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Existence, periodicity, positivity and stability of solutions by Krasnoselskii's fixed point in neutral nonlinear functional differential equations

Par:

Mr. MESMOULI Mouataz Billah

ENCADREUR :	ARDJOUNI Abdelouaheb	MCA	U.M.C.M. Souk Ahras
Co. ENCADREUR :	DJOUDI Ahcène	Prof.	U.B.M. Annaba

Devant le jury

PRESIDENT :	KELEAIAIA Smail	Prof.	U.B.M. Annaba
EXAMINATEURS : :	BENTRAD Ali	Prof.	U. Reims, France
	KOUCHE Mahieddine	MCA	U.B.M. Annaba

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Mr. M. B. Mesmouli

Supervisor Dr. A. Ardjouni and Co. Supervisor Pr. A. Djoudi

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Dedication

To my father and my mother

Acknowledgement

In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohamed His servant and messenger.

First and foremost, I must acknowledge my limitless thanks to Allah, the Ever-Magnificent; the Ever-Thankful, for His help and bless. I am totally sure that this work would have never become truth, without his guidance.

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ملخــــص

في هـذه الأطروحـة نهـتم بدراسـة الخصـائص الكميـة والنوعيـة لمجموعـة مـن المعـادلات التفاضلية غير الخطية الحيادية. نبدأ بإعطاء بعض نظريات النقطة الثابتة ونتائج حول المعادلات التفاضلية ذات تأخر. ثانيا ندرس وجود الحل الدوري ، الحل الدوري الموجب والاستقرار التقاربي للحل المعدوم لمعادلات تفاضلية غير خطية حيادية باستخدام مفهوم المقلص الموسع والنقطة الثابتة لكراسنوسلسكي-بيرتون. يتم أيضا دراسة نظريات الوجود والوحدانية للحلول الدورية لنظام غير خطي من المعادلات التفاضلية الحيادية. وفي سياق هذه الدراسة نقدم مجموعة من الأمثلة التوضيحية.

Abstract

In this thesis we study a quantitative and qualitative properties of broad classes of nonlinear delay differential equations of neutral type. We start by giving some fixed point theorems and results for delay differential equations. Second we study the existence of periodic, positive periodic solution and asymptotic stability of the zero solution for a class of nonlinear differential equations with functional delay of neutral type by using the concept of large contraction mapping in the fixed point of Krasnoselskii-Burton's. A theorems of existence and uniqueness of periodic solutions are given to a wider class of nonlinear system of differential equations with two functional delay of neutral type. Examples are also given to illustrate the claims established.

Résumé

Cette thèse est consacrée à l'étude de propriétés quantitatives et qualitatives de larges classes d'équations différentielles non linéaires à retard fonctionnel de type neutre. On commence par donner les théorèmes de point fixe et des résultats sur les équations différentielles à retard. En utilisant la technique de point fixe on établit un théorème sur la stabilité asymptotique de la solution zéro pour une équation différentielle non linéaire. Aussi des résultats d'existence de solutions périodiques et positives sont établis et démontrés. L'outil clé ici est théorème de Krasnoselskii-Burton qui utilise la notion de contraction large. D'autres théorèmes sur l'existence et l'unicité de solutions périodiques relatifs à une classe plus large d'un système non linéaire d'équations de type neutre avec deux retards fonctionnels sont donnés. Des exemples sont fournis pour illustrer les travaux établis.

CONTENTS

Introduction

Many real-life phenomena in physics, engineering, biology, medicine, economics, etc. can be modeled by an initial value problem (IVP), or Cauchy problem, for ordinary differential equations (ODEs) of the type

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq t_0, \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where the function $x(t)$, called the state variable, represents some physical quantity that evolves over time.

Nevertheless, the biological systems and processes take time delays to complete. The delays can represent gestation times, incubation periods, or transport delays. In many cases time delays can be substantial such as gestation, forestation, deforestation and maturation or can represent little lags such as acceleration and deceleration in physical processes.

Therefore, in order to make the model more consistent with the real phenomenon, it becomes natural to include time delay terms into the differential equations that model population dynamics. So, a modification of (1) by including the dependence of the derivative x' on past time values of the state variable x is needed. It is then imperative to explicitly incorporate these process times into mathematical models. Such models are referred as delay differential equation (DDE) models. Thus, it seems clear that ordinary differential models can, at best, be approximations of real word problems. That is why, investigators of all branches find in (DDE) the ultimate issue to discuss real world problems. This is due to their advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes and to the fact that such models operate on an infinite dimensional space consisting of continuous functions that accommodate high dimensional dynamics. In recent years investigators have also given special attentions to the study of equations in which the delay occurs in the derivative of the state variable as well as in the independent variable, so called neutral differential equations. It is known that such equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

Mathematically speaking, DDEs are differential equations in which the derivatives of some unknown functions at present time are dependent on the values of the functions at previous times. That is, a general delay differential equation for $x(t) \in \mathbb{R}^n$ takes the form $x'(t) = f(t, x_t)$, $t \geq t_0$ where $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, t_0]$ is a function belonging to the Banach space $C([-\tau, t_0], \mathbb{R}^n)$ of continuous functions mapping the interval $[-\tau, t_0]$ into $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ is a given function of the set $\Omega \subset [-\tau, t_0] \times C$ into $\mathbb{R}^n$. Then the initial value problem is

$$\begin{cases} x'(t) = f(t, x_t), & t \geq t_0 \\ x_{t_0} = x(t_0 + \theta) = \Phi(\theta), \end{cases} \tag{2}$$

where $\Phi(\theta) \in C$ represents the initial point or the initial data. Equation (2), also called the Volterra functional differential equation, includes both distributed delay differential equations, where f depends on x computed on a continuum, possibly unbounded ($\tau = +\infty$), set of past values, and discrete delay differential equations, where only a finite number of past values of the state variable x are involved. Despite the latter being special cases of the former, they are suitable to describe a wide class of phenomena in many branches of applied mathematics and we shall confine our interest to them.

Delay differential equations, differential integral equations and functional differential equations have been studied for at least 200 years, the general theory of DDEs is widely developed and we refer the reader to the classical books by Bellman and Cooke [24], Hale [62], Driver [54], El'sgol'ts and Norkin [55] and to the more recent books by Hale and Verduyn Lunel [63], Kolmanovskii and Myshkis [73], Kolmanovskii and Nosov [74], Diekmann, van Gils, Verduyn-Lunel and Walter [52] and Kuang [78], which also include many real-life examples of DDEs and more general retarded functional differential equations. The subject gained much momentum (especially in the Soviet Union) after 1940 due to the consideration of meaningful models of engineering systems and control. It is probably true that most engineers were well aware of the fact that hereditary effects occur in physical systems, but this effect was often ignored because there was insufficient theory to discuss such models in detail.

Delay differential equations have attracted a rapidly growing attention in the field of nonlinear dynamics and have become a powerful tool for investigating the complexities of the real-world problems such as infectious diseases, biotic population, physics, population dynamics, industrial robotics, neuronal networks, and even economics and finance. Due to their importance in numerous applications, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential equations (see [1], [3]-[22], [26]-[43], [45], [47], [48], [49], [50], [51], [53], [57]-[59], [60], [61], [63], [65], [66]-[68], [72], [74], [79], [82], [83]-[93], [94]-[97], [99], [109], [110], [111] and [114]).

More than 100 years, the world famous mathematician Lyapunov initiated what is we call the Lyapunov direct method to study stability and the existence of periodic solutions of differential and functional differential equations. But the expressions of Lyapunov functionals are very complicated and hard to construct in so many problems. May be this is due to their pointwise character. Moreover, in the study of differential equations with functional delays by using Lyapunov functionals, many difficulties arise if the delay is unbounded or if the differential equation in question has unbounded terms, (see [31], [32], [38], [64], [102]). In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [3]–[22], [26]–[44], [66]–[68], [83]–[93], [94]–[97], [56], [113], [114]). The most striking object is that the fixed point method does not only solve the problem but has a significant advantage over Liapunov's direct method. The conditions of former are always average while those of the latter are pointwise. Further, while it remains an art to construct a Liapunov's functional when it exists, a fixed point method, in one step, yields existence (sometimes uniqueness) and stability. All we need, to use the fixed point method, is a complete metric space, a suitable fixed point theorem and an elementary integral methods to solve problems that have frustrated investigators for decades.

This thesis is, what we hope, a significant contribution to this field of investigation. It contains a discussion of the existence and stability for a class of functional differential equations with delay using fixed point technique. It was motivated by a series of results obtained during the past years which bring into a single perspective the qualitative theory of ordinary differential equations, and functional differential equations with functional delay.

The particular contents of each chapter are as follows.

The first chapter was devoted to point out the tools which are needed in this project. The fixed theorem of Banach and Krasnoselkii, the functional differential equations with delay and of neutral type belong to this recall. Significant and interesting models of equations with delay emanating from biology, epidemiology and the economy are given in this part of the thesis.

The second chapter exposes results published in [84] and relates to study the asymptotic stability of the zero solution for nonlinear differential equation with functional delay of neutral type

$$\frac{d}{dt}x\left(t\right)=-a\left(t\right)h\left(x\left(t-\tau\left(t\right)\right)\right)+\frac{d}{dt}Q\left(t,x\left(t-\tau\left(t\right)\right)\right)+G\left(t,x\left(t\right),x\left(t-\tau\left(t\right)\right)\right),$$

with an assumed initial function

$$x\left(t\right)=\psi\left(t\right),\ t\in\left[m_0,0\right].$$

Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due Burton (see [26], Theorem 3) to show the stability and asymptotic stability of the zero solution.

In the chapter three, we investigate the existence of periodic or nonnegative periodic solutions of the nonlinear neutral differential equations

$$\begin{aligned}
&\frac{d}{dt}\left[x(t)-Q(t,x(t-\tau(t)))\right] \\
&=-a(t)\,h(x(t-\tau(t)))+G(t,x(t),x(t-\tau(t))),
\end{aligned}$$

where $x(t)=x(t+T)$ and a is a positive continuous real-valued function. The function $h:\mathbb{R}\to\mathbb{R}$ is continuous, $Q:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $G:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying the Caratheodory condition. The main tool here is a modification of Krasnoselskii's fixed point theorem due Burton (see [26], Theorem 3) which we use to establish the existence of periodic and nonnegative periodic solutions for this equation.

Finally, we study the existence and uniqueness of periodic solutions for the system of nonlinear differential equations with two functional delays

$$\frac{d}{dt}x(t)=A(t)\,x(t-\tau(t))+\frac{d}{dt}Q(t,x(t-g(t)))+G(t,x(t),x(t-g(t))),$$

where $A(\cdot)$ is a nonsingular $n\times n$ matrix with continuous real-valued functions as its elements. The functions $Q:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ and $G:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous in their respective arguments. In the analysis we use the fundamental matrix solution of $x'(t)=A(t)\,x(t)$ coupled with Floquet theory to invert the system into an integral system. Then we employ the Krasnoselskii's fixed point theorem to show the existence of periodic solutions and show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

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CHAPTER 1

Functional setting and delay differential equations with applications

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the thesis. Moreover, the results in this chapter may be found in most standard books on functional analysis, for example [2, 25, 30, 31, 62, 75, 76, 77, 78, 81, 100, 101, 112].

1.1 Notation and preliminaries

1.1.1 Normed and Banach space

Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+ := [0, \infty)$ a function. Then d is called a metric on X if the following properties hold.

(d_1) $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in X$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_2) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called metric space.

Example 1.1.1 *The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The metric d is called the usual metric for $\mathbb{R}$.*

Let X be a linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $N : X \to \mathbb{R}^+$ a function. Then, N is said to be a norm if the following properties hold

(N_1) $N(x) = 0$ if and only if $x = 0$;

(N_2) $N(\lambda x) = |\lambda| N(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$;

(N_3) $N(x+y) \leq N(x) + N(y)$ for all $x, y \in X$.

The ordered pair (X, N) is called a normed space.

We use the notation $\|\cdot\|$ for norm. Then every normed space $(X, \|\cdot\|)$ is a metric space (X, d) with induced metric $d(x, y) = \|x - y\|$.

Example 1.1.2 *Let $X = \mathbb{R}^n, n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:*

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \text{ for all } x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n; \\ \|x\|_p &= \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \text{ for all } x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \text{ and } p \in (1, \infty); \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i| \text{ for all } x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n. \end{aligned}$$

Definition 1.1.3 *A sequence $\{x_n\}$ in a normed space X is said to be Cauchy if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$, i.e., for $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \geq n_o$.*

Definition 1.1.4 *A normed space $(X, \|\cdot\|)$ is said to be complete if it is complete as a metric space (X, d), i.e., every Cauchy sequence is convergent in X.*

Definition 1.1.5 *A complete normed space is called a Banach space.*

Example 1.1.6 *The linear space $C([a, b])$ of continuous functions on the closed and bounded interval $[a, b]$ is a Banach space with the uniform convergence norm $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$.*

Theorem 1.1.7 *Every finite-dimensional normed space is a Banach space.*

Theorem 1.1.8 *A closed subspace of a Banach space is a Banach space.*

1.1.2 Compactness and continuity of mappings

Let (X, d) be a metric space. Recall that a subset $\mathbb{M}$ of X is called compact if every open cover of $\mathbb{M}$ has a finite subcover. Equivalently, a subset $\mathbb{M}$ of X is compact if every sequence in $\mathbb{M}$ contains a convergent subsequence with a limit in $\mathbb{M}$.

A subset $\mathbb{M}$ of X is said to be totally bounded if for each $\epsilon > 0$, there exists a finite number of elements $x_1, x_2, ..., x_n$ in X such that $\mathbb{M} \subseteq \cup_{i=1}^{n} B_\epsilon(x_i)$.

Remark 1.1.9 *1) Every subset of a totally bounded set is totally bounded.*

2) Every totally bounded set is bounded, but a bounded set need not be totally bounded.

Proposition 1.1.10 *Let X be a metric space. Then the following are equivalent*
i) X is compact.
ii) Every sequence in X has a convergent subsequence.
iii) X is complete and totally bounded.

Proposition 1.1.11 *Let X be a subset of a complete metric space X. Then we have the following*
a) $\mathbb{M}$ is compact if and only if $\mathbb{M}$ is closed and totally bounded.
b) $\overline{\mathbb{M}}$ is compact if and only if $\mathbb{M}$ is totally bounded.

A subset $\mathbb{M}$ of a topological space is said to be relatively compact if its closure is compact, i.e., $\overline{\mathbb{M}}$ is compact. In particular, we have an interesting result.

Proposition 1.1.12 *Let $\mathbb{M}$ be a closed subset of a complete metric space. Then $\mathbb{M}$ is compact if and only if it is relatively compact.*

Definition 1.1.13 *Let $\{f_n\}$ be a sequence of real valued functions with $f_n : [a, b] \to \mathbb{R}$.*
a) $\{f_n\}$ is uniformly bounded on $[a, b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a, b]$.
b) $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \epsilon$ for all n.

The following results gives the main method of proving compactness in the spaces in which we are interested.

Theorem 1.1.14 (Ascoli-Arzela) *If $\{f_n\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

But here we manipulate function spaces defined on infinite t-intervals. So, for compactness we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([31], Theorem 1.2.2 p. 20) and is as follows.

Theorem 1.1.15 *Let $q : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{f_n\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}^+$ with $|f_n(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $f(t)$ with $|f(t)| \leq q(t)$ for $t \in \mathbb{R}^+$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

Let $\mathcal{P}$ be a mapping from a metric space (X, d) into another metric space (Y, ρ). Then $\mathcal{P}$ is said to satisfy Lipschitz condition on X if there exists a constant $L > 0$ such that

$$\rho(\mathcal{P}x, \mathcal{P}y) \leq Ld(x, y) \text{ for all } x, y \in X.$$

If L is the least number for which Lipschitz condition holds, then L is called Lipschitz constant. In this case, we say that $\mathcal{P}$ is an L-Lipschitz mapping or simply a Lipschitzian mapping with Lipschitz constant L. Otherwise, it is called non-Lipschitzian mapping. An L-Lipschitz mapping $\mathcal{P}$ is said to be contraction if $L < 1$ and nonexpansive if $L = 1$. The mapping $\mathcal{P}$ is said to be contractive if

$$\rho(\mathcal{P}x, \mathcal{P}y) < d(x, y) \text{ for all } x, y \in X,\ x \neq y.$$

Definition 1.1.16 *Let (X, d) be a metric space and assume that $\mathcal{P} : X \to X$. $\mathcal{P}$ is said to be a large contraction, if for $x, y \in X$, with $x \neq y$, we have $d(\mathcal{P}x, \mathcal{P}y) < d(x, y)$, and if $\forall \epsilon > 0$, $\exists \delta < 1$ such that*

$$[x, y \in X,\ d(x, y) \geq \epsilon] \Longrightarrow d(\mathcal{P}x, \mathcal{P}y) < \delta d(x, y).$$

Now, we state an important result implying that the mapping H given by

$$H(x) = x - h(x), \tag{1.1}$$

is a large contraction on the set

$$\mathbb{M} := \{\varphi \in X,\ |\varphi(t)| \leq R,\ t \in \mathbb{R}\}.$$

This result was already obtained in [1, Theorem 3.4] and we present below its proof. We shall assume that

(H1) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-R, R]$ and differentiable on $(-R, R)$,

(H2) The function h is strictly increasing on $[-R, R]$,

(H3) $\sup_{t \in (-R,R)} h'(t) \leq 1$.

Theorem 1.1.17 *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H1)–(H3). Then the mapping H in (1.1) is a large contraction on the set $\mathbb{M}$.*

Proof. Let $\varphi, \phi \in \mathbb{M}$ with $\varphi \neq \phi$. Then $\varphi(t) \neq \phi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\varphi, \phi)$, i.e.,

$$D(\varphi, \phi) = \{t \in \mathbb{R} : \varphi(t) \neq \phi(t)\}.$$

For all $t \in D(\varphi, \phi)$, we have

$$\begin{aligned} |(H\varphi)(t) - (H\phi)(t)| &\leq |\varphi(t) - \phi(t) - h(\varphi(t)) + h(\phi(t))| \\ &\leq |\varphi(t) - \phi(t)| \left| 1 - \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} \right|. \end{aligned} \tag{1.2}$$

Since h is a strictly increasing function we have

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} > 0 \text{ for all } t \in D(\varphi, \phi). \tag{1.3}$$

For each fixed $t \in D(\varphi, \phi)$ define the interval $I_t \subset [-R, R]$ by

$$I_t = \begin{cases} (\varphi(t), \phi(t)) & \text{if } \varphi(t) < \phi(t), \\ (\phi(t), \varphi(t)) & \text{if } \phi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \phi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = h'(c_t).$$

By (H2), (H3) we have

$$0 \leq \inf_{s \in (-R,R)} h'(s) \leq \inf_{s \in I_t} h'(s) \leq h'(c_t) \leq \sup_{s \in I_t} h'(s) \leq \sup_{s \in (-R,R)} h'(s) \leq 1. \tag{1.4}$$

Hence, by (1.2)–(1.4) we obtain

$$|(H\varphi)(t) - (H\phi)(t)| \leq |\varphi(t) - \phi(t)| \left| 1 - \inf_{s \in (-R,R)} h'(s) \right|, \tag{1.5}$$

for all $t \in D(\varphi, \phi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\epsilon \in (0, 1)$ and assume that φ and ϕ are two functions in $\mathbb{M}$ satisfying

$$\epsilon \leq \sup_{t \in (-R,R)} |\varphi(t) - \phi(t)| = \|\varphi - \phi\|.$$

If $|\varphi(t) - \phi(t)| \leq \frac{\epsilon}{2}$ for some $t \in D(\varphi, \phi)$, then we get by (1.4) and (1.5) that

$$|(H\varphi)(t) - (H\phi)(t)| \leq \frac{1}{2} |\varphi(t) - \phi(t)| \leq \frac{1}{2} \|\varphi - \phi\|. \tag{1.6}$$

Since h is continuous and strictly increasing, the function $h\left(s + \frac{\epsilon}{2}\right) - h(s)$ attains its minimum on the closed and bounded interval $[-R, R]$. Thus, if $\frac{\epsilon}{2} \leq |\varphi(t) - \phi(t)|$ for some $t \in D(\varphi, \phi)$, then by (H2) and (H3) we conclude that

$$1 \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2R} \min \left\{ h\left(s + \frac{\epsilon}{2}\right) - h(s) : s \in [-R, R] \right\} > 0.$$

Hence, (1.2) implies

$$|(H\varphi)(t) - (H\phi)(t)| \leq (1 - \lambda) \|\varphi - \phi\|. \tag{1.7}$$

Consequently, combining (1.6) and (1.7) we obtain

$$|(H\varphi)(t) - (H\phi)(t)| \leq \delta \|\varphi - \phi\|, \tag{1.8}$$

where

$$\delta = \max \left\{ \frac{1}{2}, 1 - \lambda \right\}.$$

The proof is complete. ■

The following example illustrates the theorem.

Example 1.1.18 *If $\|\cdot\|$ is the supremum norm, if $\mathbb{M} = \{\varphi, \varphi : \mathbb{R} \to C, \|\varphi\| \leq \sqrt{3}/3\}$ and if $(H\varphi)(t) = \varphi(t) - \varphi^3(t)$, then H is a large contraction of the set $\mathbb{M}$.*

The following proposition guarantees the existence of Lipschitzian mappings.

Proposition 1.1.19 *Let $\mathcal{P} : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b). Suppose $\mathcal{P}$ is continuous on $[a, b]$. Then, $\mathcal{P}$ is a Lipschitz continuous function (and hence is uniformly continuous).*

Now, let X and Y be two Banach spaces and let $\mathcal{P}$ be a mapping from X into Y. Then the mapping $\mathcal{P}$ is said to be

1. bounded if $\mathbb{M}$ is bounded in X implies $\mathcal{P}(\mathbb{M})$ is bounded;
2. locally bounded if each point in X has a bounded neighborhood V such that $\mathcal{P}(V)$ is bounded;
3. closed if $x_n \to x$ in X and $\mathcal{P}x_n \to y$ in Y imply $\mathcal{P}x = y$;
4. compact if $\mathbb{M}$ is bounded implies $\mathcal{P}(\mathbb{M})$ is relatively compact ($\overline{\mathcal{P}(\mathbb{M})}$ is compact), i.e., for every bounded sequence $\{x_n\}$ in X, $\{\mathcal{P}x_n\}$ has convergent subsequence in Y;
5. completely continuous if it is continuous and compact.

In the case of linear mappings, the concepts of continuity and boundedness are equivalent, but it is not true in general.

1.2 Fixed point theorems

In this section we state some fixed point theorems that we employ to help us in proving existence and stability of solutions. We refer to the nice and concise book of Smart ([100]) or ([31]).

1.2.1 Banach fixed point

The fixed point theorem, generally known as the Banach Contraction Principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis and forms an attractive tool which facilitates the study of stability for the differential equations with or without delay.

Definition 1.2.1 *Let f be a mapping in the set $\mathbb{M}$. we call fixed point of f any point x satisfying $f(x) = x$. If there exists such x, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.*

Theorem 1.2.2 (Contraction Mapping Principle) *[30]Let (X, ρ) a complete metric space and let $\mathcal{P} : X \to X$ a contraction mapping. Then there is one and only one point $z \in X$ with $\mathcal{P}z = z$. Moreover $z = \lim z_n$ where $z_{n+1} = \mathcal{P}z_n$ and z_1 chosen arbitrarily in X.*

We give the classical Cauchy problem on existence and uniqueness of the solution to a differential equation satisfying a given initial condition.

Example 1.2.3 *Let $f(t, x)$ be a continuous real-valued function defined for t in the interval $[0, T]$, and x in $\mathbb{R}$. The Cauchy initial value problem is the problem of finding a continuously differentiable function x on $[0, T]$ satisfying the differential equation*

$$\begin{cases} x' = f\left(t, x\left(t\right)\right), & t \in [0, T]; \\ x\left(0\right) = \zeta. \end{cases} \tag{1.9}$$

Consider the space $C\left([0, T]\right)$ of continuous real-valued functions with standard supremum norm and f is L-lipschitzian with respect to x. Integrating both sides of (1.9) we obtain

$$x\left(t\right) = \zeta + \int_0^t f\left(s, x\left(s\right)\right) ds.$$

We denote the function defined by the right side of the above by $\mathcal{P}x$. Precisely,

$$\left(\mathcal{P}x\right)\left(t\right) = \zeta + \int_0^t f\left(s, x\left(s\right)\right) ds.$$

Thus $\mathcal{P} : C\left([0, T]\right) \to C\left([0, T]\right)$, and a solution to (1.9) corresponds to a fixed point x of $\mathcal{P}$. Observe that for any $x, y \in [0, T]$,

$$\begin{aligned} \left|\left(\mathcal{P}x\right)\left(t\right) - \left(\mathcal{P}y\right)\left(t\right)\right| &= \left|\int_0^t f\left(s, x\left(s\right)\right) ds - \int_0^t f\left(s, y\left(s\right)\right) ds\right| \\ &\leq \int_0^t \left|f\left(s, x\left(s\right)\right) - f\left(s, y\left(s\right)\right)\right| ds \\ &\leq L \int_0^t \left|x\left(s\right) - y\left(s\right)\right| ds \\ &= Lt \left\|x - y\right\|. \end{aligned}$$

It follows that

$$\left\|\mathcal{P}x - \mathcal{P}y\right\| \leq LT \left\|x - y\right\|.$$

If $LT < 1$ then the result is immediate via the Banach Contraction Principle.

The generalized Banach fixed point theorem for $\mathcal{P}^n$ is given below, when $\mathcal{P}^n$ is the n composed mapping of $\mathcal{P}$ with itself.

Theorem 1.2.4 *Let the operator $\mathcal{P} : \mathbb{M} \subseteq X \to \mathbb{M}$ be given on a closed nonempty set $\mathbb{M}$ in a complete metric space (X, ρ). If*

$$\rho\left(\mathcal{P}^n x, \mathcal{P}^n y\right) \leq \alpha \rho\left(x, y\right), \quad \text{for} \quad x, y \in \mathbb{M},$$

is satisfied for some fixed $\alpha \in [0, 1[$ and some fixed $n \in \mathbb{N}$, then $\mathcal{P}$ has a fixed point.

Proof. By Theorem 1.2.2, there exists exactly one $x \in \mathbb{M}$ such that $\mathcal{P}^n x = x$. This implies that

$$\mathcal{P}^n\left(\mathcal{P}x\right) = \mathcal{P}^{n+1}x = \mathcal{P}\left(\mathcal{P}^n x\right) = \mathcal{P}x.$$

Since the fixed point x of $\mathcal{P}^n$ is unique, $\mathcal{P}x = x$. ■

The term "contraction" is used in several different ways in the literature. Our use is sometimes denoted by "strict contraction." The property $\rho\left(\mathcal{P}x, \mathcal{P}y\right) \leq \rho\left(x, y\right)$ is sometimes called "contraction" but it has limited use in fixed-point theory. A concept in between these two which is frequently useful is portrayed in the next result.

Theorem 1.2.5 *[30]Let (X, ρ) a compact nonempty metric space and let $\mathcal{P} : X \to X$. If*

$$\rho\left(\mathcal{P}x, \mathcal{P}y\right) < \rho\left(x, y\right), \quad \text{for} \quad x \neq y.$$

Then $\mathcal{P}$ has a fixed point.

Proof. We have

$$\rho\left(x, \mathcal{P}x\right) \leq \rho\left(x, y\right) + \rho\left(y, \mathcal{P}x\right) + \rho\left(y, \mathcal{P}y\right) + \rho\left(\mathcal{P}y, \mathcal{P}x\right),$$

and since $\rho\left(\mathcal{P}x, \mathcal{P}y\right) \leq \rho\left(x, y\right)$ we conclude

$$\rho\left(x, \mathcal{P}x\right) - \rho\left(y, \mathcal{P}x\right) \leq 2\rho\left(x, y\right).$$

Interchanging x and y yields

$$\left|\rho\left(x, \mathcal{P}x\right) - \rho\left(y, \mathcal{P}x\right)\right| \leq 2\rho\left(x, y\right).$$

Thus the function $B : X \to [0, +\infty)$ defined by $B\left(x\right) = \rho\left(x, \mathcal{P}x\right)$ is continuous on X. The compactness of X yields $z \in X$ with $\rho\left(z, \mathcal{P}z\right) = \rho\left(\mathcal{P}z, z\right) = \inf_{x \in X} \rho\left(x, \mathcal{P}x\right)$

If $\rho\left(\mathcal{P}z, z\right) \neq 0$ then $0 \leq \rho\left(\mathcal{P}\left(\mathcal{P}x\right), \mathcal{P}x\right) \leq \rho\left(x, \mathcal{P}x\right)$ contradicting the infimum property. Thus $\rho\left(\mathcal{P}z, z\right) = 0$ and $\mathcal{P}z = z$. If there is another distinct fixed point, say $\mathcal{P}z^* = z^*$, then $\rho\left(z, z^*\right) = \rho\left(\mathcal{P}z, \mathcal{P}z^*\right) < \rho\left(z, z^*\right)$ a contradiction for $z \neq z^*$. This completes the proof. ■

Theorem 1.2.6 *[30]If (X, ρ) is a complete metric space and $\mathcal{P} : X \to X$ is a α-contraction operator with fixed point x, then for any $y \in X$ we have*

(a) $\rho\left(x, y\right) \leq \rho\left(y, \mathcal{P}y\right) / \left(1 - \alpha\right)$.

(b) $\rho\left(\mathcal{P}^n y, x\right) \leq \alpha^n \rho\left(y, \mathcal{P}y\right) / \left(1 - \alpha\right)$.

Proof. To prove (a) we note that

$$\rho(x,y) \leq \rho(y,\mathcal{P}y) + \rho(\mathcal{P}y,\mathcal{P}x) \leq \rho(y,\mathcal{P}y) + \alpha\rho(x,y),$$

so that

$$\rho(x,y)(1-\alpha) \leq \rho(y,\mathcal{P}y).$$

For (b), recall that

$$\rho(\mathcal{P}^n y,\mathcal{P}^m y) \leq \alpha^n \rho(y,\mathcal{P}y)/(1-\alpha),$$

as $m \to +\infty$, $\mathcal{P}^m y \to x$ so that we have (b). ∎

1.2.2 Krasnoselskii fixed point

The fixed point theorem of Krasnoselskii is an hybrid result and is based on Banach and Schauder theorems. Firstly, we recall the theorem of Schauder.

Definition 1.2.7 *A topological space X has the fixed-point property if, whenever $\mathcal{P} : X \to X$ is continuous, then $\mathcal{P}$ has a fixed point.*

Theorem 1.2.8 (Schauder's first fixed point theorem) *[100] Any compact convex nonempty subset $\mathbb{M}$ of a Banach space X has the fixed point property.*

It is interesting to note that this has not been the preferred form of Schauder's fixed point theorem for investigators in the area of differential equations. Most of these have used the next form, and a survey of the literature will reveal that it has caused a good bit of grief.

Definition 1.2.9 *Let $\mathbb{M}$ be a subset of a Banach space X and $\mathcal{P} : \mathbb{M} \to X$. If P is continuous and $P(\mathbb{M})$ is contained in a compact subset of X, then P is a compact mapping.*

Theorem 1.2.10 (Schauder's second fixed point theorem) *[100] Let $\mathbb{M}$ be a non-empty closed convex bounded subset of a Banach space $(X, \|\cdot\|)$. Then every continuous compact mapping $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ has a fixed point.*

It is possible to prove existence and uniqueness theorems for most nonlinear problems using contraction mappings if the functions satisfy a local Lipschitz condition. But the Schauder theorem yields existence from continuity alone. Moreover, once existence is proved it is sometimes possible to prove uniqueness with something less than a Lipschitz condition. In addition, the study of existence using Schauder's theorem can produce some interesting side results. Generally, Schauder theorem relates to nonlinear differential equations.

The fixed point theorem of Krasnoselskii is a combination of Banach theorem and that of Schauder. It was the object of several studies these last years and one meets it in several forms. In particular, the theorem of Krasnoselskii gives the existence and the stability of the solutions of the functional differential equations and the nonlinear integral equations with delay of mixed type.

In 1955 Krasnoselskii (see [99], [100]) observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,
Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

For better understanding this observation of Krasnoselskii, consider the following differential equation.

$$x'(t) = -a(t)\, x(t) - g(t, x). \tag{1.10}$$

We can transform this equation in another form while writing, formally

$$x'(t)\, e^{-\int_0^t a(s)ds} = -a(t)\, e^{-\int_0^t a(s)ds} x(t) - g(t, x)\, e^{-\int_0^t a(s)ds},$$

thus

$$x'(t)\, e^{-\int_0^t a(s)ds} + a(t)\, e^{-\int_0^t a(s)ds} x(t) = -g(t, x)\, e^{-\int_0^t a(s)ds},$$

or

$$\left(x(t)\, e^{-\int_0^t a(s)ds}\right)' = -g(t, x)\, e^{-\int_0^t a(s)ds},$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^{t} \left(x(u)\, e^{-\int_0^u a(s)ds}\right)' du = -\int_{t-T}^{t} g(u, x)\, e^{-\int_0^u a(s)ds} du,$$

what gives

$$x(t)\, e^{-\int_0^t a(s)ds} - x(T-t)\, e^{-\int_0^{T-t} a(s)ds} = -\int_{t-T}^{t} g(u, x)\, e^{-\int_0^u a(s)ds} du,$$

or

$$x(t) = x(T-t)\, e^{-\int_{T-t}^t a(s)ds} - \int_{t-T}^{t} g(u, x)\, e^{-\int_t^u a(s)ds} du. \tag{1.11}$$

If we suppose that $e^{-\int_{T-t}^t a(s)ds} := \alpha$ and if $(X, \|\cdot\|)$ is the Banach space of functions $\varphi : \mathbb{R} \to X$ continuous and T-periodic, then the equation (1.11) can be written as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (\mathcal{P}\varphi)(t).$$

where $\mathcal{B}$ is contraction provides that the constant $\alpha < 1$ and $\mathcal{A}$ is compact mapping.

This example shows the birth of the mapping $\mathrm{P}\varphi := \mathcal{B}\varphi + \mathcal{A}\varphi$ who is identified with a sum of a contraction and a compact mapping.

Thus, the search of the solution for (1.11) requires an adequate theorem which applies to this hybrid operator $\mathcal{P}$ and who can conclude the existence for a fixed point which will be, in his turn, solution of the initial equation (1.10). Krasnoselskii found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result ([31], [100]).

Theorem 1.2.11 (Krasnoselskii) *[31] Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into X such that*
(i) $\mathcal{A}$ is compact and continuous,
(ii) $\mathcal{B}$ is a contraction mapping with constant α,
(iii) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,
Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

Note that if $\mathcal{A} = 0$, the theorem become the theorem of Banach. If $\mathcal{B} = 0$, then the theorem is not other than the theorem of Schauder.
Proof. According to the condition (iii) we have

$$\begin{aligned} \|(I-\mathcal{B})\,x-(I-\mathcal{B})\,y\| &= \|(x-y)-(\mathcal{B}x-\mathcal{B}y)\| \\ &\leq \|x-y\|+\|\mathcal{B}x-\mathcal{B}y\| \\ &\leq \|x-y\|+\alpha\,\|x-y\| \\ &= (1+\alpha)\,\|x-y\|\,, \end{aligned}$$

and

$$\begin{aligned} \|(I-\mathcal{B})\,x-(I-\mathcal{B})\,y\| &= \|(x-y)-(\mathcal{B}x-\mathcal{B}y)\| \\ &\geq \|x-y\|-\|\mathcal{B}x-\mathcal{B}y\| \\ &\geq \|x-y\|-\alpha\,\|x-y\| \\ &= (1-\alpha)\,\|x-y\|\,. \end{aligned}$$

In short

$$(1-\alpha)\,\|x-y\| \leq \|(I-\mathcal{B})\,x-(I-\mathcal{B})\,y\| \leq (1+\alpha)\,\|x-y\|\,.$$

This inequality shows that $(I-\mathcal{B}) : \mathbb{M} \to (I-\mathcal{B})\,\mathbb{M}$ is continuous and one to one. Thus, $(I-\mathcal{B})^{-1}$ exist and is continuous. Let us pose $U := (I-\mathcal{B})^{-1}\mathcal{A}$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in \mathbb{M} \text{ such that } (I-\mathcal{B})^{-1}\mathcal{A}z = z.$$

This is equivalent to $z = \mathcal{A}z + \mathcal{B}z$. ■

T.A. Burton studied the theorem of Krasnoselskii (see [31], [100]) and observed (see [26]) that Krasnoselskii's result can be more interesting in applications with certain changes and formulated the Theorem 1.2.14 below (see [26] for the proof).

Burton ([26]) remarked that in certain problems the situation does not arise in contraction form. For example, if we consider the equation $x' = -x^3 = -x + (x - x^3)$.

It is proved in [26] that a large contraction defined on a bounded and complete metric space has a unique fixed point.

Theorem 1.2.12 *[26] Let (X, d) be a complete metric space and $\mathcal{P}$ be a large contraction. Suppose there is an $x \in X$ and an $L > 0$, such that $d(x, \mathcal{P}^n x) \leq L$ for all $n \geq 1$. Then $\mathcal{P}$ has a unique fixed point in X.*

Proof. Suppose there exist $x \in X$, consider $\{\mathcal{P}^n x\}$. If this is a Cauchy sequence then by the triangle inequality we have for $m \geq n$

$$\begin{aligned} d(\mathcal{P}^n x, \mathcal{P}^m x) &\leq d\left(\mathcal{P}^n x, \mathcal{P}^{n+1} x\right) + d\left(\mathcal{P}^{n+1} x, \mathcal{P}^{n+2} x\right) + \ldots + d\left(\mathcal{P}^{m-1} x, \mathcal{P}^m x\right) \\ &\leq \left(\delta^n + \delta^{n+1} + \ldots + \delta^{m-1}\right) d(x, \mathcal{P}x) \\ &\leq \frac{\delta^n}{1-\delta} d(x, \mathcal{P}x). \end{aligned}$$

Thus $d(\mathcal{P}^n x, \mathcal{P}^m x) \to 0$ if $n, m \to \infty$, since (X, d) is a complete metric space the sequence $\{\mathcal{P}^n x\}$ has a limit y in X. This fixed point is unique since $\mathcal{P}z = z$ and $\mathcal{P}w = w$ we have

$$d(z, w) = d(\mathcal{P}z, \mathcal{P}w) \leq \delta d(z, w),$$

so that $d(z, w) = 0$, that is $z = w$.

Suppose now the contradiction, if $\{\mathcal{P}^n x\}$ is not a Cauchy sequence, then there exist

$$\epsilon > 0, \;\; N_k \uparrow \infty, \;\; n_k > N_k, \;\; m_k > n_k,$$

with $d(\mathcal{P}^{m_k} x, \mathcal{P}^{n_k} x) \geq \epsilon$. Thus

$$\begin{aligned} \epsilon &\leq d(\mathcal{P}^{m_k} x, \mathcal{P}^{n_k} x) \leq d\left(\mathcal{P}^{m_k-1} x, \mathcal{P}^{n_k-1} x\right) \leq d\left(\mathcal{P}^{m_k-2} x, \mathcal{P}^{n_k-2} x\right) \\ &\leq \ldots \leq d\left(\mathcal{P}^{m_k-n_k+1} x, \mathcal{P}x\right) \leq d\left(\mathcal{P}^{m_k-n_k} x, x\right). \end{aligned}$$

Since $\mathcal{P}$ is large contraction, for this $\epsilon > 0$ there is a $\delta < 1$ such that

$$\begin{aligned} \epsilon &\leq d(\mathcal{P}^{m_k} x, \mathcal{P}^{n_k} x) \leq d\left(\mathcal{P}^{m_k-1} x, \mathcal{P}^{n_k-1} x\right) \\ &\leq \ldots \leq \delta^{n_k} d\left(\mathcal{P}^{m_k-n_k} x, x\right), \end{aligned}$$

which contradict the fact that $\epsilon > 0$ and $\delta < 1$ for $n_k \to \infty$. Then $\mathcal{P}$ has a unique fixed point in X. ■

Lemma 1.2.13 *[26] If $(X, \|\cdot\|)$ is a normed space, if $\mathbb{M} \subset \mathbb{B}$, if $\mathcal{B} : \mathbb{M} \to X$ is a large contraction, then $(I - \mathcal{B})$ is a homeomorphism of $\mathbb{M}$ onto $(I - \mathcal{B})$.*

Proof. Clearly, $I - \mathcal{B}$ is continuous. To see that if $x \neq y$, then

$$\begin{aligned} \|(I - \mathcal{B}) x - (I - \mathcal{B}) y\| &= \|(x - y) - (\mathcal{B}x - \mathcal{B}y)\| \\ &\geq \|x - y\| - \|\mathcal{B}x - \mathcal{B}y\| \\ &\geq \|x - y\| - \|x - y\| \\ &= 0. \end{aligned}$$

Hence, $I - \mathcal{B}$ is one to one and $(I - \mathcal{B})^{-1}$ exists.

Suppose that $(I - \mathcal{B})^{-1}$ is not continuous. Then $\exists (I - \mathcal{B})y$ and $(I - \mathcal{B})x_n \to (I - \mathcal{B})y$ but $x_n \to y$. Now for each $\epsilon > 0$ $\exists N$ such that $n \geq N \Rightarrow$

$$\epsilon \geq \|(I - \mathcal{B}) x_n - (I - \mathcal{B}) y\| \geq \|x_n - y\| - \|\mathcal{B}x_n - \mathcal{B}y\|. \tag{1.12}$$

Since $x_n \rightsquigarrow y$, $\exists \epsilon_0 > 0$ and $\{x_{n_k}\}$ with $\|y - x_{n_k}\| \geq \epsilon_0$; as $\mathcal{B}$ is a large contraction there is a $\delta < 1$ with $\|\mathcal{B}y - \mathcal{B}x_{n_k}\| \leq \delta$. Thus, from (1.12) we have

$$\begin{aligned}
\epsilon &\geq \|(I - \mathcal{B})\, x_n - (I - \mathcal{B})\, y\| \\
&\geq \|x_n - y\| - \delta \, \|x_n - y\| \\
&= (1 - \delta) \, \|x_n - y\| \\
&\geq (1 - \delta) \, \epsilon_0.
\end{aligned}$$

But ϵ_0 is fixed, $\delta < 1$, and a contradiction occurs as $\epsilon \to 0$; that is, as $\epsilon \to 0$, $n_k \to \infty$, but ϵ_0 remains fixed. This completes the proof. ■

Theorem 1.2.14 *[26] Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(X, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*
(i) $\mathcal{A}$ is continuous and $\mathcal{A}\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$,
(ii) $\mathcal{B}$ is large contraction,
(iii) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,
Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

Proof. For each fixed $y \in \mathbb{M}$ the mapping $\mathcal{P}z = \mathcal{B}z + \mathcal{A}y$ is a large contraction on $\mathbb{M}$ with unique fixed point z (since $\mathbb{M}$ is bounded the L is assured in Theorem 1.2.12) so that $z = \mathcal{B}z + \mathcal{A}y$ has a unique solution z. Thus, $(I - \mathcal{B})z = \mathcal{A}y$ and by the lemma 1.2.13 $\mathcal{H}y := (I - \mathcal{B})^{-1}\mathcal{A}y$ is a continuous mapping of $\mathbb{M}$ into $\mathbb{M}$. Now $\mathcal{A}\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$ and $(I - \mathcal{B})^{-1}$ is a continuous mapping of $\mathcal{A}\mathbb{M}$ into $\mathbb{M}$; it is then well-known (cf. Kreyszig [77, p. 412 and 656]) that $(I - \mathcal{B})^{-1}\mathcal{A}\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$. By Schauder's second theorem (cf. Smart [100, p. 25]) there is a fixed point $y = (I - \mathcal{B})^{-1}\mathcal{A}y$ or $y = \mathcal{A}y + \mathcal{B}y$, as required. ■

1.3 Some general results and remarks on delay differential equations

In applications, the future behavior of many phenomena are assumed to be described by the solutions of an ordinary differential equation. Implicit in this assumption is that the future behavior is uniquely determined by the present and independent of the past. In differential difference equations, or more generally functional differential equations, the past exerts its influence in a significant manner upon the future. Many models are better represented by functional differential equations, than by ordinary differential equations.

1.3.1 A general initial value problem

Given $r > 0$, denote $C([a, b], \mathbb{R}^n)$, the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a, b] = [-r, 0]$, we let $C = C([-r, 0], \mathbb{R}^n)$ and designate the norm of an element φ in C by $|\varphi| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$.

Let $\sigma \in \mathbb{R}$, $A > 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$, be defined by

$$x_t(\theta) = x(t + \theta), \text{ for } \; -r \leq \theta \leq 0.$$

Let $f : \mathbb{R} \times C \to \mathbb{R}^n$ be a given function. A functional differential equation is given by the following relation

$$\begin{cases} x'(t) = f(t, x_t), & t \geq \sigma, \\ x_\sigma = \varphi, \end{cases} \tag{1.13}$$

Definition 1.3.1 *x is said to be a solution of (1.13) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$ and x satisfies (1.13) for $t \in [\sigma, \sigma + A]$. In such a case we say that x is a solution of (1.13) on $[\sigma - r, \sigma + A]$ for a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say that $x = x(\sigma, \varphi)$, is a solution of (1.13) with initial value at σ or simply a solution of (1.13) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (1.13) on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.*

Equation (1.13) is a very general type of equation and includes differential-difference equations of the type

$$x'(t) = f(t, x(t), x(t - r(t))),$$

as well as

$$x'(t) = \int_{-r}^{0} g(t, \theta, x(t + \theta))\, d\theta.$$

If

$$f(t, \varphi) = L(t, \varphi) + h(t),$$

in which L is linear in φ and $(t, \varphi) \to L(t, \varphi)$, we say that the equation is a linear delay differential equation, it is called homogeneous if $h \equiv 0$. If $f(t, \varphi) = g(\varphi)$, equation (1.13) is an autonomous one.

Lemma 1.3.2 ([63]) *Let $\sigma \in \mathbb{R}$ and $\varphi \in C$ be given and f be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of equation (1.13) through (σ, φ) is equivalent to solving*

$$x(t) = \varphi(0) + \int_{\sigma}^{t} f(s, x_s)\, ds, \quad t \geq \sigma \quad \text{and} \quad x_\sigma = \varphi.$$

Lemma 1.3.3 ([63]) *If $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [\sigma, \sigma + A]$.*

Proof. Since x is continuous on $[\sigma - r, \sigma + A]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently for t, s in $[\sigma, \sigma + A]$, $|t - s| < \delta$, we have $|x(t + \theta) - x(s + \theta)| < \varepsilon$, $\forall \theta \in [-r, 0]$. ■

Theorem 1.3.4 (Existence, [63]) *Let $\mathbb{M}$ be an open subset of $\mathbb{R} \times C$ and $f : \mathbb{M} \to \mathbb{R}^n$ be continuous. For any $(\sigma, \varphi) \in \mathbb{M}$, there exists a solution of equation (1.13) through (σ, φ).*

Proposition 1.3.5 ([23]) *If f is at most affine i.e. $|f(t,\varphi)| \leq a|\varphi| + b$ with $a, b > 0$, then there exists a global solution i.e. $\forall\varphi$, the solution $x(\sigma,\varphi)$ is defined on $[A, +\infty)$.*

Corollary 1.3.6 ([23]) *If f is lipschitzian with respect to the second variable, then it satisfies the property in the proposition below.*

Theorem 1.3.7 (Existence and uniqueness, [63]) *Let $\mathbb{M}$ be an open subset of $\mathbb{R} \times C$ and suppose that $f : \mathbb{M} \to \mathbb{R}^n$ be continuous and $f(t,\varphi)$ be lipschitzian with respect to φ in every compact subset of $\mathbb{M}$. If $(\sigma,\varphi) \in \mathbb{M}$, then equation (1.13) has a unique solution passing through (σ,φ).*

1.3.2 Method of Steps

One way of solving DDEs is using the so-called Method of Steps. The idea is to start with the initial history on the interval $[-\tau, 0]$ and then use the differential equation to obtain a piece of solution on the next interval $[-\tau, 0]$. This process can then be repeated to generate the solution on succeeding intervals. We illustrate this method by using the following DDE [104],

$$\begin{cases} x'(t) = -x(t-1), \\ x(t) = 1, \quad \text{for } t \in [-1, 0]. \end{cases} \tag{1.14}$$

For $t \in [0, 1]$, we integrate both sides of equation (1.14) to obtain

$$x(t) - x(0) = -\int_0^t x(s-1)\,ds.$$

Now using the initial history $x(t) = 1$, for $t \in [-1, 0]$, we get

$$\begin{aligned} x(t) &= x(0) - \int_0^t x(s-1)\,ds \\ &= x(0) - \int_{-1}^{t-1} x(s)\,ds \\ &= 1 - \int_{-1}^{t-1} 1 ds \\ &= 1 - t + 1 + 1 \\ &= 1 - t. \end{aligned}$$

Thus, $x(t) = 1 - t$ on $[0, 1]$. We repeat this process now for $t \in [1, 2]$,

$$\begin{aligned} x(t) &= x(1) - \int_1^t x(s-1)\,ds \\ &= x(1) - \int_0^{t-1} x(s)\,ds \\ &= 0 - \int_0^{t-1} (1-s)\,ds \\ &= \frac{1}{2}t^2 - 2t + \frac{3}{2}. \end{aligned}$$

So $x(t) = \frac{1}{2}t^2 - 2t + \frac{3}{2}$ on $[1,2]$. Continuing, for $t \in [2,3]$, we have

$$\begin{aligned} x(t) &= x(2) - \int_2^t x(s-1)\,ds \\ &= x(2) - \int_0^{t-1} x(s)\,ds \\ &= -\frac{1}{2} - \int_1^{t-2}\left(\frac{1}{2}s^2 - 2s + \frac{3}{2}\right)ds \\ &= -\frac{1}{6}(t-1)^3 + (t-1)^2 - \frac{3}{2}(t-1)^3 + \frac{1}{6}. \end{aligned}$$

Similarly, we obtain

$$x(t) = \frac{1}{24}(t-2)^4 - \frac{1}{3}(t-2)^2 + \frac{3}{4}(t-2)^2 - \frac{1}{6}(t-2) + \frac{11}{24}.$$

for $t \in [3,4]$, and

$$x(t) = -\frac{1}{120}(t-3)^5 + \frac{1}{12}(t-3)^4 - \frac{1}{4}(t-3)^2 + \frac{1}{12}(t-3)^2 + \frac{11}{24}(t-3) - \frac{19}{120}.$$

for $t \in [4,5]$. We can still continue further to see how the solution behaves in time but the computations could become more cumbersome.

Observe that the solution is continuous on the whole interval $[-1,10]$. Moreover, we expect the solution to be smooth except on the interval $[-1,0]$ since the only assumption on the initial history is continuity. To see this, denote by $x_j(t)$ the piece of the solution on the interval $[j-1,j]$, and notice that

$$x_j(t) = x_{j-1}(j-1) - \int_{j-2}^{t-1} x_{j-1}(s)\,ds. \tag{1.15}$$

The initial history $x_0(t)$ is continuous, and in the example it is given by $x_0(t) = 1$. Using the recurrence relation (1.15), and the fact that $x_0(t)$ is continuous, we see that $x_1(t)$ is C^1. Similarly, since $x_1(t)$ is C^1, (1.15) implies that $x_2(t)$ is C^2, and as we compute succeeding pieces they are one degree smoother than the previous piece. This smoothness property applies to DDE (1.13) since finding a solution to (1.13) is equivalent to solving the integral equation

$$x(t) = \psi(t) - \int_0^t f(s, x_s)\,ds. \tag{1.16}$$

with $\psi(t) = x_0(t)$.

1.3.3 Neutral delay differential equations

Now we are ready to define an other class of delay differential equations so-called the neutral delay differential equation.

Definition 1.3.8 ([63]) *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t,φ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω. if D is continuous together with its first and second Fréchet derivatives with respect to φ and D_φ, the derivative with respect to φ, is atomic at β on Ω.*

Definition 1.3.9 ([63]) *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. The equation*

$$\frac{d}{dt}D(t, x_t) = f(t, x_t) \tag{1.17}$$

is called the neutral delay differential equation NDDE(D, f). The function D is called the difference operator for NDDE(D, f).

For a given NDDE(D, f), a function x is said to be a solution of the NDDE(D, f) if there are a $\sigma \in \mathbb{R}$, $A > 0$, such that

$$x \in C([\sigma - r, \sigma + A), \mathbb{R}^n), \quad (t, x_t) \in \Omega, \quad t \in [\sigma, \sigma + A),$$

$D(t, x_t)$ is continuously differentiable and satisfies Eq. (1.17) on $[\sigma, \sigma + A)$. For a given $\sigma \in \mathbb{R}$, $\varphi \in C$, and $(\sigma, \varphi) \in \Omega$, we say (σ, φ) is a solution of Eq. (1.17) with initial value φ at σ, or simply a solution through (σ, φ), if there is an $A > 0$ such that $x(\sigma, \varphi)$, is a solution of (1.17) on $\sigma - r, \sigma + A)$ and $x_\sigma(\sigma, \varphi) = \varphi$.

Remark 1.3.10 *If $D(t, \varphi) = D_0(t, \varphi) - g(t)$, $f(t, \varphi) = L(t, \varphi) + h(t)$, where $D_0(t, \varphi)$ and $L(t, \varphi)$ are linear in φ, then NDDE(D, f) is called linear. It is called linear homogeneous if $g \equiv 0$, $h \equiv 0$, and linear nonhomogeneous if otherwise. If both $D(t, \varphi)$ and $f(t, \varphi)$ do not depend upon t, we call NDDE(D, f) autonomous; otherwise, we call it nonautonomous.*

The following are some examples of NDDEs

Example 1.3.11 ([63]) *If $r > 0$, B is an $n \times n$ constant matrix, $D(\varphi) = \varphi(0) - B(-r)$, and $f : \Omega \to \mathbb{R}^n$ is continuous, then the pair (D, f) defines an NDDE,*

$$\frac{d}{dt}[x(t) - Bx(t - r)] = f(t, x_t). \tag{1.18}$$

Example 1.3.12 ([63]) *If $r > 0$, x is a scalar, $D(\varphi) = \varphi(0) - \sin(-r)$, and $f : \Omega \to \mathbb{R}^n$ is continuous, then the pair (D, f) defines an NDDE,*

$$\frac{d}{dt}[x(t) - \sin x(t - r)] = f(t, x_t). \tag{1.19}$$

Remark 1.3.13 *Note that when x is continuously differentiable, (1.19) is equivalent to*

$$x'(t) - (\cos x(t - r))\, x'(t - r) = f(t, x_t).$$

This shows that our definition of NDDE requires that the derivative of x enters the equation in a linear fashion. In fact, the terms involving x that multiply x' must occur with the same delay.

Now, we consider the questions of existence, uniqueness of solutions of neutral delay differential equations.

Theorem 1.3.14 ([63]) *Let Ω be an open subset of $\mathbb{R} \times C$ and $f : \Omega \to \mathbb{R}^n$ be continuous. For any $(\sigma, \varphi) \in \Omega$, there exists a solution of equation (1.17) through (σ, φ).*

Theorem 1.3.15 ([63]) *Let Ω be an open subset of $\mathbb{R} \times C$ and suppose that $f : \Omega \to \mathbb{R}^n$ be continuous and $f(t, \varphi)$ be lipschitzian with respect to φ in every compact subset of Ω. If $(\sigma, \varphi) \in \Omega$, then equation (1.17) has a unique solution passing through (σ, φ).*

1.3.4 Real examples of delay differential equations

In this section we give two examples of physical and biological systems in which the present rate of change of some unknown function depends upon past values of the same function.

Mixing of Liquids

Consider a tank containing B gallons of salt water brine. Fresh water flows in at the top of the tank at a rate of q gallons per minute (see [54]). The brine in the tank is continually stirred, and the mixed solution flows out through a hole at the bottom, also at the rate of q gallons per minute.

Let $x(t)$ be the amount (in pounds) of salt in the brine in the tank at time t. If we assume continual, instantaneous, perfect mixing throughout the tank, then the brine leaving the tank contains $x(t)/B$ lbs. of salt per gallon, and hence

$$x'(t) = -qx(t)/B.$$

But, more realistically, let us agree that mixing cannot occur instantaneously throughout the tank. Thus the concentration of the brine leaving the tank at time t will equal the average concentration at some earlier instant, say $t-r$. We shall assume that r is a positive constant, although this assumption may also be subject to improvement. The differential equation for x then becomes a delay differential equation, $x'(t) = -qx(t-r)/B$ or, setting $c = q/B$,

$$x'(t) = -cx(t-r),$$

where r is the "delay" or "time lag".

Population Growth

If $N(t)$ is the population at time t of an isolated colony of animals, the most naive model for the growth of the population is

$$N'(t) = kN(t),$$

where k is a positive constant. This implies exponential growth, $N(t) = N_0 e^{kt}$ where $N_0 = N(0)$.

A somewhat more realistic model is obtained if we admit that the growth rate coefficient k will not be constant but will diminish as $N(t)$ grows, because of overcrowding and shortage of food. This leads to the differential equation

$$N'(t) = k\left[1 - N(t)/P\right]N(t), \tag{1.20}$$

where k and P are both positive constants. This equation with $N_0 = N(0)$ can be solved by separation of variables.

Now suppose that the biological self-regulatory reaction represented by the factor $[1 - N(t)/P]$ in (1.20) is not instantaneous, but responds only after some time lag $r > 0$. Then instead of (1.20) we have the delay differential (or difference differential) equation

$$N'(t) = k\left[1 - N(t-r)/P\right]N(t). \tag{1.21}$$

This equation has been studied extensively by Wright [107], [108], Kakutani and Markus [70], Jones [69], Kaplan and Yorke [71], and others.

Many other models of population growth have been proposed both with and without time lags. For example Cooke and Yorke [46] have studied the equation

$$N'(t) = g(N(t)) - g(N(t-L)), \tag{1.22}$$

where g is a given continuous, positive function and L is the lifetime of members of the species.

1.4 Stability of delay differential equations

In 1892 the Russian mathematician Lyapunov (1857-1918) published a major work on stability of ordinary differential equations based on positive definite functions and the chain rule. It was the foundation of stability theory as we know it today for ordinary, functional, and partial differential equations, as well as overlap into control theory and integral equations. The simplest notion of stability is the one related to stability of equilibrium points.

Definition 1.4.1 *A point $x(t) = x_e$ in the state space is said to be an equilibrium point of the autonomous system $x' = f(x)$ if and only if it has the property that whenever the state of the system starts at x_e, it remains at x_e for all future time.*

According to the definition, the equilibrium points are the real roots of the equation $f(x_e) = 0$. This is made clear by noting that if $x_e' = f(x_e) = 0$, then it follows that x_e is constant and, by definition, an equilibrium point. Without loss of generality, we assume that 0 is an equilibrium point of the system. If the equilibrium point under study, x_e, is not at zero we may define a new (shifted) coordinate system $x_s(t) = x(t) - x_e$ and note that

$$x_s'(t) = x'(t) = f(x(t)) = f(x_s(t) + x_e) =: f_s(x_s(t)); \quad x_s(0) = x_0 - x_e.$$

The claim follows by noting that $f_s(0) = f(x_e) = 0$. In summary, the study of the zero equilibrium point of $x_s'(t) = f_s(x_s(t))$ is equivalent to the study of the nonzero equilibrium point x_e of $x'(t) = f(x(t))$.

We now look at the basic definition of stability, for this consider the system

$$x'(t) = f(t, x_t(t)), \quad f(t, 0) = 0, \tag{1.23}$$

where $f : (-\infty, +\infty) \times C \to \mathbb{R}^n$, with $C = C([-r, 0], \mathbb{R}^n)$ the Banach space of continuous functions $\psi : [-r, 0] \to \mathbb{R}^n$, $r > 0$ equipped with the supremum norm

$\|\psi\| = \sup_{-r\leq t\leq 0} |\psi(t)|$. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution and we define

$$E(t) = \{\psi : [t-r, t] \to \mathbb{R}^n,\ \psi \text{ is continuous}\}.$$

Definition 1.4.2 *[31]The solution $x(t) = 0$ of (1.23) is*

1. *Stable, if for every $\epsilon > 0$ and $t_0 \geq 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that*

$$[\psi \in E(t_0), \|\psi\| < \delta \text{ and } t \geq t_0] \Rightarrow |x(t, t_0, \psi)| < \epsilon,$$

2. *Uniformly stable if δ of (1) is independent of t_0;*
3. *Asymptotically stable if it is stable and if, $\forall t_1 \geq t_0$, $\exists \eta > 0$ such that*

$$[\psi \in E(t_1), \|\psi\| < \eta \text{ and } t \geq t_1] \Rightarrow |x(t, t_1, \psi)| \to 0 \text{ as } t \to +\infty,$$

4. *Asymptotically uniformly stable if it is uniformly stable and if there exist $\eta > 0$ and for $\gamma > 0$, $\exists T > 0$ such that*

$$[t_1 \geq t_0, \psi \in E(t_1), \|\psi\| < \eta \text{ and } t \geq t_1 + T] \Rightarrow |x(t, t_1, \psi)| < 0.$$

Remark 1.4.3 *If all the solutions tend to zero, then $x = 0$ is globally asymptotically stable.*

1.4.1 The Method of Liapunov Functionals

In the following, we present the method of Liapunov functional in the context of (1.23).

Definition 1.4.4 *A continuous function $V : [0, +\infty) \times \mathbb{M} \to [0, +\infty)$ locally Lipschitzian in x and checks*

$$V'(t, x) = \limsup_{h\to 0} \frac{V(t+h, x+hf(t, x)) - V(t, x)}{h} \leq 0,$$

on $[0, +\infty) \times \mathbb{M}$, is called a function of Liapounov for (1.23).

Definition 1.4.5 *A wedge is a continuous and strictly increasing function $W : [0, +\infty) \to W : [0, +\infty)$ with $W(0) = 0$.*

The next theorem contains general stability results of the method of Liapunov functional.

Theorem 1.4.6 *Let $\mathbb{M}$ an open bounded subset of $\mathbb{R}^n$ containing the zero, $V : \mathbb{R} \times \mathbb{M} \to [0, +\infty)$ a differentiable function and let $W_i, i = \overline{1, 4}$ be wedges. Suppose also $V(t, 0) = 0$ and that $W_1(\|x(t)\|) \leq V(t, x_t)$, then the following statements are true*

1. *If $V'(t, x_t) \leq 0$, then the zero solution of (1.23) is stable.*

2. If in addition to (1), $V(t, x_t) \leq W_2(\|x(t)\|)$, then the zero solution of (1.23) is uniformly stable.

3. If $f(t, x_t)$ is bounded and $V'(t, x_t) \leq -W_3(\|x(t)\|)$, then the zero solution of (1.23) is asymptotically stable.

4. If $W_1(\|x(t)\|) \leq V(t, x_t) \leq W_2(\|x(t)\|) + W_3\left(\int_{t-r}^{t} \|x(s)\| \, ds\right)$ and if $V'(t, x_t) \leq -W_4(\|x(t)\|)$, then the zero solution of (1.23) is uniformly asymptotically stable.

In the following, we illustrate the preceding results by example (see [78]). Consider the scalar equation

$$x'(t) = -a(t)x(t) - b(t)x(t - r(t)), \tag{1.24}$$

where $a(t)$, $b(t)$, and $r(t)$ are bounded continuous functions, $a(t) > 0$, $r(t) > 0$, $r'(t) < 1$.

If $b(t) = 0$, then (1.24) becomes an ordinary differential equation; a trivial Liapunov function is $V_1(x(t)) = x^2(t)/2$. In order to find a Liapunov functional V, we want to generate a term like $-x^2(t - r(t))$ in the $V_{(1.24)}$. We try

$$V(t, \varphi) = \frac{1}{2}\varphi^2(0) + \alpha \int_{-r(t)}^{0} \varphi^2(\theta) \, d\theta,$$

where α is constant, or, equivalently,

$$V(x_t) = V(t, x_t) = \frac{1}{2}x^2(t) + \alpha \int_{-r(t)}^{0} x^2(t + \theta) \, d\theta,$$

We have

$$\begin{aligned} V'(x_t) &= -(a(t) - \alpha)x^2(t) - b(t)x(t)x(t - r(t)) \\ &\quad -a(1 - r'(t))x(t - r(t)). \end{aligned}$$

Clearly, if

$$b^2(t) < 4(a(t) - \alpha)(1 - r'(t)))\alpha, \tag{1.25}$$

then $V'(x_t) < 0$. Let $r(t) < r$, where r is a positive constant; $W_1(s) = s^2/2$, $W_2(s) = ((1/2) + \alpha r)s^2$. Then,

$$W_1(\|x(t)\|) \leq V(t, x_t) \leq W_2(\|x(t)\|).$$

If $\alpha > 0$ satisfies (1.25), then there may be a positive constant $\epsilon > 0$ such that

$$V(t, x_t) \leq -\epsilon x^2(t).$$

Thus, we may take $W_3(s) = -\epsilon s^2$. By Theorem 1.4.6, we know $x = 0$ is uniformly asymptotically stable. Indeed, since (1.24) is linear, we see that all solutions of (1.24) tend to $x = 0$ if (1.25) is true for some positive constant α.

When a, b and r are constants, (1.25) reduces to

$$b^2 < 4(a - \alpha)\alpha \leq a^2,$$

which implies that if $|b| < a$, then $x = 0$ is globally asymptotically stable; i.e., $\lim_{t \to +\infty} x_t(\varphi) = 0$ for $\varphi \in C$. Note that the length of delay r is not restricted.

1.4.2 A comparison between fixed point and Liapunov theory

The paper of Burton [32] is one for a series of investigations in which you have, probably, looked at problems which were especially challenging for stability analysis using Liapunov's direct method. Burton proved that many of these problems can be solved using fixed point theory.

Let $a : [0, +\infty) \to \mathbb{R}$ be bounded and continuous function, let r be a positive constant, and let

$$x'(t) = -a(t)\, x(t - r). \tag{1.26}$$

Although we can treat solutions with any initial time, we will always look at a solution $x(t) := x(t, 0, \psi)$ where $\psi : [-r, 0] \to \mathbb{R}$ is a given continuous initial function and $x(t, 0, \psi) = \psi(t)$ on $[-r, 0]$. It is then known that there is a unique continuous solution $x(t)$ satisfying (1.26) for $t > 0$ and with $x(t) = \psi(t)$ on $[-r, 0]$.

With such ψ in mind, we can write (1.26) as

$$x'(t) = -a(t)\, x(t + r) + \frac{d}{dt} \int_{t-r}^{t} a(s + r)\, x(s)\, ds, \tag{1.27}$$

so that by the variation of parameters formula, followed by integration by parts, we obtain

$$\begin{aligned} x(t) &= x(0)\, e^{-\int_0^t a(s+r)ds} + \int_{t-r}^{t} a(u + r)\, x(u)\, du - e^{-\int_0^t a(u+r)ds} \int_{-r}^{0} a(u + r)\, x(u)\, du \\ &\quad - \int_0^t a(s + r)\, e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} a(u + r)\, x(u)\, duds, \end{aligned} \tag{1.28}$$

In a space to be defined and with a mapping defined from (1.28) we will find that we have a contraction mapping just in case there is a constant $\alpha < 1$ with

$$\int_{t-r}^{t} |a(u + r)|\, du + \int_0^t |a(s + r)|\, e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u + r)|\, duds \le \alpha, \tag{1.29}$$

As we are interested in asymptotic stability we will need

$$\int_{t-r}^{t} a(u + r)\, du \to +\infty \text{ as } t \to \infty. \tag{1.30}$$

Burton, in his paper, compared results from a certain application of fixed point theory with a certain common Liapunov functional. In theory, there is no comparison at all. It is known that if we have a strong type of stability, then there exists a Liapunov functional of a certain type. The fact that we can not find that Liapunov functional gives validity to this type of comparison. With that in mind, from (1.29) it is easy to see one of the advantages of fixed point theory over Liapunov theory. The latter requires $a(t + r) > 0$. If $a(t + r) \ge 0$, then a very good bound is obtained in (1.29) with little effort. If $a(t + r)$ changes sign then (1.29) can still hold, although a good bound on the second integral is more difficult.

Burton proved in [32] the following result but were unable to do so and he left the principle difficulty as a hypothesis.

Theorem 1.4.7 *Let (1.29) and (1.30) hold. Then for every continuous initial function* $\psi : [-r, 0] \to \mathbb{R}$ *the solution* $x(t, 0, \psi)$ *is bounded and tends to zero as* $t \to \infty$*.*

Proof. Let $(\mathbb{S}, \|\cdot\|)$ be the Banach space of bounded and continuous functions $\psi : [-r, 0] \to \mathbb{R}$ with the supremum norm. Let $(\mathbb{B}, \|\cdot\|)$ be the complete metric space with supremum norm consisting of functions $\varphi \in \mathbb{B}$ such that $\varphi(t) = \psi(t)$ on $[-r, 0]$ and $\varphi(t) \to 0$ as $t \to \infty$.

Define $\mathcal{P} : \mathbb{B} \to \mathbb{B}$ by

$$(\mathcal{P}\varphi)(t) = \psi(t) \quad \text{on } [-r, 0],$$

and

$$\begin{aligned}
(\mathcal{P}\varphi)(t) \;=\; & \psi(0)\, e^{-\int_0^t a(s+r)ds} + \int_{t-r}^{t} a(u+r)\,\varphi(u)\,du - e^{-\int_0^t a(u+r)ds} \int_{-r}^{0} a(u+r)\,\psi(u)\,du \\
& - \int_0^t a(s+r)\, e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} a(u+r)\,\varphi(u)\,duds,
\end{aligned}$$

Clearly, $\mathcal{P}$ is continuous, $(\mathcal{P}\varphi)(0) = \psi(0)$, and from (1.29) it follows that P is bounded. Also, $\mathcal{P}$ is a contraction by (1.29).

We can show that the last term tends to zero by using the classical proof that the convolution of an L_1-function with a function tending to zero, does also tend to zero. Here are the details. Let $\varphi \in \mathbb{B}$ be fixed and let $0 < T < t$. Denote the supremum of $|\varphi|$ by $\|\varphi\|$ and the supremum of $|\varphi|$ on $[T, +\infty)$ by $\|\varphi\|_{[T,+\infty)}$. Consider (1.29) and (1.30). We have

$$\begin{aligned}
& \int_0^t |a(s+r)|\, e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)\,\varphi(u)|\,duds \\
& \leq \int_0^T |a(s+r)|\, e^{-\int_s^T a(u+r)du} \int_{s-r}^{s} |a(u+r)|\,\varphi(u)\,duds\, \|\varphi\|\, e^{-\int_T^t a(u+r)du} \\
& + \int_T^t |a(s+r)|\, e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)|\,duds\, \|\varphi\|_{[T-r,+\infty)} \\
& \leq \alpha\, \|\varphi\|\, e^{-\int_T^t a(u+r)du} + \alpha\, \|\varphi\|_{[T-r,+\infty)}\,.
\end{aligned}$$

For a given $\epsilon > 0$ take T so large that $\alpha\, \|\varphi\|_{[T-r,+\infty)} < \epsilon/2$. For that fixed T, take t^* so large that $\alpha\, \|\varphi\|\, e^{-\int_T^t a(u+r)du} < \epsilon/2$ for all $t > t^*$. We then have that last term smaller ϵ than for all $t > t^*$. Thus, $\mathcal{P} : \mathbb{B} \to \mathbb{B}$ is a contraction with unique fixed point in $\mathbb{B}$. ■

Example 1.4.8 *In (1.26), let* $a(t) = 1.1 + \sin t$*. The conditions of Theorem 1.4.7 are satisfied if* $2(1.1r + 2\sin(r/2)) < 1$*. This is approximated by* $0 < r < 0.2$*.*

We will see that this can be compared to a result using a Liapunov functional and, in this case, the Liapunov functional yields a significantly better result. But the next two examples reveal something equally interesting. In a later example the fixed point result is better than that of the Liapunov functional.

1. If $a(t) \geq 0$, then the Liapunov functional fails to address the problem, while the fixed point theorem yields a result fully consistent with that of Example 1.4.8. Again, it is a good result, obtained with little effort.

2. If $a(t)$ becomes negative, then the Liapunov functional fails, while the fixed point theorem yields a stability result which is significantly poorer than in the first two cases because of inherent difficulties in estimating the integrals in (1.29).

Example 1.4.9 *In (1.26), let* $a(t) = 1 + \sin t$. *The conditions of Theorem 1.4.7 are satisfied if* $2(r + 2\sin(r/2)) < 1$. *This is approximated by* $0 < r < 0.25$.

Remark 1.4.10 *Intuitively, Example 1.4.8 should be more strongly stable than Example 1.4.9. Yet, a look at (1.29) readily reveals why our results state the opposite. Burton conjectures that a different fixed point mapping might reverse the relation.*

We return now to study the equation (1.26) by Liapunov method.

Theorem 1.4.11 *If there is a* $\delta > 0$ *with*

$$a\left(t + r\right) \geq \delta, \quad \textit{for all } t \geq 0, \tag{1.31}$$

an $\epsilon > 0$ *with*

$$a\left(t + r\right) \int_{t-r}^{t} a\left(s + r\right) ds - 2 + r \leq -\epsilon, \quad \textit{for all } t \geq 0, \tag{1.32}$$

and if there is a $\gamma > 0$ *with*

$$\gamma\left[a\left(t\right) + a\left(t + r\right)\right] \geq \left(\epsilon/2\right) a\left(t + r\right), \quad \textit{for all } t \geq 0, \tag{1.33}$$

then the zero solution of (1.26) is uniformly asymptotically stable.

Proof. From (1.27) we have

$$\frac{d}{dt}\left(x\left(t\right) - \int_{t-r}^{t} a\left(s + r\right) x\left(s\right) ds\right) = -a\left(t\right) x\left(t + r\right),$$

and we select a first Liapunov functional as

$$V_1\left(t, x_t\right) = \left(x\left(t\right) - \int_{t-r}^{t} a\left(s + r\right) x\left(s\right) ds\right)^2 + \int_{-r}^{0} \int_{t+s}^{t} a\left(u + r\right) x^2\left(u\right) du ds,$$

so that the derivative along a solution of (1.27) satisfies

$$V_1'\left(t, x_t\right) \leq -\epsilon\delta x^2.$$

Now, we need to define a second Liapunov functional and add them together to make a positive definite Liapunov functional. Define

$$V_2\left(t, x_t\right) = x^2\left(t\right) + \int_{t-r}^{t} a\left(s + r\right) x^2\left(s\right) du,$$

so that the derivative along a solution of (1.26) satisfies

$$V_2'(t, x_t) \leq (\epsilon/2)\, \delta x^2.$$

If we define

$$V_1(t, x_t) + V_2(t, x_t) := V(t, x_t),$$

then we have

$$V'(t, x_t) \leq (-\epsilon/2)\, \delta x^2.$$

We can now find wedges with

$$W_1(|x(t)|) \leq V(t, x_t) \leq W_2(\|x_t\|)$$

and, since (1.31) and (1.32) imply that $x(t)$ is bounded, conclude that the zero solution is uniformly asymptotically stable. ∎

Example 1.4.12 *In (1.26), let* $a(t) = 1.1 + \sin t$. *Theorem 1.4.11 holds if there is an* $\epsilon > 0$ *with*

$$2.1(1.1r + 2\sin(r/2)) - 2 + r < -\epsilon.$$

We make another very rough estimate by taking $\sin(r/2) = r/2$ *and say that we need* $r < 0.37$. *Then the zero solution of (1.26) is asymptotically stable.*

It seems (see [1, 3, 5, 13], [37, 39], [41, 42, 43, 47, 48, 51, 49]) that the fixed point method fits better to study stability in real world problems like those equation with delays. May be this is due to the fact that the these problems needs average conditions while the Lyapunov method deals with conditions that are pointwise.

CHAPTER 2

Study of the stability in nonlinear neutral differential equations with functional delay

Keywords. Fixed point, stability, delay, nonlinear neutral equation, large contraction mapping, integral equation.

The goal of this chapter is to present a very recent work published in [84], namely, Mesmouli M. B., Ardjouni A. and A. Djoudi., Study of the stability in nonlinear neutral differential equations with functional delay using Krasnoselskii–Burton's fixed-point, Applied Mathematics and Computation 243 (2014) 492–502.

We use, in this chapter, a modification of Krasnoselskii's fixed point theorem introduced by Burton (see [26] Theorem 3) to obtain stability results of the zero solution of a totally nonlinear neutral differential equations with functional delay.

2.1 Preliminaries and inversion of the equation

As discussed above, Lyapunov's direct method was widely used to study the stability of solutions of ordinary differential equations and functional differential equations, see e.g. [26, 28, 31, 32, 38, 39], [40, 64, 114] and the references therein. Nevertheless, the expressions of Lyapunov functional are very complicated and hard to construct.

Recently, many authors have realized that the fixed points theory can be used to study the stability of the solution. Becker, Furumochi, Zhang and Burton considered the differential equation (see [1, 3, 5, 13], [37, 39], [41, 42, 43, 47, 48, 51, 49] and [94]). The most striking object is that the fixed point method does not only solve the problem on stability but has a significant advantage over Liapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [31]). While it remains an art to construct a Liapunov's functional when it exists, a fixed point method, in one step, yields existence, uniqueness and stability. All we need, to use the fixed point method, is a complete metric space, a suitable fixed point theorem and an elementary integral methods to solve problems that have frustrated investigators for decades.

Consider now, the nonlinear neutral differential equation with functional delay expressed as follows

$$\frac{d}{dt}x(t) = -a(t)h(x(t-\tau(t))) + \frac{d}{dt}Q(t,x(t-\tau(t))) + G(t,x(t),x(t-\tau(t))), \quad (2.1)$$

with an assumed initial function

$$x(t) = \psi(t), \ t \in [m_0, 0],$$

where $\psi \in C([m_0,0],\mathbb{R})$, $m_0 = \inf\{t-\tau(t) : t \geq 0\}$. Throughout this chapter we assume that $a \in C(\mathbb{R}^+,\mathbb{R})$, $\tau \in C^1(\mathbb{R}^+,\mathbb{R})$ is bounded and $h : \mathbb{R} \to \mathbb{R}$ is continuous, $Q : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfying the Caratheodory condition with $h(0) = Q(t,0) = G(t,0,0) = 0$. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due Burton (see [28], Theorem 3) to show the stability and asymptotic stability of the zero solution of equation (2.1). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [26], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (2.1) into an integral equation written as a sum of two mappings, one is a large contraction and the other is completely continuous. After that, we use a variant of Krasnoselskii's fixed point theorem, to show the stability and asymptotic stability of the zero solution.

We present the inversion of the equation (2.1) in the following Lemma.

Lemma 2.1.1 *Let $v : [m_0,\infty) \to \mathbb{R}^+$ be an arbitrary bounded continuous function. Then x is a solution of (2.1) if and only if*

$$\begin{aligned}
&x(t) \\
&= \left[\psi(0) - Q(0,\psi(-\tau(0))) - \int_{-\tau(0)}^{0} v(s)h(\psi(s))\,ds\right] e^{-\int_0^t v(u)du} \\
&+ \int_0^t v(s)e^{-\int_s^t v(u)du} H(x(s))\,ds + Q(t,x(t-\tau(t))) \\
&+ \int_{t-\tau(t)}^{t} v(s)h(x(s))\,ds - \int_0^t v(s)e^{-\int_s^t v(u)du}\left[\int_{s-\tau(s)}^{s} v(u)h(x(u))\,du\right] ds \\
&+ \int_0^t e^{-\int_s^t v(u)du}\left[b(s)h(x(s-\tau(s))) - v(s)Q(s,x(s-\tau(s)))\right. \\
&\left.+ G(s,x(s),x(s-\tau(s)))\right] ds, \qquad (2.2)
\end{aligned}$$

where

$$H(x) = x - h(x), \quad (2.3)$$

and

$$b(s) = (1-\tau'(s))v(s-\tau(s)) - a(s). \quad (2.4)$$

Proof. Let x be a solution of (2.1). Rewrite the equation (2.1) as

$$\begin{aligned}
&\frac{d}{dt}\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]+v\left(t\right)\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]\\
&=v\left(t\right)\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]-v\left(t\right)h\left(x\left(t\right)\right)+v\left(t\right)h\left(x\left(t\right)\right)\\
&-a\left(t\right)h\left(x\left(t-\tau\left(t\right)\right)\right)+G\left(t,x\left(t\right),x(t-\tau\left(t\right)\right)\\
&=v\left(t\right)\left[x\left(t\right)-h\left(x\left(t\right)\right)\right]+\frac{d}{dt}\int_{t-\tau(t)}^{t}v\left(u\right)h\left(x\left(u\right)\right)du\\
&+\left[\left(1-\tau'\left(t\right)\right)v\left(t-\tau\left(t\right)\right)-a\left(t\right)\right]h\left(x\left(t-\tau\left(t\right)\right)\right)\\
&+G\left(t,x\left(t\right),x\left(t-\tau\left(t\right)\right)\right)-v\left(t\right)Q\left(t,x\left(t-\tau\left(t\right)\right)\right).
\end{aligned}$$

Multiply both sides of the above equation by $\exp\left(\int_0^t v\left(u\right)du\right)$ and then integrate from 0 to t, we obtain

$$\begin{aligned}
&\int_0^t\left[\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]e^{\int_0^s v(u)du}\right]'ds\\
&=\int_0^t v\left(s\right)\left[x\left(s\right)-h\left(x\left(s\right)\right)\right]e^{\int_0^s v(u)du}ds\\
&+\int_0^t\left[\frac{d}{ds}\int_{s-\tau(s)}^{s}v\left(u\right)h\left(x\left(u\right)\right)du\right]e^{\int_0^s v(u)du}ds\\
&+\int_0^t\left[b\left(s\right)h\left(x\left(s-\tau\left(s\right)\right)\right)-v\left(s\right)Q\left(s,x\left(s-\tau\left(s\right)\right)\right)\right.\\
&\left.+G\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)\right]e^{\int_0^s v(u)du}ds,
\end{aligned}$$

where $b\left(s\right)=\left(1-\tau'\left(s\right)\right)v\left(s-\tau\left(s\right)\right)-a\left(s\right)$. As a consequence, we arrive at

$$\begin{aligned}
&\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]e^{\int_0^t v(u)du}-\psi\left(0\right)+Q\left(0,\psi\left(-\tau\left(0\right)\right)\right)\\
&=\int_0^t v\left(s\right)\left[x\left(s\right)-h\left(x\left(s\right)\right)\right]e^{\int_0^s v(u)du}ds\\
&+\int_0^t\left[\frac{d}{ds}\int_{s-\tau(s)}^{s}v\left(u\right)h\left(x\left(u\right)\right)du\right]e^{\int_0^s v(u)du}ds\\
&+\int_0^t\left[b\left(s\right)h\left(x\left(s-\tau\left(s\right)\right)\right)-v\left(s\right)Q\left(s,x\left(s-\tau\left(s\right)\right)\right)\right.\\
&\left.+G\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)\right]e^{\int_0^s v(u)du}ds.
\end{aligned}$$

By dividing both sides of the above equation by $\exp\left(\int_0^t v(u)du\right)$ we obtain

$$\begin{aligned}
& x(t) - Q(t, x(t-\tau(t))) - [\psi(0) - Q(0, x(-\tau(0)))] e^{-\int_0^t v(u)du} \\
& = \int_0^t v(s)[x(s) - h(x(s))] e^{-\int_s^t v(u)du} ds \\
& + \int_0^t \left[\frac{d}{ds}\int_{s-\tau(s)}^s v(u) h(x(u)) du\right] e^{-\int_s^t v(u)du} ds \\
& + \int_0^t [b(s) h(x(s-\tau(s))) - v(s) Q(s, x(s-\tau(s))) \\
& + G(s, x(s), x(s-\tau(s)))] e^{-\int_s^t v(u)du} ds. \qquad (2.5)
\end{aligned}$$

Integration by parts

$$\begin{aligned}
& \int_0^t \left[\frac{d}{ds}\int_{s-\tau(s)}^s v(u) h(x(u)) du\right] e^{-\int_s^t v(u)du} ds \\
& = \left[\int_{s-\tau(s)}^s v(u) h(x(u)) du e^{-\int_s^t v(u)du}\right]_0^t \\
& - \int_0^t \left[\int_{s-\tau(s)}^s v(u) h(x(u)) du\right] v(s) e^{-\int_s^t v(u)du} ds \\
& = \int_{t-\tau(t)}^t v(s) h(x(s)) ds - \int_{-\tau(0)}^0 v(s) h(\psi(s)) ds e^{-\int_0^t v(u)du} \\
& - \int_0^t \left[\int_{t-\tau(t)}^t v(u) h(x(u)) du\right] v(s) e^{-\int_s^t v(u)du} ds. \qquad (2.6)
\end{aligned}$$

Then substituting (2.6) into (2.5) we obtain (2.2). The converse implication is easily obtained and the proof is complete. ■

Now, we define Carathéodory function which its to use in following it.

Definition 2.1.2 *The map $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Carathéodory conditions with respect to $L^1[0,\infty)$ if the following conditions hold.*

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Lebesgue measurable.

(ii) For almost all $t \in [0,\infty)$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0,\infty), \mathbb{R}^+)$ such that for almost all $t \in [0,\infty)$ and for all z such that $|z| < r$, we have $|f(t,z)| \leq \alpha_r(t)$.

2.2 Stability of the zero solution

From the existence theory, which can be found in [31] or [62], we conclude that for each continuous initial function $\psi \in C([m_0, 0], \mathbb{R})$, there exists a continuous solution $x(t, 0, \psi)$

which satisfies (2.1) on an interval $[0,\sigma)$ for some $\sigma>0$ and $x(t,0,\psi)=\psi(t)$, $t\in[m_0,0]$. We refer the reader to [31] for the stability definitions.

To apply Theorem 1.2.14, we need to define a Banach space X, a closed bounded convex subset $\mathbb{M}$ of X and construct two mappings; one large contraction and the other is compact operator. So, let $w:[m_0,\infty)\to[1,\infty)$ be any strictly increasing and continuous function with $w(m_0)=1$, $w(t)\to\infty$ as $t\to\infty$. Let $(\mathbb{B},|\cdot|_w)$ be the Banach space of continuous $\varphi:[m_0,\infty)\to\mathbb{R}$ for which

$$|\varphi|_w=\sup_{t\in[m_0,\infty)}\left|\frac{\varphi(t)}{w(t)}\right|<\infty.$$

Let $R\in(0,1]$ and define the set

$$\mathbb{M}:=\{\varphi\in\mathbb{B}:\varphi\text{ is Lipschitzian, }|\varphi(t,0,\psi)|\le R,\ t\in[m_0,\infty)\}. \tag{2.7}$$

Clearly, if $\{\varphi_n\}$ is a sequence of l_1-Lipschitzian functions converging to some function φ, then

$$\begin{aligned}|\varphi(t)-\varphi(s)| &= |\varphi(t)-\varphi_n(t)+\varphi_n(t)-\varphi_n(s)+\varphi_n(s)-\varphi(s)|\\ &\le |\varphi(t)-\varphi_n(t)|+|\varphi_n(t)-\varphi_n(s)|+|\varphi_n(s)-\varphi(s)|\\ &\le l_1|t-s|,\end{aligned}$$

as $n\to\infty$, which implies φ is l_1-Lipschitzian. It is clear that $\mathbb{M}$ is closed convex and bounded. For $\varphi\in\mathbb{M}$ and $t\ge0$, we define by (2.2) the mapping $\mathcal{P}:\mathbb{M}\to\mathbb{B}$ as follows

$$\begin{aligned}&(\mathcal{P}\varphi)(t)\\ &=\left[\psi(0)-Q(0,\psi(-\tau(0)))-\int_{-\tau(0)}^{0}v(s)h(\psi(s))\,ds\right]e^{-\int_0^t v(u)du}\\ &+\int_0^t v(s)e^{-\int_s^t v(u)du}H(\varphi(s))\,ds+Q(t,\varphi(t-\tau(t)))\\ &+\int_{t-\tau(t)}^{t}v(s)h(\varphi(s))\,ds-\int_0^t v(s)e^{-\int_s^t v(u)du}\left[\int_{s-\tau(s)}^{s}v(u)h(\varphi(u))\,du\right]ds\\ &+\int_0^t e^{-\int_s^t v(u)du}\left[b(s)h(\varphi(s-\tau(s)))-v(s)Q(s,\varphi(s-\tau(s)))\right.\\ &\left.+G(s,\varphi(s),\varphi(s-\tau(s)))\right]ds.\end{aligned} \tag{2.8}$$

We express the equation (2.8) as

$$\mathcal{P}\varphi=\mathcal{A}\varphi+\mathcal{B}\varphi,$$

where $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{B}$ are given by

$$\begin{aligned}(\mathcal{A}\varphi)(t) = {} & Q(t, \varphi(t-\tau(t))) + \int_{t-\tau(t)}^{t} v(s) h(\varphi(s)) du \\ & - \int_0^t v(s) e^{-\int_s^t v(u)du} \left[\int_{s-\tau(s)}^{s} v(u) h(\varphi(u)) du \right] ds \\ & + \int_0^t e^{-\int_s^t v(u)du} [b(s) h(\varphi(s-\tau(s))) - v(s) Q(s, \varphi(s-\tau(s))) \\ & + G(s, \varphi(s), \varphi(s-\tau(s)))] ds, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}(\mathcal{B}\varphi)(t) = {} & \left[\psi(0) - Q(0, \psi(-\tau(0))) - \int_{-\tau(0)}^{0} v(s) h(\psi(s)) ds \right] e^{-\int_0^t v(u)du} \\ & + \int_0^t v(s) e^{-\int_s^t v(u)du} H(\varphi(s)) ds. \end{aligned} \tag{2.10}$$

By applying Theorem 1.2.14, we need to prove that $\mathcal{P}$ has a fixed point φ on the set $\mathbb{M}$, where $\varphi(t) = x(t, 0, \psi)$ for $t \geq 0$ and $x(t, 0, \psi) = \psi(t)$ on $[m_0, 0]$, $x(t, 0, \psi)$ satisfies (2.1) and $|\varphi(t, 0, \psi)| \leq R$ with $R \in (0, 1]$. For $t \geq 0$, we will assume that the following conditions hold.
The functions h, Q are locally Lipschitz continuous, then for $t \geq 0$ and x, $y \in \mathbb{M}$ there exist a constants $E_h, E_Q > 0$, such that

$$|Q(t, x) - Q(t, y)| \leq E_Q \|x - y\|, \tag{2.11}$$

$$|h(x) - h(y)| \leq E_h \|x - y\|, \tag{2.12}$$

The functions Q, G satisfy Carathéodory conditions with respect to $L^1[0, \infty)$, such that

$$|Q(t, \varphi(t-\tau(t)))| \leq q_R(t) \leq \frac{\alpha_1}{2} R, \tag{2.13}$$

$$|G(t, \varphi(t), \varphi(t-\tau(t)))| \leq g_{\sqrt{2}R}(t) \leq \alpha_2 v(t) R, \tag{2.14}$$

$$\beta_1 \beta_2 E_h \leq \frac{\alpha_3}{2}, \tag{2.15}$$

where $\beta_1 = \sup_{t \in [0,\infty)} |\tau(t)|$, $\beta_2 = \sup_{t \in [0,\infty)} \{v(t)\}$,

$$|b(t)| E_h \leq \alpha_4 v(t), \tag{2.16}$$

$$J[\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4] \leq 1, \tag{2.17}$$

where α_i, $1 \leq i \leq 4$ are positive constants and $J > 3$. Now, assume that there are constants $l_2, l_3 > 0$ such that for $0 \leq t_1 < t_2$

$$|\tau(t_2) - \tau(t_1)| \leq l_2 |t_2 - t_1|, \tag{2.18}$$

$$\left| \int_{t_1}^{t_2} v(u) du \right| \leq l_3 |t_2 - t_1|. \tag{2.19}$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 1.2.14.

Lemma 2.2.1 *For $\mathcal{A}$ defined in (2.9), suppose that (2.11)–(2.19) hold. Then, $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ and $\mathcal{A}$ is continuous and $\mathcal{A}\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$.*

Proof. Let $\mathcal{A}$ be defined by (2.9). Observe that in view of (2.12) we have

$$\begin{aligned} |h(x)| &= |h(x) - h(0) + h(0)| \\ &\leq |h(x) - h(0)| + |h(0)| \\ &\leq E_h \|x\| . \end{aligned}$$

So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned} |\mathcal{A}\varphi(t)| \leq{}& |Q(t, \varphi(t - \tau(t)))| + \int_{t-\tau(t)}^{t} v(u) |h(\varphi(u))| \, du \\ &+ \int_0^t v(s) e^{-\int_s^t v(u)du} \left[\int_{s-\tau(s)}^{s} v(u) |h(\varphi(u))| \, du \right] ds \\ &+ \int_0^t e^{-\int_s^t v(u)du} \left[|b(s)| \, |h(\varphi(s - \tau(s)))| + v(s) |Q(s, \varphi(s - \tau(s)))| \right. \\ &\left. + |G(s, \varphi(s), \varphi(s - \tau(s)))| \right] ds \\ &\leq q_R(t) + R \int_{t-\tau(t)}^{t} v(u) E_h du \\ &+ R \int_0^t v(s) e^{-\int_s^t v(u)du} \left[\int_{s-\tau(s)}^{s} v(u) E_h du \right] ds \\ &+ R \int_0^t e^{-\int_s^t v(u)du} |b(s)| E_h ds \\ &+ R \int_0^t e^{-\int_s^t v(u)du} \left(v(s) q_R(s) + \frac{g_{\sqrt{2}R}(s)}{R} \right) ds \\ &\leq \frac{\alpha_1}{2} R + \frac{\alpha_1}{2} R + \alpha_2 R + \frac{\alpha_3}{2} R + \frac{\alpha_3}{2} R + \alpha_4 R \leq \frac{R}{J} < R. \end{aligned}$$

That is $\|\mathcal{A}\varphi\| < R$. Second we show that, for any $\varphi \in \mathbb{M}$ the function $\mathcal{A}\varphi$ is Lipschitzian.

Let $\varphi \in \mathbb{M}$, and let $0 \leq t_1 < t_2$, then

$$\begin{aligned}
&\left|\mathcal{A}\varphi(t_2) - \mathcal{A}\varphi(t_1)\right| \\
&\leq \left|Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))\right| \\
&+ \left|\int_{t_2-\tau(t_2)}^{t_2} v(s) h(\varphi(s)) du - \int_{t_1-\tau(t_1)}^{t_1} v(s) h(\varphi(s)) ds\right| \\
&+ \left|\int_0^{t_2} v(s) e^{-\int_s^{t_2} v(u)du} \left[\int_{s-\tau(s)}^{s} v(u) h(\varphi(u)) du\right] ds\right. \\
&\left. - \int_0^{t_1} v(s) e^{-\int_s^{t_1} v(u)du} \left[\int_{s-\tau(s)}^{s} v(u) h(\varphi(u)) du\right] ds\right| \\
&+ \left|\int_0^{t_2} e^{-\int_s^{t_2} v(u)du} b(s) h(\varphi(s - \tau(s))) ds\right. \\
&\left. - \int_0^{t_1} e^{-\int_s^{t_1} v(u)du} b(s) h(\varphi(s - \tau(s))) ds\right| \\
&+ \left|\int_0^{t_2} e^{-\int_s^{t_2} v(u)du} \left[-v(s) Q(s, \varphi(s - \tau(s))) + G(s, \varphi(s), \varphi(s - \tau(s)))\right] ds\right. \\
&\left. - \int_0^{t_1} e^{-\int_s^{t_1} v(u)du} \left[-v(s) Q(s, \varphi(s - \tau(s))) + G(s, \varphi(s), \varphi(s - \tau(s)))\right] ds\right|. \qquad (2.20)
\end{aligned}$$

By hypotheses (2.11), (2.12), (2.18) and (2.19), we have

$$\begin{aligned}
&\left|\int_{t_2-\tau(t_2)}^{t_2} v(s) h(\varphi(s)) ds - \int_{t_1-\tau(t_1)}^{t_1} v(s) h(\varphi(s)) ds\right| \\
&\leq E_h R\left(\int_{t_1}^{t_2} v(s) ds + \int_{t_1-\tau(t_1)}^{t_2-\tau(t_2)} v(s) ds\right) \\
&\leq E_h R\left(\int_{t_1}^{t_2} v(s) ds + \int_{t_1-\tau(t_1)}^{t_2-\tau(t_2)} v(s) ds\right) \\
&\leq E_h R l_3 |t_2 - t_1| + E_h R l_3 (1 + l_2) |t_2 - t_1| \\
&= (2E_h R l_3 + E_h R l_3 l_2) |t_2 - t_1|, \qquad (2.21)
\end{aligned}$$

and

$$\begin{aligned}
&\left|Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))\right| \\
&\leq E_Q l_1 \left|(t_2 - t_1) - (\tau(t_2) - \tau(t_1))\right| \\
&\leq (E_Q l_1 + E_Q l_1 l_2) |t_2 - t_1|, \qquad (2.22)
\end{aligned}$$

where l_1 is the Lipschitz constant of φ. By the hypotheses (2.12), (2.16) and (2.19), we have

$$\left|\int_0^{t_2} e^{-\int_s^{t_2} v(u)du} b(s)\, h(\varphi(s-\tau(s)))\, ds\right.$$
$$\left.-\int_0^{t_1} e^{-\int_s^{t_1} v(u)du} b(s)\, h(\varphi(s-\tau(s)))\, ds\right|$$
$$\leq \left|\int_0^{t_1} b(s)\, h(\varphi(s-\tau(s)))\, e^{-\int_s^{t_1} v(u)du}\left(e^{-\int_{t_1}^{t_2} v(u)du}-1\right) ds\right|$$
$$+\left|\int_{t_1}^{t_2} e^{-\int_s^{t_2} v(u)du} b(s)\, h(\varphi(s-\tau(s)))\, ds\right|$$
$$\leq \alpha_4 R\left|e^{-\int_{t_1}^{t_2} v(u)du}-1\right|\int_0^{t_1} v(s)\, e^{-\int_s^{t_1} v(u)du} ds$$
$$+E_h R\int_{t_1}^{t_2} e^{-\int_s^{t_2} v(u)du}\,|b(s)|\, ds.$$

Consequently,

$$\left|\int_0^{t_2} e^{-\int_s^{t_2} v(u)du} b(s)\, h(\varphi(s-\tau(s)))\, ds\right.$$
$$\left.-\int_0^{t_1} e^{-\int_s^{t_1} v(u)du} b(s)\, h(\varphi(s-\tau(s)))\, ds\right|$$
$$\leq \alpha_4 R\int_{t_1}^{t_2} v(u)\, du+E_h R\int_{t_1}^{t_2} e^{-\int_s^{t_2} v(u)du} d\left(\int_{t_1}^{s} |b(r)|\, dr\right) ds$$
$$=\alpha_4 R\int_{t_1}^{t_2} v(u)\, du+E_h R\left[e^{-\int_s^{t_2} v(u)du}\int_{t_1}^{s} |b(r)|\, dr\right]_{t_1}^{t_2}$$
$$+E_h R\int_{t_1}^{t_2} v(s)\, e^{-\int_s^{t_2} v(u)du}\int_{t_1}^{s} |b(r)|\, dr ds$$
$$\leq \alpha_4 R\int_{t_1}^{t_2} v(u)\, du+E_h R\int_{t_1}^{t_2} |b(s)|\, ds\left(1+\int_{t_1}^{t_2} v(s)\, e^{-\int_s^{t_2} v(u)du} ds\right)$$
$$\leq \alpha_4 R\int_{t_1}^{t_2} v(u)\, du+2E_h R\int_{t_1}^{t_2} |b(s)|\, ds$$
$$\leq \alpha_4 R\int_{t_1}^{t_2} v(u)\, du+2\alpha_4 R\int_{t_1}^{t_2} v(u)\, du\leq 3\alpha_4 R l_3\,|t_2-t_1|\,. \tag{2.23}$$

In the same way, by (2.13)–(2.15) and (2.19), we have

$$\left|\int_0^{t_2} e^{-\int_s^{t_2} v(u)du}\left[-v(s)\, Q(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right] ds\right.$$
$$\left.-\int_0^{t_1} e^{-\int_s^{t_1} v(u)du}\left[-v(s)\, Q(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right] ds\right|$$
$$\leq 3R\left(\frac{\alpha_1}{2}+\alpha_2\right) l_3\,|t_2-t_1|\,, \tag{2.24}$$

and

$$\begin{aligned}
&+\left|\int_0^{t_2} v(s)\, e^{-\int_s^{t_2} v(u)du}\left[\int_{s-\tau(s)}^{s} v(u)\, h(\varphi(u))\, du\right] ds\right. \\
&\left.-\int_0^{t_1} v(s)\, e^{-\int_s^{t_1} v(u)du}\left[\int_{s-\tau(s)}^{s} v(u)\, h(\varphi(u))\, du\right] ds\right| \\
&\leq \frac{3}{2} R\alpha_3 l_3 \left|t_2 - t_1\right|. \qquad (2.25)
\end{aligned}$$

Thus, by substituting (2.21)–(2.25) in (2.20), we obtain

$$\begin{aligned}
\left|\mathcal{A}\varphi(t_2) - \mathcal{A}\varphi(t_1)\right| &\leq \left(E_Q l_1 + E_Q l_1 l_2\right)\left|t_2 - t_1\right| + \left(2E_h R l_3 + E_h R l_3 l_2\right)\left|t_2 - t_1\right| \\
&+ 3R\left(\frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} + \alpha_4\right) l_3 \left|t_2 - t_1\right| \\
&= K\left|t_2 - t_1\right|,
\end{aligned}$$

for some constant $K > 0$. This shows that $\mathcal{A}\varphi$ is Lipschitzian if φ is. This complete to prove $\mathcal{A} : \mathbb{M} \to \mathbb{M}$.

Since $\mathcal{A}\varphi$ is Lipschitzian, then $\mathcal{A}\mathbb{M}$ is equicontinuous, which implies that the set $\mathcal{A}\mathbb{M}$ resides in a compact set in the space $(\mathbb{B}, |\cdot|_w)$.

Now, we show that $\mathcal{A}$ is continuous in the weighted norm, let $\varphi_n \in \mathbb{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&\left|\frac{\mathcal{A}\varphi_n(t) - \mathcal{A}\varphi(t)}{w(t)}\right| \\
&\leq \left|Q(t, \varphi_n(t - \tau(t))) - Q(t, \varphi(t - \tau(t)))\right|_w \\
&+ \int_{t-\tau(t)}^{t} v(s)\left|h(\varphi_n(s)) - h(\varphi(s))\right|_w ds \\
&+ \int_0^t v(s)\, e^{-\int_s^t v(u)du} \int_{s-\tau(s)}^{s} v(s)\left|h(\varphi_n(u)) - h(\varphi(u))\right|_w du ds \\
&+ \int_0^t e^{-\int_s^t v(u)du}\left|b(s)\right|\left|h(\varphi_n(s - \tau(s))) - h(\varphi(s - \tau(s)))\right|_w ds \\
&+ \int_0^t v(s)\, e^{-\int_s^t v(u)du}\left|Q(s, \varphi_n(s - \tau(s))) - Q(s, \varphi(s - \tau(s)))\right|_w ds \\
&+ \int_0^t e^{-\int_s^t v(u)du}\left|G(s, \varphi_n(s), \varphi_n(s - \tau(s))) - G(s, \varphi(s), \varphi(s - \tau(s)))\right|_w ds.
\end{aligned}$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty}\left|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)\right| = 0$. Then $\mathcal{A}$ is continuous. This complete to prove $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathcal{A}\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$. ■

The next result shows the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions, for this assume that

$$\max\left\{\left|H(-R)\right|, \left|H(R)\right|\right\} \leq \frac{2R}{J}. \qquad (2.26)$$

Choose $\gamma > 0$ small enough such that

$$\left[1 + E_Q + E_h \int_{-\tau(0)}^{0} v\left(u\right) du\right] \gamma e^{-\int_0^t v(u)du} + \frac{R}{J} + \frac{2R}{J} \leq R. \tag{2.27}$$

The chosen in the relation (2.27) will be used below in Lemma 2.2.2 and Theorem 2.2.3 to show that if $\epsilon = R$ and if $\|\psi\| < \gamma$, then the solutions satisfies $|x\left(t, 0, \psi\right)| < \epsilon$.

Lemma 2.2.2 *Let $\mathcal{B}$ be defined by (2.10), suppose (2.18), (2.19), (2.26), (2.27) and the condition of theorem 1.1.17 hold. Then $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ and $\mathcal{B}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (2.10). Obviously, $\mathcal{B}\varphi$ is continuous with the weighted norm. Let $\varphi \in \mathbb{M}$

$$\begin{aligned}
|\mathcal{B}\varphi(t)| \leq & \left|\psi\left(0\right) - Q\left(0, \psi\left(-\tau\left(0\right)\right)\right) - \int_{-\tau(0)}^{0} v(s) h\left(\psi(s)\right) ds\right| e^{-\int_0^t v(u)du} \\
& + \int_0^t v(s) e^{-\int_s^t v(u)du} \left|H\left(\varphi(s)\right)\right| ds \\
& \leq \left[1 + E_Q + E_h \int_{-\tau(0)}^{0} v\left(u\right) du\right] \gamma e^{-\int_0^t v(u)du} \\
& + \int_0^t v(s) e^{-\int_s^t v(u)du} \max\left\{\left|H\left(-R\right)\right|, \left|H\left(R\right)\right|\right\} ds < R,
\end{aligned}$$

and we use a method like in Lemma 2.2.1, we deduce that, for any $\varphi \in \mathbb{M}$ the function $\mathcal{B}\varphi$ is Lipschitzian, which implies $\mathcal{B} : \mathbb{M} \to \mathbb{M}$.

By Theorem 1.1.17, H is large contraction on $\mathbb{M}$, then for any $\varphi, \phi \in \mathbb{M}$, with $\varphi \neq \phi$ and for any $\epsilon > 0$, from the proof of that Theorem, we have found a $\delta < 1$, such that

$$\begin{aligned}
\left|\frac{\mathcal{B}\varphi\left(t\right) - \mathcal{B}\phi\left(t\right)}{w\left(t\right)}\right| & \leq \int_0^t v(s) e^{-\int_s^t v(u)du} \left|H\left(\varphi\left(u\right)\right) - H\left(\phi\left(u\right)\right)\right|_w du \\
& \leq \delta \left|\varphi - \phi\right|_w .
\end{aligned}$$

The proof is complete. ■

Theorem 2.2.3 *Assume the hypothesis of Lemmas 2.2.1 and 2.2.2. Let $\mathbb{M}$ defined by (2.7). Then the equation (2.1) has a solution in $\mathbb{M}$.*

Proof. By Lemmas 2.2.1, 2.3.1, $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 2.2.2, the mapping $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction. Next, we show that if $\varphi, \phi \in \mathbb{M}$, we have $\|\mathcal{A}\varphi + \mathcal{B}\phi\| \leq R$. Let $\varphi, \phi \in \mathbb{M}$ with $\|\varphi\|, \|\phi\| \leq$

R. By (2.13)–(2.17)

$$\begin{aligned}
\|\mathcal{A}\varphi + \mathcal{B}\varphi\| &\leq \left[1 + E_Q + E_h \int_{-\tau(0)}^{0} v(u)\,du\right] \gamma e^{-\int_0^t v(u)du} \\
&\quad + [\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4] R + \frac{2R}{J} \\
&\leq \left[1 + E_Q + E_h \int_{-\tau(0)}^{0} v(u)\,du\right] \gamma e^{-\int_0^t v(u)du} + \frac{R}{J} + \frac{2R}{J} \\
&\leq R.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 2.1.1 this fixed point is a solution of (2.1). Hence (2.1) is stable. ■

2.3 Asymptotic stability

Now, for the asymptotic stability, define $\mathbb{M}_0$ by

$$\begin{aligned}
\mathbb{M}_0 \;\; : \;\; &= \{\varphi \in \mathbb{B} : \varphi \text{ is Lipschitzian, } |\varphi(t, 0, \psi)| \leq R,\ t \in [m_0, \infty), \\
&\varphi(t) = \psi(t) \text{ if } t \in [m_0, 0] \text{ and } |\varphi(t)| \to 0 \text{ as } t \to \infty\}. \qquad (2.28)
\end{aligned}$$

All of the calculations in the proof of Theorem 2.2.3 hold with $w(t) = 1$ when $|\cdot|_w$ is replaced by the supremum norm $\|\cdot\|$. Now, assume that

$$t - \tau(t) \to \infty \text{ as } t \to \infty \text{ and } \int_0^t v(s)\,ds \to \infty \text{ as } t \to \infty, \qquad (2.29)$$

$$\frac{b(t)}{v(t)} \to 0 \text{ as } t \to \infty, \qquad (2.30)$$

$$q_R(t) \to 0 \text{ as } t \to \infty, \qquad (2.31)$$

$$\frac{g_{\sqrt{2}R}(t)}{v(t)} \to 0 \text{ as } t \to \infty. \qquad (2.32)$$

Lemma 2.3.1 *Let (2.11)–(2.19) and (2.29)–(2.32) hold. Then, the operator $\mathcal{A}$ maps $\mathbb{M}_0$ into a compact subset of $\mathbb{M}_0$.*

Proof. First, we deduce by the Lemma 2.2.1 that $\mathcal{A}(\mathbb{M}_0)$ is equicontinuous. Next, we notice that for arbitrary $\varphi \in \mathbb{M}_0$ we have

$$\begin{aligned}
|\mathcal{A}\varphi(t)| &\leq q_R(t) + E_h R \int_{t-\tau(t)}^{t} v(s)ds + E_h R \int_0^t v(s) e^{-\int_s^t v(u)du} \int_{t-\tau(t)}^{t} v(u)duds \\
&\quad + \int_0^t e^{-\int_s^t v(u)du} v(s) \left[\left|\frac{b(s)}{v(s)}\right| E_h + R q_R(s) + \frac{g_{\sqrt{2}R}(s)}{v(s)}\right] ds \\
&:= q(t).
\end{aligned}$$

We see that $q(t) \to 0$ as $t \to \infty$, which implies that the set $\mathcal{A}\mathbb{M}_0$ resides in a compact set in the space $(\mathbb{B}, \|\cdot\|)$ by Theorem 1.1.15. ∎

Theorem 2.3.2 *Assume the hypothesis of Lemmas 2.2.2 and 2.3.1 hold. Let* $\mathbb{M}_0$ *defined by (2.28). Then the equation (2.1) has a solution in* $\mathbb{M}_0$.

Proof. Note that, all of the steps in the proof of Theorem 2.2.3 hold with $w(t) = 1$ when $|\cdot|_w$ is replaced by the supremum norm $\|\cdot\|$. It is sufficient to show, for $\varphi \in \mathbb{M}_0$ then $\mathcal{A}\varphi \to 0$ and $\mathcal{B}\varphi \to 0$. Let $\varphi \in \mathbb{M}_0$ be fixed, we will prove that $|\mathcal{A}\varphi(t)| \to 0$ as $t \to \infty$, as above we have

$$\begin{aligned}
|\mathcal{A}\varphi(t)| &\leq |Q(t, \varphi(t-\tau(t)))| + \int_{t-\tau(t)}^{t} v(u)\,|h(\varphi(u))|\,du \\
&+ \int_0^t v(s)\,e^{-\int_s^t v(u)du} \left[\int_{s-\tau(s)}^{s} v(u)\,|h(\varphi(u))|\,du\right] ds \\
&+ \int_0^t e^{-\int_s^t v(u)du} \left[|b(s)|\,|h(\varphi(s-\tau(s)))| + v(s)\,|Q(s, \varphi(s-\tau(s)))|\right. \\
&\left.+ |G(s, \varphi(s), \varphi(s-\tau(s)))|\right] ds.
\end{aligned}$$

First, we have

$$|Q(t, \varphi(t-\tau(t)))| \leq q_R(t) \to 0 \text{ as } t \to \infty,$$

and

$$\int_{t-\tau(t)}^{t} v(u)\,|h(\varphi(u))|\,du \leq E_h R \int_{t-\tau(t)}^{t} v(u)\,du \to 0 \text{ as } t \to \infty.$$

Second, let $\epsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))|, |\varphi(t)| < \epsilon$, for $t \geq T$. Then we have

$$\begin{aligned}
&\int_0^t v(s)\,e^{-\int_s^t v(u)du} \left[\int_{s-\tau(s)}^{s} v(u)\,|h(\varphi(u))|\,du\right] ds \\
&= e^{-\int_T^t v(u)du} \int_0^T v(s)\,e^{-\int_s^T v(u)du} \left[\int_{s-\tau(s)}^{s} v(u)\,|h(\varphi(u))|\,du\right] ds \\
&+ \int_T^t v(s)\,e^{-\int_s^t v(u)du} \left[\int_{s-\tau(s)}^{s} v(u)\,|h(\varphi(u))|\,du\right] ds \\
&\leq e^{-\int_T^t v(u)du} \frac{\alpha_3}{2} R + \frac{\alpha_3}{2}\epsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)du} \left(|b(s)| \left|h(\varphi(s-\tau(s)))\right|\right.\\
&+ v(s)\left|Q(s,\varphi(s-\tau(s)))\right| + \left.\left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right) ds\\
&= e^{-\int_T^t v(u)du} \int_0^T e^{-\int_s^T v(u)du} \left(|b(s)| \left|h(\varphi(s-\tau(s)))\right|\right.\\
&+ v(s)\left|Q(s,\varphi(s-\tau(s)))\right| + \left.\left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right) ds\\
&+ \int_T^t e^{-\int_s^t v(u)du} \left(|b(s)| \left|h(\varphi(s-\tau(s)))\right|\right.\\
&+ v(s)\left|Q(s,\varphi(s-\tau(s)))\right| + \left.\left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right) ds\\
&\leq e^{-\int_T^t v(u)du} \left(\frac{\alpha_1}{2} + \alpha_2 + \alpha_4\right) R + \left(\frac{\alpha_1}{2} + \alpha_2 + \alpha_4\right) \epsilon.
\end{aligned}$$

By (2.29) the terms $e^{-\int_T^t v(u)du}\frac{\alpha_3}{2}R$ and $e^{-\int_T^t v(u)du}\left(\frac{\alpha_1}{2}+\alpha_2+\alpha_4\right)R$ are, as $t\to\infty$, arbitrarily small. In the same way for $\mathcal{B}\varphi \to 0$. This end the proof. ■

We give an example to illustrate the application of Theorems 2.2.3 and 2.3.2.

Example 2.3.3 *Consider the following nonlinear neutral differential equation with variable delay*

$$\frac{d}{dt}x(t) = -a(t)\,h(x(t-\tau(t))) + \frac{d}{dt}Q(t,x(t-\tau(t))) + G(t,x(t),x(t-\tau(t))), \quad (2.33)$$

where $\tau(t) = 2.10^{-2}e^{-t}$, $a(t) = \frac{1+4.10^{-4}e^{-2t}-2.10^{-2}te^{-t}}{1+t-2.10^{-2}e^{-t}}$, $Q(t,x) = \frac{x}{100e^t}$, $G(t,x,y) = \frac{x^2+y^2}{50e^t}$, $h(x) = x^3$. *Then the zero solution of (2.33) is asymptotically stable.*

Proof. We have $h:\mathbb{R}\to\mathbb{R}$ is continuous on $\left[-\sqrt{3}/3,\sqrt{3}/3\right]$, differentiable on $\left(-\sqrt{3}/3,\sqrt{3}/3\right)$, strictly increasing on $\left[-\sqrt{3}/3,\sqrt{3}/3\right]$ and $\sup_{t\in\left(-\sqrt{3}/3,\sqrt{3}/3\right)} h'(t) \leq 1$. By Theorem 1.1.17, the mapping $H(x) = x - x^3$ is a large contraction on the set

$$\begin{aligned}
\mathbb{M}_0 \ : \ &= \Big\{\varphi\in\mathbb{B} : \varphi \text{ is Lipschitzian, } |\varphi(t,0,\psi)| \leq \sqrt{3}/3,\ t\in\left[-2.10^{-2},\infty\right)\\
&\varphi(t) = \psi(t) \text{ if } t\in\left[-2.10^{-2},0\right] \text{ and } |\varphi(t)|\to 0 \text{ as } t\to\infty\Big\}.
\end{aligned}$$

Choosing $v(t) = \frac{1}{1+t}$, clearly condition (2.19) holds. Furthermore, we have $m_0 = -2.10^{-2}$, $R = \sqrt{3}/3$, $b(t) = 2.10^{-2}e^{-t}$, $h(0) = Q(t,0) = G(t,0,0) = 0$, $E_h = 1$, $E_Q = \frac{1}{100}$, $q_R(t) = \frac{\sqrt{3}e^{-t}}{300}$, $g_{\sqrt{2}R}(t) = \frac{e^{-t}}{75}$, $\alpha_1 = \frac{1}{50}$, $\alpha_2 = \frac{1}{25\sqrt{3}}$, $\alpha_3 = 4.10^{-2}$, $\alpha_4 = 2.10^{-2}$, $\beta_1 = 2.10^{-2}$, $\beta_2 = 1$, $l_2 = 2.10^{-2}$, $l_3 = 1$, $J \in \left(3, \frac{25\sqrt{3}}{2\sqrt{3}+1}\right]$.
It is easy to see that all the conditions of Theorems 2.2.3 and 2.3.2 hold. Thus, Theorem 2.3.2 implies that the zero solution of (2.33) is asymptotically stable. ■

CHAPTER 3

Study of the periodic or nonnegative periodic solutions in nonlinear neutral differential equations with functional delay

Keywords. Krasnoselskii-Burton's theorem; large contraction; neutral differential equation; integral equation; periodic solution; nonnegative solution.

The goal of this chapter is to present a recent work published in [87], namely,
Mesmouli M. B., Ardjouni A. and A. Djoudi., Study of the Periodic or Nonnegative Periodic Solutions of Functional Differential Equations via Krasnoselskii–Burton's Theorem, Differ Equ Dyn Syst, DOI 10.1007/s12591-014-0235-5.

In this chapter, we try to study the existence of periodic or nonnegative periodic solutions of the nonlinear neutral differential equation. We invert the equation to construct a sum of a compact map and a large contraction which is suitable for applying the modification of Krasnoselskii's theorem.

3.1 Introduction and preliminaries

The use of ordinary and partial differential equations to model physical or biological systems and processes has a long history, dating to Lotka and Volterra. But all processes take time delays to complete. The delays can represent gestation times, incubation periods, or transport delays. In many cases time delays can be substantial such as gestation and maturation or can represent little lags such as acceleration and deceleration in physical processes. Therefore, it become natural to include time delay terms into the differential equations that model population dynamics. The models that incorporate such delay times are referred as delay differential equation models.

In the last fifty years, delay models are becoming more common, appearing in many branches of biological, economical and physical modelling (see [1, 16, 17, 18, 19, 21, 22, 26, 31, 62, 63, 72, 95, 96, 97, 105, 109, 111, 114]). This is due to their advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes and to the fact that such models operate on an infinite dimensional space consisting of continuous functions that accommodate high dimensional dynamics (see [31], [62] and [63]).

More recently investigators have given special attentions to the study of equations in which the delay occurs in the derivative of the state variable as well as in the independent variable, so called neutral differential equations. As known in Hale [62], Hale and Lunel [63] neutral delay differential equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

Existence, uniqueness, stability and positivity of solutions of functional differential equations are of great interest in mathematics and its applications to the modeling of various practical problems (see [1, 16, 17, 18, 19, 21, 22, 26, 31, 62, 63, 72, 95, 96, 97, 105, 109, 111, 114]) and references therein. Positivity is one of the most common and most important characteristics of mathematical models. In problem of economics, the positivity is quite important for processes that model interest rate dynamics on financial market, because the interest must be positive. Also, in fluid flow problems, densities, pressures, and concentrations are always positive.

In this chapter, we study the existence of periodic or nonnegative periodic solutions of the nonlinear neutral differential equations

$$\begin{aligned}&\frac{d}{dt}\left[x(t)-Q(t,x(t-\tau(t)))\right]\\&=-a(t)\,h(x(t-\tau(t)))+G(t,x(t),x(t-\tau(t))),\end{aligned}\tag{3.1}$$

where a is a positive continuous real-valued function. The function $h:\mathbb{R}\to\mathbb{R}$ is continuous, $Q:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $G:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying the Caratheodory condition. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due Burton (see [26], Theorem 3) to show the existence of periodic and nonnegative periodic solutions for the equation (3.1). Clearly, the present problem is totally nonlinear so that the variation of parameters cannot be applied directly. Then, we resort to the idea of adding and subtracting of terms. As noted by Burton in [26], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we use the variation of parameter formula and the integration by parts to transform (3.1) into an integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, to show the existence of periodic or nonnegative periodic solutions.

For $T>0$ define $P_T=\{\phi:\phi\in C(\mathbb{R},\mathbb{R}),\ \phi(t+T)=\phi(t)\}$ where $C(\mathbb{R},\mathbb{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\|=\max_{t\in[0,T]}|x(t)|.$$

In this paper we assume that

$$a(t-T)=a(t),\quad \tau(t-T)=\tau(t),\quad \tau(t)\geq\tau^*>0,\tag{3.2}$$

with τ continuously differentiable, τ^* is constant, a is positive and

$$1 - e^{-\int_{t-T}^{t} a(s)ds} \equiv \frac{1}{\eta} \neq 0. \tag{3.3}$$

The functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T. That is

$$Q(t-T,x) = Q(t,x), \; G(t-T,x,y) = G(t,x,y). \tag{3.4}$$

The following lemma is fundamental to our results.

Lemma 3.1.1 *Suppose (3.2)–(3.4) hold. If $x \in P_T$, then x is a solution of equation (3.1) if and only if*

$$\begin{aligned}
&x(t) \\
&= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, [x(u) - h(x(u))]\, du + Q(t, x(t-\tau(t))) \\
&+ \int_{t-\tau(t)}^{t} a(u)\, h(x(u))\, du - \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\, h(x(s))\, ds du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[(1-\tau'(u))\, a(u-\tau(u)) - a(u)\right] h(x(u-\tau(u)))\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[-a(u)\, Q(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))\right] du,
\end{aligned} \tag{3.5}$$

where

$$\kappa(t,u) = e^{-\int_{u}^{t} a(s)ds}. \tag{3.6}$$

Proof. Let $x \in P_T$ be a solution of (3.1). Rewrite the equation (3.1) as

$$\begin{aligned}
&\frac{d}{dt}\left[x(t) - Q(t, x(t-\tau(t)))\right] + a(t)\left[x(t) - Q(t, x(t-\tau(t)))\right] \\
&= a(t)\left[x(t) - Q(t, x(t-\tau(t)))\right] - a(t)\, h(x(t)) + a(t)\, h(x(t)) \\
&- a(t)\, h(x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))) \\
&= a(t)\left[x(t) - h(x(t))\right] + \frac{d}{dt}\int_{t-\tau(t)}^{t} a(s)\, h(x(s))\, ds \\
&+ \left((1-\tau'(t))\, a(t-\tau(t)) - a(t)\right) h(x(t-\tau(t))) \\
&- a(t)\, Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))).
\end{aligned}$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t

to obtain

$$\begin{aligned}
&\int_{t-T}^{t}\left[\left(x\left(u\right)-Q\left(u,x\left(u-\tau\left(u\right)\right)\right)\right)e^{\int_0^u a(s)ds}\right]'du\\
&=\int_{t-T}^{t}a\left(u\right)\left[x\left(u\right)-h\left(x\left(u\right)\right)\right]e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{u}a\left(s\right)\left(x\left(s\right)\right)ds\right]e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}\left(\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)-a\left(u\right)\right)h\left(x\left(u-\tau\left(u\right)\right)\right)e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}\left[-a\left(u\right)Q\left(u,x\left(u-\tau\left(u\right)\right)\right)+G\left(u,x\left(u\right),x\left(u-\tau\left(u\right)\right)\right)\right]e^{\int_0^u a(s)ds}du.
\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&\left(x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right)e^{\int_0^t a(s)ds}\\
&-\left(x\left(t-T\right)-Q\left(t-T,x\left(t-T-\tau\left(t-T\right)\right)\right)\right)e^{\int_0^{t-T} a(s)ds}\\
&=\int_{t-T}^{t}a\left(u\right)\left[x\left(u\right)-h\left(x\left(u\right)\right)\right]e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{u}a\left(s\right)h\left(x\left(s\right)\right)ds\right]e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}\left[\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)-a\left(u\right)\right]h\left(x\left(u-\tau\left(u\right)\right)\right)e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}\left[G\left(u,x\left(u\right),x\left(u-\tau\left(u\right)\right)\right)-a\left(u\right)Q\left(u,x\left(u-\tau\left(u\right)\right)\right)\right]e^{\int_0^u a(s)ds}du.
\end{aligned}$$

By dividing both sides of the above equation by $\exp(\int_0^t a\left(s\right)ds)$ and using the fact that $x\left(t\right)=x\left(t-T\right)$, we obtain

$$\begin{aligned}
&x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\\
&=\eta\int_{t-T}^{t}a\left(u\right)\left[x\left(u\right)-h\left(x\left(u\right)\right)\right]e^{-\int_u^t a(s)ds}du\\
&+\eta\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{t}a\left(s\right)h\left(x\left(s\right)\right)ds\right]e^{-\int_u^t a(s)ds}du\\
&+\eta\int_{t-T}^{t}\left[\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)-a\left(u\right)\right]h\left(x\left(u-\tau\left(u\right)\right)\right)e^{-\int_u^t a(s)ds}du\\
&+\eta\int_{t-T}^{t}\left[G\left(u,x\left(u\right),x\left(u-\tau\left(u\right)\right)\right)-a\left(u\right)Q\left(u,x\left(u-\tau\left(u\right)\right)\right)\right]e^{\int_0^u a(s)ds}du. \qquad (3.7)
\end{aligned}$$

Integration by parts the second integral in the above expression, we obtain

$$\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{u}a(s)h(x(s))\,ds\right]e^{-\int_u^t a(s)ds}du$$
$$=\left[\int_{u-\tau(u)}^{u}a(s)h(x(s))\,dse^{-\int_u^t a(s)ds}\right]_{t-T}^{t}$$
$$-\int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u}a(s)h(x(s))\,ds\right]a(u)e^{-\int_u^t a(s)ds}du$$
$$=\left[\int_{t-\tau(t)}^{t}a(s)h(x(s))\,ds-\int_{t-T-\tau(t)}^{t-T}a(s)h(x(s))\,dse^{-\int_{t-T}^t a(s)ds}\right]$$
$$-\int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u}a(s)h(x(s))\,ds\right]a(u)e^{-\int_u^t a(s)ds}du$$
$$=-\int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u}a(s)h(x(s))\,ds\right]a(u)e^{-\int_u^t a(s)ds}du$$
$$+\frac{1}{\eta}\int_{t-\tau(t)}^{t}a(u)h(x(u))\,ds. \tag{3.8}$$

Then substituting the result of (3.8) into (3.7) to obtain (3.5). The converse implication is easily obtained and the proof is complete. ∎

3.2 Existence of periodic solutions

To apply Theorem 1.2.14, we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a completely continuous and the other is large contraction. So, we let $(\mathbb{B},\|.\|)=(P_T,\|.\|)$ and

$$\mathbb{M}=\{\varphi\in P_T,\ \|\varphi\|\leq L\}, \tag{3.9}$$

with $L\in(0,1]$. For $x\in\mathbb{M}$, let the mapping H be defined by

$$H(x)=x-h(x), \tag{3.10}$$

and by (3.5), define the mapping $\mathcal{P}:P_T\to P_T$ by

$$\begin{aligned}(\mathcal{P}\varphi)(t)&=\eta\int_{t-T}^{t}\kappa(t,u)a(u)H(\varphi(u))\,du+Q(t,\varphi(t-\tau(t)))\\&+\int_{t-\tau(t)}^{t}a(u)h(\varphi(u))\,du-\eta\int_{t-T}^{t}\kappa(t,u)a(u)\int_{u-\tau(u)}^{u}a(s)h(\varphi(s))\,dsdu\\&+\eta\int_{t-T}^{t}\kappa(t,u)\left[(1-\tau'(u))a(u-\tau(u))-a(u)\right]h(\varphi(u-\tau(u)))\,du\\&+\eta\int_{t-T}^{t}\kappa(t,u)\left[-a(u)Q(u,\varphi(u-\tau(u)))+G(u,\varphi(u),\varphi(u-\tau(u)))\right]du.\end{aligned} \tag{3.11}$$

Therefore, we express the above equation as

$$(\mathcal{P}\varphi)(t) = (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : P_T \to P_T$ are given by

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) \\
&= Q(t, \varphi(t - \tau(t))) + \int_{t-\tau(t)}^{t} a(u) h(\varphi(u)) du \\
&- \eta \int_{t-T}^{t} \kappa(t,u) a(u) \int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) [(1 - \tau'(u)) a(u - \tau(u)) - a(u)] h(\varphi(u - \tau(u))) du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) [-a(u) Q(u, \varphi(u - \tau(u))) + G(u, \varphi(u), \varphi(u - \tau(u)))] du, \qquad (3.12)
\end{aligned}$$

and

$$(\mathcal{B}\varphi)(t) = \eta \int_{t-T}^{t} \kappa(t,u) a(u) H(\varphi(u)) du. \qquad (3.13)$$

We will assume that the following conditions hold.

(H1) $a \in L^1[0,T]$ is bounded

(H2) h is locally Lipschitz continuous, then for x, $y \in \mathbb{M}$ there exist a constant $E > 0$, such that

$$|h(x) - h(y)| \leq E \|x - y\|.$$

(H3) Q, G satisfies Carathéodory conditions with respect to $L^1[0,T]$

(H4) There exists periodic functions $q_1, q_2 \in L^1[0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1(t)|x| + q_2(t).$$

(H5) There exists periodic functions $g_1, g_2, g_3 \in L^1[0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t).$$

Now, we need the following assumptions

$$\beta_1 \beta_2 (EL + |h(0)|) \leq \frac{\gamma_1}{2} L, \qquad (3.14)$$

where $\beta_1 = \max_{t\in[0,T]} |\tau(t)|$ and $\beta_2 = \max_{t\in[0,T]} \{a(t)\}$

$$q_1(t) L + q_2(t) \leq \frac{\gamma_2}{2} L, \qquad (3.15)$$

$$|(1-\tau'(t))\, a(t-\tau(t)) - a(t)|\,(EL + |h(0)|) \leq \gamma_3 La(t)\,, \tag{3.16}$$

$$g_1(t)\, L + g_2(t)\, L + g_3(t) \leq \gamma_4 La(t)\,, \tag{3.17}$$

$$J[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4] \leq 1, \tag{3.18}$$

where γ_1, γ_2, γ_3, γ_4 and J are positive constants with $J \geq 3$.

Lemma 3.2.1 *For $\mathcal{A}$ defined in (3.12), suppose that (3.2)–(3.4), (3.14)–(3.18) and (H1)–(H5) hold. Then $\mathcal{A} : \mathbb{M} \to \mathbb{M}$.*

Proof. Let $\mathcal{A}$ be defined by (3.12). First by (3.2) and (3.4), a change of variable in (3.12) shows that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. That is, if $\varphi \in P_T$ then $\mathcal{A}\varphi$ is periodic with period T. By (H2) we obtain

$$|h(x)| \leq E|x| + |h(0)|\,.$$

Then, let $\varphi \in \mathbb{M}$, by (3.14)–(3.18) and (H1)–(H5) we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)| \\
&\leq |Q(t, \varphi(t-\tau(t)))| + \int_{t-\tau(t)}^{t} a(u)\,|h(\varphi(u))|\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\,|h(\varphi(s))|\, ds du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\,[(1-\tau'(u))\, a(u-\tau(u)) + a(u)]\,|h(\varphi(u-\tau(u)))|\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\,(a(u)\,|Q(u, \varphi(u-\tau(u)))| + |G(u, \varphi u, \varphi(u-\tau(u)))|)\, du \\
&\leq q_1(t)\,|\varphi(t-\tau(t))| + q_2(t) + \beta_1\beta_2(EL + |h(0)|) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, \beta_1\beta_2(EL + |h(0)|)\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\,[(1-\tau'(u))\, a(u-\tau(u)) + a(u)]\,(EL + |h(0)|)\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\,[q_1(u)\,|\varphi(u-\tau(u))\,| + q_2(u)]\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\,[g_1(u)\,|\varphi(u)\,| + g_2(u)\,|\varphi(u-\tau(u))\,| + g_3(u)]\, du \\
&\leq \gamma_1 L + \gamma_2 L + \gamma_3 L + \gamma_4 L \leq \frac{L}{J} \leq L.
\end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}$. ■

Lemma 3.2.2 *For $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ defined in (3.12), suppose that (3.2)–(3.4), (3.14)–(3.18) and (H1)–(H5) hold. Then $\mathcal{A}$ is completely continuous.*

Proof. We show that $\mathcal{A}$ is continuous in the supremum norm, Let $\varphi_n \in \mathbb{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&\left|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)\right| \\
&\leq \left|Q(t,\varphi_n(t-\tau(t))) - Q(t,\varphi(t-\tau(t)))\right| \\
&+ \int_{t-\tau(t)}^{t} a(u)\left|h(\varphi_n(u)) - h(\varphi(u))\right| du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\left|h(\varphi_n(s)) - h(\varphi(s))\right| ds du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left[(1-\tau'(u))\, a(u-\tau(u)) - a(u)\right]\left|h(\varphi_n(u-\tau(u))) - h(\varphi(u-\tau(u)))\right| du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\left|Q(u,\varphi_n(u-\tau(u))) - Q(u,\varphi(u-\tau(u)))\right| du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left|G(u,\varphi_n(u),\varphi_n(u-\tau(u))) - G(u,\varphi u,\varphi(u-\tau(u)))\right| du.
\end{aligned}$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty}\left|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)\right| = 0$. Then $\mathcal{A}$ is continuous.

We next show that $\mathcal{A}$ is completely continuous. Let $\varphi \in \mathbb{M}$, then, by Lemma 3.2.1, we see that

$$\|\mathcal{A}\varphi\| \leq L.$$

And so the family of functions $\mathcal{A}\varphi$ is uniformly bounded. Again, let $\varphi \in \mathbb{M}$. Without loss of generality, we can pick $\omega < t$ such that $t - \omega < T$. Then

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\varphi)(\omega)\right| \\
&\leq \left|Q(t,\varphi(t-\tau(t))) - Q(\omega,\varphi(\omega-\tau(\omega)))\right| \\
&+ \left|\int_{t-\tau(t)}^{t} a(u)\, h(\varphi(u))\, du - \int_{\omega-\tau(\omega)}^{\omega} a(u)\, h(\varphi(u))\, du\right| \\
&+ \eta \left|\int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\, h(\varphi(s))\, ds du\right. \\
&\left.- \int_{\omega-T}^{\omega} \kappa(\omega,u)\, a(u) \int_{u-\tau(u)}^{u} a(s) h(\varphi(s))\, ds du\right| \\
&+ \eta \left|\int_{t-T}^{t} \kappa(t,u)\left[(1-\tau'(u))\, a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u)))\, du\right. \\
&\left.- \int_{\omega-T}^{\omega} \kappa(\omega,u)\left[(1-\tau'(u))\, a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u)))\, du\right| \\
&+ \eta \left|\int_{t-T}^{t} \kappa(t,u)\, a(u)\, Q(u,\varphi(u-\tau(u)))\, du - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, a(u)\, Q(u,\varphi(u-\tau(u)))\, du\right| \\
&+ \eta \left|\int_{t-T}^{t} \kappa(t,u)\, G(u,\varphi(u),\varphi(u-\tau(u)))\, du - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, G(u,\varphi(u),\varphi(u-\tau(u)))\, du\right|.
\end{aligned}$$

Since (H1)–(H3) and (3.14)–(3.18) hold, we can rewrite

$$
\begin{aligned}
&\eta \left| \int_{t-T}^{t} \kappa(t,u)\left[(1-\tau'(u))\,a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u)))\,du \right.\\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\left[(1-\tau'(u))\,a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u)))\,du \right|\\
&\leq \eta \int_{\omega}^{t} \kappa(t,u)\left[(1-\tau'(u))\,a(u-\tau(u)) - a(u)\right] \left|h(\varphi(u-\tau(u)))\right| du\\
&+ \eta \int_{\omega-T}^{\omega} \left|\kappa(t,u) - \kappa(\omega,u)\right| \left[(1-\tau'(u))\,a(u-\tau(u)) - a(u)\right]\\
&\times \left|h(\varphi(u-\tau(u)))\right| du\\
&+ \eta \int_{\omega-T}^{t-T} \kappa(\omega,u)\left[(1-\tau'(u))\,a(u-\tau(u)) - a(u)\right] \left|h(\varphi(u-\tau(u)))\right| du\\
&\leq 2\eta\beta_3 \int_{\omega}^{t} \gamma_3 L a(u)\,du + \eta \int_{\omega-T}^{\omega} \left|\kappa(t,u) - \kappa(\omega,u)\right| \gamma_3 L a(u)\,du\\
&\leq 2\eta\beta_3\gamma_3 L \int_{\omega}^{t} a(u)\,du + \eta\gamma_3 L \int_{0}^{T} \left|\kappa(t,u) - \kappa(\omega,u)\right| a(u)\,du,
\end{aligned}
$$

where $\beta_3 = \max_{u\in[t-T,t]}\{\kappa(t,u)\}$, and

$$
\begin{aligned}
&\eta \left| \int_{t-T}^{t} \kappa(t,u)\,a(u)\,Q(u,\varphi(u-\tau(u)))\,du \right.\\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\,a(u)\,Q(u,\varphi(u-\tau(u)))\,du \right|\\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u)\,G(u,\varphi(u),\varphi(u-\tau(u)))\,du \right.\\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\,G(u,\varphi(u),\varphi(u-\tau(u)))\,du \right|\\
&\leq 2\eta\beta_3 \int_{\omega}^{t} \left[a(u)\,q_L(u) + g_{\sqrt{2}L}(u)\right] du\\
&+ \eta \int_{0}^{T} \left|\kappa(t,u) - \kappa(\omega,u)\right| \left[a(u)\,q_L(u) + g_{\sqrt{2}L}(u)\right] du,
\end{aligned}
$$

and

$$\begin{aligned}
&\eta\left|\int_{t-T}^{t}\kappa(t,u)\,a(u)\int_{u-\tau(u)}^{u}a(s)\,h(\varphi(s))\,dsdu\right.\\
&\left.-\int_{\omega-T}^{\omega}\kappa(\omega,u)\,a(u)\int_{u-\tau(u)}^{u}a(s)h(\varphi(s))\,dsdu\right|\\
&\leq 2\eta\beta_3\int_{\omega}^{t}a(u)\,\frac{\gamma_1}{2}Ldu+\eta\int_{\omega-T}^{\omega}|\kappa(t,u)-\kappa(\omega,u)|\,a(u)\,\frac{\gamma_1}{2}Ldu\\
&\leq \eta\beta_3\gamma_1L\int_{\omega}^{t}a(u)\,du+\eta\frac{\gamma_1}{2}L\int_{0}^{T}|\kappa(t,u)-\kappa(\omega,u)|\,a(u)\,du,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{t-\tau(t)}^{t}a(u)\,h(\varphi(u))\,du-\int_{\omega-\tau(\omega)}^{\tau}a(u)\,h(\varphi(u))\,du\right|\\
&=\left|\int_{\omega}^{t}a(u)\,h(\varphi(u))\,du-\int_{\omega-\tau(\omega)}^{t-\tau(t)}a(u)\,h(\varphi(u))\,du\right|\\
&\leq (EL+h(0))\left(\int_{\omega}^{t}a(u)\,du+\int_{\omega-\tau(\omega)}^{t-\tau(t)}a(u)\,du\right),
\end{aligned}$$

which implies

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)-(\mathcal{A}\varphi)(\omega)|\\
&\leq |Q(t,\varphi(t-\tau(t)))-Q(\omega,\varphi(\omega-\tau(\omega)))|+2\eta\beta_3\gamma_3L\int_{\omega}^{t}a(u)\,du\\
&+\eta\gamma_3L\int_{0}^{T}|\kappa(t,u)-\kappa(\omega,u)|\,a(u)\,du\\
&+2\eta\beta_3\int_{\omega}^{t}\left[a(u)\,q_L(u)+g_{\sqrt{2}L}(u)\right]du\\
&+\eta\int_{0}^{T}|\kappa(t,u)-\kappa(\omega,u)|\left[a(u)\,q_L(u)+g_{\sqrt{2}L}(u)\right]du\\
&+\eta\beta_3\gamma_1L\int_{\omega}^{t}a(u)\,du+\eta\frac{\gamma_1}{2}L\int_{0}^{T}|\kappa(t,u)-\kappa(\omega,u)|\,a(u)\,du\\
&+(EL+h(0))\left(\int_{\omega}^{t}a(u)\,du+\int_{\omega-\tau(\omega)}^{t-\tau(t)}a(u)\,du\right),
\end{aligned}$$

then by the Dominated Convergence Theorem $|(\mathcal{A}\varphi)(t)-(\mathcal{A}\varphi)(\omega)|\to 0$ as $t-\omega\to 0$ independently of $\varphi\in\mathbb{M}$. Thus $(\mathcal{A}\varphi)$ is equicontinuous. Hence by Ascoli-Arzela's theorem $\mathcal{A}$ is completely continuous. ■

The next result shows the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions. Assume that

$$\max\{|H(-L)|,|H(L)|\} \leq \frac{2L}{J}. \tag{3.19}$$

Lemma 3.2.3 *Let $\mathcal{B}$ be defined by (3.13), suppose (3.19) and the condition of theorem 1.1.17 hold. Then $\mathcal{B}:\mathbb{M}\to\mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (3.13). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. Let $\varphi\in\mathbb{M}$

$$\begin{aligned}|(\mathcal{B}\varphi)(t)| &\leq \int_{t-T}^{t}\kappa(t,u)\,a(u)\max\{|H(-L)|,|H(L)|\}\,du\\ &\leq \frac{2L}{J}<L,\end{aligned}$$

which implies $\mathcal{B}:\mathbb{M}\to\mathbb{M}$.

By Theorem 1.1.17 H is large contraction on $\mathbb{M}$, then for any $\varphi,\psi\in\mathbb{M}$, with $\varphi\neq\psi$ and for any $\epsilon>0$, from the proof of that Theorem, we have found a $\delta<1$, such that

$$|(H\varphi)(t)-(H\psi)(t)|\leq\delta\,\|\varphi-\psi\|.$$

Thus,

$$\begin{aligned}|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)| &= \left|\eta\int_{t-T}^{t}\kappa(t,u)\,a(u)\,[H(\varphi(u))-H(\psi(u))]\,du\right|\\ &\leq \|\varphi-\psi\|\,\eta\int_{t-T}^{t}\kappa(t,u)\,a(u)\,du\leq\delta\,\|\varphi-\psi\|.\end{aligned}$$

The proof is complete. ■

Theorem 3.2.4 *Suppose the hypothesis of Lemmas 3.2.1, 3.2.2 and 3.2.3 hold. Let $\mathbb{M}$ defined by (3.9). Then the equation (3.1) has a T-periodic solution in $\mathbb{M}$.*

Proof. By Lemmas 3.2.1, 3.2.2, $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 3.2.3, the mapping $\mathcal{B}$ is a large contraction. Next, we show that if $\varphi,\psi\in\mathbb{M}$, we have $\|\mathcal{A}\psi+\mathcal{B}\varphi\|\leq L$. Let $\varphi,\psi\in\mathbb{M}$ with $\|\varphi\|,\|\psi\|\leq L$. By (3.14)–(3.18)

$$\begin{aligned}\|\mathcal{A}\psi+\mathcal{B}\varphi\| &\leq [\gamma_1+\gamma_2+\gamma_3+\gamma_4]\,L+\frac{2}{J}L\\ &\leq \frac{L}{J}+\frac{2L}{J}=L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z\in\mathbb{M}$ such that $z=\mathcal{A}z+\mathcal{B}z$. By Lemma 3.1.1 this fixed point is a solution of (3.1). Hence (3.1) has a T-periodic solution. ■

Example 3.2.5 *Consider the following nonlinear neutral differential equation*

$$\begin{aligned}&\frac{d}{dt}\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]\\&=-a\left(t\right)h\left(x\left(t-\tau\left(t\right)\right)\right)+G\left(t,x\left(t\right),x\left(t-\tau\left(t\right)\right)\right),\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}&T=2\pi,\ a\left(t\right)=6.10^{-2},\ \tau\left(t\right)=\tfrac{10^{-2}}{\sqrt{3}},\ h\left(x\right)=x^{3},\\&Q\left(t,x\right)=10^{-4}\sin\left(x\right),\ G\left(t,x,y\right)=2.10^{-6}\sin\left(t\right)\left(\cos\left(t\right)+\cos\left(x\right)+\sin\left(y\right)\right).\end{aligned}$$

Then the equation (3.20) has a T-periodic solution.

Proof. We have $h:\mathbb{R}\to\mathbb{R}$ is continuous on $\left[-\sqrt{3}/3,\sqrt{3}/3\right]$, differentiable on $\left(-\sqrt{3}/3,\sqrt{3}/3\right)$, strictly increasing on $\left[-\sqrt{3}/3,\sqrt{3}/3\right]$ and $\sup_{t\in\left(-\sqrt{3}/3,\sqrt{3}/3\right)}h'\left(t\right)\leq 1$. By Theorem 1.1.17, the mapping $H\left(x\right)=x-x^{3}$ is a large contraction on the set

$$\mathbb{M}=\left\{\varphi\in P_{2\pi},\ \|\varphi\|\leq\sqrt{3}/3\right\}.$$

Doing straightforward computations, we obtain

$$\begin{aligned}E&=1,\ q_{1}\left(t\right)=0,\ q_{2}\left(t\right)=10^{-4},\ g_{1}\left(t\right)=0,\ g_{2}\left(t\right)=0,\ g_{3}\left(t\right)=6.10^{-6},\\\beta_{1}&=\frac{10^{-2}}{\sqrt{3}},\ \beta_{2}=6.10^{-2},\ \gamma_{1}=\frac{12}{\sqrt{3}}.10^{-4},\ \gamma_{2}=\frac{6}{\sqrt{3}}10^{-4},\ \gamma_{3}=0,\\\gamma_{4}&=\frac{3}{\sqrt{3}}10^{-4},\ J\in\left[3,\frac{\sqrt{3}}{21}10^{4}\right].\end{aligned}$$

All hypotheses of Theorem (3.2.4) are fulfilled and so the equation (3.20) has at least a 2π-periodic solution belonging to $\mathbb{M}$. ■

3.3 Existence of nonnegative periodic solutions

In this section we obtain the existence of a nonnegative periodic solution of (3.1). By applying Theorem 1.2.14, we need to define a closed, convex, and bounded subset $\mathbb{M}$ of P_T. So, let

$$\mathbb{M}=\left\{\phi\in P_{T}:0\leq\phi\leq K\right\},\tag{3.21}$$

where K is positive constant. To simplify notation, we let

$$F\left(t,x\left(t\right)\right)=\int_{t-\tau(t)}^{t}a\left(u\right)h\left(x\left(u\right)\right)du,\tag{3.22}$$

and

$$m=\min_{u\in[t-T,t]}e^{-\int_{u}^{t}a(s)ds},\ M=\max_{u\in[t-T,t]}e^{-\int_{u}^{t}a(s)ds}.\tag{3.23}$$

It is easy to see that for all $(t,u) \in [0,2T]^2$,

$$m \leq \kappa(t,u) \leq M, \tag{3.24}$$

Then we obtain the existence of a nonnegative periodic solution of (3.1) by considering the two cases;

(1) $F(t,x(t)) \geq 0 \ \forall t \in [0,T], \ x \in \mathbb{M}$.

(2) $F(t,x(t)) \leq 0 \ \forall t \in [0,T], \ x \in \mathbb{M}$.

In the case one, we assume for all $t \in [0,T]$, $x, y \in \mathbb{M}$, that there exist positive constants c_1 and c_2 such that

$$0 \leq Q(t,y) \leq c_1 K, \tag{3.25}$$

$$0 \leq F(t,x) \leq c_2 K \tag{3.26}$$

$$c_1 + c_2 < 1, \tag{3.27}$$

$$\begin{aligned} 0 \leq & -a(t) F(t,x) + [(1-\tau'(t)) a(t-\tau(t)) - a(t)] h(y) \\ & - a(t) Q(t,y) + G(t,x,y), \end{aligned} \tag{3.28}$$

$$\begin{aligned} & -a(t) F(t,x) + [(1-\tau'(t)) a(t-\tau(t)) - a(t)] h(y) \\ & + a(t) H(x) - a(t) Q(t,y) + G(t,x,y) \ \leq \ \frac{K(1-c_1-c_2)}{M\eta T}. \end{aligned} \tag{3.29}$$

Lemma 3.3.1 *Let $\mathcal{A}$, $\mathcal{B}$ given by (3.12), (3.13) respectively, assume (3.25)–(3.29) hold. Then $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{M}$.*

Proof. Let $\mathcal{A}$ defined by (3.13). So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned} 0 \leq & (\mathcal{A}\varphi)(t) \leq Q(t,\varphi(t-\tau(t))) + F(t,x(t)) \\ & - \eta \int_{t-T}^{t} \kappa(t,u) a(u) F(u,x(u)) du \\ & + \eta \int_{t-T}^{t} \kappa(t,u) ((1-\tau'(u)) a(u-\tau(u)) - a(u)) h(\varphi(u-\tau(u))) du \\ & + \eta \int_{t-T}^{t} \kappa(t,u) [-a(u) Q(u,\varphi(u-\tau(u))) + G(u,\varphi(u),\varphi(u-\tau(u)))] du \\ & \leq \eta \int_{t-T}^{t} M \frac{K(1-c_1-c_2)}{M\eta T} du + c_1 K + c_2 K = K, \end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}$.

Now, let $\mathcal{B}$ defined by (3.13). So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned} 0 & \leq (\mathcal{B}\varphi)(t) \\ & \leq \eta \int_{t-T}^{t} M \frac{K(1-c_1-c_2)}{M\eta T} du \leq \eta M T \frac{K}{M\eta T} = K. \end{aligned}$$

That is $\mathcal{B}\varphi \in \mathbb{M}$. ∎

Theorem 3.3.2 *Suppose the hypothesis of Lemmas 3.2.2, 3.2.3 and 3.3.1 hold. Then equation (3.1) has a nonnegative T-periodic solution x in the subset $\mathbb{M}$.*

Proof. By Lemmas 3.2.2, $\mathcal{A}$ is completely continuous. Also, from Lemma 3.2.3, the mapping $\mathcal{B}$ is a large contraction. By Lemma 3.3.1, $\mathcal{A},\mathcal{B}:\mathbb{M}\to\mathbb{M}$. Next, we show that if $\varphi,\psi\in\mathbb{M}$, we have $0\leq\mathcal{A}\psi+\mathcal{B}\varphi\leq K$. Let $\varphi,\psi\in\mathbb{M}$ with $0\leq\varphi,\psi\leq K$. By (3.25)–(3.29)

$$\begin{aligned}
&\left(\mathcal{A}\psi\right)(t)+\left(\mathcal{B}\varphi\right)(t)\\
&=\eta\int_{t-T}^{t}\kappa\left(t,u\right)a\left(u\right)H\left(\varphi\left(u\right)\right)du+Q\left(t,\psi\left(t-\tau\left(t\right)\right)\right)\\
&+F\left(t,\psi\left(t\right)\right)-\eta\int_{t-T}^{t}\kappa\left(t,u\right)a(u)F\left(u,x\left(u\right)\right)du\\
&+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left[\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)-a\left(u\right)\right]h\left(\psi\left(u-\tau\left(u\right)\right)\right)du\\
&+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left[-a\left(u\right)Q\left(u,\psi\left(u-\tau\left(u\right)\right)\right)+G\left(u,\psi\left(u\right),\psi\left(u-\tau\left(u\right)\right)\right)\right]du\\
&\leq\eta\int_{t-T}^{t}\kappa\left(t,u\right)\frac{K\left(1-c_1-c_2\right)}{M\eta T}du+c_1K+c_2K\\
&\leq\eta\int_{t-T}^{t}M\frac{K\left(1-c_1-c_2\right)}{M\eta T}du+c_1K+c_2K=K.
\end{aligned}$$

On the other hand,

$$\left(\mathcal{A}\psi\right)(t)+\left(\mathcal{B}\varphi\right)(t)\geq 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z\in\mathbb{M}$ such that $z=\mathcal{A}z+\mathcal{B}z$. By Lemma 3.1.1 this fixed point is a solution of (3.1) and the proof is complete. ■

Example 3.3.3 *Consider the following nonlinear neutral differential equation*

$$\begin{aligned}
&\frac{d}{dt}\left[x\left(t\right)-Q\left(t,x\left(t-\tau\left(t\right)\right)\right)\right]\\
&=-a\left(t\right)h\left(x\left(t-\tau\left(t\right)\right)\right)+G\left(t,x\left(t\right),x\left(t-\tau\left(t\right)\right)\right),
\end{aligned}\tag{3.30}$$

where

$$\begin{aligned}
&T=2\pi,\ a\left(t\right)=6.10^{-2},\ \tau\left(t\right)=\tfrac{10^{-2}}{\sqrt{3}},\ h\left(x\right)=x^3,\\
&Q\left(t,y\right)=10^{-4}\sin\left(y\right),\ F\left(t,x\left(t\right)\right)=\textstyle\int_{t-\frac{10^{-2}}{\sqrt{3}}}^{t}x^3\left(u\right)du\geq 0,\\
&G\left(t,x,y\right)=8.10^{-2}Q\left(t,y\right)+8.10^{-2}F\left(t,x\right).
\end{aligned}$$

Then the equation (3.30) has a nonnegative T-periodic solution.

Proof. By Example 3.2.5, the mapping $H(x) = x - x^3$ is a large contraction on the set

$$\mathbb{M} = \left\{\varphi \in P_{2\pi},\ 0 \leq \varphi \leq \sqrt{3}/3\right\}.$$

A simple calculation yields

$$m = e^{-12.10^{-2}\pi},\ M = 1,\ \eta = \left(1 - e^{-12\pi.10^{-2}}\right)^{-1},\ c_1 = 10^{-4},\ c_2 = \frac{10^{-2}}{3\sqrt{3}}.$$

Then for $x, y \in \left[0, \sqrt{3}/3\right]$ we have

$$\begin{aligned} 0 \leq &-a(t) F(t,x) + \left[(1 - \tau'(t)) a(t - \tau(t)) - a(t)\right] h(y) \\ &- a(t) Q(t,y) + G(t,x,y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &- a(t) F(t,x) + \left[(1 - \tau'(t)) a(t - \tau(t)) - a(t)\right] h(y) \\ &+ a(t) H(x) - a(t) Q(t,y) + G(t,x,y) \\ &\leq 2.7 \times 10^{-2} < 2.88 \times 10^{-2} \simeq \frac{K(1 - c_1 - c_2)}{M\eta T}. \end{aligned}$$

All conditions of Theorem (3.3.2) hold and so the equation (3.30) has at least a nonnegative 2π-periodic solution belonging to $\mathbb{M}$. ■

In the case two, we substitute conditions (3.26)–(3.29) with the following conditions respectively. We assume that there exist a negative constant c_3 such that

$$c_3 K \leq F(t,x) \leq 0, \tag{3.31}$$

$$-c_3 + c_1 < 1, \tag{3.32}$$

$$\begin{aligned} \frac{-c_3 K}{M\eta T} \leq &-a(t) F(t,x) + \left[(1 - \tau'(t)) a(t - \tau(t)) - a(t)\right] h(y) \\ &- a(t) Q(t,y) + G(t,x,y), \end{aligned} \tag{3.33}$$

$$\begin{aligned} &- a(t) F(t,x) + \left[(1 - \tau'(t)) a(t - \tau(t)) - a(t)\right] h(y) \\ &+ a(t) H(x) - a(t) Q(t,y) + G(t,x,y) \ \leq \ \frac{K(1 - c_1)}{M\eta T}. \end{aligned} \tag{3.34}$$

Theorem 3.3.4 *Suppose (3.25), (3.31)–(3.34) and the hypothesis of Lemmas 3.2.1, 3.2.2 and 3.2.3 hold. Then equation (3.1) has a nonnegative T-periodic solution x in the subset $\mathbb{M}$.*

Proof. By Lemmas 3.2.1, 3.2.2, $\mathcal{A}$ is completely continuous. Also, from Lemma 3.2.3, the mapping $\mathcal{B}$ is a large contraction. To see that, it is easy to show as in Lemma 3.3.1 $\mathcal{A}, \mathcal{B} : \mathbb{M} \rightarrow \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \leq \mathcal{A}\psi + \mathcal{B}\varphi \leq K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \leq \varphi, \psi \leq K$. By (3.25) and (3.31)–(3.34) we have

$$\begin{aligned}
&(\mathcal{A}\psi)(t) + (\mathcal{B}\varphi)(t) \\
&= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \psi(t - \tau(t))) \\
&+ F(t, \psi(t)) - \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, F(u, \psi(u))\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[(1 - \tau'(u))\, a(u - \tau(u)) - a(u)\right] h(\psi(u - \tau(u)))\, du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[-a(u)\, Q(u, \psi(u - \tau(u))) + G(u, \psi(u), \psi(u - \tau(u)))\right] du \\
&\leq \eta \int_{t-T}^{t} \kappa(t,u) \frac{K(1 - c_1)}{M\eta T} du \\
&= \eta \int_{t-T}^{t} M \frac{K(1 - c_1)}{M\eta T} du + c_1 K = K.
\end{aligned}$$

On the other hand,

$$(\mathcal{A}\psi)(t) + (\mathcal{B}\varphi)(t) \geq 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 3.1.1 this fixed point is a solution of (3.1) and the proof is complete. ■

CHAPTER 4

Existence of periodic solutions for a system of nonlinear neutral functional differential equations

Keywords. Krasnoselskii's theorem; Contraction; Neutral differential equation; Integral equation; Periodic solution; Fundamental matrix solution; Floquet theory.

In this chapter, we expose the work cited in [83] as follow
Mesmouli M. B., Ardjouni A. and A. Djoudi., Existence and uniqueness of periodic solutions for a system of nonlinear neutral functional differential equations with two functional delays, Rend. Circ. Mat. Palermo (2014) 63:409-424, DOI 10.1007/s12215-014-0162-x.

The goal of the present chapter is to study the existence of periodic solutions of the nonlinear neutral system of differential equations. By using Krasnoselskii's fixed point theorem we obtain the existence of periodic solution and by contraction mapping principle we obtain the uniqueness. Our results extend and complement some earlier publications ([65], [110]).

4.1 Preliminaries, remarks and some history of the equation

A qualitative analysis such as periodicity, positivity and stability of solutions of neutral differential equations which the delay has been studied extensively by many authors, we refer the readers to [44, 31, 53], [58, 59, 60, 61, 65, 66, 67, 68, 79], [82, 109] and references therein for a wealth of reference materials on the subject.

Recently, Yankson in [110] studied the existence and uniqueness of a periodic solution of the system of differential equations

$$\frac{d}{dt}x(t) = A(t)\,x(t-\tau), \tag{4.1}$$

where $A(\cdot)$ is an $n \times n$ matrix with continuous real-valued functions as its elements and τ is a positive constant.

In 2007, Islam and Raffoul in [65] used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the system of nonlinear neutral functional differential equations

$$\frac{d}{dt}x(t) = A(t)x(t) + \frac{d}{dt}Q(t, x(t - g(t))) + G(t, x(t), x(t - g(t))). \tag{4.2}$$

where where $A(\cdot)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements. The functions $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments. Also, the authors used the contraction mapping principle to show the uniqueness of periodic solutions of (4.2).

Motivated by the works mentioned above and the references therein, we study the existence and uniqueness of periodic solutions for the system of nonlinear differential equations with two functional delays

$$\frac{d}{dt}x(t) = A(t)x(t - \tau(t)) + \frac{d}{dt}Q(t, x(t - g(t))) + G(t, x(t), x(t - g(t))), \tag{4.3}$$

where $A(\cdot)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements. The functions $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments. In the analysis we use the fundamental matrix solution of $x'(t) = A(t)x(t)$ coupled with Floquet theory to invert the system (4.3) into an integral system. Then we employ the Krasnoselskii's fixed point theorem to show the existence of periodic solutions of system (4.3). The obtained integral system is the sum of two mappings, one is a compact operator and the other is a contraction. Also, transforming system (4.3) to an integral system enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

For the definitions of the different notions used throughout this paper we refer, for example [31, 62, 63, 100, 106]. For $T > 0$ define $C_T = \{\phi : \phi \in C(\mathbb{R}, \mathbb{R}^n),\ \phi(t+T) = \phi(t),\ t \in \mathbb{R}\}$ where $C(\mathbb{R}, \mathbb{R}^n)$ is the space of all n-vector continuous functions. Then C_T is a Banach space when it is endowed with the supremum norm

$$\|x(\cdot)\| = \max_{t \in [0,T]} |x(t)|,$$

where $|\cdot|$ denotes the infinity norm for $x \in \mathbb{R}^n$. Also, if A is an $n \times n$ real matrix, then we define the norm of A by

$$|A| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Definition 4.1.1 *If the matrix $A(\cdot)$ is periodic of period T, then the linear system*

$$y'(t) = A(t)y(t), \tag{4.4}$$

is said to be noncritical with respect to T, if it has no periodic solution of period T except the trivial solution $y = 0$.

In this chapter we assume that

$$A(t+T) = A(t), \ \tau(t+T) = \tau(t) \geq \tau^* > 0, \ g(t+T) = g(t) \geq g^* > 0, \tag{4.5}$$

with τ is twice continuously differentiable and τ^*, g^* are constant. For $t \in \mathbb{R}$, $x, y, z, w \in \mathbb{R}^n$, the functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T, they are also globally Lipschitz continuous in x and in x and y, respectively. That is

$$Q(t+T,x) = Q(t,x), \ G(t+T,x,y) = G(t,x,y), \tag{4.6}$$

and there are positive constants k_1, k_2, k_3 such that

$$|Q(t,x) - Q(t,y)| \leq k_1 \|x-y\|, \tag{4.7}$$

$$|G(t,x,y) - G(t,z,w)| \leq k_2 \|x-z\| + k_3 \|y-w\|. \tag{4.8}$$

Throughout this paper it is assumed that the system (4.4) is noncritical. Now, we state some known results [62] about system (4.4). Let $K(t)$ represent the fundamental matrix of (4.4) with $K(0) = I$, where I is the $n \times n$ identity matrix. Then

a) $\det K(t) \neq 0$.

b) There exists a constant matrix B such that $K(t+T) = K(t) e^{TB}$, by Floquet theory.

c) System (4.4) is noncritical if and only if $\det(I - K(T)) \neq 0$.

Remark 4.1.2 *By preserving the notation in [110], we notice that, for the equation (4.1) Yankson assumed that there exists a nonsingular $n \times n$ matrix $G(\cdot)$ with continuous real-valued functions as its elements such that*

$$\frac{d}{dt}x(t) = G(t)x(t) - \frac{d}{dt}\int_{t-\tau}^{t} G(u)x(u)\,du + [A(t) - G(t-\tau)]x(t-\tau).$$

But this condition is not necessary and we can replace $G(\cdot)$ by $A(\cdot)$ because $A(t-\tau)$ exist. However, in the present work, this condition is removed and we assumed that $A(\cdot)$ is nonsingular $n \times n$ matrix.

The following lemma is fundamental to our results.

Lemma 4.1.3 *Suppose (4.5) and (4.6) hold. If $x \in C_T$, then x is a solution of the equation (4.3) if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)x(s)\,ds \\ &\quad + K(t)U(T)\int_{t}^{t+T} K^{-1}(s)A(s)\left(Q(s,x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)x(u)\,du\right)ds \\ &\quad + K(t)U(T)\int_{t}^{t+T} K^{-1}(s)\left[F(s)x(s-\tau(s)) + G(s,x(s),x(s-g(s)))\right]ds, \end{aligned} \tag{4.9}$$

where

$$U(T) = \left(K^{-1}(T) - I\right)^{-1},$$

and

$$F(t) = A(t) - (1 - \tau'(t)) A(t - \tau(t)).$$

Proof. Let $x \in C_T$ be a solution of (4.3) and $K(\cdot)$ is a fundamental system of solutions for (4.4). Rewrite the equation (4.3) as

$$\begin{aligned}
\frac{d}{dt}x(t) = {} & A(t)x(t) - A(t)x(t) + A(t)x(t - \tau(t)) \\
& + \frac{d}{dt}Q(t, x(t - g(t))) + G(t, x(t), x(t - g(t))) \\
= {} & A(t)x(t) - \frac{d}{dt}\int_{t-\tau(t)}^{t} A(u)x(u)\,du \\
& + [A(t) - (1 - \tau'(t))A(t - \tau(t))]x(t - \tau(t)) \\
& + \frac{d}{dt}Q(t, x(t - g(t))) + G(t, x(t), x(t - g(t))).
\end{aligned}$$

We put $A(t) - (1 - \tau'(t))A(t - \tau(t)) = F(t)$, we obtain

$$\begin{aligned}
& \frac{d}{dt}\left[x(t) - Q(t, x(t - g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right] \\
& = A(t)\left[x(t) - Q(t, x(t - g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right] \\
& + A(t)\left[Q(t, x(t - g(t))) - \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right] \\
& + F(t)x(t - \tau(t)) + G(t, x(t), x(t - g(t))).
\end{aligned}$$

Since $K(t)K^{-1}(t) = I$, it follows that

$$\begin{aligned}
0 &= \frac{d}{dt}\left[K(t)K^{-1}(t)\right] \\
&= A(t)K(t)K^{-1}(t) + K(t)\frac{d}{dt}K^{-1}(t) \\
&= A(t) + K(t)\frac{d}{dt}K^{-1}(t).
\end{aligned}$$

This implies

$$\frac{d}{dt}K^{-1}(t) = -K^{-1}(t)A(t).$$

If $x(\cdot)$ is a solution of (4.3) with $x(0) = x_0$, then

$$\begin{aligned}
&\frac{d}{dt}\left[K^{-1}(t)\left(x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)\,x(u)\,du\right)\right] \\
&= \frac{d}{dt}K^{-1}(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)\,x(u)\,du\right] \\
&+ K^{-1}(t)\frac{d}{dt}\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)\,x(u)\,du\right] \\
&= -K^{-1}(t)\,A(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)\,x(u)\,du\right] \\
&+ K^{-1}(t)\,A(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)\,x(u)\,du\right] \\
&+ K^{-1}(t)\,A(t)\left[Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(u)\,x(u)\,du\right] \\
&+ K^{-1}(t)\left(F(t)\,x(t-\tau(t)) + G(t, x(t), x(t-g(t)))\right).
\end{aligned}$$

An integration of the above equation from 0 to t yields

$$\begin{aligned}
x(t) &= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\,x(s)\,ds \\
&+ K(t)\left(x(0) - Q(0, x(0-g(0))) + \int_{-\tau(0)}^{0} A(s)\,x(s)\,ds\right) \\
&+ K(t)\int_0^t K^{-1}(s)\,A(s)\left[Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right] ds \\
&+ K(t)\int_0^t K^{-1}(s)\left(F(s)\,x(s-\tau(s)) + G(s, x(s), x(s-g(s)))\right) ds. \qquad (4.10)
\end{aligned}$$

Since $x(T) = x_0 = x(0)$, using (4.10) we get

$$\begin{aligned}
&x(0) - Q(0, x(-g(0))) + \int_{-\tau(0)}^{0} A(s)\,x(s)\,ds \\
&= (I - K(T))^{-1}\int_0^T K(T)\,K^{-1}(s)\,A(s)\left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right) ds \\
&+ (I - K(T))^{-1}\int_0^T K(T)\,K^{-1}(s)\left(F(s)\,x(s-\tau(s)) + G(s, x(s), x(s-g(s)))\right) ds. \qquad (4.11)
\end{aligned}$$

A substitution of (4.11) into (4.10) yields

$$
\begin{aligned}
&x(t) \\
&= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\,x(s)\,ds \\
&+ K(t)(I-K(T))^{-1}\int_{0}^{T} K(T)\,K^{-1}(s)\,A(s)\left(Q(s,x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right)ds \\
&+ K(t)(I-K(T))^{-1}\int_{0}^{T} K(T)\,K^{-1}(s)\,(F(s)\,x(s-\tau(s)) + G(s,x(s),x(s-g(s))))\,ds \\
&+ K(t)\int_{0}^{t} K^{-1}(s)\,A(s)\left[Q(s,x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right]ds \\
&+ K(t)\int_{0}^{t} K^{-1}(s)\,(F(s)\,x(s-\tau(s)) + G(s,x(s),x(s-g(s))))\,ds. \qquad (4.12)
\end{aligned}
$$

Now, we will show that (4.12) is equivalent to (4.9). Since

$$
\begin{aligned}
(I-K(T))^{-1} &= \left(K(T)\left(K(T)^{-1}-I\right)\right)^{-1} \\
&= \left(K(T)^{-1}-I\right)^{-1}K(T)^{-1},
\end{aligned}
$$

then the equations (4.12) becomes

$$
\begin{aligned}
&x(t) \\
&= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\,x(s)\,ds \\
&+ K(t)\left(K(T)^{-1}-I\right)^{-1}\int_{0}^{T} K^{-1}(s)\,A(s)\left(Q(s,x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right)ds \\
&+ K(t)\left(K(T)^{-1}-I\right)^{-1}\int_{0}^{T} K^{-1}(s)\,(F(s)\,x(s-\tau(s)) + G(s,x(s),x(s-g(s))))\,ds \\
&+ \int_{0}^{t} K(t)\,K^{-1}(s)\,A(s)\left(Q(s,x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right)ds \\
&+ \int_{0}^{t} K(t)\,K^{-1}(s)\,(F(s)\,x(s-\tau(s)) + G(s,x(s),x(s-g(s))))\,ds,
\end{aligned}
$$

then

$$\begin{aligned}
&x(t)\\
&= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\,x(s)\,ds\\
&+ K(t)\left(K(T)^{-1}-I\right)^{-1}\left\{\int_{t}^{T} K^{-1}(s)\,A(s)\left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right)ds\right.\\
&+ \int_{t}^{T} K^{-1}(s)\left[F(s)\,x(s-\tau(s)) + G(s, x(s), x(s-g(s)))\right]ds\\
&+ \int_{0}^{t} K(T)^{-1} K^{-1}(s)\,A(s)\left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right)ds\\
&\left.+ \int_{0}^{t} K(T)^{-1} K^{-1}(s)\left[F(s)\,x(s-\tau(s)) + G(s, x(s), x(s-g(s)))\right]ds\right\}.
\end{aligned}$$

By letting $s = v - T$ and $U(T) = \left(K(T)^{-1} - I\right)^{-1}$, the above expression yields

$$\begin{aligned}
x(t) &= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\,x(s)\,ds\\
&\quad + K(t)\,U(T)\int_{t}^{T} K^{-1}(s)\,A(s)\left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\,x(u)\,du\right)ds\\
&\quad + K(t)\,U(T)\int_{t}^{T} K^{-1}(s)\left(F(s)\,x(s-\tau(s)) + G(s, x(s), x(s-g(s)))\right)ds\\
&\quad + K(t)\,U(T)\int_{T}^{t+T} K(T)^{-1} K^{-1}(v-T)\,A(v-T)\left(Q(v-T, x(v-T-g(v-T)))\right.\\
&\quad \left.- \int_{v-T-\tau(v-T)}^{v-T} A(u)\,x(u)\,du\right)dv\\
&\quad + K(t)\,U(T)\int_{T}^{t+T} K(T)^{-1} K^{-1}(v-T)\left(F(v-T)\,x(v-T-\tau(v-T))\right.\\
&\quad \left.+ G(v-T, x(v-T), x(v-T-g(v-T)))\right)dv.
\end{aligned} \tag{4.13}$$

By (b) we have

$$K(t-T) = K(t)\,e^{-TB} \text{ and } K(T) = e^{TB}.$$

Hence,

$$K^{-1}(T)\,K^{-1}(v-T) = K^{-1}(v).$$

Consequently, since (4.5) and (4.6) hold, (4.13) becomes

$$\begin{aligned}
x(t) &= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\, x(s)\, ds \\
&\quad + K(t)\, U(T) \left[\int_t^T K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \right. \\
&\quad \left. + \int_t^T K^{-1}(s) \left(F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))) \right) ds \right] \\
&\quad + K(t)\, U(T) \left[\int_T^{t+T} K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \right. \\
&\quad \left. + \int_T^{t+T} K^{-1}(s) \left(F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))) \right) ds \right]. \qquad (4.14)
\end{aligned}$$

By combining the two integrals of the equation (4.14), we can obtained easily the equation (4.9) The converse implication is easily obtained and the proof is complete. ■

4.2 Existence and uniqueness of periodic solutions

By applying Theorems 1.2.2 and 1.2.11, we obtain in this Section the existence and the uniqueness of the periodic solution of (4.3). So, let a Banach space $(C_T, \|\cdot\|)$, a closed bounded convex subset of C_T,

$$\mathbb{M} = \{\varphi \in C_T,\ \|\varphi\| \leq L\}, \qquad (4.15)$$

with $L > 0$, and by the Lemma 4.1.3, let a mapping $\mathcal{P}$ given by

$$\begin{aligned}
(\mathcal{P}\varphi)(t) &= Q(t, \varphi(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\, \varphi(s)\, ds \\
&\quad + K(t)\, U(T) \int_t^{t+T} K^{-1}(s)\, A(s) \left(Q(s, \varphi(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, \varphi(u)\, du \right) ds \\
&\quad + K(t)\, U(T) \int_t^{t+T} K^{-1}(s) \left[F(s)\, \varphi(s-\tau(s)) + G(s, \varphi(s), \varphi(s-g(s))) \right] ds.
\end{aligned} \qquad (4.16)$$

We express equation (4.16) as

$$\mathcal{P}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned}
(\mathcal{A}\varphi)(t) &= K(t)\, U(T) \int_t^{t+T} K^{-1}(s)\, A(s) \left(Q(s, \varphi(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, \varphi(u)\, du \right) ds \\
&\quad + K(t)\, U(T) \int_t^{t+T} K^{-1}(s) \left[F(s)\, \varphi(s-\tau(s)) + G(s, \varphi(s), \varphi(s-g(s))) \right] ds,
\end{aligned} \qquad (4.17)$$

and

$$(\mathcal{B}\varphi)(t) = Q(t, \varphi(t - g(t))) - \int_{t-\tau(t)}^{t} A(s)\varphi(s)\,ds. \tag{4.18}$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 1.2.11. Since $\varphi \in C_T$, (4.5) and (4.6) hold, we have for $\varphi \in \mathbb{M}$

$$(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t) \text{ and } \mathcal{A}\varphi \in C(\mathbb{R}, \mathbb{R}^n) \Longrightarrow (\mathcal{A}\mathbb{M}) \subset \mathcal{C}_T, \tag{4.19}$$

and

$$(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t) \text{ and } \mathcal{B}\varphi \in C(\mathbb{R}, \mathbb{R}^n) \Longrightarrow (\mathcal{B}\mathbb{M}) \subset \mathcal{C}_T. \tag{4.20}$$

Lemma 4.2.1 *Suppose (4.5)–(4.8) hold. If $\mathcal{A}$ is defined by (4.17), then $\mathcal{A}$ is continuous and the image of $\mathcal{A}$ is contained in a compact set.*

Proof. Let $\varphi_n \in \mathbb{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)| \\
&\le |K(t)U(T)| \int_t^{t+T} \left|K^{-1}(s)\right| \\
&\times |A(s)| \left[\int_{s-\tau(s)}^{s} |A(u)|\,|\varphi_n(u) - \varphi(u)|\,du + |Q(s, \varphi_n(s - g(s))) - Q(s, \varphi(s - g(s)))|\right] ds \\
&+ |K(t)U(T)| \int_t^{t+T} \left|K^{-1}(s)\right| [|F(s)|\,|\varphi_n(s - \tau(s)) - \varphi(s - \tau(s))| \\
&+ |G(s, \varphi_n(s), \varphi_n(s - g(s))) - G(s, \varphi(s), \varphi(s - g(s)))|]\,ds.
\end{aligned}$$

Since Q, G are continuous, the dominated convergence theorem implies,

$$\lim_{n\to\infty} |(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)| = 0.$$

Then $\mathcal{A}$ is continuous. Next, we show that the image of $\mathcal{A}$ is contained in a compact set. Let $\mathbb{M}$ defined by (4.15), by (4.7) and (4.8), we obtain

$$\begin{aligned}
|Q(t,y)| &\le |Q(t,y) - Q(t,0) + Q(t,0)| \\
&\le k_1 \|y\| + |Q(t,0)|,
\end{aligned}$$

$$\begin{aligned}
|G(t,x,y)| &\le |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\
&\le k_2 \|x\| + k_3 \|y\| + |G(t,0,0)|.
\end{aligned}$$

Let $\varphi_n \in \mathbb{M}$ where n is a positive integer, then (4.17) is equivalent to

$$\begin{aligned}
&(\mathcal{A}\varphi_n)(t) \\
&= \int_t^{t+T} \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} A(s) \left(Q(s, \varphi_n(s - g(s))) - \int_{s-\tau(s)}^{s} A(u)\varphi_n(u)\,du\right) ds \\
&+ \int_t^{t+T} \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} [F(s)\varphi_n(s - \tau(s)) + G(s, \varphi_n(s), \varphi_n(s - g(s)))]\,ds.
\end{aligned}$$

Consequently

$$\begin{aligned} \|(\mathcal{A}\varphi_n)(\cdot)\| &\leq c\int_0^T [|A|(\alpha|A|+k_1L+\beta)+|F|L+(k_2+k_3)L+\gamma]\,ds \\ &= cT[|A|(\alpha|A|+k_1L+\beta)+|F|L+(k_2+k_3)L+\gamma] \\ &= E, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \sup_{t\in[0,T]}|\tau(t)|,\ \beta=\sup_{t\in[0,T]}|Q(t,0)|,\ \gamma=\sup_{t\in[0,T]}|G(t,0,0)|, \\ c &= \sup_{t\in[0,T]}\left(\sup_{s\in[t,t+T]}\left|\left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1}\right|\right). \end{aligned}$$

Second, we calculate $(\mathcal{A}\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (4.5) and (4.6) we obtain by taking the derivative in (4.17) that

$$\begin{aligned} &(\mathcal{A}\varphi_n)'(t) \\ &= K'(t)U(T)\int_t^{t+T}K^{-1}(s)A(s)\left(Q(s,\varphi_n(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)\varphi_n(u)\,du\right)ds \\ &+ K'(t)U(T)\int_t^{t+T}K^{-1}(s)[F(s)\varphi_n(s-\tau(s))+G(s,\varphi_n(s),\varphi_n(s-g(s)))]\,ds \\ &+ K(t)U(T)\left[K^{-1}(t+T)-K^{-1}(t)\right]A(t)\left(Q(t,\varphi_n(t-g(t)))-\int_{t-\tau(t)}^{t}A(s)\varphi_n(s)\,ds\right) \\ &+ K(t)U(T)\left[K^{-1}(t+T)-K^{-1}(t)\right][F(t)\varphi_n(t-\tau(t))+G(t,\varphi_n(t),\varphi_n(t-g(t)))]. \end{aligned} \tag{4.21}$$

Since

$$K'(t)=A(t)K(t), \tag{4.22}$$

and noting that $K^{-1}(t+T)=e^{-TB}K^{-1}(t)$, we have

$$K^{-1}(t+T)-K^{-1}(t)=e^{-TB}K^{-1}(t)-K^{-1}(t)=\left(K^{-1}(T)-1\right)K^{-1}(t). \tag{4.23}$$

A substitution of (4.22) and (4.23) into (4.21) yields

$$\begin{aligned} (\mathcal{A}\varphi_n)'(t) &= A(t)(\mathcal{A}\varphi_n)(t)+A(t)\left(Q(t,\varphi_n(t-g(t)))-\int_{t-\tau(t)}^{t}A(s)\varphi_n(s)\,ds\right) \\ &\quad +F(t)\varphi_n(t-\tau(t))+G(t,\varphi_n(t),\varphi_n(t-g(t))). \end{aligned}$$

Then

$$\left\|(\mathcal{A}\varphi_n)'(\cdot)\right\|\leq|A|E+\frac{E}{cT}.$$

Thus the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $\mathcal{A}(\mathbb{M})$ is relatively compact. ■

Lemma 4.2.2 *Suppose (4.5)–(4.7) hold and*

$$k_1 + \alpha |A| < 1. \tag{4.24}$$

If $\mathcal{B}$ is defined by (4.18), then $\mathcal{B}$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (4.18). Then for $\varphi_1, \varphi_2 \in \mathbb{M}$ we have by (4.7)

$$\begin{aligned}
&|(\mathcal{B}\varphi_1)(t) - (\mathcal{B}\varphi_2)(t)| \\
&= \left| Q(t, \varphi_1(t - g(t))) - Q(t, \varphi_2(t - g(t))) + \int_{t-\tau(t)}^{t} A(s)\varphi_1(s)\, ds - \int_{t-\tau(t)}^{t} A(s)\varphi_2(s)\, ds \right| \\
&\leq (k_1 + \alpha |A|) \|\varphi_1 - \varphi_2\|.
\end{aligned}$$

Hence $\mathcal{B}$ is contraction by (4.24). ∎

Theorem 4.2.3 *Suppose the assumptions of the Lemmas 4.2.1 and 4.2.2 hold. If there exists a constant $L > 0$ defined in $\mathbb{M}$ such that*

$$cT[|A|(\alpha|A| + k_1 L + \beta) + |F| L + (k_2 + k_3) L + \gamma] + k_1 L + \beta + \alpha |A| L \leq L.$$

Then (4.3) has a T-periodic solution.

Proof. By Lemma 4.2.1, $\mathcal{A} : \mathbb{M} \to C_T$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 4.2.2, the mapping $\mathcal{B} : \mathbb{M} \to C_T$ is a contraction. Next, we show that if $\varphi, \phi \in \mathbb{M}$, we have $\|\mathcal{A}\varphi + \mathcal{B}\phi\| \leq L$. Let $\varphi, \phi \in \mathbb{M}$ with $\|\varphi\|, \|\phi\| \leq L$. Then

$$\begin{aligned}
&\|(\mathcal{A}\varphi)(\cdot) + (\mathcal{B}\phi)(\cdot)\| \\
&\leq cT[|A|(\alpha|A| + k_1 L + \beta) + |F| L + (k_2 + k_3) L + \gamma] + k_1 L + \beta + \alpha |A| L \\
&\leq L.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 4.1.3 this fixed point is a solution of (4.3). Hence (4.3) has a T-periodic solution. ∎

Theorem 4.2.4 *Suppose (4.5)–(4.8) hold. If*

$$cT[|A|(\alpha|A| + k_1) + |F| + (k_2 + k_3)] + k_1 + \alpha |A| < 1, \tag{4.25}$$

then equation (4.3) has a unique T-periodic solution.

Proof. Let the mapping $\mathcal{P}$ be given by (4.16). For $\varphi_1, \varphi_2 \in C_T$, we have

$$
\begin{aligned}
&\left|(\mathcal{P}\varphi_1)(t) - (\mathcal{P}\varphi_2)(t)\right| \\
&\leq \left| Q(t, \varphi_1(t-g(t))) - Q(t, \varphi_2(t-g(t))) + \int_{t-\tau(t)}^{t} A(s)\varphi_1(s)\,ds - \int_{t-\tau(t)}^{t} A(s)\varphi_2(s)\,ds \right| \\
&\int_t^{t+T} \left| \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} \right| \\
&\times |A(s)| \left[\int_{s-\tau(s)}^{s} |A(u)| \left|\varphi_1(u) - \varphi_2(u)\right| du + \left|Q(s, \varphi_1(s-g(s))) - Q(s, \varphi_2(s-g(s)))\right| \right] ds \\
&+ \int_t^{t+T} \left| \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} \right| \left[|F(s)| \left|\varphi_1(s-\tau(s)) - \varphi_2(s-\tau(s))\right| \right. \\
&\left. + \left|G(s, \varphi_1(s), \varphi_1(s-g(s))) - G(s, \varphi_2(s), \varphi_2(s-g(s)))\right| \right] ds \\
&= \left[cT\left[|A|(\alpha|A| + k_1) + |F| + (k_2 + k_3)\right] + k_1 + \alpha|A|\right] \|\varphi_1 - \varphi_2\|.
\end{aligned}
$$

Since (4.25) hold, the contraction mapping principle completes the proof. ■

Remark 4.2.5 *Note that, when $Q(\cdot,\cdot) = G(\cdot,\cdot,\cdot) = 0$ and $\tau(t)$ is positive constant, the Theorems 4.2.3 and 4.2.4 reduce to the Theorems 2.7 and 2.8 respectively in [110].*

Corollary 4.2.6 *Suppose (4.5) and (4.6) hold. Let $\mathbb{M}$ defined by (4.15). Suppose there are positive constants k_1^*, k_2^* and k_3^*, such that for x, y, z and $w \in \mathbb{M}$, we have*

$$
|Q(t,x) - Q(t,y)| \leq k_1^* \|x - y\| \;\; \text{and } k_1^* + \alpha|A| < 1, \tag{4.26}
$$

$$
|G(t,x,y) - G(t,z,w)| \leq k_2^*\|x - z\| + k_3^*\|y - w\|. \tag{4.27}
$$

and

$$
cT\left[|A|(\alpha|A| + k_1^*L + \beta) + |F|L + (k_2^* + k_3^*)L + \gamma\right] + k_1^*L + \beta + \alpha|A|L \leq L. \tag{4.28}
$$

Then (4.3) has a T-periodic solution in $\mathbb{M}$. Moreover, if

$$
cT\left[|A|(\alpha|A| + k_1^*) + |F| + (k_2^* + k_3^*)\right] + k_1^* + \alpha|A| < 1,
$$

then (4.3) has a unique solution in $\mathbb{M}$.

Proof. Let the mapping $\mathcal{P}$ defined by (4.16). Then the proof follow immediately from Theorem 4.2.3 and Theorem 4.2.4. ■

Remark 4.2.7 *Note that, when $\tau(t) = 0$, the Theorems 4.2.3, 4.2.4 and Corollary 4.2.6 reduces to the Theorems 2.5, 2.6 and Corollary 2.7 respectively in [65].*

4.3 Application to second order model

Consider the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t-\tau(t))+q(t)x(t-\tau(t))=\frac{d}{dt}V(t,x(t-g(t)))+W(t,x(t),x(t-g(t))), \tag{4.29}$$

where p and q are positive periodic continuous real-valued functions with period T. The functions $V:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $W:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous in their respective arguments. $\tau(\cdot)$ and $g(\cdot)$ satisfy (4.5).

Functions $V(t,x)$ and $W(t,x,y)$ are periodic in t with period T. They are also supposed to be globally Lipschitz continuous in x and in x and y, respectively. That is,

$$V(t+T,x)=V(t,x), \quad W(t+T,x,y)=W(t,x,y), \tag{4.30}$$

and there are positive constants k_1, k_2, k_3 such that

$$|V(t,x)-V(t,y)|\le k_1\|x-y\|, \tag{4.31}$$

and

$$|V(t,x,y)-V(t,z,w)|\le k_2\|x-z\|+k_3\|y-w\|. \tag{4.32}$$

To show the existence of periodic solutions, we transform (4.29) by letting

$$\begin{cases} x_1=x, \\ x_2=x', \end{cases}$$

into a following system

$$\begin{aligned}\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}' &= \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t)\end{pmatrix}\begin{pmatrix} x_1(t-\tau(t)) \\ x_2(t-\tau(t))\end{pmatrix}+\frac{d}{dt}\begin{pmatrix} 0 \\ V(t,x_1(t-g(t)))\end{pmatrix} \\ &\quad +\begin{pmatrix} 0 \\ W(t,x_1(t),x_1(t-g(t)))\end{pmatrix},\end{aligned} \tag{4.33}$$

where

$$A(\cdot)=\begin{pmatrix} 0 & 1 \\ -q(\cdot) & -p(\cdot)\end{pmatrix},\ Q(t,x(t-g(t)))=\begin{pmatrix} 0 \\ V(t,x_1(t-g(t)))\end{pmatrix},$$

$$G(t,x(t),x(t-g(t)))=\begin{pmatrix} 0 \\ W(t,x_1(t),x_1(t-g(t)))\end{pmatrix}.$$

Example 4.3.1 *Let $q(t)=p(t)=1$, $\tau(t)=\lambda_4\cos t$, $g(\cdot)$ is nonnegative, continuous and 2π-periodic, $V(t,w)=\lambda_1\sin(t)\,w^2$, $W(t,z,w)=\lambda_2\cos(t)\,z-\lambda_3 w$.*

Since the matrix A has eigenvalues with non-zero real parts, the system $x' = Ax$ is noncritical. Consider the Banach space $(\mathcal{C}_{2\pi},\ \|\cdot\|)$,

$$C_{2\pi} = \left\{\phi : \phi \in C\left(\mathbb{R}, \mathbb{R}^2\right),\ \phi\left(t + 2\pi\right) = \phi\left(t\right),\ t \in \mathbb{R}\right\},$$

and the closed bounded convex subset of $C_{2\pi}$,

$$\mathbb{M} = \left\{\varphi \in C_{2\pi},\ \|\varphi\| \leq L\right\}.$$

Let $\varphi = (\varphi_1, \varphi_2)$, $\phi = (\phi_1, \phi_2)$. Then for $\varphi, \phi \in \mathbb{M}$ we have

$$\begin{aligned}&\left\|G\left(\cdot, \varphi\left(\cdot\right), \varphi\left(\cdot - g\left(\cdot\right)\right)\right) - G\left(\cdot, \phi\left(\cdot\right), \phi\left(\cdot - g\left(\cdot\right)\right)\right)\right\| \\ &\leq \lambda_2 \left\|\varphi - \phi\right\| + \lambda_3 \left\|\varphi - \phi\right\|.\end{aligned}$$

Hence $k_2^* = \lambda_2,\ k_3^* = \lambda_3$, in the same way $k_1^* = 2\lambda_1 L$, and

$$\alpha = \lambda_4,\ \beta = 0,\ \gamma = 0,$$

and

$$\begin{aligned} F\left(t\right) &= A\left(t\right) - \left(1 - \tau'\left(t\right)\right) A\left(t - \tau\left(t\right)\right) = \tau'\left(t\right) A\left(t\right) \\ &= -\lambda_4 \sin t \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ \left|F\right| = 2\lambda_4. \end{aligned}$$

Consequently

$$cT\left[\left|A\right|\left(\lambda_4\left|A\right| L + 2\lambda_1 L^2\right) + 2\lambda_4 L + \left(\lambda_2 + \lambda_3\right) L\right] + 2\lambda_1 L^2 + \lambda_4 \left|A\right| L \leq L,$$

for all λ_i, $1 \leq i \leq 4$ small enough. Then (4.29) has a 2π-periodic solution, by Corollary 4.2.6. Moreover,

$$cT\left[\left|A\right|\left(\lambda_4\left|A\right| + 2\lambda_1 L\right) + 2\lambda_4 + \left(\lambda_2 + \lambda_3\right)\right] + 2\lambda_1 L + \lambda_4\left|A\right| < 1,$$

is satisfied for λ_i, $1 \leq i \leq 4$ small enough. Then (4.29) has a unique 2π-periodic solution, by Corollary 4.2.6.

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