وزارة التعليم العالي والبحث العلمي

Université Badji Mokhtar Annaba

Badji Mokhtar University - Annaba

جامعة باجي مختار
عنابة

Faculté des Sciences

Département de Mathématiques

THÈSE

Présentée en vue de l'obtention du diplôme de
Doctorat Science en Mathématiques

Sélections continues sur les hyper-espaces de fermés et de convexes fermés et application aux inclusions différentielles

Option :
Topologie et analyse fonctionelle

Présentée par

Rezaiguia Ali

DIRECTEUR DE THÈSE :	Kelaiaia Smail	PROF	Univ. B. M. Annaba

Devant le jury

PRÉSIDENT :	Djoudi Ahcène	PROF	Univ.B.M. Annaba
EXAMINATEUR :	Guezane-Lakoud Assia	PROF	Univ.B.M. Annaba
EXAMINATEUR:	Bentrad Ali	PROF	Univ. de Reims (France)
EXAMINATEUR :	Ardjouni Abdelwaheb	MCA	Univ.M.C.M. S/Ahras
EXAMINATEUR :	Bouchair Abderrahmane	MCA	Univ. Univ. de Jijel

Année 2016

Sélections continues sur les hyper-espaces de fermés et de convexes fermés et application aux inclusions differentielles

Rezaiguia Ali
Université Badji Mokhtar, Annaba
Département de mathématique

Nouvembre 2016
Directeur du thèse: Pr. Kelaiaia Smail

Dedication

To my dear Mother,
and
The memory of my Father.

Acknowledgments

First, I would like to thank Allah for all that has given me strength, courage and above all knowledge.
I wish to extend my warm thanks and deep appreciation to my coach Pr. Smail Kelaiaia professor at the University of Annaba for having propose to me The subject of this thesis. This is due to his availability, advice, guidance and encouragement that I could carry out this work.
I am also very grateful to Pr. Ahcene Djoudi, professor at the University of Annaba, for having agreed to make me the honor of chairing the jury.
I would also like to thank Pr. Assia Guezane-Lakoud, Professor at the University of Annaba, Pr. Bentrad Ali, Professor at the University of Reims, Dr Bouchair Abderrahmane, professor at the University of Jijel and Dr Ardjouni Abdelwahab professor at the Mohamed Cherif Messadia University of Souk Ahras, for the honor they have done to me by kindly accept to be part of the jury.
I also want to use this occasion to thank my family for always supporting me spiritually throughout my mother.

Ali Rezaiguia

Abstract

It is well known that the theory of selection plays a very important role in solving some mathematical problems, " in topology, convex geometry, analysis..." such as the study of the fixed points of multivalued mappings, the existence of soloutions for a differential inclusions...etc.
In this thesis, by the fixed point theory for single and set-valued mapings combined with some selection theorems we have studied the following two problems:

$$\begin{aligned} &-u'''(t) \in F(t, u(t)), \; t \in (0,1), \\ &u'(0) = u'(1) = \alpha u(\eta), \; u(0) = \beta u(\eta), \end{aligned} \tag{$\mathfrak{PF}1$}$$

where α, β and η are constants in $\mathbb{R}$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.
and

$$\begin{aligned} &-u'''(t) \in F(t, u(t), u'(t)), \; t \in (0,1), \\ &u(0) = u'(0) = 0, \; u'(1) = \textstyle\sum_{i=1}^{m-2} \alpha_i u'(\eta_i). \end{aligned} \tag{$\mathfrak{PF}2$}$$

where α_i,and η_i are constants in $\mathbb{R}$ and $F : [0,1] \times E \times E \to \mathcal{P}(E)$ is a closed valued mapping, with E a Banach space.

* For the problem ($\mathfrak{PF}1$), We have established the existence of class $AC^2([0,1], \mathbb{R})$-solutions theorems when the right hand side has convex or non convex values.

* About the problem ($\mathfrak{PF}2$), we proved $W^{3,1}([0,1], E)$-solution set, is compact and is a retract of $C^1([0,1], E)$, when F is convex compact valued and satisfies a Lipschitz condition and a compactness condition.

Keywords: Selection theorems; Fixed point theory; Multi point boundary values problem, Third order diferential inclusion.

Resumé

Il est bien connu que la théorie de la sélection joue un rôle très important dans la résolution de certains problèmes mathématiques " en topologie, géométrie, analyse convexe...." par exemple l'étude des points fixes des fonctions multivoques, l'existence de soloutions pour une inclusion différentielle ... etc.

Dans cette thèse, par la théorie du point fixe pour les applications et multiapplications combinée avec quelques théorèmes de sélection, nous avons étudié les deux problèmes suivants:

$$\begin{aligned} &-u'''(t) \in F(t, u(t)), \ t \in (0,1), \\ &u'(0) = u'(1) = \alpha u(\eta), \ u(0) = \beta u(\eta), \end{aligned} \tag{$\mathfrak{PF}1$}$$

où α, β et η sont des constantes dans $\mathbb{R}$ et $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ est une fontion multivoque, et $\mathcal{P}(\mathbb{R})$ est la famille de tous les sous-ensembles de $\mathbb{R}$.

et

$$\begin{aligned} &-u'''(t) \in F(t, u(t), u'(t)), \ t \in (0,1), \\ &u(0) = u'(0) = 0, \ u'(1) = \textstyle\sum_{i=1}^{m-2} \alpha_i u'(\eta_i). \end{aligned} \tag{$\mathfrak{PF}2$}$$

où α_i, et η_i sont des constantes dans $\mathbb{R}$ et $F : [0,1] \times E \times E \to \mathcal{P}(E)$ est une fontion multivoque à valeurs dans l'ensemble des fermées de E, avec E un espace de Banach.

* Pour le problème ($\mathfrak{PF}1$), Nous avons établi des théorèmes d'existence de solutions de classe $AC^2([0,1], \mathbb{R})$ lorsque le membre droit a des valeurs convexes ou non convexes.

* Et pour le problème ($\mathfrak{PF}2$), nous avons prouvé que l'ensemble des solutions $W^{3,1}([0,1], E)$, est compact et est une rétraction pour $C^1([0,1], E)$, où F a des valeurs convexs compacts et satisfait une condition de Lipschitz et une condition de compacité.

Mots-clés: Théorème de la sélection; La théorie du point fixe; problème aux limites, inclusion différentielle de troisième ordre.

CONTENTS

Introduction

The study of selection theory is today one of the important topological problem leading to the multivalued analysis which treats of the set valued mappings and their properties as upper and lower semicontinuity which is in its part related to the theory of differential inclusions. So the theory of differential inclusion can be regarded as a part of mutivalued analysis which was intensively developped these last years using the recent developements in selection theory essentially by the authors, Andrzej Lasota, C. Castaing, T.P. Aubin....
Differential inclusions of the form

$$\mathfrak{A}y \in F(x, y)$$

where $\mathfrak{A}$ is a differential operator, are a generalizations of differential equations. So, all problems considered for differential equations, that is, existence of solutions, continuity of solutions, dependence on initial conditions and parameters, are present in the theory of differential inclusions. Since a differential inclusion hasnew issues appear, such as investigation of topological properties of the solutions set, selection of solutions with given properties, etc....
Realistic problems arising from economics, optimal control, stochastic analysis can be modelled as differential inclusions, see ([4], [50], [49], [5], [54], [14], [2], [3], [10], [26], [31], [33], [50], [43]) , and the references therein. So much attention has been paid by many autors to study this kind of problems, (see Bressan and Colombo [6], [8],Colombo [15], [16], Fryszkowsy and Gorniewicz [25], Kyritsi et al. [30], etc....[50], [49]) and the references therein.

In this thesis we have combined materials of selection theory and set valued analysis with results from non linear analysis to obtain some results on solving some problems of differential inclusions. This thesis contains four chapters which are briefly presented below.

Chapter 1. In this chapter, we introduce and state some necessary materials needed in the proof of our results, and shortly the basic results concerning the Banach spaces, the L^p spaces, Sobolev spaces and other theorems and existence and uniqueness theorems. The knowledge of all these notations and results are important for our study.

Chapter 2. we introduce two principal theorems with proof about selection theory, a "Selection Theorem Due To Bressan and Colombo " and a "Selection Theorem Due To Kuratowki, Ryll, and Nardzewski "which we need in the proof of our results in next chapters.

Chapter 3. In this chapter we discuss the existence of solutions for a third- order differential inclusion with three-point boundary conditions involving convex and nonconvex multivalued maps

$$\begin{aligned} -u'''(t) &\in F(t,u(t)),\ t\in(0,1), \\ u'(0) &= u'(1)=\alpha u(\eta),\ u(0)=\beta u(\eta), \end{aligned} \tag{$\mathfrak{PF}1$}$$

where α, β and η are constants with $\alpha \in \left[0, \frac{1}{\eta}\right)$, $0<\eta<1$, $\beta \neq 1-\alpha\eta$, $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

Our results are based on the nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory combined with some selection theorems.

Chapter 4. In this chapter, we prove that the $W_E^{3,1}([0,1])$-solutions set of the problem

$$\begin{aligned} -u'''(t) &\in F(t,u(t),u'(t)),\ t\in(0,1), \\ u(0) &= u'(0)=0,\ u'(1)=\textstyle\sum_{i=1}^{m-2}\alpha_i u'(\eta_i). \end{aligned} \tag{$\mathfrak{PF}2$}$$

is compact and a retract in $C_E^1([0,1])$, when F is convex compact valued map and satisfies a Lipschitz condition and a compactness condition.

CHAPTER 1

Preliminaries

In this Preliminary we introduce and state some necessary materials needed in the proof of our results, and shortly the basic results concerning the Hausdorff topology, the Banach spaces, the L^p spaces, $W^{k,p}$ Sobolev spaces and other theorems as existence and uniqueness theorems. The knowledge of all this notations and results are important for our study.

1.1 Hyperspace topologies - Definitions and properties

Given a topological space X, the spaces obtained by giving a topology τ (in terms of that on X) to the collection of subsets of X are called hyperspaces of X.

Hyperspace theory had its beginnings in the early 1900, with the works of **Hausdorff** and **Vietoris**. The study of hyperspaces originated with Hausdorff, who in [**Hausdorff** 1944] defined a metric, H_d, on the set of the nonempty closed subsets of a metric space X, called the

Hausdorff metric. We denote

$$\begin{aligned}
\mathcal{P}_0(X) &= \{A \in \mathcal{P}(X) : A \neq \emptyset\}, \\
\mathcal{P}_{cl}(X) &= \{A \in \mathcal{P}_0(X) : A \text{ is closed}\}, \\
\mathcal{P}_b(X) &= \{A \in \mathcal{P}_0(X) : A \text{ is bounded}\}, \\
\mathcal{P}_{comp}(X) &= \{A \in \mathcal{P}_0(X) : A \text{ is compact}\}, \\
\mathcal{P}_{cv}(X) &= \{A \in \mathcal{P}_0(X) : A \text{ is convex}\}.
\end{aligned}$$

Definition 1.1.1 *We define A to be bounded in (X, d) if there exists $x_0 \in X$ and $r > 0$ such that $A \subset B(x_0, r)$.*

In particular the metric space (X, d) is said to be bounded iff

$$\sup_{x,y \in X} d(x, y) < \infty.$$

Let (X, d) be a bounded metric space. For all $A \subseteq X$, we have

$$U(A, \varepsilon) = \bigcup_{a \in A} B_d(a, \varepsilon).$$

If $A, B \in P_0(X)$, we define

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\} \tag{1.1}$$

It will now be shown that $(P_0(X), H_d)$ forms a metric space called Hausdorff metric space determined by (X, d).

Show that H_d is a distance on $P(X)$, see [29]. It is clear that, for all A, B in $P_0(X)$,

$$H_d(A, B) = 0 \Longleftrightarrow A = B,$$

$$H_d(A, B) = H_d(B, A),$$

Let's check the triangle inequality. For every $a \in A, b \in B, c \in C$ were successively:

$$\begin{aligned}
d(a,B) &\leqslant d(a,b) \leqslant d(a,c) + d(c,b) \\
d(a,B) &\leqslant d(a,c) + d(c,B) \\
d(a,B) &\leqslant d(a,c) + H_d(C,B) \\
d(a,B) &\leqslant d(a,C) + H_d(C,B) \\
\sup_{a\in A} d(a,B) &\leqslant H_d(A,C) + H_d(C,B)
\end{aligned}$$

Similarly, we have:

$$\sup_{b\in B} d(A,b) \leqslant H_d(A,C) + H_d(C,B)$$

Hence:

$$H_d(A,B) \leqslant H_d(A,C) + H_d(C,B)$$

So

$$H_d(A,B) \leqslant H_d(A,C) + H_d(C,B).$$

Remark 1.1.1 *The Hausdorff topology τ_{H_d} on $\mathcal{P}_{cl}(X)$ is the topology induced by the Hausdorff metric H_d. However, this topology is not determined by the topology induced by the metric on X.*

Example 1.1.1 *Let X the set of positive real numbers; let $d(x,y) = \left|\frac{x}{1+x} - \frac{y}{1+y}\right|$ and $d'(x,y) =$* $\min(1, |x-y|)$. *distances d and d' induce the same topology on X, but $(\mathcal{P}_{cl}(X), \tau_{H_d})$ and $(\mathcal{P}_{cl}(X), \tau_{H_{d'}})$ are different. Indeed, we show the set $\mathcal{P}_b(\mathbb{N})$is closed for $\tau_{H_{d'}}$ but is not for $\tau_{H_{d'}}$.*

Proposition 1.1.1 *(X,d) is complete iff $(\mathcal{P}_{cl}(X), \tau_{H_d})$ is also.*

Proof. [30]. ∎

1.2 Some functional spaces needed

Definition 1.2.1 *A Banach space is a complete normed linear space E. Its dual space E' is the linear space of all continuous linear functional $f : E \to \mathbb{k}$.*

Proposition 1.2.1 [48] *E' equipped with the norm $\|\cdot\|_{E'}$ defined by*

$$\|f\|_{E'} = \sup\{|f(x)| : \|x\| \leq 1\}, \tag{1.2}$$

is also a Banach space.

Theorem 1.2.1 [48] *Let E be a Banach space. Then, E is reflexive, if and only if,*

$$B_E = \{x \in E : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(E, E')$.

Definition 1.2.2 *Let E be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in E. Then u_n converges strongly to u in E if and only if*

$$\lim_{n\to\infty} \|u_n - u\|_E = 0,$$

and this is denoted by $u_n \to u$, or $\lim\limits_{n\to\infty} u_n = u$.

In what follows, by E we will denote a Banach space over the field of real numbers $\mathbb{R}$ and let r be positive real number and by T a closed interval.

Notation 1.2.1 *The space $C(T, E)$ is the space of all continuous E-valued functions defined on T,*

$$C(T, E) = \{u : T \longrightarrow E,\ u \text{ is continuous}\}.$$

We consider the Tchebyshev norm

$$\|.\|_\infty : C(T, E) \longrightarrow [0, \infty),$$

defined by

$$\|u\|_\infty = \max\{|u(t)|, \text{ for all } t \in T\}, \tag{1.3}$$

where $|\cdot|$ stands for the norm in E. Then $(C(T, E), \|.\|_\infty)$ is a Banach space.

Notation 1.2.2 *Let $AC^i([a, b], E)$ denote the space of $i-$times differentiable functions $f : (a, b) \to E$, whose i^{th} derivative, $f^{(i)}$, is absolutely continuous.*

Definition 1.2.3 *Let $N : E \to E$ be a linear map. N said to be bounded provided there exists $r > 0$ such that*

$$|N(x)| \leqslant r\,|x|\,, \textit{for every } x \in E.$$

The following result is classical.

Proposition 1.2.2 *A linear map $N : E \to E$ is continuous if and only if N is bounded.*

Definition 1.2.4 *The space $B(E)$, is the space of all linear bounded E-valued functions defined on E,*

$$B(E) = \{\, N : E \to E| \;\; N \textit{ is linear bounded}\}\,,$$

and for $N \in B(E)$, we define the norm operator as

$$\|N\|_{B(E)} = \inf\{r > 0 \mid \forall x \in E\,|N(x)| < r|x|\}\,, \tag{1.4}$$

Then $(B(E), \|.\|_{B(E)})$ is a Banach space.

We also have

$$\|N\|_{B(E)} = \sup\{|N(x)|\,,\; |x| = 1\}\,. \tag{1.5}$$

Definition 1.2.5 *A function $f : T \to E$ is said to be measurable provided that for every open $U \subset E$,*

the set

$$f^{-1}(U) = \{t \in T \mid f(t) \in U\}\,.$$

is Lebesgue measurable.

Proposition 1.2.3 *We say that a measurable function $f : T \to E$ is Bochner integrable provided that the function $|f| : T \to [0, \infty)$ is a Lebesgue integrable function (For properties of the Bochner integral, see for instance, Yosida, see [55]).*

Definition 1.2.6 *We define the space $L^1(T, E)$, by*

$$L^1(T, E) = \{f : T \to E \mid f \textit{ is Bochner integrable}\}\,.$$

Let us add that two functions $f_1, f_2 : T \to E$ such that the set $\{f_1(t) \neq f_2(t) \mid t \in T\}$ has Lebesgue measure equal to zero are considered as equal. Then we are able to define

$$\|f\|_{L^1} = \int_a^b |f(t)|\, dt, \text{ for } T = [a, b]. \tag{1.6}$$

It is well known that $(L^1(T, E), \|.\|_{L^1})$ is a Banach space.

Definition 1.2.7 *Let $W^{3,1}([0,1], E)$. be the space of all continuous functions in $C([0,1], E)$ such that their first derivatives are continuous and their second and third weak derivatives belong to $L^1(T, E)$, $W^{3,1}([0,1], E)$ is a Banach space with its usual norm*

$$\|f\|_{W^{3,1}([0,1],E)} = \|f\|_{L^1} + \|f'\|_{L^1} + \|f''\|_{L^1} + \|f'''\|_{L^1}, \text{ for all } f \in L^1(T, E). \tag{1.7}$$

1.3 Definitions and results from multivalued analysis

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this thesis.
Let $(X, \|.\|)$ be a Banach space and Y is.

Definition 1.3.1 *A multivalued function (or a multivalued operator, multivalued map) from X into $\mathcal{P}(Y)$ is a correspondence that associates to each element $x \in X$ a subset $F(x)$ of Y. We denote this correspondence by the symbol $F : X \to \mathcal{P}(Y)$. We define:*

- *the effective domain $DomF = \{x \in X : F(x) \neq \emptyset\}$.*
- *the graph: $GraF = \{(x, y) \in X \times Y : x \in DomF, y \in F(x)\}$.*
- *the range $F(X) = \cup_{x \in X} F(x)$.*
- *the image of the set $A \in \mathcal{P}(X) : F(A) = \cup_{x \in A} F(x)$.*
- *the inverse image of the set $B \in \mathcal{P}(Y) : F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$.*
- *the strict inverse image of the set $B \in \mathcal{P}(Y) : F^+(B) = \{x \in X : F(x) \subset B\}$.*

- *the inverse multivalued operator, denoted by* $F^{-1}: Y \to P(X)$, *is defined by*

$$F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

The set $F^{-1}(y)$ *is called the fiber of* F *at the point* y.

- *A multivalued map* $G : X \to P(Y)$ *has convex (closed, compact) values if* $G(x)$ *is convex (closed, compact) for all* $x \in X$.

- *We say that* G *is bounded on bounded sets if* $G(B)$ *is bounded in* Y *for each bounded set* B *of* X, *that is,*

$$\sup_{x \in B} \{\sup \{\|y\| : y \in G(x)\}\} < \infty.$$

- *The map* G *is called upper semi continuous* (*u.s.c.*), *if for each closed set* $C \subset Y$, $G^{-}(C) = \{x \in X : G(x) \cap C \neq \emptyset\}$ *is closed in* X.

- *The map* G *is called lower semi continuous* (*l.s.c.*), *if for each open set* $O \subset Y$, $G^{-}(O) = \{x \in X : G(x) \cap O \neq \emptyset\}$ *is open in* X.

- *Also,* G *is said to be completely continuous if* $G(B)$ *is relatively compact for every bounded subset* $B \subseteq X$.

- *If the multivalued map* G *is completely continuous with nonempty compact values, then* G *is u.s.c. if and only if* G *has a closed graph* (*i.e.,* $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ *imply that* $y_* \in G(x_*)$).

- *A multivalued map* $G : T \to P_0(X)$ *is said to be measurable iff for every open* $U \subset X$ *the set* $G^{-}(U)$ *measurable set.*

- *Finally, we say that* G *has a fixed point if there exists* $x \in X$ *such that* $x \in G(x)$.

Let $\mathcal{A}$ be a subset of $T \times B$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the $\sigma-$algebra generated by all sets of the form $N \times D$, where N is Lebesgue measurable in T and D is Borel measurable in B.

Definition 1.3.2 *A subset $\mathcal{K}$ of $L^1(T,E)$ is decomposable if for all $u,v \in \mathcal{K}$ and $N \subset T$ measurable, the function $u\chi_N + v\chi_{J-N} \in \mathcal{K}$, where χ stands for the characteristic function.*

Definition 1.3.3 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}_0(L^1(T,E))$ be a multivalued operator. Say N has the property (BC) if*

1) *N is lower semi-continuous (l.s.c),*

2) *N has nonempty closed and decomposable values.*

Definition 1.3.4 *Let $F : T \times E \to \mathcal{P}_0(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator*

$$\mathcal{F} : C(T,E) \to \mathcal{P}_0(L^1(T,E))$$

by letting

$$\mathcal{F}(y) = \{v \in L^1(T,E) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in T\}. \tag{1.8}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 1.3.5 *Let $F : T \times E \to P(E)$ be a multivalued function with nonempty compact values. Say that F is of lower semicontinuous type $(l.s.c.type)$ if its associated Niemytzki operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.*

Definition 1.3.6 *The multivalued map $F : T \times E \to P(E)$ is said to be L^1-Caratheodory if ´*
(i) $t \to F(t,u)$ is measurable for each $u \in E$;
(ii) $u \to F(t,u)$ is upper semicontinuous on E for almost all $t \in T$;
(iii) for each $\rho > 0$, there exists $\phi_\rho \in L^1(T,\mathbb{R}_+)$ such that

$$\|F(t,u)\|_{P(E)} = \sup\{|v| : v \in F(t,u)\} \leqslant \phi_\rho(t)$$

for all $\|u\| \leq r$ and for a.e. $t \in T$.

Lemma 1.3.1 [38] *Let X be a Banach space. Let $F : T \times X \to P_{comp,cv}(X)$ be an L^1-Caratheodory multivalued map with*

$$S_{F,y} = \left\{ g \in L^1(T,X) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in T \right\} \neq \emptyset \tag{1.9}$$

and let Γ be a linear continuous mapping from $L^1(T,X)$ to $C(T,X)$. Then the operator

$$\Gamma \circ S_F : C(T,X) \to \mathcal{P}_{comp,cv}(C(T,X)), \quad y \to (\Gamma \circ S_F)(y) = \Gamma(S_{F,y}) \tag{1.10}$$

is a closed graph operator in $C(T,X) \times C(T,X)$.

Lemma 1.3.2 (*see* [22]). *Assume that*

H1) *$F : T \times E \to P(E)$ is a nonempty, compact-valued, multivalued map such that*

(a) $(t,u) \to F(t,u)$ is $L \otimes B$ measurable,

(b) $u \to F(t,u)$ is lower semicontinuous for a.e. $t \in T$;

H2) *for each $r > 0$, there exists a function $h_r \in L^1(T, \mathbb{R}_+)$ such that*

$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,u)\} \leq h(t)$ for each $(t,u) \in T \times E$ with $u \leq r$.

Then F is of l.s.c.type.

For more details on multivalued maps, we refer to the books of Deimling [18], Gorniewicz [26], Hu and Papageorgiou [31], and Tolstonogov [51].

1.4 Fixed point theorems

Fixed point theorems play a major role in our existence results. Therefore we state a number of fixed point theorems. We start with Schaefer's fixed point theorem.

Definition 1.4.1 *A function $f : X \to X$ is said to have a fixed point if $x_0 = f(x_0)$ for some $x_0 \in X$.*

Theorem 1.4.1 (***Schaefer's fixed point theorem,*** *see also* [20], *page* 29). *Let X be a Banach space and let $N : X \to X$ be a completely continuous map. If the set*

$$\Phi = \{x \in X : \lambda x = Nx \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

The second fixed point theorem concerns multivalued mappings.

Definition 1.4.2 *A multifunction $F : X \to \mathcal{P}(X)$ is said to have fixed point if $x_0 \in F(x_0)$ for some $x_0 \in X$.*

The next fixed point theorems are the well-known **Nonlinear alternative of Leray Schauder type and Covitz and Nadler's fixed point theorem** for multivalued contractions [17] [27] (see also Deimling [18], Theorem 11.1).
Next, we state a well-known result often referred to as the nonlinear alternative. By $\overline{U}$ and ∂U, we denote the closure of U and the boundary of U, respectively.

Theorem 1.4.2 (***Nonlinear alternative of Leray Schauder type*** [27]). *Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to \mathcal{P}(C)$ an upper semicontinuous and compact multivalued operator. Then either,*

1. *T has a fixed point in $\overline{U}$; or*
2. *There is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T(u)$.*

Before stating our next fixed point theorem, we need some preliminaries.

Definition 1.4.3 *A multivalued operator $G : X \to P_{cl}(X)$ is called*

a) *γ-Lipschitz if there exists $\gamma > 0$ such that*

$$H_d\left(G(x), G(y)\right) \leq \gamma d(x, y), \text{ for each } x, y \in X,$$

b) *a contraction if it is γ-Lipschitz with $\gamma < 1$.*

Theorem 1.4.3 (***Covitz and Nadler*** [17]). *Let (X, d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixG \neq \emptyset$.*

1.5 Some additional theorems and definitions

Let (E, Σ, μ) be a measure space and let $\mathcal{F} = \{u_\alpha\}_{\alpha \in J}$ be a family of measurable functions $u_\alpha : E \to \overline{\mathbb{R}}$. A measurable function $u_0 : E \to \overline{\mathbb{R}}$ is said to be the essential infimum of the famely $\mathcal{F}$ if

Definition 1.5.1

1- *For every $\alpha \in J$ and for μ a.e.$x \in E$, there is $u_\alpha \in \mathcal{F} : u_0(x) \leq u_\alpha(x)$,*

2- *If $u : E \to \overline{\mathbb{R}}$ is measurable function such that $u \leq u_\alpha$ for every $\alpha \in J$ and for μ a.e.$x \in E$, then $u(x) \leq u_0(x)$ for μ a.e.$x \in E$.*

Definition 1.5.2 *Suppose X is a topological space and $\mathcal{U}$ is an open cover of X. A cover $\mathcal{V}$ is a refinement of $\mathcal{U}$ if and only if*

$$\forall V \in \mathcal{V}, \exists\, U \in \mathcal{U},\ V \subseteq U.$$

Definition 1.5.3 *Suppose X is a topological space. A collection $\{A_i, i \in I\}$ of subsets of X is locally finite, if and only if for each $x \in X$ there is an open $U \ni x$ with card$\{i \in I,\ U \cap A_i \neq \emptyset\}$ finite.*

Definition 1.5.4 (*Paracompact spaces*) *A topological space X is called a paracompact space if X is a Hausdorff space and every open cover of X has a locally finite open refinement.*

Theorem 1.5.1 [21] *Every compact space is paracompact.*

Definition 1.5.5 *A nonempty partially ordered set A is said to be directed downward set, if for any $x_1, x_2 \in A$ there exists $y \in A$ such that $y \leq x_1$ and $y \leq x_2$.*

Definition 1.5.6 *A nonempty subset $A \subset C(X)$, where X is a metric space, is said to be bounded if there exists a $M > 0$ such that for all $u \in A$, $\|u\| \leq M$.*

Definition 1.5.7 *A nonempty subset $A \subset C(X)$,where X is a metric space is said to be relatively compact if $\overline{A}$ is compact.*

Definition 1.5.8 *Let A a nonempty subset in $C(X)$, then we say that A is equicontinuous at $x \in X$ if for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x)$ such that for some $y \in X$, $d(y, x) \leq \delta \Longrightarrow |u(y) - u(x)| \leq \epsilon$ for all $u \in A$.*

Theorem 1.5.2 ([9] *Ascoli–Arzela Theorem). Let X be a compact metric space. If A is an equicontinuous, bounded subset of $C(X)$, then A is relatively compact.*

Theorem 1.5.3 ([1] *Dominated convergence theorem). Let $(f_n)_{n\geq 1}$ be a sequence of strongly measurable functions and let $f : X \to E$ be a function. Suppose that*

- $\lim\limits_{n\to\infty} f_n = f$ *everywhere on X,*
- *There exists $g \in L^p(X)$ such that, for every $n \geq 1$, $\|f_n\| \leq g$ everywhere,*

then,

$$f_n \to f \text{ in } L^p(X, E).$$

In particular, in the case $p = 1$,

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu.$$

Definition 1.5.9 *A subspace A of E is called a retract of E if there is a continuous map $f : E \to E$ (called a retraction) such that for all $x \in E$ and all $a \in A$:*

$$f(x) \in A, \text{ and } f(a) = a.$$

Equivalently, a subspace A of E is called a retract of E if there is a continuous map $f : E \to E$ (called a retraction) such that for all $a \in A$:

$$f(a) = a.$$

CHAPTER 2

Selection Theory

In this chapter, we shall introduce two prancipal theorems with proof about selection theory, needed in the proof of our results in next chapters.

2.1 Continuous Selection Theorems

Before stating the continuous selection theorems Due To Bressan and Colombo, we bringing some theorems and lemma needed in the sequel.

Theorem 2.1.1 ([19],*Theorem 2.2;p.*127) *Egorov's theorem) Let* (f_n) *be a sequence of measurable functions defined on a measurable set* E *of finite measure and with values in* $\mathbb{R}^*$. *Assume that the sequence converges a.e. in* E *to a function* $f : E \to \mathbb{R}^*$, *which is finite a.e. in* E. *Then for every* $\eta > 0$, *there exists a measurable set* $E_\eta \subset E$ *such that* $\mu(E - E_\eta) < \eta$ *and* $f_n \to f$ *uniformly in* E_η.

Theorem 2.1.2 ([2], *p.*84) *approximate selection, Cellina) Let* X *be a metric space and* Z *a Bnach space. Let* $F : X \to \mathcal{P}_0(Z)$ *be an upper semicontinous map with convex values. Then for every* $\varepsilon > 0$, F *admits a continuous* $\varepsilon-$*approximate selection, i.e. a continuous function* $f_\varepsilon : X \to Z$ *such that*

$$Graf_\varepsilon \subseteq B(GraF, \varepsilon)$$

Here $B(V,\varepsilon)$ denotes the $\varepsilon-$neighborhood of a set V.

Theorem 2.1.3 [40], *Theorem 3.2, Michal's selection theorem) Let X be a paracompact topological space and Z be a Banach space, and $F : X \rightarrow \mathcal{P}_0(Z)$ a lower semicontinuous multivalued map with nonempty convex closed values. Then there exists a continuous selection $f : X \rightarrow Z$ of F.*

Lemma 2.1.1 *Let $(T, \mathcal{A}, \mu)$ be a measre space with $\sigma-$algebra $\mathcal{A}$ of subsets of T and let X be a separable metric space, and let*

$$\begin{aligned} \varphi_n &: X \rightarrow L^1(T,\mathbb{R}), \\ h_n &: X \rightarrow [0,1] \end{aligned}$$

$\forall n \geq 1$ be two sequences of continuous functions, with $\varphi_n(x)(t) \geq 0, \forall x \in X, \forall t \in T$ and such that

$$\{supp\,(h_n)\,; n \geq 1\ \} \text{ is locally finite (closed) covering of } X.$$

Then for every $\varepsilon > 0$ and every continuous strictly positive function $l : X \rightarrow \mathbb{R}^+$, there existe a continuous function $\tau : X \rightarrow \mathbb{R}^+$ and a map $\Phi : \mathbb{R}^+ \times [0,1] \rightarrow \mathcal{A}$ wich satisfy conditions

(a) $\Phi(\tau, \lambda_1) \subseteq \Phi(\tau, \lambda_2)$ if $\lambda_1 \leq \lambda_2$,

(b) $\mu(\Phi(\tau_1, \lambda_1) \triangle \Phi(\tau_2, \lambda_2)) \leq |\lambda_1 - \lambda_2| + |\tau_1 - \tau_2|$,

(c) if $h_n(x) = 1$ then

$$\left| \int_{\Phi(\tau(x),\lambda)} \varphi_n(x)\, d\mu - \lambda \int_T \varphi_n(x)\, d\mu \right| < \frac{\varepsilon}{4l(x)} \qquad (n \geq 1)$$

for all $x \in X$, and $\lambda, \lambda_1, \lambda_2 \in [0,1]$ and $\tau, \tau_1, \tau_2 \geq 0$.

Proof. (see [7], $p73$). ■

2.1.1 Selection Theorem Due To Bressan and Colombo

In what follows, the main arguments are taken from [24]. We list first some preliminary results.

Proposition 2.1.1 *Let $\mathcal{K}$ be a family defined by*

$$\mathcal{K} = \left\{ f;\ f : T \to \mathbb{R}^+ \text{ a nonegative measurable functions} \right\}$$

Then there exists a measurable functions $g : T \to \mathbb{R}^+$ such that

(I) $g \leq f$ *μ-a.e. for all $f \in \mathcal{K}$,*

(II) *if h is a measurable function such that $h \leq f$ μ-a.e. for all $f \in \mathcal{K}$, then $h \leq g$ μ-a.e*

Furthermore, there exists a sequence (f_n) in $\mathcal{K}$ such that

$$g(t) = \inf \{ f_n(t); n \geq 1 \} \quad \text{for a.e. } t \text{ in } T. \tag{2.1}$$

The sequence (f_n) can be chosen to be decreasing, if the family $\mathcal{K}$ is directed downwards.

Proof. see ([41], p.121).

By (**II**) , the function g is unique up to μ-eqivalence. it is the greatest lower bound of $\mathcal{K}$ in the sense of μ-a.e. inequality, and denoted by $ess \inf \{f;\ f \in \mathcal{K}\}$. ■

Proposition 2.1.2 *Let C be a nonempty closed decomposable subset of $L^1(T, E)$ and let $\phi(t) = ess \inf \{ \|f\|_E ; f \in F \}$, then, for every $g_0 \in L^1(T, \mathbb{R})$ such that $g_0(t) > \phi(t)$ μ-a.e., there exists an element $f_0 \in C$ such that*

$$\|f_0(t)\|_E < g_0(t) \qquad \mu\text{-a.e.} \tag{2.2}$$

Proof. The set $\mathcal{K} = \{ \|f_0(.)\|_E ;\ \ f \in C \}$ is decomposable subset of $L^1(T, \mathbb{R})$. Therefore, it is directed downwards.

By *Proposition 2.1.1,* take a sequence $(f_n)_{n\geq 1}$in C such that

$$\|f_m(t)\|_E \geq \|f_n(t)\|_E \ \ \forall m < n,\ \ t \in T, \tag{2.3}$$

$$\phi(t) = \lim_{n\to\infty} \|f_n(t)\|_E \quad \mu\text{-a.e.} \tag{2.4}$$

Let now g_0 be given, with $g_0(t) > \phi(t)$ μ-$a.e.$, and define the increasing sequense of sets

$$\begin{aligned} T_0 &= \emptyset, \\ T_n &= \{t \in T; \|f_n(t)\|_E < g_0(t)\}, n \geq 1. \end{aligned} \tag{2.5}$$

Observe that $\mu\left(T \smallsetminus \bigcup_{n\geq 0} T_n\right) = 0$.
Define the sequense (h_n) by setting

$$h_n = \begin{cases} f_k \text{ if } t \in T_k \smallsetminus T_{k-1}, \; k = 1, ..., n-1, \\ f_n \text{ if } t \in T \smallsetminus \bigcup_{k<n} T_k. \end{cases} \tag{2.6}$$

Since C is decomposable, each h_n belongs to C. Moreover, the sequence $h_n(t)$ is eventually constant for $a.e.$ $t \in T$, and $\|h_n(t)\|_E \leq \|f_1(t)\|_E$ μ-$a.e.$; hence, by the Dominated convergence theorem, h_n converges in $L^1(T, E)$ to some function f_0. Clearly, $f_0 \in C$ because C is closed. Finally, if $t \in T_n \smallsetminus T_{n-1}$ for some n, then

$$\|f_0(t)\|_E = \|f_n(t)\|_E < g_0(t). \tag{2.7}$$

Therefore, f_0 satisfies (2.2) ∎

Proposition 2.1.3 *Let X be a metric space and let $F : X \to D(L^1(T, E))$ be a l.s.c multifunction with closed decomposable values. For all $x \in X$, let $\phi_x(t) = ess \inf \{\|f(t)\|_E; f \in F(x)\}$. Then the multivalued map $P : X \to L^1(T, \mathbb{R})$ defined as*

$$P(x) = \left\{g \in L^1(T, \mathbb{R}); g(t) > \phi_x(t) \;\; \mu\text{-}a.e. \; x \in X\right\} \tag{2.8}$$

is lower semicontinuous.

Proof. Let C be an arbitrary closed subset of $L^1(T, \mathbb{R})$ it suffices to show that, if $P(x_n) \subseteq C$ for some sequence $(x_n)_{n\geqslant 1}$ converging to x_0 , then also $P(x_0) \subseteq C$. To this purpose, fix any $g_0 \in P(x_0)$ and take, by *Proposition 2.1.2* , a function $f_0 \in F(x_0)$ such that $\|f_0(t)\|_E < g_0(t)$ μ-$a.e.$ Because of the lower semicontinuity of F, there exists a sequence $f_n \in F(x_n)$ such that $f_n \to f_0$ in $L^1(T, E)$. Then , for every $n \geq 1$, the function $g_n = \|f_n\|_E + g_0 - \|f_0\|_E$ belongs to $P(x_n)$ wich is contained in C.
Since the sequence (g_n) converges to g_0 in the norm of $L^1(T, \mathbb{R})$ and C is closed, this implies $g_0 \in C$. ∎

Proposition 2.1.4 *Let X be a metric space, and let $G : X \to D(L^1(T,E))$ be a l.s.c map with closed decomposable values. Assume that there exist continuous mappings $g : X \to L^1(T,E)$ and $r : X \to L^1(T,\mathbb{R})$ such that, For every $x \in X$, the set*

$$P(x) = \{f \in G(x) \,;\, \|f(t) - g(x)(t)\|_E < r(x)(t) \;\; \mu\text{-a.e.}\} \tag{2.9}$$

is nonempty, then the map $P : X \to D(L^1(T,E))$ is l.s.c with decomposable values.

Proof. Clearly, for every $x \in X$, $P(x)$ is a decomposable set, because it is the intersection of two decomposable sets.

To check the lower semicontinuity of P, let C be any closed subset of $L^1(T,E)$. It suffices to show that, for any sequence (x_n) in X converging to a point x_0, if $P(x_n) \subseteq C$ for all $n \geq 1$, then $P(x_0) \subseteq C$.

Let $f_0 \in P(x_0)$. By the lower semicontinuity of G,there exists a sequence $f_n \in G(x_n)$, $n = 1, 2, ...$, such that $f_n \to f_0$ in $L^1(T,E)$. By possible taking a subsequence, we can assume that

$$f_n(t)\,,\, g(x_n)(t)\,,\, r(x_n)(t)$$

converge to

$$f_0(t)\,,\, g(x_0)(t)\,,\, r(x_0)(t)$$

respectively, $\mu - a.e.$in T.

Applying *Egorov's theorem* we have, for each $i \geq 1$ the existence of an increasing sequence of a measurable set $T_i \subseteq T$ such that

$$f_n,\;\; g(x_n) \;\text{ and }\; r(x_n) \;\text{ converge uniformly on }\; T_i$$

and

$$\int_{T \setminus T_i} r(x_0)\,.d\mu < \frac{1}{i}.$$

For each $k \geq 1$, consider the sets

$$T_{i,k} = \left\{ t \in T_i ; \|f_0(t) - g(x_0)(t)\|_E < r(x_0)(t) - \frac{1}{k} \right\}. \tag{2.10}$$

And notice that $\{T_{i,k}\}_{i=1}^{\infty}$ form for every i an increasing sequence of measurable sets with

$$\bigcup\nolimits_{k\geq 1} T_{i,k} = T_i$$

Hence, for every $i \geq 1$, there exists an integer $k(i)$ such that

$$\int_{T_i \setminus T_{i,k(i)}} r(x_0)\, d\mu < \frac{1}{i}. \tag{2.11}$$

Therefore, the sets $T_{i,k(i)}$ have the following properties

$$\int_{T \setminus T_{i,k(i)}} r(x_0)\, d\mu < \frac{2}{i}, \tag{2.12}$$

and

$$\|f_0(t) - g(x_0)(t)\|_E < r(x_0) - \frac{1}{k(i)} \quad \forall t \in T_{i,k(i)}. \tag{2.13}$$

By (**2.13**) and by the uniform convergence on $T_{i,k(i)}$, for all $i \geq 1$ there exists a sequence $\{n_i\}_{i\geq 1}$ such that

$$\|f_n(t) - g(x_n)(t)\|_E < r(x_n)(t) \quad \forall t \in T_{i,k(i)},\ n \geq n_i. \tag{2.14}$$

We can also assume that the sequence $(n_i)_{i\geq 1}$ is strictly increacing. For each n, choose an arbitrary $h_n \in P(x_n)$ and set, for $n_i \leq n < n_{i+1}$,

$$\mathbf{g}_n = f_n.\chi_{T_{i,k(i)}} + h_n.\chi_{T\setminus T_{i,k(i)}}. \tag{2.15}$$

Since $P(x_n)$ is decomposable, $\mathbf{g}_n \in P(x_n) \subset C$. We claim that $\mathbf{g}_n \to f_0$ in $L^1(T,E)$, which implies $f_0 \in C$.Indeed, for $n_i \leq n < n_{i+1}$, (**2.12**) and (**2.14**) yield

$$\begin{aligned}
\|\mathbf{g}_n - f_0\|_1 &\leq \int_{T\setminus T_{i,k(i)}} \|h_n - g(x_n)\|_E\, d\mu + \int_{T\setminus T_{i,k(i)}} \|g(x_n) - g(x_0)\|_E\, d\mu \\
&\quad + \int_{T\setminus T_{i,k(i)}} \|g(x_0) - f_0\|_E\, d\mu + \int_{T_{i,k(i)}} \|f_n - f_0\|_E\, d\mu \\
&\leq \int_{T\setminus T_{i,k(i)}} r(x_n)\, d\mu + \|g(x_n) - g(x_0)\|_1 + \int_{T\setminus T_{i,k(i)}} r(x_0)\, d\mu + \|f_n - f_0\|_1 \\
&\leq \|r(x_n) - r(x_0)\|_1 + \|g(x_n) - g(x_0)\|_1 + \|f_n - f_0\|_1 + \frac{4}{i}.
\end{aligned}$$

As $n \to \infty$, we also have $i \to \infty$, hence our claim is proved. ■

The next result, concerning the existence of approximate selections, is the core of the whole proof of *theorem 2.1.4.*

Proposition 2.1.5 *Let X be a separable metric space and let $G : X \to D(L^1(T,E))$ be a l.s.c map with closed decomposable values. Then, for every $\varepsilon > 0$, there exist continuous maps $f_\varepsilon : X \to L^1(T,E)$ and $\varphi_\varepsilon : X \to L^1(T,\mathbb{R})$, such that f_ε is an ε-approximate selection of G, in the sense that, for each $x \in X$, the set*

$$G_\varepsilon(x) = \{f \in G(x) ; \|f(t) - f_\varepsilon(x)(t)\|_E < \varphi_\varepsilon(x)(t) \ \mu\text{-a.e.}\} \tag{2.16}$$

is nonempty, and $\|\varphi_\varepsilon(x)\|_1 < \varepsilon$. Moreover, the map $x \to G_\varepsilon(x)$ is l.s.c. with decomposable values.

Proof. Let $\varepsilon > 0$ be fixed arbitrary. For every $\overline{x} \in X$ and $\overline{f} \in G(\overline{x})$, the multivalued map Q defined as

$$Q(x) = \left\{g \in L^1(T,\mathbb{R}); g(t) \geq ess\inf\left\{\left\|f(t) - \overline{f}(t)\right\|_E ; f \in G(x)\right\} \text{ for } a.e.\ t \in T\right\} \tag{2.17}$$

is l.s.c. with closed convex values.

To see this, define $F(x) = \left\{f - \overline{f}; f \in G(x)\right\}$. Then the map F is also l.s.c. with decomposable values. By *Proposition 2.1.3,* the multivalued map P defined in (**2.8**) is l.s.c. Hence Q is l.s.c., because $Q(x)$ is the closure of $P(x)$, for all $x \in X$.

It is therefore possible to apply Michael's theorem to Q and obtain a continuous selection $\varphi_{\overline{x},\overline{u}}$ such that $\varphi_{\overline{x},\overline{f}}(x) \in Q(x)$, for all $x \in X$ and $\varphi_{\overline{x},\overline{f}}(\overline{x}) \equiv 0$. The family of sets

$$\left\{\left\{x \in X, \left\|\varphi_{\overline{x},\overline{f}}(x)\right\|_1 < \frac{\varepsilon}{4}\right\}; \overline{x} \in X, \quad \overline{f} \in G(\overline{x})\right\} \tag{2.18}$$

is an open covering of the separable metric space X, therefore it has a countable nbd-finite open refinement $\{V_n; n \geq 1\}$. Let $\{p_n(.)\}$ be a continuous partition of unity subordinate to the covering $\{V_n\}$ and Let $\{h_n(.)\}$ be a famely of continuous functions from X into $[0,1]$ such that $h_n \equiv 1$ on $\mathbf{supp}(p_n)$ and $\mathbf{supp}(h_n) \subset V_n$. For every $n \geq 1$,choose x_n, u_n such that $V_n \subseteq \left\{x, \left\|\varphi_{x_n,u_n}(x)\right\|_1 < \varepsilon\backslash 4\right\}$ and set $\varphi_n = \varphi_{x_n,u_n}$. The function φ_n have the following properties :

$$\varphi_n(x)(t) \geq ess\inf\{\|f(t) - f_n(t)\|_E ; f \in G(x)\}, \tag{2.19}$$

$$p_n(x)\|\varphi_n(x)\|_1 \leq p_n(x) . \frac{\varepsilon}{4} \qquad (x \in X, n \geq 1). \tag{2.20}$$

lemma 2.1.1 applied to the sequences (φ_n) and (h_n), and to the function $l : l(x) = \sum_{n\geq 1} h_n$, yields a continuous function $\tau : X \to \mathbb{R}^+$and a family $\{\Phi(\tau,\lambda)\}$ of measurable subsets of T satisfying $(a), (b)$ and (c').

It is now possible to construct the function f_ε and φ_ε. Set $\lambda_0 \equiv 0$, $\lambda_n(x) = \sum_{m\leq n} p_m(x)$, and define

$$f_\varepsilon(x) = \sum\nolimits_{n\geq 1} u_n.\chi_n(x), \qquad \varphi_\varepsilon(x) = \varepsilon/4 + \sum\nolimits_{n\geq 1} \varphi_n(x).\chi_n(x), \tag{2.21}$$

where

$$\chi_n(x) = \chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))}. \tag{2.22}$$

Clearly , f_ε and φ_ε are continuous, because the above summation are locally finite.

Let G_ε be define by (**2.16**). to check that the values of G_ε are nonempty, fix any $x \in X$. For every $n \geq 1$, use *Proposition 2.1.2* and select $f_x^n \in G(x)$ such that

$$\|f_x^n(t) - f_n(t)\|_E \leq \varepsilon/4 + ess\inf\{\|f(t) - f_n(t)\|_E ; f \in G(x)\} \tag{2.23}$$

μ-$a.e.$ in T. Then

$$f_x = \sum\nolimits_{n\geq 1} f_x^n.\chi_n(x)$$

lies in $G(x)$, because $G(x)$ is decomposable, We claim that $f_x \in G_\varepsilon(x)$. Indeed, (**2.19**) and (**2.23**) yield

$$\begin{aligned} \|u_x(t) - f_\varepsilon(x)(t)\|_E &\leq \sum\nolimits_{n\geq 1} \|f_x^n(t) - f_n(t)\|_E .\chi_n(x)(t) \\ &< \varphi_\varepsilon(x)(t) \quad \mu\text{-}a.e.\text{in } T. \end{aligned}$$

Hence $G_\varepsilon(x) \neq \emptyset$. Being the intersection of two decomposable sets $G_\varepsilon(x)$ is also decomposable. The lower semicontinuity of G_ε follows from *Proposition 2.1.4.*

To conclude the proof of *Proposition 2.1.5* , it now suffices to show that $\|\varphi_\varepsilon(x)\|_1 < \varepsilon$ for every x. Set $I(x) = \{n \geq 1,\ p_n(x) > 0\}$ and notice that $1 \leq\neq I(x) \leq l(x)$. From (c') in *lemma*

2.1.1 and (**2.20**) we deduce

$$\begin{aligned}
\|\varphi_{\varepsilon}(x)\|_1 &= \varepsilon/4+\sum\nolimits_{n\geq 1}\int_T \varphi_n(x).\chi_n(x)\,d\mu,\\
&< \varepsilon/4+\sum\nolimits_{n\geq 1}\left[p_n(x)\,\|\varphi_n(x)\|_1+\varepsilon/(2l(x))\right],\\
&\leq \varepsilon/4+\left[\varepsilon/4+\frac{\neq I(x).\varepsilon}{2l(x)}\right],\\
&\leq \varepsilon
\end{aligned}$$

■

Theorem 2.1.4 *Let X be a separable metric space, and let $F : X \to D(L^1(T,E))$ be a l.s.c multifunction with closed decomposable values. Then F has continuous selection.*

Proof. Let the function F be given. Construct two sequences of continuous maps $f_n : X \to L^1(T,E)$ and $\varphi_n : X \to L^1(T,\mathbb{R})$ and a sequence of l.s.c multifunction G_n with decomposable values, such that, for all $x \in X$ and $n \geq 1$,

(**i**) $G_n(x) = \{u \in F(x)\,;\|u(t) - f_n(x)(t)\|_E < \varphi_n(x)(t)\ \ \mu\text{-}a.e.\} \neq \emptyset$,

(**ii**) $\|f_n(x)(t) - f_{n-1}(x)(t)\|_E \leq \varphi_n(x)(t) + \varphi_{n-1}(x)(t)$ μ-$a.e$ in T $\quad (n \geq 2)$,

(**iii**) $\|\varphi_n(x)\|_1 < 2^{-n}$.

To do this, define f_1 and φ_1 by applying *Proposition 2.1.5* with $G = F$, $\varepsilon = 1/2$.

Let now f_m, φ_m and G_m be defined so that (**i**) – (**iii**) hold for all $m = 1, ..., n-1$. To construct f_n and φ_n apply again *Proposition 2.1.5* with , $\varepsilon = 2^{-n}$, defining $G(x)$ to be the closure of $G_{n-1}(x)$, for all x. By induction, the maps f_n, φ_n and G_n can be defined for all $n \geq 1$. By (**ii**) , the sequence $(f_n)_{n\geq 1}$ is Cauchy in the L^1- norm, hence it converges uniformly to some continuous function $f : X \to L^1(T,E)$. By (**i**) and (**iii**) $d_{L^1}(f_n(x), F(x)) < 2^{-n}$. Since $F(x)$ is closed, this implies that $f(x) \in F(x)$ for all $x \in X$, hence f is selection of F. ■

2.2 Mesurable Selection Theorem

Let T be a set of arbitrary elements and X a metric space. Let $\mathcal{S}$ be a countably additive family of subsets of T [that is, if $A_n \in \mathcal{S}$ for $n = 1, 2, ...$ then $\cup_{n=1}^{\infty} A_n \in \mathcal{S}$]. Then the following statement is true.

Lemma 2.2.1 *Let $g_n : T \to X$ for $n = 1, 2, ...,$ and let $g(x) = \lim g_n(x)$ where the convergence is uniform.*

Suppose that

$$g_n^{-1}(U) \in \mathcal{S} \text{ whenever } U \text{ is open in } X. \tag{$\mathbf{1}_n$}$$

Then $g^{-1}(U) \in \mathcal{S}$ whenever U is open in X.

Proof. (see [43], , $p49$) ■

Let L be a field of subsets of X. [In other words, if A, B are members of L, then so are $A \cup B, A \cap B$ and $X - A$].

Denote by $\mathcal{S}$ the countably additive family induced by L, that is, the family of countable unions of members of L.

2.2.1 Selection Theorem Due To Kuratowski, Ryll, and Nardzewski

Theorem 2.2.1 *Let (T, Σ) be a measurable space, (X, d) a separable, complete metric space and $G : T \to P_{cl}(X)$ a multivalued map with nonempty closed values. If G is measurable, then it has a measurable selection. i.e. there exists a measurable map $g : T \to X$ such that $g(t) \in G(t)$, for every $t \in T$.*

Proof. Let $\mathcal{A} = (a_1, a_2, ..., a_i, ...)$ be a countable set dense in X. We may suppose of course that the diameter

$$diam\,(X) = \sup \{d\,(x, y) : x, y \in X\} \leq 1.$$

we'll define g as the limit of mappings $g_n : T \to X$, $(n = 0, 1, 2, ..)$ satisfying condition $\mathbf{1}_n$ and the two following conditions.

$$d(g_n(x), G(x)) < \frac{1}{2^n}, \tag{$\mathbf{2}_n$}$$

$$|g_n(x) - g_{n-1}(x)| < \frac{1}{2^{n-1}} \quad \text{for} \quad n > 0. \tag{$\mathbf{3}_n$}$$

Let us proceed by induction. Put $g_0(x) = r_1$, for each $x \in T$.

Thus $\mathbf{1}_0$ and $\mathbf{2}_0$ are fulfilled.

Now let us assume, for a given $n > 0$, that g_{n-1} satisfies conditions $\mathbf{1}_{n-1}$ and $\mathbf{2}_{n-1}$.

Put

$$C_{i,n} = \left\{ x : d(a_i, G(x)) < \frac{1}{2^n} \right\},$$

$$D_{i,n} = \left\{ \ x : |a_i - g_{n-1}(x)| < \frac{1}{2^{n-1}} \right\},$$

and

$$A_{i,n} = C_{i,n} \cap D_{i,n}.$$

We have, $X = A_{1,n} \cup A_{2,n} \cup ...$. For, x being a given point of T, there is by 2_{n-1}, $y \in G(x)$ such that

$$|y - g_{n-1}(x)| < \frac{1}{2^{n-1}}.$$

Since $(a_1, a_2, ..., a_i, ...)$ is dense, we can find a a_i such that

$$|a_i - y| < \frac{1}{2^n},$$

and

$$|a_i - g_{n-1}(x)| < \frac{1}{2^{n-1}}.$$

Hence, $x \in A_{i,n}$.

Denote by $B_{i,n}$ the open ball

$$B_{i,n} = \left\{ y : |y - a_i| < \frac{1}{2^n} \right\},$$

it follows that

$$C_{i,n} = \{x : G(x) \cap B_{i,n} \neq \emptyset\} \text{ and } D_{i,n} = g_{n-1}^{-1}(B_{i,n-1}).$$

Hence it follows that

$$C_{i,n} \in \mathcal{S} \text{ and } D_{i,n} \in \mathcal{S} \text{ and consequently } A_{i,n} \in \mathcal{S}.$$

Consequently, $A_i^n = \cup_{j=1}^{\infty} E_{i,j}^n$ where $E_{i,j}^n \in L$.

Arrange the double sequence (i, j) in a simple sequence (k_s, m_s) where $s = 1, 2, ...,$ and put

$$E_s^n = E_{k_s, m_s}^n.$$

We have, $T = E_1^n \cup E_2^n \cup ... \cup E_s^n \cup ...$

This identity allows us to define a mapping $g_n : T \to \mathcal{A}$ as follows:

$$g_n(x) = a_{k_s}, \;\; x \in E_s^n \backslash \left(E_1^n \cup E_2^n \cup ... \cup E_{s-1}^n \right).$$

We shall show that g_n satisfies $1_n, 2_n$ and 3_n.

By definition $g_n^{-1}\left(a_{k_s}\right) = E_s^n \backslash \left(E_1^n \cup E_2^n \cup ... \cup E_{s-1}^n \right)$. As L is a field, it follows that

$$g_n^{-1}\left(a_{k_s}\right) \in L \text{ and as } g_n^{-1}\left(a_i\right) = \cup_{k_s=j} g_n^{-1}\left(a_{k_s}\right),$$

we have

$$g_n^{-1}\left(a_i\right) \in \mathcal{S} \text{ for each } i.$$

Consequently $g_n^{-1}\left(\mathcal{Z}\right) \in \mathcal{S}$.for each $\mathcal{Z} \in \mathcal{A}$ (since $\mathcal{A}$ is countable and $\mathcal{S}$ countably additive). Thus $\mathbf{1_n}$ is satisfied. For a given x let s satisfy,

$$x \in E_s^n \backslash \left(E_1^n \cup E_2^n \cup ... \cup E_{s-1}^n \right).$$

Put $k = i$. Hence we have $x \in E_s^n \subset A_{i,n} = C_{i,n} \cap D_{i,n}$ and it is clear that g_n's satisfy $\mathbf{2}_n$ and $\mathbf{3}_n$. Thus the sequence $g_0, g_1, ..., g_n...$ has been defined according to the conditions $\mathbf{1}_n, \mathbf{2}_n$ and $\mathbf{3}_n$.

By $\mathbf{3}_n$ and by the completeness of the space X, this sequence converges uniformly to a mapping $g : T \to X$.

By *lemma 2.2.1*, it follows $g^{-1}(U) \in \mathcal{S}$ whenever U is open in X. Finally $g(x) \in G(x)$ according to $\mathbf{2}_n$. therefore the proof of *Theorem 2.2.1* is complete. ■

CHAPTER 3

Solvability Of a Three-Point Boundary Value Problem For a Third-Order Differential Inclusion

In this chapter we discuss the existence of solutions for a third- order differential inclusion with three-point boundary conditions involving convex and nonconvex multivalued maps. Our results are based on the nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory combined with some selection theorems, (see [46]).

3.1 Exposure of the problem

In [45], By using Krasnoselskii's fixed point theorem and the fixed point index theory, the authors discussed the existence of positive solutions for the problem

$$u'''(t) + a(t) f(t, u(t)) = 0, \ t \in (0,1), \qquad (\mathfrak{Pfo})$$
$$u'(0) = u'(1) = \alpha u(\eta), \ u(0) = \beta u(\eta),$$

where α, β and η are constants with $\alpha \in \left[0, \frac{1}{\eta}\right)$, and $0 < \eta < 1$ and $\beta \in [0, 1 - \alpha\eta)$.
In this chapter we investigate the solutions for a third-order differential inclusion with three-

point boundary value problem , see[46].

$$-u'''(t) \in F(t,u(t)),\ t \in (0,1), \qquad (\mathfrak{PF1})$$
$$u'(0) = u'(1) = \alpha u(\eta),\ u(0) = \beta u(\eta),$$

where α, β and η are constants with $\alpha \in \left[0, \frac{1}{\eta}\right)$, $0 < \eta < 1$, $\beta \neq 1 - \alpha\eta$, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$. The present chapter is motivated by a recent paper of Ali Rezaiguia and Smail Kelaiaia [45], where it is considered problem ($\mathfrak{Pfo}$) with $F(.,.)$ single valued and several existence results are obtained by using fixed point techniques and index theory.

3.2 Study and Discussion of Problematic

Here $C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|u\| = \sup\{|u(t)|, \text{ for all } t \in [0,1]\},$$

$L^1([0,1],\mathbb{R})$, the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by

$$\|u\|_{L^1} = \int_0^1 |u(t)|\, dt$$

and $AC^i([0,1],\mathbb{R})$ the space of $i-$times differentiable functions $u : [0,1] \to \mathbb{R}$, whose i^{th} derivative, $u^{(i)}$is absolutely continuous.

Let A be a subset of $[0,1] \times\mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{I} \times \mathcal{D}$ where $\mathcal{I}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Let a subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ be a decomposable set.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{comp}(\mathbb{R})$ be a multivalued map. Assign to F the multivalued operator

$$\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}_0(L^1([0,1],\mathbb{R}))$$

and

$$\mathcal{F}(u) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,u(t)) \text{ for a.e. } t \in [0,1]\}.$$

The operator $\mathcal{F}$ is the Niemytzki operator associated with F. (F is of the lower semi-continuous type (l.s.c type)).

Next we state a selection theorem due to Bressan and Colombo.

Lemma 3.2.1 [7] *Let Y be separable metric space and let $N : Y \to \mathcal{P}_0(L^1([0,1],\mathbb{R}))$ be a multivalued operator which has the property (BC). Then N has a continuous selection, i.e there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(u) \in N(u)$ for every $u \in Y$.*

Lemma 3.2.2 [7] *Let E be a Banach space, let $F : [0,T] \times \to \mathcal{P}_{comp,cv}(E)$ be an L^1-Caratheodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],E)$ to $C([0,1],E)$. Then the operator*

$$\Theta \circ S_F : C([0,1],E) \to \mathcal{P}_{comp,cv}(C([0,1],E)), u \to (\Theta \circ S_F)(u) = \Theta(S_{F,u})$$

is a closed graph operator in $C([0,1],E) \times C([0,1],E)$.

Lemma 3.2.3 [45] *Assume $\beta + \alpha\eta \neq 1$, then for $y \in C([0,1],R)$ the problem*

$$u'''(t) + y(t) = 0, \ t \in (0,1), \tag{3.1}$$

$$u'(0) = u'(1) = \alpha u(\eta), \ u(0) = \beta u(\eta), \tag{3.2}$$

has a unique solution

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,y(s)\,ds - \\ &\quad -\frac{1}{2}\frac{\alpha t + \beta}{(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 y(s)\,ds. \end{aligned}$$

The proof of *Lemma 3.2.3*, is given by integrating three times $u'''(t) + y(t) = 0$ over the interval $[0,t]$, we obtain

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + A_1 t^2 + A_2 t + A_3 \text{ where } A_1, A_2, A_3 \in \mathbb{R}. \tag{3.3}$$

The constants A_1, A_2 and A_3 are given by the three-point boundary conditions (3.1) and (3.2) respectively.

Lemma 3.2.4 [17] *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3.2.1 The nonconvex case

By the help of Schaefer's theorem combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we shall present first an existence result for the problem ($\mathfrak{PF}\mathfrak{1}$). Before this, let us introduce the following hypotheses which are assumed hereafter:

(H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{comp}(\mathbb{R})$ be a multivalued map verifying :

a) $(t,u) \to F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable.

b) $u \to F(t,u)$ is lower semicontinuous for a.e. $t \in [0,1]$.

(H_2) F is integrably bounded, that is, there exists a function $m \in L^1([0,1], \mathbb{R}_+)$ such that

$$\|F(t,u)\| = \sup\{\|v\| : v \in F(t,u)\} \leqslant m(t) \text{ for almost all } t \in [0,1].$$

(H_3) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{comp}(\mathbb{R})$ is such that $F(\cdot,u)\,[0,1] \times \mathbb{R} \to \mathcal{P}_{comp}(\mathbb{R})$ is measurable for each $t \in [0,1]$.

(H_4) $H_d(F(t,u), F(t,\overline{u})) \leq p(t)|u - \overline{u}|$ for almost all $t \in [0,1]$ and $u, \overline{u} \in \mathbb{R}$ with $p \in L^1([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \leq p(t)$ for almost all $t \in [0,1]$.

(H_5) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cv}(\mathbb{R})$ is Carathéodory,

(H_6) there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|w| : w \in F(t,u)\} \leq p(t)\,\psi(\|u\|) \text{ for each } (t,u) \in [0,1] \times \mathbb{R},$$

(H_7) there exists a number $M > 0$ such that

$$\left[1 + \eta^2 \frac{\alpha + |\beta|}{|1 - \alpha\eta - \beta|}\right] \psi(M)\, \|p\|_{L^1} < M.$$

Lemma 3.2.5 [23] *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{comp}(\mathbb{R})$ be a multivalued map. Assume (H_1) and (H_2) hold. Then F is of the l.s.c. type.*

Definition 3.2.1 *A function $u \in AC^2([0,1], \mathbb{R})$ is called a solution to the BVP $(\mathfrak{PF}\mathfrak{1})$ if u satisfies the differential inclusion*

$$-u'''(t) \in F(t, u(t)), \ t \in (0,1),$$

and the condition

$$u'(0) = u'(1) = \alpha u(\eta), \ u(0) = \beta u(\eta).$$

In first result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on Schaefer's fixed point theorem with the selection theorem of Bressan and Colombo [7] for lower semicontinuous maps with decomposable values.

Theorem 3.2.1 *Suppose that hypothesis (H_1) and (H_2) hold. Then the problem $(\mathfrak{PF}\mathfrak{1})$ has at least one solution.*

Proof. (H_1) and (H_2) imply by *Lemma 3.2.5* that F is of the lower semi-continuous type. Then from *Lemma 3.2.1* there exists a continuous function $g : C([0,1], \mathbb{R}) \to L^1([0,1], \mathbb{R})$ such that $g(u) \in \mathcal{F}(u)$ for all $u \in C([0,1], \mathbb{R})$.

We consider the problem

$$-u''' = g(u), \ a.e. \ t \in [0,1], \tag{3.4}$$

$$u'(0) = u'(1) = \alpha u(\eta), u(0) = \beta u(\eta). \tag{3.5}$$

If $u \in C([0,1], \mathbb{R})$ is a solution to the problem (3.4) and (3.5), then u is a solution to the problem $(\mathfrak{PF}\mathfrak{1})$.

Transform problem (3.4) and (3.5) into a fixed point problem. Consider the operator $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$, defined by

$$\begin{aligned} T(u)(t) &= -\frac{1}{2}\int_0^t (t-s)^2 g(u)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s)\,g(u)\,ds - \\ &\quad -\frac{1}{2}\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^\eta (\eta - s)^2 g(u)\,ds. \end{aligned}$$

We shall show that T is a compact operator.

Step 1: T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
|T(u_n)(t) - T(u)(t)| \leq\ & \frac{1}{2}\int_0^t (t-s)^2 |g(u_n) - g(u)|\, ds \\
&+\frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)|g(u_n) - g(u)|\, ds \\
&+\frac{1}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 |g(u_n) - g(u)|\, ds,
\end{aligned}$$

Since g is continuous, then

$$\|T(u_n) - T(u)\| \to 0 \text{ as } n \to \infty.$$

Step 2: T is bounded on bounded sets of $C([0,1],\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in T(u), u \in B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \leq r\}$ one has $\|h\| \leq c$. By (H_2) we have for each $t \in [0,1]$ that

$$|h(t)| \leq \left[1 + \frac{\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\int_0^1 m(s)\, ds + \frac{\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\int_0^\eta m(s)\, ds = c.$$

Then $\|h\| \leq c$.

Step 3: T sends bounded sets of $C([0,1],\mathbb{R})$ into equicontinuous sets.

Let $t_1, t_2 \in [0,1], t_1 < t_2$ and B_r be a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| \leq\ & \frac{1}{2}\int_{t_1}^{t_2} (t_2-s)^2 |g(u)|\, ds + \frac{1}{2}\left[t_2^2 - t_1^2 + \eta^2 \frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\right]\int_0^1 (1-s)|g(u)|\, ds \\
&+\frac{1}{2}\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\int_0^\eta (\eta-s)^2 |g(u)|\, ds + \frac{1}{2}\int_0^{t_1}\left((t_1-s)^2 - (t_2-s)^2\right)|g(u)|\, ds, \\
\leq\ & \frac{1}{2}\int_{t_1}^{t_2} (t_2-s)^2 m(s)\, ds + \left[\frac{t_2^2 - t_1^2}{2} + \frac{\alpha\eta^2(t_2-t_1)}{2|1-\alpha\eta-\beta|}\right]\int_0^1 (1-s)m(s)\, ds \\
&+\frac{1}{2}\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\int_0^\eta (\eta-s)^2 m(s)\, ds + \frac{1}{2}\int_0^{t_1}\left((t_1-s)^2 - (t_2-s)^2\right)m(s)\, ds.
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem we can onclude that T is completely continuous.

In order to apply Schaefer's theorem, it remains to show that

Step 4: The set

$$\Omega = \{u \in C([0,1], \mathbb{R}) : \lambda u = T(u) \text{ for some } \lambda > 1\}$$

is bounded.

Let $u \in \Omega$. Then $\lambda u = T(u)$ for some $\lambda > 1$ and

$$\begin{aligned} u(t) &= -\frac{\lambda^{-1}}{2}\int_0^t (t-s)^2 g(u)\,ds + \frac{\lambda^{-1}}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,g(u)\,ds \\ &\quad -\frac{\lambda^{-1}}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 g(u)\,ds, \end{aligned}$$

this implies by (H_2) that for each $t \in [0,1]$ we have

$$\begin{aligned} |u(t)| &\leq \frac{1}{2}\int_0^t (t-s)^2 m(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \left|\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right|\right]\int_0^1 (1-s)\,m(s)\,ds \\ &\quad +\frac{1}{2}\left|\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right|\int_0^\eta (\eta-s)^2 m(s)\,ds, \end{aligned}$$

thus

$$\begin{aligned} \|u\| &\leq \frac{1}{2}\int_0^1 (t-s)^2 m(s)\,ds + \frac{1}{2}\left[1 + \eta^2 \left|\frac{\alpha + \beta}{1-\alpha\eta-\beta}\right|\right]\int_0^1 (1-s)\,m(s)\,ds \\ &\quad +\frac{1}{2}\left|\frac{\alpha + \beta}{1-\alpha\eta-\beta}\right|\int_0^\eta (\eta-s)^2 m(s)\,ds = K. \end{aligned}$$

This shows that Ω is bounded.

As a consequence of Schaefer's theorem (*see* [20] *p.* 29) we deduce that T has a fixed point which is a solution to (**3.4**) and (**3.5**) and hence from Remark 2.1 a solution to the problem $(\mathfrak{PF}\mathfrak{1})$.

Now, by applying a fixed point theorem for multivalued map due to Covitz and Nadler [17].we prove the existence of solutions for the problem $(\mathfrak{PF}\mathfrak{1})$ with a non convex valued right-hand side. ■

Theorem 3.2.2 *Assume that* (H_3) *and* (H_4) *hold. Then the problem* $(\mathfrak{PF}\mathfrak{1})$ *has at least one solution on* $[0,1]$ *if*

$$\left[1 + \frac{1+\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\|p\|_{L^1} < 1.$$

Proof. For each $u \in C([0,1] \times \mathbb{R})$, define the set of selections of F by

$$S_{F,u} := \{w \in L^1([0,1], \mathbb{R}) : w(t) \in F(t, u(t)) \text{ for a.e. } t \in [0,1]\}.$$

and the multi-valued operator $\Omega : C([0,1] \times \mathbb{R}) \to \mathcal{P}_{cl}(C([0,1] \times \mathbb{R}))$ by

$$\Omega(u) = \left\{ h \in C([0,1] \times \mathbb{R}) : h(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(u)\, ds \right.$$
$$\left. +\frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right] \int_0^1 (1-s) f(u)\, ds \right\}$$
$$-\frac{1}{2}\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^\eta (\eta - s)^2 f(u)\, ds, \ t \in [0,1].$$

for $f \in S_{F,u}$. Observe that the set $S_{F,u}$ is nonempty for each $u \in C([0,1] \times \mathbb{R})$, by the assumption H_3, so F has a measurable selection (see Theorem III.6 [13]). Now we show that the operator Ω satisfies the assumptions of *Lemma 3.2.4*. To show that $\Omega(u) \in \mathcal{P}_{cl}C([0,1] \times \mathbb{R})$, for each $u \in C([0,1] \times \mathbb{R})$, let $\{v_n\}_{n \geq 0} \in \Omega(u)$ be such that $v_n \to v$ $(n \to \infty)$ in $C([0,1] \times \mathbb{R})$. Then $v \in C([0,1] \times \mathbb{R})$, and there exists $w_n \in S_{F,u}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} v_n(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w_n(s)\, ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right] \int_0^1 (1-s) w_n(s)\, ds \\ &\quad -\frac{1}{2}\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^\eta (\eta - s)^2 w_n(s)\, ds, \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that w_n converges to w in $L^1([0,1] \times \mathbb{R})$. Thus, $w \in S_{F,u}$ and, for each $t \in [0,1]$,

$$\begin{aligned} v_n(t) &\to v(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\, ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right] \int_0^1 (1-s) w(s)\, ds \\ &\quad -\frac{1}{2}\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^\eta (\eta - s)^2 w(s)\, ds, \end{aligned}$$

Hence, $v \in \Omega(u)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega u, \Omega \overline{u}) \leq \gamma \|u - \overline{u}\| \text{ for each } u, \overline{u} \in C([0,1] \times \mathbb{R}).$$

Let $u, \overline{u} \in C([0,1] \times \mathbb{R})$, and $h_1 \in \Omega(u)$. Then there exists $v_1(t) \in S_{F,u}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= -\frac{1}{2}\int_0^t (t-s)^2 v_1(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s) v_1(s)\,ds \\ & -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 v_1(s)\,ds, \end{aligned}$$

By H_4, we have

$$H_d(F(t,u), F(t,\overline{u})) \leq p(t)\,|u(t) - \overline{u}(t)|.$$

So, there exists $w \in S_{F,\overline{u}}$ such that

$$|v_1 - w| \leq p(t)\,|u - \overline{u}|,\ \ t \in [0,1].$$

Define $\mathcal{U} : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{U}(t) = \{w \in \mathbb{R} : |v_1 - w| \leq p(t)\,|u(t) - \overline{u}(t)|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}(t) \cap F(t, \overline{u}(t))$ is measurable (Proposition III.4 [13]), there exists a function $v_2(t)$ which is a measurable selection for $\mathcal{V}$. So $v_2(t) \in S_{F,\overline{u}}$, and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq p(t)\,|u(t) - \overline{u}(t)|$. For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) &= -\frac{1}{2}\int_0^t (t-s)^2 v_2(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s) v_2(s)\,ds \\ & -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 v_2(s)\,ds, \end{aligned}$$

$$\begin{aligned} h_1(t) &= -\frac{1}{2}\int_0^t (t-s)^2 v_1(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s) v_1(s)\,ds \\ & -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 v_1(s)\,ds, \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{2}\int_0^t (t-s)^2 |v_1(s) - v_2(s)|\,ds \\ & +\frac{1}{2}\left|t^2 + \eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right|\int_0^1 (1-s)\,|v_1(s) - v_2(s)|\,ds \\ & +\frac{1}{2}\left|\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right|\int_0^\eta (\eta-s)^2 |v_1(s) - v_2(s)|\,ds \\ &\leq \left[1 + \frac{1+\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\int_0^1 p(s)\,|u(s) - \overline{u}(s)|\,ds, \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le \left[1 + \frac{1+\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\|p\|_{L^1}\|u-\overline{u}\|.$$

Analogously, interchanging the roles of u and $\overline{u}$, we obtain

$$\begin{aligned} H_d(\Omega(u), \Omega(\overline{u})) &\le \gamma\|u-\overline{u}\| \\ &\le \left[1 + \frac{1+\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\|p\|_{L^1}\|u-\overline{u}\|, \end{aligned}$$

Since Ω is a contraction, it follows by *Therem 3.2.4*, that Ω has a fixed point u which is a solution of the problem ($\mathfrak{PF}\mathfrak{1}$). This completes the proof. ■

3.2.2 The convex case

Our results are based on the nonlinear alternative of Leray-Schauder type.

Theorem 3.2.3 *Assume that* $(H_5), (H_6)$ *and* (H_7) *hold. Then the boundary value problem* ($\mathfrak{PF}\mathfrak{1}$) *has at least one solution on* $[0,1]$.

Proof. Define the operator $T : C([0,1], \mathbb{R}) \to \mathcal{P}(C[0,1], \mathbb{R})$ by

$$\begin{aligned} T(u) = \Bigg\{ h \in C([0,1], \mathbb{R}) : h(t) = &-\frac{1}{2}\int_0^t (t-s)^2 f(u)\,ds + \\ &+\frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s) f(u)\,ds - \\ &-\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 f(u)\,ds \Bigg\}, \end{aligned}$$

for $f \in \mathcal{S}_{F,u}$, we will show that T satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps.

Step 1: we show that T is convex for each $u \in C([0,1], \mathbb{R})$.

Let $h_1, h_2 \in Tu$. Then there exist $w_1, w_2 \in \mathcal{S}_{F,u}$ such that, for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) = &-\frac{1}{2}\int_0^t (t-s)^2 w_i(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s) w_i(s)\,ds - \qquad i = 1,2. \\ &-\frac{1}{2}\frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 w_i(s)\,ds, \end{aligned}$$

Let $0 \leq \mu \leq 1$. So, for each $t \in [0,1]$, we have

$$\begin{aligned} \mu h_1(t) + (1-\mu) h_2(t) &= \frac{1}{2}\int_0^t (t-s)^2 (\mu w_1(s) + (1-\mu) w_2(s))\, ds + \\ &+ \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)(\mu w_1(s) + (1-\mu) w_2(s))\, ds - \\ &- \frac{1}{2}\frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 (\mu w_1(s) + (1-\mu) w_2(s))\, ds. \end{aligned}$$

Since $\mathcal{S}_{F,u}$ is convex, it follows that $\mu h_1 + (1-\mu) h_2 \in Tu$.

Step 2: we show that T maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

For a positive number r, let $B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \leq r\}$ be a bounded ball in $C([0,1],\mathbb{R})$ .So, for each $h \in Tu$, $u \in B_r$, there exists $w \in \mathcal{S}_{F,u}$ such that

$$\begin{aligned} h(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w(s)\, ds + \frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s) w(s)\, ds - \\ &- \frac{1}{2}\frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 w(s)\, ds, \end{aligned}$$

$$\begin{aligned} |h(t)| &\leq \frac{\psi(\|u\|)}{2}\left[2 + \eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\right]\int_0^1 p(s)\, ds + \\ &+ \frac{\psi(\|u\|)}{2}\eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\int_0^\eta p(s)\, ds, \end{aligned}$$

Thus,

$$\begin{aligned} \|h\| &\leq \frac{\psi(\|u\|)}{2}\left[2 + \eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\right]\int_0^1 p(s)\, ds + \\ &+ \frac{\psi(\|u\|)}{2}\eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\int_0^\eta p(s)\, ds. \end{aligned}$$

Step 3: Now we show that T maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.

Let $t_1, t_2 \in [0,1]$,with $t_1 < t_2$ and B_r be a bounded set of $C([0,1],\mathbb{R})$. So we obtain, for each $h \in Tu$, we obtain sends bounded sets of $C([0,1],\mathbb{R})$ into equicontinuous sets.

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq \frac{1}{2}\int_{t_1}^{t_2}(t_2-s)^2|w(s)|\,ds+\\
&+\frac{1}{2}\left[\left(t_2^2-t_1^2\right)+\eta^2\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\right]\int_0^1(1-s)|w(s)|\,ds+\\
&+\frac{1}{2}\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\int_0^\eta(\eta-s)^2|w(s)|\,ds+\\
&+\frac{1}{2}\int_0^{t_1}\left((t_1-s)^2-(t_2-s)^2\right)|w(s)|\,ds,\\
&\leq \frac{\psi(\|u\|)}{2}\int_{t_1}^{t_2}(t_2-s)^2p(s)\,ds+\\
&+\frac{1}{2}\left[\left(t_2^2-t_1^2\right)+\eta^2\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\right]\psi(\|u\|)\int_0^1(1-s)p(s)\,ds+\\
&+\frac{\alpha(t_2-t_1)\psi(\|u\|)}{2|1-\alpha\eta-\beta|}\int_0^\eta(\eta-s)^2p(s)\,ds+\\
&+\frac{\psi(\|u\|)}{2}\int_0^{t_1}\left((t_1-s)^2-(t_2-s)^2\right)p(s)\,ds.
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. As T satisfies the above three assumptions, it follows by the Ascoli-Arzela's theorem that $T : C([0,1],\mathbb{R}) \to P(C[0,1],\mathbb{R})$ is completely continuous.

Step 4: we show that T has a closed graph.

Let $u_n \to u_*$, $h_n \in T(u_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in Tu_*$.

Associated with $h_n \in T(u_n)$, there exists $w_n \in \mathcal{S}_{F,u_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) &= -\frac{1}{2}\int_0^t(t-s)^2w_n(s)\,ds+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)w_n(s)\,ds-\\
&-\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta(\eta-s)^2w_n(s)\,ds,
\end{aligned}$$

Thus we have to show that there exists $w_* \in \mathcal{S}_{F,u_*}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
h_*(t) &= -\frac{1}{2}\int_0^t(t-s)^2w_*(s)\,ds+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)w_*(s)\,ds-\\
&-\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta(\eta-s)^2w_*(s)\,ds,
\end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} w \quad \to \quad & \Theta w(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w(s)\,ds - \\ & -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 w(s)\,ds, \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \left\| -\frac{1}{2}\int_0^t (t-s)^2 (w_n(s) - w_*(s))\,ds + \right. \\ & +\frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)(w_n(s) - w_*(s))\,ds - \\ & \left. -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 (w_n(s) - w_*(s))\,ds \right\|. \end{aligned}$$

then $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$.

Thus, it follows by *Lemma 3.2.2* that $\Theta \circ \mathcal{F}$ is a closed graph operator.

Further, we have $h_n(t) \in \Theta(S_{F,u_n})$. Since $u_n \to u_*$, therefore, we have

$$\begin{aligned} h_*(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w_*(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w_*(s)\,ds - \\ & -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 w_*(s)\,ds. \end{aligned}$$

for some $w_* \in S_{F,u_*}$.

Step 5: we discuss a priori bounds on solutions.

Let u be a solution of $(\mathfrak{P}\mathfrak{F}1)$. So there exists $w \in L^1([0,1],\mathbb{R})$ with $w \in S_{F,u}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w(s)\,ds - \\ & -\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 w(s)\,ds, \end{aligned}$$

In view of (H_6), for each $t \in [0,1]$, we obtain

$$|u(t)| \le \psi(\|u\|)\left[1 + \eta^2 \frac{\alpha + |\beta|}{|1-\alpha\eta-\beta|}\right]\int_0^1 p(s)\,ds.$$

Consequently, we have

$$\frac{\|u\|}{\psi(\|u\|)\left[1+\eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\right]\|p\|_{L^1}} \leq 1.$$

In view of (H_7), there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in C([0,1],\mathbb{R}) : \|u\| < M+1\}.$$

Note that the operator $T : \overline{U} \to \mathcal{P}C([0,1],\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda Tx$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [27], we deduce that T has a fixed point $u \in \overline{U}$ which is a solution of the problem $(\mathfrak{PF}1)$. This completes the proof. ■

3.3 Examples

The case, when F is convex valued:

Example 3.3.1 *Consider the boundary value problem given by*

$$-u'''(t) \in F(t,u(t)),\ t \in (0,1), \tag{3.6}$$

$$u'(0) = u'(1) = \frac{1}{2}u\left(\frac{1}{2}\right),\ u(0) = \frac{1}{4}u\left(\frac{1}{2}\right), \tag{3.7}$$

where $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$u \to F(t,u) = \left[\sqrt{\frac{|u|+1}{|u|+2}}+\frac{1}{2}t^2+\frac{3}{2}, \frac{u^2}{u^2+1}+\exp(-t)-1\right]$$

For $f \in F$, *we have*

$|f| \leqslant \max\left(\sqrt{\frac{|u|+1}{|u|+2}}+\frac{1}{2}t^2+\frac{3}{2}, \frac{u^2}{u^2+1}+\exp(-t)-1\right) \leqslant 3,\ u \in \mathbb{R},$

Thus,

$\|F(t,u)\|_{\mathcal{P}} = \sup\{|w| : w \in F(t,u)\} \leq 3 = p(t)\psi(\|u\|),\ u \in \mathbb{R},$

with $p(t) = 1, \psi(\|u\|) = 3$. *Further,using the condition*

$$\left[1+\eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\right]\psi(M)\|p\|_{L^1} < M,$$

we find that $M > \frac{33}{8}$. *By* Theorem 3.2.3, *the the boundary value problem* (3.6) *and* (3.7) *has at least one solution on* $[0,1]$.

The case, when F is not necessarily convex valued:

Example 3.3.2 *If* $F(t,u) = \left[\frac{2\exp(u)}{3+\exp(u)}, t^3+t+2\right]$, *then the condition of* Theorem 3.2.1 *hold, with* $m(t) = t^3+t+2$.

Example 3.3.3 *If* $F(t,u) = \left[\frac{|u|}{1+|u|} + \frac{1}{2}t, \frac{2|u|^3}{|u|^3+1}\right] \cup \left\{\frac{|u-1|}{2+|u|} + \exp(t)\right\}$, *then the condition of* Theorem 3.2.1 *hold, with* $m(t) = e^t + 1$.

CHAPTER 4

The Topological Structure Of The Solutions Set For $(\mathfrak{PF}2)$

In this chapter, we prove that the $W^{3,1}([0,1],E)$-solution set of $(\mathfrak{PF}2)$ is compact and is retract in $C^1([0,1])$, where F is closed valued mapping.

4.1 The equation

This chapter is motivated by a recent paper [28], where it is considered problem $(\mathfrak{PF}2)$ with $F(.,.)$ single valued and several existence results are obtained by using fixed point techniques. It is a continuation of the work in [28]. Here we deal with some topological properties of the solution set for a $m-$point $(m>3)$ third order boundary value problem $(\mathfrak{PF}2)$ in a separable Banach space E of the form

$$\begin{aligned} &-u'''(t) \in F(t,u(t),u'(t)),\ t\in(0,1), \\ &u(0)=u'(0)=0,\ u'(1)=\textstyle\sum_{i=1}^{m-2}\alpha_i u'(\eta_i). \end{aligned} \tag{$\mathfrak{PF}2$}$$

with the following assumption

Assumption (A) Let $m>3$ be an integer number, $0<\eta_1<\eta_2<...<\eta_{m-2}<1$ and

$\alpha_i \in \mathbb{R}, (i = 1, 2, ..., m-2)$ satisfying the condition

$$1 - \sum_{i=1}^{m-2} \alpha_i \eta_i \neq 0,$$

and F is a closed valued mapping.

Under suitable compactness conditions on F we prove the compactness of the solution set of $(\mathfrak{PF}2)$ in $C^1_E([0,1])$ when F is convex compact valued and satisfies a Lipschitz condition and a compactness condition. Using a result due to Ricceri [47] on contractive multivalued mapping in a Banach space and these conditions on F, we also show that the solution set of problem $(\mathfrak{PF}2)$ is a retract in $C^1([0,1], E)$.

4.2 Notations and Preliminaries

Let E is a Banach space and E' its dual space, $\overline{B}_E$ is the closed unit ball of E, $\mathcal{L}([0,1])$ is the $\sigma-$algebra of Lebesgue measurable sets on $[0,1]$. $\lambda = dt$ is the Lebesgue measure on $[0,1]$, $\mathcal{B}(E)$ is the $\sigma-$algebra of Borel subsets of E.

Let $L^1([0,1], E)$, the space of all Lebesgue-Bochner integrable $E-$valued functions defined on $[0,1]$, and let $C([0,1], E)$ be the Banach space of all continuous functions u from $[0,1]$ into E endowed with the sup-norm and let $C^1([0,1], E)$ be the Banach space of all functions $u \in C([0,1], E)$ with continuous derivative, equipped with the norm

$$\|u\|_{C^1} = \max\left\{\max_{t\in[0,1]} \|u\|, \max_{t\in[0,1]} \|u'\|\right\}.$$

We also denote the space of all continuous functions in $C([0,1], E)$ such that their first derivatives are continuous and their second weak derivatives belong to $L^1([0,1], E)$ by $W^{3,1}([0,1], E)$. By $\mathcal{P}_0(E), \mathcal{P}_{comp}(E)$ and $\mathcal{P}_b(E)$, we denote the collection of all nonempty closed subsets, nonempty compact subsets and nonempty bounded closed subsets of E, respectively. If A is a subset of E then $|A| = \sup\{\|x\| : x \in A\}$ and $d(x, A) = \inf\{\|x - y\| : y \in A\}$, is the distance of a point $x \in E$ to A. The Hausdorff distance between two subsets A and B of E is $d_H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$.

We recall here some results which are directly applicable in the next sections. We begin with a lemma that summarize some properties of the Green function associated with the $m-$points boundary conditions.

Lemma 4.2.1 *Let the Assumption* (**A**) *hold. Let E be a separable Banach space and let $G : [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by*

$$G(t,s) = \frac{1}{2}\begin{cases} (1-s)\,t^2 + \mu^* t^3 \Psi_1(s), & t \leqslant s \\ (-s+2t-t^2)\,s + \mu^* t^2 (1-t)\,\Psi_2(s), & s \leqslant t \end{cases}$$

where

$$\Psi_1(s) = \sum_{i=1}^{m-2} \alpha_i (1-s) \quad \textit{and} \quad \Psi_2(s) = \sum_{i=1}^{m-2} \alpha_i s$$

and

$$\mu^* = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i}$$

Then the following assertions hold

(**i**) *For every fixed $s \in [0,1]$, the function $G(.,s)$ is right derivable on $[0,1[$ and left derivable on $]0,1]$. Its derivative is given by*

$$\left(\frac{\partial G}{\partial t}\right)_+ (t,s) = \begin{cases} (1-s)\,t + \frac{3}{2}\mu^* t^2 \Psi_1(s), & t \leqslant s \\ (1-t)\,s - \mu^* t\,(3t-2)\,\Psi_2(s), & s \leqslant t \end{cases}$$

$$\left(\frac{\partial G}{\partial t}\right)_- (t,s) = \begin{cases} (1-s)\,t + \frac{3}{2}\mu^* t^2 \Psi_1(s), & t \leqslant s \\ (1-t)\,s - \mu^* t\,(3t-2)\,\Psi_2(s), & s \leqslant t \end{cases}$$

This implies that $G(.,s)$is derivable on the intervals $[0,s]$ and $[s,1]$.

(**ii**) $G(.,.)$ and $\frac{\partial G}{\partial t}(.,.)$ satisfies

$$|G(t,s)| \leqslant M_G \quad \text{and} \quad \left|\frac{\partial G}{\partial t}(t,s)\right| \leqslant M_G \quad \forall (t,s) \in [0,1] \times [0,1]$$

where

$$M_G = 1 + \frac{3}{2}|\mu^*| \sum_{i=1}^{m-2} |\alpha_i|$$

(iii) If $u \in W^{3,1}([0,1],E)$ with $u(0) = u'(0) = 0$, and $u'(1) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i)$, then

$$u(t) = \int_0^1 G(t,s) u'''(s)\, ds, \quad \forall t \in [0,1].$$

(iv) Let $f \in L^1([0,1],E)$ and let $u_f : [0,1] \to E$ be the function defined by

$$u_f(t) = \int_0^1 G(t,s) f(s)\, ds, \quad \forall t \in [0,1].$$

Then we have

$$u_f(0) = u'_f(0) = 0, \text{ and } u'_f(1) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i)$$

Further the function u_f is derivable on $[0,1]$ and its derivative u'_f is defined by

$$u'_f(t) = \lim_{h \to 0} \frac{u_f(t+h) - u_f(t)}{h} = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)\, ds$$

(v) If $f \in L^1([0,1],E)$, the function u'_f is scalarly derivable, that is, for every $x^* \in E'$, the scalar function $\left\langle x^*, u'_f(.) \right\rangle$ is derivable and

$$-u'''_f(t) = f(t) \qquad \text{a.e. } t \in [0,1].$$

Lemma 4.2.2 *Let the Assumption* $(\mathbf{A})$ *hold and let* $f \in C([0,1],E)$ *(resp.* $f \in L^1([0,1],E)$*). Then the* $m-$*point boundary problem*

$$-u'''(t) = f(t), \ t \in (0,1), \tag{4.1}$$

$$u(0) = u'(0) = 0, \ u'(1) = \textstyle\sum_{i=1}^{m-2} \alpha_i u'(\eta_i). \tag{4.2}$$

has a unique $C^3([0,1],E)$*-solution (resp.* $W^{3,1}([0,1],E)$*-solution) defined by*

$$u(t) = \int_0^1 G(t,s) f(s)\, ds, \quad \forall t \in [0,1].$$

Theorem 4.2.1 *Let E be a Banach space, let X be a nonempty convex closed subset of E, and let φ be a contractive multivalued map with convex closed values from X into itself. Then the set*

$$Fix(\varphi) = \{x \in X : x \in \varphi(x)\}$$

is an absolute retract.

Proof. See Ricceri [47]. ■

4.3 Topological Properties of the Solutions Set

4.3.1 Compactness of the solutions set in $C^1([0,1],E)$

Theorem 4.3.1 *Let* $(\mathbf{A})$ *hold. Let* $\Gamma : [0,1] \to \mathcal{P}(E)$ *be a convex compact valued, measurable and integrably bounded multifunction. Let* $F : [0,1] \times E \times E \to \mathcal{P}_0(E)$ *be a convex compact valued multifunction satisfying the following conditions:*

$(\mathbf{A1})$ *F is* $\mathcal{L}([0.1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)-$ *measurable.*

$(\mathbf{A2})$ *There exist positive functions* $l_1, l_2 \in L^1([0,1],\mathbb{R})$ *with* $M_G||l_1+l_2||_{L^1} < 1$ *such that*

$$\begin{aligned}|d(z_1,F(t,x_1,y_1)) - d(z_2,F(t,x_2,y_2))| &\leq ||z_1-z_2|| + l_1(t)||x_1-x_2|| \\ &\quad + l_2(t)||y_1-y_2||\end{aligned}$$

for all $(t,x_1,y_1,z_1),(t,x_2,y_2,z_2) \in [0,1] \times E \times E \times E.$

$(\mathbf{A3})$ $F(t,x,y) \subset \Gamma(t)$ *, for all* $(t,x,y) \in [0,1] \times E \times E.$

Then the $W^{3,1}([0,1],E)$*-solution set,*$(\mathfrak{SPF}_2)$ *, of the problem* $(\mathfrak{PF}_2)$ *is compact in* $C^1([0,1],E)$*.*

Proof. Step 1. Clearly $\tau \to G(t,\tau)\Gamma(\tau)$ is a convex compact valued, measurable multifunction. Since Γ is integrably bounded we have

$$G(t,\tau)\Gamma(\tau) \subset G(t,\tau)\,|\Gamma(t)|\,\overline{B}_E,$$

that is, $\tau \to G(t,\tau)\Gamma(\tau)$ is also integrably bounded. By *Proposition 6.2.3* in [11] we deduce that the multivalued integral $\int_0^1 G(t,s)\Gamma(s)ds$ is norm compact in E. Similarly, it's not difficult to check that $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s)ds$ is norm compact in E.

Step 2. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $W^{3,1}([0,1],E)$-solutions in $(\mathfrak{SPF}_2)$, so

$$u_n(t) = \int_0^1 G(t,s)u_n'''(s)\,ds, \tag{4.3}$$

$$u_n'(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)u_n'''(s)\,ds \tag{4.4}$$

and

$$-u_n'''(t) \in F\left(t, u_n(t), u_n'(t)\right) \subset \Gamma(t), a.e.t \in [0,1]. \tag{4.5}$$

It's easy to see that $(u_n''')_{n\in\mathbb{N}}$ is uniformly integrable using the estimates

$$\begin{aligned}
\|u_n'''\| &\leqslant \sup\{\|z\| : z \in \Gamma(t)\} \leqslant |\Gamma(t)|, \\
\|u_n'(t)\| &\leqslant \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| \|u_n'''(s)\| \, ds \\
&\leqslant M_G \int_0^1 |\Gamma(s)| \, ds < \infty.
\end{aligned}$$

Let $t, \tau \in [0,1]$. It follows from (4.3) that

$$\begin{aligned}
\|u_n(t) - u_n(\tau)\| &= \int_0^1 |G(t,s) - G(\tau,s)| \, \|u_n'''(s)\| \, ds \\
&\leqslant \int_0^1 |G(t,s) - G(\tau,s)| \, |\Gamma(s)| \, ds. \qquad (4.6)
\end{aligned}$$

On the other hand, by the definition of the Green function G we have

$$G(t,s) - G(\tau,s) = G(t,s) = \frac{1}{2}\begin{cases} (1-s)\left(t^2-\tau^2\right) + \mu^*\left(t^3-\tau^3\right)\Psi_1(s), & t \leqslant s \\ -s\left((t-\tau)(t+\tau-2)\right) + \left[\left(t^2-t^3\right) + \left(\tau^3-\tau^2\right)\right]\mu^*\Psi_2(s), & s \leqslant t \end{cases} \tag{4.7}$$

Combining (**4.6**) and (**4.7**), it is not difficult to check that $\{u_n : n \in \mathbb{N}\}$ is equicontinuous in $C([0,1], E)$. Further, for each $t \in [0,1]$, the set $\{u_n : n \in \mathbb{N}\}$ is relatively compact in E because it is included in the norm compact set $\int_0^1 G(t,s)\Gamma(s)ds$. So by Ascoli's theorem, $\{u_n : n \in \mathbb{N}\}$ is relatively compact in $C^1([0,1], E)$.

Similarly, by using the properties of $\frac{\partial G}{\partial t}$ tand the relations

$$u_n'(t) - u_n'(\tau) = \int_0^1 \left(\frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(\tau,s)\right) u_n'''(s) \, ds,$$

$$\frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(\tau,s) = \begin{cases} \left((t-\tau)\left(1-s+\frac{3}{2}(t+\tau)\right)\right)\mu^*\Psi_1(s), & t \leqslant s \\ (\tau-t)s + (t-\tau)\left(3(t+\tau)+2\right)\mu^*\Psi_2(s), & s \leqslant t \end{cases}$$

we deduce that $\{u_n' : n \in \mathbb{N}\}$ is equicontinuous in $C_E([0,1])$. In addition, for each $t \in [0,1]$, the set $\{u_n' : n \in \mathbb{N}\}$ is included in $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s)ds$ ds which is a compact subset of E. So $\{u_n' : n \in \mathbb{N}\}$ is relatively compact in $C^1([0,1], E)$ using the Ascoli's theorem.

From the above results, we deduce that there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ still denoted by $(u_n)_{n\in\mathbb{N}}$ which converges uniformly to $u_\infty \in C^1([0,1])$ with $u_\infty(0) = u'_\infty(0) = 0$, $u'_\infty(1) = \sum_{i=1}^{m-2} \alpha_i u'_\infty(\eta_i)$. Furthermore, (u'_n) converges uniformly to u'_∞ and (u'''_n) weakly converges in $L^1([0,1],E)$ to $w_\infty \in L^1([0,1],E)$. For every $x' \in E'$ and for every $t \in [0,1]$, we have

$$\begin{aligned}
\langle x', u_\infty(t)\rangle &= \lim_{n\to\infty} \langle x', u_n(t)\rangle \\
&= \lim_{n\to\infty} \left\langle x', \int_0^1 G(t,s)u'''_n(s)\,ds \right\rangle \\
&= \lim_{n\to\infty} \int_0^1 \langle G(t,s)x', u'''_n(s)\rangle\,ds \\
&= \int_0^1 \langle G(t,s)x', w_\infty(s)\rangle\,ds \\
&= \left\langle x', \int_0^1 G(t,s)w_\infty(s)\,ds \right\rangle
\end{aligned}$$

This implies that $u_\infty(t) = \int_0^1 G(t,s)w_\infty(s)\,ds$, for a. e $t \in [0,1]$. Using the *Lemma 4.2.1 (v)* we get

$$-u'''_\infty(t) = w_\infty(t) \text{ for a.e } t \in [0,1]$$

Using (4.5) and the same arguments as in ([12]; *Corollary 5.1*), involving the lower semi continuity of integral functional for strong- weak topology in on $L^1([0,1],E) \times L^1([0,1],E)$ (see [11], Theorem 8.1.6), we conclude that $-u'''_\infty(t) \in F(t, u_\infty(t), u'_\infty(t))$, a.e. $t \in [0,1]$. The proof of our theorem is complete. ■

4.3.2 Retract of the solutions set in $C^1([0,1],E)$

Theorem 4.3.2 *Let* (**A**) *hold. Let* $\Gamma : [0,1] \to \mathcal{P}_{c,cv}(E)$ *be a convex compact valued,measurable and integrably bounded multifunction. Let* $F : [0,1] \times E \times E \to \mathcal{P}_{c,cv}(E)$ *be a convex compact valued satisfying the conditions* (**A1**) – (**A2**) *as in* Theorem 4.3.1 *and*

(**B1**) $F(t,x,y) \subset \Gamma(t)$ *, for all* $(t,x,y) \in [0,1] \times R\overline{B}_E \times \overline{B}_E$*. where* $R = M_G \int_0^1 |\Gamma(s)|\,ds$*.*

Then the $W^{3,1}([0,1],E)$*-solution set of the problem* $(\mathfrak{PF}2)$*is a retract in* $C^1([0,1],E)$*.*

Proof. *In the following we denote the set* $\left\{ y \in C^1([0,1],E) : \|y\|_{C^1([0,1],E)} \le R \right\}$ *by* $\mathcal{D}$*.*

For $u \in \mathcal{D}$, we put

$$\mathcal{M}(y) = \left\{ f \in L^1([0,1], E) : f(t) \in F(t, y(t), y'(t)), a.e. t \in [0,1] \right\} \subset S^1_\Gamma$$

and

$$\mathcal{N}(y) = \{u_f : f \in \mathcal{M}(y)\}$$

where $u_f(t) = \int_0^1 G(t,s) f(s)\, ds$, is the unique solution of the problem

$$-u'''(t) = f(t), \ \ a.e.\ \ t \in (0,1),$$
$$u(0) = u'(0) = 0,\ \ u'(1) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i).$$

It is easy to see that $\mathcal{M}(y)$ a is non-empty, convex closed and bounded subset of $L^1([0,1], E)$. Moreover $\mathcal{M}(y)$ is weakly compact in $L^1([0,1], E)$ using the weakly compactness of S^1_Γ (the set of all integrable selections of Γ).
For $f \in \mathcal{M}(y)$, we have

$$f(t) \in F(t, y(t), y'(t)) \subset \Gamma(t) \ \ a.e.\ \ t \in (0,1).$$

So

$$\|u_f(t)\| = \left\| \int_0^1 G(t,s) f(s)\, ds \right\| \leq M_G \left\| \int_0^1 f(s)\, ds \right\| \leq R$$

This implies that $\mathcal{N}(y) \subset \mathcal{D}$.
On the other hand it is easy to see that $\mathcal{N}(y)$ is a non-empty and convex subset of $C^1([0,1], E)$. Moreover by Theorem 4.3.1, *$\mathcal{N}(y)$ is compact in $C^1([0,1], E)$. So $\mathcal{N}$ defines a non-empty, convex and compact valued multifunction from $\mathcal{D}$ into itself. Let $y_1, y_2 \in \mathcal{D}$. We need to prove that there exists $\alpha \in (0,1)$ satisfying*

$$H_{d^*}(\mathcal{N}(y_1), \mathcal{N}(y_2)) \leq \alpha \|y_1 - y_2\|_{C^1([0,1],E)}, \tag{4.8}$$

where $H_{d^*}(\cdot,\cdot)$ is the Hausdorff distance on the space of compact subsets of $C^1([0,1], E)$. Let $z_1 \in \mathcal{N}(y_1)$ be arbitrary. Then $z_1 = u_{f_1}$ for some $f_1 \in \mathcal{M}(y_1)$. By using a standard measurable selection theorem, there exists a Lebesgue-measureable $f_2 : [0,1] \to E$ such that

$$f_2(t) \in F(t, y_2(t), y_2'(t)), \quad \forall t \in [0,1]$$

and

$$\|f_1(t) - f_2(t)\| = d\left(f_1(t), F(t, y_2(t), y_2'(t))\right), \quad \forall t \in [0,1]$$

As $f_1 \in \mathcal{M}(y_1)$ *we have*

$$\begin{aligned}\|f_1(t) - f_2(t)\| &= H_d\left(F(t, y_1(t), y_1'(t)), F(t, y_2(t), y_2'(t))\right) \\ &\leq l_1(t)\|y_1(t) - y_2(t)\| + l_2(t)\|y_1'(t) - y_2'(t)\| \\ &\leq (l_1(t) + l_2(t))\|y_1 - y_2\|_{C_E^1([0,1])}.\end{aligned}$$

This implies that $f_2 \in \mathcal{M}(y_2)$ *. So we have*

$$\begin{aligned}\|u_{f_1}(t) - u_{f_2}(t)\| &= \left\|\int_0^1 G(t,s)\left(f_1(s) - f_2(s)\right)ds\right\| \\ &\leq M_G \int_0^1 \|f_1(s) - f_2(s)\|\, ds \\ &\leq M_G \|l_1 + l_2\|_{L_E^1([0,1])} \|y_1 - y_2\|_{C_E^1([0,1])}\end{aligned}$$

and

$$\begin{aligned}\left\|u_{f_1}'(t) - u_{f_2}'(t)\right\| &= \left\|\int_0^1 \frac{\partial G}{\partial t}(t,s)\left(f_1(s) - f_2(s)\right)ds\right\|, \\ &\leq M_G \|l_1 + l_2\|_{L_E^1([0,1])} \|y_1 - y_2\|_{C_E^1([0,1])}.\end{aligned}$$

Hence

$$\|u_{f_1} - u_{f_2}\|_{C_E^1([0,1])} \leq M_G \|l_1 + l_2\|_{L_E^1([0,1])} \|y_1 - y_2\|_{C_E^1([0,1])},$$

and consequently

$$d\left(u_{f_1}, \mathcal{N}(y_2)\right) \leq M_G \|l_1 + l_2\|_{L_E^1([0,1])} \|y_1 - y_2\|_{C_E^1([0,1])}.$$

Whence we get

$$\sup_{z_1 \in \mathcal{N}(y_1)} d\left(z_1, \mathcal{N}(y_2)\right) \leq M_G \|l_1 + l_2\|_{L_E^1([0,1])} \|y_1 - y_2\|_{C_E^1([0,1])}.$$

From this and the analogous inequality obtained by interchanging the roles of y_1 *and* y_2 *we obtain* (4.8) *with* $\alpha = M_G \|l_1 + l_2\|_{L^1([0,1],E)}$ *. By* Theorem 4.2.1, $Fix(\mathcal{N})$ *is a retract of* $C^1([0,1], E)$ *. On the other hand, it is clear that* $Fix(\mathcal{N})$ *is also the solutions set of the problem* $(\mathfrak{PF}2)$*. The proof of the theorem is complete.* ∎

BIBLIOGRAPHY

[1] Albiac. F,Kalton. N. J, Topics in Banach Space Theory, Graduate texts in mathematics, 233. Switzerland : Springer, 2016.

[2] Aubin. J. P and Cellina. A, Differential Inclusions, Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[3] Aubin. J. P and Frankowska. H, Set-Valued Analysis, Birkhauser,Boston, 1990.

[4] Ayoola. E. O, Quantum stochastic differential inclusions satisfying a general Lipschitz condition, Dynamic Systems and Applications, vol. 17, no. 3-4, pp. 487–502, 2008.

[5] Benaim. M, Hofbauer. J, and Sorin. S, Stochastic approximations and differential inclusions. II. Applications, Mathematics of Operations Research, vol. 31, no. 4, pp. 673–695, 2006.

[6] Bressan. A and Colombo. G, Boundary value problems for lower semicontinuous differential inclusions, Funkcial. Ekvac. 36 1993, 359-373.

[7] Bressan.A and Colombo.G, Extensions and selections of maps with decomposable values, Studia Math., 90(1) (1988), 69–86.

[8] Bressan. A and Colombo. G, Generalized Baire category and differential inclusions in Banach spaces, J. Differential Equations 76 1987, 135-158.

[9] Brown. R. F, A Topological Introduction to Nonlinear Analysis, Third Edition, Springer International Publishing Birkhäuser, 2014.

[10] Brown. R. F, Furi.M, Georniewicz. L and Jiang B. , Handbook of Topological Fixed Point Theory, Springer, Dordrecht, 2005.

[11] Castaing. C, Raynaud de Fitte. P, Valadier. M, Young Measures on Topological Spaces with Applications in Control Theory and Probability Theory. Kluwer Academic Publishers, Dordrecht (2004).

[12] Castaing. C,Truong. L. X, Second order differential inclusions with m-points boundary condi tions. J. Nonlinear Convex Anal. 12(2), 199–224 (2011).

[13] Castaing. C and Valadier. M, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, Springer, Berlin, 1977.

[14] Chang.Y.K. , Li.W.T.and Nieto.J. J, Controllability of evolution differential inclusions in Banach spaces, Nonlinear Analysis: Theory, Methods & Applications, vol. 67, no. 2, pp. 623–632, 2007.

[15] Colombo. G and Goncharov. V, The sweeping processes without convexity, Set-Valued Anal. 7 1999, 357-374.

[16] Colombo. G and Monterio M. M. D. P, Sweeping by a continuous prox-regular set, J. Differential Equations 187 2003, 46-62.

[17] Covitz. H and Nadler. S. B, Multivalued contraction mappings in generalized metric spaces, Journal of Mathematics 8 (1970), 5–11.

[18] Deimling. K, Multivalued Differential Equations, de Gruyter Series in Nonlinear Analysis and Applications, vol. 1, Walter de Gruyter, Berlin, 1992.

[19] DiBenedetto. E, Real Analysis, Birkhauser Boston 2002.

[20] D. R. Smart, Fixed Point Theorems, Cambridge University Press, London, 1974.

[21] Engelking. R, General topology, Heldermann Verlag, 1989 - 529 pages.

[22] Frigon. M, Application de la théeorie de la transversalite topologique 'a des problèmes non linéaires pour des équations differentielles ordinaires, Dissertationes Mathematicae 296 (1990), 75.

[23] Frigon. M, Granas, A.: Theoremes d'existence pour des inclusions differentielles sans convexite, C. R. Acad. Sci. Paris, Ser. I., 310, (1990), 819–822.

[24] Fryszkowski. A, continuous selection for a class non-convex multivalued maps, Studia Math. 76 (1983), 163-174.

[25] Fryszkowski. A, Georniewicz. L, Mixed semicontinuous mappings and their applications to differential inclusions, Set-Valued Anal. 8 2000, 203-217.

[26] Géorniewicz. L, Topological Fixed Point Theory of Multivalued Mappings, Mathematics and Its Applications, vol. 495, Kluwer Academic, Dordrecht, 1999.

[27] Granas. A and Dugundji. J, Fixed Point Theory, Springer Monographs in Mathematics, Springer, New York, 2003.

[28] Guezane-Lakoud. A, Zenkoufi. L, Existence of Positive Solutions for a Third-Order Multi-Point Boundary Value Problem, Applied Mathematics, 2012, 3, 1008-1013.

[29] Hadj-Moussa. M, Topologies sur les Hyper-espaces Consonance et Hyperconsonance, Thèse Doctorat Mathématique; Université de Rouen; 22 Juin 1995.

[30] Hausdorff. H, Erweiterung einer Homöomorphie. Fund. Math, 16, (1930), p.883-886.

[31] Hu. S. and Papageorgiou.N, Handbook of Multivalued Analysis, Volume I: Theory, Kluwer, Dordrecht, 1997.

[32] Heikkila. S and Lakshmikantham.V, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Monographs and Textbooks in Pure and Applied Mathematics, vol. 181, Marcel Dekker, New York, 1994.

[33] Kamenskii. M, Obukhovskii.V and Zecca.P, Condensing multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, Walter de Gruyter & Co. Berlin, 2001.

[34] Katetov. M, Correction to On real valued frrnctions in topological space, Fund. Math., 40 (1953), pp. 203-205.

[35] Katetov. M, On real valued functions in topological spaces, Fund. Math., 38 (1951), pp. 85-91.

[36] Kyritsi. S, Matzakos.N and Papageorgiou.N. S, Nonlinear boundary value problems for second order differential inclusions, Czechoslovak Math. J. 55 2005, 545-579.

[37] Kunze. M, (2000). Non-Smooth Dynamical Systems. Springer Science & Business Media. ISBN 978-3-540-67993-6.

[38] Lasota. A and Opial. Z, An Application of the Kakutani-Ky Fan Theorem in the Theory of Ordinary Differential Equations, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques 13 (1965), 781–786.

[39] Li. W. S, Chang. Y. K, and Nieto. J. J, Solvability of impulsive neutral evolution differential inclusions with state-dependent delay, Mathematical and Computer Modelling, vol. 49, no. 9-10, pp. 1920–1927, 2009.

[40] Michael. E, Continuous Selections. I, The Annals of Mathematics, 2nd Ser., Vol. 63, No. 2. (Mar., 1956), pp. 361-382.

[41] Neveu. J, Discrete Parameter Martingales, North-Holland, Amsterdam 1975.

[42] Ntouyas S. K, Neumann boundary value problems for impulsive differential inclusions, Electronic Journal of Qualitative Theory of Differential Equations, no. 22, 13 pages, 2009.000.

[43] Parthasarathy.T, Selection Theorems and their Applications, Lecture Notes in Mathematics, Volume 263 1972.

[44] Remco. I, Nijmeijer.H (2013). Dynamics and Bifurcations of Non-Smooth Mechanical Systems. Springer Science & Business Media. p. V (preface). ISBN 978-3-540-44398-8.

[45] Rezaiguia. A and Kelaiaia.S, existence of a positive solution for a third-order three point boundary value problem, Matematicki Vesnik 68, 1 (2016), 12–25.

[46] Rezaiguia.A and Kelaiaia.S, existence results for third-order differential inclusion with three point boundary value problems, Acta Math. Univ. Comenianae Vol. LXXXV, 2 (2016), pp. 311–318.

[47] Ricceri. B, Une propriété topologique de l'ensemple des points fixes d'une contraction multivoque a valeurs convexes. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. 81(8), 283–286 (1987).

[48] Showalter. R. E, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, By the American Mathematical Society, (1997).

[49] Simsen. J. and Gentile.C. B, Systems of p-Laplacian differential inclusions with large diffusion, J. Math. Anal. Appl., 368(2) (2010), 525–537.

[50] Smirnov. G. V, Introduction to the Theory of Differential Inclusions, vol. 41 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, USA, 2002.

[51] Sun. J. P, A new existence theorem for right focal boundary value problems on a measure chain, Applied Mathematics Letters 18 (2005), no. 1, 41–47.38.

[52] TONG. H, Sonre characterizations of normal and perfectly normal spaces, Duke Math. J., 19 (1952), pp. 289-292.

[53] TONG. H, Some characterizations of norrnal and perfectly normal spaces, Bull. hmer. Math. Soc., 54 (1948), Abstract 46, p. 65.

[54] Tolstonogov. A. A, Differential Inclusions in Banach Spaces, Kluwer Academic Publishers, Dordrecht, 2

[55] Yosida. K, Functional Analysis, 6th ed., Fundamental Principles of Mathematical Sciences, vol. 123, Springer, Berlin, 1980.