وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR –ANNABA
UNIVERSITY
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جامعة باجي مختار
- عنابة -

Faculté des Sciences
Département de Mathématiques

THESE

Présentée en vue de l'obtention du diplôme de
DOCTORAT EN MATHEMATIQUES
Option : Analyse Numérique

A Posteriori Error Estimates combined with the asymptotic behavior for a Generalized Domain Decomposition Method in Evolutionary PDEs

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Année Universitaire 2015 / 2016

Acknowledgements

First of all, I would like to thank Allah for giving me time, strength and patience to finish this work.
Then, I would like to express my sincere gratitude to my supervisors, **Dr. Salah Boulaares** and **Pr. Mohamed Haiour** for their guidance, advice and continuous support for the work presented in this manuscript would not have been possible without the close association of them.
Sincerely, I have learned a lot from them, may Allah compensate them.
I would like also to thank deeply **Prof. Hocine Sissaoui** who accepted to preside my thesis jury.
Special thanks go to **Pr. Mahmoud Abdel-Aty**, **Dr. Mohamed Cherif Bouras** and **Pr. Bachir Djebber** for their acceptance to be members of my thesis committee.
I should express my sincere and unlimited thanks to my family who have been patient with me.
Without their support and prayers, this work would have not been accomplished.
Also, I would like to thank all the members of the mathematics department and all those who helped me to complete this work.

Contents

ملخص
في هذه المذكرة قدمنا التقدير البعدي لخطأ طريقة النطاقات المتداخلة المعممة مع شروط Dirichlet الحدية على الواجهات المطبقة على مسألة تطورية من الدرجة الثانية وذلك باستعمال مخطط نصف ضمني زمني جنبا الى جنب مع تقريب طريقة العناصر المنتهية.
و علاوة على ذلك قدمنا نتيجة للسلوك التقاربي في النظيم المنتظم وذلك باستعمال خوارزمية Benssoussan-Lions.

Abstract.

In this thesis, a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the interfaces, for parabolic variational equation with second order boundary value problems are derived using the semi-implicit time scheme combined with a finite element spatial approximation.
Furthermore a result of asymptotic behavior in uniform norm is given using Benssoussan-Lions' Algorithm.

Key words.
A posteriori error estimates; GODDM; Robin Condutions, PE.

Résumé:
Dans cette thèse, une estimations d'erreur a posteriori pour la méthode de décomposition de domaine généralisée avec recouvrement et des conditions aux limites de Dirichlet sur les interfaces, d'une équation variationnelle parabolique de second ordre, a été établie en utilisant le schéma de temps semi-implicite combiné avec une approximation par éléments finis spatiale.
En outre, un résultat de comportement asymptotique pour la norme uniforme est donnée en utilisant l'algorithme de Benssoussan-Lions.

Mots clés: Estimations d'erreur a posteriori, décomposition de domaine généralisée avec recouvrement, conditions aux limites de type Robin, EDP.

Notation

Ω : bounded domain in $\mathbb{R}^2$.

Γ : topological boundary of Ω.

$x = (x_1, x_2)$: generic point of $\mathbb{R}^2$.

$dx = dx_1 dx_2$: Lebesgue measuring on Ω.

∇u : gradient of u.

Δu : Laplacien of u.

$D(\Omega)$: space of differentiable functions with compact support in Ω.

$D'(\Omega)$: distribution space.

$C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.

$L^p(\Omega)$: space of functions p-th power integrated on with measure of dx.

$\|f\|_p = \left(\int_\Omega \left(|f|^P \right) \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \{u \in L^p(\Omega), \ \nabla u \in L^p(\Omega)\}$.

H : Hilbert space.

$H_0^1(\Omega) = W_0^{1,2}$.

If X is a Banach space

$$L^p(0, \ T; \ X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable; } \int_0^T \|f(t)\|_X^p \ dt < \infty \right\}.$$

$$L^\infty(0, \ T; \ X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable; } ess - \sup_{t \in [0, \ T]} \|f(t)\|_X^p \ < \infty \right\}.$$

$C^k([0, \ T]; \ X)$:Space of functions $k-$times continuously differentiable for $[0, \ T] \longrightarrow X$.

$D([0, \ T]; \ X)$: space of functions continuously differentiable with compact support in $[0, \ T]$.

Introduction

The thesis deals with a posteriori error estimates in H^1-norm for the generalized overlapping domain decomposition method for the following parabolic equation:

$$\text{find } u\,(t,x) \in L^2\,(0,T,D\,(\Omega)) \cap C^{\,2}\left(0,T,H^{-1}\,(\Omega)\right)\text{such that}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au - f = 0, \text{ in } \Sigma, \\ \\ u = 0 \text{ in } \Gamma, \; u\,(x,0) = u_0 \text{ in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, with sufficiently smooth boundary Γ and Σ is a set in $\mathbb{R} \times \mathbb{R}^2$ defined as $\Sigma = [0,\ T] \times \Omega$ with $T < +\infty$. A is the elliptic operator defined by

$$A = -\Delta + \alpha$$

and the functions $\alpha \in L^2\,(0,T,L^\infty\,(\Omega)) \cap C^{\,0}\,(0,T,L^\infty\,(\Omega))$ are sufficiently smooth and satisfy

$$\alpha \geq \beta > 0,\ \beta \text{ is a constant,}$$

and where $f\,(.)$ is a linear satisfies

$$f \in L^2\,(0,T,L^2\,(\Omega)) \cap C^{\,1}\,(0,T,H^{-1}\,(\Omega))\,,$$

$$f > 0 \text{ and increasing.}$$

The symbol $(.,.)_\Omega$ stands for the inner product in $L^2(\Omega)$.

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. It was invented by Herman Amandus Schwarz in 1890. This method has been used for solving the stationary or evolutionary boundary value problems on domains which consist of two or more overlapping subdomains (see [7], [8], [11], [20]–[23], [25], [27]–[38]). The solution to these qualitative problems is approximated by an infinite sequence of functions resulting from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomains. An extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in [12]–[15], [17], [27]. Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics, has been demonstrated in many papers. See the proceedings of the annual domain decomposition conference [18] and [26]–[30], [32]–[34], [31]. Moreover, a priori estimates of the errors for stationary problems is given in several papers; see for instance [29], [30] where a variational formulation of the classical Schwarz method is derived. In [26], geometry-related convergence results are obtained. In [17, 20, 22], an accelerated version of the GODDM has been treated. In addition, in [17], convergence for simple rectangular or circular geometries has been studied. However, a criterion to stop the iterative process has not been given. All these results can also be found in the recent books on domain decomposition methods [8], [25]. Recently in [20], [22], an improved version of the Schwarz method for highly heterogeneous media has been presented. The method uses new optimized boundary conditions specially designed to take into account the heterogeneity between the subdomains on the boundaries. A recent overview of the current state of the art on domain decomposition methods can be found in [1], [31].

In general, the a priori estimate for stationary problems is not suitable for assessing the quality of the approximate solutions on subdomains, since it depends mainly on the exact solution itself, which is unknown. An alternative approach is to use an approximate solution itself in order to find such an estimate. This approach, known as a posteriori estimate, became very popular in the 1990s with finite element methods; see the monographs [1], [39]. In [39], an algorithm for a nonoverlapping domain decomposition has been given. An a posteriori error analysis for the elliptic case has also been used by [1] to determine an optimal value of the penalty parameter for penalty domain decomposition methods for constructing fast solvers.

In [4], the authors derived a posteriori error estimates for the Generalized Overlapping Domain Decomposition Method (GODDM) with Robin boundary conditions on the boundaries for second order boundary value problems; they have shown that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the boundaries after a discretization of the domain by finite elements method. Also they used the techniques of the residual a posteriori error analysis to get an a posteriori error estimate for the discrete solutions on subdomains.

A numerical study of stationary and evolutionary mathematical problems of the finite element, combined with a finite difference, methods has been achieved in [4], [11]–[22], [35] and using the domain decomposition method combined with finite element method, has been treated in [8, 11, 9, 25]. In the reference [25], S.Boulaaras and M.Haiour treated the overlapping domain decomposition method combined with a finite element approximation for elliptic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions for Laplace operator Δ, where a maximum norm analysis of an overlapping Schwarz method on nonmatching grids has been used. Then, in [11] they extended the last result to the parabolic quasi variational inequalities with the similar conditions, and using the theta time scheme combined with a finite element spatial approximation, we have proved that the discretization on every subdomain converges in uniform norm. Furthermore, a result of asymptotic behavior in uniform norm has been given.

In this work, we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of parabolic equation with linear source terms using the Euler time scheme combined with a finite element spatial approximation, similar to that in [4], which investigated Laplace equation. Moreover, an asymptotic behavior in H_0^1-norm is deduced using Benssoussan–Lions' algorithms.

In the next completed work, this study can be extended by the parabolic variational equation with nonlinear source terms with similar presented conditions in the boundaries and the right hand side of the problem is contraction mapping using the new time-space discretization which is the theta time scheme combined with a finite element approximation.

The outline of the thesis is as follows:

In the first chapter, we introduce some necessary notations and we lay down some fundamental definitions and theorems on functional analysis, which will be needed some them in the body of the thesis.

In second chapter some numerical analysis of elliptic boundary value problems are given. We will explain in this chapter the main numerical methods that will be used later. Then we introduce the domain decomposition method (DDM, in short) for an elliptic boundary value problem.

In the third chapter an a posteriori error estimate is proposed for the convergence of the discrete solution using Euler time scheme combined with a finite element method on subdomains. Then, we associate with the introduced discrete problem a fixed point mapping and use that in proving the existence of a unique discrete solution. Finally, an $H_0^1(\Omega)$-asymptotic behavior estimate for each subdomain is derived.

Chapter 1

Preliminary and functional analysis

In this chapter we shall introduce and state some necessary materials needed in the proof of our results, and shortly the basic results which concerning the Banach spaces, Hilbert space, the L^p space, Sobolev spaces and other theorems. The knowledge of all these notations and results are important for our study.

1.1 Banach Spaces - Definition and Properties

We first review some basic facts from calculus in the most important class of linear spaces the "Banach spaces".

Definition 1.1.1 *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$.*

Proposition 1.1.1 *([40]) X' equipped with the norm*

$$\|f\|_{X'} = \sup \{|f(u)| : \|u\|_X \leq 1\},$$

is also a Banach space.

Definition 1.1.2 *Let X be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim_{n\longrightarrow\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \longrightarrow u$, or $\lim_{n\longrightarrow\infty} u_n = u$

Definition 1.1.3 *A sequence* (u_n) *in* X *is weakly convergent to* u *if and only if*

$$\lim_{n\longrightarrow\infty} f(u_n) = f(u),$$

for every $f \in X'$ *and this is denoted by* $\lim_{n\longrightarrow\infty} u_n = u$.

1.1.1 Banach fixed-point theorem

Definition 1.1.4 *Let* (X, d) *be a metric space. Then a map* $T : X \to X$ *is called a contraction mapping on* X *if there exists* $q \in [0, 1)$ *such that*

$$d(T(x), T(y)) \leqslant qd(x, y),$$

for all x, y *in* X.

Theorem 1.1.1 *([40]) Let* (X, d) *be a non-empty complete metric space with a contraction mapping* $T : X \to X$*. Then* T *admits a unique fixed-point* x^* *in* X *(i.e.* $T(x^*) = x^*$*). Furthermore,* x^* *can be found as follows:*
start with an arbitrary element x^0 *in* X *and define a sequence* $\{x_n\}$ *by* $x_n = T(x_{n-1})$.
Then $x_n \longrightarrow x^*$.

1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equations turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. Then, we must give some important results on these spaces here.

Definition 1.2.1 *A Hilbert space* H *is a vectorial space supplied with inner product* (u, v) *such that* $\|u\| = \sqrt{(u,\ u)}$ *is the norm which let* H *complete.*

(**The Cauchy-Schwarz inequality**) Every inner product satisfies the Cauchy-Schwarz inequality

$$|(x_1,\ x_2)| \leq \|x_1\| \, \|x_2\| \, .$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Corollary 1.2.1 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence which converges to* u*, in the weak topology and* $(v_n)_{n\in\mathbb{N}}$ *is an other sequence which converge weakly to* v*, then*

$$\lim_{n\longrightarrow\infty} (v_n, u_n) = (v, u).$$

Theorem 1.2.1 (Lax-Milgram) ([33]) *Let* V *be a real Hilbert space,* $L(.)$ *a continuous linear form on* V*,* $a(\cdot,\cdot)$ *a continuous coercive bilinear form on* V*. Then the problem*

$$\begin{cases} \text{find } u \in V \text{ such hat} \\ a(u,v) = L(v) \text{ for every } v \in V. \end{cases}$$

has a unique solution. Further, this solution depends continuously on the linear form L*.*

1.3 Functional Spaces

1.3.1 The $L^p(\Omega)$ spaces

Now we define Lebesgue spaces and collect some properties of these spaces. We consider $\mathbb{R}^2$ with the Lebesgue-measure μ.
If $\Omega \subset \mathbb{R}^2$ is a measurable set, two measurable functions $f, g : \Omega \longrightarrow \mathbb{R}$ are called equivalent, if $f = g$ a.e. (almost everywhere) in Ω.
An element of a Lebesgue space is an equivalence class.

Definition 1.3.1 *Let* $1 \leq p < \infty$*, and let* Ω *be an open domain in* $\mathbb{R}^n$*,* $n \in \mathbb{N}^*$*. Define the standard Lebesgue space* $L^p(\Omega)$*, by*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ is measurable; } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

Notation 1.3.1 *For* $p \in \mathbb{R}$*, and* $1 \leq p < \infty$ *denote by*

$$\|f\|_p = \left(\int_\Omega |f(t)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p=\infty$, *we have*

$$L^{\infty}(\Omega)=\left\{\begin{array}{l} f:\Omega\longrightarrow\mathbb{R}\ \text{is measurable and there exist a constant } C, \\ \text{such that, }\ ;\ |f(t)|<C\ \ a.e\ in\ \Omega. \end{array}\right\}$$

Also, we denote by

$$\|f\|_{\infty}=\inf\ \{C,\ |f(t)|<C\ \ a.e\ in\ \Omega\}.$$

Theorem 1.3.1 *([40])* $\left(L^{p}(\Omega),\|.\|_{p}\right),(L^{\infty}(\Omega),\|.\|_{\infty})$ *are a Banach spaces.*

Remark 1.3.1 *In particularly, when* $p=2$, $L^{2}(\Omega)$ *equipped with the inner product*

$$(f,\ g)_{\Omega}=\int_{\Omega}f(x).g(x)\,dx,$$

is a Hilbert space.

1.3.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also very useful in our next chapters.

Theorem 1.3.2 *([40]) (Hölder's inequality)*
Let $1\leq p<\infty$. *Assume that* $f\in L^{p}(\Omega)$ *and* $g\in L^{q}(\Omega)$, *then,* $fg\in L^{1}(\Omega)$ *and*

$$\int_{\Omega}|f.g|\ \ dx\leq\|f\|_{p}\ \ \|g\|_{q}\ .$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

Lemma 1.3.1 *(Minkowski inequality)*
For $1\leq p<\infty$, *we have*

$$\|u+v\|_{p}\leq\|u\|_{p}+\|v\|_{p}.$$

1.4 The Sobolev space $W^{m,\ p}(\Omega)$

Proposition 1.4.1 *Let Ω be an open domain in $\mathbb{R}^N$, then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$(T, \varphi) = \int_{\Omega} f(x)\, g(x)\ dx,\ \textit{for all}\ \varphi \in D(\Omega).$$

where $1 \leq p < \infty$, and it is well-known that f is unique.

Definition 1.4.1 *Let $m \in \mathbb{N}^*$ and $p \in [0,\ \infty[$. The $W^{m,\ p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,\ p}(\Omega) = \left\{ \begin{array}{l} f \in L^p(\Omega),\ \textit{such that}\ \partial^\alpha f \in L^p(\Omega)\ \textit{for all}\ \alpha \in \mathbb{N}^m\ \textit{such that,} \\ \\ |\alpha| = \sum_{j=1}^{n} \alpha_j \leq m,\ \textit{where},\ \partial^\alpha = \partial_1^{\alpha_1}\ \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}. \end{array} \right\}$$

Theorem 1.4.1 *$W^{m,\ p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,\ p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p\ \textit{for all}\ f \in W^{m,\ p}(\Omega).$$

Definition 1.4.2 *When $p = 2$, we prefer to denote by $W^{m,\ 2}(\Omega) = H^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$(u,\ v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_{\Omega} \partial^\alpha u \partial^\alpha v dx.$$

Definition 1.4.3 *$H_0^m(\Omega)$ is given by the completion of $D(\Omega)$ with respect to the norm $\|.\|_{H^m(\Omega)}$.*

Remark 1.4.1 *Clearly $H_0^m(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^m(\Omega)}$.*
The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega) := [H_0^m(\Omega)]^*$.

Lemma 1.4.1 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$D(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow D'(\Omega).$$

Now the smoothness of the boundary $\partial\Omega := \overline{\Omega} - \Omega$ can be described:

Definition 1.4.4 *Let Ω be an open subset of $\mathbb{R}^d$, $0 \leqslant \lambda \leqslant 1$, $m \in \mathbb{N}$. We say that its boundary that its boundary $\partial\Omega$ is of class $C^{m;\lambda}$ if the following conditions are satisfied:*
For every $x \in \partial\Omega$ there exist a neighborhood V of x in $\mathbb{R}^d$ and new orthogonal coordinates $\{y_1, ..., y_d\}$ such that V is a hypercube in the new coordinates:

$$V = \{(y_1, ..., y_d) : -a_i < y_i < a_i, i = 1, ...d\}$$

and there exists a function $\varphi \in C^{m;\lambda}(V')$ with

$$V' = \{(y_1, ..., y_{d-1}) : -a_i < y_i < a_i, i = 1, ...d-1\}$$

and such that

$$\begin{aligned} \left|\varphi(y')\right| &\leqslant \frac{1}{2}a_d, \quad \forall y' := (y_1, ..., y_{d-1}) \in V', \\ \Omega \cap V &= \left\{(y', y_d) \in V : y_d < \varphi(y')\right\}, \\ \partial\Omega \cap V &= \left\{(y', y_d) \in V : y_d = \varphi(y')\right\}. \end{aligned}$$

A boundary of class $C^{0;1}$ is called Lipschitz boundary.

1.5 The $L^p(0,\ T;\ X)$ spaces

Definition 1.5.1 *Let X be a Banach space, denote by $L^p(0,\ T; X)$ the space of measurable functions*

$$f :]0,\ T[\longrightarrow X$$

$$t \longrightarrow f(t),$$

such that

$$\int_0^T (\|f(t)\|_X^p)^{\frac{1}{p}}\, dt = \|f\|_{L^p(0,\ T;X)} < \infty.$$

If $p = \infty$

$$\|f\|_{L^\infty(0,\ T;X)} = \sup_{t\ \in]0,\ T[} ess\, \|f(t)\|_X\, .$$

Theorem 1.5.1 *The space* $L^p(0,\ T;X)$ *is a Banach space.*

Lemma 1.5.1 *Let* $f \in L^p(0,\ T;X)$ *and* $\dfrac{\partial f}{\partial t} \in L^p(0,\ T;X)\, ,\ (1 \le p \le \infty)\, ,$ *then, the function* f *is continuous from* $[0,\ T]$ *to* X. *i. e.* $f \in C^1(0,\ T;X)\, .$

1.6 Sobolev spaces of fractional order and trace theorems

In this section let $\Omega \subset \mathbb{R}^d$ is a measurable set with Lipschitz boundary $\partial\Omega$. The boundary $\partial\Omega$ of Ω will be denoted by $\Gamma := \partial\Omega$.

On the $(d-1)$-dimensional set it is also possible to define Sobolev spaces :

Definition 1.6.1 $H^{\frac{1}{2}}(\Gamma)$ *is defined by*

$$H^{\frac{1}{2}}(\Gamma) := \left\{ u \in L^2(\Gamma) : |u|_{\frac{1}{2},\Gamma} < \infty \right\}$$

where the seminorm $|.|_{\frac{1}{2},\Gamma}$ *is given by*

$$|u|_{\frac{1}{2},\Gamma} := \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|}{|x-y|^d} ds(x) ds(y), \quad u \in H^{\frac{1}{2}}(\Gamma).$$

Theorem 1.6.1 *([24])* $H^{\frac{1}{2}}(\Gamma)$ *with the scalar product*

$$(u,v)_{\frac{1}{2},\Gamma} := \int_\Gamma uv ds + \int_\Gamma \int_\Gamma \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^d} ds(x) ds(y),$$

is a Hilbert space.

Definition 1.6.2 *Let* $\Gamma_1 \subset \Gamma$ *be a proper, connected* $(d-1)$*-dimensional relative open subset. Then we define*

$$H^{\frac{1}{2}}(\Gamma_1) := \left\{ u \in L^2(\Gamma_1) : \exists \widetilde{u} \in H^{\frac{1}{2}}(\Gamma) \text{ with } u = \widetilde{u}\,|_{\Gamma 1} \right\},$$

with norm

$$\|u\|_{\frac{1}{2},\Gamma_1} := \inf_{\substack{\widetilde{u} \in H^{\frac{1}{2}}(\Gamma) \\ \widetilde{u}\,|_{\Gamma_1} = u}} \|\widetilde{u}\|_{\frac{1}{2},\Gamma}\, , \qquad u \in H^{\frac{1}{2}}(\Gamma_1).$$

Now we construct a particular subspace of $H^{\frac{1}{2}}(\Gamma_1)$. For $v \in H^{\frac{1}{2}}(\Gamma_1)$ the zero extension of v into $\Gamma - \Gamma_1$ will be denoted by $\widetilde{v}$. So we can define:

Definition 1.6.3 $H_{00}^{\frac{1}{2}}(\Gamma_1)$ *is defined by*

$$H_{00}^{\frac{1}{2}}(\Gamma_1) := \left\{ v \in L^2(\Gamma_1) : \widetilde{v} \in H^{\frac{1}{2}}(\Gamma) \right\}.$$

Notice that

$$(u,v)_{H_{00}^{\frac{1}{2}}(\Gamma_1)} := (u,v)_{\frac{1}{2},\Gamma_1} + \int\limits_{\Gamma_1} \frac{uv}{\rho(x,\partial\Gamma_1)} ds(x),$$

where $\rho(x,\partial\Gamma_1)$ is a positive function which behaves like the distance between x and $\partial\Gamma_1$, defines a scalar product in $H_{00}^{\frac{1}{2}}(\Gamma_1)$.

Remark 1.6.1 *([24]) By a direct calculation, for all $v \in L^2(\Gamma_1)$ we obtain tow positive constants c_1, c_2 such that:*

$$c_1 \|v\|_{\frac{1}{2},\Gamma_1} \leqslant \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} \leqslant c_2 \|v\|_{\frac{1}{2},\Gamma_1}.$$

Therefore $H_{00}^{\frac{1}{2}}(\Gamma_1)$ is a Hilbert space.

The dual of these spaces are denoted by

$$H^{-\frac{1}{2}}(\Gamma_1) := \left[H_{00}^{\frac{1}{2}}(\Gamma_1)\right]^*, \qquad H_{00}^{-\frac{1}{2}}(\Gamma_1) := \left[H^{\frac{1}{2}}(\Gamma_1)\right]^*.$$

Next we present some trace theorems.
Let be $u \in C(\overline{\Omega})$. Then we can define the trace of u on $\partial\Omega$:

$$\gamma_0(u) := u\,|_{\partial\Omega}\,.$$

This trace operator can be extended:

Theorem 1.6.2 *([24]) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with boundary $\partial\Omega \in C^{0;1}$. Then the trace mapping γ_0 defined on $C^{0;1}(\overline{\Omega})$ extends uniquely to a bounded, surjective linear map:*

$$\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega).$$

Moreover the right inverse of the trace operator exists:

Theorem 1.6.3 *([24]) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary $\partial\Omega$. Then there exists a linear bounded operator*

$$\begin{aligned} E &: H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega), \quad \textit{such hat} \\ \gamma_0(E(\varphi)) &= \varphi, \quad \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega). \end{aligned}$$

Note that the preceding theorems allow the definition of the following equivalent norm on $H^{\frac{1}{2}}(\partial\Omega)$:

$$\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)} := \inf_{\substack{u \in H^1(\Omega) \\ \gamma_0(u) = u}} \|u\|_{H^1(\Omega)}, \qquad \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega).$$

Sometimes the simpler notation $u\,|_{\partial\Omega} = \gamma_0(u)$ is used for functions $u \in H^1(\Omega)$.
With the trace operator γ_0 we can characterize the space $H_0^1(\Omega)$:

Theorem 1.6.4 *([24]) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with boundary $\partial\Omega \in C^{0;1}$. Then $H_0^1(\Omega)$ is the kernel of trace operator γ_0, i.e,*

$$\begin{aligned} H_0^1(\Omega) &= N(\gamma_0) = \left\{u \in H^1(\Omega) : \gamma_0(u) = 0\right\} \\ &= \left\{u \in H^1(\Omega) : u\,|_{\partial\Omega} = 0\right\}. \end{aligned}$$

Definition 1.6.4 *Let Ω is an open smooth domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Gamma_D \subsetneq \partial\Omega$ such hat $\mathrm{mes}(\Gamma_D) > 0$. We set*

$$\begin{aligned} H^1_{\Gamma_D}(\Omega) &= \left\{u \in H^1(\Omega) : \gamma_0(u) = 0 \textit{ on } \Gamma_D\right\} \\ & \left\{u \in H^1(\Omega) : u\,|_{\Gamma_D} = 0\right\}. \end{aligned}$$

Lemma 1.6.1 *$H^1_{\Gamma_D}(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^1(\Omega)}$.*

1.6.1 Inequality of Poincaré

Now we cite a variant of the inequality of Poincaré. It allows to estimate the function values of functions $u \in H^1(\Omega)$ by the first derivatives of functions $u \in H^1(\Omega)$.

Theorem 1.6.5 *([35]) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Furthermore let $\Gamma_D \subset \Omega$ be a connected part of the boundary of Ω with $mes_{d-1}(\Gamma_D) > 0$. Then the inequality*

$$\|u\|_{0,\Omega} \leqslant C(\Omega, \Gamma_D)\ \ |u|_{1,\Omega}$$

is true for all $u \in H^1(\Omega)$ *with* $\gamma_0(u)\,|_{\Gamma_D} = 0$. *The constant* $C(\Omega, \Gamma_D)$ *depends only on* Ω *and* Γ_D *and is bounded by the diameter of* Ω.

Remark 1.6.2 *By he Inequality of Poincaré we deduce that the seminorme* $|.|_{1,\Omega}$ *is an equivalent norme to* $\|.\|_{1,\Omega}$ *in* $H^1_{\Gamma_D}(\Omega)$.

1.6.2 Green's formula

Proposition 1.6.1 *([33]) Let* Ω *be an open subset of* $\mathbb{R}^d$ *, with a Lipschitz boundary. Then for all* $u, v \in H^1(\Omega)$, *we have*

$$\int_{\Omega} \left(\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u\right)\, dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \qquad i = 1, ..., d.$$

where η_i *is the* i*-th component of the outward normal vector* η.

Chapter 2

Some Numerical Analysis of Elliptic Boundary Value Problems

2.1 Introduction

In mathematics, in the field of differential equations, a **boundary value problem** is a differential equation together with a set of additional constraints, called the boundary conditions:

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = g & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where Ω is an open domain in $\mathbb{R}^N$,and $\Gamma = \partial\Omega$ is the boundary of Ω.
A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. It's called the strong solution of the problem, and (2.1) is called the strong formulation of the problem.
Baesids the boundary condition, boundary value problems are also classified according to the type of differential operator involved. For an *elliptic operator*, one discusses *elliptic boundary value problems* and for an parabolic operator, one discusses parabolic boundary value problems.
In most cases it is not possible to find analytical solutions of these problems i.e. that the explicit computation of the exact solution of such equations is often impossible to achieve. Therefore, in general, the exact problem is approached by a discrete problem that can be solved by numerical methods.
We will explain in this chapter the main numerical methods that will be used later.

2.2 The variational approach and the finite element methods

The disadvantages of the strong formulation are mainly due to the excessive regularity that one requires for different settings, so often we introduce what is called "the weak formulation" or "the variational formulation" of the problem which is written as follows:

$$\begin{cases} \text{find } u \in V \text{ such hat} \\ a(u,v) = L(v) \text{ for every } v \in V. \end{cases} \tag{2.2}$$

The principle of the variational approach for the solution of PDEs is to replace the equation by an equivalent so-called variational formulation obtained by integrating the equation multiplied by an arbitrary function, called a test function. As we need to carry out integration by parts when establishing the variational formulation, we start by recalling the

Theorem 2.2.1 ***(Green's formula)*** *([2]) Let Ω be a regular open set of class C^1. Let u be a function of $C^2(\Omega)$ and v a function of $C^1(\Omega)$, both with bounded support in the closed set Ω. Then they satisfy the integration by parts formula*

$$\int_{\Omega} \Delta uv \; dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds - \int_{\Omega} \nabla u \nabla v \; dx,$$

where $\frac{\partial u}{\partial n} = \nabla u.n$, $\dfrac{\partial u}{\partial \eta}$ is the normal derivation of u at Γ.

2.2.1 The variational formulation of some boundary value problems

Here we consider two boundary value problems that we face in the third chapter.

1) Let Ω is an open smooth domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Gamma_D \subsetneq \partial\Omega$ such that $mes(\Gamma_D) > 0$, and let be given $F \in H^1_{\Gamma_D}(\Omega), g \in H^1(\partial\Omega - \Gamma_D)$,the function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative.

Find the $u \in H^1_{\Gamma_D}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$ such hat

$$\begin{cases} -\Delta u + \alpha u = f, & \text{in } \Omega. \\ u = 0, & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} + \alpha u = \frac{\partial g}{\partial n} + \alpha g, & \text{on } \partial\Omega - \Gamma_D. \end{cases} \tag{2.3}$$

We have : $\int_{\Omega} -\Delta uv \; dx + \int_{\Omega} \alpha uv.dx = \int_{\Omega} fv.dx, \;\; \forall v \in H^1_{\Gamma_D}(\Omega)$.

By applying the Green's formula in Laplacien we get

$$\begin{aligned}
\int_{\Omega} -\Delta u v dx &= (-\Delta u, v)_{\Omega} = (\nabla u, \nabla v)_{\Omega} - \left[\left(\frac{\partial u}{\partial \eta}, v \right)_{\partial\Omega - \Gamma_D} + \left(\frac{\partial u}{\partial \eta}, v \right)_{\Gamma_D} \right] \\
&= (\nabla u, \nabla v)_{\Omega} - \left(\frac{\partial u}{\partial \eta}, v \right)_{\partial\Omega - \Gamma_D} \\
&= (\nabla u, \nabla v)_{\Omega} - \left(\frac{\partial g}{\partial n} + \alpha g - \frac{\partial u}{\partial n}, v \right)_{\partial\Omega - \Gamma_D} \\
&= (\nabla u, \nabla v)_{\Omega} + \left(\frac{\partial u}{\partial n}, v \right)_{\partial\Omega - \Gamma_D} - \left(\frac{\partial g}{\partial n} + \alpha g, v \right)_{\partial\Omega - \Gamma_D} .
\end{aligned}$$

And thus we get

$$\int_{\Omega} \nabla u \nabla v \; dx + \int_{\Omega} \alpha u v dx + \int_{\partial\Omega - \Gamma_D} \frac{\partial u}{\partial n} v ds = \int_{\Omega} f v + \int_{\partial\Omega - \Gamma_D} (\alpha g + \frac{\partial g}{\partial n}) v dx, \quad \forall v \in H^1_{\Gamma_D}(\Omega) . \quad (2.4)$$

The problem (2.4) is called the weak formulation or the variational formulation of the problem (2.3) because by applying the Lax-Milgram theorem we can prove that the problem (2.4) has a unique solution.

This solution is called the weak solution of the problem (2.3).

2) Now we will write the variational formulation of the following parabolic equation find $u \in L^2(0, T; H^1_0(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f, & \text{in } \Sigma, \\ u = 0, & \text{in } \Gamma \times [0, T], \\ u(.,0) = u_0, & \text{in } \Omega. \end{cases} , \quad (2.5)$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0, T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.

The function $\alpha \in L^{\infty}(\Omega)$ is assumed to be non-negative; f is a regular function satisfying

$$f \in L^2(0, T, L^{\infty}(\Omega)) \cap C^1\left(0, T, H^{-1}(\Omega)\right) \text{ and } f \geq 0.$$

The idea is to write a variational formulation which resembles to the first order ordinary differential system

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = 0, & \text{for } t \geqslant 0, \\ \\ u(t=0) = u_0, & \text{in } \Omega. \end{cases}$$

where $u(t)$ is a function of class C^1 from $\mathbb{R}_+$ into $\mathbb{R}^n$, and $u_0 \in \mathbb{R}^n$ and A is a real symmetric positive definite matrix of order n.

It is well known that this problem has a unique solution obtained by diagonalizing the matrix A. (for more see [2]).

For this we multiply the equation by a test function $v(x)$ which does not depend on time t. Because of the boundary condition we shall demand that v is zero on the boundary of the open set Ω. We therefore obtain, under Green's formula

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)v(x)dx + \int_\Omega \nabla u(x,t)\nabla v(x)\ dx + \int_\Omega \alpha(x)u(x,t)v(x)dx = \int_\Omega f(x,t)v(x)dx, \quad \forall v \in H_0^1(\Omega).$$

Since neither Ω nor $v(x)$ vary with time t, we can rewrite this equation in the form

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)v(x)dx + \int_\Omega \nabla u(x,t)\nabla v(x)\ dx + \int_\Omega \alpha(x)u(x,t)v(x)dx = \int_\Omega f(x,t)v(x)dx, \quad \forall v \in H_0^1(\Omega).$$

Exploiting the fact that the variables x and t play very different roles, we separate these variables by considering from now on the solution $u(t,x)$ as a function of time t with values in a space of functions defined over Ω (likewise for $f(t,x)$).

More precisely, if we are given a final time $T > 0$ (possibly equal to $+\infty$), it is considered that u is defined by

$$u :]0,T[\longrightarrow H_0^1(\Omega) \diagdown t \longmapsto u(t).$$

and we continue to use the notation $u(x,t)$ for the value $[u(t)](x)$.

The choice of the space $H_0^1(\Omega)$ is obviously dictated by the nature of the problem and can vary from one model to another. Generally it is the space which is suitable for the variational formulation of the associated stationary problem. Likewise, the source term f is from now on, considered to be a function of with values in $L^2(\Omega)$.

We thus obtain the following variational formulation of (2.5) which is an ordinary differential equation in t :

find $u(t)$, a function of $]0, T[$, with values in $H_0^1(\Omega)$, more precisely find $u \in L^2(0,\ T;\ H_0^1(\Omega)) \cap C(0,\ T;\ L^2(\Omega))$such that

$$\begin{cases} \left(\frac{\partial}{\partial t} u(t), v\right)_\Omega + a(u(t), v) =, (f(t), v)_\Omega\ ,\ \forall v \in H_0^1(\Omega)\,,\ 0 < t < T, \\ u(t=0) = u_0. \end{cases}$$

where

$$a(u(t), v) = \int_\Omega \nabla u(x,t) \nabla v(x)\ dx.$$

For solving this variational formulation we have introduce a family of functional spaces of functions of t with values in the spaces of functions of x in the section 1.5.
The proof of the existence and uniqueness of the solution of this variational formulation is carefully presented in abstract general framework in ([2]) with serval remarks.

2.2.2 The finite element method

We present here, briefly, the method of finite elements which is the numerical method of choice for the calculation of solutions of elliptic boundary value problems, but is also used for parabolic or hyperbolic problems as we shall see.
The principle of this method comes directly from the variational approach that we have given some basic concept in the preceding subsection.
Naturally, the exact resolution of the problem (2.1) is generally not possible and it is conducted to find approximations of the it's strong solution.
This is where the finite element method is shown as a numeric reference method for the calculation of solutions of boundary value problems especially those elliptical.
At present the finite element method occupies a prominent place in the world of scientific computing.
The basic idea of this method is to replace the Hilbert space$V$ on which is placed the variational formulation of the problem (2.1), i.e problem (2.2), which is typically infinite-dimensional, by a finite dimension subspaceV_h, to have a finite number of unknowns or so-called "degrees of freedom" (which are the components of the approximate solution u_h in a baseV_h, then we define this approximate solution as the solution to the following problem :

$$\begin{cases} \text{find } u_h \in V_h \text{ such hat :} \\ a(u_h, v_h) = L(v_h), \qquad \forall v_h \in V_h, \end{cases}$$

which thus reduces to the resolution of a linear system whose matrix is called the stiffness matrix.

Thus, the construction of the spacesV_h is the keystone of the variational approximation method. The object of the finite element method is precisely to build a families spaces V_h which have suitable approximation properties and lead to a satisfactory digital implementation.

In practice the familiesV_h must represent an approximation of the space$V$, in the sense that the number of degrees of freedom can be as large as possible, so as to approach the exact solution of precisely as possible. In other word :

$$\lim_{h \to 0} \left[\inf_{v_h \in V_h} \|v - v_h\| \right] = 0.$$

There are three basic aspects in the construction of the space V_h :

(FEM 1) Triangulation of the domain:

Let us consider a bounded, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$.

Then we denote by $\tau_h = \{T_i : i = 1, 2, ..., n;\ n \in \mathbb{N}^*\}$ a family of partitions of Ω i.e.

$$T_i \cap T_j = \emptyset, \quad \forall T_i, T_j \in \tau_h \qquad\qquad \bar{\Omega} = \underset{T_i \in \tau_h}{\cup} \overline{T_i}.$$

To ensure the continuity of the discrete spaces, defined with the help of the partitions, we need the following additional condition:

Definition 2.2.1 *A partition of* Ω *is called* admissible, *if two elements* T_i, T_j *are either disjoint or share a complete* k*-face,* $0 \leqslant k \leqslant d - 1$.

Remark 2.2.1 *The condition of admissibility means, that there are no hanging nodes in* Ω.

Denoting h_T as the diameter of a simplex $T \in \tau_h$ and ρ_T as the diameter of the largest ball inscribed into T, and

$$h = \max_{T \in \tau_h} h_T$$

We can formulate another important property of the partition τ_h.

Definition 2.2.2 *A partition* τ_h *is called shape regular if there exists a positive constant* C *independent of* h*, such that*

$$\sigma_T = \frac{h_T}{\rho_T} \leqslant C, \quad \forall T \in \tau_h$$

Definition 2.2.3 *A partition τ_h is called is called quasi-uniform, if there is a constant $\tau > 0$, such that*

$$\max_{T \in \tau_h} h_T \geqslant \tau h.$$

Remark 2.2.2 *The first condition ensures that asymptotically the simplices do not degenerate.*
The meaning of quasi-uniformity is, that the size of all simplices of one partition is asymptotically equal up to a constant not depending on the parameter h.

(FEM 2) Finite Element spaces:

Now we can define an example of the Finite Element spaces:

$$X_h^k = \left\{ v \in H^1(\Omega) : v\,|_T \in P_k(T), \forall T \in \tau_h \right\},$$

where P_k is the set of polynomials of degree at most k.
Sometimes we need Finite Element spaces which vanish on a part of the boundary $\Gamma_D \subsetneq \partial\Omega$ such hat $mes(\Gamma_D) > 0$,

$$X_{h,\Gamma_D}^k = \left\{ v \in X_h^k : v\,|_{\Gamma_D} = 0 \right\}.$$

(For more details see [36]).

(FEM 3)

There exists a canonical basis forV_h whose functions have small support and can be easily described.
(We can see the details in [14]).

2.3 Domain Decomposition Methods

In this section we will introduce the domain decomposition method (DDM, in short).
In numerical partial differential equations, domain decomposition methods solve a boundary value problem by splitting it into smaller boundary value problems on subdomains and iterating to coordinate the solution between adjacent subdomains.
The basic idea behind DD methods consists in subdividing the computational domain Ω, on which a boundary-value problem is set, into two or more subdomains on which discretized

problems of smaller dimension are to be solved, with the further potential advantage of using parallel solution algorithms.

There are two ways of subdividing the computational domain into subdomains: one with disjoint subdomains, the others with overlapping subdomains. In non-overlapping methods, the closure of subdomains intersect only on their interface.

Overlapping domain decomposition methods include the original Schwarz alternating method and the additive Schwarz method.

Even without conjugate gradient acceleration, the multiplicative method can take many fewer iterations than the additive version. However, the multiplicative version is not as parallelizable.

We consider in fourth chapter the tow methods : the overlapping domain decomposition method, more precisely the additive Schwarz method, and the non-overlapping method. The local problems are linked together by suitable coupling terms or transmission conditions

2.3.1 The Schwarz alternating method

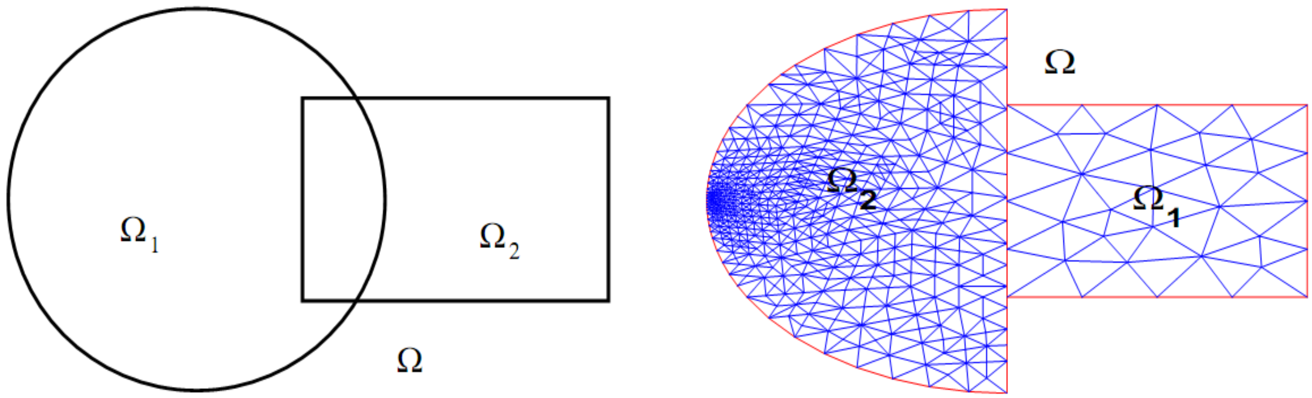
Hermann Schwarz was a German analyst of the 19th century. He was interested in proving the existence and uniqueness of the Poisson problem.

At his time, there were no Sobolev spaces nor Lax-Milgram theorem. The only available tool was the Fourier transform, limited by its very nature to simple geometries.

H.A. SCHWARZ in 1870, in order to consider more general situations, devised an iterative algorithm for solving Poisson problem set on a union of simple geometries : this is the alternating Schwarz method. (See figure 1)

The alternating Schwarz method, introduced by was probably the first example of a domain decomposition method. Starting with a decomposition into two overlapping subdomains decomposition into two overlapping subdomains and the equations are solved iteratively on the subdomains using Dirichlet values of the neighbor domains computed in the previous step. In this way H. Schwarz could show the existence of a solution of the Poisson problem for a domain with nonsmooth boundary.

Let the domain Ω be the union of a disk and a rectangle (see figure 1).



(a) The original example of H.A. SCHWARZ (b) Decomposition into simple domains

Figure 1: *The figure shows two simple decompositions. (a) is an overlapping decomposition. In (b) the meshes of Ω_1 and Ω_2 are nonmatching at the interface.*

Consider the Poisson problem which consists in finding $u : \Omega \longrightarrow \mathbb{R}$ such that:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 2.3.1 *(Original Schwarz algorithm, cf. [19]) The Schwarz algorithm is an iterative method based on solving alternatively sub-problems in domains Ω_1 and Ω_2.*
It updates $(u_1^m, u_2^m) \longrightarrow (u_1^{m+1}, u_2^{m+1})$ by

$$\begin{cases} -\Delta u_1^{m+1} = f, & \textit{in } \Omega_1 \\ u_1^{m+1} = u_2^m & \textit{on } \partial\Omega_1 \cap \overline{\Omega_2} \\ u_1 = 0 & \textit{on } \partial\Omega_1 \cap \partial\Omega. \end{cases}$$

Then,

$$\begin{cases} -\Delta u_2^{m+1} = f, & \textit{in } \Omega_2 \\ u_2^{m+1} = u_1^{m+1} & \textit{on } \partial\Omega_2 \cap \overline{\Omega_1} \\ u_2 = 0 & \textit{on } \partial\Omega_2 \cap \partial\Omega. \quad \square \end{cases}$$

H. Schwarz proved the convergence of the algorithm and thus the wellposedness of the Poisson problem in complex geometries.
With the advent of digital computers, this method also acquired a practical interest as an iterative linear solver.

Subsequently, parallel computers became available and a small modification of the algorithm (cf. [28]) makes it suited to these architectures.
We present this method in a general case :
Let given a model problem : find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} Lu = f \quad \text{in } \Omega, \\ \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

L being a generic second order elliptic operator, whose weak formulation reads

$$\text{find } u \in V = H_0^1(\Omega) \text{ such that } a(u,v) = (f,v), \quad \forall v \in V$$

being $a(\cdot,\cdot)$ the bilinear form associated with L.
Consider a decomposition of the domain Ω in two subdomains Ω_1 and Ω_2 such that

$$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}, \quad \Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset, \quad \partial\Omega_i \cap \Omega_j = \Gamma_i, \ i \neq j \ \text{ and } i,j = 1,2.$$

Consider the following iterative method. Given u_2^0 on Γ_1, solve the following problems for $m \in \mathbb{N}^*$

$$\begin{cases} Lu_1^m = f \quad \text{in } \Omega_1, \\ \\ u_1^m = u_2^{m-1} \text{ on } \Gamma_1 \\ \\ u_1^m = 0 \quad \text{on } \partial\Omega_1 - \Gamma_1, \end{cases}$$

and

$$\begin{cases} Lu_2^m = f \quad \text{in } \Omega_2, \\ \\ u_2^m = \begin{cases} u_1^{m-1} \\ \\ u_1^m \end{cases} \quad \text{on } \Gamma_2 \\ \\ u_2^m = 0 \quad \text{on } \partial\Omega_{22} - \Gamma_2, \end{cases} \tag{2.7}$$

In the case in which one chooses u_1^m on Γ_2 in (2.7) the method is named *multiplicative Schwarz* (MSM), it's algorithm is sequential. Whereas that in which we choose u_1^{m-1}, is named *additive*

Schwarz (ASM), problems in domains Ω_1 and Ω_2 may be solved concurrently. The reason of this appointment is clarified in ([35]).

Denoting the solution of iteration step i in subdomain Ω_j by u_j^i for the two-domain case the multiplicative variant can be described as follows : Starting with an initial guess, first a new solution in Ω_1 is computed. Then, already using this solution, the solution in Ω_2 is solved, and so on.

In contrast the additive algorithm uses the solution of the previous step instead of the current solution (cf. Figure 2). The second method has got the advantage that the solution of all subdomain problems can be completely done in parallel.

In the multi-domain case the multiplicative variant requires a coloring of the subdomains.

We have thus two elliptic boundary-value problems with Dirichlet conditions for the two subdomains Ω_1 and Ω_2, and we would like

the two sequences $(u_1^m)_{m\in\mathbb{N}^*}$ and $(u_2^m)_{m\in\mathbb{N}^*}$ to converge to the restrictions of the solution u of problem (2.6), that is

$$\lim_{m\to+\infty} u_1^m = u^m\,|_{\Omega_1} \quad \text{and} \lim_{m\to+\infty} u_2^m = u^m\,|_{\Omega_2}\ .$$

It can be proven that the Schwarz method applied to problem (2.6) always converges, with a rate that increases as the measure $|\Omega_{12}|$ of the overlapping region Ω_{12} increases.

It is easy to see that if the algorithm converges, the solutions $u_i^\infty, i = 1, 2$, in the intersection of the subdomains take the same values.

The original algorithms ASM and MSM are very slow. Another weakness is the need of overlapping subdomains. Indeed, only the continuity of the solution is imposed and nothing is imposed on the matching of the fluxes. When there is no overlap convergence is thus impossible.

In order to remedy the drawbacks of the original Schwarz method, Modify the original Schwarz method by replacing the Dirichlet interface conditions on $\partial\Omega_i\cap\partial\Omega$, $i = 1, 2$, by Robin interface conditions ($\partial\eta_i + \alpha$, where η_i is the outward normal to subdomain Ω_i, see [37]).

2.3.2 The generalized overlapping domain decomposition method

During the last decades, more sophisticated Schwarz methods were designed, namely the *optimized Schwarz methods or generalized overlapping domain decomposition method.*

additive Schwarz algorithm	**multiplicative Schwarz algorithm**
1. initial guess u_1^0, u_2^0	1. initial guess u_2^0
2. $i = 0$	2. $i = 0$
3. **until convergence**	3. **until convergence**
4. $i = i + 1$	4. $i = i + 1$
5. Compute u_j^i using $u_j^{i-1}, j = 1, 2$	5. Compute u_1^i using u_2^{i-1}
6. **end**	6. Compute u_2^i using u_1^i
	7. **end**

Figure 2: *Additive and multiplicative Schwarz algorithm for two subdomains.*

These methods are based on a classical domain decomposition, but they use more effective transmission conditions than the classical Dirichlet conditions at the interfaces between subdomains.

The first more effective transmission conditions were introduced by P.L. Lions (cf. [28]).

For elliptic problems, we have seen that Schwarz algorithms work only for overlapping domain decompositions and their performance in terms of iterations counts depends on the width of the overlap.

The algorithm introduced by P.L. Lions (cf. [28]) can be applied to both overlapping and non overlapping subdomains. It is based on improving

Schwarz methods by replacing the Dirichlet interface conditions by Robin interface conditions.

Let α be a positive number, the modified algorithm reads

$$\begin{cases} -\triangle u_1^m = f \quad \text{in } \Omega_1, \\ \\ \dfrac{\partial u_1^{m+1}}{\partial \eta_1} + \alpha_1 u_1^{m+1} = \dfrac{\partial u_2^m}{\partial \eta_1} + \alpha_1 u_2^m, \ \text{ on } \Gamma_1 \\ \\ u_1^m = 0 \quad \text{on } \partial\Omega_1 - \Gamma_1, \end{cases}$$

and

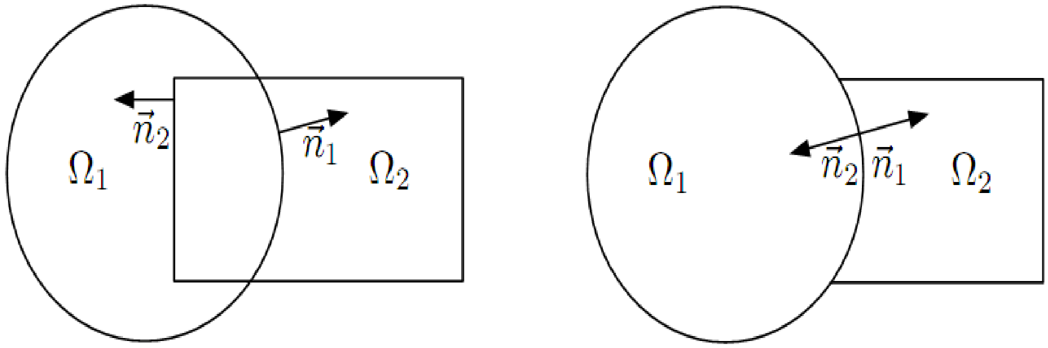


Figure 2.1: Outward normals for overlapping and non overlapping subdomains for P.L. Lions' algorithm.

$$\begin{cases} -\triangle u_2^m = f \quad \text{in } \Omega_2, \\ \dfrac{\partial u_2^{m+1}}{\partial \eta_2} + \alpha_2 u_1^{m+1} = \dfrac{\partial u_1^m}{\partial \eta_2} + \alpha_2 u_1^m, \ \text{ on } \Gamma_2 \\ u_2^m = 0 \quad \text{on } \partial\Omega_2 - \Gamma_2, \end{cases} \tag{2.8}$$

where η_1 and η_2 are the outward normals on the boundary of the subdomains.
It is also possible to consider other interface conditions than Robin conditions and optimize their choice with respect to the convergence factor.

2.3.3 Brief review of some recent results on DMM

The DD method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping sub-domains (see [3], [5], [32], [26], [34]). The solution is approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the sub-domain. The solution is approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomains. Extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in [21], [22], [16] and the references therein.

Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics, has been demonstrated in many papers. See proceedings of the annual domain decomposition conference beginning with [22] .
Moreover, the a priori estimate of the error for stationary problem is given in several papers, see for instance Lions [5] in which a variational formulation of the classical Schwarz method is derived.
In Chan et al. [16] a geometry related convergence results are obtained.
Douglas and Huang [21] studied the accelerated version of the GODDM, Engquist and Zhao [22] studied the convergence for simple rectangular or circular geometries; however, these authors did not give a criterion to stop the iterative process.
All these results can also be found in the recent books on domain decomposition methods of Quarteroni and Valli [35], Toselli and Widlund [38] .
Recently Maday and Magoules [29], [30] presented an improved version of the Schwarz method for highly heterogeneous media.
This method uses new optimized interface conditions specially designed to take into account the heterogeneity between the subdomains on the interfaces.
A recent overview of the current state of the art on domain decomposition methods can be found in two special issues of the computer methods in applied mechanics and engineering journal, edited by Farhat and Le Tallec [23], Magoules and Rixen [31] and in Nataf [32] .
In general, the a priory estimate for stationary problems is not suitable for assessing the quality of the approximate solution on subdomains since it depends mainly on the exact solution itself which is unknown.
The alternative approach is to use the approximate solution itself in order to find such an estimate.
This approach, known as a posteriori estimate, became very popular in the nineties of the last century with finite element methods, see the monographs [1], [39] and the references therein.

In their paper Otto and Lube [34] gave an a posteriori estimate for a nonoverlapping domain decomposition algorithm that said that “the better the local solutions fit together at the interface the better the errors of the subdomain solutions will be.” This error estimate enables us to know with certainty when one must stop the iterative process as soon as the required global precision is reached.
A posteriori error analysis for the elliptic case was also used by Bernardi et al. [6] to determine

an optimal value of the penalty parameter for penalty domain decomposition methods to construct fast solvers.

In recent research, in [13] the authors proved the error analysis in the maximum norm for a class of linear elliptic problems in the context of overlapping nonmatching grids and they established the optimal L^∞ error estimate between the discrete Schwarz sequence and the exact solution of the PDE.

H. Benlarbi and A.-S. Chibi in [4] derived a posteriori error estimates for the generalized overlapping domain decomposition method GODDM i.e., with Robin boundary conditions on the interfaces, for second order boundary value problems. They shown that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the interfaces.

After discretization of the domain by finite elements, they use the techniques of the residual a posteriori error analysis to get an posteriori error estimate for the discrete solutions on subdomains.

Chapter 3

An asymptotic behavior and a posteriori error estimates for the generalized overlapping domain decomposition method for parabolic equation

3.1 Introduction

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains.

Quite a few works on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems are known in the literature cf., e.g., [16], [23], [31], [4]. To prove the main result of this chapter, we proceed as in [29].

More precisely,we develop an approach which combines a geometrical convergence result due to Lions [27], [26], and a lemma which consists of estimating the error in the maximum norm between the continuous and discrete Schwarz iterates.

The optimal convergence order is then derived making use of standard finite element L^∞-error estimate for linear elliptic equations [25].

We apply the derived a posteriori error estimates for the generalized overlapping domain

decomposition method GODDM for the following evolutionary inequality:
find $u \in L^2(0,T;H_0^1(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f, \quad \text{in } \Sigma, \\ \\ u = 0, \quad \text{in } \Gamma \times [0,T], \\ \\ u(.,0) = u_0, \quad \text{in } \Omega. \end{cases}, \tag{3.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.
The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative verifies

$$\alpha \leq \beta, \quad \beta > 0.$$

f is a regular function satisfies

$$f \in L^2\left(0,T,L^2(\Omega)\right) \cap C^1\left(0,T,H^{-1}(\Omega)\right).$$

The symbol $(.,.)_\Omega$ stands for the inner product in $L^2(\Omega)$.
In first we introduce some necessary notations, then we give a variational formulation of our model.
Then a posteriori error estimate is proposed for the convergence of the discretized solution using the semi implicit-time scheme combined with a finite element method on subdomains. In addition we associate with the discrete introduced problem a fixed point mapping and we use that in proving the existence of a unique discrete solution. Finally an $H_0^1(\Omega)$-asymptotic behavior estimate for each sub-domain is derived.

3.2 The continuous problem

Using the Green Formula, the problem (3.1) can be transformed into the following continuous parabolic variational equation: find $u \in L^2(0,T;H_0^1(\Omega)) \cap C^2(0,T;H^{-1}(\Omega))$ solution to

$$\begin{cases} (u_t, v)_\Omega \quad + a(u,v) =, (f,v)_\Omega \ , \ \forall v \in H_0^1(\Omega) \\ \\ u(.,0) = u_0, \end{cases} \tag{3.2}$$

where

$$a(u,v)=\int_{\Omega}\nabla u\nabla v.dx+\int_{\Omega}\alpha uv.dx \ .$$

The symbol $(.,.)_{\Omega}$ stands for the inner product in $L^{2}\left(\Omega\right).$

3.2.1 The semi-discrete parabolic variational equation

We discretize the problem (3.2) with respect to time by using the semi–implicit scheme. Therefore, we look for a sequence of elements $u^{k}\in H_{0}^{1}\left(\Omega\right)$ which approaches $u\left(.,t_{k}\right),\ t_{k}=k\Delta t$, with initial data $u^{0}=u_{0}$.

Thus, we have for $k=1,...,n$

$$\begin{cases}\left(\dfrac{u^{k}-u^{k-1}}{\Delta t},v\right)_{\Omega}+a\left(u^{k},v\right)=\left(f^{\ k},v\right)_{\Omega},\ \forall v\in H_{0}^{1}\left(\Omega\right), \\ \\ u^{0}\left(x\right)=u_{0}\text{ in }\Omega,\end{cases} . \tag{3.3}$$

which means

$$\begin{cases}\left(\dfrac{u^{k}}{\Delta t},v\right)_{\Omega}+a\left(u^{k},v\right)=\left(f^{k}+\dfrac{u^{k-1}}{\Delta t},v\right),\ \forall v\in H_{0}^{1}\left(\Omega\right), \\ \\ u^{0}\left(x\right)=u_{0}\text{ in }\Omega.\end{cases} . \tag{3.4}$$

Then, problem (3.4) can be reformulated into the following coercive system of elliptic variational equation

$$\begin{cases}b\left(u^{k},v\right)=\left(f^{k}+\lambda u^{k-1},v\right)=\left(F\left(u^{k-1}\right),v\right),\ \forall v\in H_{0}^{1}\left(\Omega\right), \\ \\ u^{0}\left(x\right)=u_{0}\text{ in }\Omega,\end{cases} \tag{3.5}$$

such that

$$\begin{cases}b\left(u^{k},v\right)=\lambda\left(u^{k},v\right)+a\left(u^{k},v\right),\ u^{k}\in\ H_{0}^{1}\left(\Omega\right) \\ \\ \lambda=\dfrac{1}{\Delta t}=\dfrac{1}{k}=\dfrac{T}{n},\ \ k=1,...,n.\end{cases} \tag{3.6}$$

3.2.2 The space-continuous for generalized overlapping domain decomposition

Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise $C^{1,1}$ boundary $\partial\Omega$.

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12}, \qquad \partial\Omega_i \cap \Omega_j = \Gamma_i, \qquad i \neq j \text{ and } i, j = 1, 2.$$

We need the spaces

$$V_i = H^1(\Omega) \cap H^1(\Omega_i) = \left\{ v \in H^1(\Omega_i) : v_{\partial\Omega_i \cap \partial\Omega} = 0 \right\}$$

and

$$W_i = H_{00}^{\frac{1}{2}}(\Gamma_i) = \{ v\,|_{\Gamma i} : v \in V_i \text{ and } v = 0 \text{ on } \partial\Omega_i \backslash \Gamma_i \}, \tag{3.7}$$

which is a subspace of

$$H^{\frac{1}{2}}(\Gamma_i) = \left\{ \psi \in L^2(\Gamma_i) : \psi = \varphi\,|_{\Gamma i} \quad \text{for some } \varphi \in V_i,\ i = 1, 2 \right\},$$

equipped with the norm

$$\|\varphi\|_{W_i} = \inf_{v \in V_i, v = \varphi \text{ on } \Gamma_i} \|v\|_{1,\Omega}\,. \tag{3.8}$$

We define the continuous Schwarz sequences counterparts of the continuous system defined in (3.5), respectively by $u_1^{k,m+1} \in H_0^1(\Omega)$, $m = 0, 1, 2, ...$ such that

$$\begin{cases} b\left(u_1^{k,m+1}, v\right) = \left(F\left(u_1^{k-1,m+1}\right), v\right)_{\Omega_1}, \\ u_1^{k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\ \dfrac{\partial u_1^{k,m+1}}{\partial\eta_1} + \alpha_1 u_1^{k,m+1} = \dfrac{\partial u_2^{k,m}}{\partial\eta_1} + \alpha_1 u_2^{k,m}, \text{ on } \Gamma_1 \end{cases} \tag{3.9}$$

and $u_2^{k,m+1} \in H_0^1(\Omega)$ solution of

$$\begin{cases} b\left(u_2^{k,m+1}, v\right) = \left(F\left(u_2^{k-1,m+1}\right), v\right)_{\Omega_2}, \quad m = 0, 1, 2, .., \\ u_2^{k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega = \partial\Omega_2 - \Gamma_2, \\ \dfrac{\partial u_2^{k,m+1}}{\partial\eta_2} + \alpha_2 u_2^{n+1,m+1} = \dfrac{\partial u_1^{k,m}}{\partial\eta_2} + \alpha_2 u_1^{k,m}, \text{ on } \Gamma_2, \end{cases} \tag{3.10}$$

where η_i is the exterior normal to Ω_i and α_i is a real parameter, $i = 1, 2$.

Theorem 3.2.1 *cf. [25] The sequences $\left(u^{k,m+1}\right)$, $\left(u^{k,m+1}\right)$; $m \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution u of the problem (3.2). More precisely, there exist $k_1,\ k_2 \in (0, 1)$ which depend only respectively of (Ω_1, γ_2) and (Ω_2, γ_1) such that all $m \geq 0$*

$$\sup_{\bar{\Omega}_i} \left|u - u_i^{;m+1}\right| \leq k_1^n k_2^n \sup_{\gamma_1} \left|u^\infty - u^0\right|, \tag{3.11}$$

where u^∞, the asymptotic continuous solution and $\gamma_i = \partial\Omega_i \cap \Omega_j, i \neq j, i = 1, 2.$

Proof. The Schwarz alternating method converges geometrically to the solution u for the elliptic problem has been proved in [22], [27]. Then it has been updated and adapted for new bilinear parabolic form in [25].
This theorem remains true for the problem introduce in this chapter, because the introduced problem (3.2) can be reformulated as a system of elliptic variational equation (3.5). ■
Also in [13] the authors proved the error estimate for the elliptic variational inequalities using the standard nonmatching grids discretization with uniform norm and they found the following estimate

$$\left\|u - u_i^{m+1}\right\|_{L^\infty\left(\bar{\Omega}_1\right)} \leq Ch^2 \left|\log h\right|, \tag{3.12}$$

where C is a constant independent of both h and n.

Remark 3.2.1 *For the introduced problem (equation case), it can be easily noted that, $H_0^1(\Omega)$-norm remain true for (3.12), and the its proof is very similar to that in [25].*

In the next section, our main interest is to obtain an a posteriori error estimate we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplacien defined in (3.1) with the new boundary conditions of Generalized Schwarz alternating method defined in (3.9) applied to the elliptic operator Δ, we get

$$
\begin{aligned}
\left(-\Delta u_1^{k,m+1}, v_1\right)_{\Omega_1} &= \left(\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1} - \left[\left(\frac{\partial u_1^{k,m+1}}{\partial \eta_1}, v_1\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1}\right] \\
&= \left(\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1} - \left(\frac{\partial u_1^{k,m+1}}{\partial \eta_1}, v_1.\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1} - \left(\frac{\partial u_2^{k,m}}{\partial \eta_2} + \alpha_1 u_2^{k,m} - \alpha_1 u_1^{k,m+1}, v_1\right)_{\Gamma_1} \\
&= \left(\nabla u_1^{k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\alpha_1 u_1^{k,m+1}, v_1\right)_{\Gamma_1} - \left(\frac{\partial u_2^{k,m}}{\partial \eta_1} + \alpha_1 u_2^{k,m}, v_1\right)_{\Gamma_1},
\end{aligned}
$$

thus the problems (3.9), (3.10) are respectively equivalent to :
find $u_1^{k,m+1} \in V_1$ such that

$$
\begin{aligned}
b(u_1^{k,m+1}, v_1) + \left(\alpha_1 u_1^{n+1,m+1}, v_1\right)_{\Gamma_1} &= \left(F(u^{k-1}), v_1\right)_{\Omega_1} + \\
+ \left(\frac{\partial u_2^{k,m}}{\partial \eta_1} + \alpha_1 u_2^{k,m}, v_1\right)_{\Gamma_1} &, \forall v_1 \in V_1
\end{aligned}
\tag{3.13}
$$

find $u_2^{k,m+1} \in V_2$ such that

$$
\begin{aligned}
b(u_2^{k,m+1}, v_2) + \left(\alpha_2 u_2^{k,m+1}, v_2\right)_{\Gamma_2} &= \left(F(u^{k-1}), v_2\right)_{\Omega_2} + \\
+ \left(\frac{\partial u_1^{k,m}}{\partial \eta_2} + \alpha_2 u_1^{k,m}, v_2\right)_{\Gamma_2} &, \forall v_2 \in V_2.
\end{aligned}
\tag{3.14}
$$

3.3 A Posteriori Error Estimate in the Continuous Case

Since it is numerically easier to compare the subdomain solutions on the interfaces Γ_1 and Γ_2 rather than on the overlap Ω_{12}, thus we need to introduce two auxiliary problems defined on non-overlapping subdomains of Ω. This idea allows us to obtain the a posteriori error estimate by following the steps of Otto and Lube [34]. We get these auxiliary problems by coupling each one of the problems (3.9) and (3.10) with another problem in a non-overlapping way over

Ω. These auxiliary problems are needed for the analysis and not for the computation,to get the estimate.
To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing \quad \Omega = \Omega_2 \cup \Omega_4, \text{ with } \Omega_2 \cap \Omega_4 = \varnothing.$$

Let $(u_1^{k,m}, u_2^{n+1,m})$ be the solution of problems (3.13) and (3.14), we define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution of the following non-overlapping problems

$$\begin{cases} \dfrac{u_1^{k,m+1} - u_1^{k-1,m+1}}{\Delta t} - \Delta u_1^{k,m+1} = f^k \text{ in } \Omega_1, \; k = 1, ..., n, \\ \\ u_1^{k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \\ \dfrac{\partial u_1^{k,m+1}}{\partial \eta_1} + \alpha_1 u_1^{k,m+1} = \dfrac{\partial u_2^{k,m}}{\partial \eta_1} + \alpha_1 u_2^{k,m}, \text{ on } \Gamma_1 \end{cases} \tag{3.15}$$

and

$$\begin{cases} \dfrac{u_3^{k,m+1} - u^{k-1,m+1}}{\Delta t} - \Delta u_3^{k,m+1} = f^{\;k} \text{in } \Omega_3, \\ \\ u_3^{k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \\ \dfrac{\partial u_3^{k,m+1}}{\partial \eta_3} + \alpha_3 u_3^{n+1,m+1} = \dfrac{\partial u_1^{k,m}}{\partial \eta_3} + \alpha_3 u_1^{k,m}, \text{ on } \Gamma_1. \end{cases} \tag{3.16}$$

It can be taken $\epsilon_1^{n+1,m} = u_2^{n+1,m} - u_3^{n+1,m}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{n+1,m}$ and $u_3^{n+1,m}$ in problems (3.9), (3.10) and (resp.,(3.15) and (3.16)) in Ω_3. Because both overlapping and the nonoverlapping problems converge see [19] that is, $u_2^{k,m}$ and $u_3^{k,m}$ tend to u_2 (resp. u_3), $\epsilon_1^{k,m}$ should tend to zero as m tends to infinity in V_2 .

By putting

$$\begin{aligned}
\Lambda_3^{k,m} &= \frac{\partial u_2^{n+1,m}}{\partial \eta_1} + \alpha_1 u_2^{n+1,m}. \\
\Lambda_1^{k,m} &= \frac{\partial u_1^{k,m}}{\partial \eta_3} + \alpha_3 u_1^{k,m}. \\
\Lambda_3^{k,m} &= \frac{\partial u_3^{k,m}}{\partial \eta_1} + \alpha_1 u_3^{k,m} + \frac{\partial \epsilon_1^{k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{k,m} \\
\Lambda_1^{k,m} &= \frac{\partial u_1^{k,m}}{\partial \eta_3} + \alpha_3 u_1^{k,m}.
\end{aligned} \tag{3.17}$$

Under Green's formula, (3.15) and (3.16) can be reformulated into the following system of elliptic variational equations

$$\begin{aligned}
& b_1(u_1^{k,m+1}, v_1) + \left(\alpha_1 u_1^{k,m+1}, v_1\right)_{\Gamma_1} = \left(F(u^{k-1}), v_1\right)_{\Omega_1} + \\
& + \left(\Lambda_3^{k,m}, v_1\right)_{\Gamma_1}, \forall v_1 \in V_1.
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
& b_3(u_3^{k,m+1}, v_3) + \left(\alpha_3 u_3^{k,m+1}, v_3\right)_{\Gamma_1} = \left(F(u^{k-1}), v_3\right)_{\Omega 3} + \\
& + \left(\Lambda_1^{k,m}, v_3\right)_{\Gamma_1}, \forall v_3 \in V_3.
\end{aligned} \tag{3.19}$$

On the other hand by taking

$$\theta_1^{k,m} = \frac{\partial \epsilon_1^{k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{k,m}, \tag{3.20}$$

we get

$$\begin{aligned}
\Lambda_3^{k,m} &= \frac{\partial u_3^{k,m}}{\partial \eta_1} + \alpha_1 u_3^{k,m} + \frac{\partial (u_2^{k,m} - u_3^{k,m})}{\partial \eta_1} + \alpha_1 (u_2^{k,m} - u_3^{k,m}) \\
&= \frac{\partial u_3^{k,m}}{\partial \eta_1} + \alpha_1 u_3^{k,m} + \frac{\partial \epsilon_1^{k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{k,m} \\
&= \frac{\partial u_3^{k,m}}{\partial \eta_1} + \alpha_1 u_3^{k,m} + \theta_1^{k,m}.
\end{aligned} \tag{3.21}$$

Using (3.20), we have

$$\begin{aligned}\Lambda_3^{k,m+1} &= \frac{\partial u_3^{k,m+1}}{\partial \eta_1} + \alpha_1 u_3^{k,m+1} + \theta_1^{k,m+1} \\ &= -\frac{\partial u_3^{k,m+1}}{\partial \eta_3} + \alpha_1 u_3^{k,m+1} + \theta_1^{k,m+1} \\ &= \alpha_3 u_3^{k,m+1} - \frac{\partial u_1^{k,m}}{\partial \eta_3} - \alpha_3 u_1^{k,m} + \alpha_1 u_3^{k,m+1} + \theta_1^{k,m+1} \\ &= (\alpha_1 + \alpha_3) u_3^{k,m+1} - \Lambda_1^{k,m} + \theta_1^{k,m+1}\end{aligned} \tag{3.22}$$

and the last equation in (3.21), we get

$$\begin{aligned}\Lambda_1^{k,m+1} &= -\frac{\partial u_1^{k,m+1}}{\partial \eta_1} + \alpha_3 u_1^{k,m+1} \\ &= \alpha_1 u_1^{k,m+1} - \frac{\partial u_2^{k,m}}{\partial \eta_1} - \alpha_1 u_2^{k,m} + \alpha_3 u_1^{k,m+1} + \alpha_3 u_1^{k,m+1} \\ &= (\alpha_1 + \alpha_3) u_1^{k,m+1} - \Lambda_3^{k,m} + \theta_3^{k,m+1}.\end{aligned} \tag{3.23}$$

From this result we can write the following algorithm which is equivalent to the auxiliary nonoverlapping problem (3.18), (3.19). We need this algorithm and two lemmas for obtaining an a posteriori error estimate for this problem.

3.3.1 Algorithm

The sequences $(u_1^{k,m}, u_3^{k,m})_{m\in\mathbb{N}}$ solutions of (3.18), (3.19) satisfy the following domain decomposition algorithm:

Step 1: $k = 0$.

Step 2: Let $\Lambda_i^{k,0} \in W_1^*$ be an initial value, $i = 1, 3$ $\quad$ (W_1^* is the dual of W_1).

Step 3; Given $\Lambda_j^{k,m} \in W^*$ solve for $i, j = 1, 3, i \neq j$: Find $u_i^{k,m+1} \in V_i$ solution of

$$b_i(u_i^{k,m+1}, v_i) + \left(\alpha_i u_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(F(u^{k-1,m+1}), v_i\right)_{\Omega_i} + \\ + \left(\Lambda_j^{k,m+1}, v_i\right)_{\Gamma_i}, \forall v_i \in V_i.$$

Step 4: Compute

$$\theta_1^{k,m+1} = \frac{\partial \epsilon_1^{k,m+1}}{\partial \eta_1} + \alpha_1 \epsilon_1^{k,m+1}.$$

Step 5: Compute new data $\Lambda_j^{n+1,m} \in W^*$ solve for $i, j = 1, 3$, from

$$\left(\Lambda_i^{k,m+1}, \varphi\right)_{\Gamma_i} = \left((\alpha_i + \alpha_j) u_i^{k,m+1}, v_i\right)_{\Gamma_i} -$$

$$\left(\Lambda_j^{k,m+1}, \varphi\right)_{\Gamma_i} + \left(\theta_j^{k,m+1}, \varphi\right)_{\Gamma i}, \forall \varphi \in W_i, i \neq j\ .$$

Step 6: Set $m = m + 1$ go to **Step 3.**

Step 6: Set $k = k + 1$ go to **Step 2.**

Lemma 3.3.1 *Let $u_i^k = u^k|_{\Omega_i}$, $e_i^{k,m+1} = u_i^{k,m+1} - u_i^k$ and $\eta_i^{k,m+1} = \Lambda_i^{k,m+1} - \Lambda_i^k$. Then for $i, j = 1, 3, i \neq j$, the following relations hold*

$$b_i(e_i^{k,m+1}, v_i) + \left(\alpha_i e_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(\eta_j^{k,m}, v_i\right)_{\Gamma_i}, \forall v_i \in V_i \tag{3.24}$$

and

$$\left(\eta_i^{k,m+1}, \varphi\right)_{\Gamma i} = \left((\alpha_i + \alpha_j) e_i^{k,m+1}, v_1\right)_{\Gamma_i} - \left(\eta_j^{k,m}, \varphi\right)_{\Gamma_i} + \left(\theta_j^{k,m+1}, \varphi\right)_{\Gamma i}, \forall \varphi \in W_1. \tag{3.25}$$

Proof. 1. We have

$$b_i(u_i^{k,m+1}, v_i) + \left(\alpha_i u_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(F(u^{k-1,m+1}), v_i\right)_{\Omega_i} + \left\langle \Lambda_j^{k,m}, v_1 \right\rangle_{\Gamma_i}, \forall v_i \in V_i$$

and

$$b_i(u_i^k, v_i) + \left(\alpha_i u_i^k, v_i\right)_{\Gamma_i} = \left(F(u^{k-1,m+1}), v_i\right)_{\Omega_i} + \left(\Lambda_j^k, v_1\right)_{\Gamma_i}, \forall v_i \in V_i.$$

Since $b\,(.,.)$ is a coercive bilinear form; it can be deduced

$$b_i(u_i^{k,m+1} - u_i^{n+1}, v_i) + \left(\alpha_i u_i^{k,m+1} - u_i^{n+1}, v_i\right)_{\Gamma_i} = \left(\Lambda_j^{k,m} - \Lambda_i^k, v_i\right)_{\Gamma_i}, \forall v_i \in V_i$$

and so

$$b_i(e_i^{k,m+1}, v_i) + \left(\alpha_i e_i^{k,m+1}, v_i\right)_{\Gamma_i} = \left(\eta_i^{k,m}, v_1\right)_{\Gamma_i}, \forall v_i \in V_i.$$

2. We have $\lim_{m\to+\infty} \epsilon_1^{n+1,m} = \lim_{m\to+\infty} \theta_1^{n+1,m} = 0$. Then

$$\Lambda_i^k = (\alpha_1 + \alpha_3)u_i^k - \Lambda_j^k.$$

Therefore

$$\begin{aligned}
\eta_i^{k,m+1} &= \Lambda_i^{k,m+1} - \Lambda_i^{n+1} \\
&= (\alpha_1 + \alpha_3)u_i^{k,m+1} - \Lambda_j^{k,m} + \theta_j^{k,m+1} - (\alpha_1 + \alpha_3)u_i^k + \Lambda_j^k \\
&= (\alpha_1 + \alpha_3)(u_1^{k,m+1} - u_i^k) - (\Lambda_j^{k,m} - \Lambda_j^k) + \theta_j^{k,m+1}.
\end{aligned}$$

■

Lemma 3.3.2 *By letting C be a generic constant which has different values at different places one gets for $i, j = 1, 3, i \neq j$*

$$\left(\eta_i^{k,m-1} - \alpha_i e_i^{k,m}, w\right)_{\Gamma_1} \leqslant C \left\|e_i^{k,m}\right\|_{1,\Omega_i} \|w\|_{W_1} \tag{3.26}$$

and

$$\left(\alpha_i w_i + \theta_1^{k,m+1}, e_i^{k,m+1}\right)_{\Gamma_1} \leqslant C \left\|e_i^{k,m+1}\right\|_{1,\Omega_i} \|w\|_{W_1}. \tag{3.27}$$

Proof. Using Lemma 3.3.1 and the fact that the inverse of the trace mapping

$$Tr_i^{-1} : W_1 \longrightarrow V_i$$

is continuous we have for $i, j = 1, 3, i \neq j$

$$\begin{aligned}
\left(\eta_i^{k,m-1} - \alpha_i e_i^{k,m}, w\right)_{\Gamma_i} &= b_i(e_i^{k,m}, Tr^{-1}w) = \left(\nabla e_i^{k,m}, \nabla Tr^{-1}w\right)_{\Omega_i} + \\
&\quad + \left(\alpha e_i^{k,m}, Tr^{-1}w\right)_{\Omega_i} + \lambda \left(e_i^{k,m}, Tr^{-1}w\right)_{\Omega_i} \\
&\leqslant \left|e_i^{k,m}\right|_{1,\Omega_i} \left|Tr^{-1}w\right|_{1,\Omega_i} + \|\alpha\|_\infty \left\|e_i^{k,m}\right\|_{0,\Omega_i} \left\|Tr^{-1}w\right\|_{0,\Omega_i} \\
&\quad + |\lambda| \left\|e_i^{k,m}\right\|_{0,\Omega_i} \left\|Tr^{-1}w\right\|_{0,\Omega_i} \\
&\leqslant C \left\|e_i^{k,m}\right\|_{1,\Omega_i} \|w\|_{W_1}.
\end{aligned}$$

For the second estimate, we have

$$\begin{aligned}
\left(\alpha_i w_i + \theta_1^{k,m+1}, e_i^{k,m+1}\right)_{\Gamma_i} &= \left(\alpha_i w_i + \theta_1^{k,m+1}, e_i^{k,m+1}\right)_{\Gamma_i} \\
&\leqslant \left\|\alpha_i w_i + \theta_1^{k,m+1}\right\|_{0,\Gamma_1} \left\|e_i^{k,m+1}\right\|_{0,\Gamma_1} \\
&\leqslant \left(|\alpha_i| \, \|w_i\|_{0,\Gamma_1} + \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}\right) \left\|e_i^{n+1,m+1}\right\|_{0,\Gamma_1} \\
&\leq \max(|\alpha_i|, \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}) \|w_i\|_{0,\Gamma_1} \left\|e_i^{k,m+1}\right\|_{0,\Gamma_1} \\
&\leq C \left\|e_i^{k,m+1}\right\|_{0,\Gamma_1} \|w_i\|_{0,\Gamma_1} \leqslant C \left\|e_i^{k,m+1}\right\|_{0,\Gamma_1} \|w_i\|_{W_1}.
\end{aligned}$$

Thus, it can be deduceded that

$$|\alpha_i| \, \|w_i\|_{0,\Gamma_1} + \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1} \leqslant \max(|\alpha_i|, \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}) \|w_i\|_{0,\Gamma_1}.$$

■

Proposition 3.3.1 *For the sequences* $(u_1^{k,m}, u_3^{k,m})_{m\in\mathbb{N}}$ *solutions of (3.18), (3.19) we have the following a posteriori error estimation*

$$\left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_3^{k,m} - u_3^k\right\|_{3,\Omega_3} \leqslant C \left\|u_1^{k,m+1} - u_3^{k,m}\right\|_{W_1}.$$

Proof. From (3.22),(3.24), we have

$$\begin{aligned}
& b_1(e_1^{k,m+1}, v_1) + b_3(e_3^{k,m}, v_3) \\
&= \left(\eta_3^{k,m} - \alpha_1 e_1^{k,m+1}, v_1\right)_{\Gamma_1} + \left(\eta_1^{k,m-1} - \alpha_3 e_3^{k,m}, v_3\right)_{\Gamma_1} \\
&= \left(\eta_3^{n+1,m} - \alpha_1 e_1^{n+1,m+1}, v_1\right)_{\Gamma_1} + \left(\eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_3\right)_{\Gamma_1} \\
&+ \left(\eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_1\right)_{\Gamma_1} - \left(\eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_1\right)_{\Gamma_1}.
\end{aligned}$$

Thus, we have

$$b_1(e_1^{k,m+1}, v_1) + b_3(e_3^{k,m}, v_3) = \left(\eta_3^{n+1,m} - \alpha_1 e_1^{n+1,m+1} + \eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_1\right)_{\Gamma_1}$$

$$+ \left(\eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_3 - v_1\right)_{\Gamma_1}$$

$$= \left((\alpha_1 + \alpha_3)\, e_3^{n+1,m} + \theta_1^{n+1,m} - \alpha_1 e_1^{n+1,m+1} - \alpha_3 e_3^{n+1,m}, v_1\right)_{\Gamma_1}$$

$$+ \left(\eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_3 - v_1\right)_{\Gamma_1}$$

$$= \left(\alpha_1(e_3^{n+1,m} - e_1^{n+1,m+1}) + \theta_1^{n+1,m}, v_1\right)_{\Gamma_1} + \left(\eta_1^{n+1,m-1} - \alpha_3 e_3^{n+1,m}, v_3 - v_1\right)_{\Gamma_1}.$$

Taking $v_1 = e_1^{n+1,m+1}$ and $v_3 = e_3^{n+1,m}$.
Then using $\frac{1}{2}(a+b)^2 \leqslant a^2 + b^2$ and the lemma 2, we get

$$\frac{1}{2}\left(\left\|u_1^{k,m+1} - u_1^{n+1}\right\|_{1,\Omega_1} + \left\|u_3^{k,m} - u_3^{n+1}\right\|_{3,\Omega_3}\right)^2$$

$$\leqslant \left\|u_1^{k,m+1} - u_1^{k}\right\|_{1,\Omega_1}^2 + \left\|u_3^{k,m} - u_3^{k}\right\|_{3,\Omega_3}^2$$

$$\leq \left\|e_1^{k,m+1}\right\|_{1,\Omega_1}^2 + \left\|e_3^{k,m}\right\|_{3,\Omega_3}^2$$

$$\leq \left(\nabla e_1^{k,m+1}, \nabla e_1^{k,m+1}\right)_{\Omega_1} + \left(e_1^{k,m+1}, e_1^{k,m+1}\right)_{\Omega_3}$$

$$+ \left(\nabla e_3^{k,m}, \nabla e_3^{n+1,m}\right)_{\Omega_1} + \left(e_3^{k,m}, e_3^{k,m}\right)_{\Omega_3}$$

$$\leqslant \left(\nabla e_1^{k,m+1}, \nabla e_1^{k,m+1}\right)_{\Omega} + \frac{1}{\beta}\left(\alpha e_1^{k,m+1}, e_1^{k,m+1}\right)_{\Omega}$$

$$+ \left(\nabla e_3^{k,m}, \nabla e_3^{k,m}\right)_{\Omega_1} + \frac{1}{\beta}\left(\alpha e_3^{k,m}, e_3^{k,m}\right)_{\Omega_3}.$$

Then

$$\frac{1}{2}\left(\left\|u_1^{k,m+1}-u_1^{n+1}\right\|_{1,\Omega_1}+\left\|u_3^{k,m}-u_3^{n+1}\right\|_{3,\Omega_3}\right)^2$$

$$\leqslant \max(1,\tfrac{1}{\beta})\left(b_1\left(e_1^{k,m+1},e_1^{k,m+1}\right)+b_3\left(e_3^{k,m},e_3^{k,m}\right)\right)$$

$$=\max(1,\tfrac{1}{\beta})\left(\alpha_1(e_3^{k,m}-e_1^{k,m+1})+\theta_1^{k,m},e_1^{k,m+1}\right)_{\Gamma_1}$$

$$+\left(\eta_1^{k,m-1}-\alpha_3 e_3^{k,m},e_3^{k,m}-e_1^{k,m+1}\right)_{\Gamma_1}$$

$$\leqslant C_1\left\|e_1^{k,m+1}\right\|_{1,\Omega_1}\left\|e_3^{k,m}-e_1^{k,m+1}\right\|_{W_1}+C_2\left\|e_3^{k,m}\right\|_{3,\Omega_3}\left\|e_3^{k,m}-e_1^{k,m+1}\right\|_{W_1}$$

$$\leqslant \max(C_1,C_2)\left[\left\|e_1^{k,m+1}\right\|_{1,\Omega_1}+\left\|e_3^{k,m}\right\|_{3,\Omega_3}\right]\left\|e_3^{k,m}-e_1^{k,m+1}\right\|_{W_1},$$

thus

$$\left\|e_1^{n+1,m+1}\right\|_{1,\Omega_1}+\left\|e_3^{n+1,m}\right\|_{3,\Omega_3}\leqslant\left\|e_1^{n+1,m+1}-e_3^{n+1,m}\right\|_{W_1}.$$

Therefore

$$\left\|u_1^{n+1,m+1}-u_1^{n+1}\right\|_{1,\Omega_1}+\left\|u_3^{n+1,m}-u_3^{n+1}\right\|_{3,\Omega_3}\leqslant 2\max(C_1,C_2)\left\|u_1^{n+1,m+1}-u_3^{n+1,m}\right\|_{W_1}.$$

■

In similar way, we define another nonoverlapping auxiliary problem over (Ω_2,Ω_4), we get the same result.

Proposition 3.3.2 *For the sequences $(u_2^{k,m},u_4^{k,m})_{m\in\mathbb{N}}$ we get the the similar following a posteriori error estimation*

$$\left\|u_2^{k,m+1}-u_2^{k}\right\|_{2,\Omega_2}+\left\|u_4^{k,m}-u_4^{k}\right\|_{4,\Omega_4}\leqslant C\left\|u_2^{k,m+1}-u_4^{k,m}\right\|_{W_2}. \quad (3.28)$$

Proof. The proof is very similar to proof of Proposition 3.3.1. ■

Theorem 3.3.1 *Let $u_i^k=u^k\,|_{\Omega_i}$. For the sequences $(u_1^{k,m},u_2^{k,m})_{m\in\mathbb{N}}$ solutions of problems (3.15),(3.16), one have the following result*

$$\left\|u_1^{k,m+1}-u_1^{k}\right\|_{1,\Omega_1}+\left\|u_2^{k,m}-u_2^{k}\right\|_{2,\Omega_2}\leqslant C(\left\|u_1^{k,m+1}-u_2^{k,m}\right\|_{W_1}+\left\|u_2^{k,m}-u_1^{k,m-1}\right\|_{W_2}+$$

$$+\left\|e_1^{k,m}\right\|_{W_1}+\left\|e_2^{k,m-1}\right\|_{W_2}).$$

Proof. We use two nonoverlapping auxiliary problems over (Ω_1, Ω_3) and over $((\Omega_2, \Omega_4)$ resp). From the previous two propositions, we has

$$\left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_2^{k,m} - u_2^k\right\|_{2,\Omega_2}$$

$$\leqslant \left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_3^{k,m} - u_3^k\right\|_{3,\Omega_3}$$

$$+ \left\|u_2^{k,m} - u_2^{n+1}\right\|_{2,\Omega_2} + \left\|u_4^{k,m-1} - u_4^{n+1}\right\|_{4,\Omega_4}$$

$$\leqslant C\left\|u_1^{k,m+1} - u_3^{n+1,m}\right\|_{W_1} + C\left\|u_2^{k,m} - u_4^{k,m-1}\right\|_{W_2}$$

$$\leqslant C\left\|u_1^{k,m+1} - u_2^{k,m} + \epsilon_1^{n+1,m}\right\|_{W_1} + C\left\|u_2^{k,m} - u_1^{k,m-1} + \epsilon_2^{k,m-1}\right\|_{W_2}$$

$$\left\|u_1^{k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_2^{k,m} - u_2^k\right\|_{2,\Omega_2} \leqslant C(\left\|u_1^{k,m+1} - u_2^{k,m} + \epsilon_1^{k,m}\right\|_{W_1} + \left\|u_2^{k,m} - u_1^{k,m-1} + \epsilon_2^{k,m-1}\right\|_{W_2} +$$

$$\left\|\epsilon_1^{k,m}\right\|_{W_1} + \left\|\epsilon_2^{k,m-1}\right\|_{W_2}).$$

■

3.4 A Posteriori Error Estimate in the Discrete Case

In this section, we consider the discretization of the variational problems (3.9), (3.10). Let τ_h be a triangulation of Ω compatible with the discretization and $V_h \subset H_0^1$ is the subspace of continuous functions which vanish over $\partial\Omega$, we have

$$\{V_{i,h} = V_{h,\Omega_i}, \quad W_{i,h} = W_{h\Gamma_i}, \quad i = 1, 2.\}, \tag{3.29}$$

where $W_{h\Gamma_i}$ is a subspace of $H_{00}^{\frac{1}{2}}(\Gamma_i)$ which consists of continuous piecewise polynomial functions on Γ_i which vanish at the end points of Γ_i.

3.4.1 The space discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation.

Let Ω be decomposed into triangles and τ_h denote the set of all those elements where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation.

We discretize in space, i.e., that we approach the space H_0^1 by a space discretization of finite dimensional $V^h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the θ-scheme.

Therefore, we search a sequence of elements $u_h^n \in V^h$ which approaches $u^n(t_n)$, $t_n = n\Delta t$, with initial data $u_h^0 = u_{0h}$. Now we apply the θ-scheme on the following to the semi-discrete approximation for $v_h \in V^h$.

Let $u_h^{m+1} \in V_h$ be the solution of discrete problem associated with (3.5), $u_{i,h}^{m+1} = u_h^{m+1}|_{\Omega_i}$.

We construct the sequences $(u_{i,h}^{n+1,m+1})_{m\in\mathbb{N}}, u_{i,h}^{n+1,m+1} \in V_{i,h}, (i = 1, 2)$ solutions of discrete problems associated with (3.15),(3.16).

In similar manner to that of the previous section, we introduce two auxiliary problems. We define for (Ω_1, Ω_3) the following problems

$$\begin{cases} b_1(u_{1,h}^{k,m+1}, v_1) + \Big(\alpha_{1,h} u_{1,h}^{k,m+1}, v_1\Big)_{\Gamma_1} = \Big(F(u_{1,h}^{k-1,m+1}), v_1\Big)_{\Omega_1}, \\ \\ u_{1,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \\ \dfrac{\partial u_{1,h}^{k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{k,m+1} = \dfrac{\partial u_{2,h}^{k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{k,m}, \text{ on } \Gamma_1 \end{cases} \tag{3.30}$$

and

$$\begin{cases} b_1(u_{3,h}^{k,m+1}, v_1) + \left(\alpha_{3,h} u_{3,h}^{k,m+1}, v_1\right)_{\Gamma_1} = \left(F(u_{3,h}^{k-1,m+1}), v_3\right)_{\Omega_3}, \\ u_{3,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_{3,h}^{k,m+1}}{\partial \eta_3} + \alpha_3 u_{3,h}^{k,m+1} = \dfrac{\partial u_1^{k,m}}{\partial \eta_3} + \alpha_3 u_{1,h}^{k,m}, \text{ on } \Gamma_1 \end{cases} \tag{3.31}$$

and for (Ω_2, Ω_4)

$$\begin{cases} b_1(u_{2,h}^{k,m+1}, v_1) + \left(\alpha_{2,h} u_{2,h}^{k,m+1}, v_1\right)_{\Gamma_1} = \left(F(u_{2,h}^{k-1,m+1}), v_2\right)_{\Omega_2}, \\ k \\ u_{2,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \\ \dfrac{\partial u_{2,h}^{k,m+1}}{\partial \eta_2} + \alpha_2 u_{2,h}^{k,m+1} = \dfrac{\partial u_{1,h}^{k,m}}{\partial \eta_2} + \alpha_2 u_{1,h}^{k,m}, \text{ on } \Gamma_2 \end{cases}$$

and

$$\begin{cases} b_1(u_{4,h}^{k,m+1}, v_1) + \left(\alpha_{4,h} u_{4,h}^{k,m+1}, v_1\right)_{\Gamma_1} = \left(F(u_{4,h}^{k-1}), v_4\right)_{\Omega_4}, \\ u_{4,h}^{n+1,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{4,h}^{n+1,m+1}}{\partial \eta_4} + \alpha_4 u_{4,h}^{n+1,m+1} = \dfrac{\partial u_{2,h}^{n+1,m}}{\partial \eta_4} + \alpha_4 u_{2,h}^{n+1,m}, \text{ on } \Gamma_2. \end{cases} \tag{3.32}$$

We can obtain the discrete counterparts of propositions 3.3.1 and 3.3.2 by doing almost the same analysis as in the section above (i.e. passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

$$\left\| u_1^{k,m+1} - u_1^k \right\|_{1,\Omega_1} + \left\| u_3^{k,m} - u_3^k \right\|_{1,\Omega_3} \leqslant C \left\| u_1^{k,m+1} - u_3^{k,m} \right\|_{W_1} \tag{3.33}$$

and

$$\left\| u_2^{k,m+1} - u_2^{n+1} \right\|_{1,\Omega_2} + \left\| u_4^{k,m} - u_4^{n+1} \right\|_{1,\Omega_4} \leqslant C \left\| u_2^{k,m+1} - u_4^{k,m} \right\|_{W_2}. \tag{3.34}$$

Similarly as in proof of Theorem 2, we get the following discrete estimates

$$\left\|u_{1,h}^{k,m+1} - u_{1,h}^{k}\right\|_{1,\Omega_1} + \left\|u_{2,h}^{k,m} - u_{2,h}^{k}\right\|_{1,\Omega_2} \leqslant C(\left\|u_{1,h}^{k,m+1} - u_{2,h}^{k,m}\right\|_{W_1} + \left\|u_{2,h}^{k,m} - u_{1,h}^{k,m-1}\right\|_{W_2}$$

$$+ \left\|e_{1,h}^{n+1,m}\right\|_{W_1} + \left\|e_{2,h}^{n+1,m-1}\right\|_{W_2}).$$

3.5 An asymptotic behavior for the problem

3.5.1 A fixed point mapping associated with discrete problem

We define for $i = 1, 2, 3, 4$ the following mapping

$$\begin{aligned} T_h : V_{i,h} &\longrightarrow H_0^1(\Omega_i) \\ W_i &\longrightarrow TW_i = \xi_{h,i}^{k,m+1} = \partial_h (F(w_i)), \end{aligned} \tag{3.35}$$

where $\xi_{h,i}^{k}$ is the solution of the following problem

$$\begin{cases} b_i(\xi_{i,h}^{k,m+1}, v_i) + \left(\alpha_{i,h}\xi_{i,h}^{k,m+1}, v_{i,h}\right)_{\Gamma_i} = (F(w_i), v_{i,h})_{\Omega_i}, \\ \xi_{i,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ \dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} + \alpha_i \xi_{i,h}^{k,m+1} = \dfrac{\partial \xi_{j,h}^{k,m}}{\partial \eta_i} + \alpha_i \xi_{j,h}^{k,m}, \text{ on } \Gamma_i, \ i = 1, ..., 4, \ j = 1, 2. \end{cases} \tag{3.36}$$

3.5.2 An iterative discrete algorithm

Choosing $u_h^{i,0} = u_{h0}^{i}$, the solution of the following discrete equation

$$b^i\left(u_{h,i}^0, v_h\right) = \left(g_i^0, v_h\right), \ v_h \in V_h, \tag{3.37}$$

where $g^{i,0}$ is a regular function.
Now we give the following discrete algorithm

$$u_{i,h}^{k,m+1} = T_h u_{i,h}^{k-1,m+1}, k = 1, ..., n, \ i = 1, ..., 4,$$

where $u_{i,h}^{k}$ is the solution of the problem (3.36).

Proposition 3.5.1 *Let $\xi_h^{i,k}$ be a solution of the problem (3.36) with the right hand side $F^i(w_i)$ and the boundary condition* $\dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} + \alpha_i \xi_{i,h}^{k,m+1}$, $\tilde{\xi}_h^{i,k}$ *the solution for $\tilde{F}^i$ and* $\dfrac{\partial \tilde{\xi}_{i,h}^{k,m+1}}{\partial \eta_i} + \alpha_i \tilde{\xi}_{i,h}^{k,m+1}$. *The mapping T_h is a contraction in $V_{i,h}$ with the rate of contraction $\dfrac{\lambda}{\beta+\lambda}$. Therefore, T_h admits a unique fixed point which coincides with the solution of the problem (3.36).*

Proof. We note that

$$\|W\|_{H_0^1(\Omega_i)} = \|W\|_1 .$$

Setting

$$\phi = \frac{1}{\beta+\lambda} \left\| F(w_i) - F(\tilde{w}_i) \right\|_1 .$$

Then, for $\xi_{i,h}^{k,m+1} + \phi$ is solution of

$$\begin{cases} b\left(\xi_{i,h}^{k,m+1} + \phi, (v_{i,h} + \phi)\right) = \left(F(w_i) + \alpha_i \phi, (v_{i,h} + \phi)\right), \\ \\ \xi_{i,h}^{k,m+1} = 0, \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ \\ \dfrac{\partial \xi_{i,h}^{k,m+1}}{\partial \eta_i} + \alpha_i \xi_{i,h}^{k,m+1} = \dfrac{\partial \xi_{j,h}^{k,m}}{\partial \eta_i} + \alpha_i \xi_{j,h}^{k,m}, \text{ on } \Gamma_i, \; i = 1, ..., 4, \; j = 1, 2. \end{cases}$$

From assumption (2), we have

$$\begin{aligned} F(w_i) &\leq F(\tilde{w}_i) + \|F(w_i) - F(\tilde{w}_i)\|_1 \\ &\leq F(\tilde{w}_i) + \frac{\alpha}{\beta+\lambda} \|F(w_i) - F(\tilde{w}_i)\|_1 \\ &\leq F(\tilde{w}_i) + \alpha\phi. \end{aligned}$$

It is very clear that if $F^i(w_i) \geqq F^i(\tilde{w}_i)$ then $\xi_{i,h}^{k,m+1} \geqq \tilde{\xi}_{i,h}^{k,m+1}$. Thus

$$\xi_{i,h}^{k,m+1} \leq \tilde{\xi}_{i,h}^{k,m+1} + \phi.$$

But the role of w_i and $\tilde{w}_i$ are symmetrical. Thus we have the similar proof.

$$\tilde{\xi}_{i,h}^{k,m+1} \leq \xi_{i,h}^{k,m+1} + \phi,$$

yields

$$\begin{aligned}\|T(w)-T(\tilde{w})\|_1 &\leq \frac{1}{\beta+\lambda}\|F(w_i)-F(\tilde{w}_i)\|_1\\ &= \frac{1}{\beta+\lambda}\left\|f^i+\lambda w_i-f^i-\lambda\tilde{w}_i\right\|_1\\ &\leq \frac{\lambda}{\beta+\lambda}\|w_i-\tilde{w}_i\|_1 .\end{aligned}$$

■

Proposition 3.5.2 *Under the previous hypotheses and notations, we have the following estimate of convergent*

$$\left\|u_{i,h}^{n,m+1}-u_{i,h}^{\infty,m+1}\right\|_1 \leq \left(\frac{1}{1+\beta\theta(\Delta t)}\right)^n \left\|u_{i,h}^{\infty,m+1}-u_{i,h_0}\right\|_1, \; k=0,...,n, \tag{3.38}$$

where $u^{\infty,m+1}$ is an asymptotic continuous solution and u_{i,h_0} solution of (3.37).

Proof. We have

$$u_h^{i,\infty}=T_h u_h^{i,\infty},$$

$$\left\|u_{i,h}^{1,m+1}-u_{i,h}^{\infty,m+1}\right\|_1=\left\|T_h u_{i,h}^{0,m+1}-T_h u_{i,h}^{\infty,m+1}\right\|_1 \leq \left(\frac{1}{1+\beta\theta(\Delta t)}\right)\left\|u_{i,h}^{i,0}-u_{i,h}^{\infty,m+1}\right\|_1$$

and for $n+1$, we have

$$\left\|u_h^{n+1,m+1}-u_h^{i,\infty}\right\|_1=\left\|T_h u_{i,h}^{n,m+1}-T_h u_{i,h}^{\infty,m+1}\right\|_1 \leq \left(\frac{1}{1+\beta\theta(\Delta t)}\right)\left\|u_{i,h}^{n,m+1}-u_{i,h}^{i,\infty}\right\|_1,$$

then

$$\left\|u_{i,h}^{n,m+1}-u_{i,h}^{\infty}\right\|_1 \leq \left(\frac{1}{1+\beta\theta(\Delta t)}\right)^n \left\|u_{i,h}^{\infty,m+1}-u_{i,h_0}\right\|_1 .$$

■

Now we evaluate the variation in H_0^1-norm between $u(T,x)$, the discrete solution calculated at the moment $T=n\Delta t$ and u^∞, the asymptotic continuous solution of (3.2).

Theorem 3.5.1 *Under the previous hypotheses, notations, results, we have*

$$\left\|u_{i,h}^{n,m+1}-u^\infty\right\|_1 \leq C\left[\begin{array}{c}\left\|u_{1,h}^{k,m+1}-u_{2,h}^{k,m}\right\|_{W_1}+\left\|u_{2,h}^{k,m}-u_{1,h}^{k,m-1}\right\|_{W_2}+\left\|e_{1,h}^{n+1,m}\right\|_{W_1}\\ +\left\|e_{2,h}^{n+1,m-1}\right\|_{W_2}+\left(\frac{1}{1+\beta\theta(\Delta t)}\right)^n\end{array}\right] \tag{3.39}$$

and

$$\left\|u_{i,h}^{n,m+1} - u^{\infty}\right\|_1 \leq C\left[h^2\left|\log h\right| + \left(\frac{1}{1+\beta\theta\left(\Delta t\right)}\right)^n\right]. \tag{3.40}$$

Proof. Using Theorem 3.3.1 and Proposition 3.5.2, we get (3.39) and using (3.12) and Proposition 3.5.2 we get (3.40). ■

Conclusion

In this thesis, a posteriori error estimates for the generalized overlapping domain decomposition method with Robin boundary conditions on the interfaces for parabolic variational equation with second order boundary value problems are studied using the semi-implicit time scheme combined with a finite element spatial approximation. Furthermore a result of an asymptotic behavior using H_0^1-norm is given using Benssoussan-Lions Algorithm. In future completed research, the geometrical convergence of both the continuous and discrete corresponding Schwarz algorithms error estimate for linear elliptic PDEs will be established and the results of some numerical experiments will be presented to support the theory.

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