \psi, \quad u'(1) = \xi \end{aligned} \quad (2.25)$$

in Hilbert space $L_2[0,1]$ for all square integrable functions defined on $[0,1]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\left(a(x)u_x\right)_x + \delta u(x) \quad (2.26)$$

with domain

$$D(A^x) = \left\{u(x) : u, u_x, (a(x)u_x)_x \in L_2[0,1], u(0) = u(1), u'(0) = u'(1)\right\}.$$

here $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are respectively known and unknown abstract functions defined on $[0,1]$ with the values $H = L_2[0,1]$. Therefore, estimates (2.23)-(2.24) follow from estimates (2.12)-(2.13). Thus, Theorem 2.3.1 is proved. ■

Second, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\bar{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, we consider the boundary value problem for a third order partial differential

equation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} \left(a_r(x) u_{x_r}(t,x)\right)_{x_r} = f(t,x), \\ \\ x = (x_1, ..., x_n) \in \Omega,\ 0 < t < 1, \\ \\ u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \quad u_t(1,x) = \xi(x), \quad x \in \bar{\Omega}, \\ u(t,x) = 0, \quad x \in S,\ 0 \le t \le 1, \end{cases} \tag{2.27}$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$. We introduce the Hilbert space $L_2(\bar{\Omega})$, the space of integrable functions defined on $\bar{\Omega}$ equipped with norm

$$\|f\|_{L_2(\bar{\Omega})} = \left\{ \int \cdots \int_{x \in \bar{\Omega}} |f(x)|^2 \, dx_1 ... dx_n \right\}^{1/2}.$$

Theorem 2.3.2 *For the solution of the problem (2.27) the stability inequalities*

$$\max_{0 \le t \le 1} \|u(t,.)\|_{L_2(\bar{\Omega})}$$

$$\le M_2 \left[\max_{0 \le t \le 1} \|f(t,.)\|_{L_2(\bar{\Omega})} + \|f_t(1,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{L_2(\bar{\Omega})} + \|\psi\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})} \right], \tag{2.28}$$

$$\max_{0 \le t \le 1} \|u(t,.)\|_{W_2^2[0,1]} + \max_{0 \le t \le 1} \left\| \frac{\partial^3 u}{\partial t^3}(t,.) \right\|_{L_2(\bar{\Omega})}$$

$$\le M_2 \left[\max_{0 \le t \le 1} \|f_t(t,.)\|_{L_2(\bar{\Omega})} + \|f(0,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{W_2^2(\bar{\Omega})} + \|\psi\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^2(\bar{\Omega})} \right] \tag{2.29}$$

hold where M_2 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (2.27) can be written in abstract form (2.25) in Hilbert space $L_2\left(\bar{\Omega}\right)$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\sum_{r=1}^{n} \left(a_r(x) u_{x_r}\right)_{x_r} \tag{2.30}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r} \in L_2\left(\bar{\Omega}\right), 1 \le r \le n, u(x) = 0, x \in S\right\}$$

here $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown respectively abstract functions defined on $\bar{\Omega}$ with the value in $H = L_2\left(\bar{\Omega}\right)$. So estimates (2.28)-(2.29) follow from estimates (2.12)-(2.13) and from the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2\left(\bar{\Omega}\right)$. ■

Theorem 2.3.3 *For the solution of the elliptic differential problem*

$$-\sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} = w(x), x \in \Omega, u(x) = 0,\ x \in S$$

the following coercivity inequalities

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \le M \|w\|_{L_2(\overline{\Omega})}$$

are valid. Here M does not depend on $w(x)$.

Third, we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r}(t,x))_{x_r} + \delta u(t,x) = f(t,x), \\ x = (x_1, ..., x_n) \in \Omega,\ 0 < t < 1, \\ u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \quad u_t(1,x) = \xi(x), \quad x \in \bar{\Omega}, \\ \dfrac{\partial u}{\partial \overrightarrow{n}}(t,x) = 0, \quad x \in S,\ 0 \le t \le 1, \end{cases} \tag{2.31}$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t,x)$ $(t \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$, $\delta > 0$, $\overrightarrow{n}$ is the normal vector to S.

Theorem 2.3.4 *For the solution of the problem (2.31), the stability inequalities*

$$\max_{0\le t\le 1}\|u(t,.)\|_{L_2(\bar\Omega)}$$
$$\le M_3\left[\max_{0\le t\le 1}\|f(t,.)\|_{L_2(\bar\Omega)}+\|f_t(1,.)\|_{L_2(\bar\Omega)}+\|\varphi\|_{L_2(\bar\Omega)}+\|\psi\|_{L_2(\bar\Omega)}+\|\xi\|_{L_2(\bar\Omega)}\right], \tag{2.32}$$

$$\max_{0\le t\le 1}\|u(t,.)\|_{W_2^2(\bar\Omega)}+\max_{0\le t\le 1}\left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2(\bar\Omega)}$$
$$\le M_3\left[\max_{0\le t\le 1}\|f_t(t,.)\|_{L_2(\bar\Omega)}+\|f(0,.)\|_{L_2(\bar\Omega)}+\|\varphi\|_{W_2^2(\bar\Omega)}+\|\psi\|_{W_2^2(\bar\Omega)}+\|\xi\|_{W_2^2(\bar\Omega)}\right] \tag{2.33}$$

hold where M_3 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (2.31) can be written in abstract form (2.25) in Hilbert space $L_2\left(\bar\Omega\right)$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$A^x u(x)=-\sum_{r=1}^{m}\left(a_r(x)u_{x_r}\right)_{x_r}+\delta u(x) \tag{2.34}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),\left(a_r(x)u_{x_r}\right)_{x_r}\in L_2\left(\bar\Omega\right),1\le r\le m,\frac{\partial u}{\partial\vec n}=0,x\in S\right\}.$$

Here $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are respectively known and unknown abstract functions defined on $\bar\Omega$ with the value in $H=L_2\left(\bar\Omega\right)$. So estimates (2.32)-(2.33) follow from estimates (2.12)-(2.13) and from the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2\left(\bar\Omega\right)$. ∎

Theorem 2.3.5 *For the solution of the elliptic differential problem*

$$-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u(x)=w(x),x\in\Omega,\frac{\partial}{\partial\vec n}u(x)=0,\ x\in S$$

the following coercivity inequalities [69]

$$\sum_{r=1}^{n}\|u_{x_rx_r}\|_{L_2(\overline\Omega)}\le M\|w\|_{L_2(\overline\Omega)}$$

are valid. Here M does not depend on $w(x)$.

2.4 Numerical Experiments

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature (for instance, see [11], [12], [37], [40], [30]).

In the present chapter for the approximate solutions of a problem, we will use the first and high orders of accuracy difference schemes. The high order of accuracy for the approximate solution of the problem will be constructed in order to get more accurate result. We will apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first and high orders of accuracy difference schemes will be given.

2.4.1 The First Order of Accuracy Difference Scheme

We consider the local boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \\ f(t,x) = (1-t)^2 \sin x, 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = 2\sin x,\ u(1,x) = e^{-1} \sin\ x, \\ u_t(1,x) = -e^{-1} \sin x,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{2.35}$$

for a third order partial differential equation. The exact solution of problem (2.35) is

$$u(t,x) = \left(e^{-t} + (1-t)^2\right) \sin x.$$

For the approximate solutions of boundary value problem (2.35), applying the formulas

$$\frac{u(t_{k+2}) - 3u(t_{k+1}) + 3u(t_k) - u(t_{k-1})}{\tau^3} - u'''(t_k) = O(\tau),$$

$$\frac{u(1) - u(1-\tau)}{\tau} - u'(1) = O(\tau),$$

$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-u''(x_n)=O(h^2), \tag{2.36}$$

we get three-step difference scheme the first order of accuracy in t and the second order of accuracy in x

$$\begin{cases} \dfrac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^3}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}=f(t_k,x_n), \\ f(t_k,x_n)=(1-t_k)^2\sin x_n,\ t_k=k\tau, 1\le k\le N-2\ ,\ 1\le n\le M-1, \\ N\tau=1,\ x_n=nh,\ 1\le n\le M-1,\ Mh=\pi, \\ u_n^0=2\sin x_n,\ u_n^N=e^{-1}\sin x_n,\ 1\le n\le M-1, \\ u_n^N-u_n^{N-1}=-\tau e^{-1}\sin x_n, 1\le n\le M-1, \\ u_0^k=u_M^k=0, 0\le k\le N. \end{cases} \tag{2.37}$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} A\,u_{n+1}+Bu_n+Cu_{n-1}=D\varphi_n, \quad 1\le n\le M-1, \\ u_0=\vec{0},\ u_M=\vec{0}. \end{cases} \tag{2.38}$$

Here,

$$A=C=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & 3b & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & 3b & -b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & 3b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3b & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & 3b & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & 3b & -b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a=-\frac{1}{h^2},\ b=-\frac{1}{\tau^3},\ \ c=\frac{3}{\tau^3}+\frac{2}{h^2},$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},\left\{\begin{matrix} \varphi_n^k=f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ t_k=k\tau, 1\le k\le N-2,\ 1\le n\le M-1, \\ \\ \varphi_n^0=2\sin x_n, 0\le n\le M, \\ \varphi_n^{N-1}=-\tau e^{-1}\sin x_n, 0\le n\le M, \\ \varphi_n^N=e^{-1}\sin x_n, 0\le n\le M, \end{matrix}\right\}$$

and $D=I_{N+1}$ is the identity matrix,

$$u_s=\begin{bmatrix} u_s^0 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},\quad s=n-1,n,n+1.$$

This type of system was used by Samarskii and Nikolaev [82] for difference equations. For the solution of the matrix equation (2.38), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n=\alpha_{n+1}u_{n+1}+\beta_{n+1},\quad n=M-1,...2,1, \tag{2.39}$$

where $u_M=\vec{0}$, $\alpha_j\ \ (j=1,\cdots,M-1)$ are $(N+1)\times(N+1)$ square matrices, $\beta_j\ \ (j=1,...,M-1)$ are $(N+1)\times 1$ column matrices, α_1, β_1 are zero matrices, and

$$\alpha_{n+1} = -\left(B_n + C_n\alpha_n\right)^{-1} A_n, \tag{2.40}$$
$$\beta_{n+1} = \left(B_n + C_n\alpha_n\right)^{-1}\left(D_n\varphi_n - C_n\beta_n\right), n = 1,2,\dots M-1.$$

The errors are computed by

$$E_M^N = \max_{0\le k\le N, 1\le n\le M-1}\left|u(t_k,x_n) - u_n^k\right| \tag{2.41}$$

of the numerical solutions, where $u(t_k,x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k,x_n) and the results are given in the following table

Difference schemes/N,M	$20,20$	$40,40$	$80,80$
Difference scheme (2.37)	$1.5602e-02$	$7.6036e-03$	$3.7547e-03$

(2.42)

As it is seen in Table 2.42, we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately $1/2$ for first order of accuracy in t difference scheme (2.37).

2.4.2 The High Order of Accuracy Difference Schemes

Now, we will consider the high order of accuracy difference schemes for the approximate solution of the problem.

First, using formulas (2.36),

$$\frac{u(t_{k+2}) - 3u(t_{k+1}) + 3u(t_k) - u(t_{k-1})}{\tau^3} - \frac{3}{4}u'''(t_k) - \frac{1}{4}u'''(t_{k+2}) = O(\tau^3),$$
$$\frac{3u(1) - 4u(1-\tau) + u(1-2\tau)}{2\tau} = u'(1) + O(\tau^2), \tag{2.43}$$

we get three-step difference scheme the second order of accuracy in t and x

$$\begin{cases} \dfrac{u_n^{k+2}-3u_n^{k+1}+3u_n^{k}-u_n^{k-1}}{\tau^3}-3\dfrac{u_{n+1}^{k}-2u_n^{k}+u_{n-1}^{k}}{4h^2} \\ -\dfrac{u_{n+1}^{k+2}-2u_n^{k+2}+u_{n-1}^{k+2}}{4h^2} \\ =\dfrac{3f(t_k,x_n)+f(t_{k+2},x_n)}{4}, f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ t_k=k\tau,\ 1\le k\le N-2,\ N\tau=1, \\ x_n=nh,\ 1\le n\le M-1,\ Mh=\pi, \\ u_n^0=2\sin x_n, u_n^N=e^{-1}\sin x_n, 0\le n\le M, \\ 3u_n^N-4u_n^{N-1}+u_n^{N-2}=-2\tau e^{-1}\sin x_n, 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N. \end{cases} \tag{2.44}$$

This system can be written in the matrix form (2.38). Here,

$$A=C=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & d & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & d & -b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & d & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & d & -b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a=-\frac{3}{\tau^3},\ b=-\frac{1}{\tau^3},\ c=-\frac{3}{\tau^3},\ d=\frac{1}{\tau^3}+\frac{1}{2h^2},$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\left\{\begin{array}{l} \varphi_n^k=\frac{1}{4}\left(3f(t_k,x_n)+f(t_{k+2},x_n)\right), f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ \\ t_k=k\tau, 1\le k\le N-2,\ 1\le n\le M-1, \\ \\ \varphi_n^0=2\sin x_n, \varphi_n^{N-1}=-2\tau e^{-1}\sin x_n, 0\le n\le M, \\ \\ \varphi_n^N=e^{-1}\sin x_n, 0\le n\le M. \end{array}\right\}$$

Second, using formulas (2.36),

$$\frac{u(t_{k+2})-3u(t_{k+1})+3u(t_k)-u(t_{k-1})}{\tau^3}-\frac{1}{2}u'''(t_k)-\frac{1}{2}u'''(t_{k+1})=O(\tau^4), \tag{2.45}$$

$$\frac{11u(1)-18u(1-\tau)+9u(1-2\tau)-2u(1-3\tau)}{6\tau}=u'(1)+O(\tau^3),$$

we get three-step difference scheme the third order of accuracy in t and the second order of accuracy in x

$$\begin{cases} \dfrac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^3}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} \\ -\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{2h^2} \\ =\dfrac{f(t_k,x_n)+f(t_{k+1},x_n)}{2}, f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ t_k=k\tau, 1\le k\le N-2,\ N\tau=1, \\ x_n=nh,\ 1\le n\le M-1,\ Mh=\pi, \\ u_n^0=2\sin x_n, u_n^N=e^{-1}\sin x_n, 0\le n\le M, \\ 11u_n^N-18u_n^{N-1}+9u_n^{N-2}-2u_n^{N-3}=-6\tau e^{-1}\sin x_n, 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N. \end{cases} \tag{2.46}$$

This system can be written in the matrix form (2.38), too. Here,

$$A=C=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & d & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & d & -b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & d & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & d & -b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a=-\frac{1}{2h^2},\ b=-\frac{1}{\tau^3},\ c=\frac{1}{h^2}+\frac{3}{\tau^3},\ d=-\frac{3}{\tau^3}+\frac{1}{h^2},$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\left\{\begin{array}{l} \varphi_n^k=\frac{1}{2}\left(f(t_k,x_n)+f(t_{k+1},x_n)\right)\ ,f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ \\ t_k=k\tau, 1\le k\le N-2,\ 1\le n\le M-1, \\ \\ \varphi_n^0=2\sin x_n, \varphi_n^{N-1}=-6\tau e^{-1}\sin x_n, 0\le n\le M, \\ \\ \varphi_n^N=e^{-1}\sin x_n, 0\le n\le M. \end{array}\right\}$$

Third, using formulas (2.36), (2.45) and

$$\frac{25u(1)-48u(1-\tau)+36u(1-2\tau)-16u(1-3\tau)+3u(1-4\tau)}{12\tau}=u'(1)+O(\tau^4),$$

we get three-step difference scheme the fourth order of accuracy in t and the second order of accuracy in x

$$\begin{cases} \dfrac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^3}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} \\ -\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{2h^2} \\ =\dfrac{f(t_k,x_n)+f(t_{k+1},x_n)}{2}, f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ t_k=k\tau, 1\le k\le N-2, N\tau=1, \\ x_n=nh,\ 1\le n\le M-1,\ Mh=\pi, \\ u_n^0=2\sin x_n, u_n^N=e^{-1}\sin x_n, 0\le n\le M, \\ 25u_n^N-48u_n^{N-1}+36u_n^{N-2}-16u_n^{N-3}+3u_n^{N-4}=-12\tau e^{-1}\sin x_n, \\ 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N. \end{cases} \tag{2.47}$$

This system can be written in the matrix form (2.38), too. Here,

$$A=C=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & d & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & d & -b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & d & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & d & -b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a=-\frac{1}{2h^2},\ b=-\frac{1}{\tau^3},\ c=\frac{3}{\tau^3}+\frac{1}{h^2},\ d=-\frac{3}{\tau^3}+\frac{1}{h^2},$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\left\{\begin{array}{l} \varphi_n^k=\frac{1}{2}\left(f(t_k,x_n)+f(t_{k+1},x_n)\right),f(t_k,x_n)=(1-t_k)^2\sin x_n, \\ \\ t_k=k\tau,1\leq k\leq N-2,\ 1\leq n\leq M-1, \\ \\ \varphi_n^0=2\sin x_n,\varphi_n^{N-1}=-12\tau e^{-1}\sin x_n,0\leq n\leq M, \\ \\ \varphi_n^N=e^{-1}\sin x_n,0\leq n\leq M. \end{array}\right\}$$

Therefore, for the solution of the matrix equation (2.38), we will use the same formulas (2.39), (2.40) and the errors are computed by formula (2.41). Numerical results are given in following tables

Difference schemes/N,M	20, 20	40, 40	80, 80
Difference scheme (2.44)	$1.0369e-04$	$2.5593e-05$	$6.3519e-06$

(2.48)

Difference schemes/N,M	$4,30$	$8,80$	$16,200$
Difference scheme (2.37)	$9.9788e-02$	$4.2520e-02$	$1.9804e-02$
Difference scheme (2.44)	$2.1838e-03$	$4.8124e-04$	$1.1332e-04$
Difference scheme (2.46)	$4.6503e-04$	$4.9467e-05$	$5.6483e-06$
Difference scheme (2.47)	$1.2135e-04$	$7.4630e-06$	$6.6517e-07$

(2.49)

As it is seen in Table 2.48, we get some numerical results for difference scheme (2.44). Note that if N and M are doubled, the value of errors decrease by a factor of approximately $1/4$ for second order of accuracy in t difference scheme (2.44). Moreover, as it is seen in Table 2.49, if N is doubled and $M \geq N\sqrt{10N}$, the value of errors decrease by a factor of approximately $1/2^m$ for the m-th of accuracy in t difference schemes (2.37), (2.44),(2.46) and (2.47), respectively.

The errors presented in these tables indicates the accuracy of difference schemes. We conclude that, the accuracy increases with the higher order approximation.

2.5 Appendix Matlab Programing

2.5.1 Matlab Implementation of Difference Schemes

```
function RR(N,M)
    if nargin <1; end;
    close;close;
    %first order
    tau=1/N;
    h=pi/M;
    a = -1/(h^2);
    b =-1/(tau^3);
    c = -2*a-3*b;
    A=zeros(N+1,N+1);
    for i=2:N-1;
```

```
A(i,i)=a;
end;
A;
C=A;
B=zeros(N+1,N+1);
for i=2:N-1 ;
B(i,i-1)= b;
B(i,i)=c;
B(i,i+1)=3*b;
B(i,i+2)=-b ;
end;
B(1,1)=1;
B(N+1,N+1)=1;
B(N,N+1)=1;
B(N,N)=-1;
B;
D=eye(N+1,N+1);
for j=1:M+1;
for k=2:N-1;
fii(k,j) =((1-(k-1)*tau)^2)*sin((j-1)*h);
end;
fii(1,j) =2*sin((j-1)*h);
fii(N,j) =(-tau)*exp(-1)*sin((j-1)*h);
fii(N+1,j) =exp(-1)*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
```

```
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1 ;
for k=1:N+1 ;
es(k,j) =((1-(k-1)*tau)^2+exp((-k+1)*tau))*sin((j-1)*h);
end;
end;
figure ;
m(1,1)=min(min(U))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION');
figure ;
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('FIRST ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap1 = [maxerror,relativeerror] ;
%Second order
a =(3/(tau^3))+(3/(2*(h^2)));
b =-1/(tau^3);
c = (-3/(tau^3));
d=(1/(tau^3))+(1/(2*(h^2)));
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
for i=2:N-1;
A(i,i)=-3/(4*(h^2));
A(i,i+2)=-1/(4*(h^2));
end;
%A(N,N)=(tau^2)/(4*(h^2));
%A(N,N+1)=(tau^2)/(12*(h^2));
```

```
A;
C=A;
for i=2:N-1;
B(i,i-1)=b;
B(i,i)=a;
B(i,i+1)=c;
B(i,i+2)=d;
end;
B(1,1)=1;
B(N,N+1)=3%((3*tau^-1)/2)-(2*(tau^2)/12*(h^2));
B(N,N)=-4%-2*(tau^-1)-(2*(tau^2)/4*(h^2));
B(N,N-1)=1%(tau^-1)/2;
B(N+1,N+1)=1;
for j=1:M+1;
for k=2:N-1;
fii(k,j) =(3/4)*((1-(k-1)*tau)^2)*sin((j-1)*h)+(1/4)*((1-(k+1)*tau)^2)*sin((j-1)*h);
end;
fii(1,j) =2*sin((j-1)*h);
fii(N,j) =-2*tau*exp(-1)*sin((j-1)*h);
fii(N+1,j) =exp(-1)*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
figure ;
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
```

```
title('SECOND ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap2 = [maxerror,relativeerror] ;
%Third order
a = -1/(2*(h^2));
b =-1/(tau^3);
c = -2*(-1/(2*(h^2)))-3*(-1/(tau^3));
d=3*(-1/(tau^3))-2*(-1/(2*(h^2)));
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
for i=2:N-1;
A(i,i)=a;
A(i,i+1)=a;
end;
%A(N,N)=(tau^3)/(2*(h^2));
%A(N,N+1)=(tau^3)/(6*(h^2));
A;
C=A;
for i=2:N-1;
B(i,i-1)= b;
B(i,i)=c;
B(i,i+1)=d;
B(i,i+2)=-b;
end;
B(1,1)=1;
B(N,N+1)=11%-(2*(tau^3)/6*(h^2));
B(N,N)=-18%-(2*(tau^3)/2*(h^2));
B(N,N-1)=9;
B(N,N-2)=-2;
B(N+1,N+1)=1;
B;
for i=1:N+1;
D(i,i)=1;
```

```
end ;
D;
for j=1:M+1;
for k=2:N-1;
fii(k,j:j) =(1/2)*((1-(k-1)*tau)^2)*sin((j-1)*h)+(1/2)*((1-(k)*tau)^2)*sin((j-1)*h);
end;
fii(1,j) =2*sin((j-1)*h);
fii(N,j) =-(exp(-1))*6*tau*sin((j-1)*h);
fii(N+1,j) =(exp(-1))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
figure ;
m(1,1)=min(min(U))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('THIRD ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap3 = [maxerror,relativeerror] ;
%FOURTH order
a = -1/(2*(h^2));
b =-1/(tau^3);
```

```
c = -2*(-1/(2*(h^2)))-3*(-1/(tau^3));
d=3*(-1/(tau^3))-2*(-1/(2*(h^2)));
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
for i=2:N-1;
A(i,i)=a;
A(i,i+1)=a;
end;
%A(N,N)=(tau^3)/(2*(h^2));
%A(N,N+1)=(tau^3)/(6*(h^2));
A;
C=A;
for i=2:N-1;
B(i,i-1)= b;
B(i,i)=c;
B(i,i+1)=d;
B(i,i+2)=-b;
end;
B(1,1)=1;
B(N,N+1)=25%-(2*(tau^3)/6*(h^2));
B(N,N)=-48%-(2*(tau^3)/2*(h^2));
B(N,N-1)=36;
B(N,N-2)=-16;
B(N,N-3)=3;
B(N+1,N+1)=1;
B;
for i=1:N+1;
D(i,i)=1;
end ;
D;
for j=1:M+1;
for k=2:N-1;
fii(k,j:j) =(1/2)*((1-(k-1)*tau)^2)*sin((j-1)*h)+(1/2)*((1-(k)*tau)^2)*sin((j-1)*h);
end;
fii(1,j) =2*sin((j-1)*h);
fii(N,j) =-(exp(-1))*12*tau*sin((j-1)*h);
```

```
fii(N+1,j) =(exp(-1))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
figure ;
m(1,1)=min(min(U))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('FOURTH ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap4 = [maxerror,relativeerror] ;
format short e;
cevap=[cevap1,cevap2,cevap3,cevap4]
```

2.5.2 Figures Presented by Numerical Experiences of Difference Schemes

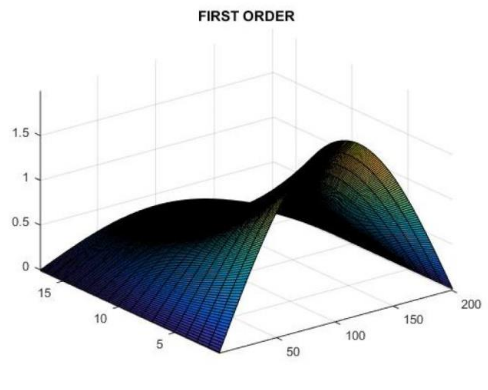

Figure 1 Solution of difference scheme (2.37)for $N = 16, M = 200$.

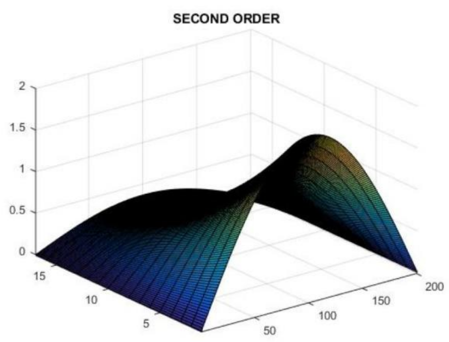

Figure 2 Solution of difference scheme (2.44)for $N = 16, M = 200$.

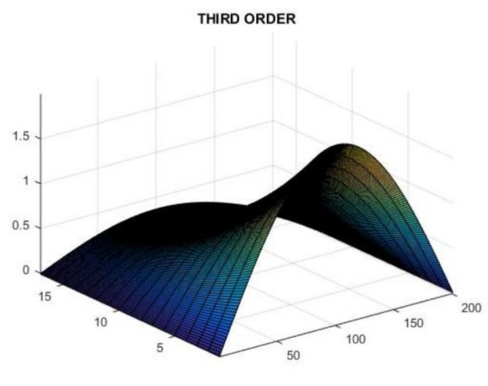

Figure 3 Solution of difference scheme (2.46)for $N = 16, M = 200$.

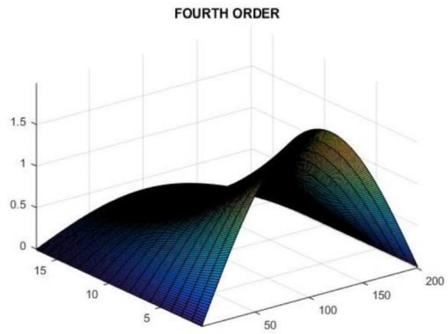

Figure 4 Solution of difference scheme (2.47)for $N = 16, M = 200$.

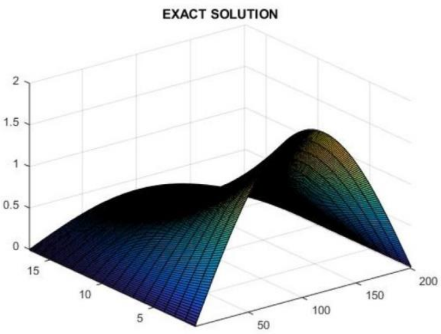

Figure 5 Exact solution of problem (2.35)for $N = 16, M = 200$.

Chapter 3

Stability of Nonlocal BVP for a Third Order Partial Differential Equation

3.1 Introduction

In Chapter 3 we consider the nonlocal boundary value problem for the third order partial differential equation

$$\begin{cases} \frac{d^3u(t)}{dt^3} + A\frac{du(t)}{dt} = f(t), \quad 0 < t < 1, \\ \\ u(0) = \gamma u(\lambda) + \varphi, \quad u'(0) = \alpha u'(\lambda) + \psi, |\gamma| < 1, \\ \\ u''(0) = \beta u''(\lambda) + \xi, \quad |1 + \beta\alpha| > |\alpha + \beta|, 0 < \lambda \leq 1 \end{cases} \tag{3.1}$$

in a Hilbert space H with a self-adjoint positive definite operator $A \geq \delta I$, where $\delta > 0$.

We are interested in studying the stability of solutions of nonlocal boundary value problem (3.1). A function $u(t)$ is a solution of problem (3.1) if the following conditions are satisfied:

(i) $u(t)$ is thrice continuously differentiable on the interval $(0,1)$ and twice continuously differentiable on the segment $[0,1]$. The derivatives at the end points of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u'(t)$ belongs to $D(A)$, $\forall t \in [0,1]$, and the function $Au'(t)$ is continuous on $[0,1]$.

(iii) $u(t)$ satisfies the equation and boundary nonlocal conditions (3.1).

Throughout this chapter, $\{C(t), t \geq 0\}$ is an operator function, $C(t) = \cos(tA^{\frac{1}{2}})$ defined by the formula

$$C(t) = \frac{e^{itA^{\frac{1}{2}}} + e^{-itA^{\frac{1}{2}}}}{2} \tag{3.2}$$

then from the following of operator function $S(t) = A^{-\frac{1}{2}} \sin(tA^{\frac{1}{2}}), S(t) = \int_0^t C(p)\, dp$, is follow that

$$S(t) = A^{-\frac{1}{2}} \frac{e^{itA^{\frac{1}{2}}} - e^{-itA^{\frac{1}{2}}}}{2i} \tag{3.3}$$

for the theory of cosine operator function, we refer to [53], [88].

Applying operator approach, stability estimates for solution of the nonlocal boundary value problem (3.1) are obtained. In practice, nonlocal boundary value problems for a third order in t partial differential equations are studied. Theorems on the stability estimates for the solutions of these problems are obtained. A first and second order of accuracy difference schemes are constructed for the approximate solution of the one dimensional partial differential equations. Numerical results are given.

The outline of Chapter 3 is as follows. The first section is introduction. In the section 2 main theorem on stability of problem (3.1) is established. Section 3 establishes the stability estimates for the solution of three problems for partial differential equations of third order in t. Section 4 numerical analysis.

3.2 Main theorem on stability

Let us give some lemmas that will be needed bellow

Lemma 3.2.1 , *For $t \geq 0$ the following estimate holds*

$$\left\|\exp\left\{\pm itA^{\frac{1}{2}}\right\}\right\|_{H\to H} \leq 1, \ \ ||C(t)||_{H\to H} \leq 1, \ \ \left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H} \leq 1. \tag{3.4}$$

Proof. Applying the spectral representation of unit self-adjoint positive definite operator A, we get

$$\left\|\exp\left\{\pm itA^{\frac{1}{2}}\right\}\right\|_{H\to H} \leq \sup_{\delta\leq\lambda<\infty} \left|\exp\left\{\pm it\lambda^{\frac{1}{2}}\right\}\right| = 1. \tag{3.5}$$

The proof of estimates $||C(t)||_{H\to H} \leq 1$, $\left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H} \leq 1$ is based on the estimate (3.5), formulas (3.2) and (3.3). ■

Lemma 3.2.2 *[12] Assume that $|1+\beta\alpha| > |\alpha+\beta|$. Then the operator Δ defined by the following formula*

$$\Delta = (1+\alpha\beta)I - (\alpha+\beta)C(\lambda), \quad 0 \leq \lambda \leq 1.$$

has a bounded inverse $T = \Delta^{-1}$ and the following estimates holds

$$\|T\|_{H\to H} \leq \frac{1}{|1+\beta\alpha| - |\alpha+\beta|}. \tag{3.6}$$

Proof. Using estimate (3.4) and triangle inequality, we can write

$$\|(1+\alpha\beta)I - (\alpha+\beta)C(\lambda)\|_{H\to H}$$

$$\geq \|(1+\alpha\beta)I\|_{H\to H} - \|(\alpha+\beta)C(\lambda)\|_{H\to H}$$

$$\geq |1+\alpha\beta| - |\alpha+\beta|\,\|C(\lambda)\|_{H\to H} \geq |1+\alpha\beta| - |\alpha+\beta|.$$

Therefore,

$$\|\Delta\|_{H\to H} \geq |1+\alpha\beta| + |\alpha+\beta| > 0.$$

From that it follows estimate (3.6). Lemma 3.2.2 proved. ■

Lemma 3.2.3 *Suppose that $\varphi \in D(A)$, $\psi \in D\left(A^{\frac{1}{2}}\right)$, $\xi \in D\left(A^{\frac{1}{2}}\right)$ and $f(t)$ is continuously differentiable on $[0,1]$. Then there is unique solution of problem (3.1) and the following formulas hold*

$$u(t) = \gamma u(\lambda) + \varphi + S(t)\left[\psi + \alpha u'(\lambda)\right] + A^{-1}(I - C(t))\left[\xi + \beta u''(\lambda)\right]$$

$$+ \int_0^t A^{-1}(I - C(t-s)) f(s)\, ds, \tag{3.7}$$

$$u(\lambda) = \frac{1}{1-\gamma}\{\varphi + S(\lambda)\left[\alpha u'(\lambda) + \psi\right] + A^{-1}(I - C(\lambda))$$

$$\times\left[\xi + \beta u''(\lambda)\right] + \int_0^\lambda A^{-1}(I - C(\lambda - s)) f(s)\, ds\bigg\}, \tag{3.8}$$

$$u'(\lambda) = T\{(I - \beta C(\lambda))[C(\lambda)\psi + S(\lambda)\xi$$

$$+ \int_0^\lambda S(\lambda - s) f(s)\, ds\bigg] + \beta S(\lambda)$$

$$\times\left[-AS(\lambda)\psi+C(\lambda)\xi+\int_0^\lambda C(\lambda-s)f(s)ds\right]\Big\},\tag{3.9}$$

$$\begin{aligned}u''(\lambda)=T\{&(I-\alpha C(\lambda))[-AS(\lambda)\psi+C(\lambda)\xi\\&+\int_0^\lambda C(\lambda-s)f(s)ds\Big]-\alpha AS(\lambda)\\&\times\left[C(\lambda)\psi+S(\lambda)\xi+\int_0^\lambda S(\lambda-s)f(s)ds\right]\Big\}.\end{aligned}\tag{3.10}$$

Proof. Obviously, we can write

$$\left(\frac{d^2}{dt^2}+A\right)\frac{d}{dt}u(t)=f(t)$$

for all $u(t)\in D(A)$. Therefore, problem (3.1) can be obviously rewritten as the equivalent nonlocal boundary value problem

$$\begin{cases}\dfrac{du(t)}{dt}=v(t),0<t<1,u(0)=\gamma u(\lambda)+\varphi,\\[2ex]\dfrac{d^2v(t)}{dt^2}+Av(t)=f(t),\ \ v(0)=\alpha v(\lambda)+\psi,v'(0)=\beta v'(\lambda)+\xi\end{cases}\tag{3.11}$$

for the system of linear differential equations. Integrating these equations, we can write

$$\begin{cases}u(t)=u(0)+\int_0^t v(s)ds,\\[2ex]v(t)=C(t)v(0)+S(t)v'(0)+\int_0^t S(t-s)f(s)ds.\end{cases}\tag{3.12}$$

Applying (3.2) and (3.3) we can write

$$\int_0^t S(s)dsu=-A^{-1}(C(t)-I)u.$$

From that and equation $\dfrac{du(t)}{dt}=v(t)$ it follows $v(0)=u'(0),v'(0)=u''(0)$ and

$$u(t)=u(0)+S(t)u'(0)-A^{-1}(C(t)-I)u''(0)+\int_0^t A^{-1}(I-C(t-s))f(s)ds.\tag{3.13}$$

Applying (3.13), nonlocal conditions $u(0)=\gamma u(\lambda)+\varphi, u'(0)=\alpha u'(\lambda)+\psi$, and $u''(0)=\beta u''(\lambda)+\xi$,we get formula (3.7). Putting $t=\lambda$ in (3.7), we get

$$u(\lambda)=\gamma u(\lambda)+\varphi+S(\lambda)\left[\alpha u'(\lambda)+\psi\right]-A^{-1}(C(\lambda)-I)\left[\beta u''(\lambda)+\xi\right]$$

$$+\int_0^\lambda A^{-1}(I-C(\lambda-s))f(s)\,ds.$$

From that it follows (3.8) for $u(\lambda)$.Therefore, we will obtain $u'(\lambda)$ and $u''(\lambda)$.Taking first and second order derivatives from (3.7) and putting $t=\lambda$, we get

$$u'(\lambda)=C(\lambda)\left[\alpha u'(\lambda)+\psi\right]+S(\lambda)\left[\beta u''(\lambda)+\xi\right]+\int_0^\lambda S(\lambda-s)f(s)\,ds,$$

$$u''(\lambda)=-AS(\lambda)\left[\alpha u'(\lambda)+\psi\right]+C(\lambda)\left[\beta u''(\lambda)+\xi\right]+\int_0^\lambda C(\lambda-s)f(s)\,ds.$$

Therefore, for obtaining $u'(\lambda)$ and $u''(\lambda)$, we have the following system of two equations

$$[I-\alpha C(\lambda)]\,u'(\lambda)-\beta S(\lambda)\,u''(\lambda)=C(\lambda)\,\psi+S(\lambda)\,\xi+\int_0^\lambda S(\lambda-s)f(s)\,ds,$$

$$\alpha AS(\lambda)\,u'(\lambda)+(I-\beta C(\lambda))\,u''(\lambda)=-AS(\lambda)\,\psi+C(\lambda)\,\xi+\int_0^\lambda C(\lambda-s)f(s)\,ds.$$

It is clear that

$$(I-\alpha C(\lambda))(I-\beta C(\lambda))+\alpha\beta AS^2(\lambda)=(1+\alpha\beta)I-(\alpha+\beta)C(\lambda).$$

By lemma 3.2.2 the operator $\Delta=(1+\alpha\beta)I-(\alpha+\beta)C(\lambda)$ has bounded inverse $T=\Delta^{-1}$. Therefore, we can get formulas (3.9) and (3.10). Lemma 3.2.3 is proved. ■

Now we will formulate the main theorem

Theorem 3.2.4 *Suppose that $\varphi\in H,\psi\in D(A)$, $\xi\in D\left(A^{1/2}\right)$ and $f(t)$ is continuously differentiable on $[0,1]$. Then there is a unique solution of problem (3.1) and the following inequalities hold*

$$\max_{0\le t\le 1}\|u(t)\|_H$$

$$\le M(\gamma)\left\{\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\left\|A^{-1}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right\}, \tag{3.14}$$

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\left\|A\frac{du}{dt}\right\|_H$$

$$\leq M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}, \tag{3.15}$$

where $M, M(\gamma)$ do not depend on $f(t)$, φ, ψ, ξ.

Proof. First, we estimate $\|u(t)\|_H$ for $t\in[0,1]$. Applying the triangle inequality, formula (3.7), and estimates (3.4), we get

$$\|u(t)\|_H\leq|\gamma|\,\|u(\lambda)\|_H+\|\varphi\|_H+\left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H}$$
$$\times\left[\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H\right]$$
$$+\|I-c(t)\|_{H\to H}\left[\left\|A^{-1}\xi\right\|_H+|\beta|\left\|A^{-1}u''(\lambda)\right\|_H\right]+\int_0^t\|I-C(t-s)\|_{H\to H}\left\|A^{-1}f(s)\right\|_H ds$$
$$\leq|\gamma|\,\|u(\lambda)\|_H+\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H+2\left\|A^{-1}\xi\right\|_H$$
$$+2|\beta|\left\|A^{-1}u''(\lambda)\right\|_H+2\max_{0\leq t\leq 1}\left\|A^{-1}f(t)\right\|_H \tag{3.16}$$

for any $t\in[0,1]$. Applying the triangle inequality, formula (3.8), and estimates (3.4), we get

$$\|u(\lambda)\|_H\leq\frac{1}{|1-\gamma|}\left\{\|\varphi\|_H+\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left[\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H\right]\right.$$
$$+\|I-c(\lambda)\|_{H\to H}\left[\left\|A^{-1}\xi\right\|_H+|\beta|\left\|A^{-1}u''(\lambda)\right\|_H\right]$$
$$\left.+\int_0^\lambda\|I-C(\lambda-s)\|_{H\to H}\left\|A^{-1}f(s)\right\|_H ds\right\}$$
$$\leq\frac{1}{|1-\gamma|}\left\{\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H+2\left\|A^{-1}\xi\right\|_H\right.$$
$$\left.+2|\beta|\left\|A^{-1}u''(\lambda)\right\|_H+2\max_{0\leq t\leq 1}\left\|A^{-1}f(t)\right\|_H\right\}. \tag{3.17}$$

Using estimates (3.16) and (3.17), we get

$$\|u(t)\|_H\leq\frac{|\gamma|}{|1-\gamma|}\left\{\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H+2\left\|A^{-1}\xi\right\|_H\right. \tag{3.18}$$
$$\left.+2|\beta|\left\|A^{-1}u''(\lambda)\right\|_H+2\max_{0\leq t\leq 1}\left\|A^{-1}f(t)\right\|_H\right\}$$

$$+\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H+2\left\|A^{-1}\xi\right\|_H$$
$$+2|\beta|\left\|A^{-1}u''(\lambda)\right\|_H+2\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H$$
$$\le\left[\frac{|\gamma|}{|1-\gamma|}+1\right]\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+|\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H\right]$$
$$+\left[\frac{|\gamma|}{|1-\gamma|}+2\right]\left[\left\|A^{-1}\xi\right\|_H+|\beta|\left\|A^{-1}u''(\lambda)\right\|_H+\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right]$$

for any $t\in[0,1]$. Therefore, the proof of estimate (3.14) is based on following estimates

$$\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H$$
$$\le M_1(\alpha,\beta,\gamma)\left\{\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\left\|A^{-1}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right\}, \tag{3.19}$$
$$\left\|A^{-1}u''(\lambda)\right\|_H$$
$$\le M_2(\alpha,\beta,\gamma)\left\{\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\left\|A^{-1}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right\}. \tag{3.20}$$

Applying formula (3.9), we get

$$A^{-\frac{1}{2}}u'(\lambda)=T\left\{A^{-\frac{1}{2}}(I-\beta C(\lambda))\left[C(\lambda)\psi+S(\lambda)\xi\right.\right.$$
$$\left.+A^{-\frac{1}{2}}\int_0^\lambda S(\lambda-s)f(s)\,ds\right]+A^{-\frac{1}{2}}\beta S(\lambda)$$
$$\left.\times\left[-AS(\lambda)\psi+C(\lambda)\xi+\int_0^\lambda C(\lambda-s)f(s)\,ds\right]\right\}.$$

Using the triangle inequality and estimates (3.4) and (3.6) , we get

$$\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H\le\|T\|_{H\to H}\left\{\|(I-\beta C(\lambda))\|_{H\to H}\left[\|C(\lambda)\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H\right.\right.$$
$$\left.+\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left\|A^{-1}\xi\right\|_H+\int_0^\lambda\left\|A^{\frac{1}{2}}S(\lambda-s)\right\|_{H\to H}\left\|A^{-1}f(s)\right\|_H ds\right]+|\beta|\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}$$
$$\times\left[\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H+\|C(\lambda)\|_{H\to H}\left\|A^{-1}\xi\right\|_H\right.$$
$$\left.\left.+\int_0^\lambda\|C(\lambda-s)\|_{H\to H}\left\|A^{-1}f(s)\right\|_H ds\right]\right\}$$

$$\leq \|T\|_{H\to H}\left\{(1+|\beta|)\left[\left\|A^{-\frac{1}{2}}\psi\right\|_{H}+\|A^{-1}\xi\|_{H}+\int_{0}^{\lambda}\|A^{-1}f(s)\|_{H}ds\right]\right.$$
$$\left.+|\beta|\left[\left\|A^{-\frac{1}{2}}\psi\right\|_{H}+\|A^{-1}\xi\|_{H}+\int_{0}^{\lambda}\|A^{-1}f(s)\|_{H}ds\right]\right\}.$$

From that it follows estimate (3.19). Applying formula (3.10), we get

$$A^{-1}u''(\lambda)=T\left\{(I-\alpha C(\lambda))\left[-S(\lambda)\psi+C(\lambda)A^{-1}\xi\right.\right.$$
$$\left.+\int_{0}^{\lambda}C(\lambda-s)A^{-1}f(s)ds\right]-\alpha S(\lambda)$$
$$\left.\times\left[C(\lambda)\psi+S(\lambda)\xi+\int_{0}^{\lambda}S(\lambda-s)f(s)ds\right]\right\}.$$

Applying the triangle inequality and estimates (3.4) and (3.6), we get

$$\|A^{-1}u''(\lambda)\|_{H}\leq\|T\|_{H\to H}\left\{\|I-\alpha C(\lambda)\|_{H\to H}\left[\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_{H}\right.\right.$$
$$\left.+\|C(\lambda)\|_{H\to H}\|A^{-1}\xi\|_{H}+\int_{0}^{\lambda}\|C(\lambda-s)\|_{H\to H}\|A^{-1}f(s)\|_{H}ds\right]$$
$$+\left\|\alpha A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left[+\|C(\lambda)\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_{H}+\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\|A^{-1}\xi\|_{H}\right.$$
$$\left.\left.+\int_{0}^{\lambda}\left\|A^{\frac{1}{2}}S(\lambda-s)\right\|_{H\to H}\|A^{-1}f(s)\|_{H}ds\right]\right\}$$
$$\leq\|T\|_{H\to H}\left\{[1+|\alpha|]\left[\left\|A^{-\frac{1}{2}}\psi\right\|_{H}+\|A^{-1}\xi\|_{H}+\int_{0}^{\lambda}\|A^{-1}f(s)\|_{H}ds\right]\right.$$
$$\left.+|\alpha|\left[\left\|A^{-\frac{1}{2}}\psi\right\|_{H}+\|A^{-1}\xi\|_{H}+\int_{0}^{\lambda}\|A^{-1}f(s)\|_{H}ds\right]\right\}$$

From that it follows estimate (3.20). Combining estimates (3.18), (3.19), and (3.20), we obtain estimate (3.4).

Second, we estimate $\left\|A\frac{du(t)}{dt}\right\|_{H}$ for $t\in[0,1]$. Applying (3.7) and taking the derivative, we get

$$A\frac{du(t)}{dt}=AC(t)\left[\psi+\alpha u'(\lambda)\right]+AS(t)\left[\xi+\beta u''(\lambda)\right]+\int_{0}^{t}AS(t-s)f(s)ds.$$

Applying formula

$$\int_{0}^{t}AS(t-s)f(s)ds=-C(t)f(0)-\int_{0}^{t}C(t-s)f'(s)ds, \tag{3.21}$$

we get

$$A\frac{du(t)}{dt} = AC(t)\left[\psi + \alpha u'(\lambda)\right] + AS(t)\left[\xi + \beta u''(\lambda)\right] - C(t)f(0) - \int_0^t C(t-s)f'(s)\,ds.$$

Using the triangle inequality and estimates (3.4), we get

$$\left\|A\frac{du(t)}{dt}\right\|_H \le \left[\|C(t)\|_{H\to H}\left[\|A\psi\|_H + |\alpha|\left\|Au'(\lambda)\right\|_H\right]\right. \tag{3.22}$$

$$+\left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H}\left[\left\|A^{\frac{1}{2}}\xi\right\|_H + |\beta|\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_H\right]$$

$$+\|C(t)\|_{H\to H}\|f(0)\|_H + \int_0^t \|C(t-s)\|_{H\to H}\left\|f'(s)\right\|_H ds$$

$$\le \|A\psi\|_H + |\alpha|\left\|Au'(\lambda)\right\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H$$

$$+|\beta|\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\left\|f'(t)\right\|_H$$

for any $t \in [0,1]$. Now, we will estimate $\|Au'(\lambda)\|_H$ and $\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_H$.Applying formulas (3.9), (3.21), and

$$\int_0^\lambda AS(\lambda-s)f(s)\,ds$$

$$= -f(\lambda) + C(\lambda)f(0) + \int_0^\lambda C(\lambda-s)f'(s)\,ds, \tag{3.23}$$

$$\int_0^\lambda C(\lambda-s)f(s)\,ds = S(\lambda)f(0) + \int_0^\lambda S(\lambda-s)f'(s)\,ds, \tag{3.24}$$

we get

$$Au'(\lambda) = T\{(I-\beta C(\lambda))[C(\lambda)A\psi + AS(\lambda)\xi$$

$$\left.-f(\lambda) + C(\lambda)f(0) + \int_0^\lambda C(\lambda-s)f'(s)\,ds\right] + \beta AS(\lambda)$$

$$\times\left[-AS(\lambda)\psi + C(\lambda)\xi + S(\lambda)f(0) + \int_0^\lambda S(\lambda-s)f'(s)\,ds\right]\Big\}.$$

Using the triangle inequality and estimates (3.4) and (3.6) , we get

$$\left\|Au'(\lambda)\right\|_H \le \|T\|_{H\to H}\Big\{\|I-\beta C(\lambda)\|_{H\to H}\Big[\|C(\lambda)\|_{H\to H}\|A\psi\|_H + \left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left\|A^{\frac{1}{2}}\xi\right\|_H$$

$$+\|f(\lambda)\|_{H\to H}+\|C(\lambda)\|_{H\to H}\|f(0)\|_{H}+\int_0^{\lambda}\|C(\lambda-s)\|_{H\to H}\left\|f'(s)\right\|_{H}\Big]$$

$$+|\beta|\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\Big[\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\|A\psi\|_{H}+\|C(\lambda)\|_{H\to H}\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\int_0^{\lambda}\left\|A^{\frac{1}{2}}S(\lambda-s)\right\|_{H\to H}\left\|f'(s)\right\|_{H}ds\Big]\Big\}$$

$$\leq\|T\|_{H\to H}\Big\{|\beta|\Big[\|A\psi\|_{H}+\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\|f(\lambda)\|_{H}+\|f(0)\|_{H}+\int_0^{\lambda}\left\|f'(s)\right\|_{H}\Big]$$

$$+|\beta|\Big[\|A\psi\|_{H}+\left\|A^{\frac{1}{2}}\xi\right\|_{H}+\int_0^{\lambda}\left\|f'(s)\right\|_{H}ds\Big\}$$

$$\leq M\left\{\|A\psi\|_{H}+\left\|A^{\frac{1}{2}}\xi\right\|_{H}+\|f(0)\|_{H}+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_{H}\right\}. \qquad (3.25)$$

Applying formulas (3.10), (3.21), (3.23), and (3.24), we get

$$A^{\frac{1}{2}}u''(\lambda)=A^{\frac{1}{2}}T\{(I-\alpha C(\lambda))[-AS(\lambda)\psi+C(\lambda)\xi$$

$$+\int_0^{\lambda}S(\lambda-s)f'(s)ds\Big]-\alpha AS(\lambda)$$

$$\times\left[C(\lambda)\psi+S(\lambda)\xi-A^{-1}f(\lambda)+A^{-1}C(\lambda)f(0)+\int_0^{\lambda}A^{-1}C(\lambda-s)f'(s)ds\right]\Big\}.$$

Using the triangle inequality and estimates (3.4) and (3.6) , we get

$$\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_{H}\leq\|T\|_{H\to H}\Big\|\Big\{\|I-\alpha C(\lambda)\|_{H\to H}\Big[\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\|A\psi\|_{H}$$

$$+\|C(\lambda)\|_{H\to H}\left\|A^{\frac{1}{2}}\xi\right\|_{H}+\int_0^{\lambda}\left\|A^{\frac{1}{2}}S(\lambda-s)\right\|_{H\to H}\left\|f'(s)\right\|_{H}ds\Big]$$

$$+|\alpha|\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\Big[\|C(\lambda)\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_{H}+\left\|A^{\frac{1}{2}}S(\lambda)\right\|_{H\to H}\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\|f(\lambda)\|_{H}+\|C(\lambda)\|_{H\to H}\|f(0)\|_{H}+\int_0^{\lambda}\|C(\lambda-s)\|_{H\to H}\left\|f'(s)\right\|_{H}ds\Big]\Big\}$$

$$\leq\|T\|_{H\to H}\Big\|\Big\{(1+|\alpha|)\Big[\|A\psi\|_{H}+\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\int_0^{\lambda}\|f(s)\|_{H}ds\Big]+|\alpha|\Big[\|A\psi\|_{H}+\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\|f(\lambda)\|_H+\|f(0)\|_H+\int_0^{\lambda}\left\|f'(s)\right\|_H ds\Big]\Big\}$$

$$\leq M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}. \tag{3.26}$$

Combining estimates (3.22), (3.25), and (3.26), we obtain estimate

$$\max_{0\leq t\leq 1}\left\|A\frac{du(t)}{dt}\right\|_H\leq M_1\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H\right.$$

$$\left.+\|f(0)\|_H+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}.$$

From that and equation (3.1) and triangle inequality it follows that

$$\max_{0\leq t\leq 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H\leq \max_{0\leq t\leq 1}\left\|A\frac{du(t)}{dt}\right\|_H+\max_{0\leq t\leq 1}\|f(t)\|_H$$

$$\leq M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}.$$

The proof of Theorem 3.2.4 is finished. ■

Moreover, we have the following theorem on stability.

Theorem 3.2.5 *Suppose that* $\psi\in D(A)$, $\xi\in D\left(A^{1/2}\right)$ *and* $f(t)$ *is continuous on* $[0,1]$ *and* $f(t)\in D\left(A^{1/2}\right)$. *Then there is a unique solution of problem (3.1) and the following estimate holds*

$$\max_{0\leq t\leq 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\leq t\leq 1}\left\|A\frac{du}{dt}\right\|_H$$

$$\leq M_1\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}}f(t)\right\|_H\right\},$$

where M_1 *does not depend on* $f(t)$, ψ, ξ.

Proof. We will estimate $\|A\frac{du}{dt}\|_H$ for $t\in[0,1]$. Applying formula (3.7), and taking the derivative, we get

$$A\frac{du(t)}{dt}=AC(t)\left[\psi+\alpha u'(\lambda)\right]+AS(t)\left[\xi+\beta u''(\lambda)\right]+\int_0^t AS(t-s)f(s)ds.$$

Applying the triangle inequality and estimates (3.4), we get

$$\left\|A\frac{du}{dt}\right\|_H\leq\|C(t)\|_{H\to H}\left[\|A\psi\|+|\alpha|\left\|Au'(\lambda)\right\|_H\right]+\left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H}\left[\left\|A^{\frac{1}{2}}\xi\right\|_H\right.$$

$$+|\beta|\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_{H}\Big]+\left\|A^{\frac{1}{2}}S(t-s)\right\|_{H\to H}\int_0^t\left\|A^{\frac{1}{2}}f(s)\right\|_{H}ds$$

$$\leq\|A\psi\|+|\alpha|\left\|Au'(\lambda)\right\|_{H}+\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+|\beta|\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_{H}+\max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}}f(t)\right\|_{H} \tag{3.27}$$

for any $t\in[0,1]$. Now, we will estimate $\|Au'(\lambda)\|_H$. Applying formula (3.9), we get

$$Au'(\lambda)=T\{(I-\beta C(\lambda))[C(\lambda)A\psi+AS(\lambda)\xi$$

$$+\int_0^{\lambda}AS(\lambda-s)f(s)ds\Big]+\beta AS(\lambda)$$

$$\times\left[-AS(\lambda)\psi+C(\lambda)\xi+\int_0^{\lambda}C(\lambda-s)f(s)ds\right]\Bigg\}.$$

Applying the triangle inequality and estimates (3.4), we get

$$\left\|Au'(\lambda)\right\|_{H}\leq\|T\|_{H\to H}\Big\{(1+|\beta|)\Big[\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\int_0^{\lambda}\left\|A^{\frac{1}{2}}f(s)\right\|_{H}ds\Big]$$

$$+|\beta|\left[\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_{H}+\int_0^{\lambda}\left\|A^{\frac{1}{2}}f(s)\right\|_{H}ds\right]\Bigg\}$$

$$\leq M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_{H}+\max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}}f(t)\right\|_{H}\right\}. \tag{3.28}$$

Now, we will estimate $\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_{H}$. Applying formula (3.10), we get

$$A^{\frac{1}{2}}u''(\lambda)=T\Big\{(I-\alpha C(\lambda))\Big[-A^{\frac{1}{2}}S(\lambda)A\psi+C(\lambda)A^{\frac{1}{2}}\xi$$

$$+\int_0^{\lambda}C(\lambda-s)A^{\frac{1}{2}}f(s)ds\Big]-\alpha A^{\frac{1}{2}}S(\lambda)$$

$$\times\left[C(\lambda)A\psi+A^{\frac{1}{2}}S(\lambda)A^{\frac{1}{2}}\xi+\int_0^{\lambda}A^{\frac{1}{2}}S(\lambda-s)A^{\frac{1}{2}}f(s)ds\right]\Bigg\}.$$

Using the triangle inequality and estimates (3.4), we get

$$\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_{H}\leq\|T\|_{H\to H}\Big\{(1+|\alpha|)\Big\|\Big[\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_{H}$$

$$+\int_0^\lambda \left\|A^{\frac{1}{2}} f(s)\right\|_H ds\Bigg]$$

$$+|\alpha|\left[\|A\psi\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \int_0^\lambda \left\|A^{\frac{1}{2}} f(s)\right\|_H ds\right]\Bigg\}$$

$$\leq M\left\{\|A\psi\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}} f(t)\right\|_H\right\}. \tag{3.29}$$

Combining estimates (3.27), (3.28), and (3.29), we obtain estimate

$$\max_{0\leq t\leq 1}\left\|A\frac{du}{dt}\right\|_H \leq M\left\{\|A\psi\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}} f(t)\right\|_H\right\}. \tag{3.30}$$

From that and equation (3.1) and triangle inequality it follows that

$$\max_{0\leq t\leq 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H \leq \max_{0\leq t\leq 1}\left\|A\frac{du}{dt}\right\|_H + \max_{0\leq t\leq 1}\|f(t)\|_H$$

$$\leq M_1\left\{\|A\psi\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}} f(t)\right\|_H\right\}.$$

Theorem 3.2.5 is proved. ■

From Theorem 3.2.4 and Theorem 3.2.5 it follows the following theorem on stability.

Theorem 3.2.6 *Assume that* $\psi \in D(A)$, $\xi \in D\left(A^{1/2}\right)$ *and* $f(t)$ *is continuously differentiable on* $[0,1]$ *and* $f(t) \in D\left(A^{1/2}\right)$. *Then there is a unique solution of problem (3.1) and the following inequalities hold*

$$\max_{0\leq t\leq 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\leq t\leq 1}\left\|A\frac{du}{dt}\right\|_H$$

$$\leq M\left\{\|A\psi\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \max_{0\leq t\leq 1}\left\|A^{\frac{1}{2}} f(t)\right\|_H\right\},$$

$$+\min\left\{\|f(0)\|_H + \max_{0\leq t\leq 1}\|f'(t)\|_H, \max_{0\leq t\leq 1}\left\|A^{1/2} f(t)\right\|_H\right\}\Bigg\},$$

where M *does not depend on* $f(t)$, ψ, ξ.

3.3 Applications

In this section we will consider three applications of main theorem 3.2.4. First, for the application of the theorem 3.2.4 we consider the nonlocal boundary value problem for a third order partial differential equation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - (a(x)u_{tx})_x + \delta u_t(t,x) = f(t,x), \;\; 0<t<1, 0<x<l, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), \quad u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), 0 \le x \le l, \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), \quad 0 \le x \le l, 0<\lambda\le 1, \\ u(t,0) = u(t,l), \quad u_x(t,0) = u_x(t,l), \quad 0 \le t \le 1. \end{cases} \tag{3.31}$$

Problem (3.31) has a unique smooth solution $u(t,x)$ for smooth $a(x) \ge a > 0$, $x \in (0,l)$, $\delta > 0$, $a(l) = a(0)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ ($x \in [0,l]$) and $f(t,x)$ ($t \in (0,1)$, $x \in (0,l)$) functions. This allows us to reduce problem (3.1) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by (3.31). Let us give a number of corollaries of abstract theorem 3.2.4

Theorem 3.3.1 *For the solution of the problem (3.31), the stability inequalities*

$$\max_{0\le t\le 1} \|u(t,.)\|_{L_2[0,1]}$$
$$\le M_1 \left[\max_{0\le t\le 1} \|f(t,.)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]} + \|\xi\|_{L_2[0,1]} \right], \tag{3.32}$$

$$\max_{0\le t\le 1} \left\| \frac{\partial u}{\partial t}(t,.) \right\|_{W_2^2[0,1]} + \max_{0\le t\le 1} \left\| \frac{\partial^3 u}{\partial t^3}(t,.) \right\|_{L_2[0,1]}$$
$$\le M_1 \left[\max_{0\le t\le 1} \|f_t(t,.)\|_{L_2[0,1]} + \|f(0,.)\|_{L_2[0,1]} + \|\psi\|_{W_2^2[0,1]} + \|\xi\|_{W_2^1[0,1]} \right] \tag{3.33}$$

hold, where M_1 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (3.31) can be written in abstract form

$$\begin{cases} \dfrac{d^3u(t)}{dt^3} + A\dfrac{du(t)}{dt} = f(t), \ \ 0 \le t \le 1, \\ \\ u(0) = \xi u(\lambda) + \varphi, \quad u_t(0) = \alpha u_t(\lambda) + \psi, \\ u_{tt}(0) = \beta u_{tt}(\lambda) + \xi \end{cases} \tag{3.34}$$

in Hilbert space $L_2[0,l]$ for all square integrable functions defined on $[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -(a(x)u_x)_x + \delta u(x) \tag{3.35}$$

with domain

$$D(A^x) = \left\{ u(x) : u, u_x, (a(x)u_x)_x \in L_2[0,l], u(0) = u(l), u'(0) = u'(l) \right\}.$$

Here $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are respectively known and unknown abstract functions defined on $[0,l]$ with the values $H = L_2[0,l]$. Therefore, estimates (3.32)-(3.33) follow from estimates (3.14)-(3.15). Thus, theorem 3.3.1 is proved. ■

Second, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\bar{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} (a_r(x)u_{tx_r})_{x_r} = f(t,x), \ \ x = (x_1, ..., x_n) \in \Omega, \ 0 < t < 1, \\ \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), \quad u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), x \in \bar{\Omega}, \\ \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), \quad x \in \bar{\Omega}, 0 < \lambda \le 1, \\ \\ u(t,x) = 0, \quad x \in S, \ 0 \le t \le 1, \end{cases} \tag{3.36}$$

where $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $\left(x \in \bar{\Omega}\right)$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$. We introduce the Hilbert space $L_2(\bar{\Omega})$, the space of integrable functions defined on $\bar{\Omega}$ equipped with norm

$$\|f\|_{L_2(\bar{\Omega})} = \left\{ \int \cdots \int_{x \in \bar{\Omega}} |f(x)|^2 \, dx_1 ... dx_n \right\}^{1/2}.$$

Theorem 3.3.2 *For the solution of the problem (3.36) the stability inequalities*

$$\max_{0\leq t\leq 1}\|u(t,.)\|_{L_2(\bar{\Omega})}$$

$$\leq M_2\left[\max_{0\leq t\leq 1}\|f(t,.)\|_{L_2(\bar{\Omega})}+\|\varphi\|_{L_2(\bar{\Omega})}+\|\psi\|_{L_2(\bar{\Omega})}+\|\xi\|_{L_2(\bar{\Omega})}\right], \tag{3.37}$$

$$\max_{0\leq t\leq 1}\|u(t,.)\|_{W_2^2[0,1]}+\max_{0\leq t\leq 1}\left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2(\bar{\Omega})}$$

$$\leq M_2\left[\max_{0\leq t\leq 1}\|f_t(t,.)\|_{L_2(\bar{\Omega})}+\|f(0,.)\|_{L_2(\bar{\Omega})}+\|\psi\|_{W_2^2(\bar{\Omega})}+\|\xi\|_{W_2^1(\bar{\Omega})}\right] \tag{3.38}$$

hold where M_2 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (3.36) can be written in abstract form (3.34) in Hilbert space $L_2\left(\bar{\Omega}\right)$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$A^x u(x)=-\sum_{r=1}^{n}\left(a_r(x)u_{x_r}\right)x_r \tag{3.39}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})\in L_2\left(\bar{\Omega}\right),1\leq r\leq n,u(x)=0,x\in S\right\}.$$

Here $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are known and unknown respectively abstract functions defined on $\bar{\Omega}$ with the value in $H=L_2\left(\bar{\Omega}\right)$. So, estimates (3.37)-(3.38) follow from estimates (3.14)-(3.15) and from the Theorem 2.3.3 on coercivity inequality for the solution of the elliptic differential problem in $L_2\left(\bar{\Omega}\right)$. Thus, theorem 3.3.2 is proved. ■

Third, we consider the nonlocal boundary value problem for a third order partial differential equation

$$\begin{cases}\dfrac{\partial^3 u(t,x)}{\partial t^3}-\sum\limits_{r=1}^{m}(a_r(x)u_{tx_r})x_r+\delta u_t(t,x)=f(t,x), \quad x=(x_1,...,x_n)\in\Omega,\ 0<t<1,\\ u(0,x)=\gamma u(\lambda,x)+\varphi(x), \quad u_t(0,x)=\alpha u_t(\lambda,x)+\psi(x),x\in\bar{\Omega},\\ u_{tt}(1,x)=\beta u_{tt}(\lambda,x)+\xi(x),\ x\in\bar{\Omega},\ 0<\lambda<1,\\ \dfrac{\partial u}{\partial \vec{m}}(0,x)=0, \quad x\in S,\ 0\leq t\leq 1,\end{cases} \tag{3.40}$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$, $\vec{m}$ is the normal vector to S .

Theorem 3.3.3 *For the solution of the problem (3.40), the stability inequalities*

$$\max_{0\leq t\leq 1} \|u(t,.)\|_{L_2(\bar{\Omega})}$$

$$\leq M_3 \left[\max_{0\leq t\leq 1} \|f(t,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{L_2(\bar{\Omega})} + \|\psi\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})} \right] \tag{3.41}$$

$$\max_{0\leq t\leq 1} \|u(t,.)\|_{W_2^2(\bar{\Omega})} \quad + \max_{0\leq t\leq 1} \left\| \frac{\partial^3 u}{\partial t^3}(t,.) \right\|_{L_2(\bar{\Omega})}$$

$$\leq M_3 \left[\max_{0\leq t\leq 1} \|f_t(t,.)\|_{L_2(\bar{\Omega})} + \|f(0,.)\|_{L_2(\bar{\Omega})} + \|\psi\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^1(\bar{\Omega})} \right] \tag{3.42}$$

hold where M_3 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (3.40) can be written in abstract form (3.34) in Hilbert space $L_2\left(\bar{\Omega}\right)$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\sum\nolimits_{r=1}^{m} (a_r(x) u_{x_r}) x_r + \delta u(x) \tag{3.43}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{k_r} \in L_2\left(\bar{\Omega}\right), 1 \leq r \leq m, \frac{\partial u}{\partial \vec{m}} = 0, x \in S \right\}.$$

Here $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are respectively known and unknown abstract functions defined on $\bar{\Omega}$ with the value in $H = L_2\left(\bar{\Omega}\right)$. So, estimates (3.41)-(3.42) follow from estimates (3.14)-(3.15) and from the Theorem 2.3.5 on coercivity inequality for the solution of the elliptic differential problem in $L_2\left(\bar{\Omega}\right)$.Thus, theorem 3.3.3 is proved. ■

3.4 Numerical Experiments

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature (for instance, see [11], [12], [37], [40], [30]).

In the present chapter for the approximate solutions of a nonlocal boundary value problem, we will use the first and second order of accuracy difference schemes. Numerically we show that the second order of accuracy for the approximate solution of the problem are more accurate than the first order of accuracy difference scheme. We apply a procedure of modified Gauss elimination method to solve the problem. Results of numerical experiences are given.

3.4.1 The First Order of Accuracy Difference Scheme

We consider the nonlocal boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial^3 u(t,x)}{\partial t \partial x^2} = f(t,x), \\ f(t,x) = -2e^{-t}\sin x, 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = \frac{1}{4}u(1,x) + \left(1 - \dfrac{1}{4e}\right)\sin x,\ 0 \le x \le \pi, \\ u_t(0,x) = \frac{1}{4}u_t(1,x) - \left(1 - \dfrac{1}{4e}\right)\sin x,\ 0 \le x \le \pi, \\ u_{tt}(0,x) = \frac{1}{4}u_{tt}(1,x) + \left(1 - \dfrac{1}{4e}\right)\sin x,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{3.44}$$

for a third order partial differential equation. The exact solution of problem (3.44) is

$$u(t,x) = e^{-t}\sin x.$$

For the approximate solutions of boundary value problem (3.44), applying formulas (2.36) and

$$\begin{cases} \dfrac{u'(t_{k+1})-2u'(t_k)+u'(t_{k-1})}{\tau^2}-u'''(t_{k+1})=O(\tau), \\ \\ \dfrac{u(\tau)-u(0)}{\tau}-u'(0)=o(\tau), \dfrac{u(1)-u(1-\tau)}{\tau}-u'(1)=O(\tau), \\ \\ \dfrac{u(2\tau)-2u(\tau)+u(0)}{\tau^2}-u''(0)=O(\tau), \\ \\ \dfrac{u(1)-2u(1-\tau)+u(1-2\tau)}{\tau^2}-u''(1)=O(\tau), \end{cases}$$

we get the first order of accuracy in t difference scheme

$$\begin{cases} \dfrac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^3} \\ \\ -\dfrac{u_{n+1}^{k+2}-u_{n+1}^{k+1}-2\left(u_n^{k+2}-u_n^{k+1}\right)+u_{n-1}^{k+2}-u_{n-1}^{k+1}}{\tau h^2}=f(t_k,x_n), \\ \\ f(t_k,x_n)=-2e^{-t_k}\sin x_n,\ t_k=k\tau,\ 1\le k\le N-2\ , \\ 1\le n\le M-1, \\ \\ N\tau=1,\ x_n=nh,\ 1\le n\le M-1,\ Mh=\pi, \\ \\ u_n^0=\frac{1}{4}\ u_n^N+\left(1-\dfrac{1}{4e}\right)\sin x_n,\ 0\le n\le M, \\ \\ \dfrac{u_n^1-u_n^0}{\tau}=\frac{1}{4}\dfrac{u_n^N-u_n^{N-1}}{\tau}-\left(1-\dfrac{1}{4e}\right)\sin x_n, 0\le n\le M, \\ \\ \dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2}=\frac{1}{4}\dfrac{u_n^N-2u_n^{N-1}+u_n^{N-2}}{\tau^2}+\left(1-\dfrac{1}{4e}\right)\sin x_n, \\ 0\le n\le M, \\ \\ u_0^k=u_M^k=0, 0\le k\le N. \end{cases} \tag{3.45}$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} A\,u_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ u_0 = \vec{0},\ u_M = \vec{0}. \end{cases} \tag{3.46}$$

Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & -a & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & -a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{4} \\ b & -3b & 3b-c & b-c & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & -3b & 3b-c & c-b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -3b & 3b-c & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3b-c & c-b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -3b & 3b-c & c-b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & -3b & 3b-c & c-b \\ -\frac{1}{\tau} & \frac{1}{\tau} & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{4\tau} & -\frac{1}{4\tau} \\ \frac{1}{\tau^2} & -\frac{2}{\tau^2} & \frac{1}{\tau^2} & 0 & 0 & \cdots & 0 & -\frac{1}{4\tau^2} & \frac{1}{2\tau^2} & -\frac{1}{4\tau^2} \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a = \frac{1}{\tau h^2},\ b = -\frac{1}{\tau^3},\ c = \frac{2}{\tau h^2},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \left\{ \begin{array}{l} \varphi_n^k = f(t_k, x_n) = -2e^{-t_k}\sin x_n, \\ t_k = k\tau, 1 \le k \le N-2,\ 1 \le n \le M-1, \\ \\ \varphi_n^0 = \left(1 - \dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^{N-1} = -\left(1 - \dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^N = \left(1 - \dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M \end{array} \right\}$$

and $D = I_{N+1}$ is the identity matrix,

$$u_s = \begin{bmatrix} u_s^0 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n, n\pm 1.$$

Therefore, for the solution of the matrix equation (3.46), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, \tag{3.47}$$

where $u_M = \vec{0}$, $\alpha_j \quad (j = 1, ..., M-1)$ are $(N+1)\times(N+1)$ square matrices, $\beta_j \quad (j = 1, ..., M-1)$ are $(N+1)\times 1$ column matrices, α_1, β_1 are zero matrices and

$$\begin{aligned} \alpha_{n+1} &= -\left(B + C\alpha_n\right)^{-1} A_n, \\ \beta_{n+1} &= \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1, ..., M-1. \end{aligned} \tag{3.48}$$

As the Chapter 2, the errors are computed by

$$E_M^N = \max_{0\le k\le N, 1\le n\le M-1} \left| u(t_k, x_n) - u_n^k \right| \tag{3.49}$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) and the results are given in the following table

Difference schemes/N,M	$20,20$	$40,40$	$80,80$
Difference scheme (3.45)	$1.3634e-02$	$6.9241e-03$	$3.4853e-03$

(3.50)

As it is seen in Table 3.50 , we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately $1/2$ for first order difference scheme.

3.4.2 The Second Order of Accuracy Difference Schemes

Now, we will consider the high order of accuracy difference schemes for the approximate solution of the problem.

First, for the approximate solution of boundary value problem (3.44), applying formulas (2.36),(2.43) and

$$\begin{cases} \dfrac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-\dfrac{1}{2}[u''(t_{k+1})+u''(t_{k-1})]=O(\tau^2), \\[2ex] \dfrac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-\dfrac{1}{2}u''(t_k)-\dfrac{1}{4}[u''(t_{k+1})+u''(t_{k-1})]=O(\tau^2), \\[2ex] \dfrac{-u(2\tau)+4u(\tau)-3u(0)}{2\tau}-u'(0)=O(\tau^2), \\[2ex] \dfrac{3u(1)-4u(1-\tau)+u(1-2\tau)}{2\tau}-u'(1)=O(\tau^2), \\[2ex] \dfrac{-3u(3\tau)+4u(2\tau)-5u(\tau)+2u(0)}{\tau^2}-u''(0)=O(\tau^2), \\[2ex] \dfrac{-3u(1)+4u(1-\tau)-5u(1-2\tau)+2u(1-3\tau)}{\tau^2}-u''(1)=O(\tau^2), \end{cases}$$

we get the second order of accuracy in t difference schemes

$$\begin{cases} \dfrac{u_n^3-3u_n^2+3u_n^1-u_n^0}{\tau^3}-\dfrac{u_{n+1}^3-2u_n^3+u_{n-1}^3+u_{n+1}^2-2u_n^2+u_{n-1}^2}{4\tau h^2} \\ +\dfrac{\left(u_{n+1}^1-2u_n^1+u_{n-1}^1\right)+\left(u_{n+1}^0-2u_n^0+u_{n-1}^0\right)}{4\tau h^2} \\ =\dfrac{f(t_1,x_n)+f(t_2,x_n)}{2}=-(e^{-t_1}+e^{-t_2})\sin x_n, \\ 1\le n\le M-1, \\ \dfrac{u_n^{k+2}-2u_n^{k+1}+2u_n^{k-1}-u_n^{k-2}}{2\tau^3} \\ -\dfrac{u_{n+1}^{k+2}-2u_n^{k+2}+u_{n-1}^{k+2}-\left(u_{n+1}^{k-2}-2u_n^{k-2}+u_{n-1}^{k-2}\right)}{4\tau h^2} \\ =f(t_k,x_n),\ f(t_k,x_n)=-2e^{-t_k}\sin x_n, \\ t_k=k\tau,\ 2\le k\le N-2\ ,\ 1\le n\le M-1, \\ N\tau=1,\ x_n=nh,\ 1\le n\le M-1,\ Mh=\pi, \\ u_n^0=\frac{1}{4}\,u_n^N+\left(1-\dfrac{1}{4e}\right)\sin x_n,\ 0\le n\le M, \\ \dfrac{-u_n^2+4u_n^1-3u_n^0}{2\tau}=\frac{1}{4}\dfrac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau}-\left(1-\dfrac{1}{4e}\right)\sin x_n, 0\le n\le M, \\ \dfrac{-u_n^3+4u_n^2-5u_n^1+2u_n^0}{\tau^2}=\frac{1}{4}\dfrac{2u_n^N-5u_n^{N-1}+4u_n^{N-2}-u_n^{N-3}}{\tau^2}+\left(1-\dfrac{1}{4e}\right)\sin x_n, \\ 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N \end{cases} \tag{3.51}$$

and

$$\begin{cases} \dfrac{u_n^3 - 3u_n^2 + 3u_n^1 - u_n^0}{\tau^3} \\ -\dfrac{u_{n+1}^3 - 2u_n^3 + u_{n-1}^3 + u_{n+1}^2 - 2u_n^2 + u_{n-1}^2}{4\tau h^2} \\ +\dfrac{\left(u_{n+1}^1 - 2u_n^1 + u_{n-1}^1\right) + \left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right)}{4\tau h^2} \\ = \dfrac{f(t_1,x_n) + f(t_2,x_n)}{2} = -(e^{-t_1} + e^{-t_2})\sin x_n, \\ 1 \le n \le M-1, \\ \dfrac{u_n^{k+2} - 2u_n^{k+1} + 2u_n^{k-1} - u_n^{k-2}}{2\tau^3} \\ -\dfrac{u_{n+1}^{k+2} - 2u_n^{k+2} + u_{n-1}^{k+2} - \left(u_{n+1}^{k-2} - 2u_n^{k-2} + u_{n-1}^{k-2}\right)}{8\tau h^2} \\ -\dfrac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} - \left(u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}\right)}{4\tau h^2} \\ = f(t_k,x_n),\ f(t_k,x_n) = -2e^{-t_k}\sin x_n, \\ t_k = k\tau,\ 2 \le k \le N-2\ ,\ 1 \le n \le M-1, \\ N\tau = 1,\ x_n = nh,\ 1 \le n \le M-1,\ Mh = \pi, \\ u_n^0 = \frac{1}{4}\, u_n^N + \left(1 - \dfrac{1}{4e}\right)\sin x_n,\ 0 \le n \le M, \\ \dfrac{-u_n^2 + 4u_n^1 - 3u_n^0}{2\tau} = \frac{1}{4}\dfrac{3u_n^N - 4u_n^{N-1} + u_n^{N-2}}{2\tau} - \left(1 - \dfrac{1}{4e}\right)\sin x_n, \\ 0 \le n \le M, \\ \dfrac{-u_n^3 + 4u_n^2 - 5u_n^1 + 2u_n^0}{\tau^2} = \frac{1}{4}\dfrac{2u_n^N - 5u_n^{N-1} + 4u_n^{N-2} - u_n^{N-3}}{\tau^2} \\ +\left(1 - \dfrac{1}{4e}\right)\sin x_n, \\ 0 \le n \le M, \\ u_0^k = u_M^k = 0, 0 \le k \le N \end{cases} \tag{3.52}$$

This system can be written in the matrix form (3.46), too. Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a & a & -a & -a & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & -a & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} X & \cdots & Z \\ \vdots & \ddots & \vdots \\ W & \cdots & Y \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2b-2a & 6b-2a & -6b+2a & 2b+2a & 0 \\ -b-c & 2b & 0 & -2b & b+c \\ 0 & -b-c & 2b & 0 & -2b \\ 0 & 0 & -b-c & 2b & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-3}{2\tau} & \frac{2}{\tau} & \frac{1}{2\tau} & 0 & 0 \\ \frac{2}{\tau^2} & -\frac{5}{\tau^2} & \frac{4}{\tau^2} & \frac{-1}{\tau^2} & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -2b & b+c & 0 & 0 \\ 2b & 0 & -2b & b+c & 0 \\ -b-c & 2b & 0 & -2b & b+c \\ 0 & 0 & -\frac{1}{8\tau} & \frac{1}{2\tau} & -\frac{3}{8\tau} \\ 0 & \frac{1}{4\tau^2} & -\frac{1}{\tau^2} & \frac{5}{4\tau^2} & -\frac{1}{2\tau^2} \end{bmatrix},$$

$$a = \frac{1}{4\tau h^2},\ b = \frac{1}{2\tau^3},\ c = \frac{1}{2\tau h^2},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \left\{ \begin{array}{l} \varphi_n^k = f(t_k,x_n) = -2e^{-t_k}\sin x_n, \\ t_k = k\tau, 1 \le k \le N-2,\ 1 \le n \le M-1, \\ \\ \varphi_n^1 = \dfrac{f(t_1,x_n)+f(t_2,x_n)}{2} = -(e^{-t_1}+e^{-t_2})\sin x_n, \\ 1 \le n \le M-1, \\ \varphi_n^0 = \left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^{N-1} = -\left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^N = \left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M \end{array} \right\}$$

for difference scheme (3.51) and

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a & a & -a & -a & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}a & a & 0 & -a & -\frac{1}{2}a & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -a & -\frac{1}{2}a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & -a & -\frac{1}{2}a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{2}a & a & 0 & -a & -\frac{1}{2}a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} X & \cdots & Z \\ \vdots & \ddots & \vdots \\ W & \cdots & Y \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -b-2a & 3b-2a & -6b+2a & 2b+2a & 0 \\ -b-a & -2b-2a & 0 & -2b+2a & b+c \\ 0 & -b-a & -2b-2a & 0 & -2b+2a \\ 0 & 0 & -b-a & -2b-2a & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-3}{2\tau} & \frac{2}{\tau} & \frac{1}{2\tau} & 0 & 0 \\ \frac{2}{\tau^2} & -\frac{5}{\tau^2} & \frac{4}{\tau^2} & \frac{-1}{\tau^2} & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -2b+2a & b+c & 0 & 0 \\ -2b-2a & 0 & -2b+2a & b+c & 0 \\ -b-a & -2b-2a & 0 & -2b+2a & b+c \\ 0 & 0 & -\frac{1}{8\tau} & \frac{1}{2\tau} & -\frac{3}{8\tau} \\ 0 & \frac{1}{4\tau^2} & -\frac{1}{\tau^2} & \frac{5}{4\tau^2} & -\frac{1}{2\tau^2} \end{bmatrix},$$

$$a = \frac{1}{4\tau h^2},\ b = \frac{1}{2\tau^3},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \left\{ \begin{array}{l} \varphi_n^k = f(t_k, x_n) = -2e^{-t_k}\sin x_n, \\ \\ t_k = k\tau, 1 \le k \le N-2,\ 1 \le n \le M-1, \\ \\ \varphi_n^1 = \dfrac{f(t_1,x_n)+f(t_2,x_n)}{2} = -(e^{-t_1}+e^{-t_2})\sin x_n, \\ \\ 1 \le n \le M-1, \\ \varphi_n^0 = \left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^{N-1} = -\left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^N = \left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M \end{array} \right\}$$

for difference scheme (3.52). Therefore, for the solution of the matrix equation (3.46), we

will use the same formulas (3.47), (3.48) and the errors are computed by formula (3.49). Numerical results are given in following tables

Difference schemes/N,M	$20,20$	$40,40$	$80,80$
Difference scheme (3.51)	$1.1098e-03$	$3.0032e-04$	$7.8051e-05$
Difference scheme (3.52)	$9.0486e-04$	$2.4530e-04$	$6.3796e-05$

(3.53)

As it is seen in Table 3.53, we get some numerical results for difference schemes (3.51),(3.52). Note that if N and M are doubled, the value of errors decrease by a factor of approximately $1/4$ for second order of accuracy in t difference schemes (3.51),(3.52).

Note that difference schemes (3.45), (3.51) and (3.52) are generated by the operator A.

Second, we consider the difference scheme generated by the operator A and A^2. For the approximate solution of boundary value problem (3.44), applying formulas (2.36),(2.43) and

$$\begin{cases} \dfrac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_k) = O(\tau^2), \\[2ex] \dfrac{-3u(3h) + 4u(2h) - 5u(h) + 2u(0)}{h^2} - u''(0) = O(h^2), \\[2ex] \dfrac{-3u(1) + 4u(1-h) - 5u(1-2h) + 2u(1-3h)}{h^2} - u''(1) = O(h^2), \\[2ex] \dfrac{u(x_{n+2}) - 4u(x_{n+1}) + 6u(x_n) - 4u(x_{n-1}) + u(x_{n-2})}{h^4} - u^{(4)}(x_n) = O(h^2), \end{cases}$$

we get the second order of accuracy in t difference scheme

$$\begin{cases} \dfrac{u_n^3-3u_n^2+3u_n^1-u_n^0}{\tau^3}-\dfrac{u_{n+1}^3-2u_n^3+u_{n-1}^3+u_{n+1}^2-2u_n^2+u_{n-1}^2}{4\tau h^2} \\ +\dfrac{\left(u_{n+1}^1-2u_n^1+u_{n-1}^1\right)+\left(u_{n+1}^0-2u_n^0+u_{n-1}^0\right)}{4\tau h^2} \\ =\dfrac{f(t_1,x_n)+f(t_2,x_n)}{2}=-(e^{-t_1}+e^{-t_2})\sin x_n,\ 1\le n\le M-1, \\ \dfrac{u_n^{k+2}-2u_n^{k+1}+2u_n^{k-1}-u_n^{k-2}}{2\tau^3} \\ -\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}-\left(u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}\right)}{2\tau h^2} \\ +\tau^2\dfrac{u_{n+2}^{k+2}-4u_{n+1}^{k+2}+6u_n^{k+2}-4u_{n-1}^{k+2}+u_{n-2}^{k+2}}{8\tau h^4} \\ -\tau^2\dfrac{u_{n+2}^{k}-4u_{n+1}^{k}+6u_n^{k}-4u_{n-1}^{k}+u_{n-2}^{k}}{8\tau h^4} \\ =f(t_k,x_n),\ f(t_k,x_n)=-2e^{-t_k}\sin x_n, \\ t_k=k\tau,\ 2\le k\le N-2\ , \\ N\tau=1,\ x_n=nh,\ 2\le n\le M-2,\ Mh=\pi, \\ u_n^0=\frac{1}{4}\,u_n^N+\left(1-\dfrac{1}{4e}\right)\sin x_n,\ 0\le n\le M, \\ \dfrac{-u_n^2+4u_n^1-3u_n^0}{2\tau}=\frac{1}{4}\dfrac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau}-\left(1-\dfrac{1}{4e}\right)\sin x_n, \\ 0\le n\le M, \\ \dfrac{-u_n^3+4u_n^2-5u_n^1+2u_n^0}{\tau^2}=\frac{1}{4}\dfrac{2u_n^N-5u_n^{N-1}+4u_n^{N-2}-u_n^{N-3}}{\tau^2} \\ +\left(1-\dfrac{1}{4e}\right)\sin x_n, 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N, \\ -u_3^k+4u_2^k-5u_1^k+2u_0^k=2u_M^k-5u_{M-1}^k+4u_{M-2}^k-u_{M-3}^k=0, \\ 0\le k\le N \end{cases} \quad (3.54)$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} F\,u_{n+2} + A\,u_{n+1} + Bu_n + Cu_{n-1} + Eu_{n-2} = D\varphi_n, \quad 2 \le n \le M-2, \\ u_0 = \vec{0},\ u_M = \vec{0}, -u_3 + 4u_2 - 5u_1 = \vec{0}, -5u_{M-1} + 4u_{M-2} - u_{M-3} = \vec{0}. \end{cases} \tag{3.55}$$

Here,

$$F = E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -b & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -b & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & -b & 0 & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a & a & -a & -a & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a & 4b & -2a & -4b & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 4b & -2a & -4b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2a & 4b & -2a & -4b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2a & 4b & -2a & -4b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} X & \cdots & Z \\ \vdots & \ddots & \vdots \\ W & \cdots & Y \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2c-2a & 6c-2a & -6c+2a & 2c+2a & 0 \\ -c & 2c-4a & -6b & -2c+4a & c+6b \\ 0 & -c & 2c-4a & -6b & -2c+4a \\ 0 & 0 & -c & 2c-4a & -6b \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-3}{2\tau} & \frac{2}{\tau} & -\frac{1}{2\tau} & 0 & 0 \\ \frac{2}{\tau^2} & -\frac{5}{\tau^2} & \frac{4}{\tau^2} & -\frac{1}{\tau^2} & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} -6b & -2c+4a & c+6b & 0 & 0 \\ 2c-4a & -6b & -2c+4a & c+6b & 0 \\ -c & 2c-4a & -6b & -2c+4a & c+6b \\ 0 & 0 & -\frac{1}{8\tau} & \frac{1}{2\tau} & -\frac{3}{8\tau} \\ 0 & \frac{1}{4\tau^2} & -\frac{1}{\tau^2} & \frac{5}{4\tau^2} & -\frac{1}{2\tau^2} \end{bmatrix},$$

$$a = \frac{1}{4\tau h^2}, b = \frac{\tau^2}{8\tau h^4},\ c = \frac{1}{2\tau^3},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \left\{ \begin{array}{l} \varphi_n^k = f(t_k, x_n) = -2e^{-t_k}\sin x_n, \\ t_k = k\tau, 2 \le k \le N-2,\ 1 \le n \le M-1, \\ \\ \varphi_n^1 = \dfrac{f(t_1,x_n)+f(t_2,x_n)}{2} = -(e^{-t_1}+e^{-t_2})\sin x_n, \\ 1 \le n \le M-1, \\ \\ \varphi_n^0 = \left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M, \\ \varphi_n^{N-1} = -\left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M \\ \varphi_n^N = \left(1-\dfrac{1}{4e}\right)\sin x_n, 0 \le n \le M. \end{array} \right\}$$

By using the modified Gauss elimination method, we can reach to the solution of $u_n^k, 0 \le k \le N,\ 0 \le n \le M$. Actually, we seek a solution of the matrix equation (3.55) by the following form

$$\begin{cases} u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1},\ n = M-2, \cdots, 2, 1, \\ \\ u_M = \widetilde{0}, \\ \\ u_{M-1} = \left[(\beta_{M-2}+5I) - (4I-\alpha_{M-2})\,\alpha_{M-1}\right]^{-1}\left[(4I-\alpha_{M-2})\,\gamma_{M-1} - \gamma_{M-2}\right], \end{cases}$$

where α_j, β_j and $\gamma_j,\ j = 1, \cdots M-1$ are calculated as

$$\alpha_{n+1} = -\left(B + C\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}\left(A + C\beta_n + E\alpha_{n-1}\beta_n\right),$$

$$\beta_{n+1} = -\left(B + C\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}(F),$$

$$\gamma_{n+1} = \left(B + C\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}\left(D\varphi_n - C\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}\right)$$

with α_1 and β_1 are $(N+1)\times(N+1)$ and γ_1 and γ_2 are $(N+1)\times 1$ zero matrices and

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & \cdots & 0 \\ 0 & \frac{4}{5} & \cdots. & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)}, \beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & \cdots & 0 \\ 0 & -\frac{1}{5} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)}.$$

Numerical results are given in following table

Difference schemes/N,M	$20,20$	$40,40$	$80,80$
Difference scheme (3.54)	$9.0743e-04$	$2.4432e-04$	$6.3629e-05$

(3.56)

As it is seen in Table 3.56, we get some numerical results for difference scheme (3.54). Note that if N and M are doubled, the value of errors decrease by a factor of approximately $1/4$ for second order of accuracy in t difference scheme (3.54).

The errors presented in these tables indicates the accuracy of difference schemes. We conclude that, the accuracy increases with the second order approximation.

3.5 Appendix Matlab Programming

3.5.1 Matlab Implementation of Difference Schemes (3.45), (3.51) and (3.52)

```
function TT(N,M)
    if nargin < 1; end;
    close;close;
    %first order
    tau=1/N;
    h=pi/M;
    a = -1/(h^2);
    b =-1/(tau^3);
    c = -2*a-3*b;
    A=zeros(N+1,N+1);
```

```
for i=2:N-1;
A(i,i+1)=1/(tau*(h^2));
A(i,i+2)=-1/(tau*(h^2));
end;
A;
C=A;
B=zeros(N+1,N+1);
for i=2:N-1 ;
B(i,i-1)= -1/(tau^3);
B(i,i)=3/(tau^3);
B(i,i+1)=(-3/(tau^3))-2/(tau*(h^2));
B(i,i+2)=(1/(tau^3))+2/(tau*(h^2)) ;
end;
B(1,1)=1;
B(1,N+1)=-1/4;
B(N,1)=-1/tau;
B(N,2)=1/tau;
B(N,N)=1/(4*tau);
B(N,N+1)=-1/(4*tau);
B(N+1,1)=1/(tau^2);
B(N+1,2)=-2/(tau^2);
B(N+1,3)=1/(tau^2);
B(N+1,N+1)=-1/(4*tau^2);
B(N+1,N)=2/(4*(tau^2));
B(N+1,N-1)=-1/(4*tau^2);
B;
D=eye(N+1,N+1);
for j=1:M+1;
for k=2:N-1;
fii(k,j) =-2*exp(-tau*(k-1))*sin((j-1)*h);
end;
fii(1,j) =(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N,j) =-(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N+1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
```

```
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1 ;
for k=1:N+1 ;
es(k,j) =(exp((-k+1)*tau))*sin((j-1)*h);
end;
end;
figure ;
m(1,1)=min(min(U))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION');
figure ;
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('FIRST ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap1 = [maxerror,relativeerror] ;
% Difference scheme 1 Second order
a =(1/4)*(1/tau)*(1/(h^2));
b =(1/2)*(1/(tau^3));
```

```
c =(1/2)*(1/tau)*(1/(h^2));
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
for i=3:N-1;
A(i,i-2)=a;
A(i,i+2)=-a;
end;
A(2,1)=a;
A(2,2)=a;
A(2,3)=-a;
A(2,4)=-a;
A;
C=A;
for i=3:N-1;
B(i,i-2)=-b-c;
B(i,i-1)=2*b;
B(i,i+1)=-2*b;
B(i,i+2)=b+c;
end;
B(1,1)=1;
B(1,N+1)=-1/4;
B(2,1)=(-1/(tau^3))-(2*a);
B(2,2)=(3/(tau^3))-(2*a);
B(2,3)=(-3/(tau^3))+(2*a);
B(2,4)=(1/(tau^3))+(2*a);
B(N,3)=-(1/2)*(1/tau);
B(N,2)=2/tau;
B(N,1)=-3/(2*tau);
B(N,N-1)=-1/(8*tau);
B(N,N)=1/(2*tau);
B(N,N+1)=-3/(8*tau);
B(N+1,1)=2/(tau^2);
B(N+1,2)=-5/(tau^2);
B(N+1,3)=4/(tau^2);
B(N+1,4)=-1/(tau^2);
B(N+1,N+1)=(-1/2)*(1/(tau^2));
```

```
B(N+1,N)=(5/4)*(1/(tau^2));
B(N+1,N-1)=-1/(tau^2);
B(N+1,N-2)=(1/4)*(1/(tau^2));
B;
for j=1:M+1;
for k=3:N-1;
fii(2,j) =-(exp(-tau)+exp(-tau*2))*sin((j-1)*h);
fii(k,j) =-2*exp(-tau*(k-1))*sin((j-1)*h);
end;
fii(1,j) =(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N,j) =-(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N+1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
figure
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('1 SECOND ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap2 = [maxerror,relativeerror] ;
% Difference scheme 2 Second order
a =(1/4)*(1/tau)*(1/(h^2));
```

```
b =(1/2)*(1/(tau^3));
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
for i=3:N-1;
A(i,i-2)=(1/2)*a;
A(i,i-1)=a;
A(i,i+1)=-a;
A(i,i+2)=-(1/2)*a;
end;
A(2,1)=a;
A(2,2)=a;
A(2,3)=-a;
A(2,4)=-a;
A;
C=A;
for i=3:N-1;
B(i,i-2)=-b-a;
B(i,i-1)=(2*b)-(2*a);
B(i,i+1)=(-2*b)+(2*a);
B(i,i+2)=b+a;
end;
B(1,1)=1;
B(1,N+1)=-1/4;
B(2,1)=-(2*b)-(2*a);
B(2,2)=(6*b)-(2*a);
B(2,3)=(-6*b)+(2*a);
B(2,4)=(2*b)+(2*a);
B(N,3)=-(1/2)*(1/tau);
B(N,2)=2/tau;
B(N,1)=-3/(2*tau);
B(N,N-1)=-1/(8*tau);
B(N,N)=1/(2*tau);
B(N,N+1)=-3/(8*tau);
B(N+1,1)=2/(tau^2);
B(N+1,2)=-5/(tau^2);
B(N+1,3)=4/(tau^2);
```

```
B(N+1,4)=-1/(tau^2);
B(N+1,N+1)=(-1/2)*(1/(tau^2));
B(N+1,N)=(5/4)*(1/(tau^2));
B(N+1,N-1)=-1/(tau^2);
B(N+1,N-2)=(1/4)*(1/(tau^2));
B;
for j=1:M+1;
for k=3:N-1;
fii(2,j) =-(exp(-tau)+exp(-tau*2))*sin((j-1)*h);
fii(k,j) =-2*exp(-tau*(k-1))*sin((j-1)*h);
end;
fii(1,j) =(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N,j) =-(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N+1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
figure
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('2 SECOND ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap3 = [maxerror,relativeerror] ;
```

```
format short e ;
cevap=[cevap1,cevap2,cevap3]
```

3.5.2 Matlab Implementation of Difference Schemes (3.54)

```
function FF(N,M)
    if nargin < 1; end;
    close;close;
    %first order
    tau=1/N;
    h=pi/M;
    a = -1/(h^2);
    b =-1/(tau^3);
    c = -2*a-3*b;
    A=zeros(N+1,N+1);
    for i=2:N-1;
    A(i,i+1)=1/(tau*(h^2));
    A(i,i+2)=-1/(tau*(h^2));
    end;
    A;
    C=A;
    B=zeros(N+1,N+1);
    for i=2:N-1 ;
    B(i,i-1)= -1/(tau^3);
    B(i,i)=3/(tau^3);
    B(i,i+1)=(-3/(tau^3))-2/(tau*(h^2));
    B(i,i+2)=(1/(tau^3))+2/(tau*(h^2)) ;
    end;
    B(1,1)=1;
    B(1,N+1)=-1/4;
    B(N,1)=-1/tau;
    B(N,2)=1/tau;
    B(N,N)=1/(4*tau);
    B(N,N+1)=-1/(4*tau);
    B(N+1,1)=1/(tau^2);
```

```
B(N+1,2)=-2/(tau^2);
B(N+1,3)=1/(tau^2);
B(N+1,N+1)=-1/(4*tau^2);
B(N+1,N)=2/(4*(tau^2));
B(N+1,N-1)=-1/(4*tau^2);
B;
D=eye(N+1,N+1);
for j=1:M+1;
for k=2:N-1;
fii(k,j) =-2*exp(-tau*(k-1))*sin((j-1)*h);
end;
fii(1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N,j)=-(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N+1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(N+1,M+1);
for j=M:-1:1
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1 ;
for k=1:N+1 ;
es(k,j) =(exp((-k+1)*tau))*sin((j-1)*h);
end;
end;
figure ;
m(1,1)=min(min(U))-0.01;
m(2,2)=nan;
surf(m);
```

```
hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION');
figure ;
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('FIRST ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap1 = [maxerror,relativeerror] ;
% Difference scheme 3 Second order
a =(1/4)*(1/tau)*(1/(h^2));
b =(tau^2)*(1/8)*(1/tau)*(1/(h^4));
c =(1/2)*(1/(tau^3));
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
F=zeros(N+1,N+1);
for i=3:N-1;
A(i,i-1)=(2*a);
A(i,i)=4*b
A(i,i+1)=-(2*a);
A(i,i+2)=-(4*b);
end;
A(2,1)=a;
A(2,2)=a;
A(2,3)=-a;
A(2,4)=-a;
A;
C=A;
for i=3:N-1;
F(i,i)=-b;
F(i,i+2)=b;
end;
```

```
F;
E=F;
for i=3:N-1;
B(i,i-2)=-c;
B(i,i-1)=(2*c)-(4*a);
B(i,i)=-(6*b);
B(i,i+1)=(-2*c)+(4*a);
B(i,i+2)=c+(6*b);
end;
B(1,1)=1;
B(1,N+1)=-1/4;
B(2,1)=-(2*a)-(2*c);
B(2,2)=(6*c)-(2*a);
B(2,3)=(-6*c)+(2*a);
B(2,4)=(2*c)+(2*a);
B(N,3)=-(1/2)*(1/tau);
B(N,2)=2/tau;
B(N,1)=-3/(2*tau);
B(N,N-1)=-1/(8*tau);
B(N,N)=1/(2*tau);
B(N,N+1)=-3/(8*tau);
B(N+1,1)=2/(tau^2);
B(N+1,2)=-5/(tau^2);
B(N+1,3)=4/(tau^2);
B(N+1,4)=-1/(tau^2);
B(N+1,N+1)=(-1/2)*(1/(tau^2));
B(N+1,N)=(5/4)*(1/(tau^2));
B(N+1,N-1)=-1/(tau^2);
B(N+1,N-2)=(1/4)*(1/(tau^2));
B;
for j=1:M+1;
for k=3:N-1;
fii(k,j) =-2*exp(-tau*(k-1))*sin((j-1)*h);
end;
fii(1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(2,j) =-(exp(-tau)+exp(-tau*2))*sin((j-1)*h);
```

```
fii(N,j)=-(1-(1/(4*exp(1))))*sin((j-1)*h);
fii(N+1,j)=(1-(1/(4*exp(1))))*sin((j-1)*h);
end;
alpha{1}=zeros(N+1,N+1);
betha{1}=zeros(N+1,N+1);
gamma{1}=zeros(N+1,1);
gamma{2}=zeros(N+1,1);
alpha{2}=(4/5)*D;
betha{2}=(-1/5)*D;
for j=3:M-1;
Q=inv(B+(C*alpha{j-1})+(E*betha{j-2})+(E*alpha{j-2}*alpha{j-1}));
alpha{j}=-Q*(A+(C*betha{j-1})+(E*alpha{j-2}*betha{j-1}));
betha{j}=-Q*F;
gamma{j}=Q*(D*(fii(:,j))-(C*gamma{j-1})-(E*alpha{j-2}*gamma{j-1})-(E*gamma{j-
2}));
end;
T=inv((betha{M-2}+(5*D))-(((4*D)-alpha{M-2})*alpha{M-1}));
U=zeros(N+1,M+1);
U(:,M)= T*((((4*D)-alpha{M-2})*gamma{M-1})-gamma{M-2});
for j=M-1:-1:1
U(:,j)=alpha{j}*U(:,j+1)+(betha{j}*U(:,j+2))+gamma{j};
end
figure
surf(m);
hold;
surf(U) ; rotate3d ;axis tight;
title('3 SECOND ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap2 = [maxerror,relativeerror]
```

3.5.3 Figures Presented by Numerical Experiences of Difference Schemes

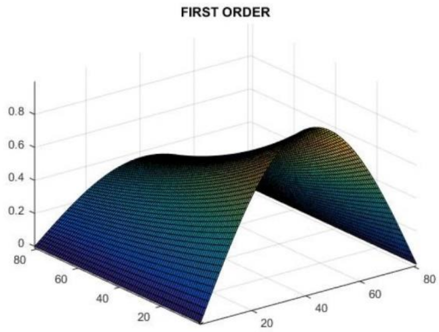

Figure 1-3 Solution of difference scheme (3.45)for $N = 80, M = 80$.

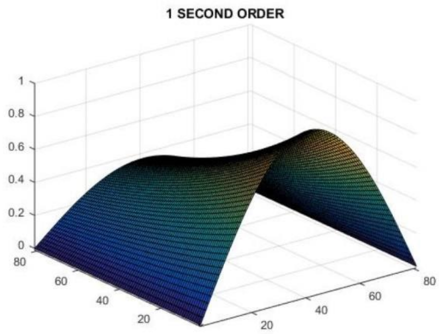

Figure 2-3 Solution of difference scheme (3.51)for $N = 80, M = 80$.

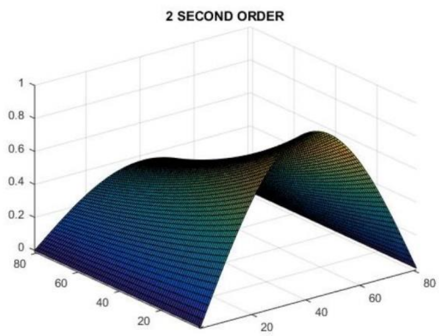

Figure 3-3 Solution of difference scheme (3.52)for $N = 80, M = 80$.

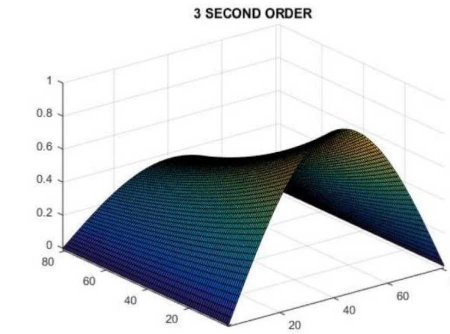

Figure 4-3 Solution of difference scheme (3.54)for $N = 80, M = 80$.

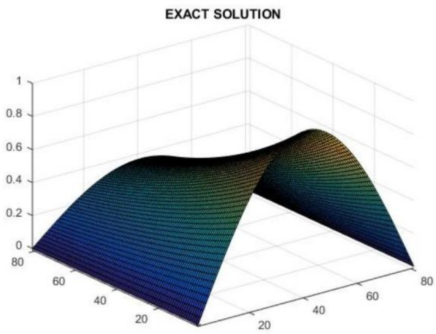

Figure 5-3 Exact solution of problem (3.44) for $N = 80, M = 80$.

Chapter 4

Conclusion

In the present thesis, well-posedness of a local boundary value problem

$$\begin{cases} \frac{d^3u(t)}{dt^3} + Au(t) = f(t), \quad 0 < t < 1, \\ \\ u(0) = \varphi, \quad u(1) = \psi, \quad u'(1) = \xi \end{cases} \tag{4.1}$$

and a nonlocal boundary value problem

$$\begin{cases} \frac{d^3u(t)}{dt^3} + A\frac{du(t)}{dt} = f(t), \quad 0 < t < 1, \\ \\ u(0) = \gamma u(\lambda) + \varphi, \quad u'(0) = \alpha u'(\lambda) + \psi, |\gamma| < 1, \\ u''(0) = \beta u''(\lambda) + \xi, \quad |1 + \beta\alpha| > |\alpha + \beta|, 0 < \lambda \leq 1 \end{cases} \tag{4.2}$$

for third order partial differential equations in a Hilbert space H with a self-adjoint positive definite operator A is investigated. It is well-known that various boundary value problems for partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Applying all these analytical methods, exact solutions of several problems for third order partial differential equations with constant coefficients are presented. Moreover, operator approach permit us to study local and nonlocal boundary value problems for third order partial differential equations in arbitrary Hilbert space with a self-adjoint positive definite operator. Theorems on stability estimates for the solution of these boundary value problems are established. In practice, stability estimates for the solution of several problems for third order partial differential equations are obtained. The difference schemes for the numerical solution of one-dimensional third order partial differential equations are presented. Numerical results are given.

Thesis consists of four chapters. Chapter 1 consists of introduction and application of Fourier series, Laplace transform and Fourier transform methods to getting analytically exact solution of six problems for third order partial differential equations with constant coefficients.

In Chapter 2 boundary value problem (4.1) for a third order partial differential equations in an arbitrary Hilbert space with a self-adjoint positive definite operator A is investigated. The following main theorems are proved.

Theorem 4.0.1 *Assume that* $\delta > \left(\frac{1}{3}\ln 4\right)^3, \varphi \in D(A), \psi \in D(A), \xi \in D\left(A^{2/3}\right)$ *and* $f(t)$ *is continuously differentiable on* $[0,1]$. *Then there is a unique solution of problem (4.1) and the following inequalities hold*

$$\max_{0\le t\le 1} \|u(t)\|_H$$

$$\le M\left\{\|\varphi\|_H + \|\xi\|_H + \left\|f'(1)\right\|_H + \|\psi\|_H + \max_{0\le t\le 1}\|f(t)\|_H\right\},$$

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\le t\le 1}\|Au(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H + \|A\psi\|_H + \left\|A^{2/3}\xi\right\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\},$$

where M *does not depend on* $f(t)$, φ, ψ, ξ.

Theorem 4.0.2 *Assume that* $\delta > \left(\frac{1}{3}\ln 4\right)^3, \varphi \in D(A), \psi \in D(A), \xi \in D\left(A^{2/3}\right)$ *and* $f(t)$ *is continuous on* $[0,1]$ *and there exists* $f'(1)$ *and* $f(t) \in D\left(A^{1/3}\right)$. *Then there is a unique solution of problem (4.1) and the following inequalities hold*

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\le t\le 1}\|Au(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H + \|A\psi\|_H + \left\|A^{2/3}\xi\right\|_H + \left\|f'(1)\right\|_H + \max_{0\le t\le 1}\left\|A^{1/3}f(t)\right\|_H\right\},$$

where M *does not depend on* $f(t)$, φ, ψ, ξ.

From Theorem 4.0.1 and Theorem 4.0.2 it follows the following theorem on stability.

Theorem 4.0.3 *Assume that* $\delta > \left(\frac{1}{3}\ln 4\right)^3, \varphi \in D(A), \psi \in D(A), \xi \in D\left(A^{2/3}\right)$ *and* $f(t)$ *is continuously differentiable on* $[0,1]$ *and* $f(t) \in D\left(A^{1/3}\right)$. *Then there is a unique solution of problem (4.1) and the following inequalities hold*

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\le t\le 1}\|Au(t)\|_H \le M\left\{\|A\varphi\|_H + \|A\psi\|_H + \left\|A^{2/3}\xi\right\|_H\right.$$

$$+\min\left\{\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H,\left\|f'(1)\right\|_H+\max_{0\le t\le 1}\left\|A^{1/3}f(t)\right\|_H\right\}\Bigg\},$$

where M does not depend on $f(t)$, φ, ψ, ξ.

Three applications of main theorems are given. First, for the application of the theorem 4.0.1 we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3}-(a(x)u_x(t,x))_x+\delta u(t,x)=f(t,x),\ \ 0<t,x<1, \\ u(0,x)=\varphi(x),\ u(1,x)=\psi(x),\ u_t(t,0)=\xi(x),\ \ 0\le x\le 1, \\ u_t(t,0)=u(t,1),\quad u_x(t,0)=u_x(x,1),\quad 0\le t\le 1. \end{cases} \tag{4.3}$$

Problem (4.3) has a unique smooth solution $u(t,x)$ for smooth $a(x)\ge a>0$, $x\in(0,1)$, $\delta>0$, $a(1)=a(0)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ $(x\in[0,1])$ and $f(t,x)$ $(t\in(0,1),x\in(0,1))$ functions. This allows us to reduce problem (4.1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (4.3). Let us give a number of corollaries of abstract theorem 4.0.1.

Theorem 4.0.4 *For the solution of the problem (4.3), the stability inequalities*

$$\max_{0\le t\le 1}\|u(t,.)\|_{L_2[0,1]}$$

$$\le M\left[\max_{0\le t\le 1}\|f(t,.)\|_{L_2[0,1]}+\|f_t(1,.)\|_{L_2[0,1]}+\|\varphi\|_{L_2[0,1]}+\|\psi\|_{L_2[0,1]}+\|\xi\|_{L_2[0,1]}\right],$$

$$\max_{0\le t\le 1}\|u(t,.)\|_{W_2^2[0,1]}+\max_{0\le t\le 1}\left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2[0,1]}$$

$$\le M\left[\max_{0\le t\le 1}\|f_t(t,.)\|_{L_2[0,1]}+\|f(0,.)\|_{L_2[0,1]}+\|\varphi\|_{W_2^2[0,1]}+\|\psi\|_{W_2^2[0,1]}+\|\xi\|_{W_2^2[0,1]}\right]$$

hold where M does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Second, let $\Omega\subset\mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\bar{\Omega}=\Omega\cup S$. In $[0,1]\times\Omega$, we consider the boundary value problem for a third order partial differential

equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x)) x_r = f(t,x), \\ x = (x_1, ..., x_n) \in \Omega, \ 0 < t < 1, \\ u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \quad u_t(1,x) = \xi(x), \quad x \in \bar{\Omega}, \\ u(t,x) = 0, \quad x \in S, \ 0 \le t \le 1, \end{cases} \tag{4.4}$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$.

Theorem 4.0.5 *For the solution of the problem (4.4) the stability inequalities*

$$\max_{0 \le t \le 1} \|u(t,.)\|_{L_2(\bar{\Omega})}$$

$$\le M_2 \left[\max_{0 \le t \le 1} \|f(t,.)\|_{L_2(\bar{\Omega})} + \|f_t(1,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{L_2(\bar{\Omega})} + \|\psi\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})} \right],$$

$$\max_{0 \le t \le 1} \|u(t,.)\|_{W_2^2[0,1]} + \max_{0 \le t \le 1} \left\| \frac{\partial^3 u}{\partial t^3}(t,.) \right\|_{L_2(\bar{\Omega})}$$

$$\le M_2 \left[\max_{0 \le t \le 1} \|f_t(t,.)\|_{L_2(\bar{\Omega})} + \|f(0,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{W_2^2(\bar{\Omega})} + \|\psi\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^2(\bar{\Omega})} \right]$$

hold where M_2 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Third, we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x))_{x_r} + \delta u(t,x) = f(t,x), \\ x = (x_1, ..., x_n) \in \Omega, \ 0 < t < 1, \\ u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \quad u_t(1,x) = \xi(x), \quad x \in \bar{\Omega}, \\ \frac{\partial u}{\partial \vec{n}}(t,x) = 0, \quad x \in S, \ 0 \le t \le 1, \end{cases} \tag{4.5}$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$, $\overrightarrow{n}$ is the normal vector to S.

Theorem 4.0.6 *For the solution of the problem (4.5), the stability inequalities*

$$\max_{0\leq t\leq 1} \|u(t,.)\|_{L_2(\bar{\Omega})} \leq M_3 \left[\max_{0\leq t\leq 1} \|f(t,.)\|_{L_2(\bar{\Omega})} + \|f_t(1,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{L_2(\bar{\Omega})} + \|\psi\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})}\right],$$

$$\max_{0\leq t\leq 1} \|u(t,.)\|_{W_2^2(\bar{\Omega})} + \max_{0\leq t\leq 1} \left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2(\bar{\Omega})}$$
$$\leq M_3 \left[\max_{0\leq t\leq 1} \|f_t(t,.)\|_{L_2(\bar{\Omega})} + \|f(0,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{W_2^2(\bar{\Omega})} + \|\psi\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^2(\bar{\Omega})}\right]$$

hold where M_3 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. For the numerical experience we consider the local boundary value problem

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \\ f(t,x) = (1-t)^2 \sin x, 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = 2\sin x,\ u(1,x) = e^{-1} \sin\ x, \\ u_t(1,x) = -e^{-1} \sin x,\ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \leq t \leq 1 \end{cases} \tag{4.6}$$

for one-dimensional a third order partial differential equation. The first and high orders of accuracy difference schemes for numerical solution of problem (4.6) are presented. We apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first and high orders of accuracy difference schemes are given.

In Chapter 3 nonlocal boundary value problem (4.2) for a third order partial differential equations in an arbitrary Hilbert space with a self-adjoint positive definite operator A is investigated. The following main theorems are proved.

Theorem 4.0.7 *Suppose that $\psi \in D(A)$, $\xi \in D\left(A^{1/2}\right)$ and $f(t)$ is continuously differentiable on $[0,1]$. Then there is a unique solution of problem (4.2) and the following inequalities* *hold*

$$\max_{0\le t\le 1}\|u(t)\|_H$$

$$\le M(\gamma)\left\{\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\left\|A^{-1}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right\},$$

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\left\|A\frac{du}{dt}\right\|_H$$

$$\le M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\},$$

where $M, M(\gamma)$ do not depend on $f(t)$, φ, ψ, ξ.

Theorem 4.0.8 *Suppose that $\psi\in D(A)$, $\xi\in D\left(A^{1/2}\right)$ and $f(t)$ is continuous on $[0,1]$ and $f(t)\in D\left(A^{1/2}\right)$. Then there is a unique solution of problem (4.2) and the following estimate holds*

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\left\|A\frac{du}{dt}\right\|_H$$

$$\le M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{\frac{1}{2}}f(t)\right\|_H\right\},$$

where M does not depend on $f(t)$, ψ, ξ.

From Theorem 4.0.7 and Theorem 4.0.8 it follows the following theorem on stability.

Theorem 4.0.9 *Assume that $\psi\in D(A)$, $\xi\in D\left(A^{1/2}\right)$ and $f(t)$ is continuously differentiable on $[0,1]$ and $f(t)\in D\left(A^{1/2}\right)$. Then there is a unique solution of problem (4.2) and the following inequalities hold*

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\left\|A\frac{du}{dt}\right\|_H$$

$$\le M\left\{\|A\psi\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{\frac{1}{2}}f(t)\right\|_H\right\},$$

$$+\min\left\{\|f(0)\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H,\max_{0\le t\le 1}\left\|A^{1/2}f(t)\right\|_H\right\}\Bigg\},$$

where M does not depend on $f(t)$, ψ, ξ.

Three applications of main theorems are given. First, for the application of the theorem 4.0.7 we consider the nonlocal boundary value problem for a third order partial differential equationation

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - (a(x)u_{tx})_x + \delta u_t(t,x) = f(t,x), \ \ 0 < t < 1, 0 < x < l, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), \quad u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), 0 \leq x \leq l, \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), \quad 0 \leq x \leq l, 0 < \lambda \leq 1 \\ u(t,0) = u(t,l), \quad u_x(t,0) = u_x(t,l), \quad 0 \leq t \leq 1. \end{cases} \tag{4.7}$$

Problem (4.7) has a unique smooth solution $u(t,x)$ for smooth $a(x) \geq a > 0$, $x \in (0,l)$, $\delta > 0$, $a(l) = a(0)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ ($x \in [0,l]$) and $f(t,x)$ ($t \in (0,1)$, $x \in (0,l)$) functions. This allows us to reduce problem (4.2) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by (4.7). Let us give a number of corollaries of abstract theorem 3.2.4

Theorem 4.0.10 *For the solution of the problem (4.7), the stability inequalities*

$$\max_{0 \leq t \leq 1} \|u(t,.)\|_{L_2[0,1]}$$

$$\leq M_1 \left[\max_{0 \leq t \leq 1} \|f(t,.)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]} + \|\xi\|_{L_2[0,1]} \right],$$

$$\max_{0 \leq t \leq 1} \left\| \frac{\partial u}{\partial t}(t,.) \right\|_{W_2^2[0,1]} + \max_{0 \leq t \leq 1} \left\| \frac{\partial^3 u}{\partial t^3}(t,.) \right\|_{L_2[0,1]}$$

$$\leq M_1 \left[\max_{0 \leq t \leq 1} \|f_t(t,.)\|_{L_2[0,1]} + \|f(0,.)\|_{L_2[0,1]} + \|\psi\|_{W_2^2[0,1]} + \|\xi\|_{W_2^1[0,1]} \right]$$

hold, where M_1 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Second, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\bar{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, we consider the boundary value problem for a third order partial differential

equation

$$
\begin{cases}
\dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n} (a_r(x)u_{tx_r})_{x_r} = f(t,x), \ \ x=(x_1,...,x_n)\in\Omega, \ 0<t<1, \\
u(0,x)=\gamma u(\lambda,x)+\varphi(x), \quad u_t(0,x)=\alpha u_t(\lambda,x)+\psi(x), x\in\bar{\Omega}, \\
u_{tt}(0,x)=\beta u_{tt}(\lambda,x)+\xi(x), \quad x\in\bar{\Omega}, 0<\lambda\leq 1, \\
u(t,x)=0, \quad x\in S, \ 0\leq t\leq 1,
\end{cases} \tag{4.8}
$$

where $a_r(x)$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $\left(x\in\bar{\Omega}\right)$ and $f(t,x)$ $(x\in[0,1])$, $x\in\Omega$ are given smooth functions and $a_r(x)>0$.

Theorem 4.0.11 *For the solution of the problem (4.8) the stability inequalities*

$$
\max_{0\leq t\leq 1}\|u(t,.)\|_{L_2(\bar{\Omega})}
$$

$$
\leq M_2\left[\max_{0\leq t\leq 1}\|f(t,.)\|_{L_2(\bar{\Omega})}+\|\varphi\|_{L_2(\bar{\Omega})}+\|\psi\|_{L_2(\bar{\Omega})}+\|\xi\|_{L_2(\bar{\Omega})}\right],
$$

$$
\max_{0\leq t\leq 1}\|u(t,.)\|_{W_2^2[0,1]}+\max_{0\leq t\leq 1}\left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2(\bar{\Omega})}
$$

$$
\leq M_2\left[\max_{0\leq t\leq 1}\|f_t(t,.)\|_{L_2(\bar{\Omega})}+\|f(0,.)\|_{L_2(\bar{\Omega})}+\|\psi\|_{W_2^2(\bar{\Omega})}+\|\xi\|_{W_2^1(\bar{\Omega})}\right]
$$

hold where M_2 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Third, we consider the nonlocal boundary value problem for a third order partial differential equation

$$
\begin{cases}
\dfrac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{m} (a_r(x)u_{tx_r})x_r+\delta u_t(t,x) = f(t,x), \qquad x=(x_1,...,x_n)\in\Omega, \ 0<t<1, \\
u(0,x)=\gamma u(\lambda,x)+\varphi(x), \quad u_t(0,x)=\alpha u_t(\lambda,x)+\psi(x), x\in\bar{\Omega}, \\
u_{tt}(1,x)=\beta u_{tt}(\lambda,x)+\xi(x), \ x\in\bar{\Omega}, \ 0<\lambda<1, \\
\dfrac{\partial u}{\partial \overrightarrow{m}}(0,x)=0, \quad x\in S, \ 0\leq t\leq 1,
\end{cases} \tag{4.9}
$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$, $\vec{m}$ is the normal vector to S .

Theorem 4.0.12 *For the solution of the problem (4.9), the stability inequalities*

$$\max_{0 \le t \le 1} \|u(t,.)\|_{L_2(\bar{\Omega})}$$

$$\le M_3 \left[\max_{0 \le t \le 1} \|f(t,.)\|_{L_2(\bar{\Omega})} + \|\varphi\|_{L_2(\bar{\Omega})} + \|\psi\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})} \right],$$

$$\max_{0 \le t \le 1} \|u(t,.)\|_{W_2^2(\bar{\Omega})} \quad + \max_{0 \le t \le 1} \left\| \frac{\partial^3 u}{\partial t^3}(t,.) \right\|_{L_2(\bar{\Omega})}$$

$$\le M_3 \left[\max_{0 \le t \le 1} \|f_t(t,.)\|_{L_2(\bar{\Omega})} + \|f(0,.)\|_{L_2(\bar{\Omega})} + \|\psi\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^1(\bar{\Omega})} \right]$$

hold where M_3 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

For the numerical experience we consider the boundary value problem

$$\begin{cases} \dfrac{\partial^3 u(t,x)}{\partial t^3} - \dfrac{\partial u^3(t,x)}{\partial t \partial x^2} = f(t,x), \\ f(t,x) = -2e^{-t} \sin x, 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = \frac{1}{4} u(1,x) + \left(1 - \dfrac{1}{4e}\right) \sin x,\ 0 \le x \le \pi, \\ u_t(0,x) = \frac{1}{4} u_t(1,x) - \left(1 - \dfrac{1}{4e}\right) \sin x,\ 0 \le x \le \pi, \\ u_{tt}(0,x) = \frac{1}{4} u_{tt}(1,x) + \left(1 - \dfrac{1}{4e}\right) \sin x,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{4.10}$$

for one-dimensional a third order partial differential equation. The first and second order of accuracy difference schemes for numerical solution of problem (4.10) are presented. We apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first and second order of accuracy difference schemes are given.

Bibliography

[1] Necmettin Aggez and Maral Ashyralyyewa. Numerical solution of stochastic hyperbolic equations. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[2] Sherif Amirov and Aleksandr Ivanovich Kozhanov. Mixed boundary value problem for a class of strongly nonlinear sobolev-type equations of higher order. In *Doklady Mathematics*, volume 88, pages 446–448. Pleiades Publishing, Ltd., 2013.

[3] Sherif Amirov and Aleksandr Ivanovich Kozhanov. Global solvability of initial boundary-value problems for nonlinear analogs of the boussinesq equation. *Mathematical Notes*, 99(1-2):183–191, 2016.

[4] Yusufjon Apakov. On the solution of a boundary-value problem for a third-order equation with multiple characteristics. *Ukrainian Mathematical Journal*, pages 1–12, 2012.

[5] Yusufjon Apakov and B Irgashev. Boundary-value problem for a degenerate high-odd-order equation. *Ukrainian Mathematical Journal*, 66(10), 2015.

[6] Yusufjon Apakov and Stasys Rutkauskas. On a boundary value problem to third order pde with multiple characteristics. *Nonlinear Analysis*, 16(3):255–269, 2011.

[7] Maksat Ashyraliyev. A note on the stability of the integral-differential equation of the hyperbolic type in a hilbert space. *Numerical Functional Analysis and Optimization*, 29(7-8):750–769, 2008.

[8] Maksat Ashyraliyev, JG Blom, and JG Verwer. On the numerical solution of diffusion–reaction equations with singular source terms. *Journal of Computational and Applied Mathematics*, 216(1):20–38, 2008.

[9] Allaberen Ashyralyev. Nonlocal boundary-value problems for abstract parabolic equations: well-posedness in bochner spaces. *Journal of Evolution Equations*, 6(1):1–28, 2006.

[10] Allaberen Ashyralyev. A note on the bitsadze-samarskii type nonlocal boundary value problem in a banach space. *Journal of Mathematical Analysis and Applications*, 344(1):557–573, 2008.

[11] Allaberen Ashyralyev and Necmettin Aggez. A note on the difference schemes of the nonlocal boundary value problems for hyperbolic equations. *Numerical Functional Analysis and Optimization*, 25(5-6):439–462, 2004.

[12] Allaberen Ashyralyev and Necmettin Aggez. Finite difference method for hyperbolic equations with the nonlocal integral condition. *Discrete Dynamics in Nature and Society*, 2011, 2011.

[13] Allaberen Ashyralyev and Necmettin Aggez. On the solution of nbvp for multidimensional hyperbolic equations. *The Scientific World Journal*, 2014, 2014.

[14] Allaberen Ashyralyev, Necmettin Aggez, and Fatih Hezenci. Boundary value problem for a third order partial differential equation. In *AIP Conference Proceedings*, volume 1470, pages 130–133. AIP, 2012.

[15] Allaberen Ashyralyev and Muzaffer Akat. An approximation of stochastic hyperbolic equations: case with wiener process. *Mathematical Methods in the Applied Sciences*, 36(9):1095–1106, 2013.

[16] Allaberen Ashyralyev and Doghonay Arjmand. A note on the taylor's decomposition on four points for a third-order differential equation. *Applied mathematics and computation*, 188(2):1483–1490, 2007.

[17] Allaberen Ashyralyev, Doghonay Arjmand, and Muhammet Koksal. Taylor's decomposition on four points for solving third-order linear time-varying systems. *Journal of the Franklin Institute*, 346(7):651–662, 2009.

[18] Allaberen Ashyralyev and Charyyar Ashyralyyev. On the problem of determining the parameter of an elliptic equation in a banach space. *Nonlinear Anal., Model. Control*, 19(3):350–366, 2014.

[19] Allaberen Ashyralyev and Okan Gercek. Nonlocal boundary value problems for elliptic-parabolic differential and difference equations. *Discrete Dynamics in Nature and Society*, 2008:1–16, 2008.

[20] Allaberen Ashyralyev and Okan Gercek. On second order of accuracy difference scheme of the approximate solution of nonlocal elliptic-parabolic problems. In *Abstract and Applied Analysis*, volume 2010. Hindawi Publishing Corporation, 2010.

[21] Allaberen Ashyralyev and Okan Gercek. On the second order of accuracy stable implicit difference scheme for elliptic-parabolic equations. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[22] Allaberen Ashyralyev, I Karatay, and PE Sobolevskii. On well-posedness of the nonlocal boundary value problem for parabolic difference equations. *Discrete Dynamics in Nature and Society*, 2004(2):273–286, 2004.

[23] Allaberen Ashyralyev and Mehmet Emir Koksal. On the second order of accuracy difference scheme for hyperbolic equations in a hilbert space. *Numerical Functional Analysis and Optimization*, 26(7-8):739–772, 2005.

[24] Allaberen Ashyralyev and Mehmet Emir Koksal. Stability of a second order of accuracy difference scheme for hyperbolic equation in a Hilbert space. *Discrete Dynamics in Nature and Society*, 2007, 2007.

[25] Allaberen Ashyralyev and Mehmet Emir Koksal. On the numerical solution of hyperbolic pdes with variable space operator. *Numerical Methods for Partial Differential Equations*, 25(5):1086–1099, 2009.

[26] Allaberen Ashyralyev, Mehmet Emir Koksal, and Ravi P Agarwal. A difference scheme for cauchy problem for the hyperbolic equation with self-adjoint operator. *Mathematical and Computer Modelling*, 52(1):409–424, 2010.

[27] Allaberen Ashyralyev, Mehmet Emir Koksal, and Ravi P Agarwal. An operator-difference scheme for abstract cauchy problems. *Computers & Mathematics with Applications*, 61(7):1855–1872, 2011.

[28] Allaberen Ashyralyev and Y Ozdemir. Stability of difference schemes for hyperbolic-parabolic equations. *Computers & Mathematics with Applications*, 50(8):1443–1476, 2005.

[29] Allaberen Ashyralyev and Yildirim Ozdemir. On nonlocal boundary value problems for hyperbolic-parabolic equations. In *Mathematical Methods in Engineering*, pages 131–140. Springer, 2007.

[30] Allaberen Ashyralyev and Yildirim Ozdemir. A note on difference schemes of nonlocal boundary value problems for hyperbolic-parabolic equations. In *AIP Conference Proceedings*, volume 1309, pages 725–736. AIP, 2010.

[31] Allaberen Ashyralyev and Yildirim Ozdemir. On numerical solutions for hyperbolic–parabolic equations with the multipoint nonlocal boundary condition. *Journal of the Franklin Institute*, 351(2):602–630, 2014.

[32] Allaberen Ashyralyev, Javier Pastor, Sergey Piskarev, and HA Yurtsever. Second order equations in functional spaces: qualitative and discrete well-posedness. In *Abstract and Applied Analysis*, volume 2015. Hindawi Publishing Corporation, 2015.

[33] Allaberen Ashyralyev and Sinem Nur Simsek. Nonlocal boundary value problems for a third order partial differential equation. In *AIP Conference Proceedings*, volume 1611, pages 234–240. AIP, 2014.

[34] Allaberen Ashyralyev and Pavel E Sobolevskii. *New Difference Schemes for Partial Differential Equations*, volume 148. Birkhäuser, 2012.

[35] Allaberen Ashyralyev and Ozgur Yildirim. On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations. *Taiwanese Journal of Mathematics*, pages 165–194, 2010.

[36] Allaberen Ashyralyev and Ozgur Yildirim. Stable difference schemes for the hyperbolic problems subject to nonlocal boundary conditions with self-adjoint operator. *Applied Mathematics and Computation*, 218(3):1124–1131, 2011.

[37] Allaberen Ashyralyev and Ozgur Yildirim. A note on the second order of accuracy stable difference schemes for the nonlocal boundary value hyperbolic problem. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[38] Allaberen Ashyralyev and Ozgur Yildirim. On stability of a third order of accuracy difference scheme for hyperbolic nonlocal bvp with self-adjoint operator. In *Abstract and Applied Analysis*, volume 2013. Hindawi Publishing Corporation, 2013.

[39] Allaberen Ashyralyev and Ozgur Yildirim. On the numerical solution of hyperbolic ibvp with high-order stable finite difference schemes. *Boundary Value Problems*, 2013(1):29, 2013.

[40] Allaberen Ashyralyev and Serhat Yilmaz. Modified crank–nicholson difference schemes for ultra-parabolic equations. *Computers & Mathematics with Applications*, 64(8):2756–2764, 2012.

[41] Charyyar Ashyralyyev. High order of accuracy difference schemes for the inverse elliptic problem with dirichlet condition. *Boundary Value Problems*, 2014(1):5, 2014.

[42] Charyyar Ashyralyyev and Yasar Akkan. Numerical solution to inverse elliptic problem with neumann type overdetermination and mixed boundary conditions. *Electronic Journal of Differential Equations*, 2015(188):1–15, 2015.

[43] Charyyar Ashyralyyev and Mutlu Dedeturk. Approximate solution of inverse problem for elliptic equation with overdetermination. In *Abstract and Applied Analysis*, volume 2013. Hindawi Publishing Corporation, 2013.

[44] Charyyar Ashyralyyev and Mutlu Dedeturk. Approximation of the inverse elliptic problem with mixed boundary value conditions and overdetermination. *Boundary Value Problems*, 2015(1):51, 2015.

[45] Charyyar Ashyralyyev and Oznur Demirdag. The difference problem of obtaining the parameter of a parabolic equation. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[46] Charyyar Ashyralyyev, Ayfer Dural, and Yasar Sozen. Finite difference method for the reverse parabolic problem. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[47] Kheireddine Belakroum, Allaberen Ashyralyev, and Assia Guezane-Lakoud. A note on the nonlocal boundary value problem for a third order partial differential equation. In *AIP Conference Proceedings*, volume 1759, page 020021. AIP Publishing, 2016.

[48] Abdelkader Boucherif and Radu Precup. Semilinear evolution equations with nonlocal initial conditions. *Dynamic Systems and Applications*, 16(3):507, 2007.

[49] Regimantas Ciupaila, Mifodijus Sapagovas, and Olga Stikoniene. Numerical solution of nonlinear elliptic equation with nonlocal condition. *Nonlinear Analysis-Modelling and Control*, 18(4):412–426, 2013.

[50] Ahmet Ümid Çoskun. Investigation of microscale heat transfer effects in nanoscale electronic devices by spectral methods. In *International Conference on Mathematical Modeling and Scientific Computing*, 2001.

[51] Zilal Direk and Maksat Ashyraliyev. Fdm for the integral-differential equation of the hyperbolic type. *Advances in Difference Equations*, 2014(1):132, 2014.

[52] TD Dzhuraev and Yusufjon Apakov. On the theory of the third-order equation with multiple characteristics containing the second time derivative. *Ukrainian Mathematical Journal*, 62(1):43–55, 2010.

[53] Hector O Fattorini. *Second Order Linear Differential Equations in Banach Spaces*, volume 108. Elsevier, 2011.

[54] Sergei A Gabov and Aleksei G Sveshnikov. Problems in the dynamics of stratified fluids. *Moscow Izdatel Nauka*, 1986.

[55] Okan Gercek. Well-posedness of the first order of accuracy difference scheme for elliptic-parabolic equations in hölder spaces. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[56] Okan Gercek and Allaberen Ashyralyev. On a difference scheme of the second order of accuracy for elliptic-parabolic equations. *Boundary Value Problems*, 2012(1):1–13, 2012.

[57] Jerome A Goldstein. *Semigroups of Linear Operators and Applications*. Oxford University Press, USA, 1985.

[58] Natalia Dmitrievna Golubeva and Ludmila Stepanovna Pul'kina. A nonlocal problem with integral conditions. *Mathematical Notes*, 59(3):326–328, 1996.

[59] MR Grossinho, FM Minhos, and AI Santos. A third-order boundary value problem with one-sided nagumo condition. *Nonlinear Analysis: Theory, Methods & Applications*, 63(5):e247–e256, 2005.

[60] Assia Guezane-Lakoud and Assia Frioui. Nonlinear three-point boundary-value problem. *Sarajevo J. Math*, 8(20):1–6, 2012.

[61] Assia Guezane-Lakoud, Nacira Hamidane, and Rabah Khaldi. On a third-order three-point boundary value problem. *International Journal of Mathematics and Mathematical Sciences*, 2012, 2012.

[62] Assia Guezane-Lakoud and Rabah Khaldi. Study of a third-order three-point boundary value problem. In *AIP Conference Proceedings*, volume 1309, pages 329–335. AIP, 2010.

[63] Lijun Guo, Jian-Ping Sun, and Ya-Hong Zhao. Multiple positive solutions for nonlinear third-order three-point boundary-value problems. *Electronic Journal of Differential Equations*, 2007(112):1–7, 2007.

[64] Feliksas Ivanauskas, T Meškauskas, and Mifodijus Sapagovas. Stability of difference schemes for two-dimensional parabolic equations with non-local boundary conditions. *Applied mathematics and computation*, 215(7):2716–2732, 2009.

[65] FF Ivanauskas, Yu A Novitski, and MP Sapagovas. On the stability of an explicit difference scheme for hyperbolic equations with nonlocal boundary conditions. *Differential equations*, 49(7):849–856, 2013.

[66] Justina Jachimaviciene, Mifodijus Sapagovas, A Štikonas, and O Štikoniene. On the stability of explicit finite difference schemes for a pseudoparabolic equation with nonlocal conditions. *Nonlinear Anal. Model. Control*, 19(2):225–240, 2014.

[67] Belakroum Kheireddine, Ashyralyev Allaberen, and Guezane-Lakoud Assia. A note on the nonlocal boundary value problem for a third order partial differential equation. *Filomat*, (just-accepted), 2018.

[68] VI Korzyuk and VB Kiselev. A boundary value problem for a third-order linear nonclassical equation generated by the heat equation, 2001.

[69] Aleksandr Ivanovich Kozhanov. Mixed boundary value problem for some classes of third order differential equations. *Matematicheskii Sbornik*, pages 504–522, 1982.

[70] Aleksandr Ivanovich Kozhanov. Mixed problem for one class of quasilinear equation of third order. *Boundary Value Problems for Nonlinear Equations, Novosibirsk*, pages 118–128, 1982.

[71] Aleksandr Ivanovich Kozhanov. Parabolic equations with an unknown absorption coefficient. In *Doklady Mathematics*, volume 74, pages 573–576. Springer, 2006.

[72] Aleksandr Ivanovich Kozhanov. Solvability of some spatially nonlocal boundary value problems for second-order linear hyperbolic equations. In *Doklady Mathematics*, volume 80, pages 599–601. Springer, 2009.

[73] Aleksandr Ivanovich Kozhanov. Solvability of boundary value problems with the nonlocal bitsadze-samarskii condition for linear hyperbolic equations. In *Doklady Mathematics*, volume 81, pages 467–470. Springer, 2010.

[74] Aleksandr Ivanovich Kozhanov. On the solvability of certain spatially nonlocal boundary-value problems for linear hyperbolic equations of second order. *Mathematical Notes*, 90(1-2):238, 2011.

[75] Aleksandr Ivanovich Kozhanov. Problems with integral-type conditions for some classes of nonstationary equations. In *Doklady Mathematics*, volume 90, pages 440–443. Springer, 2014.

[76] Aleksandr Ivanovich Kozhanov. On the solvability of spatially nonlocal problems with conditions of integral form for some classes of nonstationary equations. *Differential Equations*, 51(8):1043–1050, 2015.

[77] Aleksandr Ivanovich Kozhanov and Ludmila S Pulkina. On the solvability of boundary value problems with a nonlocal boundary condition of integral form for multidimensional hyperbolic equations. *Differential equations*, 42(9):1233–1246, 2006.

[78] Aleksandr Ivanovich Kozhanov and Regina Rifkhatovna Safiullova. Linear inverse problems for parabolic and hyperbolic equations. *Journal of Inverse and Ill-Posed Problems*, 18(1):1–24, 2010.

[79] Mustafa Kudu and Ilhame Amirali. Method of lines for third order partial differential equations. *Journal of Applied Mathematics and Physics*, 2(02):33, 2014.

[80] Chahla Latrous and Ameur Memou. A three-point boundary value problem with an integral condition for a third-order partial differential equation. In *Abstract and Applied Analysis*, volume 2005, pages 33–43. Hindawi Publishing Corporation, 2005.

[81] Zeqing Liu, Shin Min Kang, and Jeong Sheok Ume. Triple positive solutions of nonlinear third order boundary value problems. *Taiwanese Journal of Mathematics*, pages 955–971, 2009.

[82] Evgenii S Nikolaev and Aleksandr A Samarskii. *Numerical Methods for Grid Equations: Iterative Methods*. Birkhäuser, 1989.

[83] Jing Niu and Ping Li. Numerical algorithm for the third-order partial differential equation with three-point boundary value problem. In *Abstract and Applied Analysis*, volume 2014. Hindawi Publishing Corporation, 2014.

[84] M Aslam Noor and EA Al-Said. Finite-difference method for a system of third-order boundary-value problems. *Journal of optimization theory and applications*, 112(3):627–637, 2002.

[85] Yildirim Ozdemir and Mehmet Kucukunal. A note on nonlocal boundary value problems for hyperbolic schrödinger equations. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation, 2012.

[86] Alex P Palamides and Anastasia N Veloni. A singular third-order 3-point boundary-value problem with nonpositive green's function. *Electronic Journal of Differential Equations*, 2007(151):1–13, 2007.

[87] Panos K Palamides and Alex P Palamides. A third-order 3-point bvp. applying krasnosel'skii's theorem on the plane without a green's function. *Electron. J. Qual. Theory Differ. Equ*, 2008, 2008.

[88] S Piskarev and Shaw. On certain operator families related to cosine operator functions. *Taiwanese Journal of Mathematics*, pages 527–546, 1997.

[89] Ludmila S Pulkina. A nonclassical problem for a degenerate hyperbolic equation. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (11):48–51, 1991.

[90] Ludmila S Pulkina. Certain nonlocal problem for a degenerate hyperbolic equation. *Mathematical Notes*, 51(3):286–290, 1992.

[91] Ludmila S Pulkina. Problems with nonlinear boundary conditions for a hyperbolic equation. *Proceedings of the Steklov Institute of Mathematics*, 278(1):199–207, 2012.

[92] Ludmila S Pulkina. Solution to nonlocal problems of pseudohyperbolic equations. *Electronic Journal of Differential Equations*, 2012(116):1–9, 2012.

[93] Ludmila S Pulkina. Nonlocal problems for hyperbolic equations with degenerate integral conditions. *Electronic Journal of Differential Equations*, 2016(193):1–12, 2016.

[94] Haidong Qu. Positive solutions of boundary value problems of nonlinear third-order differential equations. *Int. Journal of Math. Analysis*, 4:855–860, 2010.

[95] John M Rassias and Erkinjon T Karimov. Boundary-value problems with non-local condition for degenerate parabolic equations. *arXiv preprint arXiv:1301.3208*, 2013.

[96] Svajūnas Sajavičius. Stability of the weighted splitting finite-difference scheme for a two-dimensional parabolic equation with two nonlocal integral conditions. *Computers & Mathematics with Applications*, 64(11):3485–3499, 2012.

[97] Aleksandr A Samarskij and Evgenii S Nikolaev. *Numerical Methods for Grid Equations: Volume II Iterative Methods*. Birkhäuser, 2012.

[98] Mifodijus Sapagovas and Kristina Jakubeliene. Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition. *Nonlinear Anal. Model. Control*, 17(1):91–98, 2012.

[99] Sergey Smirnov. On the third order boundary value problems with asymmetric nonlinearity. *Nonlin. Analysis Model. Control*, 16(2):231–241, 2011.

[100] Sergey Smirnov. On some spectral properties of third order nonlinear boundary value problems. *Mathematical Modelling and Analysis*, 17(1):78–89, 2012.

[101] PE Sobolevskii. *Difference Methods for the Approximate Solution of Differential Equations*. Voronezh State University Press, Voronezh, Russia, 1975.

[102] Hadjimamed Soltanov. A note on the goursat problem for a multidimensional hyperbolic equation. *Contemporary Analysis and Applied Mathematics*, 1(2):98–106, 2013.

[103] Jian-Ping Sun and Qiu-Yan Ren. Existence of solution for third-order m-point boundary value problem. *Applied Mathematics E-Notes*, 10:268–274, 2010.

[104] R Suryanarayana. A third order boundary value problem with nonlinear terms existence and uniqueness. *Global Journal of Mathematical Sciences-Theory and Practical*, 3:181–187, 2011.

[105] Feng Wang, Hai-hua Lu, and Fang Zhang. Multiple positive solutions for nonlinear third-order boundary value problems in banach spaces. *Turkish Journal of Mathematics*, 33(1):55–63, 2009.

[106] Yuandi Wang and Shengzhou Zheng. The existence and behavior of solutions for nonlocal boundary problems. *Boundary Value Problems*, 2009(1):484879, 2009.

[107] Ozgur Yildirim and Meltem Uzun. On the numerical solutions of high order stable difference schemes for the hyperbolic multipoint nonlocal boundary value problems. *Applied Mathematics and Computation*, 254:210–218, 2015.

[108] Ozgur Yildirim and Meltem Uzun. On third order stable difference scheme for hyperbolic multipoint nonlocal boundary value problem. *Discrete Dynamics in Nature and Society*, 2015, 2015.