وزارة التعليم العالي والبحث العلمي

Université Badji Mokhtar Annaba
Badji Mokhtar University - Annaba

جامعة باجي مختار عنابة

Faculté des Sciences
Département de Mathématiques

THESE

Présentée en vue de l'obtention du diplôme de
Doctorat en Sciences
Option : Mathématiques Appliquées

Existence and stability of solutions for some functional differential and delay integro-differential equations by the fixed point technique

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Année : 2018

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A Doctoral Thesis,

By Faycal Bouchelaghem

Advisors: Dr. A. Ardjouni and Pr. A. Djoudi

Dedication

I dedicate this modest work

- To the memory of my dear grandfather, grandmother and my mother who miss me at all times.

- To my father who sacrificed himself to see me one day succeed, and to whom I owe everything.

- To my dear wife and my dearest children Mohamed, Adem and Meriem.

- To my dear brothers and dear sisters.

- To all my friends.

- To all my family.

Acknowledgement

First, I must thank Allah the Almighty for giving me enough courage to do this work

This thesis could not have been completed without the valuable help of several who contributed directly or indirectly to the outcome of this work of research.

I take this page to try to express a fragment of my gratitude to the people who contributed to the existence of this thesis.

First, I want to thank my advisors Dr. Abdelouaheb Ardjouni (Souk Ahras University), His expertise, his sense of perfection and his availability have allowed me to have quality coaching. A. Ardjouni has never hesitated to take the time to explain, discuss, exchange and share his great expertise in research. Thanks to him, I was able to successfully complete my Ph.D. I can not thank him enough for everything he did for me. I wish to express my deep gratitude to him.

These thanks are to be shared with Prof. Ahcène Djoudi (Badji-Mokhtar University, Annaba) my co-supervisor, for his supervision and advice during these four years of thesis. He always knew how to make himself available to guide me and clarify me about our research problem. I owe him a lot.

I address my thanks to the president of jury Prof. Smail Kelaiaia and the members of jury, Prof. Abderrezak Chaoui, Dr. Abdelouahab Salmi and Dr. Hamid Boulares for the honor that they made me while taking part in this thesis, for their time and valuable comments on my thesis.

I also express my sincere gratitude to my friends, I thank those who encouraged me during these years of thesis. In particular, Faouzi, Kamel, Saleh, Hamid and Malik.

Abstract

In this thesis, we study qualitative properties of broad classes of nonlinear delay dynamic and delay integro-dynamic equations. We start by giving some fixed point theorems, results for delay differential equations and elements of calculus on time scales. Second, by using the Schauder and Krasnoselskii's fixed point theorems, we study the periodicity and positivity of solutions for a class of nonlinear delay dynamic equations, neutral dynamic equations, delay integro-dynamic equations and difference equations with summation boundary conditions. Finally, by applying the contraction mapping principle, we show the existence of a unique periodic solution and the asymptotic stability of the zero solution.

Keywords: Delay dynamic equations, Delay integro-dynamic equations, Fixed point theory, Existence, Periodicity, Positivity, Stability.

Mathematics Subject Classification: 34K13, 34K20, 34K30, 34K40, 45D05, 45J05, 47H10.

ملخـــــص

في هذه الأطروحة نهتم بدراسة الخصائص النوعية لمجموعة من المعادلات الديناميكية غير الخطية والمعادلات الديناميكية التكاملية ذات تأخر. نبدأ بإعطاء بعض نظريات النقطة الثابتة وبعض النتائج حول المعادلات التفاضلية ذات تأخر وعناصر الحساب على أزمنة سلمية. ثانيا، ندرس الدورية والايجابية لحلول مجموعة من المعادلات الديناميكية ذات تأخر، المعادلات الديناميكية الحيادية، المعادلات الديناميكية التكاملية ذات تأخر ومعادلات الفروق مع شروط حدية تجميعية، باستخدام نظريات النقطة الثابتة لشودار ولكراسنوسلسكي. وأخيرًا، من خلال مبدأ التقليص، نبرهن على وجود حل دوري وحيد وعلى الاستقرار المقارب للحل الصفري.

الكلمات المفتاحية: معادلات ديناميكية ذات تأخر، معادلات ديناميكية تكاملية ذات تأخر، نظريات النقطة الثابتة، الدورية، الايجابية، الاستقرار.

Résumé

Dans cette thèse, nous étudions les propriétés qualitatives de larges classes d'équations dynamiques non linéaires à retard et d'équations intégro-dynamiques à retard. Nous commençons par donner quelques théorèmes de points fixes, des résultats pour des équations différentielles à retard et des éléments de calcul sur des échelles de temps. Deuxièmement, en utilisant les théorèmes des points fixes de Schauder et Krasnoselskii, nous étudions la périodicité et la positivité des solutions pour une classe d'équations dynamiques à retard non linéaire, les équations dynamiques neutrales, les équations intégro-dynamiques à retard et les équations de différence avec conditions aux limites de sommations. Enfin, en appliquant le principe de la contraction, nous montrons l'existence d'une solution périodique unique et la stabilité asymptotique de la solution zéro.

Mots-clés: Equations différentielles à retard, Equations intégro-différentielles à retard, Théorèmes des points fixes, Existence, Périodicité, Positivité, Stabilité.

Mathematics Subject Classification: 34K13, 34K20, 34K30, 34K40, 45D05, 45J05, 47H10.

Contents

Chapter 1

Introduction

The theory of fixed points is a promising area in mathematics especially in nonlinear functional analysis because it has wide applicability in various fields of pure and applied mathematics as well as in other fields like physical science, life science and economics. Classical and major results in these areas are Banach's fixed point theorem, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem.

Banach in 1922 states that a contraction mapping on a complete metric space has a unique fixed point. However, on historical point of view, the major classical result in fixed point theory is due to Brouwer given in 1912, which states that a continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point. An extension of this result is the Schauder's fixed point theorem in 1930 which states that a continuous map on a convex compact subspace of a Banach space has a fixed point. Thereafter, Krasnoselskii in 1955 studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name (see [17], [28], [30], [33], [71], [98]).

Time scales calculus was initiated in 1988 by Stefan Hilger. It bridges the gap between continuous and discrete analysis and expands on both theories. Differential equations are defined on an interval of the set of real numbers while difference equations are defined on discrete sets. However, some physical systems are modeled by what is called dynamic equations because they are either differential equations, difference equations or a combination of both. This means that dynamic equations are defined on connected, discrete or combination of both types of sets. Hence, time scales calculus provides a generalization of differential and difference analysis (see [19], [20], [63], [74]).

Delay dynamic and delay integro-dynamic equations arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of delay dynamic and delay integro-dynamic equations have received the attention of many authors (see [1]–[16], [18]–[27], [29]–[45], [47]–[69], [72]–[97], [99]–[108], [110]).

One of the most important qualitative aspects of delay dynamic and delay integro-dynamic equations is determining the stability of a given model. Stability results by using the fixed point theory can sometimes provide better conditions for convergence to zero of solutions, than Lyapunov methods. The advantages of this particular fixed point method have been achieved thanks to fixed point methods requiring averaging conditions of the vector field, by using appropriately chosen variation of parameters type formulas to invert the delay dynamic equation into an integral form. As is known in dynamic equations theory, a common method for proving existence of solutions is through fixed point methods. However, in fairly recently times, the fixed point theory have been used to obtain further properties of the solution, namely attractively of solutions to an equilibrium, and not merely the existence of these solution curves, as is normally done in classical dynamic equations theory. The aforementioned method for stability of dynamic equations has been applied successfully for delay dynamic equations and delay integro-dynamic equations (see [11], [22], [23], [29]–[31], [33], [36], [47], [49], [87], [108]).

We have been interested in the use of fixed point theory to problem of periodicity and positivity and stability for delay dynamic and delay integro-dynamic equations. We have studied and contributed to it and have obtained interesting results. In this thesis we present a collection of results to some problems of delay dynamic and delay integro-dynamic equations by using fixed point theory (see [21]–[26]).

This thesis contains eight chapters which are briefly presented below. Chapter two is essentially an introduction to the fixed point theory, delay differential equations, elements of calculus on time scales, where we fix notations, terminology to be used. It is a survey aimed at recalling some basic definitions and theory. While some of the classical and recent results about fixed point theory, delay differential equations and elements of calculus on time scales are also presented in this chapter. Fixed point theorems frequently call for compact sets in Banach spaces which may be subsets of continuous functions. For that purpose, we give topologies which will provide many of those compact sets.

In chapter 3, we study the existence of positive periodic solutions for the dynamic equations on time scales

$$x^{\triangle}(t) + p(t)\, x^{\sigma}(t) + q(t)\, x(\tau(t)) = 0,\ t \geq t_0. \tag{E}$$

The main tool employed here is the Schauder's fixed point theorem. Two examples are also given to illustrate this work (see [24]).

In chapter 4, we study the existence and stability of positive periodic solutions for the delay nonlinear dynamic equation on time scales

$$x^{\triangle}(t) + p(t)\, x^{\sigma}(t) - \sum_{i=1}^{n} q_i(t)\, f_i(x(\tau_i(t))) = 0,\ t \geq t_0.$$

An examples is also given to illustrate this work (see [23]).

In chapter 5, we study the existence of positive solutions for (E). The asymptotic properties of solutions are also treated. Three examples are also given to illustrate this work (see [25]).

In chapter 6, we study the existence of positive periodic and positive solutions for the integro-dynamic equations on time scales

$$x^{\triangle}(t) + \int_{t-\tau}^{t} p(t-s)g(x(s))\Delta s,\ t \geq T.$$

The main tool employed here is the Schauder's fixed point theorem. The exponential stability of positive solutions is also treated (see [21]).

In chapter 7, the nonlinear neutral dynamic equation with periodic coefficients

$$\begin{aligned}&[u(t) - g(u(t - \tau(t)))]^{\Delta}\\&= p(t) - a(t)u^{\sigma}(t) - a(t)g(u^{\sigma}(t - \tau(t))) - h(u(t), u(t - \tau(t))),\end{aligned}$$

is considered in this work. By using Krasnoselskii's fixed point theorem we obtain the existence of periodic and positive periodic solutions and by contraction mapping principle we obtain the uniqueness. Stability results of this equation are analyzed (see [22]).

In chapter 8, we study the existence of positive solutions for the second-order difference equation with summation boundary conditions

$$\begin{cases} \triangle^2 u\left(t-1\right) + a\left(t\right) f\left(u\left(t\right)\right) = 0,\ t \in \left\{1, 2, \ldots, T\right\}, \\ \triangle u\left(0\right) = 0, \quad u\left(T+1\right) = \alpha \sum\limits_{s=1}^{\eta} u\left(s\right). \end{cases}$$

The main tool employed here is the Krasnoselskii's fixed point theorem in a cone (see [26]).

Chapter 2

Preliminaries

2.1 Functional analysis

In this section we discus compact sets. The following elements have been gathered from several analysis and some specialized books on the functional analysis and the most is in the following bibliography ([33], [70], [109]).

Definition 2.1 (Metric) Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. Then d is called a metric on X if the following properties hold

i) $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$,
ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called metric space.

Definition 2.2 (Norm) Let $(X, +, .)$ be a linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $N : X \to [0, \infty)$ a function. Then, N is said to be a norm if the following properties hold

i) $N(x) = 0$ if and only if $x = 0$,
ii) $N(\lambda x) = |\lambda| \, N(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$,
iii) $N(x + y) \leq N(x) + N(y)$ for all $x, y \in X$.

The ordered pair(X, N) is called a normed space.

We use the notation $\|.\|$ for norm. Then every normed space $(X, \|.\|)$ is a vector space and it is a metric space (X, d) with induced metric $d(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition 2.3 Let X be a nonempty set and d is a metric on X, a sequence $\{x_n\} \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0$, i.e $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

Definition 2.4 The metric space (X, d) is complete if every Cauchy sequence in (X, d) has a limit in that space.

Definition 2.5 A Banach space is a complete normed space.

We often say a Banach space is a complete normed vector space.

Theorem 2.1 *A closed subspace of a Banach space is a Banach space.*

Example 2.1 The linear space $C([a, b])$ of continuous functions on the closed and bounded interval $[a, b]$ is a Banach space with the uniform convergence norm $\|f\|_\infty = \sup\limits_{t\in[a,b]} |f(t)|$.

Definition 2.6 Let (X, d) be a metric space. Recall that a subset Ω of X is called compact if every open cover of Ω has a finite subcover. Equivalently, a subset Ω of X is compact if every sequence in Ω contains a convergent subsequence with a limit in Ω.

Example 2.2 Let $\phi : [a, b] \to \mathbb{R}^n$ be continuous and let X de the set of continuous functions $f : [a, c] \to \mathbb{R}^n$ with $c > b$ and $f(t) = \phi(t)$ for $a \leq t \leq b$. Define $d(f, g) = \sup_{a\leq t\leq c} |f(t) - g(t)|$ for $f, g \in X$. Then (X, d) is complete metric space but not a Banach space because $f + g$ is not in X.

Definition 2.7 Let (X, d) be a metric space. A subset Ω of X is said to be totally bounded if for each $\varepsilon > 0$, there exists a finite number of elements $x_1, x_2, ..., x_n$ in X such that $\Omega \subseteq U_{i=1}^n B_\varepsilon(x_i)$.

Proposition 2.1 *Let (X, d) be a metric space. Then the following are equivalent*

i) X is compact.

ii) Every sequence in X has a convergent subsequence.

iii) X is complete and totally bounded.

Definition 2.8 A set Ω in a metric space (X, d) is relatively compact if its closure is compact, i.e., $\overline{\Omega}$ is compact.

Proposition 2.2 *Let Ω be a closed subset of a complete metric space (X, d). Then Ω is compact if and only if it is relatively compact.*

Definition 2.9 Let X be a compact metric space and Ω be a subset of $C(X)$.

a) Ω is uniformly bounded if there exists $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$.

b) Ω is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in X$ and $d(t_1, t_2) < \delta$ imply $|u(t_1) - u(t_2)| < \varepsilon$ for all $u \in \Omega$.

The following result gives the main method of proving compactness in the spaces in which we are interested.

Theorem 2.2 (Ascoli-Arzela [109]) *Let X be a compact metric space. If Ω is an equicontinuous, uniformly bounded subset of $C(X)$, then Ω is relatively compact.*

Definition 2.10 Let S be a mapping from a metric space (X, d) into another metric space (Y, d'). Then S is said to satisfy Lipschitz condition on X if there existe a constant $L > 0$ such that

$$d'(Sx, Sy) \leq Ld(x, y) \text{ for all } x, y \in X.$$

If L is the least number for which Lipschitz condition holds, then L is called Lipschitz constant. In this case, we say that S is an L-Lipschitz mapping or simply a Lipschitzian mapping with Lipschitz constant L. Otherwise, it is called non-Lipschitzian mapping. An L-Lipschitz mapping S is said to be contraction if $L < 1$ and nonexpansive if $L = 1$.

The mapping S is said to be contractive if

$$d'(Sx, Sy) < d(x, y) \text{ for all } x, y \in X, \ x \neq y.$$

The following proposition guarantees the existence of Lipschitzian mappings.

Proposition 2.3 *Let $S : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a,b). Suppose S is continuous on $[a,b]$. Then, S is a Lipschitz continuous function (and hence is uniformly continuous).*

Definition 2.11 Let X and Y be two Banach spaces and let the mapping $S : X \to X$. Then

1) S is said to be bounded if Ω is bounded in X implies $S(\Omega)$ is bounded.

2) S is said to be closed if $x_n \to x$ in X and $Sx_n \to y$ in Y imply $Sx = y$.

3) S is said to be compact if Ω is bounded in X implies $S(\Omega)$ is relatively compact ($\overline{S(\Omega)}$ is compact), i.e., for every bounded sequence $\{x_n\}$ in X, $\{Sx_n\}$ has convergent subsequence in Y.

4) S is said to be completely continuous if it is continuous and compact.

In the case of linear mappings, the concepts of continuity and boundedness are equivalent, but it is not true in general.

2.2 Fixed point theory

In this section we state some fixed point theorems that we employ to help us in proving existence and stability of solutions (see [17], [28], [30], [33], [71], [98]).

Definition 2.12 Let S be a mapping in the set Ω. we call fixed point of S any point x satisfying $S(x) = x$. If there exists such x, we say that S has a fixed point, which is equivalent to saying that the equation $S(x) - x = 0$ has a null solution.

Fixed point theorems guarantee the existence of a fixed point under appropriate conditions on the map S and the set Ω.

2.2.1 Banach fixed point theorem

Let the classical Cauchy problem on existence and uniqueness of the solution to the differential equation satisfying a given initial condition,

$$\begin{cases} x' = f(t,x), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

can be expressed as an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{2.2}$$

from which a sequence of functions $\{x_n\}$ may be inductively defined by

$$x_0(t) = x_0, \; x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0)ds,$$

and, in general

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds. \tag{2.3}$$

This is called Picard method of successive approximations and, under liberal conditions on f, one can show that $\{x_n\}$ converges uniformly on some interval $|t - t_0| \leq k$ to some continuous function, say x. Taking the limit in the equation defining x_{n+1}, we pass the limit through the integral and have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

so that $x(t_0) = x_0$ and, upon differentiation, we obtain $x'(t) = f(t, x(t))$. Thus, x is a solution of the initial value problem.

Banach realized that this was actually a fixed-point theorem with wide application. For if we define an operator S on a complete metric space $C([t_0, t_0 + k], \mathbb{R})$ with the supremum norm $\|.\|$ by $x \in C$ implies

$$(Sx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{2.4}$$

then a fixed point of S, say $S\phi = \phi$, is a solution of the initial value problem.

The idea had two outstanding features. First, it had application to problems in every area of mathematics which used complete metric spaces. And it was clean. For example, the standard muddy and shaky proofs of implicit function theorems became clear and solid using the fixed-point theory. We will use it here to prove existence of solutions of various kinds of differential equations.

Theorem 2.3 (Contraction mapping principle [98]) *Let (X, d) be a complete metric space and $S : X \to X$ a contraction operator. Then there is a unique point $x \in X$ with $Sx = x$.*

Furthermore, if $y \in X$ and if $\{y_n\}$ is defined inductively by $y_1 = Sy$ and $y_{n+1} = Sy_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Sx = x$ has one and only one solution.

In applying this result to (2.1), a distressing event occurred which we now briefly describe. Assume that f is continuous and satisfies a global Lipschitz condition in x, say

$$|f(t, x_1) - f(t, x_2)| \leq \alpha \, |x_1 - x_2| \, ,$$

for $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. Then by (2.4) we obtain (for $t \geq t_0$)

$$\begin{aligned} |Sx_1(t) - Sx_2(t)| &= \left| \int_{t_0}^{t} [f(s, x_1(s)) - f(s, x_2(s))] \, ds \right| \\ &\leq \int_{t_0}^{t} \alpha \, |x_1(s) - x_2(s)| \, ds, \end{aligned}$$

so that if $\|.\|$ is the sup norm on continuous functions on $[t_0, t_0 + k]$, then

$$\|Sx_1 - Sx_2\| \leq \alpha k \, \|x_1 - x_2\| \, .$$

This is a contraction if $\alpha k = \lambda < 1$. Now α is fixed and we take k small enough that $\alpha k < 1$. This gives a fixed point which is a solution of (2.1) on $[t_0, t_0 + k]$.

2.2.2 Schauder's theorem

Denote the unit ball in $\mathbb{R}^n$ by $B^n := \overline{B(0,1)} = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the unit sphere (the boundary of the unit ball) by $D^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\} = \partial B$.

Definition 2.13 Let A be a subset of a topological space X. A retraction is a map $r : X \to A$ such that $r(x) = x$ for all $x \in A$. If there exists a retraction from X to A, we say A is a retract of X.

Lemma 2.1 (No-retraction theorem [98]) *There is no continuous retraction* $r : B^n \to D^{n-1}$.

Theorem 2.4 (Brouwer's fixed point theorem [28]) *Every continuous map* $S : B^n \to B^n$ *has a fixed point.*

Corollary 2.1 ([98]) *Let* Ω *be a nonempty, compact and convex subset of* $\mathbb{R}^n$. *Every continuous map* $S : \Omega \to \Omega$ *has a fixed point.*

Definition 2.14 Let X be a normed vector space and $F = \{x_1, x_2, ..., x_n\}$ a finite subset of X. Then $conv(F)$, the convex hull of F, is defined by

$$conv(F) = \left\{ \sum_{j=1}^{n} t_j x_j : \sum_{j=1}^{n} t_j = 1,\ t_j \geq 0 \right\}.$$

For future applications, we will need a more general definition to handle the case in which F is infinite.

Definition 2.15 Let X be a normed vector space and F a subset of X. The convex hull $conv(F)$ is the intersection of all convex sets $H \subseteq X$ such that $F \subseteq H$.

Proposition 2.4 ([98]) *Definitions 2.14 and 2.15 are equivalent for finite sets.*

Lemma 2.2 (Schauder projection lemma [98]) *Let* Ω *be a compact subset of a normed vector space* X, *with metric* d *induced by the norm* $\|.\|$. *Given* $\epsilon > 0$, *there exists a finite subset* $F \subseteq X$ *and a map* $P : \Omega \to conv(F)$ *such that* $d(P(x), x) < \epsilon$ *for all* $x \in \Omega$. *This map is called the Schauder projection.*

Theorem 2.5 (Schauder's fixed point theorem [98]) *Let* Ω *be a closed, convex and nonempty subset of a Banach space* X. *Let* $S : \Omega \to \Omega$ *be a continuous mapping such that* $S\Omega$ *is a relatively compact subset of* X. *Then* S *has at least one fixed point in* Ω. *That is there exists an* $x \in \Omega$ *such that* $Sx = x$.

2.2.3 Krasnoselskii's fixed point theorem

Definition 2.16 A topological space X has the fixed-point property if, whenever $S : X \to X$ is continuous, then S has a fixed point.

Definition 2.17 Let Ω be a subset of a Banach space X and $S:\Omega\rightarrow X$ application. If S is continuous and $S\left(\Omega\right)$ is contained in a compact set in X, then we say that S is a compact application "we also say that S is completely continuous".

In 1955 Krasnoselskii's [33, 98] observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,

Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

For better understanding this observation of Krasnoselskii's, consider the following differential equation.

$$x'\left(t\right)=-a\left(t\right)x\left(t\right)-g\left(t,x\right). \tag{2.5}$$

We can transform this equation in another form while writing, formally

$$x'\left(t\right)e^{\int_0^t a(s)ds}=-a\left(t\right)e^{\int_0^t a(s)ds}x\left(t\right)-g\left(t,x\right)e^{\int_0^t a(s)ds},$$

thus

$$x'\left(t\right)e^{\int_0^t a(s)ds}+a\left(t\right)e^{\int_0^t a(s)ds}x\left(t\right)=-g\left(t,x\right)e^{\int_0^t a(s)ds},$$

or

$$\left(x\left(t\right)e^{\int_0^t a(s)ds}\right)'=-g\left(t,x\right)e^{\int_0^t a(s)ds},$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^{t}\left(x\left(u\right)e^{\int_0^u a(s)ds}\right)'du=-\int_{t-T}^{t}g\left(u,x(u)\right)e^{\int_0^u a(s)ds}du,$$

that gives

$$x\left(t\right)e^{\int_0^t a(s)ds}-x\left(t-T\right)e^{\int_0^{t-T} a(s)ds}=-\int_{t-T}^{t}g\left(u,x(u)\right)e^{\int_0^u a(s)ds}du,$$

or

$$x\left(t\right)=x\left(t-T\right)e^{-\int_{t-T}^t a(s)ds}-\int_{t-T}^{t}g\left(u,x(u)\right)e^{-\int_u^t a(s)ds}du. \tag{2.6}$$

If we suppose that $e^{-\int_{t-T}^t a(s)ds}:=\lambda<1$ and $(X,\|.\|)$ is the Banach space of functions $\varphi:\mathbb{R}\rightarrow\mathbb{R}$ continuous, then the Equation (2.6) can be written as

$$\varphi\left(t\right)=\left(A\varphi\right)\left(t\right)+\left(B\varphi\right)\left(t\right):=\left(S\varphi\right)\left(t\right).$$

where A is contraction provides that the constant $\lambda < 1$ and B is compact mapping.

This example shows the birth of the mapping $S\varphi := A\varphi + B\varphi$ which is identified with a sum of a contraction and a compact mapping.

Thus, the search of the solution for Equation (2.6) requires an adequate theorem which applies to this hybrid operator S and which can conclude the existence for a fixed point which will be, in its turn, solution of the initial Equation (2.5). Krasnoselskii's found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result [98].

Theorem 2.6 (Krasnoselskii's fixed point theorem [98]) *Let Ω be a closed bounded convex nonempty subset of a Banach space $(X, \|.\|)$. Suppose that A and B map Ω into X such that*

(i) A is a contraction mapping,

(ii) B is compact and continuous,

(iii) $x, y \in \Omega$, implies $Ax + By \in \Omega$,

Then there exists $z \in \Omega$ with $z = Az + Bz$.

Remark 2.1 Note that if $B = 0$, the theorem becomes the theorem of Banach. If $A = 0$, then the theorem is not other than the theorem of Schauder.

Theorem 2.7 (Krasnoselskii's cone fixed point theorem [71]) *Let X be a Banach space, and let $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K,$$

be a completely continuous operator such that

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, or

(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

2.3 Retarded functional differential equations

Delayed problems are innumerable in literature. Some of these problems have been of particular interest. We have chosen in this thesis some models intervening in different real world domains (see [52, 58, 59, 72]).

2.3.1 Delay differential equations

Suppose $\tau \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping in the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-\tau, 0]$ we let $C = C([-\tau,0], \mathbb{R}^n)$ and designate the norm of an element ψ in C by $\|\psi\| = \sup_{-\tau<s<0} |\psi(s)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in C\left([t_0 - \tau, t_0 + A], \mathbb{R}^n\right)$, then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$.

Definition 2.18 If Ω is a subset of $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t), \tag{2.7}$$

where

$$x_t(s) = x(t+s), \quad -\tau \leq s \leq 0,$$

is a retarded functional differential equation on Ω and will denote this equation by RFDE. If we wish to emphasize that the equation is defined by f, we write the RFDE(f).

A function x is said to be a solution of Equation (2.7) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C\left([t_0 - \tau, t_0 + A], \mathbb{R}^n\right)$, $(t, x_t) \in \Omega$ and $x(t)$ satisfies Equation (2.7) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}$, $\psi \in C$, we say $x(t, t_0, \psi)$ is a solution of Equation(2.7) with initial value ψ at t_0 or simply a solution through (t_0, ψ) if there is an $A > 0$ such that $x(t, t_0, \psi)$ is a solution of Equation (2.7) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t, t_0, \psi) = \psi$.

If $\tau = 0$, Equation (2.7) is a very general type of equation and includes ordinary differential equations.

If $f(t,\psi) = L(t,\psi) + h(t)$ where $L(t,\psi)$ is linear in ψ, we say Equation (2.7) is linear and is homogeneous if $h \equiv 0$ and nonhomogeneous if $h \neq 0$.

If $f(t,\psi) = g(\psi)$ where g does not depend on t, we claim Equation (2.7) is autonomous.

Example 2.3 The following equations are delay differential equations

$$x'(t) = 3x(t) + 2x(t-5), \tag{2.8}$$

$$x'(t) = a(t)x(t) + b(t)x'(t-\tau(t)) + h(t), \tag{2.9}$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)ds, \tag{2.10}$$

where a, b, τ are continuous functions.

Equation (2.8) is an linear autonomous delay differential equation with constant $\tau = 5$. Equation (2.9) is nonhomogeneous, linear nonautonomous delay functional differential equations and Equation (2.10) is a delay linear integro-differential equation.

If $t_0 \in \mathbb{R}$, $\psi \in C$ are given and $f(t,\psi)$ is continuous, then finding a solution of Equation (2.7) through (t_0,ψ) is equivalent to solving the integral equation

$$\begin{aligned} x_{t_0} &= \psi, \\ x(t) &= \psi(0) + \int_{t_0}^{t} f(s,x_s)ds, \ t \geq t_0. \end{aligned} \tag{2.11}$$

We define Sx by

$$\begin{aligned} (Sx)(t) &= \psi(0) + \int_{t_0}^{t} f(s,x_s)ds, \ t \geq t_0, \\ x_{t_0} &= \psi. \end{aligned}$$

To prove the existence of the solution through a point $(t_0,\psi) \in \mathbb{R} \times C$, we consider an $\eta > 0$ and all functions x on $[t_0-\tau, t_0+A]$ which are continuous and coincide with ψ on $[t_0-\tau, t_0]$, that is, $x_{t_0} = \psi$. The values of these functions on $[t_0, t_0+\eta]$ are restricted to the class of x such that $|x(t)-\psi(0)| < \delta$ for $t \in [t_0, t_0+\eta]$. The usual mapping S obtained from the corresponding integral equation is defined and it is then shown that η and δ can be so chosen that S maps this class into itself and is completely continuous. Thus, Schauder's fixed-point theorem implies existence (for examples details see the books [58, 59, 72]).

Theorem 2.8 (Existence [59]) *In (2.7), suppose Ω is an open subset in $\mathbb{R} \times C$ and f is continuous on Ω. If $(t_0, \psi) \in \Omega$, then there is a solution of (2.7) passing through (t_0, ψ).*

Definition 2.19 We say $f(t, \psi)$ is Lipschitz in ψ in a compact set X of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \psi_i) \in K$, $i = 1, 2$,

$$|f(t, \psi_1) - f(t, \psi_2)| \leq k \, |\psi_1 - \psi_2| \,. \tag{2.12}$$

Theorem 2.9 (Uniqueness [59]) *Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω. If $(t_0, \psi) \in \Omega$, then there is a unique solution of (2.7) through (t_0, ψ).*

2.3.2 Neutral differential equations

In order to define a general class of neutral delay differential equations NDDEs (or neutral functional differential equations NFDEs), we need the definition of atomic.

Definition 2.20 Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, ψ). A function $\Psi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Ψ is continuous together with its first and second Fréchet derivatives with respect to ψ and $\Psi\psi$, the derivative with respect to ψ, is atomic at β on Ω.

Definition 2.21 Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $\Psi : \Omega \to \mathbb{R}^n$ are given continuous functions with Ψ atomic at zero. The equation

$$\frac{d}{dt}\Psi(t, x_t) = f(t, x_t), \tag{2.13}$$

is called the neutral delay differential equation $NDDE(\Psi, f)$.

Definition 2.22 A function x is said to be a solution of the $NDDE(\Psi, f)$ or Equation (2.13), if there are $t_0 \in \mathbb{R}$, $A > 0$, such that $x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n)$, $(t, x_t) \in \Omega$, $t \in [t_0, t_0 + A)$, $\Psi(t, x_t)$ is continuously differentiable and satisfies (2.13) on $[t_0, t_0 + A)$. For a given $t_0 \in \mathbb{R}$, $\psi \in C$, and $(t_0, \psi) \in \Omega$, we say $x(t_0, \psi)$ is a solution of (2.13) with initial value ψ at t_0, or simply a solution through (t_0, ψ) ,if there is an $A > 0$ such that $x(t_0, \psi)$, is a solution of (2.13) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \psi) = \psi$.

Theorem 2.10 (Existence [59]) *If Ω is an open set in $\mathbb{R} \times C$ and $(t_0, \psi) \in \Omega$, then there exists a solution of the $NDDE(\Psi, f)$ through (t_0, ψ).*

Theorem 2.11 (Uniqueness [59]) *If $\Omega \subseteq \mathbb{R} \times C$ is open and $f : \Omega \to \mathbb{R}^n$ as Lipschitz in ψ on compact sets of Ω, then, for any $(t_0, \psi) \in \Omega$, there exists a unique solution of the $NDDE(\Psi, f)$ through (t_0, ψ).*

Example 2.4 The equations

$$\begin{aligned} x'(t) &= 3x'(t-5), \\ x'(t) &= -x(t) + \left[x'(t-4) + 1\right]^2, \\ x'(t) &= x(t) + x'(t-2) - x'(t-7), \end{aligned}$$

are neutral differential equations.

2.3.3 Method of Steps

The method of steps is an elementary method that can be used to solve some $DDEs$ analytically. This method is usually discarded as being too tedious, but in some cases the tedium can be removed by using computer algebra (see [61]). Consider the following general DDE

$$x'(t) = a_0 x(t) + a_1 x(t - w_1) + ... + a_m x(t - w_m), \tag{2.14}$$

where $x(t) = \psi(t)$ on the initial interval $-\max(w_i) \leq t \leq 0$. Let $b = \min(w_i)$. Then it is clear that the values of $x(t - w_m)$ are known in the interval $0 \leq t \leq b$. These values are $\psi(t - w_m)$. Thus, for the interval $0 \leq t \leq b$ we have

$$x'(t) = a_0 x(t) + a_1 \psi(t - w_1) + ... + a_m \psi(t - w_m),$$

and so

$$x(t) = \int_0^t \left(a_0 x(v) + a_1 \psi(v - w_1) + ... + a_m \psi(v - w_m)\right) dv + x(0).$$

Now that we know $x(t)$ on $[0, b]$ we can repeat this procedure to obtain $x(t)$ on the interval $b \leq t \leq 2b$. This is given by

$$x(t) = \int_b^t \left(a_0 x(v) + a_1 \psi(v - w_1) + ... + a_m \psi(v - w_m)\right) dv + x(b). \tag{2.15}$$

This process can be continued indefinitely, so long as the integrals that occur can be evaluated without too much effort. It is this last restriction that usually causes people to give up on this method, because the tedium and length of the method quickly overwhelms a human computer. However, it turns out that for certain classes of problems, where the phenomenon of "expression swell" is not too serious, we can take the method quite far, with a computer algebra system to automate the solution of the tedious sub-problems.

Example 2.5 Consider the following delay differential equation

$$x'(t) = 3x(t-5),\ t \geq t_0. \tag{2.16}$$

For example, assume that $t_0 = 0$. If we want to solve the system on time $t = 0$, it is clear that we will need to know the value of the system at time $t = -5$. To know the system at time $t = 5$, it is essential to know its value at time $t = 0$.

Therefore, to have a complete solution of $t = 0$ to $t = 5$, we are obliged to know a whole initial solution of t $t = -5$ to $t = 0$. So, we must have an initial function defined on $[-5, 0]$. Let ψ be an initial datum and assume that $\psi(t) = 2,\ t \in [-5, 0]$. Now, we have enough information to find a solution, $x(t, 0, \psi)$, on $[0, 5]$. Now $0 \leq t \leq 5$ implies $-5 \leq t - 5 \leq 0$ and therefore $x(t-5) = \psi(t-5) = 2$.

For $0 \leq t \leq 5$ the (2.16) becomes $x'(t) = 6$ and the general solution will be

$$x(t) = 6t + c, \quad c = \text{constant}.$$

By replacing t by zero in this last equation, we find that $c = 2$. Thus, we will have

$$x(t) = 6t + 2,\ 0 \leq t \leq 5.$$

Now, we can use this information to progress and solve (2.16) for future times, for example for $t \in [5, 10]$. Here we see that

$$\begin{aligned} x'(t) &= 3(6(t-5)+2) \\ &= 18t - 84. \end{aligned}$$

She has the general solution

$$x(t) = 9t^2 - 84t + d, \quad d = \text{constant}.$$

Using the solution we found for $t \in [0,5]$, and exactly in $t = 5$ we get the value $d = 227$. Therefore

$$x(t) = 9t^2 - 84t + 227, \; 5 \leq t \leq 10.$$

We can progress in this direction and solve the delay equation as an infinite series of EDO. This method can be programmed in Maple using a simple for loop.

2.3.4 Problems with a delay

In this section we give two examples of physical and biological systems in which the present rate of change of some unknown function depends upon past values of the same function.

Epidemics (Cooke and Yorke)

In the work of Cooke and Yorke (1973) the Lotka assumption is changed so that the number of births per unit time is a function only of the population size, not of the age distribution (see [33]). Under this assumption, we let $x(t)$ be the population size and let the number of births be $B(t) = g(x(t))$. Assume each individual has life span L so that the number of deaths per unit time is $g(x(t-L))$. Then the population size is described by

$$x'(t) = g(x(t)) - g(x(t-L)), \tag{2.17}$$

where g is some differentiable function. We note that every constant function is a solution of (2.17).

The following model for the spread of gonorrhea is considered by Cooke and Yorke (1973). The population is divided into two classes:

(a) $S(t)$ the number of susceptible, and

(b) $x(t)$ the number of infectious.

The rate of new infection depends only on contacts between susceptible and infectious individuals. Since $S(t)$ equals the constant total population minus $x(t)$, the rate is some function $g(x(t))$. Assume that an exposed individual is immediately infectious and stays infectious for a period L (the time for treatment and cure). Then x also satisfies (2.17) holds. Now, at any time t, $x(t)$ equals the sum of capital produced over the period $[t-L,t]$ plus a constant c denoting the value of nondepreciating assets. Thus,

$$\begin{aligned} x(t) &= \int_0^L P(s)g\left(x(t-s)\right)ds + c \\ &= \int_{t-L}^t P(t-u)g\left(x(u)\right)du + c. \end{aligned} \tag{2.18}$$

Controlling a ship

Minorsky (1962) designed an automatic steering device for the battleship New Mexico. The following is a sketch of the problem (see [33]). Let the rudder of the ship have angular position $x(t)$ and suppose there is a friction force proportional to the velocity, say $-cx'(t)$. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected by a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h > 0$ between the time the ship gets off course and the time the electric motor activates the restoring force. The equation for $x(t)$ is

$$x''(t) + cx'(t) + g(x(t-h)) = 0, \tag{2.19}$$

where $xg(x) > 0$ if $x \neq 0$ and c is a positive constant. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course.

2.4 Stability of delay differential equations

The theory of stability was created by the Russian mathematician Lyapunov (1857-1918). This theory has found wide application in various fields of physics and mathematical sciences. From a mathematical point of view, the theory of stability presents a particular case of the qualitative theory of differential equations. Lyapunov's method was the ultimate object for studying stability for differential equations and partial differential equations. Nevertheless, this method has encountered serious obstacles and there are still a lot of problems that resist this technique.

The simplest notion of stability is the one related to stability of equilibrium points.

Definition 2.23 A point $x(t) = x_e$ in the state space is said to be an equilibrium point of the autonomous system $x' = f(x)$ if and only if it has the property that whenever the state of the system starts at x_e, it remains at x_e for all future time.

According to the definition, the equilibrium points are the real roots of the equation $f(x_e) = 0$. This is made clear by noting that if $x'_e = f(x_e) = 0$, then it follows that x_e is constant and, by definition, an equilibrium point. Without loss of generality, we assume that 0 is an equilibrium point of the system. If the equilibrium point under study, x_e, is not at zero we may define a new (shifted) coordinate system $x_s(t) = x(t) - x_e$ and note that

$$x'_s(t) = x'(t) = f(x(t)) = f(x_s(t) + x_e) =: f_s(x_s(t)), x_s(0) = x(0) - x_e.$$

The claim follows by noting that $f_s(0) = f(x_e) = 0$. In summary, the study of the zero equilibrium point of $x'_s(t) = f_s(x_s(t))$ is equivalent to the study of the nonzero equilibrium point x_e of $x'(t) = f(x(t))$.

We consider the system

$$x' = f(t, x_t) \text{ for all } t \geq t_0, \tag{2.20}$$

$$x(t) = \psi(t) \text{ for all } t_0 - \tau \leq t \leq t_0, \tag{2.21}$$

where $f : (-\infty, +\infty) \times C \to \mathbb{R}^n$, with $C = C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions $\psi : [t_0 - \tau, t_0] \to R^n$, $\tau > 0$ equipped with the supremum norm $\|\psi\| = \sup_{-\tau \leq t \leq 0} |\psi(t)|$. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution $x(t, t_0, \psi)$ through of the problem (2.20)-(2.21) and to be continuous in (t, t_0, ψ) of the definition domain of f (see Hale [58]).

Definition 2.24 Suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}$.

1) The trivial solution $x(t) = 0$ of (2.20) is Stable in t_0 ($t_0 \in \mathbb{R}$), if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that if $\|\psi\| < \delta$, the solution of (2.20)-(2.21) exists on $[t_0 - \tau, \infty)$ and $|x(t, t_0, \psi)| < \epsilon$ for all $t \geq t_0$. Otherwise we will say that the solution is unstable in t_0.

2) The solution $x = 0$ of (2.20) is said to be uniformly stable if the number $\delta = \delta(\epsilon)$ is independent of t_0.

3) The trivial solution $x(t) = 0$ of (2.20) is Asymptotically stable in t_0 if it is stable in t_0 and if there exists $\delta_1 = \delta_1(t_0) > 0$ such that whenever all $\|\psi\| < \delta_1$, the solution of the problem (2.20)-(2.21) satisfies

$$|x(t, t_1, \psi)| \to 0 \text{ as } t \to \infty.$$

4) The trivial solution $x(t) = 0$ of (2.20) is Asymptotically uniformly stable in t_0 if it is uniformly stable and if there exist $\delta_1 > 0$ (independent of t_0) as for all t_0 and $\|\psi\| < \delta_1$ the solution x of the problem (2.20)-(2.21) satisfies the condition $x(t, t_0, \psi) \to 0$ when $t \to \infty$, in the following way: for all $\eta > 0$ there is a $T = T(\eta) > 0$ such that $|x(t, t_0, \psi)| < \eta$ for all $t \geq t_0 + T$.

Remark 2.2 If all the solutions tend to zero, then $x = 0$ is globally asymptotically stable.

Example 2.6 Consider, for $t \geq 1$, the differential delay equation

$$\begin{aligned} x' &= \frac{1}{t}x(t) - \frac{27t}{(t+2)^3}x^3\left(\frac{t+2}{3}\right) \\ &= \frac{1}{t}x(t) - \frac{27t}{(t+2)^3}x^3(t - \tau(t)), \end{aligned}$$

with $\tau(t) = \frac{2}{3}(t-1)$, with the initial condition $x(1) = x_0$. We easily check that the unique solution of this problem is

$$x(t) = x_0 t \exp\left(-x_0^2(t-1)\right), \text{ for all } t \geq 1.$$

Thus, $\lim_{t\to\infty} x(t) = 0$, for all x_0. Suppose that $|x_0| = \delta$, then

$$\left|x\left(1 + \frac{1}{\delta^2}\right)\right| = \frac{1}{e}\left(\delta + \frac{1}{\delta}\right) \geq \frac{2}{e}.$$

Therefore, for all δ the solution is outside the ball $|x| < \dfrac{2}{e}$, in times $t = 1 + \dfrac{1}{\delta^2}$, and the solution is unstable.

2.4.1 The method of Lyapunov functionals

If $V : \mathbb{R} \times C \to \mathbb{R}$ is continuous and $x(t, t_0, \psi)$ is the solution of Equation (2.20) through (t_0, ψ), we define

$$V'(t, \psi) = \lim_{h \to 0^+} \sup \frac{1}{h} \left[V(t+h, x_{t+h}(t, \psi)) - V(t, \psi) \right].$$

The function $V'(t, \psi)$ is the upper right-hand derivative of $V(t, \psi)$ along the solution of Equation (2.20).

Theorem 2.12 ([58]) *Suppose $f : \mathbb{R} \times C \to \mathbb{R}$ takes $\mathbb{R}\times$(bounded sets of C) of $\mathbb{R} \times C$ into bounded sets of $\mathbb{R}^n$, and $u, v, w : [0, +\infty) \to [0, +\infty)$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times C \to \mathbb{R}$ such that*

$$\begin{aligned} u(|\psi(0)|) &\leq V(t, \psi) \leq v(|\psi|), \\ V'(t, \psi) &\leq -w(|\psi(0)|), \end{aligned}$$

then the solution $x = 0$ of Equation (2.20) is uniformly stable. If $w(s) > 0$ for $s > 0$, then $x = 0$ is uniformly asymptotically stable.

Definition 2.25 (Exponentially stable [36]) The trivial solution of (2.20) is said to be exponentially stable if it is stable and there exists a positive constant λ such that for any $\psi \in C = C([-\tau, 0], \mathbb{R}^n)$ there exists K (which may depend on ψ) such that $|x(t, t_0, \psi)| \leq Ke^{-\lambda t}$ for $t > 0$.

2.4.2 The method of fixed point theory

When one wants to study the stability of the trivial solution of a differential equation with delay by the method of fixed point one will have to proceed as follows

1) A delay differential equation requires primarily a an initial function defined on an appropriate initial interval I_{t_0} i.e. $\psi : I_{t_0} \to \mathbb{R}^n$. We must fall immediately after a suitable space C of functions $\varphi : I_{t_0} \cup [t_0, +\infty) \to \mathbb{R}^n$ which coincide on I_{t_0} with ψ. According to the case of needs we can always add other restrictions to the functions φ of C such as the

boundary or the condition $\varphi(t) \to \infty$ when $t \to \infty$. This last condition is necessary if we wish to study asymptotic stability.

2) Then we have to invert the differential equation to define what we call a fixed point application i.e., a mapping $S : C \to C$ whose fixed point is the solution of the given delay equation (the original equation).Nevertheless, this inversion can be a delicate task in many cases. For example if the equation does not have a linear term in its structure we will not be able to use the variation of the parameters. It is therefore essential to act differently and to try if a transformation of this equation is possible.

3) A fixed point theorem must be chosen allowing the equation $S(x) = x$ to have a solution. Especially if S is a contraction we can apply the Banach fixed point theorem, if S is compact then we will apply the theorem of Schauder or Schaeffer and if S is puts in the form of a sum of a contraction and a compact application then the Krasnoselskii hybrid theorem can give satisfaction. It thus becomes clear that the stability method by the fixed point method relies on three essential things, the variation of the parameters, a complete space and a fixed point theorem.n one stage we can conclude the existence (or even uniqueness) and stability.n addition, it will be seen that this method always requires conditions on average however the conditions of the Lyapunov method are always punctual.

2.4.3 A comparison between fixed point and Lyapunov theory

Burton has proved that many of these problems can be solved using fixed point theory. we will recall some examples of comparison methods from the paper of Burton [31].

Let $a : [0, +\infty) \to \mathbb{R}$ be bounded and continuous function, let τ be a positive constant, and let

$$x'(t) = -a(t)x(t - \tau). \tag{2.22}$$

Although we can treat solutions with any initial time, we will always look at a solution $x(t) := x(t, 0, \psi)$ where $\psi : [-\tau, 0] \to \mathbb{R}$ is a given continuous initial function and $x(t, 0, \psi) = \psi(t)$ on $[-\tau, 0]$. It is then known that there is a unique continuous solution $x(t)$ satisfying (2.22) for $t > 0$ and with $x(t) = \psi(t)$ on $[-\tau, 0]$. With such ψ in mind, we can write (2.22) as

$$x'(t) = -a(t)x(t+\tau) + \frac{d}{dt}\int_{t-\tau}^{t} a(s+\tau)x(s)ds, \tag{2.23}$$

so that by the variation of parameters formula, followed by integration by parts, we obtain

$$\begin{aligned} x(t) &= x(0)\, e^{-\int_0^t a(s+\tau)ds} + \int_{t-\tau}^{t} a(u+\tau)x(u)du - e^{-\int_0^t a(u+\tau)du}\int_{-\tau}^{0} a(u+\tau)x(u)du \\ &\quad - \int_0^t a(s+\tau)e^{-\int_s^t a(u+\tau)du}\int_{s-\tau}^{s} a(u+\tau)x(u)duds. \end{aligned} \tag{2.24}$$

In a space to be defined and with a mapping defined from (2.24) we will find that we have a contraction mapping just in case there is a constant $\alpha < 1$ with

$$\int_{t-\tau}^{t} |a(u+\tau)|\, du + \int_0^t |a(s+\tau)|\, e^{-\int_s^t a(u+\tau)du}\int_{s-\tau}^{s} |a(u+\tau)|\, duds \le \alpha. \tag{2.25}$$

As we are interested in asymptotic stability we will need

$$\int_{t-\tau}^{t} a(u+\tau)du \to 0 \text{ as } t \to \infty. \tag{2.26}$$

Burton, in his paper, compared results from a certain application of fixed point theory with a certain common Lyapunov functional. In theory, there is no comparison at all. It is known that if we have a strong type of stability, then there exists a Lyapunov functional of a certain type. The fact that we can not find that Lyapunov functional gives validity to this type of comparison. With that in mind, from (2.25) it is easy to see one of the advantages of fixed point theory over Lyapunov theory. The latter requires $a(t+\tau) > 0$. If $a(t+\tau) \ge 0$, then a very good bound is obtained in (2.25) with little effort. If $a(t+\tau)$ changes the sign then (2.25) can still hold, although a good bound on the second integral is more difficult.

Burton proved in [31] the following result but were unable to do so and he left the principle difficulty as a hypothesis.

Theorem 2.13 ([31]) *Let (2.25) and (2.26) hold. Then for every continuous initial function $\psi : [-\tau, 0] \to \mathbb{R}$, the solution $x(t, 0, \psi)$ is bounded and tends to zero as $t \to \infty$.*

Proof. Let $(\mathbb{B}, \|.\|)$ be the Banach space of bounded and continuous functions $\psi : [-\tau, 0] \to \mathbb{R}$ with the supremum norm. Let $(X, \|.\|)$ be the complete metric space with supremum norm consisting of functions $\varphi \in X$ such that $\varphi(t) = \psi(t)$ on $[-\tau, 0]$ and $\varphi(t) \to 0$ as $t \to \infty$. Define $S : X \to X$ by

$$(S\varphi)(t) = \psi(t), \ t \in [-\tau, 0],$$

and

$$(S\varphi)(t) = \psi(0) e^{-\int_0^t a(s+\tau)ds} + \int_{t-\tau}^{t} a(u+\tau)\varphi(u)du - e^{-\int_0^t a(u+\tau)du} \int_{-\tau}^{0} a(u+\tau)\psi(u)du$$
$$- \int_0^t a(s+\tau)e^{-\int_s^t a(u+\tau)du} \int_{s-\tau}^{s} a(u+\tau)\varphi(u)duds,$$

Clearly, S is continuous, $(S\varphi)(0) = \psi(0)$, and from (2.25) it follows that S is bounded. Also, S is a contraction by (2.25).

We can show that the last term tends to zero by using the classical proof that the convolution of an L_1-function with a function tending to zero, does also tend to zero.

Here are the details. Let $\varphi \in X$ be fixed and let $0 < T < t$. Denote the supremum of $|\varphi|$ by $\|\varphi\|$ and the supremum of $|\varphi|$ on $[T, +\infty)$ by $\|\varphi\|_{[T,+\infty)}$. Consider (2.25) and (2.26).We have

$$\int_0^t |a(s+\tau)| e^{-\int_s^t a(u+\tau)du} \int_{s-\tau}^{s} |a(u+\tau)\varphi(u)| duds$$
$$\leq \int_0^T |a(s+\tau)| e^{-\int_s^T a(u+\tau)du} \int_{s-\tau}^{s} |a(u+\tau)\psi(u)| duds \|\varphi\| e^{-\int_T^t a(u+\tau)du}$$
$$+ \int_T^t |a(s+\tau)| e^{-\int_s^T a(u+\tau)du} \int_{s-\tau}^{s} |a(u+\tau)| duds \|\varphi\|_{[T-\tau,+\infty)}$$
$$\leq \alpha \|\varphi\| e^{-\int_T^t a(u+\tau)du} + \alpha \|\varphi\|_{[T-\tau,+\infty)}.$$

For a given $\epsilon > 0$ take T so large that $\alpha \|\varphi\|_{[T-\tau,+\infty)} < \frac{\epsilon}{2}$. For that fixed T, take t^* so large that $\alpha \|\varphi\|_{[T-\tau,+\infty)} < \frac{\epsilon}{2}$ for all $t > t^*$. We then have that last term smaller ϵ than for all $t > t^*$. Thus, $S : X \to X$ is a contraction with unique fixed point in X. ■

Example 2.7 ([31]) In (2.22) let

$$a(t) = 1.1 + \sin t.$$

The conditions of Theorem 2.13 are satisfied if

$$2(1.1\tau + 2\sin\left(\frac{\tau}{2}\right)) < 1.$$

2.4. Stability of delay differential equations

This is approximated by $0 < \tau < 0.2$. We obtain the conclusion of Theorem 2.13, i.e the zero solution of (2.22) is asymptotically stable.

Example 2.8 ([31]) In (2.22), let

$$a(t) = 1 + \sin t.$$

The conditions of Theorem 2.13 are satisfied if

$$2(\tau + 2\sin(\tau/2)) < 1.$$

This is approximated by $0 < \tau < 0.25$, i.e the zero solution of (2.22) is asymptotically stable.

Example 2.9 ([31]) In (2.22), let

$$a(t) = 1 + 2\sin t,$$

with $0 < \tau < 1$. Then the conditions of Theorem 2.13 are satisfied if

$$(\tau + 4\sin(\tau/2))(2 + 2e^2) < 1. \tag{2.27}$$

This is approximated by $0 \leq \tau < 0.02$, i.e the zero solution of (2.22) is asymptotically stable.

Equation (2.22) can be written in the form

$$x'(t) = -a(t+\tau)x(t) + \frac{d}{dt}\int_{t-\tau}^{t} a(s+\tau)x(s)ds. \tag{2.28}$$

Then Equation (2.22) is equivalent to

$$\left(x(t) - \int_{t-\tau}^{t} a(s+\tau)x(s)ds\right)' = -a(t+\tau)x(t).$$

Let's choose the functional Lyapunov $V(t, x_t) = V_1(t, x_t) + V_2(t, x_t)$, where

$$\begin{aligned} V_1(t, x_t) = &\left(x(t) - \int_{t-\tau}^{t} a(s+\tau)x(s)ds\right)^2 \\ &- \int_{-\tau}^{0}\left(\int_{t+\tau}^{t} a(u+\tau)x^2(u)du\right)ds, \end{aligned}$$

and $\lambda > 0$

$$V_2(t, x_t) = \lambda\left[x^2(t) + \int_{t-\tau}^{t} a(s+\tau)x^2(s)ds\right].$$

Theorem 2.14 ([31]) *Suppose that*

$$a(t+\tau) \geq 0,\ \textit{for all } t \geq 0 \textit{ and } \int_0^\infty a(s)ds = \infty, \tag{2.29}$$

and there exist $\varepsilon > 0$ *with*

$$a(t+\tau)\int_{t-\tau}^{t} a(s+\tau)ds - 2 + \tau \leq \varepsilon \textit{ for all } t \geq 0, \tag{2.30}$$

and there exist $\lambda > 0$ *where*

$$\lambda\left[a(t) + a(t+\tau)\right] \leq \frac{\varepsilon}{2}a(t+\tau) \textit{ for all } t \geq 0, \tag{2.31}$$

then the zero solution of (2.22) is asymptotically stable.

Example 2.10 ([31]) In (2.22), let

$$a(t) = 1.1 + \sin t.$$

then the equation (2.22) becomes

$$x'(t) = -\left(1.1 + \sin t\right) x(t-\tau).$$

Theorem 2.14 remains valid if there exist $\varepsilon > 0$ checking

$$2.1(1.1 + 2\sin(\frac{\tau}{2})) - 2 + \tau < \varepsilon.$$

Taking sin $\sin\left(\frac{\tau}{2}\right) = \frac{\tau}{2}$, we get from the last inequality the estimate $\tau < 0.37$. So, if $\tau < 0.37$ the zero solution of (2.22) is asymptotically stable.

Remark 2.3 In Example 2.7 we saw that the solution of equation (2.22) with $a(t) = 1.1 + \sin t$ is asymptotically stable if $\tau < 0.2$ with the fixed point method.However, the same conclusion is obtained with $\tau < 0.37$ by the method of Lyapunov according to Example 2.10. with have $1 + 2\sin t$ changes sign for $t \geq 0$ and consequently Theorem 2.14 relating to the Lyapunov method is not applicable under these conditions for this example.However, according to Example 2.9 one obtains by the method of fixed point the asymptotic stability of the trivial solution of the equation (2.22) if $(\tau + 4\sin(\tau/2))(2 + 2e^2) < 1$.

On top of that we also know that the construction of functional Lyapounov is not an easy thing.There is not a general method valid for all differential equations. For more information and examples of this comparison, see Burton's book ([30]).

2.5 Elements of calculus on time scales

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing. The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [19], [20], [74] most of the material needed to read this work. We start by giving some definitions necessary for our work. All the definitions, theorem, notations, and basic results that are used in this section can be found in [20].

2.5.1 Description of time scales

Definition 2.26 A time scale is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$ is denoted by $\mathbb{T}$.

Example 2.11 The reals $\mathbb{R}$, the integers $\mathbb{Z}$, the positive integers $\mathbb{N}$, and the nonnegative integers $\mathbb{N}_0$ are a time scales . The most common time scales are $\mathbb{T} = \mathbb{R}$ for continuous calculus, $\mathbb{T} = \mathbb{Z}$ for discrete calculus, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$, for quantum calculus.

Example 2.12 The rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R}\diagdown\mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval $(0, 1)$, are not time scales.

We Assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

Definition 2.27 The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined, respectively, by

$$\begin{aligned} \sigma(t) &= \inf \{s \in \mathbb{T} : s > t\}, \\ \rho(t) &= \sup \{s \in \mathbb{T} : s < t\}. \end{aligned}$$

A point $t \in \mathbb{T}$ is called right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, and left-scattered if $\rho(t) < t$.

Here it is assumed that $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ contains the maximal element t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ contains the minimal element t).

Points that are right-scattered and left-scattered at the same time are called isolated. Points that are right-dense and left-dense at the same time are called dense.

In addition to the set $\mathbb{T}$, the set $\mathbb{T}^k$ is defined as follows. If $\mathbb{T}$ contains the left scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$, and $\mathbb{T}^k = \mathbb{T}$ in the other cases. Therefore,

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash\{\rho(\sup \mathbb{T}), \sup \mathbb{T}\} & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

The difference from an arbitrary element $t \in \mathbb{T}$ to its forward is called the graininess of the time scale $\mathbb{T}$ and is given by the formula

$$\mu(t) = \sigma(t) - t.$$

Example 2.13 1) If $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\rho(t) = t$ and $\mu(t) = 0$ for all $t \in \mathbb{T}$. Hence every point $t \in \mathbb{R}$ is dense.

2) If $\mathbb{T} = h\mathbb{Z}$ $(h \neq 0)$, $\sigma(t) = t + h$, $\rho(t) = t - h$ and $\mu(t) = h$ for all $t \in \mathbb{T}$. Hence if $h > 0$ every point $t \in \mathbb{Z}$ is isolated.

3) If $\mathbb{T} = q^{N_0} = \{q^n : n \in \mathbb{N}_0\} \cup \{0\}$ $(h \neq 0)$, $\sigma(t) = qt$, $\rho(t) = \dfrac{1}{q}t$ and $\mu(t) = q\,(t-1)$ for all $t \in \mathbb{T}$. Hence if $q > 1$ every point $t \in q^{N_0}$ is isolated.

4) If $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, $\sigma(t) = \dfrac{t}{1-t}$, $\rho(t) = \dfrac{t}{t+1}$ and $\mu(t) = \dfrac{t^2}{1-t}$ for all $t \in \mathbb{T} - \{1\}$.

5) If $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}_0\}$, $\sigma(t) = \left(\sqrt{t}+1\right)^2$, $\rho(t) = \left(\sqrt{t}-1\right)^2$ and $\mu(t) = 2\sqrt{t}+1$ for all $t \in \mathbb{T}$.

Definition 2.28 If $f : \mathbb{T} \to \mathbb{R}$ we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T},$$

i.e., $f^\sigma = f \circ \sigma$.

The notion of periodic time scales is introduced in Kaufmann and Raffoul [69]. The following definitions are borrowed from [69].

Definition 2.29 We say that a time scale $\mathbb{T}$ is periodic if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Example 2.14 ([69]) The following time scales are periodic.

1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih],\ h > 0$ has period $p = 2h$.
2) $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.
3) $\mathbb{T} = \mathbb{R}$.
4) $\mathbb{T} = \{t = k - q^m : k \in Z,\ m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $p = 1$.

Remark 2.4 ([69]) All periodic time scales are unbounded above and below.

Definition 2.30 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t \pm \omega) = f(t)$ for all $t \in \mathbb{T}$ and ω is the smallest number such that $f(t \pm \omega) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t \pm \omega) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.5 ([69]) If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

2.5.2 Differentiation

The theory of dynamic equations at time scales was introduced in 1988 by Stefan Hilger in his doctoral dissertation where he defined the Δ-derivated as follows.

Definition 2.31 ([20]) For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)\,(\sigma(t) - s)\right| \leq \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

If f is derivative for all $t \in \mathbb{T}$, then the function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of f on $\mathbb{T}^k$.

Example 2.15 (1) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = a$ for all $t \in \mathbb{T}$, where $a \in \mathbb{R}$ is constant, then $f^{\Delta}(t) = 0$. This is clear because for any $\varepsilon > 0$,

$$|(f(\sigma(t)) - f(s)) - 0\,(\sigma(t) - s)| = |a - a| = 0 \le \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

(2) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) = 1$. This is clear because for any $\varepsilon > 0$,

$$|(f(\sigma(t)) - f(s)) - 1\,(\sigma(t) - s)| = 0 \le \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

Theorem 2.15 ([20]) *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t with

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = \mu(t) f^{\Delta}(t) + f(t).$$

Example 2.16 1) If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(x)$ for all $t \in \mathbb{R}$.

2) If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \dfrac{f(t+h) - f(t)}{h} = \Delta f$ for all $t \in \mathbb{R}$.

3) If $\mathbb{T} = q^{\mathbb{N}_0}$ and $f(t) = t^2$, then $f^{\Delta}(t) = qt + t$ for all $t \in \mathbb{T}$.

4) If $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, and $f(t) = \log(t)$, then $f^{\Delta}(t) = \dfrac{\log q}{t(q-1)}$ for all $t \in \mathbb{T}$.

5) If $\mathbb{T} = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$, and $f(t) = \log(t)$, then $f^{\Delta}(t) = \dfrac{t-1}{t^2} \log(1-t)$ for all $t \in \mathbb{T} - \{1\}$.

Theorem 2.16 ([20]) *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}$. Then*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) *For any constant a, $af : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(af)^{\Delta}(t) = af^{\Delta}(t).$$

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(\sigma(t)) + f(t)\, g^{\Delta}(t) \\ &= f^{\Delta}(t)g(t) + f(\sigma(t))\, g^{\Delta}(t).\end{aligned}$$

(iv) *If $f(t)f(\sigma(t)) \neq 0$, then $\dfrac{1}{f}$ is differentiable at t with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) *If $g(t).g(\sigma(t)) \neq 0$, then $\dfrac{f}{g}$ is differentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 2.17 1) Let $a \in \mathbb{R}$ and $m \in \mathbb{N}$, $f : \mathbb{T} \to \mathbb{R}$ defined by $f(t) = (t-a)^m$ we have

$$f^{\Delta}(t) = \sum_{k=0}^{m-1} (\sigma(t) - a)^k (t-a)^{m-k-1}.$$

2) Let $a \in \mathbb{R}$ and $m \in \mathbb{N}$, $g : \mathbb{T} - \{a\} \to \mathbb{R}$ defined by $g(t) = \dfrac{1}{(t-a)^m}$ provided $(t-a)(\sigma(t)-a) \neq 0$ we have

$$g^{\Delta}(t) = \sum_{k=0}^{m-1} \frac{1}{(\sigma(t)-a)^{m-k}(t-a)^{k+1}}.$$

Remark 2.6 Assume $f, g : \mathbb{Z} \to \mathbb{Z}$ are defined by $f(t) = t^2$ and $g(t) = 2t$, then

$$(fog)^{\Delta}(t) = \Delta(fog)(t) = 4(t+1)^2 - 4t^2 = 8t + 4,$$

and

$$f^{\Delta}(g(t))g^{\Delta}(t) = (2g(t)+1)(2(t+1) - 2t) = 8t + 2.$$

Thus

$$(fog)^{\Delta}(t) \neq f^{\Delta}(g(t))g^{\Delta}(t).$$

Theorem 2.17 ([20]) *Suppose* $f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t_0 \in \mathbb{T}\backslash\{\max \mathbb{T}\}$. *If* $f^{\Delta}(t_0) > 0$, *then* f *is right-increasing. If* $f^{\Delta}(t_0) < 0$, *then* f *is right-decreasing.*

Let $f : \mathbb{T} \to \mathbb{R}$ be a strictly increasing function such that $\widetilde{\mathbb{T}} = f(\mathbb{T})$ is also a time scale. By $\widetilde{\sigma}$ we denote the jump function on $\widetilde{\mathbb{T}}$ and by $\widetilde{\Delta}$ we denote the derivative on $\widetilde{\mathbb{T}}$. then $f \circ \sigma = \widetilde{\sigma} \circ f$.

Theorem 2.18 ([20]) *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is strictly increasing,* $\widetilde{\mathbb{T}} = f(\mathbb{T})$ *is a time scale and* $g : \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $f^{\Delta}(t)$ *and* $g^{\widetilde{\Delta}}(f(t))$ *exist for* $t \in \mathbb{T}^k$, *then*

$$(g \circ f)^{\Delta} = \left(g^{\widetilde{\Delta}} \circ f\right) f^{\Delta}.$$

2.5.3 Integration

Definition 2.32 A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.33 A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Theorem 2.19 (Existence of pre-antiderivatives [20]) *Let* f *be regulated. Then there exists a function* F *which is re-differentiable with region of differentiation* D *such that*

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in D.$$

Definition 2.34 Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f^{\Delta}(t)\Delta t = F(t) + c,$$

where c is an arbitrary constant and F is a pre-antiderivative of f. For all $a, b \in \mathbb{T}$, the Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}.$$

Theorem 2.20 (Existence of antiderivatives [20]) *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then an antiderivative of f is F defined by*

$$F(t) = \int_{t_0}^t f(r)\Delta r \text{ for all } t \in \mathbb{T}.$$

The following theorem gives several elementary properties of the delta integral.

Theorem 2.21 ([20]) *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

(i) $\int_a^b [f(r) + g(r)]\,\Delta r = \int_a^b f(r)\Delta r + \int_a^b g(r)\Delta r$,

(ii) $\int_a^b [\alpha f(r)]\,\Delta r = \alpha \int_a^b f(r)\Delta r$,

(iii) $\int_a^b f(r)\Delta r = -\int_b^a f(r)\Delta r$,

(iv) $\int_a^b f(r)\Delta r = \int_a^c f(r)\Delta r + \int_c^b f(r)\Delta r$,

(v) $\int_a^b f(\sigma(r))g^{\Delta}(r)\Delta r = (fg)(b) - (fg)(a) + \int_a^b f^{\Delta}(r)g(r)\Delta r$,

(vi) $\int_a^b f(r)g^{\Delta}(r)\Delta r = (fg)(b) - (fg)(a) + \int_a^b f^{\Delta}(r)g(\sigma(r))\Delta r$,

(vii) $\int_a^a f(r)\Delta r = 0$,

(viii) if $f(t) \geq 0$ *for all* $a \leq t < b$*, then* $\int_a^a f(r)\Delta r \geq 0$,

(ix) if $f(t) \leq g(t)$ *on* $[a, b)$*, then* $\left|\int_a^b f(r)\Delta r\right| \leq \int_a^b g(r)\Delta r$.

Definition 2.35 Infinite integrals are defined as

$$\int_a^{\infty} f(r)\Delta r = \lim_{t\to\infty} \int_a^t f(r)\Delta r.$$

Theorem 2.22 ([20]) *If $f : \mathbb{T} \to \mathbb{R}$ is an arbitrary function and $t \in \mathbb{T}$, then*

$$\int_t^{\sigma(t)} f(r)\Delta r = \mu(t)f(t).$$

Theorem 2.23 ([20]) *Let $a, b \in \mathbb{T}^k$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t, .)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $r \in [a, \sigma(t)]$, such that*

$$\left|f(\sigma(t), r) - f(s, \tau) - f^\Delta(t, r)(\sigma(t) - s)\right| \leq \varepsilon \left|\sigma(t) - s\right| \text{ for all } s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

$$(i)\ g(t) := \int_a^t f(t, r)\Delta r \text{ implies } g^\Delta(t) := \int_a^t f^\Delta(t, r)\Delta r + f(\sigma(t), t),$$

$$(ii)\ h(t) := \int_t^b f(t, r)\Delta r \text{ implies } h^\Delta(t) := \int_t^b f^\Delta(t, r)\Delta r - f(\sigma(t), t).$$

Theorem 2.24 ([20]) *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(r)\Delta r = \int_a^b f(r)dr.$$

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(r)\Delta r = \begin{cases} \sum\limits_{r\in[a,b)} \mu(r)f(r) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum\limits_{r\in[b,a)} \mu(r)f(r) & \text{if } a > b. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, then

$$\int_a^b f(r)\Delta r = \begin{cases} \sum\limits_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum\limits_{k=\frac{b}{h}}^{\frac{a}{h}-1} hf(kh) & \text{if } a > b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(r)\Delta r = \begin{cases} \sum\limits_{k=a}^{b-1} f(r) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum\limits_{k=b}^{a-1} f(r) & \text{if } a > b. \end{cases}$$

Example 2.18 1) For $a, b \in \mathbb{T}$, $\int_a^b \alpha \log k \Delta t = \alpha \log k \int_a^b \Delta t = t|_a^b \alpha \log k = (b-a)\,\alpha \log$.

2) For $t \in \mathbb{T} = h\mathbb{Z}$ $(h > 0)$, $\int_t^{t+h} \cos r \Delta r = \int_t^{\sigma(t)} \cos r \Delta r = h \cos t$.

3) Let $f : \mathbb{T} \to \mathbb{R}$ $(\mu(t) \neq 0)$, $f(t) = \dfrac{\sin(\sigma(t)) - \sin t}{\mu(t)}$, then for $a, b \in \mathbb{T}$,

$\int_a^b f(r) \Delta r = \int_a^b \dfrac{\sin(\sigma(r)) - \sin r}{\mu(r)} \Delta r = \sin r|_a^b = \sin b - \sin a$.

2.5.4 The regressive group

Definition 2.36 A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$.

The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \ \forall t \in \mathbb{T}\}.$$

Definition 2.37 If p,$q \in \mathcal{R}$, then we defined the the function $p \oplus q$ by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \text{ for all } t \in \mathbb{T}^k,$$

and the function $\ominus p$ defined by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \text{ for all } t \in \mathbb{T}^k.$$

Lemma 2.3 ([20]) *the functions are also elements of $\mathcal{R}$.*

Definition 2.38 We define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t) \text{ for all } t \in \mathbb{T}^k.$$

Lemma 2.4 ([20]) *Suppose $p, q \in \mathcal{R}$. Show directly from the definition that*

(i) $p \ominus p = 0$,

(ii) $\ominus(\ominus p) = p$,

(iii) $p \ominus p \in \mathcal{R}$,

(iv) $p \ominus q = \dfrac{p - q}{1 + \mu(t)p(t)}$,

(v) $\ominus(p \ominus q) = q \ominus p$,

(vi) $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Lemma 2.5 ([20]) 1) $(\mathcal{R}, \oplus)$ *is an Abelian group. This group is called the regressive group.*

2) $(\mathcal{R}^+, \oplus)$ *is a subgroup of the regressive group.*

Definition 2.39 Let $p \in \mathcal{R}$, then the generalized exponential function e_p is defined as the unique solution of the initial value problem

$$x^{\Delta}(t) = p(t)x(t), \quad x(s) = 1, \text{ where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \zeta_{\mu(r)}(p(r))\Delta r\right), \text{ for all } s,t \in \mathbb{T}, \tag{2.32}$$

with

$$\zeta_h(r) = \begin{cases} \frac{\log(1+hr)}{h} & \text{if } h > 0, \\ r & \text{if } h = 0, \end{cases}$$

where log is the principal logarithm function. $\zeta_h(r)$ we called the cylinder transformation.

Lemma 2.6 ([20]) *Let $p \in \mathcal{R}$, then*

(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$,

(ii) $[e_p(., s)]^{\Delta} = p(t)e_p(., s)$,

(iii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))\, e_p(t, s)$,

(iv) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$,

(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$,

(vi) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$,

(vii) $\left(\dfrac{1}{e_p(., s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(., s)}$,

(ix) $[e_p(c, .)]^{\Delta} = -p(t)\, [e_p(c, .)]^{\sigma}$ *where* $c \in \mathbb{T}$.

Example 2.19 ([20]) Let $t_0 = 0$, $p(t) = 1$, then,

1) if $\mathbb{T} = \mathbb{R}$, $e_p(t, t_0) = e^t$,

2) if $\mathbb{T} = \mathbb{Z}$, $e_p(t, t_0) = 2^t$,

3) if $\mathbb{T} = h\mathbb{Z}$, $e_p(t, t_0) = (1 + h)^{\frac{t}{h}}$.

Lemma 2.7 ([1]) *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(r)\Delta r\right), \text{ for all } t \in [s, \infty)_{\mathbb{T}}. \tag{2.33}$$

Corollary 2.2 ([1]) *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(r)\Delta r\right) < 1 \text{ for all } t \in [s,\infty)_{\mathbb{T}}.$$

In terms of the exponential function (2.32), there are two variation of constants formulas that read as follows.

Theorem 2.25 (Variation of constants [20]) *Let $f \in C_{rd}$, $t_0 \in \mathbb{T}$, $p \in \mathcal{R}$, and $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$x^{\Delta} = -p\,(t)\,x(\sigma(t)) + f\,(t)\,,\ x\,(t_0) = x_0,$$

is given by

$$x\,(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t,s)f\,(s)\,\Delta s,$$

and the unique solution of the initial value problem

$$x^{\Delta} = p\,(t)\,x(t) + f\,(t)\,,\ x\,(t_0) = x_0,$$

is given by

$$x\,(t) = e_p(t,t_0)x_0 + \int_{t_0}^t e_p(t,\sigma\,(s))f\,(s)\,\Delta s.$$

Chapter 3

Existence of positive periodic solutions for delay dynamic equations

Keywords. Positive periodic solutions, Schauder's fixed point theorem, dynamic equations, time scales.

The goal of this chapter is to present a very recent work published in [24], namely, F. Bouchelaghem, A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for delay dynamic equations*, Proyecciones Journal of Mathematics, 36(3) (2017), 449–460.

In this chapter, we study the existence of positive periodic solutions for a dynamic equations on time scales. The main tool employed here is the Schauder's fixed point theorem.

3.1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $t_0 \in \mathbb{T}$. In this chapter, we consider the following delay dynamic equation

$$x^{\triangle}(t) + p(t) x^{\sigma}(t) + q(t) x(\tau(t)) = 0, \; t \geq t_0. \tag{3.1}$$

Throughout this chapter we assume that $p, q : [t_0, \infty) \cap \mathbb{T} \to \mathbb{R}$ are rd-continuous, $\tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $x(\tau(t))$ is well defined over $\mathbb{T}$. We also assume that $\tau : [t_0, \infty) \cap \mathbb{T} \to [0, \infty) \cap \mathbb{T}$ is rd-continuous, $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$. To reach our desired end we have to transform (3.1) into an integral equation and then use Schauder's fixed point theorem to show the existence of positive periodic solutions.

The organization of this chapter is as follows. In Section 2, we establish our main results for positive periodic solutions by applying the Schauder's fixed point theorem. In Section 3, we present two examples to illustrate our results. The results presented in this chapter extend the main results in [90].

3.2 The existence of periodic solutions

In this the section we will study existence of positive ω-periodic solution of (3.1). In the next lemma and theorem we choose $T \in \mathbb{T}$ sufficiently large that $\tau(t) \geq t_0$ for $t \geq T$.

Lemma 3.1 *Suppose that there exists a rd-continuous function $k : [T, \infty) \cap \mathbb{T} \to (0, \infty)$ such that*

$$\begin{aligned} &p + qk \in \mathcal{R}^+, \\ &\int_t^{t+\omega} \xi_{\mu(s)} \left[\ominus \left(p(s) + q(s)k(s)\right)\right] \Delta s = 0, \ t \geq T. \end{aligned} \tag{3.2}$$

Then the function

$$f(t) = \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s\right), \ t \geq T,$$

is ω-periodic.

Proof. For $t \geq T$ we obtain

$$\begin{aligned} &f(t+\omega) \\ &= \exp\left(\int_T^{t+\omega} \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s\right) \\ &= \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s)\right] \Delta s\right) \exp\left(\int_t^{t+\omega} \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s\right) \\ &= f(t). \end{aligned}$$

Thus the function f is ω-periodic. ∎

Theorem 3.1 *Suppose that there exists a rd-continuous function* $k : [T, \infty) \cap \mathbb{T} \to (0, \infty)$ *such that (3.2) holds and*

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s = \log(k(t)), \ \ \tau(t) \geq T. \tag{3.3}$$

Then (3.1) has a positive ω-periodic solution.

Proof. Let $X = C_{rd}([t_0, \infty) \cap \mathbb{T}, \mathbb{R})$ be the Banach space with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$. With regard to Lemma 3.1 we define

$$\begin{aligned} M &= \max_{t \in [T,\infty)\cap\mathbb{T}} \left\{ \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s \right) \right\}, \\ m &= \min_{t \in [T,\infty)\cap\mathbb{T}} \left\{ \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s \right) \right\}. \end{aligned} \tag{3.4}$$

We now define a closed, bounded and convex subset Ω of X as follows

$$\begin{aligned} \Omega = \{x \in X : x(t+\omega) = x(t), \ t \geq T, \\ m \leq x(t) \leq M, \ t \geq T, \\ k(t)x^\sigma(t) = x(\tau(t)), \ t \geq T, \\ x(t) = 1, \ t_0 \leq t \leq T\}. \end{aligned}$$

Define the operator $S : \Omega \longrightarrow X$ as follows

$$(Sx)(t) = \begin{cases} \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus \left(p(s) + q(s) \dfrac{x(\tau(s))}{x^\sigma(s)} \right) \right] \Delta s \right), & t \geq T, \\ 1, & t_0 \leq t \leq T. \end{cases}$$

We will show that for any $x \in \Omega$ we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$ we get

$$\begin{aligned} (Sx)(t) &= \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus \left(p(s) + q(s) \frac{x(\tau(s))}{x^\sigma(s)} \right) \right] \Delta s \right) \\ &= \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s \right) \leq M. \end{aligned}$$

Furthermore for $x \in \Omega$ and $t \geq T$ we obtain

$$(Sx)(t) = \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s \right) \geq m.$$

For $t \in [t_0, T] \cap \mathbb{T}$ we have $(Sx)(t) = 1$, that is $(Sx)(t) \in \Omega$.

Further for every $x \in \Omega$ and $\tau(t) \geq T$ we get

$$\begin{aligned}(Sx)(\tau(t)) &= \exp\left(\int_T^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\\ &= (Sx)^\sigma(t)\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right). \qquad (3.5)\end{aligned}$$

With regard to (3.3) and (3.5) for $\tau(t) \geq T$ it follows that

$$(Sx)(\tau(t)) = (Sx)^\sigma(t)\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right) = k(t)(Sx)^\sigma(t).$$

Finally we will show that for $x \in \Omega$, $t \geq T$ the function Sx is ω-periodic. For $x \in \Omega$, $t \geq T$ and according to (3.2) we have

$$\begin{aligned}&(Sx)(t+\omega)\\ &= \exp\left(\int_T^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\\ &= \exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\\ &\times \exp\left(\int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\\ &= (Sx)(t).\end{aligned}$$

So Sx is ω-periodic on $[T,\infty)\cap\mathbb{T}$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is completely continuous. Let $x_i \in \Omega$ be such that $x_i \longrightarrow x \in \Omega$ as $i \longrightarrow \infty$. For $t \geq T$, we have

$$\begin{aligned}&|(Sx_i)(t)-(Sx)(t)|\\ &= \left|\exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x_i(\tau(s))}{x_i^\sigma(s)}\right)\right]\Delta s\right)\right.\\ &\left. - \exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\right|.\end{aligned}$$

By applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{i\longrightarrow\infty} \|Sx_i - Sx\| = 0.$$

For $t \in [t_0, T]\cap\mathbb{T}$ the relation above is also valid. This means that S is continuous.

3.2. The existence of periodic solutions

We now show that $S\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of function $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty) \cap \mathbb{T}$. The uniform boundedness follows from the definition of Ω. With regard to (3.4) for $t \geq T$, $x \in \Omega$ we get

$$\begin{aligned}
&\left|(Sx)^{\Delta}(t)\right| \\
&= \left|-\left(p(t) + q(t)\frac{x(\tau(t))}{x^{\sigma}(t)}\right)\right| \\
&\times \exp\left(\int_{T}^{\sigma(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^{\sigma}(s)}\right)\right]\Delta s\right) \\
&= |p(t) + q(t)k(t)| \exp\left(\int_{T}^{\sigma(t)} \xi_{\mu(s)}\left[\ominus(p(s) + q(s)k(s))\right]\Delta s\right) \\
&\leq M_1,\ \ M_1 > 0.
\end{aligned}$$

For $t \in [t_0, T] \cap \mathbb{T}$, $x \in \Omega$, we have

$$\left|(Sx)^{\Delta}(t)\right| = 0.$$

This shows the equicontinuity of the family $S\Omega$. Hence $S\Omega$ is relatively compact and therefore S is completely continuous. By Theorem 2.5 there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that x_0 is a positive ω-periodic solution of (3.1). The proof is complete.

■

3.3 Two examples

In this section, we give two examples to illustrate the applications of Theorem 3.1.

Example 3.1 Consider the delay dynamic equation on $\mathbb{T}$ with $\mu(t) \neq 0$,

$$x^{\triangle}(t) - \frac{1}{\mu(t)}x^{\sigma}(t) + \frac{e^{(\cos\sigma(t) - \cos(t))}}{\mu(t)}x(\sigma(t) - 2\pi) = 0,\ \ t \geq 0. \tag{3.6}$$

We take $k(t) = 1$. Then for conditions (3.2), (3.3) and $\omega = 2\pi$ we obtain

$$\begin{aligned}
&\int_t^{t+\omega} \xi_{\mu(s)} \left[\ominus \left(p(s) + q(s)k(s)\right)\right] \Delta s \\
&= \int_t^{t+2\pi} \frac{1}{\mu(s)} \log \left[\ominus \left(p(s) + q(s)k(s)\right) \mu(s) + 1\right] \Delta s \\
&= \int_t^{t+2\pi} \frac{1}{\mu(s)} \log \left[\frac{1}{1 + \mu(s) \left(p(s) + q(s)\right)}\right] \Delta s \\
&= \int_t^{t+2\pi} -\frac{1}{\mu(s)} \log \left[1 + \mu(t) \left(p(s) + q(s)\right)\right] \Delta s \\
&= \int_t^{t+2\pi} -\frac{(\cos \sigma(s) - \cos(s))}{\mu(s)} \Delta s \\
&= -\cos s \,|_t^{t+2\pi} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s \\
&= \int_{\sigma(t)}^{\sigma(t)-2\pi} \xi_{\mu(s)} \left[\ominus(p(s) + q(s)k(s))\right] \Delta s = 0,\ t \geq 0.
\end{aligned}$$

All conditions of Theorem 3.1 are satisfied. Thus Eq. (3.6) has a positive $\omega = 2\pi$-periodic solution

$$x(t) = \exp\left(\int_T^t -\frac{(\cos \sigma(s) - \cos(s))}{\mu(s)} \Delta s\right) = e^{\cos(T) - \cos(t)},\ t \geq T.$$

Example 3.2 Consider the delay differential equation on $\mathbb{T} = \mathbb{R}$,

$$x'(t) - (\frac{1}{2} \sin t + e^{-t}) x(t) + e^{-t - \cos t} x(t - \pi) = 0,\ t \geq 0. \tag{3.7}$$

We choose $k(t) = e^{\cos t}$. Then for conditions (3.2), (3.3) and $\omega = 2\pi$ we have

$$\begin{aligned}
\int_t^{t+\omega} \left[p(s) + q(s)k(s)\right] ds &= -\frac{1}{2} \int_t^{t+2\pi} \sin(s) ds = 0, \\
\int_{\tau(t)}^t \left[p(s) + q(s)k(s)\right] \Delta s &= -\frac{1}{2} \int_{t-\pi}^t \sin(s) ds = \cos t,\ t \geq 0.
\end{aligned}$$

All conditions of Theorem 3.1 are satisfied. Thus (3.7) has a positive $\omega = 2\pi$-periodic solution

$$x(t) = \exp\left(\int_T^t \left(\frac{1}{2} \sin s\right) ds\right) = e^{\frac{1}{2}(\cos(T) - \cos(t))},$$

for $t \geq T$.

Chapter 4

Existence and stability of positive periodic solutions for delay nonlinear dynamic equations

Keywords. Positive periodic solutions, Schauder's fixed point theorem, dynamic equations, time scales

The goal of this chapter is to present a very recent work published in [23], namely, F. Bouchelaghem, A. Ardjouni and A. Djoudi, *Existence and stability of positive periodic solutions for delay nonlinear dynamic equations*, Nonlinear Studies, 25(1) (2018), 191–202.

In this chapter, we study the existence and stability of positive periodic solutions for a delay nonlinear dynamic equation on time scales.

4.1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $t_0 \in \mathbb{T}$. In this chapter, we consider the following delay nonlinear dynamic equation

$$x^{\triangle}(t) + p(t)\, x^{\sigma}(t) - \sum_{i=1}^{n} q_i(t)\, f_i(x(\tau_i(t))) = 0,\ t \geq t_0, \tag{4.1}$$

Throughout this chapter we assume that $p, q_i : [t_0, \infty) \cap \mathbb{T} \to \mathbb{R}$ are rd-continuous, $f_i : \mathbb{R} \to \mathbb{R}$ is differentiable with $f_i(x) > 0$ for $x > 0$, $\tau_i : \mathbb{T} \to \mathbb{T}$ is increasing so that the function

$x(\tau_i(t))$ is well defined over $\mathbb{T}$, $i = 1, ..., n$. We also assume that $\tau_i : [t_0, \infty) \cap \mathbb{T} \to [0, \infty) \cap \mathbb{T}$ is rd-continuous, $\tau_i(t) < t$ and $\lim_{t\to\infty} \tau_i(t) = \infty$, $i = 1, ..., n$. To reach our desired end we have to transform (4.1) into an integral equation and then use Schauder's fixed point theorem to show the existence of positive periodic solutions. The sufficient conditions for the exponential stability of positive periodic solutions are also considered.

The organization of this chapter is as follows. In Section 2, we establish our main results for positive periodic solutions by applying the Schauder's fixed point theorem. The exponential stability of the positive periodic solutions is the topic of Section 3. The model for the survival of red blood cells is treated in Section 4. An examples is also given to illustrate our results. The results presented in this chapter extend the main results in [49].

4.2 Existence of periodic solutions

In this the section we will study existence of positive ω-periodic solution of (4.1). In the next lemma and theorem we choose $T \in \mathbb{T}$ sufficiently large that $\tau_i(t) \geq t_0$ for $t \geq T$, $i = 1, ..., n$.

Lemma 4.1 *Suppose that there exist rd-continuous functions $k_i : [T, \infty) \cap \mathbb{T} \to (0, \infty)$, $i = 1, ..., n$ such that*

$$\begin{aligned} & p - \sum_{i=1}^{n} q_i k_i \in \mathcal{R}^+, \\ & \int_t^{t+\omega} \xi_{\mu(s)} \left[\ominus \left(p(s) - \sum_{i=1}^{n} q_i(s) k_i(s) \right) \right] \Delta s = 0, \ t \geq T. \end{aligned} \tag{4.2}$$

Then the function

$$\psi(t) = \exp\left(\int_T^t \xi_{\mu(s)} \left[\ominus (p(s) - \sum_{i=1}^{n} q_i(s) k_i(s) \right] \Delta s \right), \ t \geq T,$$

is ω-periodic.

Proof. For $t \geq T$ we obtain

$$\begin{aligned}
&\psi(t+\omega)\\
&= \exp\left(\int_T^{t+\omega} \xi_{\mu(s)} \left[\ominus(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right] \Delta s\right)\\
&= \exp\left(\int_T^{t} \xi_{\mu(s)} \left[\ominus(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right] \Delta s\right)\\
&\times \exp\left(\int_t^{t+\omega} \xi_{\mu(s)} \left[\ominus(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right] \Delta s\right)\\
&= \psi(t).
\end{aligned}$$

Thus the function ψ is ω-periodic. ■

Theorem 4.1 *Suppose that there exist rd-continuous functions $k_i : [T,\infty) \cap \mathbb{T} \to (0,\infty)$, $i = 1, ..., n$ such that (4.2) holds and*

$$\begin{aligned}
&f_i\left(\exp\left(\int_T^{\tau_j(t)} \xi_{\mu(s)} \left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\Delta s\right)\right]\right)\right)\\
&\times \exp\left(\int_{\sigma(t)}^{T} \xi_{\mu(s)} \left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\Delta s\right)\right]\right)\\
&= k_j(t),\ \ \tau_j(t) \geq T,\ \ j = 1, ..., n. \qquad (4.3)
\end{aligned}$$

Then (4.1) has a positive ω-periodic solution.

Proof. Let $X = C_{rd}([t_0,\infty) \cap \mathbb{T}, \mathbb{R})$ be the Banach space with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$. With regard to Lemma 4.1 we define

$$\begin{aligned}
M &= \max_{t\in[T,\infty)\cap\mathbb{T}} \left\{\exp\left(\int_T^{t} \xi_{\mu(s)} \left[\ominus(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right] \Delta s\right)\right\},\\
m &= \min_{t\in[T,\infty)\cap\mathbb{T}} \left\{\exp\left(\int_T^{t} \xi_{\mu(s)} \left[\ominus(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right] \Delta s\right)\right\}. \qquad (4.4)
\end{aligned}$$

We now define a closed, bounded and convex subset Ω of X as follows

$$\begin{aligned}\Omega = \{x \in X : x(t+\omega) = x(t),\ t \geq T, \\ m \leq x(t) \leq M,\ t \geq T, \\ k_i(t)x^\sigma(t) = f_i\left(x(\tau_i(t))\right),\ t \geq T,\ i = 1, ..., n, \\ x(t) = 1,\ t_0 \leq t \leq T\}.\end{aligned}$$

Define the operator $S : \Omega \longrightarrow X$ as follows

$$(Sx)(t) = \begin{cases} \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right), & t \geq T, \\ 1, & t_0 \leq t \leq T. \end{cases}$$

We will show that for any $x \in \Omega$ we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$ we get

$$\begin{aligned}(Sx)(t) &= \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right) \\ &= \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right)\right]\Delta s\right) \leq M.\end{aligned}$$

Furthermore for $x \in \Omega$ and $t \geq T$ we obtain

$$(Sx)(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)k_i(s)\right)\right]\Delta s\right) \geq m.$$

For $t \in [t_0, T] \cap \mathbb{T}$ we have $(Sx)(t) = 1$, that is $(Sx)(t) \in \Omega$.

Further for every $x \in \Omega$ and $\tau_j(t) \geq T$, $j = 1, ..., n$, we get

$$\begin{aligned}f_j\left((Sx)(\tau_j(t))\right) &= f_j\left(\exp\left(\int_T^{\tau_j(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right)\right) \\ &= f_j\left(\exp\left(\int_T^{\tau_j(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right)\right) \\ &\times \exp\left(\int_{\sigma(t)}^T \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right) \\ &\times \exp\left(\int_T^{\sigma(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) - \sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right). \qquad (4.5)\end{aligned}$$

With regard to (4.3) and (4.5) for $\tau_j(t) \geq T$, $j = 1, ..., n$, it follows that

$$\begin{aligned}
f_j\left((Sx)(\tau_j(t))\right) &= f_j\left(\exp\left(\int_T^{\tau_j(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+\sum_{i=1}^n q_i(s)k_i(s)\right)\right]\Delta s\right)\right) \\
&\quad \times \exp\left(\int_{\sigma(t)}^{\tau_j(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+\sum_{i=1}^n q_i(s)k_i(s)\right)\right]\Delta s\right) S(x)^\sigma(t) \\
&= k_j(t)(Sx)^\sigma(t),\ j = 1, ..., n.
\end{aligned}$$

Finally we will show that for $x \in \Omega$, $t \geq T$ the function Sx is ω-periodic. For $x \in \Omega$, $t \geq T$ and according to (4.2) we have

$$\begin{aligned}
(Sx)(t+\omega) &= \exp\left(\int_T^{t+\omega} \xi_{\mu(s)}\left[\ominus\left(p(s)-\sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right) \\
&= \exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus\left(p(s)-\sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right) \\
&\times \exp\left(\int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus\left(p(s)-\sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right) \\
&= (Sx)(t)\exp\left(\int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s)-\sum_{i=1}^n q_i(s)k_i(s))\right]\Delta s\right) = (Sx)(t).
\end{aligned}$$

So Sx is ω-periodic on $[T, \infty) \cap \mathbb{T}$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is completely continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \longrightarrow x(t) \in \Omega$ as $k \longrightarrow \infty$. For $t \geq T$, we have

$$\begin{aligned}
|(Sx_k)(t) - (Sx)(t)| = \Bigg| &\exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus\left(p(s)-\sum_{i=1}^n q_i(s)\frac{f_i\left(x_k(\tau_i(s))\right)}{x_k^\sigma(s)}\right)\right]\Delta s\right) \\
&- \exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus\left(p(s)-\sum_{i=1}^n q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^\sigma(s)}\right)\right]\Delta s\right)\Bigg|.
\end{aligned}$$

Since $f_i\left(x_k(\tau_i(s))\right)/x_k^\sigma(s) \to f_i\left(x(\tau_i(s))\right)/x^\sigma(s)$ as $k \to \infty$ for $i = 1, 2, ..., n$,by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k \longrightarrow \infty} \left\|\left(Sx_k\right)(t) - (Sx)(t)\right\| = 0.$$

For $t \in [t_0, T] \cap \mathbb{T}$ the relation above is also valid. This means that S is continuous.

We now show that $S\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of function $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)\cap\mathbb{T}$. The uniform boundedness follows from the definition of Ω. With regard to (4.4) for $t\geq T$, $x\in\Omega$ we get

$$\begin{aligned}
&\left|(Sx)^{\Delta}(t)\right| \\
&= \left|\ominus\left(p(t)-\sum_{i=1}^{n} q_i(t)\frac{f_i\left(x(\tau_i(t))\right)}{x^{\sigma}(t)}\right)\right| \\
&\times \exp\left(\int_T^{\sigma(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+\sum_{i=1}^{n} q_i(s)\frac{f_i\left(x(\tau_i(s))\right)}{x^{\sigma}(s)}\right)\right]\Delta s\right) \\
&\leq \left|p(t)-\sum_{i=1}^{n} q_i(t)k_i(t)\right| \exp\left(\int_T^{\sigma(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+\sum_{i=1}^{n} q_i(s)k_i(s)\right)\right]\Delta s\right) \\
&\leq M_1,\ M_1>0.
\end{aligned}$$

For $t\in[t_0,T]\cap\mathbb{T}$, $x\in\Omega$, we have

$$\left|(Sx)^{\Delta}(t)\right| = 0.$$

This shows the equicontinuity of the family $S\Omega$. Hence $S\Omega$ is relatively compact and therefore S is completely continuous. By Theorem 2.5 there is an $x_0\in\Omega$ such that $Sx_0=x_0$. We see that x_0 is a positive ω-periodic solution of (4.1). The proof is complete.

■

4.3 Stability of positive periodic solutions

In this section, we consider the exponential stability of the positive periodic solution of (4.1). Let $r=\min_{1\leq i\leq n}\{\inf_{t\geq T}\tau_i(t)\}$. We denote $x\left(t,T,\psi\right),\ t\geq r,\ \psi\in C_{rd}\left([r,T]\cap\mathbb{T},(0,\infty)\right)$ for a solution of (4.1) satisfying the initial condition $x\left(t,T,\psi\right)=\psi(t)$, $t\in[r,T]\cap\mathbb{T}$, where T is the initial point. Let $x\left(t\right)=x\left(t,T,\psi\right),\ \widetilde{x}\left(t\right)=x\left(t,T,\widetilde{\psi}\right)$ and $y(t)=x\left(t\right)-\widetilde{x}\left(t\right),\ t\in[r,\infty)\cap\mathbb{T}$. By (4.1), we get

$$y^{\triangle}\left(t\right)+p\left(t\right)y^{\sigma}\left(t\right)-\sum_{i=1}^{n} q_i\left(t\right)\left[f_i\left(x\left(\tau_i\left(t\right)\right)\right)-f_i\left(\widetilde{x}\left(\tau_i\left(t\right)\right)\right)\right]=0,\ t\geq T.$$

By the mean value theorem, we obtain

$$y^{\triangle}(t) + p(t)\, y^{\sigma}(t) - \sum_{i=1}^{n} q_i(t)\, f_i'(x_i^*)\left[x(\tau_i(t)) - \widetilde{x}(\tau_i(t))\right] = 0,\ f_i'(x) = \frac{df_i(x)}{dx},$$

$$y^{\triangle}(t) + p(t)\, y^{\sigma}(t) - \sum_{i=1}^{n} q_i(t)\, f_i'(x_i^*)\, y(\tau_i(t)) = 0,\ t \geq T. \tag{4.6}$$

Lemma 4.2 *Assume that* $x \in (0,\infty)$, $|f_i'(x)| \leq a$, $t - \tau_i^{\sigma}(t) \leq b$, $t \geq T$, $i = 1, ..., n$ *and*

$$\sup_{t\geq T}\left\{-p(t) + a\sum_{i=1}^{n} q_i^{\sigma}(t)\right\} < 0.$$

Then there exists $\lambda \in (0,1]$ *such that*

$$\lambda - p(t) + a e_{\lambda}(b,0)\sum_{i=1}^{n} q_i^{\sigma}(t) < 0 \text{ for } t \geq T.$$

Proof. Define a continuous function $H(u)$ by

$$H(u) = \sup_{t\geq T}\left\{u - p(t) + a e_{u}(b,0)\sum_{i=1}^{n} q_i^{\sigma}(t)\right\},\ \ u \in [0,1].$$

By hypothesis, we get

$$H(0) = \sup_{t\geq T}\left\{-p(t) + a\sum_{i=1}^{n} q_i^{\sigma}(t)\right\} < 0.$$

According to the continuity of $H(u)$ and $H(0) < 0$, there exists $\lambda \in (0,1]$ such that $H(\lambda) < 0$, that is

$$\lambda - p(t) + a e_{\lambda}(b,0)\sum_{i=1}^{n} q_i^{\sigma}(t) < 0 \text{ for } t \geq T.$$

We have achieved the desired result. ■

Next we will assume that the function

$$F(t, x, x_1, ..., x_n) = -p(t)\, x^{\sigma}(t) + \sum_{i=1}^{n} q_i(t)\, f_i(x_i(t)),\ t \geq r,$$

satisfies Lipschitz-type condition with respect to $x, x_i > 0$, $i = 1,...,n$.

Definition 4.1 Let $\widetilde{x}$ be a positive solution of (4.1). If there exist constants $T_{\psi,\widetilde{x}}$, $K_{\psi,\widetilde{x}}$ and $\lambda > 0$ such that for every solution $x(t, T, \psi)$ of (4.1)

$$|x(t, T, \psi) - \widetilde{x}(t)| < K_{\psi,\widetilde{x}}\, e_{\ominus\lambda}(t,0) \text{ for all } t \geq T_{\psi,\widetilde{x}}.$$

Then $\widetilde{x}$ is said to be exponentially stable.

4.3. Stability of positive periodic solutions

In the next lemma, we establish sufficient conditions for the exponential stability of the positive solution $\widetilde{x}(t) = x\left(t, T, \widetilde{\psi}\right)$ of (4.1).

Lemma 4.3 *Assume that* $x \in (0, \infty)$, $|f_i'(x)| \leq a$, $t - \tau_i^\sigma(t) \leq b$, $t \geq T$, $i = 1, ..., n$ *and*

$$\sup_{t \geq T} \left\{ -p(t) + a \sum_{i=1}^{n} q_i^\sigma(t) \right\} < 0.$$

Then there exists $\lambda \in (0, 1]$ *such that*

$$\left| x(t, T, \psi) - x\left(t, T, \widetilde{\psi}\right) \right| < K_{\psi,\widetilde{x}} e_{\ominus\lambda}(t, 0) \text{ for all } t \geq T_{\psi,\widetilde{x}}.$$

where

$$K_{\psi,\widetilde{x}} = \max_{t \in [r,T] \cap \mathbb{T}} e_\lambda(T, 0) |y(t)| + 1.$$

Proof. We consider the Lyapunov function

$$L(t) = |y(t)| e_\lambda(t, 0), \ t \geq r, \ \lambda \in (0, 1].$$

We claim that $L(t) < K_{\psi,\widetilde{x}}$ for $t > T$. In order to prove it, suppose that there exists $t_* > T$ such that $L(t_*) = K_{\psi,\widetilde{x}}$ and $L(t) < K_{\psi,\widetilde{x}}$ for $t \in [r, \sigma(t_*)) \cap \mathbb{T}$. Calculating the upper left derivative of L along the solution y of (4.6), we obtain

$$\begin{aligned}
(L(t))^{\Delta^-} &\leq -p(t) |y^\sigma(t)| e_\lambda(t, 0) + e_\lambda(t, 0) \sum_{i=1}^{n} q_i^\sigma(t) |f_i'(x_i^*)| |y(\tau_i^\sigma(t))| + \lambda |y(t)| e_\lambda(t, 0) \\
&= [\lambda |y(t)| - p(t) |y^\sigma(t)|] e_\lambda(t, 0) + e_\lambda(t, 0) \sum_{i=1}^{n} q_i^\sigma(t) |f_i'(x_i^*)| |y(\tau_i^\sigma(t))| \\
&\leq [\lambda - p(t)] |y(t)| e_\lambda(t, 0) + a e_\lambda(t, 0) \sum_{i=1}^{n} q_i^\sigma(t) |y(\tau_i^\sigma(t))|, \ t \geq T.
\end{aligned}$$

For $t = t_*$ and applying Lemma 4.2, we get

$$\begin{aligned}
0 \leq (L(t))^{\Delta^-} &\leq [\lambda - p(t_*)] |y(t_*)| e_\lambda(t_*, 0) + a e_\lambda(t_*, 0) \sum_{i=1}^{n} q_i^\sigma(t_*) |y(\tau_i^\sigma(t_*))| \\
&= [\lambda - p(t_*)] |y(t_*)| e_\lambda(t_*, 0) + a \sum_{i=1}^{n} q_i^\sigma(t_*) |y(\tau_i^\sigma(t_*))| e_\lambda(\tau_i^\sigma(t_*), 0) e_\lambda(t_* - \tau_i^\sigma(t_*), 0) \\
&= [\lambda - p(t_*)] K_{\psi,x_1} + a \sum_{i=1}^{n} q_i^\sigma(t_*) L(\tau_i^\sigma(t_*)) e_\lambda(t_* - \tau_i^\sigma(t_*), 0) \\
&\leq \left[\lambda - p(t_*) + a e_\lambda(b, 0) \sum_{i=1}^{n} q_i^\sigma(t_*) \right] K_{\psi,x_1} < 0,
\end{aligned}$$

which is a contradiction. Therefore we obtain

$$L(t) = |y(t)|\, e_{\lambda}(t,0) < K_{\psi,\widetilde{x}} \text{ for } t > T, \text{ and for some } \lambda \in (0,1]\,.$$

The proof is complete. ■

Theorem 4.2 *Assume that* $x \in (0,\infty)$, $|f_i'(x)| \le a$, $t - \tau_i^{\sigma}(t) \le b$, $t \ge T$, $i = 1,...,n$ *and*

$$\sup_{t\ge T}\left\{-p(t) + a\sum_{i=1}^{n} q_i^{\sigma}(t)\right\} < 0,$$

and there exist functions $k_i \in C_{rd}\left([T,\infty)\cap\mathbb{T},(0,\infty)\right)$, $i = 1,...,n$ *such that (4.2) and (4.3) hold. Then (4.1) has a positive* ω*-periodic solution which is exponentially stable.*

Proof. *The proof follows from the Theorem 4.1 and Lemma 4.3.* ■

4.4 Model for the survival of red blood cells

In this section, we consider the existence of positive ω-periodic solutions for the nonlinear delay dynamic equation of the form

$$x^{\triangle}(t) + p(t)\,x^{\sigma}(t) - q(t)\,e_{\ominus\gamma}\left(x\left(\tau\left(t\right)\right),0\right) = 0,\ \ t \ge t_0, \tag{4.7}$$

which is a special case of (4.1), where $q_1(t) = q(t)$, $q_i(t) = 0$, $i = 2,...,n$, $f_1\left(x\left(\tau_1\left(t\right)\right)\right) = e_{\ominus\gamma}\left(x\left(\tau\left(t\right)\right),0\right)$, $f_i\left(x\right) = 0$ and $\tau_i\left(t\right) = 0$, $i = 2,...,n$, $\gamma \in \mathcal{R}^{+}$. We will also establish the sufficient conditions for the exponential stability of the positive periodic solution.

Rewriting the Theorem 4.2 to the equation (4.7) we obtain the next result.

Theorem 4.3 *Suppose that* $\gamma > 0$, $t - \tau^{\sigma}(t) \le b$, $t \ge T$,

$$\sup_{t\ge T}\left\{-p(t) + \gamma q^{\sigma}(t)\right\} < 0, \tag{4.8}$$

and there exists function $k : C_{rd}\left([T,\infty)\cap\mathbb{T},(0,\infty)\right)$ *such that*

$$\begin{aligned} &p - qk \in \mathcal{R}^{+}, \\ &\int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus\left(p(s) - q(s)k(s)\right)\right]\Delta s = 0,\ \ t \ge T. \end{aligned} \tag{4.9}$$

and

$$f\left(\exp\left(\int_T^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)-q(s)k(s)\right)\right]\Delta s\right)\right)$$
$$\times\exp\left(\int_{\sigma(t)}^{T} \xi_{\mu(s)}\left[\ominus\left(p(s)-q(s)k(s)\right]\Delta s\right) = k(t),\ \tau(t)\geq T. \tag{4.10}$$

Then (4.7) has a positive ω-periodic solution which is exponentially stable.

Example 4.1 Consider the nonlinear delay dynamic equation on $\mathbb{T}$ with $\mu(t)\neq 0$,

$$x^{\triangle}(t)+p(t)x^{\sigma}(t)-q(t)e_{\ominus\gamma}\left(x\left(\tau(t)\right),0\right)=0,\ t\geq t_0, \tag{4.11}$$

where $\gamma\in\mathcal{R}^+$, $\tau(t)=t-\pi$,

$$p(t)=1+\frac{e^{(\cos\sigma(t)-\cos t)}-1}{\mu(t)},$$
$$q(t)=e^{(\cos T-\cos\sigma(t))}e_{\gamma}\left(e^{(\cos T-\cos\tau(t))},0\right).$$

We choose

$$k(t)=e^{(\cos\sigma(t)-\cos T)}e_{\ominus\gamma}\left(e^{(\cos T-\cos\tau(t))},0\right).$$

Then for conditions (4.9), (4.10) and $\omega=2\pi$, we get

$$\begin{aligned}1+\mu(t)\left(p(t)-q(t)k(t)\right)&=1+\mu(t)\left(\frac{e^{(\cos\sigma(t)-\cos t)}-1}{\mu(t)}\right)\\&=e^{(\cos\sigma(t)-\cos t)}>0,\ \forall t\in\mathbb{T},\end{aligned}$$

then $p-qk\in\mathcal{R}^+$, and

$$\begin{aligned}\int_t^{t+\omega}\xi_{\mu(s)}\left[\ominus\left(p(s)-q(s)k(s)\right)\right]\Delta s&=-\int_t^{t+2\pi}\frac{\cos\left(\sigma(s)\right)-\cos(s)}{\mu(s)}\Delta s\\&=-\cos(s)\big|_t^{t+2\pi}\\&=0.\end{aligned}$$

Therefore

$$f\left(\exp\left(\int_T^{\tau(t)}\xi_{\mu(s)}\left[\ominus\left(p(s)-q(s)k(s)\right)\right]\Delta s\right)\right)\exp\left(\int_{\sigma(t)}^{T}\xi_{\mu(s)}\left[\ominus\left(p(s)-q(s)k(s)\right)\right]\Delta s\right)$$
$$=e_{\ominus\gamma}\left(\exp\left(\cos(T)-\cos\left(\tau(t)\right)\right),0\right)\exp(\cos(\sigma(t))-\cos(T))$$
$$=k(t).$$

4.4. Model for the survival of red blood cells

The conditions (4.9) and (4.10) of Theorem 4.3 are satisfied and (4.11) has a positive $\omega = 2\pi$ periodic solution

$$\begin{aligned} x(t) &= \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) - q(s)k(s)\right)\right]\Delta s\right), \ t \geq T, \\ &= \exp\left(-\int_T^t \frac{\cos\left(\sigma(s)\right) - \cos(s)}{\mu(s)}\Delta s\right) \\ &= \exp(\cos(T) - \cos(t)). \end{aligned}$$

If we put $\gamma = 0.04$, $T = \frac{\pi}{2}$, $\mu(t) \geq \frac{1-e^{-1}}{1-0.04e^{1.04}}$, we get

$$\begin{aligned} &-p(t) + 0.04q^{\sigma}(t) \\ &= -(1 + \frac{e^{(\cos\sigma(t) - \cos t)} - 1}{\mu(t)}) + \gamma e^{(\cos T - \cos\sigma(\sigma(t)))} e_{\gamma}\left(e^{(\cos T - \cos\sigma(\tau(t)))}, 0\right) \\ &= -(1 + \frac{e^{(\cos\sigma(t) - \cos t)} - 1}{\mu(t)}) + 0.04e^{-\cos\sigma(\sigma(t)))} e_{0.04}\left(e^{-\cos\sigma(\tau(t)))}, 0\right) \\ &\leq -(1 + \frac{e^{-1} - 1}{\mu(t)}) + 0.04e^{1.04} \\ &\leq 0. \end{aligned}$$

Then, the condition (4.8) is satisfied and solution x is exponentially stable.

Chapter 5

Existence of positive solutions of delay dynamic equations

Keywords. Positive solutions, asymptotic properties, Schauder's fixed point theorem, dynamic equations, time scales.

The goal of this chapter is to present a very recent work published in [25], namely, F. Bouchelaghem, A. Ardjouni and A. Djoudi, *Existence of positive solutions of delay dynamic equations*, Positivity, 21(4) (2017), 1483–1493.

In this Chapter, we study the existence of positive solutions for a dynamic equations on time scales. The main tool employed here is the Schauder's fixed point theorem. The asymptotic properties of solutions are also treated.

5.1 Introduction

Let $\mathbb{T}$ be a time scale such that $t_0 \in \mathbb{T}$. In this chapter, we consider the following delay dynamic equation

$$x^{\triangle}(t) + p(t)\, x^{\sigma}(t) + q(t)\, x(\tau(t)) = 0,\ \ t \geq t_0, \tag{5.1}$$

Throughout this chapter we assume that $p : [t_0, \infty) \cap \mathbb{T} \to \mathbb{R}$ and $q : [t_0, \infty) \cap \mathbb{T} \to (0, \infty)$ are rd-continuous, $\tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $x(\tau(t))$ is well defined over $\mathbb{T}$. We also assume that $\tau : [t_0, \infty) \cap \mathbb{T} \to [0, \infty) \cap \mathbb{T}$ is rd-continuous, $\tau(t) < t$ and

$\lim_{t\to\infty} \tau(t) = \infty$. To reach our desired end we have to transform (5.1) into an integral equation and then use Schauder's fixed point theorem to show the existence of solutions which are bounded by positive functions. The asymptotic properties of solutions are also treated.

The organization of this chapter is as follows. In Section 2, we establish our main results for positive solutions by applying the Schauder's fixed point theorem. In Section 3, we present the asymptotic properties of solutions. In Section 4, we give three examples to illustrate our results. The results presented in this chapter extend the main results in [49].

5.2 Existence of positive solutions

In this section we shall investigate the existence of positive solutions for equation (5.1). The main result is in the following theorem.

Theorem 5.1 *Suppose that for* $t \geq t_0$

$$1 < k_1 \leq k_2,\ \ p(t) + k_1 q(t) \geq 0,$$

and

$$\begin{aligned} &\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1 q(s)\right)\right]\Delta s \geq \log k_1, \\ &\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2 q(s)\right)\right]\Delta s \leq \log k_2. \end{aligned} \tag{5.2}$$

Then equation (5.1) has a solution which is bounded by positive functions.

Proof. *We choose* $T \geq t_0 + \tau(T)$ *and set*

$$\begin{aligned} u(t) &= \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2 q(s)\right)\right]\Delta s\right), \\ v(t) &= \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1 q(s)\right)\right]\Delta s\right), \quad t \geq T. \end{aligned}$$

Let $C_{rd}\left([t_0, \infty), \mathbb{R}\right)$ *be the set of all bounded rd-continuous functions with the norm*

$$\|x\| = \sup_{t\geq t_0} |x(t)| < \infty.$$

Then $C_{rd}\left([t_0,\infty),\mathbb{R}\right)$ is a Banach space. We define a close, bounded and convex subset Ω of $C_{rd}\left([t_0,\infty),\mathbb{R}\right)$ as follows

$$\begin{aligned}\Omega=\{u(t)\le x(t)\le v(t), &\qquad t\ge T,\\ x(\tau(t))\le k_2x^\sigma(t), &\qquad t\ge T,\\ x(\tau(t))\ge k_1x^\sigma(t), &\qquad t\ge T,\\ x(t)=1, &\qquad \tau(T)\le t\le T\}.\end{aligned}$$

Define the map $S:\Omega\to C_{rd}\left([t_0,\infty),\mathbb{R}\right)$ as follows

$$(Sx)(t)=\begin{cases}\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right), & t\ge T,\\ 1, & \tau(T)\le t\le T.\end{cases}$$

We shall show that for any $x\in\Omega$ we have $Sx\in\Omega$. For every $x\in\Omega$ and $t\ge T$ we get

$$(Sx)(t)\le \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s)+k_1q(s)\right)\right]\Delta s\right)=v(t).$$

Furthermore for $t\ge T$ we have

$$(Sx)(t)\ge \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s)+k_2q(s)\right)\right]\Delta s\right)=u(t).$$

For $t\in[\tau(T),T]$ we obtain $(Sx)(t)=1$. Further for every $x\in\Omega$ and $\tau(t)\ge T$ we get

$$\begin{aligned}(Sx)(\tau(t))&=\exp\left(\int_T^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\\ &=(Sx)(\sigma(t))\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right).\end{aligned}\tag{5.3}$$

With regard to (5.2) and (5.3) we have

$$\begin{aligned}(Sx)(\tau(t))&\le (Sx)(\sigma(t))\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+k_2q(s)\right)\right]\Delta s\right)\\ &\le k_2(Sx)(\sigma(t)),\quad \tau(t)\ge T,\end{aligned}$$

and

$$\begin{aligned}(Sx)(\tau(t))&\ge (Sx)(\sigma(t))\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+k_1q(s)\right)\right]\Delta s\right)\\ &\ge k_1(Sx)(\sigma(t)),\quad \tau(t)\ge T.\end{aligned}$$

For $\tau(T) \leq \tau(t) \leq T$ we obtain $(Sx)(\tau(t)) = 1$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is continuous. Let $x_i \in \Omega$ be such that $x_i \to x$ as $i \to \infty$. Because Ω is closed, $x \in \Omega$. For $t \geq T$ we have

$$\begin{aligned}
&|(Sx_i)(t) - (Sx)(t)| \\
&= \Bigg|\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x_i(\tau(s))}{x_i^\sigma(s)}\right)\right]\Delta s\right) \\
&- \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\Bigg|.
\end{aligned}$$

So we conclude that

$$\lim_{i\to\infty} \|Sx_i - Sx\| = 0.$$

This means that S is continuous.

The family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded on $[\tau(T), \infty)$. It follows from the definition of Ω. This family is also equicontinuous on $[\tau(T), \infty)$. Then by Arsela-Ascoli theorem the $S\Omega$ is relatively compact subset of $C_{rd}\left([t_0, \infty), \mathbb{R}\right)$. By Theorem 2.5 there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that x_0 is a positive solution of the equation (5.1). The proof is complete. ∎

Corollary 5.1 *Suppose that $k > 1$, $p(t) + kq(t) \geq 0$ and*

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + kq(s)\right)\right]\Delta s = \log k, \quad t \geq t_0.$$

Then equation (5.1) has a solution

$$x(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + kq(s)\right)\right]\Delta s\right), \quad t \geq T.$$

5.3 Asymptotic properties

In this section some asymptotic properties of positive solutions of (5.1) are treated.

Theorem 5.2 *Suppose that $0 < \alpha < 1$, $p(t) + \alpha q(t) \geq 0$, $t \geq t_0$ and*

$$\lim_{t\to\infty} \frac{1}{k-\alpha}\int_{\tau(t)}^{\sigma(t)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s > \max_{k>1}\frac{\log k}{k-\alpha}. \tag{5.4}$$

Then equation (5.1) has not a positive solution x with the property

$$x(\tau(t)) \geq kx^{\sigma}(t),\ t \geq T \geq t_0,\ k \in (1, \infty)\,.$$

Proof. *Assume to the contrary that (5.1) has a positive solution x with the property $x(\tau(t)) \geq kx^{\sigma}(t)$, $t \geq T \geq t_0$ for some $k \in (1, \infty)$. Then*

$$\begin{aligned} x^{\Delta}(t) + q(t)\left[-\alpha x^{\sigma}(t) + x(\tau(t))\right] &\leq 0, \\ x^{\Delta}(t) + q(t)\left[-\alpha + \frac{x(\tau(t))}{x^{\sigma}(t)}\right] x^{\sigma}(t) &\leq 0, \\ x^{\Delta}(t) + q(t)\left[k - \alpha\right] x^{\sigma}(t) &\leq 0, \\ x^{\Delta}(t) e_{q(k-\alpha)}(t,s) + x^{\sigma}(t) q(t)\left[k-\alpha\right] e_{q(k-\alpha)}(t,s) &\leq 0, \quad t \geq T. \end{aligned} \tag{5.5}$$

Integrating the last inequality we get

$$\begin{aligned} &\int_{\tau(t)}^{\sigma(t)} \left(x(v) e_{q(k-\alpha)}(v,s)\right)^{\Delta} \Delta v \;\leq\; 0, \\ &\frac{x(\tau(t))}{x^{\sigma}(t)} \;\geq\; e_{q(k-\alpha)}(\sigma(t), \tau(t)), \\ &\frac{x(\tau(t))}{x^{\sigma}(t)} \;\geq\; \exp\left(\int_{\tau(t)}^{\sigma(t)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s\right), \quad t \;\geq\; T. \end{aligned}$$

With regard to condition (5.4) there exists a constant $c > 0$ such that

$$\frac{1}{k-\alpha}\int_{\tau(t)}^{\sigma(t)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s \geq c > \max_{k>1} \frac{\log k}{k-\alpha}, \quad t \geq t_1 \geq T. \tag{5.6}$$

Then we obtain

$$\frac{x(\tau(t))}{x^{\sigma}(t)} \geq \exp\left((k-\alpha)c\right) = c_1, \quad t \geq t_1,$$

where $c_1 > k$. Repeating this process by using (5.5) we get

$$\frac{x(\tau(t))}{x^{\sigma}(t)} \geq \exp\left((c_i-\alpha)c\right) = c_{i+1}, \quad t \geq t_{i+1}, \tag{5.7}$$

where $c_{i+1} > c_i$, $i = 1, 2, ...,$ since $(c_i - \alpha)c > \log c_i$. We see that $c_i \to \infty$ as $i \to \infty$. On the other hand for given $\epsilon > 0$ and sufficiently large i we have

$$c_{i+1} < c_i + \epsilon, \quad (c_i - \alpha)c = \log c_{i+1} \leq \log(c_i + \epsilon), \quad c \leq \frac{\log(c_i+\epsilon)}{c_i - \alpha},$$

which is a contradiction with (5.6) far small $\epsilon > 0$.

According to (5.6) for arbitrary $t \geq t_1$ we can choose $t^* > t$ such that

$$\int_{\tau(t^*)}^{\sigma(t)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s \geq \frac{c}{2}(k-\alpha) \text{ and } \int_{\sigma(t)}^{\sigma(t^*)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s \geq \frac{c}{2}(k-\alpha).$$

For Lemma 2.7 we have

$$\int_{\tau(t^*)}^{\sigma(t)} q(s)\Delta s \geq \frac{1}{k-\alpha}\int_{\tau(t^*)}^{\sigma(t)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s \geq \frac{c}{2},$$

and

$$\int_{\sigma(t)}^{\sigma(t^*)} q(s)\Delta s \geq \frac{1}{k-\alpha}\int_{\sigma(t)}^{\sigma(t^*)} \xi_{\mu(s)}\left[(k-\alpha)q(s)\right]\Delta s \geq \frac{c}{2}.$$

Then

$$-x^{\Delta}(t) \geq -\alpha q(t)x^{\sigma}(t) + q(t)x(\tau(t)),$$

Integrating the inequality from $\tau(t^*)$ to $\sigma(t)$ and then from $\sigma(t)$ to $\sigma(t^*)$ and using the non increasing character of x we obtain

$$\begin{aligned}
x(\tau(t^*)) - x^{\sigma}(t) &\geq \int_{\tau(t^*)}^{\sigma(t)} q(s)x(\tau(s))\Delta s - \alpha\int_{\tau(t^*)}^{\sigma(t)} q(s)x^{\sigma}(s)\Delta s \\
&\geq \int_{\tau(t^*)}^{t} q(s)x(\tau(s))\Delta s - \alpha\int_{\tau(t^*)}^{\sigma(t)} q(s)x(s)\Delta s \\
&\geq \frac{c}{2}x(\tau(t)) - \frac{c\alpha}{2}x(\tau(t^*)), \\
x(\tau(t^*)) &\geq \frac{c}{2+c\alpha}x(\tau(t)), \quad t \geq t_1.
\end{aligned}$$

Further we have

$$\begin{aligned}
x^{\sigma}(t) - x^{\sigma}(t^*) &\geq \int_{\sigma(t)}^{\sigma(t^*)} q(s)x(\tau(s))\Delta s - \alpha\int_{\sigma(t)}^{\sigma(t^*)} q(s)x^{\sigma}(s)\Delta s \\
&\geq \int_{\sigma(t)}^{t^*} q(s)x(\tau(s))\Delta s - \alpha\int_{\sigma(t)}^{\sigma(t^*)} q(s)x(s)\Delta s \\
&\geq \frac{c}{2}x(\tau(t^*)) - \frac{c\alpha}{2}x^{\sigma}(t), \\
x^{\sigma}(t) &\geq \frac{c}{2+c\alpha}x(\tau(t^*)), \quad t \geq t_1.
\end{aligned}$$

It follows that

$$x^{\sigma}(t) \geq \frac{c}{2+c\alpha}x(\tau(t^*)) \text{ and } x(\tau(t^*)) \geq \frac{c}{2+c\alpha}x(\tau(t)), \quad t \geq t_1.$$

Hence we have

$$\frac{x(\tau(t))}{x^{\sigma}(t)} \leq \left(\frac{2+c\alpha}{c}\right)^{2}, \quad t \geq t_{1.}$$

This is contradiction with (5.7). The proof of Theorem 5.2 is complete. ■

5.4 Examples

Example 5.1 Consider the delay dynamic equation on $\mathbb{T} = h\mathbb{Z} = \{hk,\ k \in \mathbb{Z}\}$ where $h > 1$,

$$x^{\Delta}(t) + \frac{3}{10h}x^{\sigma}(t) + \frac{1}{6h}x(t-h) = 0, \quad t \geq h. \tag{5.8}$$

If we take $k_1 = \frac{3}{2}$ and $k_2 = 6$, we get

$$\begin{aligned}
&\int_{\sigma(t)}^{t-h} \xi_{\mu(s)}\left[\ominus\left(p(s)+k_1 q(s)\right)\right]\Delta s \\
&= \int_{t-h}^{t+h} \frac{1}{h}\log\left[1+h\left(\frac{3}{10h}+\frac{3}{2}\frac{1}{6h}\right)\right]\Delta s \\
&= \frac{1}{h}\log\left(\frac{31}{20}\right)\int_{t-h}^{t+h}\Delta s \\
&= 2\log\left(\frac{31}{20}\right) \geq \log k_1,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\sigma(t)}^{t-h} \xi_{\mu(s)}\left[\ominus\left(p(s)+k_2 q(s)\right)\right]\Delta s \\
&= \int_{t-h}^{t+h} \frac{1}{h}\log\left[1+h\left(\frac{3}{10h}+6\frac{1}{6h}\right)\right]\Delta s \\
&= \int_{t-h}^{t+h} \frac{1}{h}\log\left(\frac{23}{10}\right)\Delta s \\
&= 2\log\left(\frac{23}{10}\right) \leq \log k_2.
\end{aligned}$$

Then all conditions of Theorem 5.1 are satisfied and equation (5.8) has a solution which is bounded by functions

$$\begin{aligned}
u(t) &= \exp\left(\int_h^t \frac{1}{h}\log\left(\frac{23}{10}\right)\Delta s\right) = \exp\left(\log\left(\frac{23}{10}\right)\left(\frac{t-h}{h}\right)\right), \\
v(t) &= \exp\left(\int_h^t \frac{1}{h}\log\left(\frac{31}{20}\right)\Delta s\right) = \exp\left(\log\left(\frac{31}{20}\right)\left(\frac{t-h}{h}\right)\right), \quad t \geq h.
\end{aligned}$$

Example 5.2 Consider the delay dynamic equation on $\mathbb{T} = \mathbb{Z}$,

$$x^{\Delta}(t) + (1 - \log 2)x^{\sigma}(t) + \frac{\log 2}{16}x(t-3) = 0, \quad t \geq 3. \tag{5.9}$$

If we take $k = 16$, we have

$$\begin{aligned}
&\int_{\sigma(t)}^{t-3} \xi_{\mu(s)} \left[\ominus \left(p(s) + kq(s)\right)\right] \Delta s \\
&= \int_{t-3}^{t+1} \log \left[1 + \left(1 - \log 2 + 16\frac{\log 2}{16}\right)\right] \Delta s \\
&= 4 \log 2 = \log k, \quad t \geq 3.
\end{aligned}$$

Then all conditions of Corollary 5.1 are satisfied and equation (5.9) has a solution

$$x(t) = \exp(-\log 2(t-3)), \quad t \geq 3.$$

Example 5.3 Consider the delay dynamic equation on $\mathbb{T} = \mathbb{R}$,

$$x^{\Delta}(t) + \frac{1}{7t}x(t) + \frac{2}{7t}x\left(\frac{2}{3}t\right) = 0, \quad t \geq 1. \tag{5.10}$$

If we take $k_1 = \frac{6}{5}$ and $k_2 = 3$ we obtain

$$\begin{aligned}
\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)} \left[\ominus \left(p(s) + k_1 q(s)\right)\right] \Delta s &= \int_{\frac{2}{3}t}^{t} \left(\frac{1}{7s} + \frac{6}{5}\frac{2}{7s}\right) ds \geq \log k_1, \\
\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)} \left[\ominus \left(p(s) + k_2 q(s)\right)\right] \Delta s &= \int_{\frac{2}{3}t}^{t} \left(\frac{1}{7s} + 3\frac{2}{7s}\right) ds \leq \log k_2.
\end{aligned}$$

Then all conditions of theorem 5.1 are satisfied and equation (5.10) has a positive solution.

Chapter 6

Existence and stability of positive periodic solutions for nonlinear delay integro-dynamic equations

Keywords. Positive periodic solutions, Stability, Schauder's fixed point theorem, integro-dynamic equations, time scales.

The goal of this chapter is to present a very recent work [21], namely,
F. Bouchelaghem, A. Ardjouni and A. Djoudi, *Existence and stability of positive periodic solutions for nonlinear delay integro-dynamic equations*, Submitted.

In this chapter, we study the existence of positive periodic and positive solutions for a integro-dynamic equations on time scales. The main tool employed here is the Schauder's fixed point theorem. The exponential stability of positive solutions is also treated.

6.1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this chapter, we consider the following nonlinear delay integro-dynamic equation

$$x^{\triangle}(t) + \int_{t-\tau}^{t} p(t-s)g(x(s))\Delta s, \ t \geq T. \tag{6.1}$$

Throughout this chapter we assume that $p \in C_{rd}([0,\tau),\mathbb{R})$, $g \in C((0,\infty),(0,\infty))$, $\tau, T \in \mathbb{T}$ are positive constants. To reach our desired end we have to transform (6.1) into an integral equation and then use Schauder's fixed point theorem to show the existence of positive periodic and positive solutions. The sufficient conditions for the exponential stability of positive solutions are also considered.

The organization of this chapter is as follows. In Section 2 and 3, we establish our main results for positive periodic and positive solutions by applying the Schauder's fixed point theorem. The exponential stability of the positive periodic solutions is the topic of Section 4. The results presented in this chapter extend the main results in [50].

6.2 Existence of periodic solutions

In this section we will study the existence of positive ω-periodic solutions of (6.1). In the next lemma and theorems we choose $T > \tau > 0$.

Lemma 6.1 *Suppose that there exists a positive continuous function $k(t,s)$, $t-\tau \le s \le t$, such that*

$$\begin{aligned}&\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v \in \mathcal{R}^+,\\ &\int_{t}^{t+\omega} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u = 0,\ t \ge T.\end{aligned} \tag{6.2}$$

Then the function

$$f(t) = \exp\left(\int_{T}^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right),\ t \ge T,$$

is ω-periodic.

Proof. For $t \ge T$ we obtain

$$\begin{aligned} f(t+\omega) &= \exp\left(\int_{T}^{t+\omega} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)\\ &= \exp\left(\int_{T}^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)\\ &\times \exp\left(\int_{t}^{t+\omega} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)\\ &= f(t). \end{aligned}$$

Thus the function f is ω-periodic. ■

Theorem 6.1 *Suppose that there exists a positive continuous function $k(t,s)$, $t-\tau \leq s \leq t$, such that (6.2) holds and*

$$\exp\left(\int_T^{\sigma(t)} \xi_{\mu(u)}\left[\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)$$
$$\times g\left(\exp\left(\int_T^{s} \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)\right) = k(t,s) \quad t \geq T. \tag{6.3}$$

Then (6.1) has a positive ω-periodic solution.

Proof. Let $X = C_{rd}\left([T-\tau,\infty),\mathbb{R}\right)$ be the Banach space with the norm $\|x\| = \sup_{t\geq T-\tau}|x(t)|$. We set

$$f(t) = \exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right),\ t \geq T.$$

With regard to Lemma 6.1 we have $m \leq f(t) \leq M$, where

$$m = \min_{t\in[T,\infty)}\left\{\exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)\right\},$$
$$M = \max_{t\in[T,\infty)}\left\{\exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right)\right\}. \tag{6.4}$$

We now define a closed, bounded and convex subset Ω of X as follows

$$\begin{aligned}\Omega = \{x \in X : & x(t+\omega) = x(t),\ t \geq T,\\ & m \leq x(t) \leq M,\ t \geq T,\\ & k(t,s)x^{\sigma}(t) = g\left(x(s)\right),\ t \geq T,\ t-\tau \leq s \leq t,\\ & x(t) = 1,\ T-\tau \leq t \leq T\}.\end{aligned}$$

Define the operator $S : \Omega \longrightarrow X$ as follows

$$(Sx)(t) = \begin{cases} \exp\left(\int_T^t \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)\frac{g(x(v))}{x^{\sigma}(u)}\Delta v\right]\Delta u\right), & t \geq T,\\ 1, & T-\tau \leq t \leq T.\end{cases}$$

We will show that for any $x \in \Omega$ we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$ we get

$$\begin{aligned}(Sx)(t) &= \exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)\frac{g(x(v))}{x^{\sigma}(u)}\Delta v\right]\Delta u\right)\\ &= \exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right) \leq M,\end{aligned}$$

and $(Sx)(t) \geq m$. For $t \in [T-\tau, T]$ we have $(Sx)(t) = 1$, that is $(Sx)(t) \in \Omega$.

Further for every $x \in \Omega$ and $t \geq T$, $T - \tau \leq s \leq t$, according to (6.3) it follows

$$\begin{aligned}
g\left((Sx)(s)\right) &= g\left(\exp\left(\int_T^s \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right)\right) \\
&= \exp\left(\int_T^{\sigma(t)} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right) \\
&\times \exp\left(\int_T^{\sigma(t)} \xi_{\mu(u)}\left[\int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right) \\
&\times g\left(\exp\left(\int_T^s \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right)\right) \\
&= \exp\left(\int_T^{\sigma(t)} \xi_{\mu(u)}\left[\int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right) \\
&\times g\left(\exp\left(\int_T^s \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right)\right)(Sx)^\sigma(t) \\
&= k(t,s)(Sx)^\sigma(t).
\end{aligned}$$

Finally we will show that for $x \in \Omega$, $t \geq T$ the function Sx is ω-periodic. For $x \in \Omega$, $t \geq T$ and with regard to (6.2) we get

$$\begin{aligned}
(Sx)(t+\omega) &= \exp\left(\int_T^{t+\omega} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right) \\
&= \exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right) \\
&\times \exp\left(\int_t^{t+\omega} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right) \\
&= (Sx)(t)\exp\left(\int_t^{t+\omega} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right) = (Sx)(t).
\end{aligned}$$

So Sx is ω-periodic on $[T, \infty)$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is completely continuous. First we will show that S is continuous. Let $x_i \in \Omega$ be such that $x_i \to x \in \Omega$ as $i \to \infty$. For $t \geq T$ we have

$$\begin{aligned}
&|(Sx_i)(t) - (Sx)(t)| \\
&= \left|\exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x_i(v))}{x_i^\sigma(u)}\Delta v\right]\Delta u\right)\right. \\
&\left. - \exp\left(\int_T^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right)\right|.
\end{aligned}$$

By applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{i\to\infty} \|Sx_i - Sx\| = 0.$$

For $t \in [T-\tau, T]$, the relation above is also valid. This means that S is continuous.

We now show that $S\Omega$ is relatively compact. It is sufficient to show by the Arzela–Ascoli theorem that the family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[T-\tau, \infty)$. The uniform boundedness follows from the definition of Ω. According to (6.4) for $t \geq T$, $x \in \Omega$ we get

$$\begin{aligned}
\left|(Sx)^{\Delta}(t)\right| &= \left|\ominus \int_{t-\tau}^{t} p(t-v)\frac{g(x\,(v))}{x^{\sigma}(t)}\Delta v\right| \\
&\times \exp\left(\int_{T}^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)\frac{g(x\,(v))}{x^{\sigma}(u)}\Delta v\right]\Delta u\right) \\
&= \left|\ominus \int_{t-\tau}^{t} p(t-v)k(t,v)\Delta v\right| \\
&\times \exp\left(\int_{T}^{t} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right) \\
&\leq M_1,\ \ M_1 > 0.
\end{aligned}$$

For $t \in [T-\tau, T]\,$, $x \in \Omega$ we have

$$\left|(Sx)^{\Delta}(t)\right| = 0.$$

This shows the equicontinuity of the family $S\Omega$. Hence $S\Omega$ is relatively compact and therefore S is completely continuous. By Theorem 2.5 there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that x_0 is a positive ω-periodic solution of (6.1). The proof is complete. ∎

Corollary 6.1 *Suppose that there exists a positive continuous function $k(t,s)$, $t-\tau \leq s \leq t$, such that (6.2) holds and*

$$\exp\left(\int_{s}^{\sigma(t)} \xi_{\mu(u)}\left[\int_{u-\tau}^{u} p(u-v)k(u,v)\Delta v\right]\Delta u\right) = k(t,s),\ \ t \geq T. \tag{6.5}$$

Then the equation

$$x^{\triangle}\,(t) + \int_{t-\tau}^{t} p(t-s)x(s)\Delta s,\ \ t \geq T, \tag{6.6}$$

has a positive ω-periodic solution

$$x(t) = \exp\left(\int_T^t \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right),\ t \geq T.$$

6.3 Existence of positive solutions

In this section we will investigate the existence of positive solutions of nonlinear integro-dynamic equation (6.1).

Theorem 6.2 *Suppose that there exists a positive continuous function $k(t,s)$, $t-\tau \leq s \leq t$, such that (6.3) holds and*

$$\int_{t-\tau}^t p(t-s)k(t,s)\Delta s > 0,\ t \geq T. \tag{6.7}$$

Then (6.1) has a positive solution

$$x(t) = \exp\left(\int_T^t \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right),\ t \geq T.$$

Proof. Let $X_1 = x \in C_{rd}([T-\tau,\infty),\mathbb{R})$ be the set of all rd-continuous bounded functions. Then X_1 is a Banach space with the norm $\|x\| = \sup_{t\geq T-\tau}|x(t)|$. We set

$$w(t) = \exp\left(\int_T^{\sigma(t)} \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right),\ t \geq T.$$

We define a closed, bounded and convex subset Ω_1 of X_1 as follows

$$\begin{aligned}\Omega_1 = \{x \in X_1 : w(t) \leq x(t) \leq 1,\ t \geq T,&\\ k(t,s)x^\sigma(t) = g(x(s)),\ t \geq T,\ t-\tau \leq s \leq t,&\\ x(t) = 1,\ T-\tau \leq t \leq T\}.&\end{aligned}$$

Define the operator $S_1 : \Omega_1 \longrightarrow X_1$ as follows

$$(S_1x)(t) = \begin{cases} \exp\left(\int_T^t \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)\frac{g(x(v))}{x^\sigma(u)}\Delta v\right]\Delta u\right), & t \geq T,\\ 1, & T-\tau \leq t \leq T.\end{cases}$$

For every $x \in \Omega_1$ and $t \geq T$ we obtain

$$(S_1x)(t) = \exp\left(\int_T^t \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right) \leq 1,$$

and $(S_1x)(t) \geq w(t)$. For $t \in [T-\tau, T]$ we get $(S_1x)(t) = 1$, that is $(S_1x)(t) \in \Omega$. Now we can proceed by the similar way as in the proof of Theorem 6.1. We omit the rest of the proof. ■

Corollary 6.2 *Assume that there exists a positive and continuous function $k(t,s)$, $t-\tau \leq s \leq t$, such that (6.5) and (6.7) hold. Then (6.6) has a positive solution*

$$x(t) = \exp\left(\int_T^t \xi_{\mu(u)}\left[\ominus \int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u\right), \ t \geq T.$$

Corollary 6.3 *Assume that there exists a positive continuous function $k(t,s)$, $t-\tau \leq s \leq t$, such that (6.3) and (6.7) hold and*

$$\lim_{t\to\infty}\int_T^t \xi_{\mu(u)}\left[\int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u = \infty.$$

Then (6.1) has a positive solution which tends to zero.

Corollary 6.4 *Assume that there exists a positive continuous function $k(t,s)$, $t-\tau \leq s \leq t$, such that (6.3) and (6.7) hold and*

$$\lim_{t\to\infty}\int_T^t \xi_{\mu(u)}\left[\int_{u-\tau}^u p(u-v)k(u,v)\Delta v\right]\Delta u = a.$$

Then (6.1) has a positive solution which tends to constant $e_{\ominus a}$.

6.4 Stability of positive solutions

In this section we consider the exponential stability of positive solution of (6.1). We denote $x(t,T,\psi)$, $t \geq T-\tau$, for a solution of (6.1) satisfying the initial condition $x(s,T,\psi) = \psi(s) > 0$ for $s \in [T-\tau, T]$. Let $x(t) = x(t,T,\psi)$, $x_1(t) = x(t,T,\psi_1)$ and $y(t) = x(t) - x_1(t)$, $t \in [T-\tau, \infty)$. Then we get

$$y^{\Delta}(t) + \int_{t-\tau}^t p(t-s)\left[g(x(s)) - g(x_1(s))\right]\Delta s = 0, \ t \geq T.$$

By the mean value theorem we obtain

$$y^{\Delta}(t) + \int_{t-\tau}^t p(t-s)g'(x_*)\left[x(t) - x_1(t)\right]\Delta s = 0, \quad g'(x) = \frac{dg(x)}{dx},$$

$$y^{\Delta}(t) + \int_{t-\tau}^t p(t-s)g'(x_*)y(s)\Delta s = 0, \ t \geq T. \tag{6.8}$$

Definition 6.1 Let x_1 be a positive solution of (6.1) and there exist constants T_{ψ,x_1}, K_{ψ,x_1} and $\lambda > 0$ such that for every solution $x(t,T,\psi)$ of (6.1)

$$|x(t,T,\psi) - x_1(t)| \leq K_{\psi,x_1} e_{\ominus\lambda}(t,0),\ t \geq T_{\psi,x_1}.$$

Then x_1 is said to be exponentially stable.

We assume that the function

$$F(t,x) = \int_{t-\tau}^{t} p(t-s)g(x(s))\Delta s,\ t \geq T,$$

is Lipschitzian in second argument.

In the next theorem we establish sufficient conditions for the exponential stability of positive solution $x_1(t) = x(t,T,1)$ of (6.1).

Theorem 6.3 *Suppose that (6.3) and (6.7) hold and*

$$p \in C_{rd}\left([0,\tau], (0,\infty)\right),\quad g \in C^1\left((0,\infty), (0,\infty)\right),\quad g'(x) \geq c > 0.$$

Then (6.1) has a positive solution which is exponentially stable.

Proof. We will show that there exists a positive λ such that

$$|x(t,T,\psi) - x_1(t)| \leq K_{\psi,x_1} e_{\ominus\lambda}(t,0),\ t \geq T_1 = T + 2\tau,$$

where $K_{\psi,x_1} = \max_{t\in[T-\tau,T_1]} |y(t)|\, e_\lambda(T_1,0) + 1$. Consider the Lyapunov function

$$L(t) = |y(t)|\, e_\lambda(t,0),\ t \geq T_1.$$

We claim that $L(t) \leq K_{\psi,x_1}$ for $t \geq T_1$. On the other hand there exists $t_* > T_1$ such that $L(t_*) \leq K_{\psi,x_1}$. Calculating the upper left derivative of $L(t)$ along the solution y of (6.8) we obtain

$$(L(t))^{\Delta^-} = -e_\lambda(t,0)\int_{t-\tau}^{t} p(t-s)g'(x_*)\,|y(s)|\,\Delta s + \lambda e_\lambda(t,0)\,|y^\sigma(t)|\,,\ t \geq T_1.$$

For $t = t_*$ t we get

$$0 \leq (L(t_*))^{\Delta^-} = -ce_\lambda(t_*,0)\int_{t_*-\tau}^{t_*} p(t_*-s)\,|y(s)|\,\Delta s + \lambda e_\lambda(t_*,0)\,|y^\sigma(t_*)|\,.$$

If $y(t) > 0$, $t \geq T$ then from (6.8) it follows that for $t \geq T+\tau$ the function y is decreasing and if $y(t) < 0$, $t \geq T$ then y is increasing for $t \geq T+\tau$. We conclude that $|y(t)|$, $t \geq T+\tau$ has decreasing character. Then we obtain

$$\begin{aligned}
0 \leq (L(t_*))^{\Delta^-} &\leq -c\,|y(t_*)|\, e_\lambda(t_*,0)\int_{t_*-\tau}^{t_*} p(t_*-s)\Delta s + \lambda e_\lambda(t_*,0)\,|y^\sigma(t_*)| \\
&= \left(-c\,|y(t_*)|\int_0^\tau p(u)\Delta u + \lambda\,|y^\sigma(t_*)|\right) e_\lambda(t_*,0) \\
&\leq \left(-c\int_0^\tau p(u)\Delta u + \lambda\right)|y(t_*)|\, e_\lambda(t_*,0).
\end{aligned}$$

For $0 < \lambda < c\int_0^\tau p(u)\Delta u$ we have a contradiction. Thus $|y(t)|\, e_\lambda(t,0) \leq K_{\psi,x_1}$ for $t \geq T_1$ and $0 < \lambda < c\int_0^\tau p(u)\Delta u$. The proof is complete. ■

Chapter 7

Existence, uniqueness and stability of periodic solutions for nonlinear neutral dynamic equations

Keywords. Fixed point, Periodic solutions, stability, dynamic equations, time scales.

The goal of this chapter is to present a very recent work [22], namely,
F. Bouchelaghem, A. Ardjouni and A. Djoudi, *Existence, uniqueness and stability of periodic solutions for nonlinear neutral dynamic equations*, Kragujevac Journal of Mathematics, Accepted.

In this chapter, a nonlinear neutral dynamic equation with periodic coefficients is considered. By using Krasnoselskii's fixed point theorem we obtain the existence of periodic and positive periodic solutions and by contraction mapping principle we obtain the uniqueness. Stability results of this equation are analyzed.

7.1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this chapter, we consider the following nonlinear neutral dynamic equation

$$\begin{aligned}&[u(t) - g(u(t-\tau(t)))]^{\Delta}\\&= p(t) - a(t)u^{\sigma}(t) - a(t)g(u^{\sigma}(t-\tau(t))) - h(u(t), u(t-\tau(t))).\end{aligned} \tag{7.1}$$

Throughout this chapter we assume that a, p and τ are real valued rd-continuous functions with a and τ are positive functions, $id - \tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $u(t-\tau(t))$ is well defined over $\mathbb{T}$. The functions g and h are continuous in their respective arguments. To reach our desired end we have to transform (7.1) into an integral equation written as a sum of two mapping, one is a contraction and the other is continuous and compact. After that, we use Krasnoselskii's fixed point theorem, to show the existence of periodic and positive periodic solutions. We also obtain the existence of a unique periodic solution by employing the contraction mapping principle. In addition to the study of existence and uniqueness, in this research we obtain sufficient conditions for the stability of the periodic solution by using the contraction mapping principle.

The organization of this chapter is as follows. In Section 2, we establish the existence and uniqueness of periodic solutions. In Section 3, we give sufficient conditions to ensure the existence of positive periodic solutions. The stability of the periodic solution is the topic of Section 4. The results presented in this chapter extend the main results in [87].

7.2 Existence and uniqueness of periodic solutions

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = n\omega$ for some $n \in \mathbb{N}$. By the notation $[a, b]$ we mean

$$[a,b] = \{t \in \mathbb{T},\ a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a, b)$, $(a, b]$ and (a, b) are defined similarly.

Define $C_T = \{\varphi \in C(\mathbb{T}, \mathbb{R}) : \varphi(t+T) = \varphi(t)\}$ where $C(\mathbb{T}, \mathbb{R})$ is the space of all real-valued rd-continuous functions. Then $(C_T, \|.\|)$ is a Banach space when it is endowed with the supremum norm

$$\|\varphi\| = \max_{t\in[0,T]} |\varphi(t)| .$$

We will need the following lemma whose proof can be found in [69].

Lemma 7.1 *Let $x \in C_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this chapter we assume that $a \in \mathcal{R}^+$, $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a(t+T) = a(t),\ p(t+T) = p(t),\ \ (id-\tau)(t+T) = (id-\tau)(t), \tag{7.2}$$

with $\tau(t) \geq \tau^* > 0$ and

$$e_a(T,0) > 1. \tag{7.3}$$

The functions $g(x)$, $h(x,y)$ are also globally Lipschitz continuous in x and in x and y, respectively. That, there are positive constants k_1, k_2 and k_3 such that

$$|g(x) - g(y)| \leq k_1 \|x-y\| \text{ and } k_1 < 1, \tag{7.4}$$

and

$$|h(x,y) - h(z,w)| \leq k_2 \|x-z\| + k_3 \|y-w\| . \tag{7.5}$$

Lemma 7.2 *Suppose (7.2) and (7.3) hold. If $u \in C_T$, then u is a solution of (7.1) if and only if*

$$\begin{aligned} u(t) &= g(u(t-\tau(t))) \\ &\quad + \gamma \int_t^{t+T} [p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h(u(s), u(s-\tau(s)))]\, e_{\ominus a}(t,s)\Delta s, \end{aligned} \tag{7.6}$$

where

$$\gamma = (e_a(T,0) - 1)^{-1}.$$

Proof. Let $u \in C_T$ be a solution of (7.1). Multiply both sides of (7.1) by $e_a(t,0)$ and then integrate from t to $t+T$, to obtain

$$\begin{aligned} &\int_t^{t+T} \left[(u(s) - g(u(s-\tau(s))))^\Delta e_a(s,0)\right] \Delta s \\ &= -\int_t^{t+T} a(s)\left[u^\sigma(s) - g(u^\sigma(s-\tau(s))\right] e_a(s,0)\Delta s \\ &+ \int_t^{t+T} [p(s) - 2a(s)g(u^\sigma(s-\tau(s)) - h(u(s), u(s-\tau(s))))]\, e_a(s,0)\Delta s. \end{aligned}$$

Performing an integration by part, we obtain

$$\begin{aligned}
&[u(t)-g(u(t-\tau(t)))]\,e_a(t,0)\,(e_a(T,0)-1)\\
&-\int_t^{t+T} a(s)\,[u^\sigma(s)-g(u^\sigma(s-\tau(s)))]\,e_a(s,0)\Delta s\\
&=-\int_t^{t+T} a(s)\,[u^\sigma(s)-g\,(u^\sigma(s-\tau\,(s))]\,e_a(s,0)\Delta s\\
&+\int_t^{t+T}[p(s)-2a(s)g\,(u^\sigma(s-\tau\,(s))-h\,(u(s),u(s-\tau(s))))]\,e_a(s,0)\Delta s.
\end{aligned}$$

By dividing both sides of the above equation by $e_a(t,0)\,(e_a(T,0)-1)$, we arrive at

$$\begin{aligned}
u(t)&=g(u(t-\tau(t)))+(e_a(T,0)-1)^{-1}\\
&\times\int_t^{t+T}[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h(u(s),u(t-\tau(s)))]\,e_{\ominus a}(t,s)\Delta s.
\end{aligned}$$

The converse implication is easily obtained and the proof is complete. ■

By applying Theorems 2.3 and 2.6, we obtain in this Section the existence and the uniqueness of periodic solution of (7.1). So, let a Banach space $(C_T,\|.\|)$, a closed bounded convex subset of C_T,

$$\Omega=\{\varphi\in C_T,\ \ \|\varphi\|\leq L\},\tag{7.7}$$

with $L>0$, and by the Lemma 7.2, we define the mapping S given by

$$\begin{aligned}
&(S\varphi)\,(t)\\
&=g\,(\varphi\,(t-\tau\,(t)))\\
&+\gamma\int_t^{t+T}[p(s)-2a\,(s)\,g\,(\varphi^\sigma\,(s-\tau\,(s)))-h\,(\varphi\,(s)\,,\varphi\,(s-\tau\,(s)))]\,e_{\ominus a}(t,s)\Delta s.
\end{aligned}\tag{7.8}$$

Therefore, we express (7.8) as

$$S\varphi=A\varphi+B\varphi,$$

where A and B are given by

$$\begin{aligned}
&(A\varphi)\,(t)\\
&=\gamma\int_t^{t+T}[p(s)-2a\,(s)\,g\,(\varphi^\sigma\,(s-\tau\,(s)))-h\,(\varphi\,(s)\,,\varphi\,(s-\tau\,(s)))]\,e_{\ominus a}(t,s)\Delta s,
\end{aligned}\tag{7.9}$$

and

$$(B\varphi)(t) = g(\varphi(t-\tau(t))). \tag{7.10}$$

Since $\varphi \in C_T$ and (7.2) holds, we have for any $\varphi \in \Omega$

$$\begin{aligned}
&(A\varphi)(t+T)\\
&= \gamma \int_{t+T}^{t+T+T} [p(s) - 2a(s)\, g(\varphi^{\sigma}(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))]\, e_{\ominus a}(t+T, s)\Delta s\\
&= \gamma \int_{t}^{t+T} [p(s+T) - 2a(s+T)\, g(\varphi^{\sigma}(s+T-\tau(s+T)))\\
&- h(\varphi(s+T), \varphi(s+T-\tau(s+T)))]\, e_{\ominus a}(t+T, s+T)\Delta s\\
&= (A\varphi)(t),
\end{aligned}$$

and

$$(B\varphi)(t+T) = g(\varphi(t+T-\tau(t+T))) = g(\varphi(t-\tau(t))) = (B\varphi)(t).$$

Then

$$A\Omega, B\Omega \subset C_T. \tag{7.11}$$

Theorem 7.1 *Assume that (7.2)–(7.5) hold. Let a constant $L > 0$ defined in Ω such that*

$$k_1 L + |g(0)| + \gamma\beta T\,(\mu + 2\lambda k_1 L + |g(0)| + k_2 L + k_3 L + |h(0,0)|) \le L, \tag{7.12}$$

where

$$\beta = e_a(T, 0),\ \lambda = \sup_{t\in[0,T]} \{a(t)\},\ \mu = \sup_{t\in[0,T]} |p(t)|.$$

Then (7.1) has a T-periodic solution.

Proof. First, let A defined by (7.9), we show that A is continuous in the supremum norm and the image of A is contained in a compact set. Let $\varphi_n \in \Omega$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&|(A\varphi_n)(t) - (A\varphi)(t)|\\
&\le 2\gamma \int_{t}^{t+T} a(s)\, |g(\varphi_n^{\sigma}(s-\tau(s))) - g(\varphi^{\sigma}(s-\tau(s)))|\, e_{\ominus a}(t, s)\Delta s\\
&+ \gamma \int_{t}^{t+T} |h(\varphi_n(s), \varphi_n(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))|\, e_{\ominus a}(t, s)\Delta s.
\end{aligned}$$

Since g and h are continuous, the dominated convergence theorem implies,

$$\lim_{n\to\infty} \left|\left(A\varphi_n\right)(t) - \left(A\varphi\right)(t)\right| = 0,$$

then A is continuous. Now, by (7.4) and (7.5), we obtain

$$|g(y)| \leq k_1 |y| + |g(0)|,$$
$$|h(x,y)| \leq k_2 |x| + k_3 |y| + |h(0,0)|.$$

Then, let $\varphi_n \in \Omega$ where n is a positive integer, we have

$$\begin{aligned}
&\left|\left(A\varphi_n\right)(t)\right| \\
&\leq \gamma \int_t^{t+T} \left[|p(s)| + 2a(s)\left|g\left(\varphi_n^\sigma\left(s-\tau(s)\right)\right)\right| + \left|h\left(\varphi_n(s), \varphi_n\left(s-\tau(s)\right)\right)\right|\right] e_{\ominus a}(t,s)\Delta s \\
&\leq \gamma \int_t^{t+T} \left[p(s) + 2a(s)\left(k_1\left\|\varphi_n^\sigma\right\| + |g(0)|\right) + k_2\left\|\varphi_n\right\| + k_3\left\|\varphi_n\right\| + |h(0,0)|\right] e_{\ominus a}(t,s)\Delta s \\
&\leq \gamma\beta T\left(\mu + 2\lambda\left(k_1 L + |g(0)|\right) + k_2 L + k_3 L + |h(0,0)|\right) \leq L,
\end{aligned}$$

by (7.12). Next, we calculate $(A\varphi_n)^\Delta(t)$ and show that it is uniformly bounded. By making use of (7.2) we obtain by taking the derivative in (7.9) that

$$(A\varphi_n)^\Delta(t) = -a(t)\left(A\varphi_n\right)^\sigma(t) + p(t) - 2a(t)g(\varphi_n^\sigma(t-\tau(t))) - h(\varphi_n(t), \varphi_n(t-\tau(t))).$$

Then, by (7.5) and (7.12) we have

$$\left|(A\varphi_n)^\Delta(t)\right| \leq \lambda L + \mu + 2\lambda\left(k_1 L + |g(0)|\right) + k_2 L + k_3 L + |h(0,0)| = Q,$$

Thus the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $\overline{A\Omega}$ is compact.

Second, let B be defined by (7.10). Then for $\varphi_1, \varphi_2 \in \Omega$ we have by (7.4)

$$\begin{aligned}
|(B\varphi_1)(t) - (B\varphi_2)(t)| &= \left|g\left(\varphi_1\left(t-\tau(t)\right)\right) - g\left(\varphi_2\left(t-\tau(t)\right)\right)\right| \\
&\leq k_1 \left\|\varphi_1 - \varphi_2\right\|.
\end{aligned}$$

Hence B is contraction because $k_1 < 1$.

7.2. Existence and uniqueness of periodic solutions

Finally, we show that if $\varphi, \phi \in \Omega$, then$\|A\varphi + B\phi\| \leq L$. Let $\varphi, \phi \in \Omega$ with $\|\varphi\|, \|\phi\| \leq L$, then

$$\begin{aligned}
&\|A\varphi + B\phi\| \\
&\leq k_1 \|\phi\| + |g(0)| \\
&+ \gamma \int_t^{t+T} [p(s) + 2a(s)(k_1 \|\varphi^\sigma\| + |g(0)|) + k_2 \|\varphi\| + k_3 \|\varphi\| + |h(0,0)|] e_{\ominus a}(t,s)\Delta s \\
&\leq k_1 L + |g(0)| + \gamma\beta T(\mu + 2\lambda(k_1 L + |g(0)|) + k_2 L + k_3 L + |h(0,0)|) \leq L,
\end{aligned}$$

by (7.12). Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \Omega$ such that $z = Az + Bz$. By Lemma 7.2 this fixed point is a solution of (7.1). Hence (7.1) has a T-periodic solution. ■

Theorem 7.2 *Suppose (7.2)–(7.5) hold. If*

$$k_1 + \gamma\beta T(2\lambda k_1 + k_2 + k_3) < 1, \tag{7.13}$$

then (7.1) has a unique T-periodic solution.

Proof. Let the mapping S be given by (7.8). For any $\varphi_1, \varphi_2 \in C_T$, we have

$$\begin{aligned}
&|(S\varphi_1)(t) - (S\varphi_2)(t)| \\
&\leq |g(\varphi_1(t-\tau(t))) - g(\varphi_2(t-\tau(t)))| \\
&+ 2\gamma \int_t^{t+T} a(s)|g(\varphi_1^\sigma(s-\tau(s))) - g(\varphi_2^\sigma(s-\tau(s)))| e_{\ominus a}(t,s)\Delta s \\
&+ \gamma \int_t^{t+T} |h(\varphi_1(s), \varphi_1(s-\tau(s))) - h(\varphi_2(s), \varphi_2(s-\tau(s)))| e_{\ominus a}(t,s)\Delta s \\
&\leq k_1 \|\varphi_1 - \varphi_2\| + \gamma \int_t^{t+T} (2\lambda k_1 + k_2 + k_3)\|\varphi_1 - \varphi_2\| e_{\ominus a}(t,s)\Delta s \\
&\leq [k_1 + \gamma\beta T(2\lambda k_1 + k_2 + k_3)] \|\varphi_1 - \varphi_2\|.
\end{aligned}$$

Since (7.13) hold, the contraction mapping principle completes the proof. ■

Corollary 7.1 *Suppose (7.2)–(7.5) hold and let β, λ and μ be constants defined in Theorem 7.1. Let Ω defined by (7.7). Suppose there are positive constants k_1^*, k_2^* and k_3^* such that for any $x, y, z, w \in \Omega$, we have*

$$|g(x) - g(y)| \leq k_1^* \|x - y\| \text{ and } k_1^* < 1, \tag{7.14}$$

$$|h(x,y) - h(z,w)| \leq k_2^* \|x - z\| + k_3^* \|y - w\|, \tag{7.15}$$

and

$$k_1^* L + |g(0)| + \gamma\beta T \left(\mu + 2\lambda \left(k_1^* L + |g(0)|\right) + k_2^* L + k_3^* L + |h(0,0)|\right) \leq L. \tag{7.16}$$

Then (7.1) has a T-periodic solution in Ω. Moreover, if

$$k_1^* + \gamma\beta T(2\lambda k_1^* + k_2^* + k_3^*) < 1, \tag{7.17}$$

then (7.1) has a unique T-periodic solution in Ω.

Proof. Let the mapping S defined by (7.8). Then the proof follow immediately from Theorem 7.1 and Theorem 7.2. ■

Notice that the constants k_1^*, k_2^* and k_3^* may depend on L.

7.3 Existence of positive periodic solutions

It is for sure that existence of positive solutions is important for many applied problems. In this Section, by applying the Krasnoselskii's fixed point theorem and some techniques, to establish a set of sufficient conditions which guarantee the existence of positive periodic solutions of (7.1). So, we let $(X, \|.\|) = (C_T, \|.\|)$ and $\Omega(E, K) = \{\varphi \in C_T : E \leq \varphi(t) \leq K \text{ for all } t \in [0, T]\}$, for any $0 < E < K$. We assume that, there exist constants a_1, a_2, g_1 and g_2 such that for all $(t, (x, y, z)) \in [0, T] \times [E, K]^3$ we have

$$0 \leq g_1, \ 0 \leq g_2 < 1, \ -g_1 y \leq g(y) \leq g_2 y, \tag{7.18}$$

$$0 < a_1 \leq a(t) \leq a_2, \tag{7.19}$$

$$(E + g_1 K)\, a_2 \leq p(t) - 2a(t)\, g(z) - h(x, y) \leq (1 - g_2)\, K a_1. \tag{7.20}$$

Theorem 7.3 *Assume that (7.2)–(7.5) and (7.18)–(7.20) hold. Then (7.1) has at least one positive T-periodic solution in $\Omega(E, K)$.*

Proof. By Lemma 7.2, it is obvious that (7.1) has a solution φ if and only if $S\varphi = \varphi$ has a solution φ. Let A, B defined by (7.9), (7.10) respectively. A change of variable

$t \longmapsto t+T$ in (7.9) and (7.10) show that for any $\varphi \in \Omega(E,K)$ and $t \in \mathbb{R}$

$$A\left(\Omega\left(E,K\right)\right) \subseteq C_T, \quad B\left(\Omega\left(E,K\right)\right) \subseteq C_T. \tag{7.21}$$

Arguing as in the Theorem 7.1, the operator A is continuous. Next, we claim that A is compact. It is sufficient to show that $A\left(\Omega\left(E,K\right)\right)$ is uniformly bounded and equicontinuous in $[0,T]$. Notice that (7.19) and (7.20) ensure that

$$\begin{aligned}
&\|A\varphi\| \\
&\leq \sup_{t\in[0,T]}\left|\gamma\int_t^{t+T}\left[p(s)-2a\left(s\right)g\left(\varphi^\sigma\left(s-\tau\left(s\right)\right)\right)-h\left(\varphi\left(s\right),\varphi\left(s-\tau\left(s\right)\right)\right)\right]e_{\ominus a}(t,s)\Delta s\right| \\
&\leq \left(1-g_2\right)K\gamma a_1 \sup_{t\in[0,T]}\int_t^{t+T} e_{\ominus a}(t,s)\Delta s \\
&\leq \left(1-g_2\right)K \text{ for all } \varphi \in [E,K],
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(A\varphi\right)^\Delta\left(t\right)\right| \\
&\leq a\left(t\right)\left|\left(A\varphi\right)^\sigma\left(t\right)\right| + \left|p(t)-2a\left(t\right)g\left(\varphi^\sigma\left(t-\tau\left(t\right)\right)\right)-h\left(\varphi\left(t\right),\varphi\left(s-\tau\left(t\right)\right)\right)\right| \\
&\leq a_2\left(1-g_1\right)K + \left(1-g_1\right)a_1K \\
&= \left(a_2+a_1\right)\left(1-g_1\right)k \text{ for all } \left(t,\varphi\right)\in[0,T]\times[E,K],
\end{aligned}$$

which give that $A(\Omega(E,K))$ is uniformly bounded and equicontinuous in $[0,T]$. Hence by Ascoli-Arzela's theorem A is compact. Next, let B defined by (7.10), for all $\varphi_1,\varphi_2 \in \Omega(E,K)$ and $t\in\mathbb{R}$, we obtain by (7.4)

$$\|B\varphi_1 - B\varphi_2\| \leq k_1\left\|\varphi_1-\varphi_2\right\|.$$

Thus B is a contraction. Moreover, by (7.18)–(7.20), we infer that for all $\varphi,\phi\in\Omega(E,K)$ and $t\in\mathbb{R}$

$$\begin{aligned}
&\left(A\varphi\right)\left(t\right)+\left(B\phi\right)\left(t\right) \\
&= g\left(\phi\left(t-\tau\left(t\right)\right)\right) \\
&+\gamma\int_t^{t+T}\left[p(s)-2a\left(s\right)g\left(\varphi^\sigma\left(s-\tau\left(s\right)\right)\right)-h\left(\varphi\left(s\right),\varphi\left(s-\tau\left(s\right)\right)\right)\right]e_{\ominus a}(t,s)\Delta s \\
&\leq g_2K+\left(1-g_2\right)K\gamma\int_t^{t+T} a\left(s\right)e_{\ominus a}(t,s)\Delta s = K,
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&(A\varphi)(t)+(B\phi)(t)\\
&\geq g\left(\phi\left(t-\tau\left(t\right)\right)\right)\\
&+\gamma\int_{t}^{t+T}\left[p(s)-2a\left(s\right)g\left(\varphi^{\sigma}\left(s-\tau\left(s\right)\right)\right)-h\left(\varphi\left(s\right),\varphi\left(s-\tau\left(s\right)\right)\right)\right]e_{\ominus a}(t,s)\Delta s\\
&\geq -g_{1}K+\left(E+g_{1}\right)K\gamma\int_{t}^{t+T}a\left(s\right)e_{\ominus a}(t,s)\Delta s=E,
\end{aligned}$$

which imply that

$$A\varphi+B\phi\in\Omega(E,K)\text{ for all }\varphi,\phi\in\Omega(E,K)\text{ and }t\in\mathbb{R}. \tag{7.22}$$

Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z\in\Omega(E,K)$ such that $z=Az+Bz$. By Lemma 7.2 this fixed point is a solution of (7.1). Hence (7.1) has a positive T-periodic solution. This completes the proof. ■

Theorem 7.4 *Assume that (7.2)–(7.5) hold. Suppose that there exist constants E, K, a_1, a_2, g_1, g_2 and $t_0\in[0,T]$ satisfying (7.18)–(7.20) with*

$$0\leq E<K, \tag{7.23}$$

and either

$$\left(E+g_{1}K\right)a_{2}<p\left(t_{0}\right)-2a\left(t_{0}\right)g\left(z\right)-h\left(x,y\right)\ \text{ for all } x,y,z\in[E,K], \tag{7.24}$$

or

$$a\left(t_{0}\right)<a_{2}. \tag{7.25}$$

Then (7.1) has at least one positive T-periodic solution in $\Omega(E,K)$ with $E<u\leq K$ for each $t\in[0,T]$.

Proof. As in the proof of Theorem 7.3, we conclude similarly that (7.1) has an T-periodic solution $u\in\Omega(E,K)$. Now we assert that $u(t)>E$ for all $t\in[0,T]$. Otherwise, there exists $t^* \in [0, T]$ satisfying $u(t^*) = E$. In view of (7.6), (7.8), (7.18) and (7.23), we have

$$\begin{aligned}
E &= g\left(u\left(t^* - \tau\left(t^*\right)\right)\right) \\
&+ \gamma \int_{t^*}^{t^*+T} \left[p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s - \tau\left(s\right)\right)\right)\right] e_{\ominus a}(t^*, s)\Delta s \\
&\geq \gamma \int_{t^*}^{t^*+T} \left[p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s - \tau\left(s\right)\right)\right)\right] e_{\ominus a}(t^*, s)\Delta s \\
&- g_1 K,
\end{aligned}$$

which implies that

$$\begin{aligned}
0 &\geq \gamma \int_{t^*}^{t^*+T} \left[p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s - \tau\left(s\right)\right)\right)\right] e_{\ominus a}(t^*, s)\Delta s \\
&- (E + g_1 K) \\
&= \gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s - \tau\left(s\right)\right)\right) \\
&-(E + g_1 K)a(s)]\, e_{\ominus a}(t^*, s)\Delta s. \qquad (7.26)
\end{aligned}$$

Assume that (7.24) holds. By means of (7.19), (7.20), (7.24) and the continuity of h, g, a, p, τ and u, we get that

$$\begin{aligned}
&\gamma \int_{t^*}^{t^*+T} e_{\ominus a}(t^*, s)\, [p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s - \tau\left(s\right)\right)\right) \\
&-(E + g_1 K)a(s)]\, \Delta s \\
&\geq \int_{t^*}^{t^*+T} e_{\ominus a}(t^*, s)\, [p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s - \tau\left(s\right)\right)\right) \\
&-(E + g_1 K)a_2]\, \Delta s \\
&> 0,
\end{aligned}$$

which contradicts (7.26).

Assume that (7.25) holds. In light of (7.19), (7.20), (7.25) and the continuity of h, g, a, p, τ and u, we get that

$$\begin{aligned}
&\gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s-\tau\left(s\right)\right)\right) \\
&-(E+g_1K)a(s)]\, e_{\ominus a}(t^*, s)\Delta s \\
&> \int_{t^*}^{t^*+T} e_{\ominus a}(t^*, s)\Delta s \int_{t^*}^{t^*+T} e_{\ominus a}(t^*, s)\, [p(s) - 2a(s)g(u^\sigma(s-\tau(s))) \\
&-h\left(u\left(s\right), u\left(s-\tau\left(s\right)\right)\right) - (E+g_1K)a_2]\, \Delta s \\
&> 0,
\end{aligned}$$

which contradicts (7.26). This completes the proof. ■

Example 7.1 Consider (7.1), where

$$\mathbb{T} = \mathbb{R},\ p(t) = 3 + \frac{\sin t}{5},\ a(t) = 1 + \frac{\cos t}{4},\ \tau(t) = 2\cos^2 t,$$

$$g(x) = -\frac{x \sin x}{20} \text{ for all } x \in \mathbb{R},$$

$$h(x, y) = 1 + \sin^2 x + \cos^2 y \text{ for all } (x, y) \in \mathbb{R}^2.$$

Let $T = 2\pi$, $K = 10$, $E = 1$, $g_1 = g_2 = \frac{1}{20}$, $a_1 = \frac{3}{4}$, $a_2 = \frac{5}{4}$, $k_1 = \frac{11}{20}$. It is easy to see that (7.4), (7.5) hold. Notice that

$$\begin{aligned}
(E + g_1 K)\, a_2 = \frac{15}{8} < \frac{195}{40} &= 3 - \frac{1}{5} + 2\left(1 - \frac{1}{4}\right)\frac{1}{20} + 2 \\
&\leq p\left(t\right) - 2a\left(t\right) g\left(z\right) - h\left(x, y\right) \\
&\leq 3 + \frac{1}{5} + 2(\frac{5}{4})(\frac{1}{20}) + 3 = \frac{253}{40} \\
&< \frac{285}{40} = \left(1 - g_2\right) K a_1 \text{ for all } (t, x, y, z) \in \mathbb{R}^4.
\end{aligned}$$

That is, (7.20) is satisfied. Thus Theorem 7.3 yields that (7.1) has a positive 2π-periodic solution in $\Omega(1, 10)$.

7.4 Stability of periodic solutions

This Section concerned with the stability of a T-periodic solution u^* of (7.1). Let $v = u - u^*$ then (7.1) is transformed as

$$\begin{aligned}&(v(t) - G(v(t-\tau(t))))^{\Delta}\\&= -a(t)v^{\sigma}(t) - a(t)G(v^{\sigma}(t-\tau(t))) - H(v(t), v(t-\tau(t))),\end{aligned} \tag{7.27}$$

where

$$G(v(t-\tau(t))) = g(u^*(t-\tau(t)) + v(t-\tau(t))) - g(u^*(t-\tau(t)),$$

and

$$\begin{aligned}&H(v(t), v(t-\tau(t)))\\&= h(u^*(t) + v(t)), u^*(t-\tau(t)) + v(t-\tau(t)) - h(u^*(t), u^*(t-\tau(t))).\end{aligned}$$

Clearly, (7.27) has trivial solution $v \equiv 0$, and the conditions (7.4) and (7.5) hold for G and H respectively. To arrive at the Lemma 7.2, as in the proof of this Lemma, multiply both sides of (7.27) by $e_a(t,0)$ and then integrate from 0 to t, to obtain

$$\begin{aligned}v(t) &= (v(0) - G(v(-\tau(0))))e_{\ominus a}(t,0) + G(v(t-\tau(t)))\\&\quad - \int_0^t [2a(s)G(v^{\sigma}(s-\tau(s))) + H(v(s), v(s-\tau(s)))]e_{\ominus a}(t,s)\Delta s.\end{aligned} \tag{7.28}$$

Thus, we see that v is a solution of (7.27) if and only if it satisfies (7.28). Assumed initial function

$$v(t) = \psi(t), \quad t \in [m_0, 0],$$

with $\psi \in C([m_0,0], R)$, $[m_0, 0] = \{s \leq 0 \mid s = t - \tau(t),\ t \geq 0\}$.

Define the set Ω_ψ by

$$\Omega_\psi = \{\varphi \in C_T,\ \|\varphi\| \leq R,\ \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0],\ \varphi(t) \to 0 \text{ as } t \to \infty\}, \tag{7.29}$$

for some positive constant R. Then, $(\Omega_\psi, \|.\|)$ is a complete metric space where $\|.\|$ is the supremum norm.

Theorem 7.5 *If (7.2), (7.4), (7.5) and*

$$e_{\ominus a}(t,0) \to 0 \ \textit{as} \ t \to \infty, \tag{7.30}$$

$$t - \tau(t) \to \infty \ \textit{as} \ t \to \infty, \tag{7.31}$$

$$k_1 + \int_0^t (2\lambda k_1 + k_2 + k_3)\, e_{\ominus a}(t,s)\Delta s \le \alpha < 1, \tag{7.32}$$

hold. Then every solution $v(t,0,\psi)$ of (7.27) with small continuous initial function ψ, is bounded and asymptotically stable.

Proof. Let the mapping $\mathcal{S}$ defined by $\psi(t)$ if $t \in [m_0, 0]$ and

$$\begin{aligned}(\mathcal{S}\varphi)(t) &= (\psi(0) - G(\psi(-\tau(0))))\, e_{\ominus a}(t,0) + G(\varphi(t-\tau(t))) \\ &\quad - \int_0^t [2a(s)G(\varphi^\sigma(s-\tau(s))) + H(\varphi(s), \varphi(s-\tau(s)))]\, e_{\ominus a}(t,s)\Delta s, \end{aligned} \tag{7.33}$$

if $t \ge 0$. Since G and H are continuous, it is easy to show that $\mathcal{S}\varphi$ is. Let ψ be a small given continuous initial function with $\|\psi\| < \delta$ $(\delta > 0)$. Then using the condition (7.32) and the definition of $\mathcal{S}$ in (7.33), we have for $\varphi \in \Omega_\psi$

$$\begin{aligned}|(\mathcal{S}\varphi)(t)| &\le |\psi(0) - G(\psi(-\tau(0)))|\, e_{\ominus a}(t,0) + k_1 R \\ &\quad + R\int_0^t (2\lambda k_1 + k_2 + k_3)\, e_{\ominus a}(t,s)\Delta s \\ &\le (1+k_1)\delta + k_1 R + R\int_0^t (2\lambda k_1 + k_2 + k_3)\, e_{\ominus a}(t,s)\Delta s \\ &\le (1+k_1)\delta + \alpha R \le R,\end{aligned}$$

which implies $\|\mathcal{S}\varphi\| \le R$, for the right δ. Next we show that $(\mathcal{S}\varphi)(t) \to 0$ as $t \to \infty$. The first term on the right side of (7.33) tends to zero, by condition (7.30). Also, the second term on the right side tends to zero, because of (7.31) and the fact that $\varphi \in \Omega_\psi$. Let $\epsilon > 0$ be given, then there exists a $t_1 > 0$ such that for $t > t_1$, $\varphi(t - \tau(t)) < \epsilon$. By the condition (7.30), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$e_{\ominus a}(t, t_2) < \frac{\epsilon}{\alpha R}.$$

Thus for $t > t_2$, we have

$$\begin{aligned}
&\left| \int_0^t \left[2a(s)G(\varphi^\sigma(s-\tau(s))) + H\left(\varphi\left(s\right), \varphi\left(s-\tau\left(s\right)\right)\right)\right] e_{\ominus a}(t,s)\Delta s \right| \\
&\leq R \int_0^{t_1} \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t,s)\Delta s + \epsilon \int_0^t \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t,s)\Delta s \\
&\leq R e_{\ominus a}(t,t_2) \int_0^t \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t_2,s)\Delta s + \alpha\epsilon \\
&\leq \alpha R e_{\ominus a}(t,t_2)\alpha + \alpha\epsilon < \alpha\epsilon + \epsilon.
\end{aligned}$$

Hence, $(\mathcal{S}\varphi)(t) \to 0$ as $t \to \infty$. It is natural now to prove that $\mathcal{S}$ is contraction under the supremum norm. Let $\varphi_1, \varphi_2 \in \Omega_\psi$. Then

$$\begin{aligned}
&\left|\left(\mathcal{S}\varphi_1\right)(t) - \left(\mathcal{S}\varphi_2\right)(t)\right| \\
&\leq \left|G\left(\varphi_1\left(t-\tau\left(t\right)\right)\right) - G\left(\varphi_2\left(t-\tau\left(t\right)\right)\right)\right| \\
&+ 2\lambda \int_0^t \left|G(\varphi_1^\sigma(s-\tau(s))) - G(\varphi_2^\sigma(s-\tau(s)))\right| e_{\ominus a}(t,s)\Delta s \\
&+ \int_0^t \left|H\left(\varphi_1\left(s\right), \varphi_1\left(s-\tau\left(s\right)\right)\right) - H\left(\varphi_2\left(s\right), \varphi_2\left(s-\tau\left(s\right)\right)\right)\right| e_{\ominus a}(t,s)\Delta s \\
&\leq k_1 \left\|\varphi_1 - \varphi_2\right\| + \int_0^t \left(2\lambda k_1 + k_2 + k_3\right) \left\|\varphi_1 - \varphi_2\right\| e_{\ominus a}(t_2,s)\Delta s \\
&\leq \left[k_1 + \int_0^t \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t_2,s)\Delta s\right] \left\|\varphi_1 - \varphi_2\right\| \\
&\leq \alpha \left\|\varphi_1 - \varphi_2\right\|.
\end{aligned}$$

Hence, the contraction mapping principle implies, $\mathcal{S}$ has a unique fixed point in Ω_ψ which solves (7.27), bounded and asymptotically stable. ■

Theorem 7.6 *If (7.2), (7.4), (7.5) and (7.32) hold. Then, the zero solution is stable.*

Proof. The stability of the zero solution of (7.27) follows simply by replacing R by ϵ in the above Theorem. ■

Chapter 8

Positive solutions for a second-order difference equation with summation boundary conditions

Keywords. Positive solutions, Krasnoselskii's fixed point theorem, difference equations, three-point summation boundary value problems, cones.

The goal of this chapter is to present a very recent work published in [26], namely, F. Bouchelaghem, A. Ardjouni and A. Djoudi, *Positive solutions for a second-order difference equation with summation boundary conditions*, Kragujevac Journal of Mathematics, 41(2) (2017), 166–177.

In this chapter we study the existence of positive solutions for a second-order difference equation with summation boundary conditions. The main tool employed here is the Krasnoselskii's fixed point theorem in a cone.

8.1 Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second order ordinary differential and difference equations was initiated by Ilin and Moiseev [65]. Then Gupta [56] studied three-point boundary value problems for nonlinear second order ordinary differential equations. Since then, the existence of positive solutions for nonlinear second-order three point boundary value problems have also been studied by many authors by using the fixed point theorems or coincidence degree theory, one may see the text books [2], [3] and the papers [16], [62], [76], [110].

Liu [84] proved the existence of single and multiple positive solutions for the three-point boundary value problem

$$\begin{cases} u''(t) + a(t) f(u(t)) = 0, \ t \in (0,1), \\ u'(0) = 0, \quad u(1) = \beta u(\eta), \end{cases}$$

where $0 < \eta < 1$ and $0 < \beta < 1$.

In [97], Sitthiwirattham and Ratanapun considered the following three-point summation boundary value problem

$$\begin{cases} \triangle^2 u(t-1) + a(t) f(u(t)) = 0, \ t \in \{1, 2, \ldots, T\}, \\ u(0) = 0, \quad u(T+1) = \alpha \sum\limits_{s=1}^{\eta} u(s), \end{cases}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \ldots, T-1\}$, $0 < \alpha < \dfrac{2T+2}{\eta(\eta+1)}$. They obtained the existence of positive solutions by using the Krasnoselskii's fixed point theorem in cones.

In this chapter, we are interested in the analysis of qualitative theory of the problems of the existence of positive solutions to second-order difference equations. Inspired and motivated by the works mentioned above and the references therein, we concentrate on the existence of positive solutions for the following second-order difference equation with three-point summation boundary value problem

$$\begin{cases} \triangle^2 u(t-1) + a(t) f(u(t)) = 0, \ t \in \{1, 2, \ldots, T\}, \\ \triangle u(0) = 0, \quad u(T+1) = \alpha \sum\limits_{s=1}^{\eta} u(s), \end{cases} \tag{8.1}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \ldots, T-1\}$.

The aim of this chapter is to give some results for existence of positive solutions to (8.1), assuming that $0 < \alpha < \dfrac{1}{\eta}$ and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

Let $\mathbb{N}$ be the non negative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} : i \le k \le j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (8.1), we mean that a function $u : \mathbb{N}_{T+1} \to [0, \infty)$ and satisfies the problem (8.1).

Throughout this chapter, we suppose the following conditions hold.

(A1) $f \in C\left([0, \infty), [0, \infty)\right)$,

(A2) $a \in C\left(\mathbb{N}_{T+1}, [0, \infty)\right)$ and there exists $t_0 \in \mathbb{N}_{\eta, T+1}$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone (Theorem 2.7).

8.2 Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 8.1 *Let $\alpha\eta \neq 1$. Then, for $y \in C\left(\mathbb{N}_{T+1}, [0, \infty)\right)$, the problem*

$$\begin{cases} \triangle^2 u(t-1) + y(t) = 0, \ t \in \mathbb{N}_{1,T}, \\ \triangle u(0) = 0, \quad u(T+1) = \alpha \sum\limits_{s=1}^{\eta} u(s), \end{cases} \tag{8.2}$$

has a unique solution

$$\begin{aligned} u(t) = & \frac{1}{1-\alpha\eta} \sum_{s=1}^{T} (T-s+1)\, y(s) \\ & - \frac{\alpha}{2(1-\alpha\eta)} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)\, y(s) \\ & - \sum_{s=1}^{t-1} (t-s)\, y(s), \ t \in \mathbb{N}_{T+1}. \end{aligned}$$

Proof. From (8.2), we get

$$\begin{aligned} \triangle u(t) - \triangle u(t-1) &= -y(t) \\ \triangle u(t-1) - \triangle u(t-2) &= -y(t-1) \\ &\vdots \\ \triangle u(1) - \triangle u(0) &= -y(1). \end{aligned}$$

We sum the above equations to obtain

$$\triangle u(t) = \triangle u(0) - \sum_{s=1}^{t} y(s), \ t \in \mathbb{N}_T,$$

from $\triangle u(0) = 0$, we have

$$\triangle u(t) = -\sum_{s=1}^{t} y(s), \ t \in \mathbb{N}_T. \tag{8.3}$$

We denote $\sum_{s=p}^{q} y(s) = 0$, if $p > q$. Similarly, we sum (8.3) from $t = 0$ to $t = h$, we get

$$u(h+1) = u(0) - \sum_{s=1}^{h} (h+1-s)\, y(s), \ h \in \mathbb{N}_T,$$

by changing the variable from $h+1$ to t, we have

$$u(t) = u(0) - \sum_{s=1}^{t-1} (t-s)\, y(s), \ t \in \mathbb{N}_{T+1}. \tag{8.4}$$

We sum (8.4) from $s = 1$ to $s = \eta$, we obtain

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \eta u(0) - \sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} (l-s)\, y(s) \\ &= \eta u(0) - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)\, y(s). \end{aligned}$$

By (8.4) from $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$, we get

$$(1-\alpha\eta)\, u(0) = \sum_{s=1}^{T} (T-s+1)\, y(s) - \frac{\alpha}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)\, y(s).$$

Therefore,

$$u(0) = \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,y(s) - \frac{\alpha}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)\,y(s).$$

Hence, (8.2) has a unique solution

$$u(t) = \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,y(s) - \frac{\alpha}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)\,y(s) - \sum_{s=1}^{t-1}(t-s)\,y(s),\ t \in \mathbb{N}_{T+1}.$$

■

Lemma 8.2 *Let $0 < \alpha < \frac{1}{\eta}$. If $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $y(t) \geq 0$ for $t \in \mathbb{N}_{T+1}$, then the unique solution u of (8.2) satisfies $u(t) \geq 0$ for $t \in \mathbb{N}_{T+1}$.*

Proof. From the fact that $\triangle^2 u(t-1) = u(t+1) - 2u(t) + u(t-1) = -y(t) \leq 0$, we know that $\triangle u(t)$ is a monotone decreasing sequence. Thus $\triangle u(t) \leq \triangle u(0) = 0$ and $u(t)$ is a monotone decreasing sequence, this is $u(t) \geq u(T+1)$. So, if $u(T+1) \geq 0$, then $u(t) \geq 0$ for $t \in \mathbb{N}_{T+1}$.

If $u(T+1) < 0$, then $\sum_{s=1}^{\eta} u(s) < 0$. Since $\sum_{s=1}^{\eta} u(s) \geq \eta u(\eta)$, we get

$$u(T+1) = \alpha\sum_{s=1}^{\eta} u(s) > \frac{1}{\eta}\sum_{s=1}^{\eta} u(s) \geq u(\eta),$$

that is

$$u(T+1) > u(\eta),$$

which contradicts the fact that $u(t)$ is a monotone decreasing sequence. ■

Lemma 8.3 *Let $\alpha > \frac{1}{\eta}$. If $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $y(t) \geq 0$ for $t \in \mathbb{N}_{T+1}$, then (8.2) has no positive solution.*

Proof. Assume (8.2) has a positive solution u.

If $u(T+1) > 0$, then $\sum_{s=1}^{\eta} u(s) > 0$. It implies

$$u\left(T+1\right) = \alpha \sum_{s=1}^{\eta} u(s) > u(\eta),$$

that is

$$u(T+1) > u(\eta),$$

which is a contradiction to the fact that $u(t)$ is a monotone decreasing sequence.

If $u\left(T+1\right) = 0$, then $\sum_{s=1}^{\eta} u(s) = 0$, this $u\left(t\right) = 0$ for all $t \in \mathbb{N}_{\eta}$. If there exists $t_0 \in \mathbb{N}_{\eta+1,T}$ such that $u\left(t_0\right) > 0$, the $u\left(0\right) = u\left(\eta\right) < u\left(t_0\right)$, a contradiction with the fact that $u(t)$ is a monotone decreasing sequence. Therefore, no positive solutions exist. ■

In the rest of the chapter, we assume that $0 < \alpha < \frac{1}{\eta}$. Moreover, we will work in the Banach space $C\left(\mathbb{N}_{T+1}, [0, \infty)\right)$, and only the sup norm is used.

Lemma 8.4 *Let $0 < \alpha < \frac{1}{\eta}$. If $y \in C\left(\mathbb{N}_{T+1}, [0, \infty)\right)$ and $y\left(t\right) \geq 0$ for $t \in \mathbb{N}_{T+1}$, then the unique solution u of (8.2) satisfies*

$$\min_{t \in \mathbb{N}_{T+1}} u(t) \geq \gamma \left\| u \right\|, \tag{8.5}$$

where

$$\gamma = \frac{\alpha\eta\left(T+1-\eta\right)}{T+1-\alpha\eta^2}. \tag{8.6}$$

Proof. By Lemma 8.2, we know that

$$u\left(T+1\right) \leq u\left(t\right) \leq u\left(0\right).$$

So

$$\min_{t \in \mathbb{N}_{T+1}} u\left(t\right) = u\left(T+1\right), \ \max_{t \in \mathbb{N}_{T+1}} u\left(t\right) = u\left(0\right). \tag{8.7}$$

From the fact that $u(t)$ is monotone decreasing, we get

$$u\left(T+1\right) = \alpha \sum_{s=1}^{\eta} u(s) \geq \alpha\eta u(\eta). \tag{8.8}$$

Since $\Delta^2 u\left(t\right) \leq 0$ and $\Delta u\left(t\right) \leq 0$ for $t \in \mathbb{N}_{T+1}$, we have

$$\frac{u(0) - u(T+1)}{-(T+1)} \geq \frac{u(T+1) - u(\eta)}{(T+1) - \eta}.$$

By (8.8), we get

$$
\begin{aligned}
u(0) &\leq u(T+1) - \frac{u(T+1) - u(\eta)}{(T+1) - \eta}(T+1) \\
&\leq u(T+1)\left(1 - \frac{1 - \frac{1}{\alpha\eta}}{T+1-\eta}(T+1)\right) \\
&= u(T+1)\frac{T+1-\alpha\eta^2}{\alpha\eta\,(T+1-\eta)}.
\end{aligned}
$$

Combining this with (8.7), we obtain

$$
\min_{t\in\mathbb{N}_{T+1}} u\,(t) \geq \frac{\alpha\eta\,(T+1-\eta)}{T+1-\alpha\eta^2}\,\|u\|\,.
$$

■

8.3 Positive solutions

Now we are in the position to establish the main results.

Theorem 8.1 *Assume (A1) and (A2) hold. Then the problem (8.2) has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. It is known that $0 < \alpha < \frac{1}{\eta}$. From Lemma 8.1, u is a solution to the boundary value problem (8.2) if and only if u is a fixed point of operator A, where A is defined by

$$
\begin{aligned}
u\,(t) =& \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a\,(s)\,f(u(s)) \\
&- \frac{\alpha}{2\,(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta-s)\,(\eta-s+1)\,a\,(s)\,f(u(s)) \\
&- \sum_{s=1}^{t-1}(t-s)\,a\,(s)\,f(u(s)) \\
&:= (Au)(t).
\end{aligned}
$$

Denote

$$
K = \left\{u \mid u \in C\left(\mathbb{N}_{T+1}, [0,\infty)\right),\ u \geq 0,\ \ \min_{t\in\mathbb{N}_{T+1}} u(t) \geq \gamma\,\|u\|\right\}.
$$

where γ is defined in (8.6).

It is obvious that K is a cone in $C(\mathbb{N}_{T+1}, [0,\infty))$. By Lemma 8.4, $A(K) \subseteq K$. It is also easy to check that $A : K \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $L_1 > 0$ so that $f(u) \le \varepsilon u$, for $0 < u \le L_1$, where $\varepsilon > 0$ satisfies

$$\frac{\varepsilon}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a(s) \le 1.$$

Thus, if we let

$$\Omega_1 = \{u \in C(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < L_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned} Au(t) &\le \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a(s)\,f(u(s)) \\ &\le \frac{\varepsilon}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a(s)\,u(s) \\ &\le \frac{\varepsilon}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a(s)\,\|u\| \\ &\le \|u\|. \end{aligned}$$

Thus $\|Au(t)\| \le \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $L_2 > 0$ such that $f(u) \ge \rho u$, for $u \ge L_2$, where $\rho > 0$ is chosen so that

$$\frac{\rho\gamma}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a(s) \ge 1.$$

Let $L = \max\left\{2L_1, \frac{L_2}{\gamma}\right\}$ and $\Omega_2 = \{u \in C(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < L\}$. Then $u \in K \cap \partial\Omega_2$ implies

$$\min_{t\in\mathbb{N}_{T+1}} u(t) \ge \gamma\|u\| = \gamma L \ge L_2,$$

and so

$$\begin{aligned}
Au(\eta) &= \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a(s)\,f(u(s)) \\
&- \frac{\alpha}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)\,a(s)\,f(u(s)) \\
&- \sum_{s=1}^{\eta-1}(\eta-s)\,a(s)\,f(u(s)) \\
&= \frac{1}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a(s)\,f(u(s)) + \frac{1}{1-\alpha\eta}\sum_{s=1}^{\eta-1}(T-s+1)\,a(s)\,f(u(s)) \\
&- \frac{\alpha}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta^2+\eta-2\eta s+s^2-s)a(s)\,f(u(s)) \\
&- \sum_{s=1}^{\eta-1}(\eta-s)\,a(s)\,f(u(s)) \\
&= \frac{1}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a(s)\,f(u(s)) \\
&+ \frac{2(T-\eta)+2-\alpha\eta}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}a(s)\,f(u(s)) + \frac{\alpha}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta(\eta-s)+s)a(s)\,f(u(s)) \\
&+ \frac{\alpha}{2(1-\alpha\eta)}\sum_{s=1}^{\eta-1}s(\eta-s)a(s)\,f(u(s)).
\end{aligned}$$

Hence

$$\begin{aligned}
Au(\eta) &\geq \frac{1}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a(s)\,f(u(s)) \\
&\geq \frac{\rho}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a(s)\,u(s) \\
&\geq \frac{\rho\gamma}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a(s)\,\|u\| \geq \|u\|\,.
\end{aligned}$$

Hence,$\|Au\| \geq \|u\|$, $u \in K\cap\partial\Omega_2$. Bay the first part of Theorem 2.7, A has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$ such that $L_1 \leq \|u\| \leq L$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, we may choose $L_3 > 0$ so that $f(u) \geq Mu$, for $0 < u \leq L_3$, where $M > 0$ satisfies

$$\frac{M\gamma}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a\,(s) \geq 1.$$

Thus, if we let

$$\Omega_3 = \{u \in C\,(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < L_3\}\,,$$

then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}
Au\,(\eta) &= \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a\,(s)\,f(u(s)) \\
&\quad - \frac{\alpha}{2\,(1-\alpha\eta)}\sum_{s=1}^{\eta-1}(\eta-s)\,(\eta-s+1)\,a\,(s)\,f(u(s)) \\
&\quad - \sum_{s=1}^{\eta-1}(\eta-s)\,a\,(s)\,f(u(s)) \\
&\geq \frac{1}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a\,(s)\,f(u(s)) \\
&\geq \frac{M}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a\,(s)\,u(s) \\
&\geq \frac{M\gamma}{1-\alpha\eta}\sum_{s=\eta}^{T}(T-s+1)\,a\,(s)\,\|u\| \geq \|u\|\,.
\end{aligned}$$

Thus, $\|Au(t)\| \geq \|u\|$, $u \in K \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $L_4 > 0$ such that $f(u) \leq \lambda u$, for $u \geq L_4$, where $\lambda > 0$ satisfies

$$\frac{\lambda}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a\,(s) \leq 1.$$

Choose $L' = \max\left\{2L_3, \frac{L_4}{\gamma}\right\}$. Let

$$\Omega_4 = \{u \in C\,(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < L'\}\,.$$

Then $u \in K \cap \partial\Omega_4$ implies

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geq \gamma\,\|u\| = \gamma L' \geq L_4.$$

Therefore,

$$\begin{aligned}Au(t) &\leq \frac{1}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a\,(s)\,f(u(s))\\&\leq \frac{\lambda}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a\,(s)\,u(s)\\&\leq \frac{\lambda}{1-\alpha\eta}\sum_{s=1}^{T}(T-s+1)\,a\,(s)\,\|u\|\\&\leq \|u\|\,.\end{aligned}$$

Thus $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_4$. By the second part of Theorem 2.7, A has a fixed point in $K \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $L_3 \leq \|u\| \leq L'$. This completes the sublinear part of the theorem. Therefore, the problem (8.2) has at least one positive solution. ■

8.4 Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 8.1 Consider the BVP

$$\begin{cases}\triangle^2 u\,(t-1) + te^{3t}(1+\sin{(u)}) = 0,\; t \in \mathbb{N}_{1,5},\\ \triangle u\,(0) = 0, \quad u\,(6) = \frac{2}{7}\sum\limits_{s=1}^{3} u\,(s)\,.\end{cases} \tag{8.9}$$

Set $\alpha = \frac{2}{7}$, $\eta = 3$, $T = 5$, $a(t) = te^{3t}$, $f(u) = 1 + \sin{(u)}$.

We can show that

$$0 < \alpha = \frac{2}{7} < \frac{1}{3} = \frac{1}{\eta}.$$

A simple calculation we get $f_0 = \infty$, $f_\infty = 0$ and (ii) of Theorem 8.1 holds. Then BVP (8.9) has at least one positive solution.

Example 8.2 Consider the BVP

$$\begin{cases}\triangle^2 u\,(t-1) + (3t^2+2t+1)(u^2\ln(u+1)) = 0,\; t \in \mathbb{N}_{1,7},\\ \triangle u\,(0) = 0, \quad u\,(8) = \frac{3}{17}\sum\limits_{s=1}^{5} u\,(s)\,.\end{cases} \tag{8.10}$$

Set $\alpha = \frac{3}{17}$, $\eta = 5$, $T = 7$, $a(t) = 3t^2 + 2t + 1$, $f(u) = u^2 \ln(u+1)$.

We can show that

$$0 < \alpha = \frac{3}{17} < \frac{1}{5} = \frac{1}{\eta}.$$

A simple calculation we get $f_0 = 0$, $f_\infty = \infty$ and (i) of Theorem 8.1 holds. Then BVP (8.10) has at least one positive solution.

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