وزارة التعليم العالي و البحث العلمي

UNIVERSITE BADJI MOKHTAR ANNABA
BADJI MOKHTAR UNIVERSITY ANNABA

جامعة باجي مختار
عنابة

Faculté des Sciences

Année : 2018

Département de Mathématiques

THESE

Présenté en vue de l'obtention du diplôme de DOCTORAT en Sciences

Option
Mathématiques Appliquées

Titre

Résolution de quelques problèmes aux limites d'ordre fractionnaire par la méthode de sous et sur solutions

Par
Djalal BOUCENNA

DIRECTRICE DE THESE : Guezane-Lakoud Assia Prof. U.B.M. ANNABA

CO-DIRECTEUR : Khaldi Rabah Prof U.B.M. ANNABA

Devant le jury

PRESIDENT :	Djebabla Abdelhak	M.C.A.	U.B.M. ANNABA
EXAMINATEURS :	Hitta Amara	Prof.	U. GUELMA
	Chaoui Abdrezak	Prof.	U. GUELMA
	Friou Assia	M.C.A.	U. GUELMA

CONTENTS

ACKNOWLEDGMENTS

First, I am truly grateful to my parents for their love and guidance.

I would like to express my sincere gratitude to my supervisor Prof. Assia Guezane-Lakoud and my co-advisor Prof. Rabah Khaldi. I thank them for their support, their advises, their motivation and their patience. It would never have been possible for me to carry out this work without their incredible support and encouragement.

I have benefited from many fruitful discussions with Professors Juan J. Nieto and Rosana Rodríguez-López, Mouafek Benchohra, Mohammed Ali Hammami, Abdellatif Ben-Makhlouf and Omar Naifar. I can not forget their help, motivation and hospitality.

I would also like to thank Professors Abderrazek Chaoui, Abdelhak Djebabla and Assia Frioui for agreeing to be members of the jury. At last, I wish to thank all my colleagues and my friends for their encouragement.

ملخص

الهدف الرئيسي من هذه الرسالة هو دراسة بعض المعادلات التفاضلية الكسرية في فضاءات بناخ باستعمال نظريات النقطة الصامدة. يتم التحقق في الأسئلة المتعلقة بوجود و استقرار الحلول من خلال تقنيات مختلفة مثل بناء الحلول العلوية والسفلية، تطبيق نظرية شودر، نظرية كراسنوسلسكي للنقطة الصامدة.

أولاً، نبرهن وجود وموقع الحلول الموجبة في فضاء الدوال المستمرة لمشكلة ذات نقاط متعددة تحتوي على مشتقات ريمان- ليوفيل وشرط تكاملي، وهذا من خلال إعطاء صيغ الحلول العليا و السفلى، ثم باستخدام نظرية شودر. باستعمال نفس التقنيات ندرس في فضاء سوبوليف الكسري مشكلة ذات نقاط متعددة تحتوي على مشتقات ريمان- ليوفيل وشروط تكاملية كسرية. بالنسبة للمشكلة الثالثة درسنا وجود واستقرار مشكلة كوشي مع مشتقات كسرية من نوع كابوتو- كاتوغامبولا باستخدام نظرية كراسنوسلسكي للنقطة الصامدة.

الكلمات الرئيسية:

معادلة تفاضلية كسرية، وجود الحل، حل موجب، نظرية شودر للنقطة الصامدة ، نظرية كراسنوسلسكي للنقطة الصامدة ، المشتقة الكسرية من نوع كابوتو- كاتوغامبولا.

ABSTRACT

The principal objective of this thesis is the study of some class of fractional differential equations in Banach spaces by fixed point theorems. The questions related to the existence and stability of solutions are investigate by different techniques such, by construction the upper and lower solutions, applying Schauder fixed point theorem and Krasnoselskii fixed point theorem.

First, we establish the existence and localization of positive solutions in the space of continuous functions for a multipoint Riemann-Liouville fractional boundary value problem with integral conditions, by finding the explicit expressions of the upper and lower solutions, then using the Schauder fixed point theorem. Thanks to the same techniques, we treat in a fractional Sobolev space a multipoint Riemann-Liouville fractional boundary value problem with fractional integral conditions. For the third studied problem, we establish the existence and stability for Caputo-Katugampola fractional problem by using Krasnoselski fixed point theorem.

Keywords: Fractional differential equation; Existence of solution; Positive solution; Stability of solution; Schauder fixed point theorem; Krasnoselskii fixed point theorem; Caputo-Katugampola fractional derivative.

RÉSUMÉ

L'objectif principal de cette thèse est l'étude d'une classe d'équations différentielles fractionnaires dans l'espace de Banach par des théorèmes du point fixe. Les questions liées à l'existence et la stabilité des solutions sont étudiées par différentes techniques telles que la construction des sous et sur solutions et l'application des théorème de point fixe de Schauder et de Krasnoselskii.

On établit d'abord l'existence et la localisation de solutions positives dans l'espace des fonctions continues pour un problème aux limites multipoint contenant les derivées fractionnaires de type Riemann-Liouville et des conditions intégrales, et ceci en donnant les expressions explicites des sous et sur solutions et en utilisant le théorème de Schauder. Grâce aux mêmes techniques, on traite dans un espace de Sobolev fractionnaire un problème aux limites multipoints avec les derivées fractionnaires de type Riemann-Liouville et des conditions intégrales fractionnaires. Pour le troisième problème étudié, ont établit l'existence et la stabilité d'un problème avec des derivées fractionnaires de Caputo-Katugampola en utilisant le théorème du point fixe de Krasnoselskii.

Mots-clés: Equation différentielle fractionnaire, Existence de la solution, Solution positive, Stabilité de la solution, Théorème du point fixe de Schauder, Théorème du point fixe de Krasnoselskii , Dérivée fractionnaire de Caputo-Katugampola.

INTRODUCTION

Fractional calculus can be seen as a generalization of classical calculus. It should be noted that the fractional calculus is now more attractive and many monographs and conferences are devoted to this subject, although it is an old subject and known since the 17th century. The advantage of fractional derivatives is that they are nonlocal operators describing the memory and hereditary properties of many materials and processes. Recently, fractional calculus is introduced in mathematical psychology to describe human behavior since the manner he reacts to external influences depends on the experiences he had in the past [15].

Many authors have shown that derivatives of fractional order are better suited to the description of various real materials and that the introduction of the fractional calculation in the modeling reduces the number of parameters required. While fractional integral can be used for example in order to better describe the accumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of the regression model [55].

Due to these facts, differential equations involving fractional derivatives are more adequate to describe many phenomena in different fields of applied sciences and engineering such as in control, signal processing, electrochemistry, viscoelasticity, rheology, chaotic dynamics, statistical physics, biosciences, [15, 31].

We must mention that there is no general applicable method to discuss the classical questions related to an arbitrary given fractional differential equation and that to study the existence, uniqueness and properties of solutions, different methods are used. This includes the upper and lower solutions method, the Mawhin theory, the decomposition method, the variational iteration method, the homotopy method... [4, 16, 17, 53, 52, 51, 22, 23, 24, 25, 26, 27, 28, 29, 30, 38, 39, 40]

Another important question regarding solutions for fractional differential equations is their stability. Note that the analysis of the stability of fractional differential equations is more complex than ordinary differential equations, due to the fact that fractional derivatives are nonlocal and have a singular kernel. The literature on the stability of fractional differential equations is limited and concentrated on a fractional order between zero and one. We can cite some articles dealing with the stability of solutions for systems of fractional differential equations or for fractional differential equations [11, 12, 20, 44, 49, 50]. Most of them used Lyapunov direct or indirect method without finding the explicit form of the solution.

This thesis is devoted to the study of some nonlinear fractional differential equations by using fixed point theorems. Since Riemann-Liouville and Caputo fractional derivatives are the most used in differential equations, we investigate, in the second and third chapters, the existence of solutions for differential equations involving Riemann-Liouville type derivative. As it is natural to look for generalizations of fractional derivatives and integrals, for which the known ones are particular cases, Katugampola introduced in [37] a new type of fractional derivative that generalizes both the Riemann–Liouville and Hadamard fractional derivatives. Following this idea, Almeida et all [2] presented the so called Caputo–Katugampola derivative, that generalizes the concept of Caputo and Caputo–Hadamard fractional derivatives. The new operator is the left inverse of the Katugampola fractional integral and keeps some of the fundamental properties of the Caputo and Caputo–Hadamard fractional derivatives. A Caputo–Katugampola fractional differential equation is treated in the fourth chapter. Let us give the review of each chapter of the thesis.

In chapter 1, we introduce some functions that are of fundamental importance in the theory of fractional differential equations, Gamma function and Beta function. We provide some basic knowledge about fractional integrals and derivatives, such Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, Katugampola derivative and Caputo–Katugampola derivative. We give a characterization of a compact set in the space of continuous functions and in the space of p-integrable functions that is in Lp, some fixed point theorems and the theory on stability of solutions for fractional differential equations.

In chapter 2, we study the existence of solutions for a multipoint fractional higher order boundary value problem with integral conditions (P1):

$$\begin{aligned} D_{0+}^{\alpha} y(t) + f(t, y(t)) &= 0, \ t \in (0,1) \\ y^{(i)}(0) &= 0 \ \ i = 0, ..., n-2 \\ y(1) &= \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds, \end{aligned}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, $n-1 \leq \alpha < n$, $n \geq 2$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R}_{+})$ is a given function, $0 < \eta_k < 1$, $\lambda_k > 0$, $k = 0, ..., m$. By Schauder fixed point theorem, we prove the existence of solution for problem (P1), then we show that the solution is lying between the lower and upper solutions that we construct explicitly. The results of this chapter are accepted for publication:

D. Boucenna, A. Guezane-Lakoud, Juan J. Nieto, R. Khaldi, On a multipoint fractional boundary value problem with integral conditions, J. Nonlinear Funct. Anal. Vol. 2017, Article ID 53, 1–13 (2017).

Chapter 3, concerns the existence of positive solutions for a multipoint fractional boundary value problem with fractional integral conditions in a fractional Sobolev space (P2):

$$\begin{aligned} D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\alpha} u(t)) &= 0, \ n-1 \leq \alpha \leq n, \ t \in (0,1) \\ D_{0+}^{\alpha-i} u(0) &= 0, \ \ \forall i = 2, ..., n, \\ u(1) &= \sum_{k=0}^{m} \lambda_k I_{0+}^{\beta} u(\eta_k), \ \lambda_k > 0, \end{aligned}$$

where D_{0+}^{α} and D_{0+}^{α} denote the Riemann-Liouville fractional derivatives of order α and γ respectively, $n-1 \leq \alpha < n$, $n \geq 2$, $0 < \gamma < 1$, I_{0+}^{β} denotes the Riemann-Liouville fractional integral of order $\beta > 0$, f is a given function, $0 < \eta_k < 1$, $\lambda_k > 0$, $k = 0, ..., m$. By constructing the upper and lower control functions, we prove the existence of positive solution lying between the upper and lower solutions.

In chapter 4, we study the existence and stability of solution for the following fractional initial value problem (P3):

$$\begin{aligned} {}^{C}D_{t_0^+}^{\alpha,\rho} x(t) &= f(t, x(t)) \qquad t \geq t_0, \\ x(t_0) &= x_0, \quad x'(t_0) = x_1, \end{aligned}$$

where ${}^{C}D_{t_0^+}^{\alpha,\rho}$ is the Caputo–Katugampola fractional derivative, $1 < \alpha < 2$, $\rho > 0$, $\rho \neq 1$, $x_0, x_1 \in \mathbb{R}$ and $f : [t_0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We give sufficient conditions to guarantee the stability of the zero solution of problem (P3). The main results are obtained by using Krasnoselskiis fixed point theorem in a weighted Banach space. The results of this chapter are submitted for publication.

CHAPTER 1

PRELIMINARIES

We introduce some important functions which are used in fractional calculus. The Gamma function plays the role of the generalized factorial and the Beta function occur when computing fractional derivatives of power functions. We give some necessary concepts on the fractional calculus theory, namely the Riemann-Liouville integral and derivative, the Caputo derivative and Caputo Katugampola fractional derivative. We cite their basic properties including the rules for their compositions and the conditions for the equivalence of various definitions. More information about these functions can be found in [2, 3, 36, 37].

1.1 Gamma and Beta functions

Definition 1 *The Gamma function* $\Gamma(.)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt,$$

which converges in the right half of the complex plane, that is, $\mathrm{Re}(z) > 0$.

The Gamma function satisfies

$$\Gamma(z+1) = z\Gamma(z), \ \operatorname{Re}(z) > 0$$

and for any integer $n \geq 0$, we have

$$\Gamma(n+1) = n!.$$

A limit definition of the Gamma function is given by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^2}{z(z+1)\dots(z+n)}, \ \ Re(z) > 0,$$

Definition 2 *For every z, w such that $\operatorname{Re}(z) > 0$, $\operatorname{Re}(w) > 0$, the Beta function is defined by*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt.$$

An interesting formula relating the Gamma and Beta functions is

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \operatorname{Re}(z) > 0, \operatorname{Re}(w) > 0$$

1.2 Fractional integrals and fractional derivatives

In this section, we focus on the Riemann-Liouville integrals and derivatives and the Caputo derivative since they are the most used ones in applications. We will formulate the conditions of their equivalence and derive the most important properties. There are several types of fractional derivatives such the Grunwald Letnikov fractional derivative, Riesz fractional derivative, Hadamard fractional derivative. The choice of the appropriate fractional derivative or integral depends on the considered problem since each of them has its own advantages and disadvantages.

Definition 3 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a func-*

tion $f:(a,+\infty)\rightarrow\mathbb{R}$ *is given by*

$$I_{0^+}^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_{0}^{t}\left(t-s\right)^{\alpha-1}f\left(s\right)ds$$

provided that the right side is pointwise defined on $(a,+\infty)$.

Definition 4 *The Riemann-Liouville fractional derivative of order* $\alpha>0$ *of a function* $f:(a,+\infty)\rightarrow\mathbb{R}$ *is given by*

$$D_{a^+}^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{ds}\right)^{n}\int\limits_{a}^{x}\frac{f\left(s\right)}{\left(t-s\right)^{\alpha-n+1}}ds=\left(\frac{d}{ds}\right)^{n}I_{a^+}^{n-\alpha}f\left(s\right),$$

provided that the right side is pointwise defined on $(a,+\infty)$, *where* $n=[\alpha]+1$, $[\alpha]$ *denotes the integer part of* α.

Lemma 5 *Let* $\alpha\geq\beta>0$, *then for* $f\in L^{P}\left[a,b\right]\left(1\leq p\leq\infty\right)$ *the relation*

$$\left(D_{a^+}^{\beta}I_{a^+}^{\alpha}f\right)\left(t\right)=I_{a^+}^{\alpha-\beta}f(t)$$

holds almost everywhere on $[a,b]$. *In particular if* $\alpha=\beta$ *we get*

$$\left(D_{a^+}^{\alpha}I_{a^+}^{\alpha}f\right)\left(t\right)=f(t)$$

Lemma 6 *The fractional integral operator* $I_{a^+}^{\alpha}$ *is bounded from* $L^p(a,b)$ $(1\leq p\leq\infty)$ *into itself*

$$\left\|I_{a^+}^{\alpha}f\right\|_{L^P}\leq k\left\|f\right\|_{L^p},\qquad k=\frac{\left(b-a\right)^{\alpha}}{\Gamma(\alpha+1)}$$

Definition 7 *Let* $\alpha>0$ *and* $n=[\alpha]+1$, *for a function* $f\in AC^{n}\left(\left[a,b\right],\mathbb{R}\right)$ *the Caputo fractional derivative of order* α *of* f *is defined by*

$$\begin{aligned}\left({}^{C}D_{a^+}^{\alpha}f\right)\left(t\right) &= I^{n-\alpha}D^{(n)}f\left(t\right)\\ &= \frac{1}{\Gamma\left(n-\alpha\right)}\int\limits_{a}^{x}\left(t-s\right)^{n-\alpha-1}f^{(n)}\left(s\right)ds.\end{aligned}$$

where $D=\frac{d}{dt}$ *denotes the classical derivative and* $AC^n[a,b]=\{f\in C^{n-1}[a,b], f^{(n-1)}$ *absolutely continuous function*$\}$.

Properties. Let $\alpha,\beta>0$ and $n=[\alpha]+1$, then the following relations hold:

$$I_{a^+}^{\alpha}(x-a)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}.$$

$$D_{a^+}^{\alpha}(x-a)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}.$$

$${}^{C}D_{a^+}^{\alpha}(x-a)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \beta>n.$$

On the other hand, for $k=1,2,...,n$, we have

$$D_{a^+}^{\alpha}(x-a)^{\alpha-k}(t)=0,$$

and for $k=0,1,...,n-1$

$${}^{C}D_{a^+}^{\alpha}(x-a)^{k}(t)=0,$$

in particular,

$${}^{C}D_{a^+}^{\alpha}(1)=0.$$

The Riemann-Liouville fractional derivative of a constant is in general not equal to zero, in fact

$$D_{a^+}^{\alpha}(1)=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, 0<\alpha<1.$$

Lemma 8 *Let* $\alpha>0$, $n=[\alpha]+1$ *and* $f:[a,b]\to\mathbb{R}$ *be a given fonction. Assume that* $D_{a^+}^{\alpha}f$ *and* ${}^{C}D_{a^+}^{\alpha}f$ *exist. Then*

$${}^{C}D_{a^+}^{\alpha}f(t)=D_{a^+}^{\alpha}f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}.$$

Lemma 9 *Let* $\alpha>0$, *then the fractional differential equation*

$$D_{0^+}^{\alpha}f(t)=0.$$

has $f(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+...+c_nt^{\alpha-n}$, $c_i\in\mathbb{R}, i=1,2,...,n$ *as solution.*

Lemma 10 *Let $\alpha > 0$, $n = [\alpha] + 1$. If $f \in L^1[a,b]$ and $f_{n-\alpha} \in AC^n[a,b]$, then the equality*

$$\left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f\right)(t) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha-j}.$$

holds almost everywhere on $[a,b]$. In particular, if $0 < \alpha < 1$, then

$$\left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f\right)(t) = f(t) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1},$$

where $f_{n-\alpha} = I_{a^+}^{n-\alpha} f$ and $f_{1-\alpha} = I_{a^+}^{1-\alpha} f$.

Theorem 11 *Let $\beta > \alpha > 0$, then we have*

$$\left(I_{a^+}^{\alpha C} D_{a^+}^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^{k}.$$

$$\left(D_{a^+}^{\alpha} I_{a^+}^{\beta} f\right)(t) = I_{a^+}^{\beta-\alpha} f(t).$$

$$D^m D_{a^+}^{\alpha} f(t) = D_{a^+}^{\alpha+m} f(t), m \in N.$$

Definition 12 *The Hadamard fractional integral of order $\alpha > 0$ of a function f is defined by*

$$\boldsymbol{I}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, a < t < b.$$

A more general fractional integral referred as Hadamard fractional integral of order α is given by

$$\boldsymbol{I}_{a^+}^{\alpha,\mu} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{s}{t}\right)^{\mu} \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, a < t < b, \mu \in \mathbb{R}.$$

Definition 13 *The Hadamard fractional derivative of order $\alpha > 0$ of a function*

f is defined by

$$\boldsymbol{D}_{a^+}^{\alpha} f(t) = \left(t\frac{d}{dt}\right)^n \boldsymbol{I}_{a^+}^{n-\alpha} f(t), a < t < b, n = [\alpha] + 1$$

A more general fractional derivative referred as Hadamard fractional derivative of order α is given by

$$\boldsymbol{D}_{a^+}^{\alpha,\mu} f(t) = t^{-\mu} \left(t\frac{d}{dt}\right)^n t^{\mu} \boldsymbol{I}_{a^+}^{n-\alpha,\mu} f(t), a < t < b, n = [\alpha] + 1$$

1.3 Generalized fractional integrals and fractional derivatives

Katugampola in [37] introduced a new type of fractional derivative generalizing Riemann-Liouville and Hadamard fractional derivatives. Later, Almeida and all in [2], introduced a generalization of the derivative as the left inverse of Katugampola's fractional integral and which retains some of the fundamental properties of the fractional derivatives of Caputo and Caputo-Hadamard, the new derivative is called Caputo-Katugampola fractional derivative [2, 3, 36, 37].

Definition 14 *(Katugampola fractional integrals) Let a, b be two real and $f : [a, b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional integrals of order $\alpha > 0$, parameter $\rho > 0$, of f is defined as*

$$I_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s)\, ds.$$

Definition 15 *(Katugampola fractional derivative) Let $0 < a < b < \infty$ be two real $f : [a, b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional derivative of order $\alpha > 0$, and parameter $\rho > 0$, is defined as*

$$\begin{aligned} D_{a^+}^{\alpha,\rho} f(t) &= \left(t^{1-\rho} \frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\rho} f(t) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{n-\alpha-1} f(s)\, ds. \end{aligned}$$

Proposition 16 *We have the following properties for Katugompola fractional integral and derivative.*

$$\begin{aligned} D_{a^+}^{\alpha,\rho} \left(I_{a^+}^{\alpha,\rho}\right) f(t) &= f(t), \\ I_{a^+}^{\alpha,\rho} \left(I_{a^+}^{\beta,\rho}\right) f(t) &= I_{a^+}^{\alpha+\beta,\rho} f(t) \\ \lim_{\rho \to 1} I_{a^+}^{\alpha,\rho} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \\ \lim_{\rho \to 0^+} I_{a^+}^{\alpha,\rho} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \\ \lim_{\rho \to 0^+} D_{a^+}^{\alpha,\rho} f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}, \\ \lim_{\rho \to 1} D_{a^+}^{\alpha,\rho} f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds. \end{aligned}$$

1.4 Caputo-Katugampola fractional derivative

Definition 17 *(Caputo-Katugampola fractional derivative) Let* $0 < a < b < \infty$ *be two real,* $\rho > 0$ *be a positive real number and* $f \in AC^n([a,b], \mathbb{R})$. *The Caputo-Katugampola fractional derivative of order* $\alpha > 0$ *of the function* f *is defined by*

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha,\rho} f(t) &= I_{a^+}^{n-\alpha,\rho} \left(t^{1-\rho} \frac{d}{dt} \right)^n f(t) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{n-\alpha-1} \left(t^{1-\rho} \frac{d}{dt} \right)^n f(s)\, ds \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t \frac{s^{(\rho-1)(1-n)} f^{(n)}(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds, \end{aligned}$$

where n *is the smallest integer greater than* α.

Properties.

1- When $\rho = 1$, the Caputo-Katugampola derivative coincides with Caputo derivative.

2- In the case $0 < \alpha < 1$ and $\rho > 0$, then

$${}^{C}D_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{(\rho-1)} (f(s) - f(a))}{(t^\rho - s^\rho)^\alpha} ds,$$

3- If $f \in C[a,b]$ then

$${}^{C}D_{a^+}^{\alpha,\rho} I_{a^+}^{\alpha,\rho} f(t) = f(t),$$

and if $f \in C^1[a,b]$ then

$$I_{a^+}^{\alpha,\rho} {}^{C}D_{a^+}^{\alpha,\rho} f(t) = f(t) - f(a).$$

4- If $f(a) = 0$, then the Caputo Katugampola and the Katugampola fractional derivatives coincide. Moreover if both types of derivatives exist then

$$ {}^{C}D_{a^+}^{\alpha,\rho} f(t) = D_{a^+}^{\alpha,\rho} f(t) - \frac{f(a)\,\rho^{\alpha}\left(t^{\rho} - s^{\rho}\right)^{-\alpha}}{\Gamma(1-\alpha)}. $$

1.5 Fixed point theorems

Fixed point theory is an important topic with a big number of applications in various fields of mathematics. The fixed point theorems concern a function f satisfying some conditions and admits a fixed point, that is $f(x) = x$. Knowledge of the existence of fixed points has pertinent applications in many branches of analysis and topology. Following if the conditions are imposed on the function or on the set, different fixed point theorems are given, we cite the following.

Theorem 18 *(Banach contraction principle) Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

Theorem 19 *(Schauder fixed point theorem) Let Ω be a nonempty closed bounded and convex subset of a normed space. Let N be a continuous mapping from Ω into a compact subset of Ω, then N has a fixed point in Ω.*

Theorem 20 *(Krasnoselskii fixed point theorem) [42]. Let Ω be a closed bounded and convex nonempty subset of a Banach space X. Suppose that A and B map Ω into X such that*
(i) A is continuous and compact.
(ii) B is a contraction mapping.
(iii) $x, y \in \Omega$, implies $Ax + By \in \Omega$.
Then there exists $x \in \Omega$ with $x = Ax + Bx$.

The criteria for compactness for sets in the space of continuous functions $C([a,b])$ is the following.

Theorem 21 *(Arzela-Ascoli theorem)*. *A set $\Omega \subset C([a,b])$ is relatively compact in $C([a,b])$ iff the functions in Ω are uniformly bounded and equicontinuous on $[a,b]$.*

We recall that a family Ω of continuous functions is uniformly bounded if there exists $M > 0$ such that

$$\|f\| = \max_{x\in[a,b]} |f(x)| \leq M, \ f \in \Omega.$$

The family Ω is equicontinuous on $[a, b]$, if $\forall\, \varepsilon > 0$, $\exists\delta > 0$ such that $\forall t_1, t_2 \in [a, b]$ and $\forall f \in \Omega$, we have

$$|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon.$$

The criteria for compactness for sets in the space of integrable functions $L^p(0, 1)$ is the following.

Lemma 22 *[6]. Let F be a bounded set in $L^p(0, 1)$, $1 \leq p < \infty$. Assume that*

(i) $\lim\limits_{|h|\to 0} \|\tau_h f - f\|_p = 0$ *uniformly on F,*
(ii) $\lim\limits_{\varepsilon\to 0} \int_{1-\varepsilon}^{1} |f(t)|^p\, dt = 0,$ *uniformly on F.*

Then F is relatively compact in $L^p(0, 1)$. Where $\tau_h f(t) = f(t + h)$.

The following theorem gives a necessary and sufficient condition for a set of functions to be relatively compact in an L^p space.

Theorem 23 ***(Riesz-Kolmogorov)****. Let F be a bounded subset in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, The subset F is relatively compact if and only if the following properties hold.*
1. $\lim_{x\to 0} \int_{\mathbb{R}^n} |f(y + x) - f(y)|^p dy = 0$ *uniformly on F.*

2. $\lim_{R\to\infty} \int\limits_{|x|>R} |f(x)|^p\, dx = 0$ *uniformly on F.*

1.6 Stability of solutions

Recently, several methods are introduced to study the stability of nonlinear fractional differential equations, but most of them are devoted to the case $0 < \alpha < 1$. We can cite the Mittag–Leffler stabilities, Ulam stability, generalized Gronwall-Bellman inequality... However, when the fractional order is greater than one, it is difficult to apply these methods and finding another effective method to investigate the stability of nonlinear fractional differential equations is actually the purpose of many research works. Let us consider in the Banach space X the differential equation

$$\frac{dx}{dt} = f\left(t, x\right), t_0 \leq t < \infty \tag{1.1}$$

where $f : [t_0, \infty) \times X \to X$.

Definition 24 *[13] A solution $x = \varphi\left(t\right)$ of equation (1.1) is said to be stable if for any $\varepsilon > 0$ and any $t_1 \geq t_0$, there exists $\delta > 0$ such that every other solution $x = \psi\left(t\right)$ defined in a neighborhood of t_1 and satisfying $\left\|\varphi(t_1) - \psi(t_1)\right\| < \delta$, exists for all $t \geq t_1$ and satisfies $\left\|\varphi(t) - \psi(t)\right\| < \varepsilon, \forall t \geq t_1$.*

A solution $x = \varphi\left(t\right)$ of equation (1.1) is said to be uniformly stable if the constant δ can be chosen independly of $t_1 \geq t_0$.

A solution $x = \varphi\left(t\right)$ of equation (1.1) is said to be asymptotically stable if it's stable and for any $t_1 \geq t_0$, there exists $\delta > 0$ such that $\left\|\varphi(t_1) - \psi(t_1)\right\| < \delta$ implies $lim_{t\to+\infty} \left\|\varphi(t) - \psi(t)\right\| = 0$.

Remark 25 *Note that by setting $y\left(t\right) = x\left(t\right) - \varphi\left(t\right)$, where $\varphi\left(t\right)$ is a solution of equation (1.1), the study of the stability of a solution can be reduced to the investigation of the stability of the zero solution of an auxiliary equation.*

CHAPTER 2

POSITIVE SOLUTIONS FOR A MULTIPOINT FRACTIONAL BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS

2.1 Introduction

Fractional differential equations as generalization of differential equations of integer order can modelize many phenomena in different fields of applied sciences and engineering such as viscoelasticity, rheology, thermodynamics, biosciences, bioengineering, etc. Several methods are involved in the study of the existence of solutions, one can quote the upper and lower solutions method, the theory of Mawhin and the method of successive approximation .... In particular, some fixed point theorems, such Banach fixed point theorem, Schauder fixed point theorem, Leray-Schauder fixed point theorem and Guo-Krasnoselski theorem are used in the study of the existence of solutions or positive solutions for boundary value problems for nonlinear fractional differential equations, see [1,4,15-19, 21-23,31-36,42,46,48,52,54-56]. Furthermore, the use of the upper and lower solutions method provides information on the existence and localization of solutions.

In [45], Liang et al. studied by means of lower and upper solutions method and Schauder fixed point theorem the existence of positive solutions for the following problem

$$\begin{aligned} D_{0^+}^{\alpha} y + f(t,y) &= 0, \quad 3<\alpha<4, \; 0<t<1, \\ y(0) &= y'(0) = y''(0) = y''(1) = 0. \end{aligned}$$

Chen et al in [10], investigate the existence of multiple positve solutions for the following fractional boundary value problem

$$\begin{aligned} D_{0^+}^{\alpha} y + f(t,y) &= 0, \; 2<\alpha<3, \; 0<t<1, \\ y(0) &= y'(0) = y'(1) = 0. \end{aligned}$$

By the properties of the Green function, the lower and upper solutions method and Schauder fixed point theorem, the authors proved the existence of at least one positive solution. Then using Leggett Williams fixed point theorem, they showed the existence of multiple positive solutions.

This chapter concerns the existence of positive solutions for the following boundary value problem for nonlinear fractional differential equation.

$$D_{0+}^{\alpha} y(t) + f(t, y(t)) = 0, n-1 \leq \alpha < n,\ 0 <\ t < 1 \tag{2.1}$$

$$\begin{aligned} y^{(i)}(0) &= 0, \quad i = 0, ..., n-2, \\ y(1) &= \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds, \end{aligned} \tag{2.2}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R}_+)$ is a given function, $n \in \mathbb{N}$, $n \geq 2$, $0 < \eta_k < 1$, $\lambda_k > 0$, $\forall k = 0, ..., m$.

Firstly we solve the linear problem, then we construct the lower and upper solutions. By Schauder fixed point theorem, we establish the existence of at least one positive solution for the boundary value problem (2.1)-(2.2) lying between these two functions. The obtained results are illustrated by an example.

2.2 Existence of positive solutions

Lemma 26 *Assume that $h \in C(0,1) \cap L^1(0,1)$ and $n-1 \leq \alpha < n$, $n \geq 2$. Then the solution to the boundary value problem*

$$D_{0+}^{\alpha} y(t) + h(t) = 0, t \in (0,1), \tag{2.3}$$

$$\begin{gathered} y^{(i)}(0) = 0 \ \ i = 0, ..., n-2, \\ y(1) = \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds,\ \lambda_k > 0. \end{gathered} \tag{2.4}$$

is given by

$$y(t) = \int_0^1 G(t,s)\, h(s)\, ds + \frac{t^{\alpha-1}}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k, s)\, h(s)\, ds.$$

where $\xi = 1 - \frac{1}{\alpha}\sum\limits_{k=0}^{m} \lambda_k \eta_k^\alpha > 0$ and

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, 0 \le s \le t \le 1 \\ t^{\alpha-1}(1-s)^{\alpha-1} .0 \le t \le s \le 1. \end{cases}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha+1)} \begin{cases} t^{\alpha}(1-s)^{\alpha-1} - (t-s)^{\alpha}, 0 \le s \le t \le 1 \\ t^{\alpha}(1-s)^{\alpha-1}, 0 \le t \le s \le 1. \end{cases}$$

Proof. Let y be a solution of the fractional boundary value problem (2.3)-(2.4). By Lemma 9, it yields

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds. \quad (2.5)$$

Taking conditions (2.4) into account, we obtain

$$c_2 = c_3 ... = c_n = 0,$$

and then

$$\begin{aligned} y(1) &= c_1 - I^\alpha h(1) = \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds \\ &= \sum_{k=0}^{m} \lambda_k \left(-I^{\alpha+1} h(\eta_k) + \frac{c_1}{\alpha} \eta_k^\alpha \right) \\ &= -\sum_{k=0}^{m} \lambda_k I^{\alpha+1} h(\eta_k) + c_1 \sum_{k=0}^{m} \frac{\lambda_k}{\alpha} \eta_k^\alpha, \end{aligned}$$

that implies

$$c_1 = \frac{1}{\xi\Gamma(\alpha)} \left(\int_0^1 (1-s)^{\alpha-1} h(s)\, ds - \frac{1}{\alpha} \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^\alpha h(s)\, ds \right).$$

Hence the solution of problem (2.3)-(2.4) is the following

$$\begin{aligned}
y(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds \\
&\quad +\frac{t^{\alpha-1}}{\xi\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-1} h(s)\,ds \right.\\
&\quad \left. -\frac{1}{\alpha}\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}(\eta_k-s)^{\alpha} h(s)\,ds\right].
\end{aligned}$$

Finally, some computations give

$$\begin{aligned}
y(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds \\
&\quad +\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+t^{\alpha-1}\frac{\frac{1}{\alpha}\sum\limits_{k=0}^{m}\lambda_k\eta_k^{\alpha}}{\Gamma(\alpha)\left(1-\frac{1}{\alpha}\sum\limits_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)}\right)\int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\
&\quad -\frac{t^{\alpha-1}\sum\limits_{k=0}^{m}\lambda_k}{\xi\alpha\Gamma(\alpha)}\int_0^{\eta_k}(\eta_k-s)^{\alpha} h(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left(t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right) h(s)\,ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} h(s)\,ds \\
&\quad +\frac{t^{\alpha-1}}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}\left(\eta_k^{\alpha}(1-s)^{\alpha-1}-(\eta_k-s)^{\alpha}\right) h(s)\,ds
\end{aligned}$$

$$+\frac{t^{\alpha-1}}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\int_{\eta_k}^{1}\eta_k^{\alpha}(1-s)^{\alpha-1}h(s)\,ds$$

$$=\int_0^1 G(t,s)\,h(s)\,ds+\frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)\,h(s)\,ds.$$

The proof is completed. ■

Lemma 27 *The functions G and H are continuous nonnegative and satisfy*

$$0\le G(t,s)\le\frac{1}{\Gamma(\alpha)},\ \ 0\le H(t,s)\le\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ \ 0\le t,s\le 1.$$

Now we define the concept of upper and lower solutions for the fractional boundary value problem (2.1)-(2.2).

Definition 28 *A function $\beta\in C[0,1]$ is called a lower solution of the fractional boundary value problem (2.1)-(2.2), if*

$$\begin{aligned}-D_{0^+}^{\alpha}\beta(t)&\le f(t,\beta(t)),\ t\in(0,1),\\ \beta^{(i)}(0)&\le 0,\quad i=0,...,n-2,\\ \beta(1)&\le\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}y(s)\,ds.\end{aligned}$$

Definition 29 *A function $\gamma\in C[0,1]$ is called an upper solution of the fractional boundary value problem (2.1)-(2.2), if*

$$\begin{aligned}-D_{0^+}^{\alpha}\gamma(t)&\ge f(t,\gamma(t)),t\in(0,1),\\ \gamma^{(i)}(0)&\ge 0,\quad i=0,...,n-2,\\ \gamma(1)&\ge\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}y(s)\,ds,\ \lambda_k>0.\end{aligned}$$

Define the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$Fy(t) = \int_0^1 G(t,s) f(s,y(s))\,ds + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s) f(s,y(s))\,ds,$$

then y is a solution of problem (2.1)-(2.2) if and only if y is a fixed point of F. Setting

$$\begin{aligned}
p(t) &= \int_0^1 G(t,s)\,ds + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)\,ds,\\
&= \frac{-t^{\alpha}+t^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\frac{\lambda_k}{\Gamma(\alpha+1)}\left(\frac{-\eta_k^{\alpha+1}}{\alpha+1}+\frac{\eta_k^{\alpha}}{\alpha}\right)\\
&= \frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1}}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha}-\frac{\eta_k}{\alpha+1}\right)\\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha+1)}\left[(1-t)+\frac{1}{\xi}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha}-\frac{\eta_k}{\alpha+1}\right)\right],
\end{aligned}$$

then the function

$$g(t) = \int_0^1 G(t,s) f(s,p(s))\,ds + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s) f(s,p(s))\,ds, t\in[0,1]$$

is a positive solution of the following problem

$$-D_{0^+}^{\alpha} g(t) = f(t,p(t)), t\in(0,1)$$

$$g^{(i)}(0) = 0 \quad \forall i = 0,...,n-2,$$

$$g(1) = \sum_{k=0}^{m}\lambda_k\int_0^{\eta_k} g(s)\,ds,$$

consequently

$$a_1 p(t) \leq g(t) \leq a_2 p(t), \qquad 0<t<1, \tag{2.6}$$

where

$$a_1 = \min\left\{1, \min_{t\in[0,1]} f\left(t, p\left(t\right)\right)\right\}, \ a_2 = \max\left\{1, \max_{t\in[0,1]} f\left(t, p\left(t\right)\right)\right\}. \tag{2.7}$$

We have the following results:

Theorem 30 *Assume that the following conditions are satisfied*
$(H1)$ *There exist a function* $\varphi \in L^1\left([0,1], \mathbb{R}_+\right)$ *and a continuous nondecreasing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+,$ *such that*

$$f\left(t, y\right) \leq \varphi\left(t\right)\psi\left(\left|y\right|\right),$$

for all $t \in [0,1]$ *and all* $y \in \mathbb{R}$.
$(H2)$ *There exists a constant* $\rho > 0$ *such that*

$$\psi\left(\rho\right)\left(\frac{1}{\Gamma\left(\alpha\right)} + \frac{1}{\xi\Gamma\left(\alpha+1\right)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\left\|\varphi\right\|_{L^1} < \rho. \tag{2.8}$$

$(H3)$ $f\left(t, p\left(t\right)\right) \neq 0,$ *for* $t \in [0,1]$ *and there exists a constant* μ, $0 < \mu < 1$ *such that for all* k, $0 < k < 1$, *we have*

$$k^{\mu} f\left(t, u\right) \leq f\left(t, ku\right), \ u \in \mathbb{R}_+. \tag{2.9}$$

Then the problem (2.1)-(2.2) has at least one positive solution $y \in C[0,1]$ *satisfying*

$$\beta\left(t\right) \leq y\left(t\right) \leq \gamma\left(t\right), t \in [0,1].$$

Where β *and* γ *are respectively the lower and upper solutions for problem (2.1)-(2.2), defined as*

$$\beta\left(t\right) = k_1 g\left(t\right), \gamma\left(t\right) = k_2 g\left(t\right), \tag{2.10}$$

$$k_1 = \min\left(1, r\right)k_3, \ k_2 = \max\left(1, R\right)k_4, \tag{2.11}$$

$$r = \min\left(f\left(t, y(t)\right), t \in [0,1], \left\|y\right\| \leq \rho\right),$$

$$R=\max\left(f\left(t,y(t)\right),t\in\left[0,1\right],\left\Vert y\right\Vert \leq\rho\right),$$

$$k_{3}=\min\left(\frac{1}{a_{2}},a_{1}^{\frac{\mu}{1-\mu}}\right),\ k_{4}=\max\left(\frac{1}{a_{1}},a_{2}^{\frac{\mu}{1-\mu}}\right). \tag{2.12}$$

Proof. Let us prove that F is completely continuous operator.
Let $y\in B_{\rho}=\left\{y\in C\left[0,1\right]:\left\Vert y\right\Vert \leq\rho\right\}$, we have

$$\begin{aligned} Fy\left(t\right) &\leq \frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{1}\left(1-s\right)^{\alpha-1}f\left(s,y\left(s\right)\right)ds \\ &\quad+\frac{1}{\xi\Gamma\left(\alpha+1\right)}\sum_{k=0}^{m}\lambda_{k}\eta_{k}^{\alpha}\int_{0}^{1}\left(1-s\right)^{\alpha-1}f\left(s,y\left(s\right)\right)ds. \end{aligned}$$

thanks to Condition $(H1)$ we obtain

$$Fy\left(t\right)\leq\psi\left(\rho\right)\left(\frac{1}{\Gamma\left(\alpha\right)}+\frac{1}{\xi\Gamma\left(\alpha+1\right)}\sum_{k=0}^{m}\lambda_{k}\eta_{k}^{\alpha}\right)\int_{0}^{1}\left(1-s\right)^{\alpha-1}\varphi\left(s\right)ds,$$

in view of (2.8) it yields

$$\left\Vert Fy\right\Vert \leq\psi\left(\rho\right)\left(\frac{1}{\Gamma\left(\alpha\right)}+\frac{1}{\xi\Gamma\left(\alpha+1\right)}\sum_{k=0}^{m}\lambda_{k}\eta_{k}^{\alpha}\right)\left\Vert \varphi\right\Vert _{L^{1}}<\rho.$$

thus F is uniformly bounded on B_{ρ} and $F\left(B_{\rho}\right)\subset B_{\rho}$.

Let $t_1, t_2 \in [0,1], t_1 < t_2$, then

$$
\begin{aligned}
|Fy(t_2) - Fy(t_1)| \;\le\; & \frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right] f(s, y(s))\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s))\, ds \\
& + \frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k, s) f(s, y(s))\, ds.
\end{aligned}
$$

By the help of Condition $(H1)$ we obtain

$$
\begin{aligned}
|Fy(t_2) - Fy(t_1)| \le \psi(\rho) \|\varphi\|_{L^1} \Bigg(& \frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\Gamma(\alpha)} \\
& + \frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\Gamma(\alpha)} + \frac{(t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\xi \Gamma(\alpha+1)} \sum_{k=0}^{m} \lambda_k \eta_k^{\alpha} \Bigg) \\
\le \; & \frac{\psi(\rho) \|\varphi\|_{L^1} \left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{\Gamma(\alpha)} \left(2 + \frac{1}{\alpha\xi} \sum_{k=0}^{m} \lambda_k \eta_k^{\alpha}\right) \\
& + \frac{\psi(\rho) \|\varphi\|_{L^1} (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}
$$

that tends to 0 as $t_2 \to t_1$. Hence $F(B_\rho)$ is equicontinuous. By Arzela-Ascoli Theorem we conclude that F is completely continuous. Applying Schauder fixed point Theorem 19, it follows that F has a fixed point $y \in B_\rho$. Let us remark that the solution y satisfies

$$
rp(t) \le y(t) \le Rp(t) \;\; \forall t \in (0,1). \tag{2.13}
$$

Now we prove that $\beta(t) \le y(t) \le \gamma(t), t \in [0,1]$. Combining (2.6) and (2.10), we get the following estimates for $t \in (0,1)$

$$k_1 a_1 \leq \frac{\beta(t)}{p(t)} \leq k_1 a_2, \tag{2.14}$$

$$\frac{1}{k_2 a_2} \leq \frac{p(t)}{\gamma(t)} \leq \frac{1}{k_2 a_1}, \tag{2.15}$$

furthermore (2.11) implies

$$k_3 a_2 \leq 1, \ \ k_4 a_1 \geq 1. \tag{2.16}$$

Let $t \in (0,1)$, from (2.11),(2.14)-(2.16) we obtain

$$\frac{\beta(t)}{p(t)} \leq k_1 a_2 \leq \min(1,r)\, k_3 a_2 \leq r,$$

$$\frac{p(t)}{\gamma(t)} \leq \frac{1}{k_2 a_1} \Rightarrow \frac{\gamma(t)}{p(t)} \geq k_2 a_1 = \max(R,1)\, k_4 a_1 \geq R,$$

hence

$$\beta(t) \leq rp(t)\,, \ \ \gamma(t) \geq Rp(t) \ \text{ for any } t \in [0,1] \tag{2.17}$$

From (2.13) and (2.17), it yields

$$\beta(t) \leq y(t) \leq \gamma(t) \ \text{ for any } t \in [0,1]\,.$$

Finally we shall prove that $\beta(t) = k_1 g(t)\,,\ \gamma(t) = k_2 g(t)$ are respectively lower and upper solutions for problem (2.1)-(2.2). Thanks to (2.12), we get the following estimates

$$(k_3 a_1)^{\mu} \geq k_3, \text{and } \ (k_4 a_2)^{\mu} \leq k_4, \tag{2.18}$$

$$(k_1 a_1)^{\mu} \geq k_1, \text{and } \ (k_2 a_2)^{\mu} \leq k_2. \tag{2.19}$$

Using (2.9), (2.14) and (2.19), we get for any $t \in (0,1)$

$$\begin{aligned} f(t,\beta(t)) &= f\left(t, \frac{\beta(t)}{p(t)} p(t)\right) \geq \left(\frac{\beta(t)}{p(t)}\right)^{\mu} f(t,p(t)) \\ &\geq (k_1 a_1)^{\mu} f(t,p(t)) \geq k_1 f(t,p(t))\,, \end{aligned}$$

and

$$\begin{aligned} k_2 f(t, p(t)) &= k_2 f\left(t, \frac{p(t)}{\gamma(t)}\gamma(t)\right) \geq k_2 \left(\frac{p(t)}{\gamma(t)}\right)^{\mu} f(t, \gamma(t)) \\ &\geq k_2 (k_2 a_2)^{-\mu} f(t, \gamma(t)) \geq f(t, \gamma(t)), \end{aligned}$$

consequently

$$\begin{cases} -D_{0+}^{\alpha}\beta(t) = k_1 f(t, p(t)) \leq f(t, \beta(t)) & t \in (0,1), \\ -D_{0+}^{\alpha}\gamma(t) = k_2 f(t, p(t)) \geq f(t, \gamma(t)) & t \in (0,1), \end{cases}$$

and

$$\begin{aligned} \beta^{(i)}(0) &\leq 0 \quad \forall i = 0, ..., n-2, \;\; \beta(1) \leq \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds, \\ \gamma^{(i)}(0) &\geq 0 \quad \forall i = 0, ..., n-2, \;\; \gamma(1) \geq \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} y(s)\, ds, \end{aligned}$$

thus $\beta(t) = k_1 g(t)$ and $\gamma(t) = k_2 g(t)$ are respectively lower and upper solutions of problem (2.1)-(2.2). The proof of Theorem 30 is achieved. ■

2.3 Example

Consider the fractional boundary value problem (2.1)-(2.2) with

$$\begin{aligned} \alpha &= 2.5, \lambda_1 = 0.25, \lambda_2 = 0.75, \eta_1 = 0.5, \eta_2 = 0.25, \\ f(t,y) &= 1 + t + \frac{1}{100}\left(\frac{\Gamma(\alpha+1)|y|}{(1-t) + \frac{1}{\xi}\sum\limits_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha} - \frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}} \end{aligned}$$

that we denote by (P), then

$$\begin{aligned} p(t) &= 0.30009t^{\frac{3}{2}}(1.019595 - t), \\ f(t,p(t)) &= 1 + t + 0.01t^{\frac{3}{4}}. \\ \xi &\approx 0.97295 > 0. \end{aligned}$$

For $\mu = \frac{1}{2}$ and for $0 < k < 1$, it is easy to verify that

$$\begin{aligned} k^{\frac{1}{2}} f(t,y) &= k^{\frac{1}{2}} + k^{\frac{1}{2}} t + \frac{1}{100}\left(\frac{\Gamma(\alpha+1)k|y|}{(1-t) + \frac{1}{\xi}\sum\limits_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha} - \frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}} \\ &\leq 1 + t + \frac{1}{100}\left(\frac{\Gamma(\alpha+1)k|y|}{(1-t) + \frac{1}{\xi}\sum\limits_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha} - \frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}} \\ &\leq f(t,ky) \end{aligned}$$

Moreover

$$|f(t,y(t))| \leq 2\left(1 + \frac{1}{200}\left(\frac{\Gamma(\alpha+1)}{(1-t) + \frac{1}{\xi}\sum\limits_{k=0}^{m}\lambda_k\eta_k^{\alpha}\left(\frac{1}{\alpha} - \frac{\eta_k}{\alpha+1}\right)}\right)^{\frac{1}{2}} |y|^{\frac{1}{2}}\right)$$

$$\leq 2\left(1 + 0.0751886\,|y|^{\frac{1}{2}}\right) = \varphi(t)\,\psi(|y|),$$

thus

$$\varphi(t) = 2, \psi(|y|) = \left(1 + 0.0751886\,|y|^{\frac{1}{2}}\right).$$

If we choose $\rho = 2$, it yields

$$\psi(\rho)\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\xi\Gamma(\alpha+1)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha}\right)\|\varphi\|_{L^1} - \rho = -0.3008 < 0,$$

thus (3.3) is satisfied. Since $f(t, p(t))$ is increasing in t then

$$\begin{aligned}
\min_{t\in[0,1]} f(t, p(t)) &= \min_{t\in[0,1]}\left(1 + t + 10^{-2}t^{\frac{3}{4}}\right) = 1,\\
\max_{t\in[0,1]} f(t, p(t)) &= \max_{t\in[0,1]}\left(1 + t + 10^{-2}t^{\frac{3}{4}}\right) = 2.01,
\end{aligned}$$

hence, $f(t, p(t)) \neq 0$. Since all conditions of Theorem 30 are satisfied, thus the fractional boundary value problem (P) has at least one positive solution such that $\beta(t) \leq y(t) \leq \gamma(t)$, $t \in [0, 1]$. Let us find the explicit forms of the functions g, β and γ. By computation, we get

$$\begin{aligned}
a_1 &= 1,\ a_2 = 2.01, r = 1, R = 2.184174, k_1 = k_3 = 0.49751,\\
k_4 &= 2.01,\ k_2 = 4.390189
\end{aligned}$$

$$g(t) = 0.365\,06t^{1.5} - 0.752\,25t^{2.5}\left(0.4 + 0.114\,29t + 0.842\,63 \times 10^{-3}t^{0.75}\right),$$

$$\begin{aligned}
\beta(t) &= 0.181621t^{1.5} - 0.374\,251t^{2.5}\left(0.114\,29t + 1.474\,6 \times 10^{-3}t^{0.75} + 0.4\right),\\
\gamma(t) &= 1.602682t^{1.5} - 3.302519t^{2.5}\left(0.114\,29t + 1.474\,6 \times 10^{-3}t^{0.75} + 0.4\right).
\end{aligned}$$

CHAPTER 3

POSITIVE SOLUTIONS FOR A MULTIPOINT FRACTIONAL BOUNDARY VALUE PROBLEM IN A FRACTIONAL SOBOLEV SPACE

3.1 Introduction

The L^p-solutions of fractional differential equations are discussed by Burton et al. in [8]. The authors considered a Caputo fractional differential equation of the form

$$\begin{aligned} {}^{C}D_{0+}^{\alpha}u(t) &= f(t,u(t)), 0<\alpha<1, t>0, \\ u(0) &\in \mathbb{R}. \end{aligned}$$

Using a variety of techniques, they showed that the solutions belong to $L^p(\mathbb{R}_+)$, $p \geq 1$.

In [34], Karoui et al. considered the following nonlinear quadratic integral equations

$$x(t) = a(t) + f(t, x(t)) \int_0^{+\infty} h(t, s, x(s))\, ds, t > 0$$

and

$$x(t) = a(t) + x(t) \int_0^{+\infty} k(t, s)\, h(s, x(s))\, ds, t > 0,$$

that contain many special cases of nonlinear integral equations. The authors proved the existence of solutions in $L^p(\mathbb{R}_+)$ by using a result of noncampactness combined with Schauder and Darbo fixed point theorems. Following similar ideas, Karoui et al., in [35], proved the existence of $L^p[a,b]$-solutions and $C([a,b])$-solutions of Hammerstein and Volterra types nonlinear integral equations by means of Shaefer's and Schauder's fixed point theorems and a generalized Gronwall's inequality.

Motived by the the above papers and the reference [5], we investigate the existence of $L^p(0,1)$-solutions of a Riemann-Liouville fractional boundary value problem with integral conditions, that is

$$D_{0+}^{\alpha}u(t) + f(t, u(t), D_{0+}^{\gamma}u(t)) = 0, 0 < t < 1, \tag{3.1}$$

$$\begin{aligned} D_{0^+}^{(\alpha-i)}u(0) &= 0 \ \forall i = 2, ..., n, \\ u(1) &= \sum_{k=0}^{m} \lambda_k I_{0^+}^{\beta} u(\eta_k), \ \lambda_k > 0, \end{aligned} \tag{3.2}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\gamma}$ denote the Riemann-Liouville fractional derivatives of order α and γ respectively, $n-1 \leq \alpha < n$, $n \geq 4$, $0 < \gamma < 1$, $I_{0^+}^{\beta}$ denotes the Riemann-Liouville fractional integral of order $\beta > 0$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}_+$ is a given function, $0 < \eta_k < 1$, $\lambda_k > 0$, $k = 0, ..., m$. Set $\xi = 1 - \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \sum_{k=0}^{m} \lambda_k \eta_k^{\alpha+\beta-1} > 0$.
By means of the upper and lower solutions method and Schauder fixed point theorem, the existence of at least one positive solution for the boundary value problem (3.1) and (3.2) in a fractional Sobolev space is established.

3.2 Riemann-Liouville fractional Sobolev spaces

Let us introduce the Riemann-Liouville fractional Sobolev spaces. In what follows we denote by $[a,b]$ a non empty interval of $\mathbb{R}$. Let

$$W^{1,1}(a,b) = \left\{ u \in L^1(a,b), u' \in L^1(a,b) \right\},$$

be the Sobolev space endowed with the norm

$$\|u\|_{W^{1,1}} = \|u\|_{L^1} + \|u'\|_{L^1}.$$

here u' denote the distributional derivative of u. The Sobolev space $W^{1,1}(a,b)$ coincides with the space of absolutely continuous functions $AC(a,b)$. Moreover we have $W^{1,1}(a,b) \subset C[a,b]$.

Definition 31 *[5] The Riemann-Liouville fractional Sobolev space is defined by*

$$W_{RL,a^+}^{s,1} = \left\{ u \in L^1(a,b), I_{a^+}^{1-s} u \in W^{1,1}(a,b) \right\}, 0 < s < 1.$$

$W_{RL,a^+}^{s,1}$ *is a Banach space endowed with the norm*

$$\|u\|_{W_{RL,a^+}^{s,1}} = \|u\|_{L^1} + \left\| I_{a^+}^{1-s} u) \right\|_{W^{1,1}}.$$

For more details on the Sobolev space $W^{s,1}_{RL,a^+}$, we refer to [5].

3.3 Existence of solutions

First, we solve the correspondant linear problem.

Lemma 32 *Assume that $h \in L^1(0,1)$ and $n-1 \leq \alpha \leq n$, $n \geq 4$, then The unique solution of linear boundary value problem*

$$D^{\alpha}_{0^+}u(t) + h(t) = 0, 0 < t < 1, \tag{3.3}$$

$$\begin{aligned} D^{(\alpha-i)}_{0^+}u(0) &= 0 \ \forall i = 2, ..., n, \\ u(1) &= \sum_{k=0}^{m} \lambda_k I^{\beta}_{0^+}u(\eta_k), \ \lambda_k > 0, \end{aligned} \tag{3.4}$$

is given by

$$u(t) = \int_0^1 G(t,s)\, h(s)\, ds + \frac{t^{\alpha-1}}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k, s)\, h(s)\, ds$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] & 0 \leq s \leq t \leq 1 \\ t^{\alpha-1}(1-s)^{\alpha-1} & 0 \leq t \leq s \leq 1 \end{cases} \tag{3.5}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} \left[t^{\alpha+\beta-1}(1-s)^{\alpha-1} - (t-s)^{\alpha+\beta-1}\right] & 0 \leq s \leq t \leq 1 \\ t^{\alpha+\beta-1}(1-s)^{\alpha-1} & 0 \leq t \leq s \leq 1 \end{cases} \tag{3.6}$$

Proof. Let u be a solution of the problem (3.3)-(3.4). By Lemma 9, we get

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds$$

Taking conditions (3.4) into account, it yields

$$c_2 = c_3 = ... = c_n = 0.$$

then

$$\begin{aligned} u(1) &= c_1 - I_{0^+}^{\alpha} h(1) = \sum_{k=0}^{m} \lambda_k I_{0^+}^{\beta} u(\eta_k) \\ &= -\sum_{k=0}^{m} \lambda_k I_{0^+}^{\alpha+\beta} h(\eta_k) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \sum_{k=0}^{m} \lambda_k \eta_k^{\alpha+\beta-1}, \end{aligned}$$

that implies

$$c_1 = \frac{1}{\Gamma(\alpha)\xi} \left(\int_0^1 (1-s)^{\alpha-1} h(s)\, ds - \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^{\alpha+\beta-1} h(s)\, ds \right).$$

Hence the solution of problem (3.3)-(3.4) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)\xi} \left[\int_0^1 (1-s)^{\alpha-1} h(s)\, ds \right. \\ &\quad \left. - \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^{\alpha+\beta-1} h(s)\, ds \right]. \end{aligned}$$

Now, using some computations, we obtain

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds \\
&\quad + \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha+\beta-1}}{\Gamma(\alpha)\left(1-\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha+\beta-1}\right)}\right)\int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\
&\quad - \frac{t^{\alpha-1}\sum_{k=0}^{m}\lambda_k}{\Gamma(\alpha+\beta)\left(1-\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\eta_k^{\alpha+\beta-1}\right)}\int_0^{\eta_k} (\eta_k-s)^{\alpha+\beta-1} h(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left(t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) h(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} h(s)\,ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha+\beta)\xi}\sum_{k=0}^{m}\lambda_k\int_0^{\eta_k}\left(\eta_k^{\alpha+\beta-1}(1-s)^{\alpha-1} - (\eta_k-s)^{\alpha+\beta-1}\right) h(s)\,ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha+\beta)\xi}\sum_{k=0}^{m}\lambda_k\int_{\eta_k}^1 \eta_k^{\alpha+\beta-1}(1-s)^{\alpha-1} h(s)\,ds \\
&= \int_0^1 G(t,s)\,h(s)\,ds + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)\,h(s)\,ds
\end{aligned}$$

■

Lemma 33 *The functions G and H are continuous nonnegative and satisfy*

$$\begin{aligned}
G(t,s) &\le \frac{1}{\Gamma(\alpha)}, 0 \le t, s \le 1. \\
H(t,s) &\le \frac{1}{\Gamma(\alpha+\beta)}, 0 \le t, s \le 1.
\end{aligned}$$

Denote by G^* and H^* the nonnegative constants

$$\begin{aligned} G^* &= \sup\{G(t,s),\ 0 \le t,s \le 1\}, \\ H^* &= \sup\{H(t,s),\ 0 \le t,s \le 1\}. \end{aligned}$$

Let $a,b,c,d \in \mathbb{R}_+$, define the upper and lower control functions respectively by

$$U(t,u,v) = \sup\{f(t,\lambda,\mu) : a \le \lambda \le u,\ b \le \mu \le v\},$$

and

$$L(t,u,v) = \inf\{f(t,\lambda,\mu) : u \le \lambda \le c,\ v \le \mu \le d\}, 0 < t < 1.$$

We have $L(t,u,v) \le f(t,u,v) \le U(t,u,v)$ for $0 < t < 1$, $a \le u \le c$ and $b \le v \le d$. Set the cone $K = \left\{u \in W^{1-\gamma,1}_{RL,0^+},\ u(t) \ge 0, 0 < t < 1\right\}$. Note that the norm in the space $W^{1-\gamma,1}_{RL,0^+}$ is

$$\|u\|_{W^{1-\gamma,1}_{RL,0^+}} = \|u\|_{L^1} + \left\|I^{1-\gamma}_{0^+}u\right\|_{L^1} + \left\|D^{\gamma}_{0^+}u\right\|_{L^1}.$$

We make the following hypothesis:
$(H1)$ There exist $u^*, u_* \in K$, such that $a \le u^*(t) \le u_*(t) \le c$,
$b \le D^{\gamma}_{0^+}u^*(t) \le D^{\gamma}_{0^+}u_*(t) \le d$ and for all $t \in [0,1]$, we have

$$\begin{aligned} u^*(t) &\ge \int_0^1 G(t,s)\,U(s,u^*(s),D^{\gamma}u^*(s))\,ds \\ &\quad + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s)\,U(s,u^*(s),D^{\gamma}u^*(s))\,ds \end{aligned}$$

$$\begin{aligned} u_*(t) &\le \int_0^1 G(t,s)\,L\left(s,u_*(s),D^{\gamma}_{0^+}u_*(s)\right)ds \\ &\quad + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s)\,L\left(s,u_*(s),D^{\gamma}_{0^+}u_*(s)\right)ds. \qquad (3.7) \end{aligned}$$

$$
\begin{aligned}
D_{0^+}^{\gamma}u^*(t) &\geq \frac{1}{\Gamma(\alpha-\gamma)}\int_0^1 G_1(t,s)\,U(s,u^*(s),D^{\gamma}u^*(s))\,ds \\
&\quad + \frac{\Gamma(\alpha)\,t^{\alpha-\gamma-1}}{\xi\Gamma(\alpha-\gamma)\,\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\int_0^1 H_1(\eta_k,s)\,U(s,u^*(s),D^{\gamma}u^*(s))\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
D_{0^+}^{\gamma}u_*(t) &\leq \int_0^1 G_1(t,s)\,U(s,u_*(s),D^{\gamma}u_*(s))\,ds \\
&\quad + \frac{\Gamma(\alpha)\,t^{\alpha-\gamma-1}}{\xi\Gamma(\alpha-\gamma)\,\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)\,U(s,u_*(s),D^{\gamma}u_*(s))\,ds,
\end{aligned}
$$

where

$$
G_1(t,s) = \frac{1}{\Gamma(\alpha-\gamma)}\begin{cases} t^{\alpha-\gamma-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-\gamma-1}, & s \leq t \\ t^{\alpha-\gamma-1}(1-s)^{\alpha-1}, & s \geq t. \end{cases}
$$

We recall the definition of a Caratheodory function:

Definition 34 *A map $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is said to be Caratheodory if:*
(a) $t\to f(t,u,v)$ is measurable for each $u,v\in\mathbb{R}$.
(b) $(u,v)\to f(t,u,v)$ is continuous for almost all $t\in[0,1]$.

Theorem 35 *Assume $(H1)$ and the following hypotheses hold*
$(H2)$ $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}_+$ is a Caratheordory function.

$(H3)$ There exist a nonnegative function $g\in L^1[0,1]$, two constants $C>0$ and $R>0$ such that

$$
f(t,u,v) \leq g(t) + C(|u|+|v|),\ 0\leq t\leq 1,\ u,v\in\mathbb{R}. \tag{3.8}
$$

and

$$
\left(\|g\|_{L^1} + CR\right)\left[\left(G^* + \frac{H^*}{\xi}\sum_{k=0}^{m}\lambda_k\right)\left(1 + \frac{1}{\Gamma(2-\gamma)}\right) + \frac{2}{\xi\Gamma(\alpha-\gamma)}\right] \leq R \tag{3.9}
$$

Then the boundary value problem (3.1)-(3.2) has at least one positive solution, such that $u_*(t) \leq u(t) \leq u^*(t)$ *and* $D_{0^+}^{\gamma} u^*(t) \leq D_{0^+}^{\gamma} u(t) \leq D_{0^+}^{\gamma} u_*(t)$, *for all* $t \in [0,1]$.

Proof. Transform the problem (3.1)-(3.2) into a fixed point problem. Denote by D_R the set

$$D_R = \Big\{ u \in K, \ \|u\|_{W_{RL,0^+}^{1-\gamma,1}} \leq R, \ u_*(t) \leq u(t) \leq u^*(t),$$

$$D_{0^+}^{\gamma} u^*(t) \leq D_{0^+}^{\gamma} u(t) \leq D_{0^+}^{\gamma} u_*(t), t \in [0,1] \Big\},$$

where R is defined in hypothesis $(H3)$. It is clear that D_R is a bounded, closed and convex subset of $W_{RL,0^+}^{1-\gamma,1}$. Define the operator $N : D_R \to W_{RL,0^+}^{1-\gamma,1}$ by

$$\begin{aligned} Nu(t) &= \int_0^1 G(t,s) f(s, u(s), D^{\gamma} u(s)) \, ds \\ &\quad + \frac{t^{\alpha-1}}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k, s) f(s, u(s), D^{\gamma} u(s)) \, ds \end{aligned}$$

We shall show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be done in some steps.

Claim1: N is continuous in $W_{RL,0^+}^{1-\gamma,1}$. Let be a sequence such that $u_n \to u$ in

$W^{1-\gamma,1}_{RL,0^+}$. from (3.8) and Lemma 33 we get

$$\begin{aligned}
& \left|Nu_n(t) - Nu(t)\right| \\
\leq & \int_0^1 G(t,s) \left|f(s,u_n(s), D^\gamma u_n(s)) - f(s,u(s), D^\gamma u(s))\right| ds \\
& + \frac{t^{\alpha-1}}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k, s) \\
& \times \left|f(s,u_n(s), D^\gamma u_n(s)) - f(s,u(s), D^\gamma u(s))\right| ds \\
\leq & \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^{m} \lambda_k\right) \left\|f(.,u_n(.), D^\gamma u_n(.)) - f(.,u(.), D^\gamma u(.))\right\|_{L^1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \left\|Nu_n - Nu\right\|_{L^1} \leq \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^{m} \lambda_k\right) \times \\
& \left\|f(.,u_n(.), D^\gamma u_n(.)) - f(.,u(.), D^\gamma u(.))\right\|_{L^1}. \qquad (3.10)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \left|I^{1-\gamma}_{0^+} Nu_n(t) - I^{1-\gamma}_{0^+} Nu(t)\right| \\
\leq & \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \left|Nu_n(s) - Nu(s)\right| ds \\
\leq & \frac{1}{\Gamma(2-\gamma)} \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^{m} \lambda_k\right) \times \\
& \left\|f(.,u_n(.), D^\gamma u_n(.)) - f(.,u(.), D^\gamma u(.))\right\|_{L^1}, \qquad (3.11)
\end{aligned}$$

hence

$$\begin{aligned}
& \left\|I^{1-\gamma}_{0^+} Nu_n - I^{1-\gamma}_{0^+} Nu\right\|_{L^1(0,1)} \leq \\
& \frac{1}{\Gamma(2-\gamma)} \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^{m} \lambda_k\right) \\
& \times \left\|f(.,u_n(.), D^\gamma u_n(.)) - f(.,u(.), D^\gamma u(.))\right\|_{L^1}. \qquad (3.12)
\end{aligned}$$

Remarking that Nu can be written as

$$Nu(t) = -I_{0^+}^{\alpha} f(t, u(t), D^{\gamma} u(t)) + t^{\alpha-1} H_u,$$

where

$$\begin{aligned} H_u &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s), D^{\gamma} u(s))\, ds \\ &- \frac{1}{\Gamma(\alpha+\beta)\xi} \sum_{k=0}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^{\alpha+\beta-1} f(s, u(s), D^{\gamma} u(s))\, ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)\xi} \sum_{k=0}^{m} \lambda_k \eta_k^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} f(s, u(s), D^{\gamma} u(s))\, ds, \end{aligned}$$

then

$$\begin{aligned} & \left| D_{0^+}^{\gamma} N u_n(t) - D_{0^+}^{\gamma} N u(t) \right| = \\ & \Big| I_{0^+}^{\alpha-\gamma} f(t, u_n(t), D^{\gamma} u_n(t)) - I_{0^+}^{\alpha-\gamma} f(t, u(t), D^{\gamma} u(t)) \\ & + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} t^{\alpha-\gamma-1} (H_u - H_{u_n}) \Big| \\ \leq \;& \frac{2}{\xi \Gamma(\alpha-\gamma)} \left\| f(., u_n(.), D^{\gamma} u_n(.)) - f(., u(.), D^{\gamma} u(.)) \right\|_{L^1}. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \left\| D_{0^+}^{\gamma} N u_n(t) - D_{0^+}^{\gamma} N u(t) \right\|_{L^1} &\leq \frac{2}{\xi \Gamma(\alpha-\gamma)} \\ \times \left\| f(., u_n(.), D^{\gamma} u_n(.)) - f(., u(.), D^{\gamma} u(.)) \right\|_{L^1}. \end{aligned} \tag{3.13}$$

Thanks to inequalities (3.10)-(3.13), the operator N is continuous in $W_{RL,0^+}^{1-\gamma,1}$.

Claim2: $N(D_R) \subset D_R$. Let $u \in D_R$, we have

$$\begin{aligned} |Nu(t)| &\leq \int_0^1 G(t,s) f(s,u(s), D^\gamma u(s)) \, ds \\ &\quad + \frac{t^{\alpha-1}}{\xi} \sum_{k=0}^m \lambda_k \int_0^1 H(\eta_k, s) f(s,u(s), D^\gamma u(s)) \, ds \\ &\leq \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^m \lambda_k \right) \left(\|g\|_{L^1} + C \left(\|u\|_{L^1} + \|D^\gamma u\|_{L^1} \right) \right). \end{aligned} \tag{3.14}$$

Hence,

$$\|Nu\|_{L^1} \leq \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^m \lambda_k \right) \left(\|g\|_{L^1} + CR \right). \tag{3.15}$$

Similarly, we obtain

$$\left\| I_{0^+}^{1-\gamma} Nu \right\|_{L^1} \leq \frac{1}{\Gamma(2-\gamma)} \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^m \lambda_k \right) \left(\|g\|_{L^1} + CR \right), \tag{3.16}$$

and

$$\left\| D_{0^+}^\gamma Nu \right\|_{L^1} \leq \frac{2}{\xi \Gamma(\alpha-\gamma)} \left(\|g\|_{L^1} + CR \right). \tag{3.17}$$

Taking (3.15)-(3.17) and (3.9) into account, we get

$$\|Nu\|_{W_{RL,0^+}^{1-\gamma,1}} \leq R.$$

Let $u \in D_R$, then $u_*(t) \le u(t) \le u^*(t)$. By hypothesis $(H1)$ we have

$$\begin{aligned}
Nu(t) &\le \int_0^1 G(t,s)\,U(s,u(s),(D^\gamma u)(s))\,ds \\
&\quad + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s)\,U(s,u(s),(D^\gamma u)(s))\,ds \\
&\le \int_0^1 G(t,s)\,U(s,u^*(s),D^\gamma u^*(s))\,ds \\
&\quad + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s)\,U(s,u^*(s),D^\gamma u^*(s))\,ds \\
&\le u^*(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Nu(t) &\ge \int_0^1 G(t,s)\,L(s,u^*(s),(D^\gamma u)^*(s))\,ds \\
&\quad + \frac{t^{\alpha-1}}{\xi}\sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s)\,L(s,u^*(s),(D^\gamma u)^*(s))\,ds \\
&\ge u^*(t),
\end{aligned}$$

thus, $u_*(t) \le Nu(t) \le u^*(t)$, for all $u \in D_R$, $t \in [0,1]$.

Remarking that $Nu(t)$ can be writen as

$$\begin{aligned}
Nu(t) &= -I_{0^+}^{\alpha} f(t,u(t),D^\gamma u(t)) + \frac{t^{\alpha-1}}{\xi}\left(I_{0^+}^{\alpha} f(1,u(1),D^\gamma u(1))\right. \\
&\quad - \sum_{k=0}^{m}\lambda_k I_{0^+}^{\alpha+\beta} f(\eta_k,u(\eta_k),D^\gamma u(\eta_k)),
\end{aligned}$$

by calculus, we get

$$\begin{aligned}
D_{0^+}^{\gamma}Nu(t) &= -I_{0^+}^{\alpha-\gamma}f(t,u(t),D^{\gamma}u(t)) + \frac{\Gamma(\alpha)}{\xi\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}\left(I_{0^+}^{\alpha}f(1,u(1),D^{\gamma}u(1))\right. \\
&\quad -\sum_{k=0}^{m}\lambda_k I_{0^+}^{\alpha+\beta}f(\eta_k,u(\eta_k),D^{\gamma}u(\eta_k)))
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 G_1(t,s)f(s,u(s),D^{\gamma}u(s))\,ds + \frac{\Gamma(\alpha)t^{\alpha-\gamma-1}}{\xi\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta)} \\
&\quad \times \sum_{k=0}^{m}\lambda_k \int_0^1 H(\eta_k,s)f(s,u(s),D^{\gamma}u(s))\,ds.
\end{aligned}$$

Since the functions G_1 and H are continuous and nonnegative on $[0,1]$, then

$$\begin{aligned}
D_{0^+}^{\gamma}Nu(t) &\leq \frac{1}{\Gamma(\alpha-\gamma)}\int_0^1 G_1(t,s)U(s,u(s),D^{\gamma}u(s))\,ds \\
&\quad +\frac{\Gamma(\alpha)t^{\alpha-\gamma-1}}{\xi\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)U(s,u(s),D^{\gamma}u(s))\,ds \\
&\leq \frac{1}{\Gamma(\alpha-\gamma)}\int_0^1 G_1(t,s)U(s,u^*(s),D^{\gamma}u^*(s))\,ds \\
&\quad +\frac{\Gamma(\alpha)t^{\alpha-\gamma-1}}{\xi\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta)}\sum_{k=0}^{m}\lambda_k\int_0^1 H(\eta_k,s)U(s,u^*(s),D^{\gamma}u^*(s))\,ds \\
&\leq D_{0^+}^{\gamma}u^*(t)
\end{aligned}$$

Similarly we prove that $D_{0^+}^{\gamma}Nu(t) \geq D_{0^+}^{\gamma}u_*(t)$ and then $N(D_R) \subset D_R$.

Claim 3: $N(D_R)$ is relatively compact in $W^{1-\gamma,1}_{RL,0^+}$. To this end, we show that the two statement of Lemma 22 hold. Let $u \in D_R$. From (3.5) we have

$$\begin{aligned}
& \left|Nu(t+h) - Nu(t)\right| \\
\leq & \int_0^1 \left|G(t+h,s) - G(t,s)\right| \left|f(s,u(s),D^\gamma u(s))\right| ds \\
& + \frac{(t+h)^{\alpha-1} - t^{\alpha-1}}{\xi} \sum_{k=0}^{m} \lambda_k \int_0^1 H(\eta_k,s) \left|f(s,u(s),D^\gamma u(s))\right| ds
\end{aligned}$$

$$\leq \left(3h(\alpha-1) + h^{\alpha-1} + \frac{1}{\xi} h(\alpha-1) H^* \sum_{k=0}^{m} \lambda_k\right) \left(\|g\|_{L^1} + CR\right) \to 0 \text{ as } h \to 0, \tag{3.18}$$

In view of (3.14), it yields

$$\begin{aligned}
& \left|I^{1-\gamma}_{0^+} Nu(t+h) - I^{1-\gamma}_{0^+} Nu(t)\right| \\
= & \frac{1}{\Gamma(1-\gamma)} \left| \int_0^{t+h} (t+h-s)^{-\gamma} Nu(s)\, ds - \int_0^t (t-s)^{-\gamma} Nu(s)\, ds \right| \\
\leq & \frac{1}{\Gamma(1-\gamma)} \left(\int_0^t \left((t-s)^{-\gamma} - (t+h-s)^{-\gamma}\right) \left|Nu(s)\right| ds \right. \\
& \left. + \int_t^{t+h} (t+h-s)^{-\gamma} \left|Nu(s)\right| ds \right) \\
\leq & \frac{\left(\|g\|_{L^1} + CR\right)}{\Gamma(2-\gamma)} \left(G^* + \frac{H^*}{\xi} \sum_{k=0}^{m} \lambda_k\right) \left(t^{1-\gamma} + 2h^{1-\gamma} - (t+h)^{1-\gamma}\right) \\
\to & 0 \text{ as } h \to 0.
\end{aligned} \tag{3.19}$$

Moreover, we have

$$\begin{aligned}
&\left|D_{0^+}^{\gamma}Nu\left(t+h\right)-D_{0^+}^{\gamma}Nu\left(t\right)\right| \\
&\Big|I_{0^+}^{\alpha-\gamma}f\left(t+h,u\left(t+h\right),D^{\gamma}u\left(t+h\right)\right)-I_{0^+}^{\alpha-\gamma}f\left(t,u\left(t\right),D^{\gamma}u\left(t\right)\right) \\
&+\frac{\Gamma\left(\alpha\right)}{\Gamma\left(\alpha-\gamma\right)}\left(\left(t+h\right)^{\alpha-\gamma-1}-t^{\alpha-\gamma-1}\right)H_u\Bigg| \\
\leq\ &\frac{\left(\|g\|_{L^1}+CR\right)}{\Gamma\left(\alpha-\gamma\right)}\left(\frac{2\left(\alpha-\gamma-1\right)h}{\xi}+h^{\alpha-\gamma-1}\right)\to 0\text{ as }h\to 0. \qquad (3.20)
\end{aligned}$$

From (3.18)-(3.20), we obtain that $\|\tau_h Nu-Nu\|_{W_{RL,0^+}^{1-\gamma,1}}\to 0$ as $h\to 0$ for any $u\in D_R$ and then the first statement of Lemma 22 is proved. Now let us show the second statement of Lemma 22. From the proof of Claim 2, we get

$$\begin{aligned}
&\int_{1-\epsilon}^{1}\left|Nu\left(t\right)\right|dt+\int_{1-\epsilon}^{1}\left|I_{0^+}^{1-\gamma}Nu\left(t\right)\right|dt+\int_{1-\epsilon}^{1}\left|D_{0^+}^{\gamma}Nu\left(t\right)\right|dt\leq \\
&\epsilon\left(\|g\|_{L^1}+CR\right)\times \\
&\left(\left(1+\frac{1}{\Gamma\left(2-\gamma\right)}\right)\left(G^*+\frac{H^*}{\xi}\sum_{k=0}^{m}\lambda_k\right)+\frac{2}{\xi\Gamma\left(\alpha-\gamma\right)}\right)\to 0, \qquad (3.21)
\end{aligned}$$

uniformly on D_R. Since all hypotheses of Lemma 22 are satisfied then N is relatively compact on D_R. Thanks to Schauder's fixed point Theorem, N has a fixed point $u\in D_R$, which is a positive solution of the problem (3.1)-(3.2). ■

CHAPTER 4

EXISTENCE AND STABILITY FOR CAPUTO-KATUGAMPOLA FRACTIONAL DIFFERENTIAL EQUATIONS

4.1 Introduction

A generalized fractional derivative was recently proposed by Katugampola in [36] which generalizes the concept of Caputo and Caputo-Hadamard fractional derivatives. In [2], Almeida et al, gave a new type of fractional operator, that is called Caputo–Katugampola derivative and studied the existence and uniqueness of solution for the following fractional Cauchy type problem involving the Caputo–Katugampola derivative

$$^{C}D_{a^{+}}^{\alpha,\rho}x\left(t\right)=f\left(t,x\left(t\right)\right),\qquad a\leq t\leq b,$$

$$x\left(a\right)=x_{a},\ x_{a}\in\mathbb{R}\quad,$$

where $0<\alpha<1$, $\rho>0$, $\rho\neq 1$, $f\in C\left(\left[a,b\right]\times\mathbb{R},\mathbb{R}\right)$. The authors proved an approximation formula for the Caputo–Katugampola fractional derivative that depends only on the first-order derivative of a given function, then they solved numerically the above problem.

Baleanu et al. in [3] considered chaotic behavior and Lyapunov stability of fractional differential equations containing the Caputo–Katugampola fractional derivative of order $0<\alpha\leq 1$. The proofs are based on Adomian polynomials and a fractional Taylor series.

In [20], Ge and Kou investigated the stability of the solutions of the following nonlinear Caputo fractional differential equation

$$^{C}D^{\alpha}x\left(t\right)=f\left(t,x\left(t\right)\right)\qquad t\geq 0,$$

$$x\left(0\right)=x_{0},\quad x'\left(0\right)=x_{1},$$

where $^{C}D^{\alpha}x$ is the standard Caputo's fractional derivative of order $1<\alpha<2$. By employing the Krasnoselskii's fixed point theorem in a weighted Banach space, the authors obtained stability results.

Motivated by the mentioned papers, we focus, in this chapter, on the existence and stability of solutions for a Caputo–Katugampola fractional nonlinear differential equation of the form

$$ {}^{C}D_{t_0^+}^{\alpha,\rho} x(t) = f(t, x(t)) \qquad t \geq t_0, \tag{4.1} $$

$$ x(t_0) = x_0, \quad x'(t_0) = x_1, \tag{4.2} $$

where $1 < \alpha < 2$, $\rho > 0$, $\rho \neq 1$, $x_0, x_1 \in \mathbb{R}$ and $f : [t_0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(t, 0) = 0$, $\forall t \geq t_0$.
To prove the existence and stability of the solutions, we transform the problem (4.1)-(4.2) into an integral equation that returns to a sum of two mappings, one is a contraction and the other is compact, and then we apply Krasnoselskii's fixed point theorem.

4.2 Existence and stability of solutions

We recall the definition Caputo-Katugampola fractional derivative.

Definition 36 *The Caputo-Katugampola fractional derivative of order* $\alpha > 0$ *of the function* f *is defined by*

$$
\begin{aligned}
{}^{C}D_{a^+}^{\alpha,\rho} f(t) &= I_{a^+}^{n-\alpha,\rho} \left(t^{1-\rho} \frac{d}{dt} \right)^n f(t) \\
&= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{n-\alpha-1} \left(t^{1-\rho} \frac{d}{dt} \right)^n f(s)\, ds \\
&= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t \frac{s^{(\rho-1)(1-n)} f^{(n)}(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds.
\end{aligned}
$$

where $n = 1 + [\alpha]$.

Next, we give a relation between the Caputo–Katugampola fractional derivatives of order α and $(\alpha - 1)$.

Lemma 37 *Let $1 < \alpha < 2$. The following relationship holds*

$$^{C}D_{t_0^+}^{\alpha,\rho}u(t) = {}^{C}D_{t_0^+}^{\alpha-1,\rho}\left[\int_{t_0}^{t} s^{1-\rho}u^{(2)}(s)\,ds\right].$$

Proof. We have

$$\begin{aligned}
& {}^{C}D_{t_0^+}^{\alpha-1,\rho}\left[\int_{t_0}^{t} s^{1-\rho}u^{(2)}(s)\,ds\right] \\
= & \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)}\int_{t_0}^{t}\frac{d}{ds}\left(\int_{t_0}^{s}\tau^{1-\rho}u^{(2)}(\tau)\,d\tau\right)\frac{1}{(t^{\rho}-s^{\rho})^{\alpha-1}}ds \\
= & \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)}\int_{t_0}^{t}\frac{s^{(1-\rho)}u^{(2)}(s)}{(t^{\rho}-s^{\rho})^{\alpha-1}}ds \\
= & {}^{C}D_{t_0^+}^{\alpha,\rho}u(t).
\end{aligned}$$

■

Let $\rho > 0$, $\rho \neq 1$ and $g : [t_0, +\infty) \to \mathbb{R}^+$ such that for all $t, s \geq t_0$, $g(t) \geq t^{\alpha\rho+3}$ and $g\left(\frac{t}{s}\right) g(s) \leq g(t)$.
Let

$$E = \left\{ x \in C[t_0, +\infty), \mathbb{R} : \sup_{t \geq t_0} \frac{|x(t)|}{g(t)} < \infty \right\}.$$

Note that E is a Banach space equipped with the norm $\|x\| = \sup_{t\geq t_0} \frac{|x(t)|}{g(t)}$. For more properties of the Banach space E, we refer to [11].

We recall the definition of stability.

Definition 38 *The trivial solution $x = 0$ of (4.1)-(4.2) is said to be stable in the Banach space E, if for every $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ such that $|x_0| + |x_1| < \delta$ implies that the solution $x(t)$ exists for all $t \geq t_0$ and satisfies $\|x\| \leq \varepsilon$.*

Let $\mathcal{F}(\varepsilon) = \{x : x \in E, \ \|x\| \leq \varepsilon\}$. We will use the following modified compactness criterion:

Lemma 39 *[41] Let $\mathcal{F}$ be a subset of the Banach space E. Then $\mathcal{F}$ is relatively compact in E if the following conditions are satisfied*
(i) $\left\{\frac{x(t)}{g(t)} : x(t) \in \mathcal{F}\right\}$ *is uniformly bounded,*
(ii) $\left\{\frac{x(t)}{g(t)} : x(t) \in \mathcal{F}\right\}$ *is equicontinuous on any compact interval of* $\mathbb{R}^{+}$,
(iii) $\left\{\frac{x(t)}{g(t)} : x(t) \in \mathcal{F}\right\}$ *is equiconvergent at infinity. i.e. for any given $\varepsilon > 0$,there exists a $T > t_0$ such that for all $x \in \mathcal{F}$ and $t_1, t_2 > T$ it holds* $\left|\frac{x(t_1)}{g(t_1)} - \frac{x(t_2)}{g(t_2)}\right| < \varepsilon$.

Next, we solve the corresponding linear problem.

Lemma 40 *Let $r \in C([t_0, +\infty), \mathbb{R})$, $1 < \alpha < 2$ and $\rho > 0$, $\rho \neq 1$. A function x is a solution of the fractional initial value problem*

$$ {}^{C}D_{t_0^+}^{\alpha,\rho} x(t) = r(t) \qquad t \geq t_0, \tag{4.3}$$

$$ x(t_0) = x_0, \quad x'(t_0) = x_1, \tag{4.4}$$

if and only if, x is a solution of the fractional integral equation

$$\begin{aligned} x(t) &= x_0 \left(\left(\frac{t}{t_0} \right)^{1-\rho} - (1-\rho) t_0^{-\rho} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds \right) + x_1 t_0^{1-\rho} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds \\ &\quad + \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} x(\tau) \, ds \right] d\tau \\ &\quad + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \frac{\tau^{\rho-1}}{(s^{\rho} - \tau^{\rho})^{2-\alpha}} r(\tau) \, ds \right] d\tau. \end{aligned} \tag{4.5}$$

Proof. Assume that $x(t)$ is a solution of the initial value problem (4.1)-(4.2). From Lemma 37, we have

$$ {}^{C}D_{t_0^+}^{\alpha,\rho} x(t) = {}^{C}D_{t_0^+}^{\alpha-1,\rho} \left[\int_{t_0}^{t} s^{1-\rho} x^{(2)}(s) \, ds \right]. \tag{4.6}$$

By integration by parts, it yields

$$\begin{aligned}\int_{t_0}^{t} s^{1-\rho}x^{(2)}(s)\,ds &= \left[s^{1-\rho}x'(s)\right]_{t_0}^{t} - \int_{t_0}^{t}(1-\rho)\,s^{-\rho}x'(s)\,ds \\ &= \left[t^{1-\rho}x'(t) - t_0^{1-\rho}x'(t_0)\right] - (1-\rho)\int_{t_0}^{t} s^{-\rho}x'(s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\int_{t_0}^{t} s^{-\rho}x'(s)\,ds &= \left[s^{-\rho}x(s)\right]_{t_0}^{t} + \rho\int_{t_0}^{t} s^{-\rho-1}x(s)\,ds \\ &= \left[t^{-\rho}u(t) - t_0^{-\rho}x(t_0)\right] + \rho\int_{t_0}^{t} s^{-\rho-1}x(s)\,ds.\end{aligned}$$

Using the Leibniz integral law, we obtain

$$\left[\int_{t_0}^{t} s^{1-\rho}x^{(2)}(s)\,ds\right] = I_{t_0^+}^{\alpha-1,\rho}r(t) = \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_0}^{t}\frac{s^{\rho-1}r(s)}{(t^\rho - s^\rho)^{2-\alpha}}ds.$$

Then,

$$\begin{aligned}& t^{1-\rho}x'(t) + (\rho-1)\,t^{-\rho}x(t) - t_0^{1-\rho}x_1 - (\rho-1)\,t_0^{-\rho}x_0 \\ & + (\rho-1)\,\rho\int_{t_0}^{t} s^{-\rho-1}x(s)\,ds \\ = \;& \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_0}^{t}\frac{s^{\rho-1}r(s)}{(t^\rho - s^\rho)^{2-\alpha}}ds.\end{aligned}$$

Hence,

$$x'(t) = \frac{1}{t}(1-\rho)\,x(t) + \frac{1}{t^{1-\rho}}\left[t_0^{1-\rho}x_1 + (\rho-1)\,t_0^{-\rho}x_0\right.$$

$$+ (1-\rho)\,\rho \int_{t_0}^{t} s^{-\rho-1} x(s)\,ds + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_{t_0}^{t} \frac{s^{\rho-1} r(s)}{(t^{\rho}-s^{\rho})^{2-\alpha}} ds \Bigg] .$$

By the variation of constants formula, we get

$$\begin{aligned} x(t) &= x_0 \left(\left(\frac{t}{t_0}\right)^{1-\rho} + (\rho-1)\, t_0^{-\rho} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds \right) + x_1 \left(t_0^{1-\rho} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds \right) \\ &\quad + \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} x(\tau)\,ds \right] d\tau \\ &\quad + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \frac{\tau^{\rho-1}}{(s^{\rho}-\tau^{\rho})^{2-\alpha}} r(\tau)\,ds \right] d\tau. \end{aligned}$$

Conversely, if we assume that $x(t)$ is a solution of (4.3), then using the fact that ${}^{C}D_{t_0^+}^{\alpha,\rho}$ is the left inverse of $I_{t_0^+}^{\alpha,\rho}$, we get that x is the solution of (4.3)-(4.4). ■

Lemma 41 *Let* $\rho > 0$, $\rho \neq 1$, $1 < \alpha < 2$. *Then, there exist*

$$M_1 = M_1(\rho) > 0,\ \ 0 < M_2 = M_2(\rho) < 1,\ \ M_3 = M_3(\rho) > 0,$$

such that for all $t \geq t_0$, *we have*

$$\frac{1}{g(t)} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds \leq M_1, \tag{4.7}$$

$$\frac{1}{g(t)} \left| \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} x(\tau)\,ds \right] d\tau \right| \leq M_2 \|x\|, \tag{4.8}$$

$$\frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \leq M_3 t^{-1} \tau^{\alpha\rho}, \tag{4.9}$$

where

$$k(t,\tau) = \begin{cases} \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_\tau^t \frac{t^{1-\rho}}{s^{2-2\rho}} \frac{\tau^{\rho-1}}{(s^\rho-\tau^\rho)^{2-\alpha}} ds, \ t > \tau, \\ 0, \ t \le \tau. \end{cases}$$

Proof. Let $\rho > 0$, $\rho \neq 1$ and $1 < \alpha < 2$, we have two cases:
Case1: For $\rho \in (0,1)$, we have

$$\begin{aligned} \frac{1}{g(t)} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds &\le \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \frac{1}{t^{\alpha\rho+3}} ds \\ &\le \frac{1}{t_0^{2-2\rho}} \int_{t_0}^{t} \frac{1}{t^{\alpha\rho+\rho+2}} ds \\ &\le \frac{1}{t_0^{\alpha\rho-\rho+3}}, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{g(t)} \left| \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} x(\tau)\, ds \right] d\tau \right| \\ \le & \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{1}{g(t)} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} |x(\tau)|\, ds \right] d\tau \\ \le & \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{\tau^{\alpha\rho+3}}{t^{\alpha\rho+3}} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} \frac{|x(\tau)|}{g(\tau)} ds \right] d\tau \\ \le & \ \|x\| \int_{t_0}^{t} \left[t^{-\alpha\rho-\rho-1} \frac{\tau^{\alpha\rho-\rho+2}}{\tau^{2-2\rho}} \right] d\tau \\ \le & \ \|x\| \int_{t_0}^{t} \left[t^{-\alpha\rho-\rho-1} \tau^{\alpha\rho+\rho} \right] d\tau \le \frac{\|x\|}{\alpha\rho+\rho+1} \\ < & \ \|x\|, \end{aligned}$$

some computations give

$$\begin{aligned}
\frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} &\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{1}{g\left(\frac{t}{\tau}\right)}\frac{t^{1-\rho}}{s^{2-2\rho}}\frac{\tau^{\rho-1}}{\left(s^{\rho}-\tau^{\rho}\right)^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{\tau^{\alpha\rho+3}}{t^{\alpha\rho+3}}\frac{t^{1-\rho}}{s^{2-2\rho}}\frac{\tau^{\rho-1}}{\left(s^{\rho}-\tau^{\rho}\right)^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{t^{-\alpha\rho-\rho-2}}{\rho s^{1-\rho}}\frac{\rho s^{\rho-1}\tau^{\alpha\rho+\rho+2}}{\left(s^{\rho}-\tau^{\rho}\right)^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}t^{-\alpha\rho-\rho-2}}{\rho\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{\tau^{\alpha\rho+2\rho+1}\rho s^{\rho-1}}{\left(s^{\rho}-\tau^{\rho}\right)^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}t^{-\alpha\rho-\rho-2}}{\rho\Gamma(\alpha)}\frac{\tau^{\alpha\rho+2\rho+1}}{\left(t^{\rho}-\tau^{\rho}\right)^{1-\alpha}} \\
&\leq \frac{\rho^{1-\alpha}t^{-1}}{\Gamma(\alpha)}\tau^{\alpha\rho}.
\end{aligned}$$

Case2: Similarly, for $\rho > 1$, we have the following estimates

$$\begin{aligned}
\frac{1}{g(t)}\int_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}ds &\leq \int_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\frac{1}{t^{\alpha\rho+3}}ds \\
&\leq \int_{t_0}^{t}\frac{1}{s^{2-2\rho}t^{\alpha\rho+\rho+2}}ds \\
&\leq \frac{1}{t_0^{\alpha\rho-\rho+3}},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{g(t)}\left|\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}x(\tau)\,ds\right]d\tau\right| \\
&\leq \int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{1}{g(t)}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}\left|x(\tau)\right|ds\right]d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{\tau^{\alpha\rho+3}}{t^{\alpha\rho+3}}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}\frac{|x(\tau)|}{g(\tau)}ds\right]d\tau \\
&\leq \|x\|\int_{t_0}^{t}\left[t^{-\alpha\rho+\rho-3}\tau^{\alpha\rho-\rho+2}\right]d\tau \\
&\leq \frac{\|x\|}{\alpha\rho-\rho+3}<\frac{\|x\|}{3},
\end{aligned}
$$

moreover,

$$
\begin{aligned}
\frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} &\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{1}{g\left(\frac{t}{\tau}\right)}\frac{t^{1-\rho}}{s^{2-2\rho}}\frac{\tau^{\rho-1}}{(s^{\rho}-\tau^{\rho})^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{\tau^{\alpha\rho+3}}{t^{\alpha\rho+3}}\frac{t^{1-\rho}}{s^{2-2\rho}}\frac{\tau^{\rho-1}}{(s^{\rho}-\tau^{\rho})^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{t^{-\alpha\rho-\rho-2}}{\rho s^{1-\rho}}\frac{\rho s^{\rho-1}\tau^{\alpha\rho+\rho+2}}{(s^{\rho}-\tau^{\rho})^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}t^{-\alpha\rho-3}}{\rho\Gamma(\alpha-1)}\int_{\tau}^{t}\frac{\tau^{\alpha\rho+\rho+2}\rho s^{\rho-1}}{(s^{\rho}-\tau^{\rho})^{2-\alpha}}ds \\
&\leq \frac{\rho^{2-\alpha}t^{-\alpha\rho-3}}{\rho\Gamma(\alpha)}\frac{\tau^{\alpha\rho+\rho+2}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}\leq\frac{\rho^{1-\alpha}t^{-1}}{\Gamma(\alpha)}\tau^{\alpha\rho}.
\end{aligned}
$$

The proof is completed. ■

Theorem 42 *Let $1<\alpha<2$, $\rho>0$, $\rho\neq1$. Suppose that there exist constants $\eta>0$, $\beta_1>0$ and a continuous function $\psi:\mathbb{R}^+\times(0,\eta]\to\mathbb{R}^+$ such that*

$$
\frac{|f(t,yg(t))|}{g(t)}\leq\psi(t,|y|), \tag{4.10}
$$

for all $t\geq t_0$, $0<|y|\leq\eta$, and

$$
\sup_{t\geq t_0}\int_{t_0}^{t}\frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)}\frac{\psi(\tau,r)}{r}d\tau\leq\beta_1<1-M_2, \tag{4.11}
$$

holds for every $0 < r \leq \eta$, *where* M_2 *is given in Lemma (41),* $\psi(t,r)$ *is nondecreasing in* r *for fixed* t *and* $t^{\alpha\rho}\psi(t,r) \in L^1[t_0,+\infty)$ *in* t *for fixed* r. *Then*
(i) The generalized nonlinear FDEs (4.1)-(4.2) is stable in the Banach space E.
(ii) The generalized nonlinear FDEs (4.1)-(4.2) has at least one solution such that $\lim_{t\to+\infty}\frac{x(t)}{g(t)} = 0$.

Proof. (i) Define two mappings A, B on $F(\varepsilon)$ as follows:

$$\begin{aligned} Ax(t) &= \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\frac{\tau^{\rho-1}}{(s^\rho-\tau^\rho)^{2-\alpha}}f(\tau,x(\tau))\,ds\right]d\tau \\ &= \int_{t_0}^{t}k(t,\tau)f(\tau,x(\tau))\,d\tau, \end{aligned} \tag{4.12}$$

$$\begin{aligned} Bx(t) &= x_0\left(\left(\frac{t}{t_0}\right)^{1-\rho}+(\rho-1)t_0^{-\rho}\int_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}ds\right)+x_1\left(t_0^{1-\rho}\int_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}ds\right) \\ &\quad+\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}x(\tau)\,ds\right]d\tau. \end{aligned} \tag{4.13}$$

Obviously, for $x \in F(\varepsilon)$ both Ax and Bx are continuous on $[t_0,+\infty)$. Let $x \in F(\varepsilon)$. By (4.10),(4.11), we have for any $t \geq t_0$:

$$\frac{|Ax(t)|}{g(t)} = \frac{1}{g(t)}\left|\int_{t_0}^{t}k(t,\tau)f(\tau,x(\tau))\,d\tau\right|$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t} \frac{k(t,\tau)\left|f(\tau, x(\tau))\right|}{g(t)} d\tau \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \frac{\left|f\left(\tau, \frac{x(\tau)}{g(\tau)} g(\tau)\right)\right|}{g(\tau)} d\tau \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \psi\left(\tau, \frac{|x(\tau)|}{g(\tau)}\right) d\tau \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \psi(\tau, \varepsilon)\, d\tau \\
&\leq \beta_1 \varepsilon < +\infty.
\end{aligned}
$$

Then

$$\frac{|Ax(t)|}{g(t)} \leq \beta_1 \varepsilon. \tag{4.14}$$

On the other hand, there exists $M_4 = M_4(\rho) = \frac{1}{t_0^{\alpha\rho+2}}$, such that

$$\frac{\left(\frac{t}{t_0}\right)^{1-\rho}}{g(t)} \leq \frac{\left(\frac{t}{t_0}\right)^{1-\rho}}{g(t)} \leq \frac{t^{1-\rho}}{t_0^{1-\rho} t^{\alpha\rho+2}} \leq \frac{1}{t_0^{\alpha\rho+2}} := M_4. \tag{4.15}$$

From (4.7),(4.8) and (4.15), we get

$$
\begin{aligned}
\frac{|Bx(t)|}{g(t)} \leq\ & \frac{|x_0|}{g(t)} \left| \left(\frac{t}{t_0}\right)^{1-\rho} + (\rho - 1)\, t_0^{-\rho} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds \right| \\
& + \frac{|x_1|}{g(t)} t_0^{1-\rho} \int_{t_0}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} ds
\end{aligned}
$$

$$+\frac{1}{g(t)}\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}\left|x(\tau)\right|ds\right]d\tau$$
$$\leq \left|x_0\right|\left(M_4+\left|\rho-1\right|t_0^{-\rho}M_1\right)+\left|x_1\right|t_0^{1-\rho}M_1+M_2\varepsilon<+\infty. \tag{4.16}$$

Then $AF(\varepsilon)\subset E$ and $BF(\varepsilon)\subset E$. Next, we shall use Lemma 39 to prove there exists at least one fixed point of the operator $(A+B)$ in $F(\varepsilon)$. Here, we divide the proof into three steps.

Step 1. We will prove that $Ax+By\in F(\varepsilon)$ for all $x,y\in F(\varepsilon)$.

Take $\delta\leq\frac{(1-M_2-\beta_1)}{M_4+\left(\left|\rho-1\right|t_0^{-\rho}+t_0^{1-\rho}\right)M_1}\varepsilon$. Let $x,y\in F(\varepsilon)$, from (4.14),(4.16) we obtain

$$\begin{aligned}&\frac{\left|Ax(t)+By(t)\right|}{g(t)}\\ \leq\ &\left|x_0\right|\left(M_4+\left|\rho-1\right|t_0^{-\rho}M_1\right)+\left|x_1\right|t_0^{1-\rho}M_1+M_2\varepsilon+\beta_1\varepsilon\\ \leq\ &\left(M_4+\left(\left|\rho-1\right|t_0^{-\rho}+t_0^{1-\rho}\right)M_1\right)\delta+M_2\varepsilon+\beta_1\varepsilon\\ \leq\ &\varepsilon.\end{aligned}$$

Which implies that $Ax+By\in F(\varepsilon)$ for all $x,y\in F(\varepsilon)$.

Step 2. We will prove that A is continuous and $A\mathcal{F}(\varepsilon)$ is a relatively compact in E.

Firstly, we will show that A is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n\to x$ in $\mathcal{F}(\varepsilon)$.

Using (4.10), we get

$$\begin{aligned}&\tau^{\alpha\rho}\frac{\left|f(\tau,x_n(\tau))-f(\tau,x(\tau))\right|}{g(\tau)}\\ \leq\ &\tau^{\alpha\rho}\frac{\left|f(\tau,x_n(\tau))\right|+\left|f(\tau,x(\tau))\right|}{g(\tau)}\\ \leq\ &\tau^{\alpha\rho}\left(\psi\left(\tau,\frac{\left|x_n(\tau)\right|}{g(\tau)}\right)+\psi\left(\tau,\frac{\left|x(\tau)\right|}{g(\tau)}\right)\right)\\ \leq\ &2\tau^{\alpha\rho}\psi(\tau,\varepsilon)\in L^1[t_0,+\infty).\end{aligned}$$

It follows from (4.9) that for any $t \geq t_0$, we have

$$\begin{aligned}
\frac{|Ax_n(t) - Ax(t)|}{g(t)} &= \frac{1}{g(t)} \left| \int_{t_0}^{t} k(t,\tau)\left[f(\tau, x_n(\tau)) - f(\tau, x(\tau))\right] d\tau \right| \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{g(\tau)} d\tau \\
&\leq \frac{M_3}{t_0} \int_{t_0}^{t} \tau^{\alpha\rho} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{g(\tau)} d\tau \\
&\leq \frac{M_3}{t_0} \int_{t_0}^{+\infty} \tau^{\alpha\rho} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{g(\tau)} d\tau.
\end{aligned}$$

Then

$$\|Ax_n - Ax\| \leq \frac{M_3}{t_0} \int_{t_0}^{+\infty} \tau^{\alpha\rho} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{g(\tau)} d\tau.$$

We have for any $\tau \geq t_0$,

$$\frac{|x_n(\tau) - x(\tau)|}{g(\tau)} \leq \|x_n - x\|,$$

so

$$\lim_{n\to+\infty} |x_n(\tau) - x(\tau)| = 0 \text{ for all } \tau \geq t_0,$$

then

$$\lim_{n\to+\infty} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{g(\tau)} = 0 \text{ for all } \tau \geq t_0,$$

since f is continuous in $[t_0, +\infty) \times \mathbb{R}$. Thus, it follows from the dominated convergence Theorem, that $\|Ax_n - Ax\| \to 0$ as $n \to +\infty$. Therefore A is continuous. Secondly, we will prove that $A\mathcal{F}(\varepsilon)$ is a relatively compact in E. From (4.9) it follows that there exists a constant $M_3 = M_3(\rho)$, such that for all $t \geq t_0$

$$\frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \leq M_3 t^{-1} \tau^{\alpha\rho}.$$

And for any $T \geq t_0$, the function $\frac{k(t,\tau)g(\tau)}{g(t)}$ is uniformly continuous on

$$\left\{(t,\tau) : t_0 \leq \tau \leq t \leq T\right\}.$$

For any $x \in F(\varepsilon)$, and for any $t_1, t_2 \in [t_0, T]$, $t_1 < t_2$ we have

$$\begin{aligned}
\left|\frac{Ax(t_2)}{g(t_2)} - \frac{Ax(t_1)}{g(t_1)}\right| &\leq \left|\int_{t_0}^{t_2} \frac{k(t_2,\tau)}{g(t_2)} f(\tau, x(\tau))\, d\tau - \int_{t_0}^{t_1} \frac{k(t_1,\tau)}{g(t_1)} f(\tau, x(\tau))\, d\tau\right| \\
&\leq \int_{t_0}^{t_1} \left|\frac{k(t_2,\tau)}{g(t_2)} - \frac{k(t_1,\tau)}{g(t_1)}\right| |f(\tau, x(\tau))|\, d\tau \\
&\quad + \int_{t_1}^{t_2} \frac{k(t_2,\tau)}{g(t_2)} f(\tau, x(\tau))\, d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_0}^{t_1} \left|\frac{k(t_2,\tau) g(\tau)}{g(t_2)} - \frac{k(t_1,\tau) g(\tau)}{g(t_1)}\right| \psi(\tau,\varepsilon)\, d\tau + \int_{t_1}^{t_2} \frac{k_1(t_1,\tau) g(\tau)}{g(t_1)} \psi(\tau,\varepsilon)\, d\tau \\
&\leq \int_{t_0}^{t_1} \left|\frac{k(t_2,\tau) g(\tau)}{g(t_2)} - \frac{k(t_1,\tau) g(\tau)}{g(t_1)}\right| \psi(\tau,\varepsilon)\, d\tau + \frac{M_3}{t_0} \int_{t_1}^{t_2} \tau^{\alpha\rho} \psi(\tau,\varepsilon)\, d\tau,
\end{aligned}$$

as $t_2 \to t_1$ which means that $\left\{\frac{Ax(t)}{g(t)},\ x(t) \in \mathcal{F}(\varepsilon)\right\}$, is equicontinuous on any compact interval of $[t_0, +\infty)$. By Lemma 39, in order to show that $AF(\varepsilon)$ is a relatively compact set of E.
We only need to prove that $\left\{\frac{Ax(t)}{g(t)}, x(t) \in \mathcal{F}(\varepsilon)\right\}$ is equiconvergent at infinity.

From (4.9) , we have

$$\begin{aligned}\frac{|Ax(t)|}{g(t)} &\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g(t)} |f(\tau, x(\tau))| \, d\tau \\ &\leq \int_{t_0}^{t} \left| \frac{k(t,\tau)}{g\left(\frac{t}{\tau}\right)} \right| \psi(\tau, \varepsilon) \, d\tau \\ &\leq \frac{M_3}{t} \int_{t_0}^{t} \tau^{\alpha\rho} \psi(\tau, \varepsilon) \, d\tau \to 0, \text{ as } t \to +\infty. \end{aligned} \tag{4.17}$$

Thus $\left\{ \frac{Ax(t)}{g(t)}, x(t) \in \mathcal{F}(\varepsilon) \right\}$ is equiconvergent at infinity. Hence the required conclusion is true.

Step 3. We claim that $B : F(\varepsilon) \to E$ is a contraction mapping. In fact, for any $x_1, x_2 \in F(\varepsilon)$ from (4.8), it follows that

$$\begin{aligned}\sup_{t \geq t_0} \left| \frac{Bx_1(t)}{g(t)} - \frac{Bx_2(t)}{g(t)} \right| &\leq \frac{1}{g(t)} \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} |x_1(\tau) - x_2(\tau)| \, ds \right] d\tau \\ &\leq M_2 \|x_1 - x_2\| .\end{aligned}$$

By Krasnoselskii fixed point theorem, we conclude that there exists at least one fixed point of the operator $A + B$ in $F(\varepsilon)$, which is a solution of (4.1)-(4.2). Then the generalized nonlinear FDEs (4.1)-(4.2) is stable in the Banach space E.

(ii) For any $0 < \varepsilon < \eta$, set

$$\mathcal{F}^*(\varepsilon) = \left\{ x \in \mathcal{F}(\varepsilon), \ \lim_{t \to +\infty} \frac{x(t)}{g(t)} = 0 \right\}.$$

We will show that $Ax + By \in F^*(\varepsilon)$ for any $x, y \in F^*(\varepsilon)$, i.e. $\frac{Ax(t)+By(t)}{g(t)} \to 0$ as $t \to +\infty$.

$$\frac{Ax(t) + Bx(t)}{g(t)} = \frac{1}{g(t)} \left[\int_{t_0}^{t} k(t,\tau) f(\tau, x(\tau)) \, d\tau \right.$$

$$+x_0\left(\left(\frac{t}{t_0}\right)^{1-\rho}+(\rho-1)\,t_0^{-\rho}\int\limits_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}ds\right)$$
$$+\,x_1\left(t_0^{1-\rho}\int\limits_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}ds\right)+\int\limits_{t_0}^{t}\left[\int\limits_{\tau}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}x(\tau)\,ds\right]d\tau\Bigg].$$

By the fact that, $g(t)\geq t^{\alpha\rho+3}$, we obtain,

$$\frac{t^{1-\rho}}{g(t)}\to 0 \text{ as } t\to+\infty \tag{4.18}$$

and

$$\frac{1}{g(t)}\int\limits_{t_0}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}ds \text{ as } t\to+\infty. \tag{4.19}$$

Moreover, we have

$$\frac{1}{g(t)}\int\limits_{t_0}^{t}\left[\int\limits_{\tau}^{t}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}x(\tau)\,ds\right]d\tau\leq\int\limits_{t_0}^{t}\left[\int\limits_{\tau}^{t}\frac{1}{g\left(\frac{t}{\tau}\right)}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}\frac{x(\tau)}{g(\tau)}ds\right]d\tau.$$

Let us consider the following cases:

Case 1. If $\rho\in(0,1)$, we have

$$\int\limits_{t_0}^{t}\left[\int\limits_{\tau}^{t}\frac{1}{g\left(\frac{t}{\tau}\right)}\frac{t^{1-\rho}}{s^{2-2\rho}}\tau^{-\rho-1}\frac{x(\tau)}{g(\tau)}ds\right]d\tau\leq\int\limits_{t_0}^{t}\left[t^{-\alpha\rho-\rho-1}\tau^{\alpha\rho+\rho}\right]\frac{x(\tau)}{g(\tau)}d\tau.$$

It follows from $\lim_{\tau\to+\infty}\frac{x(\tau)}{g(\tau)}=0$, that there exists $T_1>t_0$, such that for $t\geq T_1$, it yields

$$\frac{|x(t)|}{g(t)}<(\alpha\rho+\rho+1)\,\frac{\varepsilon}{2}.$$

Moreover, there exists $T_2>T_1$, such that for $t\geq T_2$ we get

$$\frac{1}{t}\int\limits_{t_0}^{T_1}\frac{|x(t)|}{g(t)}<\frac{\varepsilon}{2}.$$

Consequently,

$$
\begin{aligned}
& \int_{t_0}^{t} \left[t^{-\alpha\rho-\rho-1}\tau^{\alpha\rho+\rho}\right] \frac{|x(\tau)|}{g(\tau)} d\tau \\
= & \int_{t_0}^{T_1} \left[t^{-\alpha\rho-\rho-1}\tau^{\alpha\rho+\rho}\right] \frac{|x(\tau)|}{g(\tau)} d\tau \\
+ & \int_{T_1}^{t} \left[t^{-\alpha\rho-\rho-1}\tau^{\alpha\rho+\rho}\right] \frac{|x(\tau)|}{g(\tau)} d\tau \\
< & \frac{\varepsilon}{2} + (\alpha\rho+\rho+1) \frac{\varepsilon}{2(\alpha\rho+\rho+1)} < \varepsilon.
\end{aligned}
$$

Case 2. If $\rho > 1$, we have

$$
\int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{1}{g\left(\frac{t}{\tau}\right)} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} \frac{x(\tau)}{g(\tau)} ds \right] d\tau \leq \int_{t_0}^{t} \left[t^{-\alpha\rho+\rho-3}\tau^{\alpha\rho-\rho+2}\right] \frac{x(\tau)}{g(\tau)} d\tau.
$$

It follows from $\lim_{\tau\to+\infty} \frac{x(\tau)}{g(\tau)} = 0$, that for all $\varepsilon > 0$, there exists $T_1 > t_0$, such that for
$t \geq T_1$

$$
\frac{|x(t)|}{g(t)} < (\alpha\rho - \rho + 3) \frac{\varepsilon}{2}.
$$

Furthermore, there exists $T_2 > T_1$ such that for $t \geq T_2$, it yields

$$
\frac{1}{t} \int_{t_0}^{T_1} \frac{|x(t)|}{g(t)} < \frac{\varepsilon}{2},
$$

so,

$$
\begin{aligned}
& \int_{t_0}^{t} \left[t^{-\alpha\rho+\rho-3}\tau^{\alpha\rho-\rho+2}\right] \frac{x(\tau)}{g(\tau)} d\tau \\
= & \int_{t_0}^{T_1} \left[t^{-\alpha\rho+\rho-3}\tau^{\alpha\rho-\rho+2}\right] \frac{|x(\tau)|}{g(\tau)} d\tau \\
& + \int_{T_1}^{t} \left[t^{-\alpha\rho+\rho-3}\tau^{\alpha\rho-\rho+2}\right] \frac{|x(\tau)|}{g(\tau)} d\tau \\
< & \frac{\varepsilon}{2} + (\alpha\rho-\rho+3) \frac{\varepsilon}{2(\alpha\rho-\rho+3)} < \varepsilon.
\end{aligned}
$$

From the previous cases, we obtain

$$
\frac{1}{g(t)} \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{t^{1-\rho}}{s^{2-2\rho}} \tau^{-\rho-1} x(\tau)\, ds \right] d\tau \to 0 \text{ as } t \to +\infty. \tag{4.20}
$$

By the help of (4.17), we get

$$
\frac{1}{g(t)} \int_{t_0}^{t} k(t,\tau) f(\tau, x(\tau))\, d\tau \to 0 \text{ as } t \to +\infty. \tag{4.21}
$$

Thanks to (4.18)-(4.21), it yields $\frac{Ax(t)+By(t)}{g(t)} \to 0$ as $t \to +\infty$.
Thus, there exists at least one solution of problem (4.1),(4.2) such that

$$
\lim_{t\to+\infty} \frac{x(t)}{g(t)} = 0.
$$

■

4.3 Example

Let us consider the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D_{1}^{\frac{3}{2},2}x\left(t\right)=\frac{1}{4}\left(xe^{-t}+\frac{x^{2}}{1+t^{12}}\right), \\ x\left(1\right)=x_{1},x'\left(1\right)=x_{2}. \end{cases} \tag{4.22}$$

Choosing $g\left(t\right)=t^{6}$, then the Banach space E is

$$E=\left\{x\in C\left[1,+\infty\right),\mathbb{R},\sup_{t\geq 1}\frac{\left|x\left(t\right)\right|}{t^{6}}<\infty\right\}.$$

We have

$$\frac{\left|f\left(t,yg\left(t\right)\right)\right|}{g\left(t\right)}=\frac{1}{4}\frac{\left|yt^{6}e^{-t}+\frac{y^{2}t^{12}}{1+t^{12}}\right|}{t^{6}}\leq\frac{1}{4}\left(\left|y\right|e^{-t}+\frac{y^{2}}{t^{6}}\right).$$

Taking

$$\psi\left(t,r\right)=\frac{1}{4}\left(re^{-t}+\frac{r^{2}}{t^{6}}\right),$$

thus, we obtain

$$\int_{1}^{+\infty}t^{2}\psi\left(t,r\right)dt=r\frac{5e^{-1}}{4}+\frac{r}{3}$$

and

$$\sup_{t\geq t_{0}}\int_{t_{0}}^{t}\frac{k\left(t,\tau\right)}{g\left(\frac{t}{\tau}\right)}\frac{\psi\left(\tau,r\right)}{r}d\tau\leq\frac{\sqrt{2}}{\sqrt{\pi}}\frac{5e^{-1}}{4}+\frac{r}{3}.$$

Then there exists $\eta>0$, such that

$$\sup_{t\geq t_{0}}\int_{t_{0}}^{t}\frac{k\left(t,\tau\right)}{g\left(\frac{t}{\tau}\right)}\frac{\psi\left(\tau,r\right)}{r}d\tau\leq\frac{1}{2}<1-\frac{1}{4},\ \forall\ 0<r\leq\eta.$$

Moreover, $t^{3}\psi\left(t,r\right)=\frac{1}{4}\left(rt^{3}e^{-t}+\frac{r^{2}}{t^{3}}\right)\in L^{1}\left[1,+\infty\right)$, for fixed r.
Then, by Theorem 42, we conclude that the nonlinear fractional differential equation (4.22) is stable and there exists at least one solution such that $\lim_{t\to+\infty}\frac{x(t)}{g(t)}=0$.

CONCLUSION

This thesis can be divided in two main parts. The first parts is devoted to the study of the existence of positive solutions for higher order Riemann-Liouville fractional differential equations subject to integral conditions. The functional spaces were either the space of continuous functions $C([0,1])$ or the Sobolev space $W^{s,1}_{RL,0^+}$, $0 < s < 1$. The mean tools are Schauder fixed point theorem and lower and upper solutions method. In the second part, we investigated the existence and stability of solutions for a Caputo–Katugampola fractional nonlinear differential equation of order $1 < \alpha < 2$, jointly with initial conditions. Employing Krasnoselskii fixed point Theorem and a modified compactness criterion in the space of integrable functions $L^p(0,1)$, we obtained the stability results in a weignted space.

These studies can be extend to more general boundary value problems involving other types of fractional derivatives and using numerical methods.

BIBLIOGRAPHY

[1] R. P. Agarwal, Formulation of Euler-Lagrange equations for fractional variational problems. J. Math. Anal. Appl. 272, 368–379 (2002).

[2] R. Almeida, A. B. Malinowska, T. Odzijewicz, Fractional differential equations with dependence on the Caputo Katugampola derivative, J. Comput. Nonlinear Dynam, Vol. 11, 061017-1, (2016).

[3] D. Baleanu, G. C. Wu, S. D. Zeng, Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, Chaos, Solitons and Fractals, 1–7 (2017).

[4] M. Benchohra, Juan . J. Nieto, A. Ouahab, Existence results for functional integral inclusions of Volterra type, Dynam. Systems Appl. 14 (2005), 57–69.

[5] M. Bergounioux, A. Leaci, G. Nardi, F. Tomarelli, Fractional Sobolev spaces and functions of bounded variation of one variable. Fractional Calculus & Applied Analysis, Vol 20 No 4, 936–962 (2017).

[6] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer New York Dordrecht Heidelberg London (2010).

[7] D. Boucenna, A. Guezane-Lakoud, Juan J. Nieto, R. Khaldi, On a multipoint fractional boundary value problem with integral conditions, Vol. 2017, Article ID 53, 1–13 (2017).

[8] T. A. Burton, B. Zhang, L_p-solutions of fractional differential equations, Nonlinear Stud. 19, No. 2, 161–177 (2012).

[9] A. Boroomand, M. B. Menhaj, Fractional-order Hopfeld neural networks," Lecture Notes in Computer Science, Vol. 5509, 883–890 (2009).

[10] W. Chen,Y. Zhao, Solvability of boundary value problems of nonlinear fractional differential equations, Chen and Zhao Advances in Difference Equations (2015) 2015:36.

[11] C. Corduneanu, Integral equations and stability of feedback systems , Academic Press, New York and London (1973).

[12] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos and R. Castro-Linares, Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems, Commun. Nonlinear Sci. Numer. Simul., Vol. 22, No. 1–3, 650–659 (2015).

[13] Ju. L. Daleckii, M. G. A. Krein, Stability of solutions of differential equations in Banach space, AMS 1974.

[14] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. 204, 609–625 (1996).

[15] K. Diethelm, The Analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type. Springer, Heidelberg, 2010.

[16] C. De Coster, P. Habets, Two-point boundary value problems: Lower and upper solutions, Mathematics in Science and Engineering , Vol 205, Series Editor: C.K. Chui, Elsevier, 2006.

[17] D. Franco, Juan J. Nieto, Donal O'Regan, Upper and lower solutions for first order problems with nonlinear boundary conditions, Extracta Mathematicae,18 (2003), 153–160.

[18] W. Feng,S. Sun, Z. Han, Y. Zhao, Existence of solutions for a singular system of nonlinear fractional differential equations. Comput. Math. Appl. 62, 1370–1378 (2011).

[19] A. Frioui, A. Guezane-Lakoud, R. Khaldi, Fractional boundary value problems on the half line. Opuscula Math. 37 (2017), no. 2, 265–280.

[20] F. Ge, C. Kou, Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations, Appl. Math. Comput , Vol. 257, pp. 308–316 (2015).

[21] A. Guezane-Lakoud, Initial value problem of fractional order. Cogent Math. 2 (2015), Art. ID 1004797, 6 pp

[22] A. Guezane-Lakoud, R. Khaldi, Upper and lower solutions method for higher order fractional initial value problems. J. Dyn. Syst. Geom. Theor. 15 (2017), no. 1, 29–35.

[23] A. Guezane-Lakoud, R. Khaldi, On a boundary value problem at resonance on the half line. J. Fract. Calc. Appl. 8 (2017), no. 1, 159–167.

[24] A. Guezane-Lakoud, R. Khaldi, A. Kılıçman, Existence of solutions for a mixed fractional boundary value problem, Adv. Differ. Equ. 2017, 2017:164

[25] A. Guezane-Lakoud, A., Ashyralyev, Positive solutions for a system of fractional differential equations with nonlocal integral boundary conditions, Differ Equ Dyn Syst (2017) 25: 519.

[26] A. Guezane-Lakoud, R. Khaldi, A. Kılıçman, Solvability of a boundary value problem at resonance, SpringerPlus (2016) 5:1504.

[27] A. Guezane-Lakoud, R. Khaldi and D. F. M. Torres, Lyapunov-type inequality for a fractional boundary value problem with natural conditions, SeMA Journal

[28] A. Guezane-Lakoud, G. Rebiai, and R. Khaldi, Existence of solutions for a nonlinear fractional system with nonlocal boundary conditions. Proyecciones (Antofagasta), Dec 2017, vol.36, no.4, p.727–737.

[29] A. Guezane-Lakoud, R. Khaldi, On a boundary value problem at resonance on the half line, Journal of fractional calculus and application, Vol, 8(1) Jan. 2017, pp.159-167

[30] A. Guezane-Lakoud, R. Khaldi, Successive approximations to solve higher order fractional differential equations, Journal of Nonlinear Functional Analysis 2016 (2016), Article ID 29

[31] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[32] H. Jafari, V. Daftardar-Gejji, Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method. Appl. Math. Comput. 180, 700–706 (2006).

[33] A.A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[34] A. Karoui, H. BenAouicha, A. Jawahdou, Existence and numerical solutions of nonlinear quadratic integral equations defined on unbounded intervals. Numer. Funct. Anal. Opt. 31, No. 6, 691–714 (2010).

[35] A. Karoui, A. Jawahdou, Existence and approximate L_p and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types. Appl. Math. Comput. 216, No. 7, 2077–2091 (2010).

[36] U. N. Katugampola, Existence and uniqueness results for a class of generalized fractional differential equations, Bull. Math. Anal. Appl., 6(4), (2014) pp. 1–15.

[37] U. N. Katugampola, New approach to a generalized fractional integral, Applied Mathematics and Computation 218 (2011) 860–865.

[38] R. Khaldi, A. Guezane-Lakoud, Lyapunov inequality for a boundary value problem involving conformable derivative, Progr. Fract. Differ. Appl. 3, No. 4, 323–329 (2017).

[39] R. Khaldi and A. Guezane-Lakoud, Higher order fractional boundary value problem for mixed type derivatives, Journal of Nonlinear Functional Analysis 2017 (2017), Article ID 30.

[40] R. Khaldi, A. Guezane-Lakoud, Upper and lower solutions method for higher order boundary value problems, Progr. Fract. Differ. Appl. 3 (2017), No. 1, 53—57.

[41] C. Kou, H. Zhou and Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, Nonlinear Anal., Vol. 74, pp. 5975–5986 (2011).

[42] M. A. Krasnoselskii, Some problems of nonlinear analysis, Amer. Math. Soc. Transl., Vol. 10, pp. 345–409 (1958).

[43] R. W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. 28, 673–688 (1979).

[44] Y. Li, Y. Q. Chen and I. Podlubny, Mittag-Leffler stability of fractional order nonlinear dynamic systems, Automatica, Vol. 45, No. 8, 1965–1969 (2009).

[45] S. Liang, J. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equation, Nonlinear Analysis 71 (2009) 5545–5550.

[46] D. Matignon, Stability result on fractional differential equations with applications to control processing, In IMACS. SMC Proc., Lille, France, pp. 963–968 (1996).

[47] K.S. Miller, B. Ross, : An Introduction to the fractional calculus and fractional differential equation. Wiley, New York (1993).

[48] R. Magin, Fractional calculus in bioengineering, Critical Reviews in Biomedical Engineering, Vol. 32, 195–377 (2004).

[49] O. Naifar, A. Ben Makhlouf, M. A. Hammami, Comments on Mittag-Leffler stability of fractional order nonlinear dynamic systems , Automatica, Vol. 75, 329 (2017).

[50] O. Naifar, A. Ben Makhlouf and M. A. Hammami, Comments on Lyapunov stability theorem about fractional system without and with delay, Commun. Nonlinear Sci. Numer. Simul., Vol. 30, 360–361 (2016).

[51] M. Jia, X. Liu, Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. Appl. Math. Comput. 232 (2014), 313–323.

[52] Juan J. Nieto, R. Rodríguez-López, Monotone method for first-order functional differential equations, Comput. Math. Appl. 52 (2006), 471–484.

[53] Juan J. Nieto, Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions, Proceedings of the American Mathematical society, 125 (1997) 2599–2604.

[54] K.B. Oldham, J. Spanier, The fractional calculus. Academic Press, New York (1974).

[55] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[56] K. R. Prasad , B. M. B. Krushna: Lower and upper solution for general two-point fractional orded boundary value problems. J. App. Eng. Math. V.5, N.1, 2015, 80–87.

[57] T. Qiu, Z. Bai, Existence of positive solutions for singular fractional equations. Electron. J. Differ. Equ. 2008, 146 (2008).

[58] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integral and derivative. Theory and Applications. Gordon & Breach, Yverdon (1993).

[59] S. Sun, Y. Zhao, Z. Han, M. Xu, Uniqueness of positive solutions for boundary value problems of singular fractional differential equations. Inverse Probl. Sci. Eng. 20 (3), 299–309 (2012).

[60] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics, Vol. 23, 397–404 (2002).

[61] A. Souahi, A. Guezane-Lakoud, A. Hitta, Positive solutions for higher-order nonlinear fractional differential equations. Vietnam J. Math. 45 (2017), no. 3, 441–450.

[62] D. Tripathil, S. Pandey and S. Das, Peristaltic flow of viscoelastic fluid with fractional Maxwell model through a channel, Applied Mathematics and Computation, Vol. 215, 3645–3654 (2010).

[63] G. Wang, S. Liu, R. P. Agarwal, L. Zhang, Positive Solutions of Integral Boundary Value Problem Involving Rimann-Liouville Fractional Derivative. J. Frac. Calc. and App. 2013, 312–321.

[64] X. Xu, D. Jiang, C.Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. Nonlinear Anal. 71, 4676–4688 (2009).

[65] Y. Yu, D. Jiang, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. Northeast Normal University (2009).

[66] S. Zhang, The existence of a positive solution for nonlinear fractional differential equation. J. Math. Anal. Appl. 252, 804–812 (2000).

[67] S. Zhang, Existence of positive solutions for some class of nonlinear fractional equation. J. Math. Anal. Appl. 278, 136–148 (2003).

[68] Y. Zhao, S. Sun, Z. Han, W. Feng, Positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders. Adv. Differ. Equ. 2011, 10 (2011).

[69] Y. Zhao, S. Sun, Z. Han, Q. Li, Theory of fractional hybrid differential equations. Comput. Math. Appl. 62, 1312–1324 (2011).omat. Contr., Vol. 29, No. 5, 441–444 (1984).