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Study of the existence, uniqueness and stability of certain fractional differential equations

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A Doctoral Thesis, By Hamid Boulares
Advisors: Pr. Y. Laskri and Dr. A. Ardjouni

Dedication

To My wife, and My childrens, Miled, and Abdelmouiz

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Abstract

Existence, uniqueness and stability of solutions in delay fractional differential equations play an important role in the qualitative analysis of delay fractional differential equations. In this thesis, we have used a fixed point technique to prove existence, uniqueness and stability results of the solutions of a class of nonlinear fractional differential equations with functional delay. This class of equations has proved very challenging in the theory of Liapunov's direct method. The stability results are exclusively obtained by fixed point theorems.

Keywords: Delay fractional differential equations, Fixed point theory, Stability, Krasnoselskii fixed point theorem.

Mathematics Subject Classification: 34K20, 34K30, 34K40.

Résumé

L'existence, l'unicité et la stabilité de solutions d'équations différentielles fractionnaire à retard joue un rôle important dans l'analyse qualitative des équations différentielles fractionnaires à retard. Dans cette thèse, nous avons utilisé une technique de point fixe pour prouver des résultats d'existence, d'unicité et de stabilité de solutions d'une classe d'équations non linéaires à retard fonctionnel. Cette classe d'équations fait partie du nombre de problèmes qui ont résisté à la méthode directe de Liapounov. Les résultats de stabilité sont exclusivement obtenus par le théorème de point fixe.

Mots-clés: Point fixe, stabilité, théorème de Krasnoselskii, contraction, équation differentiales fractionaires à retard.

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Chapter 1

Introduction

The theory of fixed point is one of the most powerful tools of modern mathematics. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology and geometry. In particular fixed point theorem has been applied in such field as mathematics engineering, physics, economics, game theory, biology and chemistry etc. Classical and major results in these areas are: Banach's fixed point theorem, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem.

In 1886, Poincare [50] was the first to work in this field. Then Brouwer [11] in 1912, proved fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani [31]. Mean while Banach principle came in to existence which was considered as one of the fundamental principle in the field of functional analysis. In 1922, Banach [8] proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point.

An important generalization of Brouwer's theorem was discovered in 1930 by Schauder it may be stated as follows: any non empty, compact convex subset of a Banach space has the topological fixed point property. The compactness condition on subset is a stronger one. It is natural to modify the theorem by relaxing the condition of compactness. Schauder also proved a theorem for a compact map which is known as second form of above stated theorem. Second fixed point theorem of Schauder stated that, every compact self mapping of a closed bounded convex subset of a Banach space has at least one fixed point.

In 1932, Krasnoselskii [55] studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name.

The term fractional calculus is more than 300 years old. It is a generalization of the ordinary differentiation and integration to non-integer (arbitrary) order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. In a letter to L'Hospital in 1695 Leibniz raised the following question (Miller and Ross, 1993): "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" The story goes that L'Hospital was somewhat curious about that question and replied by another question to Leibniz. "What if the order will be 1/2?" Leibniz in a letter dated September 30, 1695 replied: "It will lead to a paradox, from which one day useful consequences will be drawn." The question raised by Leibniz for a fractional derivative was an ongoing topic in the last 300 years. Several mathematicians contributed to this subject over the years. People like Liouville, Riemann, and Weyl made major contributions to the theory of fractional calculus. The story of the fractional calculus continued with contributions from Fourier, Abel, Leibniz, Grünwald, and Letnikov. Nowadays, the fractional calculus attracts many scientists and engineers. There are several applications of this mathematical phenomenon in mechanics, physics, chemistry, control theory and so on (Caponetto et al., 2010; Magin, 2006; Monje et al., 2010; Oldham and Spanier, 1974; Oustaloup, 1995; Podlubny, 1999). It is natural that many authors tried to solve the fractional derivatives, fractional integrals and fractional differential equations in Matlab.

Delay differential equations are differential equations in which the unknown function and its derivatives enter, generally speaking, under different values of the argument (see [10], [13], [20], [29]). For example, $x'(t) = f(t, x(t), x(t-\tau))$ with $\tau > 0$ is an example of a such equation.

Delay equations describe many processes with an aftereffect or delayed response phenomenon. Such equations appear, for example, any time when in physics or technology we consider a problem of a force, acting on a material point, that depends on the velocity and position of the point not only at the given moment but at some moment preceding the given moment.

Any investigation of the stability of an equation, or a system of equations, using Lyapunov's direct method requires the construction of a suitable Lyapunov function or functional as the case may be. This can be a hard task. In a parallel way, the study of the stability of an equation using fixed point technic involves the construction of a suitable fixed point mapping. This can, in so many cases, be an arduous task too. For example, the absence of linear terms in a an equation makes it difficult to build a fixed point mapping. So one may begin by transforming the given equation into a more tractable one that does not change the basic structure and properties of the original equation. Although the transformation can tends to be difficult, but having such a transformation is fundamental to invert the given equation and employ the fixed point theory.

We have been interested in the use of fixed point theory to problem of stability. We have studied and contributed to it and have obtained interesting results. In this thesis we present a collection of results to some problems that have offered resistance to Lyapunov's direct method. Here, we give, what we hope, a very detailed work which clearly establish fixed point theory as a viable tool in stability theory.

The thesis consists of four chapters. Chapter 2, is essentially an introduction to the fixed point theory, delay fractional differential equations and stability theory, where we fix notations, terminology to be used. It is a survey aimed at recalling some basic definitions and theory. While some of the classical and recent results about fixed point theory, delay fractional differential equations and stability theory are also presented in this chapter. Fixed point theorems frequently call for compact sets in weighted Banach spaces which may be subsets of continuous functions on finite or infinite intervals on $\mathbb{R}$. For that purpose, we give topologies which will provide many of those compact sets.

In chapter three, we present work of Gao, Liu and Luo [23] in which they established sufficient conditions for the existence, uniqueness and stability of solutions for the non-local non-autonomous system of fractional order differential equations with delays

$$\begin{aligned} D^{\alpha}x(t) &= \sum_{j=1}^{n} a_j(t) f(t, x(t), x(t-\tau_j)),\ t > 0, \\ x(t) &= \phi(t) \text{ for } t < 0 \text{ and } \lim_{t\to 0^-} \phi(t) = 0, \\ I^{1-\alpha}x(t)|_{t=0} &= 0, \end{aligned}$$

by appealing to the contraction mapping principle in a weighted Banach space.

Finally in chapter four, we discuss standard approaches to the problem of stability and asymptotic stability of the zero solution to the delay fractional differential equations

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = f\left(t, x(t), x(t-\tau(t))\right) + {}^{C}D_{0+}^{\alpha-1}g(t, x(t-\tau(t))),\ t \geqslant 0, \\ x(t) = \phi(t),\ t \in [m_0, 0]\,,\ x'(0) = x_1, \end{cases}$$

where $1 < \alpha < 2$ and $g(t,0) = f(t,0,0) = 0$. By first converting the nonlinear delay fractional differential equation to an ordinary delay differential equation with a fractional integral perturbation. Our main results are obtained via the Krasnoselskii's fixed point theorem in a weighted Banach space, which surely provides a new way to the stability analysis (see [9]).

Chapter 2

Preliminaries

2.1 Functional analysis

Questions concerning existence of solutions of differential equations and the existence of periodic solutions can be well formulated in terms of fixed points of mappings. In fact, fixed-point theory was developed, in large measure, as a means of answering such questions. All but one of the fixed point theorems which we consider here require a setting in a compact subset of a metric space. We consider a variety of differential equations and as the equations become more general it becomes increasingly difficult to find a space in which the set in question is compact. In this section we discuss six compact sets which are central to this book [13].

Definition 2.1 A pair (E, ρ) is a metric space if E is a set and $\rho : E \times E \to [0, \infty)$ such that when y, z, and u are in E then

(a) $\rho(y, z) \geq 0$, $\rho(y, y) = 0$ and $\rho(y, z) = 0$ implies $y = z$,

(b) $\rho(y, z) = \rho(z, y)$, and

(c) $\rho(y, z) \leq \rho(y, u) + \rho(u, z)$.

The metric space is complete if every Cauchy sequence in (E, ρ) has a limit in that space. A sequence $\{x_n\} \subset E$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $\rho(x_n, x_m) < \varepsilon$.

Definition 2.2 A set $\mathcal{M}$ in a metric space (E, ρ) is compact if each sequence $\{x_n\} \subset \mathcal{M}$ has a subsequence with limit in $\mathcal{M}$.

Definition 2.3 Let $\{f_n\}$ be a sequence of functions with $f_n : [a, b] \to \mathbb{R}$, the reals.

(a) $\{f_n\}$ is uniformly bounded on $[a, b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a, b]$.

(b) $\{f_n\}$ is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \varepsilon$, for all n.

The first result gives the main method of proving compactness in the spaces in which we are interested.

Theorem 2.1 *[13][Ascoli-Arzela] If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

Proof. As the rational numbers are countable, we may let $t_1, t_2, ...$ be a sequence of all rational numbers on $[a, b]$ taken in any fixed order. Consider the sequence $\{f_n(t_1)\}$. This sequence is bounded so it contains a convergent subsequence, say $\{f_n^1(t_1)\}$ with limit $\phi(t_1)$ The sequence $\{f_n^1(t_2)\}$ also has a convergent subsequence, say $\{f_n^2(t_2)\}$, with limit $\phi(t_2)$. If we continue in this way we obtain a sequence of sequences (there will be one sequence for each value of m):

$$f_n^m(t), \; m = 1, 2, ...; \; n = 1, 2, ...,$$

each of which is a subsequence of all the preceding ones, and such that for each m we have

$$f_n^m(t_m) \to \phi(t_m) \text{ as } n \to \infty.$$

We select the diagonal. That is, consider the sequence of functions

$$F_k(t) = f_k^k(t).$$

It is a subsequence of the given sequence and is, in fact, a subsequence of each of the sequences $\{f_n^m(t)\}$, for n large. As $f_n^m(t_m) \to \phi(t_m)$, it follows that $F_k(t_m) \to \phi(t_m)$ as $k \to \infty$ for each m.

We now show that $\{F_k(t)\}$ converges uniformly on $[a, b]$. Let $\varepsilon_1 > 0$ be given, and let $\varepsilon = \varepsilon_1/3$. Denote by δ the number with the property described in the definition of equicontinuity for the number ε. Now, divide the interval $[a, b]$ into p equal parts, where p is any integer larger than $(b - a)/\delta$. Let ζ_j be a rational number in the j th part $(j = 1, ..., p)$; then $\{F_k(t)\}$ converges at each of these points. Hence, for each j there exists an integer M_j such that $\left|F_r(\zeta_j) - F_s(\zeta_j)\right| < \delta$ if $r > M_j$ and $s > M_j$. Let M be the largest of the numbers M_j.

If t is in the interval $[a, b]$, it is in one of the p parts, say the jth; so $\left|t - \zeta_j\right| < \delta$, and $\left|F_k(t) - F_k(\zeta_j)\right| < \varepsilon$ for every k. Also, if $r > M_2 \geq M_j$ and $s > M$, then $\left|F_r(\zeta_j) - F_s(\zeta_j)\right| < \delta$. Hence, if $r > M$ and $s > M$ then

$$\begin{aligned} |F_r(t) - F_s(t)| &= \left|(F_r(t) - F_r(\zeta_j)) + (F_r(\zeta_j) - F_s(\zeta_j)) - (F_s(t) - F_s(\zeta_j))\right| \\ &\leq \left|F_r(t) - F_r(\zeta_j)\right| + \left|F_r(\zeta_j) - F_s(\zeta_j)\right| + \left|F_s(t) - F_s(\zeta_j)\right| \\ &< 3\varepsilon = \varepsilon_1. \end{aligned}$$

By the Cauchy criterion for uniform convergence, the sequence $\{F_k(t)\}$ converges uniformly to some function $\phi(t)$. As each $F_k(t)$ is continuous, so is $\phi(t)$. This completes the proof. ■

Definition 2.4 A linear space $(E, +, .)$ is a normed space if for each $x \in E$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|\lambda x\| = |\lambda| \, \|x\|$ for each $\lambda \in \mathbb{R}$, and
(3) $\|x + y\| \leq \|x\| + \|y\|$.

Note a normed space is a vector space and it is a metric space with $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition 2.5 A Banach space is a complete normed space.

We often say a Banach space is a complete normed vector space.

Example 2.1 (a) The space $\mathbb{R}^n$ over the field $\mathbb{R}$ is a vector space and there are many suitable norms for it. For example, if $x = (x_1, ..., x_n)$ then

(1) $\|x\| = \max_i |x_i|$,
(2) $\|x\| = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$, or
(3) $\|x\| = \sum_{i=1}^n |x_i|$,

are all suitable norms. Norm (2) is the Euclidean norm. Notice that the square root is required in order that $\|\lambda x\| = |\lambda| \, \|x\|$.

(b) With any of these norms, $(\mathbb{R}^n, \|.\|)$ is a Banach space. It is complete because the real numbers are complete.

(c) A set $\mathcal{M}$ in $(\mathbb{R}^n, \|.\|)$ is compact if and only if it is closed and bounded, as is seen in any text on advanced calculus.

Example 2.2 (a) The space $C([a, b], \mathbb{R}^n)$ consisting of all continuous functions $f : [a, b] \to \mathbb{R}^n$ is a vector space over the reals.

(b) If $\|f\| = \max_{a \leq t \leq b} |f(t)|$, where $|.|$ is a norm in $\mathbb{R}^n$, then it is a Banach space.

(c) For a given pair of positive constants M and K, the set $\mathcal{M} = \{f \in C([a, b], \mathbb{R}^n) | \ \|f\| \leq M, \ |f(u) - f(v)| \leq K |u - v|\}$ is compact. To see this, note first that Ascoli's theorem is also true for vector sequences; apply it to each component successively. If $\{f_n\}$ is any sequence in $\mathcal{M}$, then it is uniformly bounded and equicontinuous. By Ascoli's theorem it has a subsequence converging uniformly to a continuous function $f : [a, b] \to \mathbb{R}^n$. But $|f_n(t)| \leq M$ for any fixed t, so $\|f\| \leq M$. Moreover, if we denote the subsequence by $\{f_n\}$ again, then for fixed u and v there exist $\varepsilon_n > 0$ and $\delta_n > 0$ with

$$\begin{aligned}|f(u) - f(v)| &\leq |f(u) - f_n(u)| + |f_n(u) - f_n(v)| + |f_n(v) - f(v)| \\ &\stackrel{def}{=} \varepsilon_n + |f_n(u) - f_n(v)| + \delta_n \\ &\leq \varepsilon_n + \delta_n + K\,|u - v| \to K\,|u - v|\,,\end{aligned}$$

as $n \to \infty$. Hence, $f \in \mathcal{M}$ and $\mathcal{M}$ is compact.

Example 2.3 (a) Let $\phi : [a, b] \to \mathbb{R}^n$ be continuous and let E be the set of continuous functions $f : [a, c] \to \mathbb{R}^n$ with $c > b$ and with $f(t) = \phi(t)$ for $a \leq t \leq b$. Define $\rho(f, g) = \sup_{a\leq t\leq c} |f(t) - g(t)|$ for $f, g \in E$.

(b) Then (E, ρ) is a complete metric space but not a Banach space because $f + g$ is not in E.

Example 2.4 (a) Let (E, ρ) denote the space of bounded continuous functions $f : (-\infty, 0] \to \mathbb{R}^n$ with $\rho(\phi, \psi) = \|\phi - \psi\| = \sup_{-\infty<s\leq 0} |\phi(s) - \psi(s)|$ where $|.|$ is the Euclidean norm in $\mathbb{R}^n$.

(b) Then (E, ρ) is a Banach space.

(c) The set

$$L = \{f \in E \mid \|f\| \leq 1,\ |f(u) - f(v)| \leq |u - v|\},$$

is not compact in (E, ρ). To see this, consider the sequence of functions $\{f_n\}$ from $(-\infty, 0]$ into $[0, 1]$ with $f_n(t) = 0$ for $t \leq -n$, $f_n(t)$ is the straight line between the points $(-n, 0)$ and $(0, 1)$. Any subsequence of $\{f_n\}$ converges pointwise to $f \equiv 1$.But $\rho(f_n, 1) = 1$ for all n. Thus, there is no subsequence of $\{f_n\}$ with a limit in (E, ρ).

Example 2.5 (a) Let (E, ρ) denote the space of continuous functions $f : (-\infty, 0] \to \mathbb{R}^n$ with

$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(f, g)/ \{1 + \rho_n(f, g)\}\,,$$

where

$$\rho_n(f, g) = \max_{-n\leq s\leq 0} |f(s) - g(s)|\,,$$

and $|.|$ is the Euclidean norm on $\mathbb{R}^n$.

(b) Then (E, ρ) is a complete metric space. The distance between all functions is bounded by 1.

(c) And $(E, +, .)$ is a vector space over $\mathbb{R}$.

(d) But (E, ρ) is not a Banach space because ρ does not define a norm; $\rho(x, 0) = \|x\|$ will not satisfy $\|\lambda x\| = |\lambda|\,\|x\|$.

(e) The space (E, ρ) is a locally convex topological vector space. For details and properties see [18, 21, 51]. The reason we need to identify the space here is that we shall later use the Schauder-Tychonov fixed-point theorem which is applicable to this particular space.

(f) Let M and K be given positive constants. The set

$$\mathcal{M} = \{f \in E \mid |f(t)| \le M, \text{ on } (-\infty, 0], \ |f(u) - f(v)| \le K\,|u - v|\},$$

is compact. To see this, let $\{f_n\}$ be a sequence in $\mathcal{M}$. We must show that there is an $f \in \mathcal{M}$ and a subsequence, say $\{f_n\}$ again, such that $\rho(f_n, f) \to 0$ as $n \to \infty$. If we examine ρ we see that $\rho(f_n, f) \to 0$ as $n \to \infty$ just in case f_n converges to f uniformly on compact subsets of $(-\infty, 0]$. Consider $\{f_n\}$ on $[-1, 0]$; it is uniformly bounded and equicontinuous so there is a subsequence, say $\{f_n^1\}$ converging uniformly to some continuous f on $[-1, 0]$. Moreover, the argument in Example 2.2 shows that $|f(t)| \le M$ and $|f(u) - f(v)| \le K\,|u - v|$. Next, consider $\{f_n^1\}$ on $[-2, 0]$; it is uniformly bounded and equicontinuous so there is a subsequence $\{f_n^2\}$ converging uniformly to a continuous function, say f again, on $[-2, 0]$. Continue in this way and have $F_n = f_n^n$: which is a subsequence of $\{f_n\}$ and it converges uniformly on compact subsets of $(-\infty, 0]$ to a function $f \in \mathcal{M}$. Thus, $\mathcal{M}$ is compact.

Example 2.6 Let $(E, |.|_h)$ be as in Example 2.7 with (a) and (b) holding. Then the set

$$\mathcal{M} = \left\{f \in E \mid |f(t)| \le \sqrt{h\,(t)}, \text{ on } (-\infty, 0], \ |f(u) - f(v)| \le |u - v|\right\},$$

is compact. To prove this, let $\{f_n\} \subset \mathcal{M}$. We must show that there is $f \in \mathcal{M}$ and a subsequence $\{f_{n_k}\}$ such that $|f_{n_k} - f|_h \to 0$ as $k \to \infty$. Use Ascoli's theorem repeatedly as before and obtain a subsequence $\{f_k^k\}$ converging uniformly to a continuous f on compact subsets of $(-\infty, 0]$ and $|f(t)| \le \sqrt{h\,(t)}$. Let $\varepsilon > 0$ be given and find $K > 0$ with $2/\sqrt{h(-K)} < \varepsilon/2$. Find N such that

$$\max_{-K \le t \le 0} \left|f_k^k(t) - f(t)\right| \le \varepsilon/2,$$

if $k > N$. Then $k > N$ implies

$$\begin{aligned}
\left|f_k^k - f\right|_h &\le \sup_{-\infty \le t \le -K} \left|(f_k^k(t) - f(t))/h(t)\right| + \max_{-K \le t \le 0} \left|(f_k^k(t) - f(t))/h(t)\right| \\
&\le \left[\sup_{-\infty \le t \le -K} 2\sqrt{h\,(t)}/h(t)\right] + \max_{-K \le t \le 0} \left|f_k^k(t) - f(t)\right| \\
&< \varepsilon.
\end{aligned}$$

It is readily established that

$$|f(u) - f(v)| \le |u - v|\,.$$

Lemma 2.1 ([35]) *Let*

$$E = \left\{ x(t) \mid x(t) \in C(\mathbb{R}^+),\ \lim_{t\to\infty} x(t)/h(t) = 0 \right\},$$

with the norm

$$\|x\| = \sup_{t\geq 0} x(t)/h(t).$$

Then $(E, \|.\|)$ *is a Banach space.*

Proof. We only prove that the space E is complete. Let $\{x_n\}$ be a Cauchy sequence in E. Then, for any given $\varepsilon > 0$ and any $t \in [0, +\infty)$, there exists a constant $N > 0$, such that for $n, m \geq N$,

$$\frac{|x_n(t) - x_m(t)|}{h(t)} \leq \|x_n - x_m\| < \varepsilon, \tag{2.1}$$

i.e.,$\{x_n(t)/h(t)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus there exists a $y(t)$ such that

$$\lim_{n\to\infty} \frac{x_n(t)}{h(t)} = y(t),\ t \geq 0.$$

This means that a function $y : [0, +\infty) \to \mathbb{R}$ is well defined. In (2.1), let $m \to \infty$, we have

$$\left| \frac{x_n(t)}{h(t)} - y(t) \right| \leq \varepsilon,\ t \in \mathbb{R}^+,\ n \geq N. \tag{2.2}$$

Next, we show that $y(t) \in C(\mathbb{R}^+)$. In fact, for any given $t_0 \in [0, +\infty)$, by (2.2) and the continuity of $\lim_{t\to\infty} x_{_N}(t)/h(t)$ on $[0, +\infty)$, we could know that there exists a $\delta > 0$ such that $|t - t_0| < \delta$ implies

$$\begin{aligned} |y(t) - y(t_0)| &\leq \left| y(t) - \frac{x_{_N}(t)}{h(t)} \right| + \left| \frac{x_{_N}(t)}{h(t)} - \frac{x_{_N}(t_0)}{h(t_0)} \right| + \left| \frac{x_{_N}(t_0)}{h(t_0)} - y(t_0) \right| \\ &\leq 2\sup_{s\geq 0} \left| y(s) - \frac{x_{_N}(s)}{h(s)} \right| + \varepsilon \\ &\leq 3\varepsilon. \end{aligned}$$

Therefore, $y(t) \in C(\mathbb{R}^+)$. Let $x(t) = h(t)y(t)$, then $x(t) \in C(0, \infty)$, $x(t) \in C(\mathbb{R}^+)$. Note that

$$\frac{|x(t)|}{h(t)} \leq \sup_{t\geq 0} \left| y(t) - \frac{x_{_N}(t)}{h(t)} \right| + \frac{|x_{_N}(t)|}{h(t)}. \tag{2.3}$$

Then by (2.3) together with $\lim_{t\to+\infty} x_{_N}(t)/h(t) = 0$, we can see that $\lim_{t\to+\infty} x(t)/h(t) = 0$. The proof is complete. ■

Theorem 2.2 ([35]) *Let $\mathcal{M}$ be a subset of the Banach space E. Then $\mathcal{M}$ is relatively compact in E if the following conditions are satisfied*

(*i*) $\{x(t)/h(t) : x \in \mathcal{M}\}$ *is uniformly bounded,* *(ii)* $\{x(t)/h(t) : x \in \mathcal{M}\}$ *is equicontinuous on any compact interval of* $\mathbb{R}^+$*,*

(iii) $\{x(t)/h(t) : x \in \mathcal{M}\}$ *is equiconvergent at infinity i.e. for any given* $\varepsilon > 0$*, there exists a* $T_0 > 0$ *such that for all* $x \in \mathcal{M}$ *and* $t_1, t_2 > T_0$*, if holds*

$$|x(t_2)/h(t_2) - x(t_1)/h(t_1)| < \varepsilon.$$

Proof. "$\Leftarrow$" From Lemma (2.1), we know E is a Banach space. In order to prove that the subset $\mathcal{M}$ is precompact in E, we only need to show $\mathcal{M}$ is totally bounded in E, that is for all $\varepsilon > 0$, $\mathcal{M}$ has a finite ε-net.

For any given $\varepsilon > 0$, by (iii), there exists a $T > 0$, for all $x \in \mathcal{M}$ and $t \geq T$, we have

$$\frac{|x(t)|}{h(t)} < \frac{\varepsilon}{2}.$$

Let

$$\mathcal{M}_{[0,T]} = \left\{ \frac{x(t)}{h(t)} : 0 \leq t \leq T,\ x \in \mathcal{M} \right\},$$

with the norm

$$\|x\|_{\mathcal{M}_{[0,T]}} = \max_{t \in [0,T]} \left| \frac{x(t)}{h(t)} \right|.$$

In view of (i), (ii) and Ascoli–Arzela theorem, we can know that $\mathcal{M}_{[0,T]}$ is precompact in $C[0,T]$. Thus, there exist $x_1,\ x_2, ..., x_k \in \mathcal{M}$ such that, for any $x(t)/h(t) \in \mathcal{M}_{[0,T]}$, we have

$$\|x - x_i\|_{\mathcal{M}_{[0,T]}} = \max_{t \in [0,T]} \left| \frac{x(t)}{h(t)} - \frac{x_i(t)}{h(t)} \right| \leq \varepsilon,$$

for some i, $1 \leq i \leq k$. Therefore,

$$\|x - x_i\| = \max \left\{ \max_{t \in [0,T]} \frac{|x(t) - x_i(t)|}{h(t)}, \sup_{t \geq T} \frac{|x(t) - x_i(t)|}{h(t)} \right\} \leq \varepsilon.$$

So, for any $\varepsilon > 0$, $\mathcal{M}$ has a finite ε-net $\{x_1, x_2, ..., x_k\} \subset \mathcal{M}$, that is, $\mathcal{M}$ is totally bounded in E. Hence $\mathcal{M}$ is precompact in E.

"$\Rightarrow$" Assume that $\mathcal{M}$ is precompact, then for any $\varepsilon > 0$, there exists a finite ε-net of $\mathcal{M}$. Let the finite ε-net be $\{x_1, x_2, ..., x_k\} \subset \mathcal{M}$. Then for any $x \in \mathcal{M}$, $\exists x_i \in \{x_1, x_2, ..., x_k\}$ such that

$$\left| \frac{x(t)}{h(t)} \right| \leq \left| \frac{x(t)}{h(t)} - \frac{x_i(t)}{h(t)} \right| + \left| \frac{x_i(t)}{h(t)} \right| \leq \varepsilon + \|x_i\| \leq \varepsilon + \max_{1 \leq i \leq k} \{\|x_i\|\}. \tag{2.4}$$

By virtue of (2.4), it is easy to know (i) (iii) are satisfied.

Finally, for any $\varepsilon > 0$, there exists a $T > 0$, such that

$$\left| \frac{x_i(t_1)}{h(t_1)} - \frac{x_i(t_2)}{h(t_2)} \right| < \varepsilon,\ t_1, t_2 > T,\ i \in \{1, 2, ..., k\}.$$

On the other hand, we know $x_i(t)/h(t) \in C[0, T+1]$. Thus, there exists a $\delta > 0 (\delta < 1)$, such that for any $t_1, t_2 \in C[0, T+1]$ and $i \in \{1, 2, ..., k\}$, if $|t_1 - t_2| < \delta$, then

$$\left| \frac{x_i(t_1)}{h(t_1)} - \frac{x_i(t_2)}{h(t_2)} \right| < \varepsilon.$$

Therefore, for any $x \in \mathcal{M}$, we have

(1) there exists $i \in \{1, 2, ..., k\}$ such that $\|x - x_i\| < \varepsilon$;

(2) if $t_1, t_2 \in [0, +\infty)$ and $|t_1 - t_2| < \delta$, then

$$\begin{aligned} \left| \frac{x(t_1)}{h(t_1)} - \frac{x(t_2)}{h(t_2)} \right| &\leq \left| \frac{x(t_1)}{h(t_1)} - \frac{x_i(t_1)}{h(t_1)} \right| + \left| \frac{x_i(t_1)}{h(t_1)} - \frac{x_i(t_2)}{h(t_2)} \right| + \left| \frac{x_i(t_2)}{h(t_2)} - \frac{x(t_2)}{h(t_2)} \right| \\ &\leq 2\|x - x_i\| + \varepsilon \leq 3\varepsilon. \end{aligned}$$

This means that (ii) is satisfied. Consequently, the Theorem is proved. ■

Example 2.7 Let $h : (-\infty, 0] \to [1, \infty)$ be a continuous strictly decreasing function with $h(0) = 1$ and $h(r) \to \infty$ as $r \to -\infty$.

(a) Let $(E, |.|_h)$ be the space of continuous functions $f : (-\infty, 0] \to \mathbb{R}^n$ for which

$$\sup_{-\infty < t \leq 0} |f(t)/h(t)| \overset{def}{=} |f|_h .$$

exists.

(b) Then $(E, |.|_h)$ is a Banach space.

(c) For positive constants M and K the set

$$\mathcal{M} = \{f \in E \mid |f(t)| \leq M, \text{ on } (-\infty, 0], \; |f(u) - f(v)| \leq K|u - v|\},$$

is compact. Let $\{f_n\}$ be a sequence in $\mathcal{M}$ and construct the subsequence of Example 2.5 so that $\{F_n\}$ converges to $f \in \mathcal{M}$ uniformly on compact subsets of $(-\infty, 0]$. We need to show that if

$$\delta_n = \sup_{-\infty < t \leq 0} |(F_n(t) - f(t))/h(t)|,$$

then $\delta_n \to 0$ as $n \to \infty$. For a given $\varepsilon > 0$ there exists $T > 0$ such that $2M/h(-T) < \varepsilon/2$. Thus,

$$\delta_n \leq (\varepsilon/2) + \max_{-T \leq t \leq 0} |F_n(t) - f(t)|.$$

Since the convergence is uniform on $[-T, 0]$, there is an N such that $n \geq N$ implies $\max_{-T \leq t \leq 0} |F_n(t) - f(t)| \leq \varepsilon/2$.

2.2 Fixed point theory

2.2.1 Banach fixed point theorem

Recall that an initial value problem

$$x' = f(t, x), \; x(t_0) = x_0, \tag{2.5}$$

can be expressed as an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{2.6}$$

from which a sequence of functions $\{x_n\}$ may be inductively defined by

$$x_0(t) = x_0, \; x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0)ds,$$

and, in general,

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds. \tag{2.7}$$

This is called Picard's method of successive approximations and, under liberal conditions on f, one can show that $\{x_n\}$ converges uniformly on some interval $|t - t_0| \leq k$ to some continuous function, say $x(t)$. Taking the limit in the equation defining $x_{n+1}(t)$, we pass the limit through the integral and have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

so that $x(t_0) = x_0$ and, upon differentiation, we obtain $x'(t) = f(t, x(t))$. Thus, $x(t)$ is a solution of the initial value problem.

Banach realized that this was actually a fixed-point theorem with wide application. For if we define an operator B on a complete metric space $C([t_0, t_0 + k], \mathbb{R})$ with the supremum norm $\|.\|$ (see Example (2.2)) by $x \in C$ implies

$$(Bx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{2.8}$$

then a fixed point of B, say $B\phi = \phi$, is a solution of the initial value problem. The idea had two outstanding features. First, it had application to problems in every area of mathematics which used complete metric spaces. And it was clean. For example, the standard muddy and shaky proofs of implicit function theorems became clear and solid using the fixed-point theory. We will use it here to prove existence of solutions of various kinds of differential equations.

Definition 2.6 Let (E, ρ) be a complete metric space and $B : E \to E$. The operator B is a contraction operator if there is an $\lambda \in (0, 1)$ such that $x, y \in E$ imply

$$\rho(Bx, By) \leq \lambda \rho(x, y).$$

Theorem 2.3 *[Contraction Mapping Principle] Let (E, ρ) be a complete metric space and $B : E \to E$ a contraction operator. Then there is a unique $x \in E$ with $Bx = x$. Furthermore, if $y \in E$ and if $\{y_n\}$ is defined inductively by $y_1 = By$ and $y_{n+1} = By_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Bx = x$ has one and only one solution.*

Proof. Let $x_0 \in E$ and define a sequence $\{x_n\}$ in E by $x_1 = Bx_0$, $x_2 = Bx_1 = B^2x_0, ..., x_n = Bx_{n-1} = B^n x_0$. To see that $\{x_n\}$ is a Cauchy sequence, note that if $m > n$ then

$$\begin{aligned}
\rho(x_n, y_m) &= \rho(B^n x_0, B^m x_0) \\
&\leq \lambda \rho(B^{n-1} x_0, B^{m-1} x_0) \\
&\vdots \\
&\leq \lambda^n \rho(x_0, x_{m-n}) \\
&\leq \lambda^n \{\rho(x_0, x_1) + \rho(x_1, x_2) + ... + \rho(x_{m-n-1}, x_{m-n})\} \\
&\leq \lambda^n \left\{\rho(x_0, x_1) + \lambda \rho(x_0, x_1) + ... + \alpha^{m-n-1} \rho(x_0, x_1)\right\} \\
&= \lambda^n \rho(x_0, x_1) \left\{1 + \lambda + ... + \lambda^{m-n-1}\right\} \\
&\leq \lambda^n \rho(x_0, x_1) \{1/ (1 - \lambda)\}.
\end{aligned}$$

Because $\lambda < 1$, the right side tends to zero as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence and (E, ρ) is complete so it has a limit $x \in E$. Now B is certainly continuous so

$$Bx = B\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (Bx_n) = \lim_{n\to\infty} x_{n+1} = x,$$

and x is a fixed point. To see that x is the unique fixed point, let $Bx = x$ and $By = y$. Then

$$\rho(x, y) = \rho(Bx, By) \leq \lambda \rho(x, y),$$

and, because $\lambda < 1$, we conclude that $\rho(x, y) = 0$ so that $x = y$. This completes the proof.

In applying this result to (2.5), a distressing event occurred which we now briefly describe. Assume that f is continuous and satisfies a global Lipschitz condition in x, say

$$|f(t, x_1) - f(t, x_2)| \leq L |x_1 - x_2|,$$

for $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. Then by (2.8) we obtain (for $t \geq t_0$)

$$\begin{aligned}
|Bx_1(t) - Bx_2(t)| &= \left|\int_{t_0}^{t} [f(s, x_1(s)) - f(s, x_2(s))]\, ds\right| \\
&\leq \int_{t_0}^{t} L |x_1(s) - x_2(s)|\, ds,
\end{aligned}$$

so that if $\|.\|$ is the sup norm on continuous functions on $[t_0, t_0 + k]$, then

$$\|Bx_1 - Bx_2\| \leq Lk \|x_1 - x_2\| .$$

This is a contraction if $Lk = \lambda < 1$. Now L is fixed and we take k small enough that $Lk < 1$. This gives a fixed point which is a solution of (2.5) on $[t_0, t_0 + k]$. ■

But the distressing part is that this interval is shorter than the one given by the results of Picard's successive approximations. While this can be satisfactorily dealt with in most cases of interest, it is upsetting. Fortunately there are two ways to cure it. Hale [28] adopts a different metric which resolves the discrepancy. A different way is through use of asymptotic fixed-point theorems. We shall see two other asymptotic fixed-point theorems, Browder's and Horn's, in addition to the following one.

Theorem 2.4 *Let (E, ρ) be a complete metric space and suppose that $B : E \to E$ such that B^m is a contraction for some fixed positive integer m. Then B has a fixed point in E.*

Proof. Let x be the unique fixed point of B^m, $B^m x = x$. Then $BB^m x = Bx$ and $BB^m x = B^m Bx$ so $B^m Bx = Bx$. Thus, Bx is also a fixed point of B^m and so, by uniqueness, $Bx = x$. Thus, x is a fixed point of B. Moreover, it is unique because if $By = y$, then $B^m y = y$ so $x = y$. This completes the proof. ■

The term "contraction" is used in several different ways in the literature. Our use is sometimes denoted by "strict contraction." The property $\rho(Bx, By) \leq \rho(x, y)$ is sometimes called "contraction" but it has limited use in fixed-point theory. A concept in between these two which is frequently useful is portrayed in the next result.

Theorem 2.5 *Let (E, ρ) be a compact nonempty metric space,*

$$B : E \to E \text{ and } \rho(Bx, By) < \rho(x, y),$$

for $x \neq y$. Then B has a unique fixed point.

Proof. We have

$$\rho(x, Bx) \leq \rho(x, y) + \rho(y, Bx) \leq \rho(x, y) + \rho(y, By) + \rho(By, Bx),$$

and since $\rho(By, Bx) \leq \rho(x, y)$ we conclude

$$\rho(x, Bx) - \rho(y, By) \leq 2\rho(x, y).$$

Interchanging x and y yields

$$|\rho(x, Bx) - \rho(y, By)| \leq 2\rho(x, y).$$

Thus the function $\vartheta : E \to [0,\infty)$ defined by $\vartheta(x) = \rho(x, Bx)$ is continuous on E. The compactness of E yields $z \in E$ with $\rho(z, Bz) = \rho(Bz, z) = \inf_{x\in E} \rho(x, Bx)$. If $\rho(Bz, z) \neq 0$ then $0 \leq \rho(B(Bz), Bz) < \rho(Bz, z)$ contradicting the infimum property. Thus $\rho(Bz, z) = 0$ and $Bz = z$. If there is another distinct fixed point, say $By = y$, then $\rho(y, z) = \rho(By, Bz) < \rho(y, z)$, a contradiction for $y \neq z$, This completes the proof. ■

Notice that the successive approximations are constructive in spirit. At least in theory one may begin with $x_0 \in E$, compute $x_1, ..., x_n$. Frequently one is interested in determining just how near x_0 and x_n are to that unique fixed point x. The next result gives an approximation.

Theorem 2.6 *If (E, ρ) is a complete metric space and $B : E \to E$ is a contraction operator with fixed point x, then for any $y \in E$ we have*
(a) $\rho(x, y) \leq \rho(By, y)/(1 - \lambda)$
and
(b) $\rho(B^n y, x) \leq \lambda^n \rho(By, y)/(1 - \lambda)$.

Proof. To prove (a) we note that

$$\rho(y, x) \leq \rho(y, By) + \rho(By, Bx) \leq \rho(y, By) + \lambda\rho(y, x),$$

so that

$$\rho(y, x)(1 - \lambda) \leq \rho(y, By).$$

For (b), recall that in the proof of Theorem 2.3 we had

$$\rho(B^n y, B^m y) \leq \lambda^n \rho(By, y)/(1 - \lambda).$$

As $m \to \infty, B^m y \to x$ so that

$$\rho(B^n y, x) \leq \lambda^n \rho(y, By)/(1 - \lambda).$$

This completes the proof. ■

2.2.2 Krasnoselskii fixed point theorem

Definition 2.7 Let $\mathcal{M}$ be a subset of a Banach space and et $A : \mathcal{M} \to E$ application. If A is continuous and $A\mathcal{M}$ is contained in a compact set in E, then we say that A is a compact application (we also say that A is completely continuous).

Theorem 2.7 *[Schauder][13], [55], [62], Let $\mathcal{M}$ be a convex set in a Banach space E and $A : \mathcal{M} \to E$ a compact application. Then A has a fixed point.*

In 1955 Krasnoselskii (see [13], [55]) observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,

Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

For better understanding this observation of Krasnoselskii, consider the following differential equation.

$$x'(t) = -a(t)\,x(t) - g(t,x). \tag{2.9}$$

We can transform this equation in another form while writing, formally

$$x'(t)\,e^{-\int_0^t a(s)ds} = -a(t)\,e^{-\int_0^t a(s)ds}x(t) - g(t,x)\,e^{-\int_0^t a(s)ds},$$

thus

$$x'(t)\,e^{-\int_0^t a(s)ds} + a(t)\,e^{-\int_0^t a(s)ds}x(t) = -g(t,x)\,e^{-\int_0^t a(s)ds},$$

or

$$\left(x(t)\,e^{-\int_0^t a(s)ds}\right)' = -g(t,x)\,e^{-\int_0^t a(s)ds},$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^{t}\left(x(u)\,e^{-\int_0^u a(s)ds}\right)' du = -\int_{t-T}^{t} g(u,x)\,e^{-\int_0^u a(s)ds}du,$$

what gives

$$x(t)\,e^{-\int_0^t a(s)ds} - x(T-t)\,e^{-\int_0^{T-t} a(s)ds} = -\int_{t-T}^{t} g(u,x)\,e^{-\int_0^u a(s)ds}du,$$

or

$$x(t) = x(T-t)\,e^{-\int_{T-t}^t a(s)ds} - \int_{t-T}^{t} g(u,x)\,e^{-\int_t^u a(s)ds}du. \tag{2.10}$$

If we suppose that $e^{-\int_{T-t}^t a(s)ds} := \lambda$ and if $(E, \|.\|)$ is the Banach space of functions $\varphi : \mathbb{R} \to E$ continuous, then the Equation (2.10) can be written as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (P\varphi)(t).$$

where B is contraction provides that the constant $\lambda < 1$ and A is compact mapping.

This example shows the birth of the mapping $P\varphi := B\varphi + A\varphi$ who is identified with a sum of a contraction and a compact mapping.

Thus, the search of the solution for (2.10) requires an adequate theorem which applies to this hybrid operator P and who can conclude the existence for a fixed point which will be, in his turn, solution of the initial Equation (2.9). Krasnoselskii found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result [55].

Theorem 2.8 (Krasnoselskii [55]) *Let $\mathcal{M}$ be a non-empty closed convex subset of a Banach space $(E, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into E such that*
(i) $Ax + By \in \mathcal{M}$ for all $x, y \in \mathcal{M}$,
(ii) A is continuous and $A\mathcal{M}$ is contained in a compact set of E,
(iii) B is a contraction with constant $\lambda < 1$.
Then there is a $x \in \mathcal{M}$ with $Ax + Bx = x$.

Note that if $A = 0$, the theorem become the theorem of Banach. If $B = 0$, then the theorem is not other than the theorem of Schauder.
Proof. According to the condition (iii) we have

$$\begin{aligned} \|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\ &\leq \|x-y\|+\|Bx-By\| \\ &\leq \|x-y\|+\lambda\,\|x-y\| \\ &= (1+\lambda)\,\|x-y\|\,, \end{aligned}$$

and

$$\begin{aligned} \|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\ &\geq \|x-y\|-\|Bx-By\| \\ &\geq \|x-y\|-\lambda\,\|x-y\| \\ &= (1-\lambda)\,\|x-y\|\,. \end{aligned}$$

In short

$$(1-\lambda)\,\|x-y\| \leq \|(I-B)\,x-(I-B)\,y\| \leq (1+\lambda)\,\|x-y\|\,.$$

This inequality shows that $(I-B):\mathcal{M}\to(I-B)\,\mathcal{M}$ is continuous and bijective. Thus, $(I-B)^{-1}$ exist and is continuous. Let us pose $U := (I-B)^{-1}A$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in \mathcal{M} \text{ such that } (I-B)^{-1}Az = z.$$

This is equivalent to $z = Az + Bz$. ■

2.3 Fractional calculus

Some special functions, important for the fractional calculus, as Gamma and Beta functions, the complementary error function, Mittag-Leffler function, are summarized in this section see [7, 32, 34, 45, 48, 52, 54].

Furthermore, fractional integration see [47, 48].

2.3.1 Special Functions

The Gamma Function

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$, i.e., $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \ \operatorname{Re} z > 0.$$

By analytic continuation the function is extended to the whole complex plane except for the points $0, -1, -2, -3, ...$, where it has simple poles. Thus, $\Gamma : \mathbb{C}\backslash \{0, -1, -2, ...\} \to \mathbb{C}$. Some of the most properties are

$$\begin{aligned} \Gamma(1) &= \Gamma(2) = 1, \\ \Gamma(z+1) &= z\Gamma(z), \\ \Gamma(n) &= (n-1)!, \ n \in \mathbb{N}, \\ \Gamma(1/2) &= \sqrt{\pi}, \\ \Gamma(n+1/2) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!!, \ n \in \mathbb{N}. \end{aligned} \tag{2.11}$$

The Gamma function is studied by many mathematicians. There is a long list of well-known properties (see, for example, [27]) but in this survey formulas (2.11) are sufficient.

The Beta function

The Beta function is defined by the integral

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \ \operatorname{Re} z > 0, \ \operatorname{Re} w > 0.$$

In addition, $B(z,w)$ is used sometimes for convenience to replace a combination of Gamma functions. This relation between the Gamma and Beta function [27],

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{2.12}$$

is used later on.

Equation (2.12) provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the Gamma function. It should also be mentioned that the Beta function is symmetric, i. e.,

$$B(z,w) = B(w,z).$$

The complementary error function (erfc)

The complementary error function (see http://mathworld.wolfram.com/Erfc.html and [48]) is an entire function, defined as

$$\operatorname{erf} \mathrm{c}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.$$

Special values of the complementary error function are

$$\begin{aligned} \operatorname{erf} \mathrm{c}(-\infty) &= 2, \\ \operatorname{erf} \mathrm{c}(0) &= 1, \\ \operatorname{erf} \mathrm{c}(+\infty) &= 0. \end{aligned}$$

The following relations are interesting to be mentioned

$$\begin{aligned} \operatorname{erf} \mathrm{c}(-t) &= 2 - \operatorname{erf} \mathrm{c}(t), \\ \int_0^\infty \operatorname{erf} \mathrm{c}(t) dt &= \frac{1}{\sqrt{\pi}}, \\ \int_0^\infty \operatorname{erf} \mathrm{c}^2(t) dt &= \frac{2-\sqrt{2}}{\sqrt{\pi}}. \end{aligned}$$

The Mittag-Lefflar function

While the Gamma function is a generalization of factorial function, the Mittag-Lefflar function is a generalization of the exponential function, first introduced as a one-parameter function by the series [48]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ \alpha > 0, \ \alpha \in \mathbb{R}, \ z \in \mathbb{C}.$$

Later, the two-parameter generalization is introduced by Agarwal

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta > 0, \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{C}, \tag{2.13}$$

which is of great importance for the fractional calculus. It is called two-parameter function of Mittag-Lefflar type.Some of its interesting properties are [48]

$$\begin{aligned} E_{1,1}(z) &= e^z, \\ E_{2,1}(z^2) &= \cosh(z), \\ E_{2,2}(z^2) &= \frac{\sinh(z)}{z}, \\ E_{\alpha,1}(z) &= E_\alpha(z), \\ E_{1/2,1}(z) &= e^{z^2} \operatorname{erf} \mathrm{c}(-z). \end{aligned} \tag{2.14}$$

2.3. Fractional calculus

2.3.2 Fractional integral according to Riemmann-Liouville

Cauchy's formula for repeated integration [47, 48]

$$I^n x(t) := \int_a^t \int_a^{s_1} ... \int_a^{s_{n-1}} x(s)ds...ds_2 ds_1 = \frac{1}{(n-1)!}\int_a^t x(s)(t-s)^{n-1}ds, \tag{2.15}$$

holds for $n \in \mathbb{N},\ a,t \in \mathbb{R},\ t > a$. If n is substituted by a positive real number α and $(n-1)!$ by its generalization $\Gamma(\alpha)$, a formula for fractional integration is obtained.

Definition 2.8 Suppose that $\alpha > 0, t > a,\ \alpha, a, t \in \mathbb{R}$. then the fractional operator

$$I^\alpha x(t) := \frac{1}{\Gamma(\alpha)}\int_a^t x(s)(t-s)^{\alpha-1}ds, \tag{2.16}$$

is referred to as Riemmann-Liouville fractional integral of order α.

After the introduction of the fractional integration operator it is reasonable to define also the fractional differentiation operator. There are different definitions, which do not coincide in general. This survey regards two of them, namely, the Riemann-Liouville and the Caputo fractional operator see [26, 48].

Definition 2.9 Suppose that $\alpha > 0,\ t > a,\ \alpha, a, t \in \mathbb{R}$. Then

$$D^\alpha x(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{x(s)}{(t-s)^{\alpha+1-n}}ds, & n-1 < \alpha < n \in \mathbb{N}, \\ \frac{d^n}{dt^n}x(t), & \alpha = n \in \mathbb{N}. \end{cases} \tag{2.17}$$

is called the Riemmann-Liouville fractional derivative or the Riemmann-Liouville fractional differential operator of order α.

We have the following properties of the Riemmann-Liouville integral operator:
(i) The Riemmann-Liouville integral operator I^α of order α is a linear operator.

$$I^\alpha(\lambda x(t) + \mu y(t)) = \lambda I^\alpha(x(t)) + \mu I^\alpha(y(t)),\ \lambda, \mu \in \mathbb{R},\ \alpha \in \mathbb{R}^+.$$

(ii) Semigroup properties:

$$I^\alpha(I^\beta x(t)) = I^{\alpha+\beta}(x(t)),\ \alpha, \beta \in \mathbb{R}^+.$$

(iii) Commutative property:

$$I^\alpha(I^\beta x(t)) = I^\beta(I^\alpha x(t)),\ \alpha, \beta \in \mathbb{R}^+.$$

(iv) Introduce the following causal function (vanishing for $t < 0$)

$$\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)},\ \alpha > 0.$$

then, we have that

1) $\Phi_\alpha(t) * \Phi_\beta(t) = \Phi_{\alpha+\beta}(t), \ \alpha, \beta \in \mathbb{R}^+$.

2) $I^\alpha x(t) = \Phi_\alpha(t) * x(t), \ \alpha \in \mathbb{R}^+$.

The Laplace transform:

$$L\left\{I^\alpha x(t)\right\} = L\left\{\Phi_\alpha(t)\right\} L\left\{x(t)\right\} = \frac{1}{s^\alpha} L\left\{x(t)\right\}.$$

(v) Effect on power functions

$$I^\alpha(t^\beta) = \frac{t^{\beta+\alpha}}{\Gamma(\beta+1+\alpha)}\Gamma(\beta+1) \text{ for all } \alpha > 0 \text{ and } \beta > -1, \ t > 0.$$

Proof of (v). *Using the definition of fractional integral and the property of* $B(\alpha, \beta)$ *function, we get*

$$\begin{aligned} I^\alpha(t^\beta) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{s^\beta}{(t-s)^{1-\alpha}} ds = \frac{t^{\beta+\alpha}}{\Gamma(\alpha)} \int_0^1 p^\beta (1-p)^{\alpha-1} dp \\ &= \frac{t^{\beta+\alpha}}{\Gamma(\alpha)} B(\beta+1, \alpha) = \frac{t^{\beta+\alpha}}{\Gamma(\beta+1+\alpha)} \Gamma(\beta+1). \end{aligned}$$

Proposition 2.1 *1)* $I^\alpha(1) = \frac{1}{\Gamma(1+\alpha)} t^\alpha$ *for all* $\alpha > 0$, $t > 0$.

2) Let x *analytic function, then*

$$\begin{aligned} I^\alpha x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-u)^{1-\alpha}} x(u) du \\ &= x(0)\frac{t^\alpha}{\Gamma(1+\alpha)} + x'(0)\frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + ... + x^{(n)}(0)\frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)} + ... \text{ for all } \alpha > 0, \end{aligned}$$

Applying this formula, we can obtain the fractional integral of order $\alpha > 0$ *from elementary functions.*

$$\begin{aligned} I^\alpha(e^{at}) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{e^{au} du}{(t-u)^{1-\alpha}} = e^{at} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{e^{-as} ds}{s^{1-\alpha}} = e^{at} \frac{a^{-\alpha}}{\Gamma(\alpha)} \int_0^{at} p^{\alpha-1} e^{-p} dp \\ &= a^{-\alpha} e^{at} \frac{\gamma(\alpha, at)}{\Gamma(\alpha)} = E_t(\alpha, a) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{a t^{1+\alpha}}{\Gamma(2+\alpha)} + ... + \frac{a^n t^{n+\alpha}}{\Gamma(n+1+\alpha)} + ... \end{aligned}$$

for all $\alpha > 0, \ t \geq 0$.

The special operator D^α that we choose to use, which requires the dependent variable x to be continuous and $\lceil \alpha \rceil-$times differentiable in the independent variable t, is defined by

$$D^\alpha x(t) := D^{\lceil \alpha \rceil} I^{\lceil \alpha \rceil - \alpha} x(t),$$

such that $\lim_{\alpha \to n^+} D^\alpha x(t) = D^n x(t)$ for any $n \in N$, $D^0 x(t) = x(t)$, where $\lceil \alpha \rceil$ is the ceiling function giving the smallest integer greater than or equal to α, and $\alpha \to n^+$ means

α approaches n from above. It is accepted practice to call D^α the Riemann-Liouville fractional differential operator of order α.

We have the following properties of the Riemann-Liouville fractional differential operator D^α of order α :

(i) Non-semigroup and non-commutative properties:

$$D^\alpha D^\beta x(t) \neq D^{\alpha+\beta} x(t),\ \ \alpha, \beta \in \mathbb{R}^+.$$

Suppose that $n-1<\alpha<n,\ \ m, n \in N, \alpha \in \mathbb{R}^+$. Then in general

$$D^m D^\alpha x(t)=D^{\alpha+m} x(t) \neq D^\alpha D^m x(t).$$

(ii)

$$D^\alpha(c)=\frac{c}{\Gamma(1-\alpha)} t^{-\alpha} \text{ for any constant } c.$$

2.3.3 The Caputo fractional differential operator

In this subsection an alternative operator to the Riemann-Liouville operator (2.17) is considered see [16].

Definition 2.10 Suppose that $\alpha>0, t>a,\ \ \alpha, a, t \in \mathbb{R}$. the fractional operator

$$^{C}D^\alpha x(t):=\left\{\begin{array}{c}\frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}} d s,\ \ n-1<\alpha<n \in \mathbb{N}, \\ \frac{d^n}{d t^n} x(t),\ \ \alpha=n \in \mathbb{N}.\end{array}\right. \tag{2.18}$$

is called the Caputo fractional derivative or Caputo fractional operator of order α. this operator is introduced by the Italian mathematician Caputo in 1967 [16].

Example 2.8 Let $a=0,\ \ \alpha=1/2,\ \ (n=1),\ \ x(t)=t$. Then, applying formula (2.18) we get

$$^{C}D^{1/2} t=\frac{1}{\Gamma(1/2)} \int_0^t \frac{1}{(t-s)^{1/2}} d s.$$

Taking into account the properties of the Gamma function and using the substitution $u:=(t-s)^{1/2}$ the final result for the Caputo fractional derivative of the function $x(t)=t$ is obtained as

$$\begin{aligned} ^{C}D^{1/2} t &= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{1/2}} d(t-s)=-\frac{1}{\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{1}{u} d u^2 \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2 u}{u} d u=\frac{2 \sqrt{t}}{\sqrt{\pi}}. \end{aligned}$$

Note ${}^{C}Dt = 1,\ {}^{C}D(1) = 0,\ {}^{C}D^{3/2}t = 0.$
Therefore,

$$\begin{aligned} {}^{C}D^{1/2}\, {}^{C}D(t) &= 0,\ {}^{C}D^{C}D^{1/2}(t) = \frac{1}{\sqrt{t}\sqrt{\pi}},\ \text{and} \\ {}^{C}D^{1/2}\, {}^{C}D(t) &= {}^{C}D^{3/2}(t) = 0,\ {}^{C}D^{C}D^{1/2}(t) \neq^{C} D^{3/2}(t). \end{aligned}$$

then, The Caputo fractional differential operator ${}^{C}D^{\alpha}$ of order α is non-semigroup properties.

If $x(t)$ and $y(t)$ are sufficiently smooth functions. Then
(i) ${}^{C}D^{\alpha}$ linear operator

$${}^{C}D^{\alpha}(\lambda x(t) + \mu y(t)) = \lambda^{C}D^{\alpha}(x(t)) + \mu^{C}D^{\alpha}\left(y(t)\right),\ \lambda, \mu \in \mathbb{R},\ \alpha \in \mathbb{R}^{+}.$$

(ii) Non commutative property: Suppose that $n-1 < \alpha < n,\ m, n \in \mathbb{N},\ \alpha \in \mathbb{R}^{+}$ and ${}^{C}D^{\alpha}f(t)$ exists. Then in general

$${}^{C}D^{\alpha}D^{m}x(t) =^{C} D^{\alpha+m}x(t) \neq D^{mC}D^{\alpha}x(t).$$

(iii)

$${}^{C}D^{\alpha}(c) = 0 \text{ for any constant } c.$$

(iv) The Laplace transform:

$$L\left\{{}^{C}D^{\alpha}x(t)\right\} = s^{\alpha}L\left\{x(t)\right\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}x^{(k)}(0).$$

(v) The Riemann-Liouville integral operator I^{α} and the Caputo fractional differential operator ${}^{C}D^{\alpha}$ are inverse operator in the sense that

$${}^{C}D^{\alpha}I^{\alpha}x(t) = x(t) \text{ and } I^{\alpha C}D^{\alpha}x(t) = x(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{t^{k}}{k!}x^{(k)}(0^{+}),\ \alpha \in \mathbb{R}^{+}. \tag{2.19}$$

where $\lfloor\alpha\rfloor$ is the floor function giving the largest integer less than or equal to α. The classic n-fold integral and differential operators of integer order satisfy like formula

$$D^{n}I^{n}x(t) = x(t) \text{ and } I^{n}D^{n}x(t) = x(t) - \sum_{k=0}^{n-1} \frac{t^{k}}{k!}x^{(k)}(0^{+}),\ n \in \mathbb{N}.$$

In general the two operator Riemann-Liouville and Caputo, do not coincide,i.e.,

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &\neq D^{\alpha}x(t), \\ {}^{C}D^{\alpha}x(t) &= D^{\alpha}\left(x(t) - \sum_{k=0}^{n-1} \frac{t^{k}}{k!}x^{(k)}(0)\right). \end{aligned}$$

Remark 2.1 Suppose that $n-1<\alpha<n,\ n\in N$. Let x be an analytic function, then
(i) ${}^{C}D^{\alpha}x(t)=x^{(n)}(0)\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}+...,$
(ii) $D^{\alpha}x(t)=x(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}+x'(0)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+...+x^{(n)}(0)\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}+...$
From that it follows that
(i) ${}^{C}D^{\alpha}e^{t}=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}+...+\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}+...$ for all $\alpha\in(0,1)$,
(ii) $D^{\alpha}e^{t}=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+...+\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}+...$ for all $\alpha\in(0,1)$.

In this sense, the Caputo derivative and Riemann-Liouville integral are analytic continuations of the well-known operator $n-$fold derivative and integral from the classical calculus. The Caputo fractional derivative ${}^{C}D^{\alpha}$ defined by

$${}^{C}D^{\alpha}x(t):=I^{\lceil\alpha\rceil-\alpha C}D^{\lceil\alpha\rceil}x(t), \tag{2.20}$$

such that $\lim\limits_{\alpha\to n^{-}}\ {}^{C}D^{\alpha}x(t)=D^{n}x(t)$ for any $n\in\mathbb{N}$, ${}^{C}D^{0}x(t)=x(t)$, where $\lceil\alpha\rceil$ is the ceiling function giving the smallest integer greater than or equal to α, and $\alpha\to n^{-}$ means α approaches n from blew.

The Caputo fractional derivative ${}^{C}D^{\alpha}$ defined in (2.20) can be expressed in a more explicit notation as the integral

$${}^{C}D^{\alpha}x(t)=\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_{0}^{t}\frac{1}{(t-u)^{1+\alpha-\lceil\alpha\rceil}}(D^{\lceil\alpha\rceil}x)(u)du,\ \alpha,t\in\mathbb{R}^{+}, \tag{2.21}$$

where the weak singularity caused by the Abel kernel of the integral operator is readily observed. This singularity can be removed through an integration by parts

$${}^{C}D^{\alpha}x(t)=\frac{1}{\Gamma(1+\lceil\alpha\rceil-\alpha)}\left(t^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}f(0^{+})+\int_{0}^{t}(t-u)^{\lceil\alpha\rceil-\alpha}(D^{1+\lceil\alpha\rceil}x)(u)du\right),$$

proved that the dependent variable x is continuous and $(1+\lceil\alpha\rceil)$-times differentiable in the independent variable t over the integral of differentiation (integration) $[0,t]$.

Example 2.9 Let $N_{0}=N\cup\{0\}$. If $\beta\in N_{0}$ and $\beta<\lceil\alpha\rceil$, then $D^{\lceil\alpha\rceil}(u^{\beta})=0$ and using formula (2.21), we get ${}^{C}D^{\lceil\alpha\rceil}x(t)=0$. If $\beta\in N_{0}$ and $\beta\geq\lceil\alpha\rceil$ or $\beta\notin N$ and $\beta>\lceil\alpha\rceil$, then

$$\begin{aligned}
{}^{C}D^{\alpha}(t^{\beta}) &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_{0}^{t}\frac{1}{(t-u)^{1+\alpha-\lceil\alpha\rceil}}D^{\lceil\alpha\rceil}(u^{\beta})du\\
&= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_{0}^{t}\frac{1}{(t-u)^{1+\alpha-\lceil\alpha\rceil}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\lceil\alpha\rceil)}u^{\beta-\lceil\alpha\rceil}du\\
&= \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(\beta+1-\lceil\alpha\rceil)}\int_{0}^{1}p^{\beta-\lceil\alpha\rceil}(1-p)^{-1-\alpha+\lceil\alpha\rceil}dp\\
&= \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(\beta+1-\lceil\alpha\rceil)}B(\beta+1-\lceil\alpha\rceil,\lceil\alpha\rceil-\alpha)\\
&= \frac{t^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)}\Gamma(\beta+1).
\end{aligned}$$

therefore

$$^{C}D^{\alpha}(t^{\beta})=\begin{cases} 0 \text{ if } \beta\in N_0 \text{ and } \beta<\lceil\alpha\rceil, \\ \frac{t^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)}\Gamma(\beta+1) \text{ if } \beta\in N_0 \text{ and } \beta\geq\lceil\alpha\rceil \text{ or } \beta\notin N \text{ and } \beta>\lceil\alpha\rceil. \end{cases}$$

Lemma 2.2 *Let $r(t)\in C\,[0,+\infty)$. Then $x(t)\in C\,[0,+\infty)$ is a solution of the Cauchy type problem*

$$\begin{cases} {}^{C}D^{\alpha}_{0+}x(t)=r(t),\ t\in\mathbb{R}^{+},\ 1<\alpha<2, \\ x(0)=x_0,\ x'(0)=x_1, \end{cases} \tag{2.22}$$

if and if $x(t)$ is a solution of the Cauchy type problem

$$\begin{cases} x(t)=I^{\alpha-1}_{0+}r(t)+x_1, \\ x(0)=x_0. \end{cases} \tag{2.23}$$

Proof. To begin with, we claim that for any $0<\gamma<1$, if $\psi\in C\,[0,+\infty)$, then $\left(I^{\gamma}_{0+}\psi\right)(0)=0$. In fact, since

$$I^{\gamma}_{0+}\psi(t)=\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\psi(s)ds,$$

we can conclude that

$$\left|I^{\gamma}_{0+}\psi(t)\right|=\frac{1}{\Gamma(\gamma)}\left|\int_0^t (t-s)^{\gamma-1}\psi(s)ds\right|\leqslant\frac{\|\psi\|_t}{\Gamma(\gamma+1)}t^{k}\to 0 \text{ as } t\to 0.$$

(1) Let $x\in C\,[0,+\infty)$ be a solution of the problem (2.22).
For any $t\in\mathbb{R}^{+}$, shows that

$$^{C}D^{\alpha}_{0+}x(t)=\left({}^{C}D^{\alpha-1}_{0+}D^{1}x\right)(t)=r(t).$$

According to (2.19), we have

$$x'(t)=x'(0)+I^{\alpha-1}_{0+}r(t)=I^{\alpha-1}_{0+}r(t)+x_1,$$

which means that $x(t)$ is a solution of the problem (2.23).
(2) Let $x(t)$ be a solution of the problem (2.23).
For any $t\in\mathbb{R}^{+}$, it is easy to see that

$$^{C}D^{\alpha}_{0+}x(t)={}^{C}D^{\alpha-1}_{0+}x'(t)=\left({}^{C}D^{\alpha-1}_{0+}I^{\alpha-1}_{0+}r\right)(t)+{}^{C}D^{\alpha-1}_{0+}x_1=r(t).$$

Besides, note that $r(t)\in C\,[0,+\infty)$, we have $x'(0)=I^{\alpha-1}_{0+}r(0)+x_1=x_1$. ■

2.4 Retarded functional differential equations

As has been asked by many students in many classrooms, "Why study this subject?" Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier? The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality see [20, 29, 36].

2.4.1 Delay differential equations

Suppose $\tau \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-\tau, 0]$ we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-\tau<\theta<0} |\phi(\theta)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If

$$t_0 \in \mathbb{R}, A \geq 0 \text{ and } x \in C\left([t_0 - \tau, t_0 + A], \mathbb{R}^n\right),$$

then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta), -\tau \leq \theta \leq 0$.

Definition 2.11 If Ω is a subset of $\mathbb{R} \times C, f : \Omega \to \mathbb{R}^n$ is a given function and "$'$" represents the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t), \tag{2.24}$$

is a retarded functional differential equation on Ω and will denote this equation by $RFDE$. If we wish to emphasize that the equation is defined by f,we write the $RFDE\,(f)$. A function x is said to be a solution of Equation (2.24) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C\left([t_0 - \tau, t_0 + A], \mathbb{R}^n\right), (t, x_t) \in \Omega$ and $x(t)$ satisfies Equation (2.24) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}, \phi \in C$, we say $x(t_0, \phi, f)$ is a solution of Equation (2.24) with initial value ϕ at t_0 or simply a solution through (t_0, ϕ) if there is an $A > 0$ such that $x(t_0, \phi, f)$ is a solution of Equation (2.24) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \phi, f) = \phi$.

Equation (2.24) is a very general type of equation and includes ordinary differential equations ($\tau = 0$).

We say Equation (2.24) is linear if $f(t, \phi) = L(t, \phi) + h(t)$ where $L(t, \phi)$ is linear in ϕ; is homogeneous if $h \equiv 0$ and nonhomogeneous $h \neq 0$. We claim Equation (2.24) is autonomous if $f(t, \phi) = g(\phi)$ where g does not depend on t.

For example, the following equations are delay differential equations

$$x'(t) = 2x(t) + 5x(t-1), \tag{2.25}$$

$$x'(t) = a(t)x(t) + b(t)x'(t-\tau(t)) + h(t), \tag{2.26}$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)ds. \tag{2.27}$$

a, b, τ are continuous functions. Equation (2.25) is an linear autonomous delay differential equation with constant $\tau = 1$, Equation (2.26) is nonhomogeneous, linear nonautonomous delay functional differential equations and Equation (2.27) is a delay linear integro-differential equation.

If $t_0 \in \mathbb{R}, \phi \in C$ are given and $f(t, \phi)$ is continuous, then finding a solution of Equation (2.24) through (t_0, ϕ) is equivalent to solving the integral equation

$$\begin{aligned} x_{t_0} &= \phi, \\ x(t) &= \phi(0) + \int_{t_0}^{t} f(s, x_s)ds, \ t \geq t_0. \end{aligned} \tag{2.28}$$

we define Tx by

$$\begin{aligned} Tx(t) &= \phi(0) + \int_{t_0}^{t} f(s, x_s)ds, \ t \geq t_0, \\ x_{t_0} &= \phi. \end{aligned}$$

To prove the existence of the solution through a point $(t_0, \phi) \in \mathbb{R} \times C$, we consider an $\eta > 0$ and all functions x on $[t_0 - \tau, t_0 + A]$ which are continuous and coincide with ϕ on $[t_0 - \tau, t_0]$; that is, $x_{t_0} = \phi$. The values of these functions on $[t_0, t_0 + \eta]$ are restricted to the class of x such that $|x(t) - \phi(0)| < \delta$ for $t \in [t_0, t_0 + \eta]$. The usual mapping T obtained from the corresponding integral equation is defined and it is then shown that η and δ can be so chosen that T maps this class into itself and is completely continuous. Thus, Schauder's fixed-point theorem implies existence (for examples details see the books [29, 36]).

Theorem 2.9 *(Existence) In (2.24), suppose Ω is an open subset in $\mathbb{R} \times C$ and f is continuos on Ω. If $(t_0, \phi) \in \Omega$, then there is a solution of (2.24) passing through (t_0, ϕ).*

Definition 2.12 We say $f(t, \phi)$ is Lipschitz in ϕ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \phi_i) \in K$, $i = 1, 2$,

$$|f(t, \phi_1) - f(t, \phi_2)| \leq k \left|\phi_1 - \phi_2\right|. \tag{2.29}$$

Theorem 2.10 *(Uniqueness) Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitz in ϕ in each compact set in Ω. If $(t_0, \phi) \in \Omega$, then there is a unique solution of Eq. (2.24) through (t_0, ϕ).*

Neutral delay differential equations

In order to define a general class of neutral delay differential equations ($NDDEs$) (or neutral functional differential equations ($NFDEs$)), we need the definition of atomic.

Definition 2.13 Suppose $\Omega \subseteq \mathbb{R}\times C$ is open with elements (t, ϕ). A function $\Psi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Ψ is continuous together with its first and second Fréchet derivatives with respect to ϕ: and Ψ_ϕ, the derivative with respect to ϕ, is atomic at β on Ω.

Definition 2.14 Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $\Psi : \Omega \to \mathbb{R}^n$ are given continuous functions with Ψ atomic at zero. The equation

$$\frac{d}{dt}\Psi(t, x_t) = f(t, x_t), \tag{2.30}$$

is called the neutral delay differential equation $NDDE(\Psi, f)$.

Definition 2.15 A function x is said to be a solution of the $NDDE(\Psi, f)$ or Equation (2.30), if there are $t_0 \in \mathbb{R}, A > 0$, such that $x \in C([t_0 - \tau, t_0 + A), \mathbb{R}^n), (t, x_t) \in \Omega, t \in [t_0, t_0 + A)$,

$\Psi(t, x_t)$ is continuously differentiable and satisfies Eq. (2.30) on $[t_0, t_0 + A)$. For a given $t_0 \in \mathbb{R}$, $\phi \in C$, and $(t_0, \phi) \in \Omega$, we say $x(t_0, \phi)$ is a solution of Eq. (2.30) with initial value ϕ at t_0, or simply a solution through (t_0, ϕ) ,if there is an $A > 0$ such that $x(t_0, \phi)$, is a solution of (2.30) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \phi) = \phi$.

Theorem 2.11 *(Existence) if Ω is an open set in $\mathbb{R} \times C$ and $(t_0, \phi) \in \Omega$, t hen there exists a solution of the $NDDE(\Psi, f)$ through (t_0, ϕ).*

Theorem 2.12 *(Uniqueness). If $\Omega \subseteq \mathbb{R} \times C$ is open and $f : \Omega \to \mathbb{R}^n$ as Lipschitz in ϕ on compact sets of Ω, then, for any $(t_0, \phi) \in \Omega$, there exists a unique solution of the $NDDE(\Psi, f)$ through (t_0, ϕ).*

For example

$$\begin{aligned} x'(t) &= -x'(t-1), \\ x'(t) &= x(t-1) + [x'(t-3)+1]^3, \\ x''(t) &= x(\frac{t}{2}) + x'(t-1) - x'(t-3), \end{aligned}$$

are neutral delay differential equations.

2.4.2 Method of Steps

The method of steps is an elementary method that can be used to solve some $DDEs$ analytically. This method is usually discarded as being too tedious, but in some cases the tedium can be removed by using computer algebra see [30]. Consider the following general DDE :

$$y'(t) = a_0 y(t) + a_1 y(t - w_1) + ... + a_m y(t - w_m), \tag{2.31}$$

where $y(t) = H(t)$ on the initial interval $-\max(w_i) \leq t \leq 0$. Let $b = \min(w_i)$. Then it is clear that the values of $y(t - w_m)$ are known in the interval $0 \leq t \leq b$. These values are $H(t - w_m)$. Thus, for the interval $0 \leq t \leq b$ we have

$$y'(t) = a_0 y(t) + a_1 H(t - w_1) + ... + a_m H(t - w_m),$$

and so

$$y(t) = \int_0^t (a_0 y(v) + a_1 H(v - w_1) + ... + a_m H(v - w_m))\, dv + y(0).$$

Now that we know $y(t)$ on $[0, b]$ we can repeat this procedure to obtain $y(t)$ on the interval $b \leq t \leq 2b$. This is given by:

$$y(t) = \int_b^t (a_0 y(v) + a_1 H(v - w_1) + ...)\, dv + y(b). \tag{2.32}$$

This process can be continued indefinitely, so long as the integrals that occur can be evaluated without too much effort. It is this last restriction that usually causes people to give up on this method, because the tedium and length of the method quickly overwhelms a human computer. However, it turns out that for certain classes of problems, where the phenomenon of "expression swell" is not too serious, we can take the method quite far, with a computer algebra system to automate the solution of the tedious sub-problems.

Example 2.10 For an example of this method we look first at a very simple DDE :

$$y'(t) = -y(t - 1),$$

with $y(t) = H(t) = 1$ for $-1 \leq t \leq 0$. The solution in the interval $0 \leq t \leq 1$ is given by:

$$y(t) = \int_0^t -H(x - 1)dx + y(0) = 1 - t.$$

Now we can solve for the solution in the interval $1 \leq t \leq 2$. This solution is given by:

$$y(t) = \int_1^t -H(t - 1)dx + y(1) = \frac{t^2}{2} - 2t + \frac{3}{2}.$$

This method can be programmed in Maple using a simple for loop.

2.4.3 Problems with a delay

In this subsection we introduce a large number of problems, both old and new, which are treated using the general theory of differential equations. We attempt to give sufficient description concerning the derivation, solution, and properties of solutions so that the reader will be able to appreciate some of the flavor of the problem. In none of the cases do we give a complete treatment of the problem, but offer references for further study.

Economics models

The following problem is copied from an elementary text on differential equations by Boyce and DiPrima [10]: "A young person with no initial capital invests k dollars per year at an annual interest rate τ. Assume that investments are made continuously and that interest is compounded continuously. If $\tau = 7.5\%$, determine k so that one million dollars will be available at the end of forty years."

It is solved by writing

$$S' = 0.075S + k, \;\; S(0) = 0,$$

and solving for $S(40)$. Several things are idealized in the problem, but still it is a fair model. It is noted there that in certain contexts continuous investment yields roughly the same as daily investment and it allows the student the opportunity to see the power of differential equations in giving a simple solution to an otherwise tedious problem.

Now the forty years is up and for computational convenience instead of the one million dollars let us say that the person has $\$900,000$ to invest and to live off the proceeds. During times of low interest rates a financial advisor may recommend bank certificates of deposit of 90-day maturity, automatically renewed at the existing interest rate, but laddered so that $\$10,000$ of the total matures every day and both principal and interest are reinvested. This enables the investor to quickly take advantage of rising rates and to lock in high interest long-term instruments if they become available. We imagine that this is changed to continuous reinvestment, just as the elementary problem imagined continuous investment of k dollars per year. If the total value is again $S(t)$, then from just the investment we would have

$$S'(t) = b(t)S(t - (1/4)).$$

The $b(t)$ represents a product. One factor is the fraction of the total amount of $S(t-1/4)$ which was invested three months earlier and matured today. The other factor is the interest being offered at that time. In addition, the person withdraws a percentage of the total $S(t)$ continuously for living expenses, resulting in an equation

$$S'(t) = -a(t)S(t) + b(t)S(t - 1/4), \;\; S(t) = \psi(t) \text{ for } -1/4 \le t \le t_0.$$

Here, the initial condition is an initial function $\psi : [-1/4, 0] \to \mathbb{R}$ with $\psi(t)$ being exactly that amount $S(t)$ which was invested at time t.

We can draw several conclusions of the following type. First, if the solutions are bounded, then times are likely to become difficult since inflation will eat away at the value and medical bills will increase with time; at this time, some studies have shown that those retiring with income sufficient to meet three times their current need approach desperate conditions within fifteen years. Next, we can ask if solutions will tend to zero. If they do, the person will be destined for the poor farm. At a minimum, the retiree must adjust the withdrawals so that the conditions of our theorem are not met.

Clearly, in this example it will make sense for both $a(t)$ and $b(t)$ to vary; $a(t)$ can be negative the day the income tax refund check arrives, and $b(t)$ can be negative when the bank fails and the FDIC assumes control see [14].

Controlling a ship

Minorsky (1962) designed an automatic steering device for the battleship New Mexico. The following is a sketch of the problem see [13].

Let the rudder of the ship have angular position $x(t)$ and suppose there is a friction force proportional to the velocity, say $-cx'(t)$. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected by a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h > 0$ between the time the ship gets off course and the time the electric motor activates the restoring force. The equation for $x(t)$ is

$$x''(t) + cx'(t) + g(x(t-h)) = 0, \tag{2.33}$$

where $xg(x) > 0$ if $x \neq 0$ and c is a positive constant. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course.

Epidemics (Cooke and Yorke)

In the work of Cooke and Yorke (1973) the Lotka assumption is changed so that the number of births per unit time is a function only of the population size, not of the age distribution see [13]. Under this assumption, we let $x(t)$ be the population size and let the number of births be $B(t) = g(x(t))$. Assume each individual has life span L so that the number of deaths per unit time is $g(x(t-L))$. Then the population size is described by

$$x'(t) = g(x(t)) - g(x(t-L)), \tag{2.34}$$

where g is some differentiable function. We note that every constant function is a solution of (2.34).

The following model for the spread of gonorrhea is considered by Cooke and Yorke (1973). The population is divided into two classes:

(a) $S(t)$ =the number of susceptibles, and

(b) $x(t)$ =the number of infectious.

The rate of new infection depends only on contacts between susceptible and infectious individuals. Since Set) equals the constant total population minus $x(t)$, the rate is some function $g(x(t))$. Assume that an exposed individual is immediately infectious and stays infectious for a period L (the time for treatment and cure). Then x also satisfies (2.34) holds.Now, at any time t, $x(t)$ equals the sum of capital produced over the period $[t-L, t]$ plus a constant c denoting the value of nondepreciating assets. Thus,

$$\begin{aligned} x(t) &= \int_0^L P(s)g[x(t-s)]ds + c \qquad (2.35) \\ &= \int_{t-L}^t P(t-u)g[x(u)]du + c. \end{aligned}$$

Some models of war and peace

L. F. Richardson (1881-1953, see [13]), a British Quaker, observed two world wars and was concerned about them (cf. Richardson, 1960; Jacobson, 1984). He speculated that wars begin where arms races end and he felt that international dynamics could be modeled mathematically because of human motivations. He claimed that men are guided by "their traditions, which are fixed, and their instincts which are mechanical"; thus, on a grand scale they are incapable of good and evil. He sought to develop a theory of international dynamics to guide statesmen with domestic and foreign policy, much as dynamics guides machine design.

Let X and Y be nations suspicious of each other. Suppose X and Y create stocks of arms x and y, respectively; more generally, x and y represent "threats minus cooperation" so that negative values have meaning. At least three things affect the arms buildup of X;

(a) economic burden;

(b) terror at the sight of $y(t)$ (or national pride);

(c) grievances and suspicions of y.

The same will, of course, apply to Y.

Richardson assumed that each side had complete and instantaneous knowledge of the arms of the other side and that each side could react instantaneously. He reasoned from (a) that

$$dx/dt = -a_1 x,$$

because the burden is proportional to the size x, and he argued from (b) that

$$dx/dt = -a_1 x + b_1 y,$$

because the terror is proportional to the size y. Finally, Richardson assumed constant standing grievances, say g_i so that the complete system is

$$\begin{aligned} x' &= -a_1 x + b_1 y + g_1, \\ y' &= -a_2 y + b_2 x + g_2. \end{aligned} \tag{2.36}$$

with a_i, b_i, and g_i , $i = 1, 2$ being positive constants. Domestic and foreign policy will set the a_i and b_i, although Richardson maintained a more mechanical view.

Hill (1978) recognized deficiencies in Richardson's model. He reasoned that it takes time to respond to an observed situation and, therefore, proposed the model

$$\begin{aligned} x' &= -a_1 x(t-T) + b_1 y(t-T) + g_1, \\ y' &= -a_2 y(t-T) + b_2 x(t-T) + g_2. \end{aligned}$$

where T is a positive constant.

Prey-predator population models (Lotka-Voltera)

Let $x(t)$ be the population at time t of some species of animal called prey and let $y(t)$ be the population of a predator species which lives off these prey. We assume that $x(t)$ would increase at a rate proportional to $x(t)$ if the prey were left alone, i.e., we would have $x'(t) = a_1 x(t)$, where $a_1 > 0$. However the predators are hungry, and the rate at which each of them eats prey is limited only by his ability to find prey. (This seems like a reasonable assumption as long as there are not too many prey available.) Thus we shall assume that the activities of the predators reduce the growth rate of $x(t)$ by an amount proportional to the product $x(t)y(t)$, i.e.,

$$x'(t) = a_1 x(t) - b_1 x(t) y(t),$$

where b_1 is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume $y'(t) = -a_2 y(t)$, where $a_2 > 0$, i.e., the predator species would die out exponentially. However, given food the predators breed at a rate proportional to their number and to the amount of food available to them. Thus we consider the pair of equations

$$\begin{aligned} x'(t) &= a_1 x(t) - b_1 x(t) y(t), \\ y'(t) &= -a_2 y(t) + b_2 x(t) y(t), \end{aligned} \tag{2.37}$$

where a_1, a_2, b_1, and b_2 are positive constants. This well-known model was invented and studied by Lotka [1920], [1925] and Volterra [1928], [1931].

Vito Volterra was trying to understand the observed fluctuations in the sizes of populations $x(t)$ of commercially desirable fish and $y(t)$ of larger fish which fed on the smaller ones in the Adriatic Sea in the decade from 1914 to 1923 see [20].

The sunflower equation

Somolinos (1978) has considered the equation

$$x'' + (a/r)x' + (b/r)\sin x(t-r) = 0,$$

and has obtained interesting results on the existence of periodic solutions. The study of this problem goes back to the early 1800s and has attracted much attention. It involves the motion of a sunflower plant see [13].

2.5 Stability in delay fractional differential equations

In this section, we present work of Sadati, Ghaderi and Ranjbar [53] on some fractional comparison results and stability theorem for fractional time delay systems.

2.5.1 Fractional functional differential equations

Let $C([a,b],\mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a maximum time delay τ of a system. In this case, we are often interested in the set of continuous function mapping $[-\tau,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $C = C([-\tau,0],\mathbb{R}^n)$. For any $A > 0$ and any continuous function of time $\psi \in C([t_0-\tau, t_0+A],\mathbb{R}^n),\ t_0 \le t \le t_0 + A$, let $x_t(\theta) \in C$ be a segment of function x defined as $x_t(\theta) = x(t+\theta),\ -\tau \le \theta \le 0$.

Consider Caputo fractional nonlinear time-delay system

$$ {}_{t_0}^{c}D_t^{\alpha} x(t) = f(t, x_t). \tag{2.38}$$

where $x(t) \in \mathbb{R}^n,\ 0 < \alpha \le 1$ and $f : \mathbb{R} \times C \to \mathbb{R}^n$. Equation (2.38) indicates Caputo derivatives of the state variable x on $[t_0, t]$ and $x(\zeta)$ for $t-\tau \le \zeta \le t$. A such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length τ, say, from $t_0-\tau$ to t_0, i.e.,

$$x_t = \phi, \tag{2.39}$$

where $\phi \in C$ is given. In other words we have $x(t_0+\theta) = \phi(\theta), -\tau \le \theta \le 0$. Throughout the manuscript we will use the Euclidean norm for vectors denoted by $\|.\|$. The space of continuous initial functions $C([-\tau,0],\mathbb{R}^n)$ is provided with the supremum norm

$$\|\phi\|_0 = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|. \tag{2.40}$$

Let $\rho > 0$ be a given constant, and let

$$C_\rho = \{\phi \in C\colon \|\phi\|_0 < \rho\}, \tag{2.41}$$

and

$$S_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}. \tag{2.42}$$

2.5.2 Generalization of some comparison results in fractional functional differential equations

In this subsection we will develop and generalize some basic comparison results in theory of functional differential equation to the fractional case. The integer order derivative version of these theorems can be found in [37].

Let $0 < \alpha < 1$ and $p = 1 - \alpha$. Denote by $C_p([t_0, T], \mathbb{R})$, the function space as follows:

$$C_p([t_0, T], \mathbb{R}) = \{u \in C((t_0, T], \mathbb{R}) \text{ and } (t - t_0)^p u(t) \in C([t_0, T], \mathbb{R})\}. \tag{2.43}$$

Lemma 2.3 *Let $m \in C_p([t_0, T], \mathbb{R})$ be locally Hölder continuous with exponent $\lambda > \alpha$ and $t_1 \in (t_0, T]$, we have*

$$m(t_1) = 0 \text{ and } m(t) \leq 0 \text{ for } t_0 \leq t \leq t_1. \tag{2.44}$$

Then it follows that,

$$D^\alpha m(t_1) \geq 0. \tag{2.45}$$

For proof of Lemma 2.3, please see in [38].

Definition 2.16 A function $a(r)$ is said to belong to the class $\mathfrak{K}$, if $a \in C([0, \rho), \mathbb{R}^+)$, $a(0) = 0$ and $a(r)$ is strictly monotone increasing in r.

Definition 2.17 A function $V(t, x)$ with $V(t, 0) \equiv 0$ is said to positive definite if there exist a function $b \in \mathfrak{K}$ such that

$$V(t, x) \geq b(\|x\|), \tag{2.46}$$

is satisfied for $(t, x) \in \mathfrak{R}_+ \times S_\rho$, and it is said to be decrescent if a function $a \in \mathfrak{K}$ exist such that

$$V(t, x) \leq a(\|x\|),\ (t, x) \in \mathbb{R}^+ \times S_\rho. \tag{2.47}$$

Definition 2.18 (Lyapunov-like function).Let $V \in C([-\tau, \infty) \times S_\rho, R_+)$, and let $\phi \in C_\rho$. We define the fractional-order Dini derivatives in Caputo's sence ${}^cD_+^\alpha V(t, \phi(0), \phi)$ and ${}^cD_-^\alpha V(t, \phi(0), \phi)$ with respect to the functional differential system (2.38) as follows:

$${}^cD_+^\alpha V(t, \phi(0), \phi) = \lim_{h \to 0^+} \sup \frac{1}{h^\alpha} \left[V(t, \phi(0)) - V(t - h, \phi(0) - h^\alpha f(t, \phi))\right], \tag{2.48}$$

$${}^cD_-^\alpha V(t, \phi(0), \phi) = \lim_{h \to 0^-} \inf \frac{1}{h^\alpha} \left[V(t, \phi(0)) - V(t - h, \phi(0) - h^\alpha f(t, \phi))\right], \tag{2.49}$$

we need, consequently, the following subsets of C defined by

$$\Omega_1 = \{\phi \in C_\rho : |V_t|_0 = V(t, \phi(0)),\ t \in [t_0, \infty]\}\,, \tag{2.50}$$

$$\Omega_0 = \{\phi \in C_\rho : V(t+\theta, \phi(\theta)) < L\left(V(t, \phi(0))\right),\ t \in [t_0, \infty]\}\,, \tag{2.51}$$

where $L(u)$ is continuous on $\mathbb{R}^+$, non-decreasing in u, and $L(u) < u$, for $u > 0$; and

$$|V_t|_0 = \sup_{-\tau \le \theta \le 0} V(t+\theta, \phi(\theta)). \tag{2.52}$$

In fact, these Lyapunov-like functions act like a transformation of (2.38) into a relatively simple fractional differential equation, the properties of solutions of this simple system can be transferred back to the original more complicated system. This is known as the comparison principle, in general.We can now state some fractional comparison results via the Lyapunov-like function.

Theorem 2.13 *Let $V \in C([-\tau, \infty) \times S_\rho, R_+)$ and $V(t, x)$ be locally Lipschitzian in x. Assume that functional ${}^cD_-^\alpha V(t, \phi(0), \phi)$, defined by (2.49). verifies the inequality*

$${}^cD_-^\alpha V(t, \phi(0), \phi) \le g(t, V(t, \phi(0))),\ t > t_0,\ \phi \in \Omega_1, \tag{2.53}$$

where $g \in C([t_0, \infty) \times R_+, R_+)$ and $r(t, t_0, u_0)$ is the maximal solution of the scalar differential equations of

$${}^cD^\alpha u = g(t, u),\ u(t_0) = u_0 \ge 0, \tag{2.54}$$

existing on $[t_0, \infty)$. Let $x(t_0, \phi_0)$ be any solution of (2.38) defined in the future, satisfying

$$\sup_{-\tau \le s \le 0} V(t_0, \phi_0(s)) \le u_0. \tag{2.55}$$

Then,

$$V(t, x(t_0, \phi_0(t))) \le r(t, t_0, u_0),\ t \ge t_0. \tag{2.56}$$

Proof. Let $x(t_0, \phi_0)$ be any solution of (2.38) with an initial function $\phi_0 \in C_\rho$ at $t = t_0$. Define the function

$$m(t) = V(t, x(t_0, \phi_0)(t)).$$

For $\varepsilon > 0$ sufficiently small, consider the fractional differential equation

$${}^cD^\alpha u = g(t, u) + \varepsilon,\ u(t_0) = u_0 \ge 0, \tag{2.57}$$

whose solutions $u(t, \varepsilon) = u(t, t_0, u_0, \varepsilon)$ exists as far as $r(t, t_0, u_0)$ to the right of t_0. Since

$$\lim_{\varepsilon \to 0} u(t, \varepsilon) = r(t, t_0, u_0), \tag{2.58}$$

it is enough to show that

$$m(t) \leq u(t, \varepsilon), \ t \geq t_0. \tag{2.59}$$

If this inequality is not true, let t_1 be greatest lower bound of numbers $t > t_0$ for which (2.59) is false. The continuity of the functions $m(t)$ and $u(t, \varepsilon)$ implies that

(i) $m(t) \leq u(t, \varepsilon), \ t_0 \leq t \leq t_1$;

(ii) $m(t) = u(t, \varepsilon), \ t = t_1$.

Now by Lemma 2.3 we have

$$D^{\alpha} m(t_1) \geq D^{\alpha} u(t_1, t_0, u_0, \varepsilon) = g(t_1, u(t_1, t_0, u_0, \varepsilon)) + \varepsilon. \tag{2.60}$$

Since $g(t, u) + \varepsilon \geq 0, \ u(t, t_0, u_0, \varepsilon)$ is non-decreasing in t; and this implies, from (i) and (ii) that

$$|m_{t_1}|_0 = u(t_1, t_0, u_0, \varepsilon) = m(t_1). \tag{2.61}$$

Setting $\phi = x_{t_1}(t_0, \phi_0)$ and noting that $\phi(0) = x(t_0, \phi_0)(t_1)$, it follows that

$$|V_{t_1}|_0 = V(t_1, \phi(0)). \tag{2.62}$$

This means that $\phi \in \Omega$, and, consequently. using the Lipschitzian character of $V(t, x)$ in x and the relation (2.53), we obtain the inequality

$${}^{c}D^{\alpha} m(t_1) \leq g(t_1, m(t_1)). \tag{2.63}$$

which is incompatible with (2.60). Hence (2.56) is valid and the proof is complete. ■

Corollary 2.1 *Let $V \in C([-\tau, \infty) \times S_{\rho}, R_+)$ and $V(t, x)$ be locally Lipschitz in x. Assume that, for $t > t_0, \phi \in \Omega_0$,*

$${}^{c}D_{+}^{\alpha} V(t, \phi(0), \phi) \leq 0. \tag{2.64}$$

Let $x(t_0, \phi_0)$ be any solution of (2.38) such that $x(t_0, \phi_0)(t) \in S_{\rho}$ for $t \in [t_0, t_1]$. Then

$$V(t, x(t_0, \phi_0)(t)) \leq \sup_{-\tau \leq s \leq 0} V(t_0, \phi_0(s)), \ t \in [t_0, t_1]. \tag{2.65}$$

Proof. If we set $g \equiv 0$ in Theorem 2.13, we obtain the inequality

$$V(t, x(t_0, \phi_0)(t)) \leq V(t_2, x(t_0, \phi_0)(t_2)), \tag{2.66}$$

where $t_2 \in (t_0, t_1)$. Since $V(t_2, x(t_0, \phi_0)(t_2)) > 0$, the assumption on $L(u)$ implies that

$$V(t_2 + s, x(t_0, \phi_0)(t_2 + s)) \leq L\left(V(t_2, x(t_0, \phi_0)(t_2))\right), \tag{2.67}$$

which shows that $x_t(t_0, \phi_0) \in \Omega_0, \ t_0 \leq t \leq t_2$. The remain of the proof is similar to the proof of Theorem 2.13. ■

The next comparison theorem gives a better estimate.

Theorem 2.14 *Let the assumptions of Theorem 2.13 hold except that inequality (2.53) is replaced by*

$$^{c}D_{+}^{\alpha}V(t,\phi(0),\phi)+d\left(\|\phi(0)\|\right)\leq g(t,V(t,\phi(0))), \tag{2.68}$$

for $t\geq t_0,\ \phi\in\mathcal{C}_\rho$, where the function $d\in\mathfrak{K}$. Assume further that $g(t,u)$ is monotone non-decreasing in u for each t.Then (2.55) implies

$$V(t,x(t_0,\phi_0)(t))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}d(\|x(t_0,\phi_0)(s)\|)ds\leq r(t). \tag{2.69}$$

Proof. Consider

$$m(t)=V(t,x(t_0,\phi_0)(t))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}d(\|x(t_0,\phi_0)(s)\|)ds, \tag{2.70}$$

for $t\geq t_0$ and set $\phi=x_t(t_0,\phi_0)$ so that $\phi(0)=x(t_0,\phi(0))(t)$. We obtain, using the condition (2.68), the inequality

$$^{c}D_{+}^{\alpha}m(t_1)\leq g(t_1,m(t_1)). \tag{2.71}$$

Here, we have used the monotonicity of $g(t,u)$ in u and the fact that

$$V(t,x(t_0,\phi_0)(t))\leq m(t), \tag{2.72}$$

while applying the assumption (2.69). Therefore, it is eazy to prove the result of Theorem 2.14, following the statements in the proof of Theorem 2.13. ■

2.5.3 Stability

Let us consider the fractional functional differential system (2.38). We will assume that $f(t,0)\equiv 0$, so that the system (2.38) possesses the trivial solution ($x=0$). Let us also suppose that the solutions $x(t_0,\phi_0)$ of (2.38) exist in the future.

Definition 2.19 The trivial solution of (2.38) is said to be stable if, for each $\varepsilon>0$, t_0, there exists a positive function $\delta=\delta(t_0,\varepsilon)$ that is continuous in t_0 for each ε, such that, whenever

$$\|\phi_0\|\leq\delta, \tag{2.73}$$

we have

$$\|x(t_0,\phi_0)(t)\|<\varepsilon,\ t\geq t_0. \tag{2.74}$$

Definition 2.20 The solution $x=0$ is said to be uniformly stable if the number δ in the previous definition is independent of t_0.

Theorem 2.15 *Assume that there exists a function $V(t,x)$ satisfying the following conditions:*

(i) $V \in C([-\tau,\infty)\times S_\rho,\mathbb{R}^+)$, $V(t,x)$ is positive definite, decrescent, and locally Lipschitzian in x;

(ii) for $t>t_0, \phi\in\Omega_0$,

$$^cD_+^\alpha V(t,\phi(0),\phi)\leq 0. \tag{2.75}$$

Then, the trivial solution of (2.38) is uniformly stable.

Proof. Since V is positive definite and decrescent, there exist functions $a,b\in\mathfrak{K}$ satisfying

$$b\left(\|x\|\right)\leq V(t,x)\leq a\left(\|x\|\right),\ (t,x)\in[t_0,\infty)\times S_\rho. \tag{2.76}$$

Let $0<\varepsilon<\rho,\ t_0\in\mathbb{R}^+$ be given. Choose $\delta=\delta(\varepsilon)>0$ such that

$$a(\delta)<b(\varepsilon). \tag{2.77}$$

We claim that, if $\|\phi_0\|\leq\delta$, then $\|x(t_0,\phi_0)\|<\varepsilon,\ t\geq t_0$. Suppose that this is not true. Then, there exists a solution $x(t_0,\phi_0)$ of (2.38) with $\|\phi_0\|_0\leq\delta$ such that

$$\|x(t_0,\phi_0)(t_2)\|=\varepsilon, \tag{2.78}$$

and

$$\|x(t_0,\phi_0)(t)\|\leq\varepsilon,\ t\in[t_0,t_2], \tag{2.79}$$

so that

$$V(t_2,x(t_0,\phi_0)(t_2))\geq b(\varepsilon), \tag{2.80}$$

because of (2.76). Furthermore, this means that $x(t_0,\phi_0)(t_2)\in S_\rho,\ t\in[t_0,t_2]$. Hence, the choise $u_0=a\left(\|\phi_0\|_0\right)$ and the condition

$$^cD_-^\alpha V(t,\phi(0),\phi)\leq 0,\ t\in[t_0,t_2],\ \phi\in\Omega_0, \tag{2.81}$$

give the estimate

$$V(t,x(t_0,\phi_0)(t))\leq a(\|\phi_0\|_0),\ t\in[t_0,t_2], \tag{2.82}$$

because of Corollary 2.1. Now the relations (2.80), (2.82), and (2.77) lead to contradiction

$$b(\varepsilon)\leq V(t_2,x(t_0,\phi_0)(t_2))\leq a(\|\phi_0\|_0)\leq a(\delta)<b(\varepsilon). \tag{2.83}$$

This proves that the trivial solution of (2.38) is uniformly stable. ■

Chapter 3

Existence and uniqueness for delay fractional differential equations

In this chapter, we establish sufficient conditions for the existence, uniqueness and stability of solutions for nonlinear fractional differential equations with delays and integral boundary conditions (see [23]).

3.1 Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. Fractional differential equations appear naturally in a number of fields such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electron-analytical chemistry, biology, control theory, etc. An excellent account of the study of fractional differential equations can be found in [2, 3, 19, 34, 39, 44, 45, 48, 49, 52, 54, 61] and the references therein. Boundary value problems for fractional differential equations have been discussed in [5, 6, 15, 43, 56, 57, 58, 60, 61, 63, 64]. By contrast, the development of stability for solutions of fractional differential equations is a bit slow. El-Sayed, Gaafar and Hamadalla [22] discuss the existence, uniqueness and stability of solutions for the non-local non-autonomous system of fractional order differential equations with delays

$$D^\alpha x_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j) + h_i(t),\ t > 0,$$

where D^α denotes the Riemann-Liouville derivative of order α.

We consider nonlinear fractional differential equations with delay and integral boundary conditions of the form

$$D^{\alpha}x(t) = \sum_{j=1}^{n} a_j(t) f(t, x(t), x(t-\tau_j)), \; t > 0, \tag{3.1}$$

$$x(t) = \phi(t) \text{ for } t < 0 \text{ and } \lim_{t\to 0^-} \phi(t) = 0, \tag{3.2}$$

$$I^{1-\alpha}x(t)|_{t=0} = 0, \tag{3.3}$$

where $\alpha \in (0,1)$, $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $a_j(t), \phi(t)$ are given continuous functions, $\tau_j \geq 0, \; j = 1, 2, ..., n$ are constants.

In this chapter our aim is to show the existence of a unique solution for (3.1)-(3.3) and its uniform stability.

Let $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and satisfy the Lipschitz conditions

$$|f(t, x, y_j) - f(t, u, v_j)| \leq k\,|x - u| + k_j\,|y_j - v_j|\,, \; k > 0, \; k_j > 0, \; j = 1, 2, ..., n,$$

for all $x, y_j, u, v_j \in \mathbb{R}$.

3.2 Existence of a unique solution for nonlinear fractional differential equations

Let X be the class of all continuous functions defined on $\mathbb{R}^+$ with the norm

$$\|x\| = \sup\left\{e^{-Nt}\,|x(t)|\right\}, \; x \in X.$$

Theorem 3.1 *Let $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition: if*

$$\frac{\sum_{j=1}^{n} a_j (k + k_j e^{-N\tau_j})}{N^{\alpha}} < 1,$$

where $a_j = \max_{t\in\mathbb{R}^+}\{|a_j(t)|\}$, then nonlinear fractional differential equations (3.1)-(3.3) have a unique positive solution.

Proof. For $t > 0$, equation (3.1) can be written as

$$\frac{d}{dt} I^{1-\alpha}x(t) = \sum_{j=1}^{n} a_j(t) f(t, x(t), x(t-\tau_j)).$$

Integrating both sides of the above equation, we obtain

$$I^{1-\alpha}x(t) - I^{1-\alpha}x(t)|_{t=0} = \sum_{j=1}^{n} \int_0^t a_j(s) f(s, x(s), x(s-\tau_j))ds,$$

then

$$I^{1-\alpha}x(t) = \sum_{j=1}^{n} \int_0^t a_j(s) f(s, x(s), x(s-\tau_j))ds.$$

Applying the operator by I^{α} on both sides,

$$Ix(t) = \sum_{j=1}^{n} I^{\alpha+1} a_j(t) f(t, x(t), x(t-\tau_j)),$$

differentiating both sides, we obtain

$$x(t) = \sum_{j=1}^{n} I^{\alpha} a_j(t) f(t, x(t), x(t-\tau_j)). \tag{3.4}$$

Now, let $F : X \to X$ be defined by

$$Fx\,(t) = \sum_{j=1}^{n} I^{\alpha} a_j(t) f(t, x(t), x(t-\tau_j)).$$

then

$$\begin{aligned}
|Fx(t) - Fy(t)| &= \left| \sum_{j=1}^{n} I^{\alpha} a_j(t) f(t, x(t), x(t-\tau_j)) - \sum_{j=1}^{n} I^{\alpha} a_j(t) f(t, y(t), y(t-\tau_j)) \right| \\
&= \left| \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ a_j(s) f(s, x(s), x(s-\tau_j)) - a_j(s) f(s, y(s), y(s-\tau_j)) \right\} ds \right| \\
&\leq \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| a_j(s) f(s, x(s), x(s-\tau_j)) - a_j(s) f(s, y(s), y(s-\tau_j)) \right| ds \\
&\leq \sum_{j=1}^{n} a_j \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ k\,|x(s) - y(s)| + k_j\,|x(s-\tau_j) - y(s-\tau_j)| \right\} ds \\
&\leq \sum_{j=1}^{n} a_j k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - y(s)|\, ds \\
&+ \sum_{j=1}^{n} a_j k_j \int_0^{\tau_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s-\tau_j) - y(s-\tau_j)|\, ds \\
&+ \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s-\tau_j) - y(s-\tau_j)|\, ds.
\end{aligned}$$

By conditions (3.2), we have

$$\begin{aligned}
|Fx(t) - Fy(t)| \leq &\sum_{j=1}^{n} a_j k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - y(s)|\, ds \\
&+ \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s-\tau_j) - y(s-\tau_j)|\, ds,
\end{aligned}$$

and

$$
\begin{aligned}
& e^{-Nt}\left|Fx(t)-Fy(t)\right| \\
& \le \sum_{j=1}^{n} a_j k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} \left|x(s)-y(s)\right| ds \\
& + \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau_j)} e^{-N(s-\tau_j)} \left|x(s-\tau_j)-y(s-\tau_j)\right| ds \\
& \le \sum_{j=1}^{n} a_j k \sup_{t\in\mathbb{R}^+} \left\{e^{-Nt}\left|x(t)-y(t)\right|\right\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} ds \\
& + \sum_{j=1}^{n} a_j k_j \int_0^{t-\tau_j} \frac{(t-\theta-\tau_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} e^{-N\theta} \left|x(\theta)-y(\theta)\right| d\theta \\
& \le \sum_{j=1}^{n} a_j k \sup_{t\in\mathbb{R}^+} \left\{e^{-Nt}\left|x(t)-y(t)\right|\right\} \int_0^{Nt} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)} du \\
& + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+} \left\{e^{-Nt}\left|x(t)-y(t)\right|\right\} \int_0^{t-\tau_j} \frac{(t-\theta-\tau_j)^{\alpha-1} e^{-N(t-\theta)}}{\Gamma(\alpha)} d\theta \\
& \le \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k \left\|x-y\right\| + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+} \left\{e^{-Nt}\left|x(t)-y(t)\right|\right\} \int_0^{t-\tau_j} \frac{u^{\alpha-1}e^{-Nu}}{\Gamma(\alpha)} e^{-N\tau_j} du \\
& \le \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k \left\|x-y\right\| + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+} \left\{e^{-Nt}\left|x(t)-y(t)\right|\right\} \frac{e^{-N\tau_j}}{N^\alpha} \int_0^{N(t-\tau_j)} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)} du \\
& \le \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k \left\|x-y\right\| + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+} \left\{e^{-Nt}\left|x(t)-y(t)\right|\right\} \frac{e^{-N\tau_j}}{N^\alpha} \\
& \le \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k \left\|x-y\right\| + \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k_j e^{-N\tau_j} \sup_{t\in\mathbb{R}^+} e^{-Nt}\left|x(t)-y(t)\right| \\
& \le \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j \left(k+k_j e^{-N\tau_j}\right) \left\|x-y\right\|.
\end{aligned}
$$

Now, choose N large enough such that $\frac{1}{N^\alpha}\sum_{j=1}^{n} a_j \left(k+k_j e^{-N\tau_j}\right) < 1$. So, the map $F: X \to X$ is a contraction and it has a fixed point $x = Fx$, and hence there exists a unique $x \in X$ which is a solution of integral equation (3.4).

We now prove the equivalence between integral equation (3.4) and nonlinear fractional differential equations (3.1)-(3.3). Indeed, since$x \in X$ and $I^{1-\alpha}x(t) \in C(X)$, applying the operator $I^{1-\alpha}$ on both sides of (3.4), we obtain

$$\begin{aligned} I^{1-\alpha}x(t) &= \sum_{j=1}^{n} I^{1-\alpha}I^{\alpha}a_j(t)f(t,x(t),x(t-\tau_j)) \\ &= \sum_{j=1}^{n} Ia_j(t)f(t,x(t),x(t-\tau_j)). \end{aligned}$$

Differentiating both sides,

$$DI^{1-\alpha}x(t) = \sum_{j=1}^{n} DIa_j(t)f(t,x(t),x(t-\tau_j)),$$

we get

$$D^{\alpha}x(t) = \sum_{j=1}^{n} a_j(t)f(t,x(t),x(t-\tau_j)),\ t>0,$$

which proves the equivalence of (3.4) and (3.1).We want to prove that $\lim_{t\to 0^+} x = 0$. Since $a_j(t)f(t,x(t),x(t-\tau_j))$ are continuous on $[0,T]$, there exist constants m, M such that $m \le a_j(t)f(t,x(t),x(t-\tau_j)) \le M$. We have

$$I^{\alpha}a_j(t)f(t,x(t),x(t-\tau_j)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a_j(s)f(s,x(s),x(s-\tau_j))ds,$$

which implies

$$m\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \le I^{\alpha}f(t,x(t),x(t-\tau_j)) \le M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds,$$

$$nm\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \le \sum_{j=1}^{n} I^{\alpha}a_j(t)f(t,x(t),x(t-\tau_j)) \le nM\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds,$$

which in turn implies

$$nm\frac{t^{\alpha}}{\Gamma(\alpha+1)} \le \sum_{j=1}^{n} I^{\alpha}a_j(t)f(t,x(t),x(t-\tau_j)) \le nM\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

and

$$\lim_{t\to 0^+}\sum_{j=1}^{n} I^{\alpha}a_j(t)f(t,x(t),x(t-\tau_j)) = 0.$$

Then from (3.4) $\lim_{t\to 0^+} x(t) = 0$ and from (3.2), we have $\lim_{t\to 0^+}\phi(t) = 0$. ■

Now, for $t \in (-\infty, T], T < \infty$, the solution of nonlinear fractional differential equations (3.1)-(3.3) takes the form

$$x(t) = \begin{cases} \phi(t), & t<0, \\ 0, & t=0, \\ \sum\limits_{j=1}^{n} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a_j(s)f(s,x(s),x(s-\tau_j))ds, & t>0. \end{cases}$$

3.3 Stability of a unique solution for nonlinear fractional differential equations

In this section, we study the stability of the solution of nonlinear fractional differential equations (3.1)-(3.3).

The $\widetilde{x}(t)$ is a solution of the nonlinear fractional differential equations

$$\left(\widetilde{P}\right)\begin{cases} D^{\alpha}\widetilde{x}\left(t\right)=\sum_{j=1}^{n}a_{j}\left(t\right)f\left(t,\widetilde{x}\left(t\right),\widetilde{x}\left(t-\tau_{j}\right)\right), \ t>0, \\ \widetilde{x}\left(t\right)=\widetilde{\phi}\left(t\right) \text{ for } t<0 \text{ and } \lim_{t\to 0^{-}}\widetilde{\phi}\left(t\right)=0, \\ I^{1-\alpha}\widetilde{x}\left(t\right)|_{t=0}=0. \end{cases}$$

Definition 3.1 The solution of nonlinear fractional differential equation (3.1) is stable if for any $\epsilon>0$, there exists $\delta>0$ such that for any two solutions $x\left(t\right)$ and $\widetilde{x}\left(t\right)$ of nonlinear fractional differential equations (3.1)-(3.3) and $\widetilde{P}$ respectively, one has $\left\|\phi\left(t\right)-\widetilde{\phi}\left(t\right)\right\|\leq\delta$, then $\left\|x\left(t\right)-\widetilde{x}\left(t\right)\right\|<\epsilon$ for all $t\geq 0$.

Theorem 3.2 *The solution of nonlinear fractional differential equations (3.1)-(3.3) is uniformly stable.*

Proof. Let $x\left(t\right)$ and $\widetilde{x}\left(t\right)$ be the solutions of nonlinear fractional differential equations (3.1)-(3.3) and $\widetilde{P}$ respectively, then for $t>0$, from (3.4), we have

$$\begin{aligned} \left|x\left(t\right)-\widetilde{x}\left(t\right)\right| &= \left|\sum_{j=1}^{n}I^{\alpha}a_{j}\left(t\right)f(t,x\left(t\right),x(t-\tau_{j}))-\sum_{j=1}^{n}I^{\alpha}a_{j}\left(t\right)f(t,\widetilde{x}\left(t\right),\widetilde{x}(t-\tau_{j}))\right| \\ &\leq \sum_{j=1}^{n}\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|a_{j}(s)f(s,x(s),x(s-\tau_{j}))-a_{j}(s)f(s,\widetilde{x}(s),\widetilde{x}(s-\tau_{j}))\right|ds \\ &\leq \sum_{j=1}^{n}a_{j}k\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x(t)-\widetilde{x}(t)\right|ds \\ &+\sum_{j=1}^{n}a_{j}k_{j}\int_{0}^{\tau_{j}}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|\phi(s-\tau_{j})-\widetilde{\phi}(s-\tau_{j})\right|ds \\ &+\sum_{j=1}^{n}a_{j}k_{j}\int_{\tau_{j}}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x(s-\tau_{j})-\widetilde{x}(s-\tau_{j})\right|ds. \end{aligned}$$

and

$$
\begin{aligned}
& e^{-Nt}\left|x(t)-\widetilde{x}(t)\right| \\
& \leq \sum_{j=1}^{n} a_j k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns}\left|x(s)-\widetilde{x}(s)\right| ds \\
& +\sum_{j=1}^{n} a_j k_j \int_0^{\tau_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau_j)} e^{-N(s-\tau_j)}\left|\phi(s-\tau_j)-\widetilde{\phi}(s-\tau_j)\right| ds \\
& +\sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau_j)} e^{-N(s-\tau_j)}\left|x(s-\tau_j)-\widetilde{x}(s-\tau_j)\right| ds \\
& \leq \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k\left\|x(t)-\widetilde{x}(t)\right\| \int_0^{Nt} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)} du \\
& +\sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|\phi(t)-\widetilde{\phi}(t)\right|\right\} \int_{-\tau_j}^{0} \frac{\left(t-\theta-\tau_j\right)^{\alpha-1} e^{-N(t-\theta)}}{\Gamma(\alpha)} d\theta \\
& +\sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|x(t)-\widetilde{x}(t)\right|\right\} \int_{0}^{t-\tau_j} \frac{\left(t-\theta-\tau_j\right)^{\alpha-1} e^{-N(t-\theta)}}{\Gamma(\alpha)} d\theta \\
& \leq \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k\left\|x(t)-\widetilde{x}(t)\right\| \\
& +\sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|\phi(t)-\widetilde{\phi}(t)\right|\right\} \frac{e^{-N\tau_j}}{N^\alpha} \int_{N(t-\tau_j)}^{Nt} \frac{u^{\alpha-1}e^{-Nu}}{\Gamma(\alpha)} du \\
& +\sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|x(t)-\widetilde{x}(t)\right|\right\} \frac{e^{-N\tau_j}}{N^\alpha} \int_{0}^{N(t-\tau_j)} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)} du \\
& \leq \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k\left\|x(t)-\widetilde{x}(t)\right\| + \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k_j e^{-N\tau_j} \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|x(t)-\widetilde{x}(t)\right|\right\} \\
& +\frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k_j e^{-N\tau_j} \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|\phi(t)-\widetilde{\phi}(t)\right|\right\} \\
& \leq \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j\left(k+k_j e^{-N\tau_j}\right)\left\|x(t)-\widetilde{x}(t)\right\| + \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}\left\|\phi(t)-\widetilde{\phi}(t)\right\|.
\end{aligned}
$$

Then

$$
\left[1-\frac{1}{N^\alpha}\sum_{j=1}^{n} a_j\left(k+k_j e^{-N\tau_j}\right)\right]\left\|x(t)-\widetilde{x}(t)\right\| \leq \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}\left\|\phi(t)-\widetilde{\phi}(t)\right\|
$$

and

$$
\left\|x(t)-\widetilde{x}(t)\right\| \leq \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}\left[1-\frac{1}{N^\alpha}\sum_{j=1}^{n} a_j\left(k+k_j e^{-N\tau_j}\right)\right]^{-1}\left\|\phi(t)-\widetilde{\phi}(t)\right\|,
$$

therefore, for $\epsilon > 0$, we can find $\delta = \left(\frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha}\right)^{-1} \left[1 - \frac{1}{N^\alpha}\sum_{j=1}^{n} a_j \left(k + k_j e^{-N\tau_j}\right)\right] \epsilon$ such that $\left\|\phi(t) - \widetilde{\phi}(t)\right\| < \delta$. Then $\|x(t) - \widetilde{x}(t)\| \leq \epsilon$, which proves that the solution $x(t)$ is uniformly stable. ■

Chapter 4

Stability in delay nonlinear fractional differential equations

In this chapter, we give sufficient conditions to guarantee the asymptotic stability of the zero solution to a kind of delay nonlinear fractional differential equations of order α $(1 < \alpha < 2)$. By using the Krasnoselskii's fixed point theorem in a weighted Banach space, we establish new results on the asymptotic stability of the zero solution provided that $g(t, 0) = f(t, 0, 0) = 0$, which include and improve some related results in the literature (see [9]).

4.1 Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1, 2, 12, 17, 24, 25, 34, 35, 40, 41, 42, 46, 48], [59] and the references therein.

Recently, Agarwal, Zhou and He [2] discussed the existence of solutions for the neutral fractional differential equation with bounded delay

$$\begin{cases} {}^{C}D^{\alpha}\left(x(t) - g(t, x_t)\right) = f\left(t, x_t\right), \ t \geqslant t_0, \\ x_{t_0} = \phi, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $0 < \alpha < 1$. By employing the Krasnoselskii's fixed point theorem, the authors obtained existence results.

The delay fractional differential equation

$$\begin{cases} \dfrac{d^\alpha}{dt^\alpha}x(t) = f\left(t, x(t), x(t-\tau)\right), \ t \in [0, T], \\ x\left(t\right) = \phi\left(t\right), \ t \in [-\tau, 0], \ 0 < \alpha < 1, \end{cases}$$

has been investigated in [1], where $\dfrac{d^\alpha}{dt^\alpha}$ denotes Riemann-Liouville fractional derivative of order $0 < \alpha < 1$. By using the Krasnoselskii's fixed point theorem, the existence of solutions has been established.

In [24], Ge and Kou investigated the asymptotic stability of the zero solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = f\left(t, x(t)\right), \ t \geqslant 0, \\ x(0) = x_0, \ x'(0) = x_1, \end{cases}$$

where ${}^{C}D_{0+}^{\alpha}$ is the standard Caputo's fractional derivative of order $1 < \alpha < 2$. By employing the Krasnoselskii's fixed point theorem in a weighted Banach space, the authors obtained stability results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the asymptotic stability of the zero solution to delay fractional differential equations. Inspired and motivated by the works mentioned above and the papers [1, 2, 12, 17, 24, 25, 34, 35, 40, 41, 42, 46, 48], [59] and the references therein, we concentrate on the asymptotic stability of the zero solution for the nonlinear fractional differential equation with variable delay

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = f\left(t, x(t), x(t-\tau(t))\right) + \ {}^{C}D_{0+}^{\alpha-1}g(t, x(t-\tau(t))), \ t \geqslant 0, \\ x(t) = \phi(t), \ t \in [m_0, 0], \ x'(0) = x_1, \end{cases} \tag{4.1}$$

where $1 < \alpha < 2$, $\mathbb{R}^+ = [0, +\infty)$, $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $t - \tau(t) \to \infty$ as $t \to \infty$, $m_0 = \inf_{t \geqslant 0}\{t - \tau(t)\}$, $x_1 \in \mathbb{R}$, $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $g(t, 0) = f(t, 0, 0) = 0$, ${}^{C}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative and we denote the solution of (4.1) by $x\left(t, \phi, x_1\right)$. To show the asymptotic stability of the zero solution, we transform (4.1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation is the sum of two mappings, one is a contraction and the other is compact.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (4.1) and the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [55]. In Section 3, we give and prove our main results on stability.

4.2 Preliminaries and inversion of the equation

The following Banach space plays a fundamental role in our discussion. Let $h : [m_0, +\infty) \to [1, +\infty)$ be a strictly increasing continuous function with $h(m_0) = 1$, $h(t) \to \infty$ as $t \to \infty$, $h(s)h(t-s) \leqslant h(t)$ for all $m_0 \leqslant s \leqslant t \leqslant \infty$. Let

$$E = \left\{ x \in C\left([m_0, +\infty)\right) : \sup_{t \geqslant m_0} |x(t)| / h(t) < \infty \right\}.$$

Then E is a Banach space equipped with the norm $\|x\| = \sup_{t \geqslant m_0} \frac{|x(t)|}{h(t)}$. For more properties of this Banach space, see [35]. Moreover, let

$$\|\varphi\|_t = \max\left\{|\varphi(s)| : m_0 \leqslant s \leqslant t\right\},$$

for any $t \geqslant m_0$, any given $\varphi \in C\left([m_0, +\infty)\right)$ and let $\Im(\varepsilon) = \{x \in E : \|x(t)\| \leqslant \varepsilon$ for $t \in [m_0, +\infty)$ and $x(t) = \phi(t)$ if $t \in [m_0, 0]\}$ for any $\varepsilon > 0$.

Lemma 4.1 ([24]) *Let $r \in C\left([m_0, +\infty)\right)$. Then $x \in C\left([m_0, +\infty)\right)$ is a solution of the Cauchy type problem*

$$\begin{cases} {}^{C}D_{0+}^{\alpha} x(t) = r(t),\ t \in \mathbb{R}^+,\ 1 < \alpha < 2, \\ x(t) = \phi(t),\ t \in [m_0, 0],\ x'(0) = x_1, \end{cases} \tag{4.2}$$

if and only if x is a solution of the Cauchy type problem

$$\begin{cases} x'(t) = I_{0+}^{\alpha-1} r(t) + x_1,\ t \in \mathbb{R}^+, \\ x(t) = \phi(t),\ t \in [m_0, 0]. \end{cases} \tag{4.3}$$

Lemma 4.2 *Let $k \in \mathbb{R}$. Then $x \in C\left([m_0, +\infty)\right)$ is a solution of 4.1 if and only if*

$$\begin{aligned} x(t) &= \phi(0)e^{-kt} + \frac{1 - e^{-kt}}{k}\left(x_1 - g(0, \phi(-\tau(0)))\right) \\ &+ \int_0^t e^{-k(t-s)}\left(kx(s) + g(s, x(s - \tau(s)))\right) ds \\ &+ \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds f(u, x(u), x(u - \tau(u))) du. \end{aligned} \tag{4.4}$$

Proof. Let $x \in C\left([m_0, +\infty)\right)$ be a solution of (4.1). From Lemma 4.1, we have

$$\begin{cases} x'(t) = I_{0+}^{\alpha-1}\left(f\left(t, x(t), x(t - \tau(t))\right) + {}^{C}D_{0+}^{\alpha-1} g(t, x(t - \tau(t)))\right) + x_1,\ t \in \mathbb{R}^+, \\ x(t) = \phi(t),\ t \in [m_0, 0]. \end{cases}$$

Then

$$\begin{cases} x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x(s - \tau(s))) ds \\ \qquad\qquad + g(t, x(t - \tau(t))) - g(0, \phi(-\tau(0))) + x_1,\ t \in \mathbb{R}^+, \\ x(t) = \phi(t),\ t \in [m_0, 0]. \end{cases} \tag{4.5}$$

Rewrite (4.5) as

$$\begin{cases} x'(t) + kx\,(t) = kx\,(t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x(s-\tau(s)))ds \\ \qquad +g(t, x(t-\tau(t))) - g(0, \phi(-\tau(0))) + x_1,\ t \in \mathbb{R}^+, \\ x(t) = \phi(t),\ t \in [m_0, 0]\,. \end{cases}$$

By the variation of constants formula, we obtain (4.4). Since each step is reversible, the converse follows easily. This completes the proof. ■

Definition 4.1 The trivial solution $x = 0$ of (4.1) is said to be

(i) stable in Banach space E, if for every $\varepsilon > 0$, there exists a $\delta = \delta\,(\varepsilon) > 0$ such that $|\phi(t)| + |x_1| \leqslant \delta$ implies that the solution $x(t) = x(t, \phi, x_1)$ exists for all $t \geqslant m_0$ and satisfies $\|x\| \leqslant \varepsilon$.

(ii) asymptotically stable,if it is stable in Banach space E and there exists a number $\sigma > 0$ such that $|\phi(t)| + |x_1| \leqslant \sigma$ implies $\lim_{t\to\infty} \|x(t)\| = 0$.

4.3 Stability and asymptotic stability

Before stating and proving the main results, we introduce the following hypotheses.

$(h1)$ g and f are continuous functions and $g(t, 0) = f(t, 0, 0) = 0$. g is also supposed to be locally Lipschitz continuous in x. That is, there is a $L_g > 0$ so that if $|x|\,, |y| \leq l$ then

$$|g\,(t, x) - g\,(t, y)| \leq L_g\,\|x - y\|\,. \tag{4.6}$$

$(h2)$ There exists a constant $\beta_1 \in (0, 1)$ such that

$$\beta_1 \left(1 + \frac{L_g}{|k|}\right) < 1, \tag{4.7}$$

and

$$e^{-kt}/h(t) \in BC\,([m_0, +\infty)) \cap L^1\,([m_0, +\infty))\,,\ |k| \int_0^t e^{-ku}/h(u)du \leqslant \beta_1 < 1. \tag{4.8}$$

$(h3)$ There exists constants $\eta > 0$, $\beta_2 \in (0, 1-\beta_1)$ and a continuous function $\tilde{f}$: $[0,\infty) \times (0, \eta] \times (0, \eta] \to \mathbb{R}^+$ such that

$$\frac{|f(t, \upsilon_1 h(t), \upsilon_2 h(t-\tau(t)))|}{h(t)} \leqslant \tilde{f}(t, |\upsilon_1|\,, |\upsilon_2|), \tag{4.9}$$

holds for all $t \geqslant 0$, $0 < |\upsilon_1|\,, |\upsilon_2| \leqslant \eta$ and

$$\sup_{t\geqslant 0} \int_0^t \frac{K(t-u)}{h(t-u)} \tilde{f}(u, r_1, r_2)du \leqslant \beta_2 < 1 - \beta_1, \tag{4.10}$$

holds for every $0 < r_1, r_2 \leqslant \eta$, where $\tilde{f}(t, r_1, r_2)$ is nondecreasing in r_1 and r_2 for fixed t, $\tilde{f}(t, r_1, r_2) \in L^1([0, +\infty))$ in t for fixed r_1 and r_2, and

$$K(t-u) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_u^t e^{-k(t-s)} (s-u)^{\alpha-2} ds, & t-u \geqslant 0, \\ 0, & t-u < 0. \end{cases} \tag{4.11}$$

Theorem 4.1 *Suppose that $(h1)-(h3)$ hold. Then the trivial solution $x = 0$ of (4.1) is stable in Banach space E.*

Proof. For any given $\varepsilon > 0$, we first prove the existence of $\delta > 0$ such that

$$|\phi(t)| + |x_1| < \delta \text{ implies } \|x\| \leqslant \varepsilon.$$

In fact, according to (4.8), there exists a constant $M_1 > 0$ such that

$$\frac{e^{-kt}}{h(t)} \leqslant M_1. \tag{4.12}$$

Let $0 < \delta \leqslant \frac{\left(1-\beta_1\left(1+\frac{L_g}{|k|}\right)-\beta_2\right)|k|}{M_1|k|+(1+M_1)(1+L_g)}\varepsilon$. Consider the non-empty closed convex subset $\Im(\varepsilon) \subseteq E$, for $t \geqslant 0$, we denote two mapping A and B on $\Im(\varepsilon)$ as follows:

$$\begin{aligned} Ax(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_u^t e^{-k(t-s)} (s-u)^{\alpha-2} ds f(u, x(u), x(u-\tau(u))) du \\ &= \int_0^t K(t-u) f(u, x(u), x(u-\tau(u))) du, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} Bx(t) &= \phi(0)e^{-kt} + \frac{1-e^{-kt}}{k} \left(x_1 - g(0, \phi(-\tau(0)))\right) \\ &+ \int_0^t e^{-k(t-s)} \left(kx(s) + g(s, x(s-\tau(s)))\right) ds. \end{aligned} \tag{4.14}$$

Obviously, for $x \in \Im(\varepsilon)$, both Ax and Bx are continuous functions on $[m_0, +\infty)$. Furthermore, for $x \in \Im(\varepsilon)$, by (4.8)-(4.10) for any $t \geqslant 0$, we have

$$\begin{aligned} \frac{|Ax(t)|}{h(t)} &\leqslant \int_0^t \frac{K(t-u)}{h(t-u)} \frac{|f(u, x(u), x(u-\tau(u)))|}{h(u)} du \\ &\leqslant \int_0^t \frac{K(t-u)}{h(t-u)} \tilde{f}\left(u, \frac{|x(u)|}{h(u)}, \frac{|x(u-\tau(u))|}{h(u-\tau(u))}\right) du \\ &\leqslant \beta_2 \|x\| \leqslant \beta_2 \varepsilon < \infty, \end{aligned} \tag{4.15}$$

and

$$
\begin{aligned}
\frac{|Bx(t)|}{h(t)} &= \left|\phi(0)\frac{e^{-kt}}{h(t)} + \frac{1-e^{-kt}}{kh(t)}\left(x_1 - g(0,\phi(-\tau(0)))\right)\right. \\
&\quad \left. + \int_0^t \frac{e^{-k(t-s)}}{h(t)}\left(kx(s) + g(s, x(s-\tau(s)))\right) ds\right| \\
&\leqslant M_1 |\phi(0)| + \frac{1+M_1}{|k|}\left(|x_1| + |g(0,\phi(-\tau(0)))|\right) + |k|\int_0^\infty \frac{e^{-ku}}{h(u)}du(1+\frac{L_g}{|k|})\,\|x\| \\
&\leqslant M_1 |\phi(0)| + \frac{1+M_1}{|k|}\left(|x_1| + L_g\,|\phi(-\tau(0))|\right) + \beta_1(1+\frac{L_g}{|k|})\varepsilon < \infty. \qquad (4.16)
\end{aligned}
$$

Then $A\Im(\varepsilon) \subseteq E$ and $B\Im(\varepsilon) \subseteq E$. Next, we shall use Theorem 4.1 to prove there exists at least one fixed point of the operator $A+B$ in $\Im(\varepsilon)$. Here, we divide the proof into three steps.

Step1. We prove that $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.

For any $x, y \in \Im(\varepsilon)$, from (4.15) and (4.16), we obtain that

$$
\begin{aligned}
\sup_{t\geqslant 0}\frac{|Ax(t) + By(t)|}{h(t)} &= \sup_{t\geqslant 0}\left\{\left|\phi(0)\frac{e^{-kt}}{h(t)} + \frac{1-e^{-kt}}{kh(t)}\left(x_1 - g(0,\phi(-\tau(0)))\right)\right.\right. \\
&\quad + \int_0^t \frac{e^{-k(t-s)}}{h(t)}\left(ky(s) + g(s, y(s-\tau(s)))\right) ds \\
&\quad \left.\left. + \int_0^t \frac{K(t-u)}{h(t)} f(u, x(u), x(u-\tau(u)))du\right|\right\} \\
&\leqslant M_1 |\phi(0)| + \frac{1+M_1}{|k|}\left(|x_1| + L_g\delta\right) \\
&\quad + |k|\int_0^\infty \frac{e^{-ku}}{h(u)}du\left(1+\frac{L_g}{|k|}\right)\|y\| + \beta_2\|x\| \\
&\leqslant \frac{M_1|k| + (1+M_1)(1+L_g)}{|k|}\delta + \beta_1\left(1+\frac{L_g}{|k|}\right)\varepsilon + \beta_2\varepsilon \leqslant \varepsilon,
\end{aligned}
$$

which implies $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.

Step 2. It is easy to see that A is continuous. Now we only prove that $A\Im(\varepsilon)$ is a relatively compact in E. In fact, from (4.15), we get that $\{x(t)/h(t) : x \in \Im(\varepsilon)\}$ is uniformly bounded in E. Moreover, a classical theorem states the fact that the convolution of an L^1-function with a function tending to zero, does also tend to zero. Then we conclude that for $t-u \geqslant 0$, we have

$$
\begin{aligned}
0 \leqslant \lim_{t\to\infty}\frac{K(t-u)}{h(t-u)} &\leqslant \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_u^t \frac{e^{-k(t-s)}}{h(t-u)}\frac{(s-u)^{\alpha-2}}{h(s-u)}ds \\
&= \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_0^t \frac{e^{-k(t-u-s)}}{h(t-u-s)}\frac{s^{\alpha-2}}{h(s)}ds = 0, \qquad (4.17)
\end{aligned}
$$

due to the fact $\lim_{t\to\infty}\frac{t^{\alpha-2}}{h(t)}=0$. Together with the continuity of K and h, we get that there exists a constant $M_2>0$ such that

$$\left|\frac{K(t-u)}{h(t-u)}\right|\leqslant M_2, \tag{4.18}$$

and for any $T_0\in\mathbb{R}^+$, the function $K(t-u)h(u)/h(t)$ is uniformly continuous on $\{(t,u): 0\leqslant u\leqslant t\leqslant T_0\}$. For any $t_1,t_2\in[0,T_0]$, $t_1<t_2$, we have

$$\begin{aligned}\left|\frac{Ax(t_2)}{h(t_2)}-\frac{Ax(t_1)}{h(t_1)}\right| &= \left|\int_0^{t_2}\frac{K(t_2-u)}{h(t_2)}f(u,x(u),x(u-\tau(u)))du\right.\\ &\quad\left.-\int_0^{t1}\frac{K(t_1-u)}{h(t_1)}f(u,x(u),x(u-\tau(u)))du\right|\\ &\leqslant \int_0^{t_1}\left|\frac{K(t_2-u)}{h(t_2)}-\frac{K(t_1-u)}{h(t_1)}\right|\left|f(u,x(u),x(u-\tau(u)))\right|du\\ &\quad+\int_{t_1}^{t_2}\frac{K(t_2-u)}{h(t_2-u)}\tilde f(u,\varepsilon,\varepsilon)\,du\\ &\leqslant \int_0^{t_1}\left|\frac{K(t_2-u)h(u)}{h(t_2)}-\frac{K(t_1-u)h(u)}{h(t_1)}\right|\tilde f(u,\varepsilon,\varepsilon)\,du\\ &\quad+M_2\int_{t_1}^{t_2}\tilde f(u,\varepsilon,\varepsilon)\,du\ \to\ 0,\end{aligned}$$

as $t_2\to t_1$, which means that $\{x(t)/h(t): x\in\Im(\varepsilon)\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$. By Theorem 2.2, in order to show that $A\Im(\varepsilon)$ is a relatively compact set of E, we only need to prove that $\{x(t)/h(t): x\in\Im(\varepsilon)\}$ is equiconvergent at infinity. In fact, for any $\varepsilon_1>0$, there exists a $L>0$ such that

$$M_2\int_L^\infty\tilde f(u,\varepsilon,\varepsilon)\,du\leqslant\frac{\varepsilon_1}{3}.$$

According to (4.17), we get that

$$\lim_{t\to\infty}\sup_{u\in[0,L]}\frac{K(t-u)}{h(t-u)}\leqslant\max\left\{\lim_{t\to\infty}\frac{K(t-L)}{h(t-L)},\lim_{t\to\infty}\frac{K(t)}{h(t)}\right\}=0.$$

Thus, there exists $T>L$ such that $t_1,t_2\geqslant T$, we have

$$\begin{aligned}\sup_{u\in[0,L]}\left|\frac{K(t_2-u)h(u)}{h(t_2)}-\frac{K(t_1-u)h(u)}{h(t_1)}\right| &\leqslant \sup_{u\in[0,L]}\left|\frac{K(t_2-u)}{h(t_2-u)}\right|+\sup_{u\in[0,L]}\left|\frac{K(t_1-u)}{h(t_1-u)}\right|\\ &\leqslant\frac{\varepsilon_1}{3}\left(\int_0^\infty\tilde f(u,\varepsilon,\varepsilon)\,du\right)^{-1}\end{aligned}$$

Therefore, for $t_1, t_2 \geqslant T$,

$$\begin{aligned}
\left|\frac{Ax(t_2)}{h(t_2)} - \frac{Ax(t_1)}{h(t_1)}\right| &= \left|\int_0^{t_2} \frac{K(t_2-u)}{h(t_2)} f(u, x(u), x(u-\tau(u)))du \right. \\
&\quad \left. - \int_0^{t1} \frac{K(t_1-u)}{h(t_1)} f(u, x(u), x(u-\tau(u)))du \right| \\
&\leqslant \int_0^L \left|\frac{K(t_2-u)h(u)}{h(t_2)} - \frac{K(t_1-u)h(u)}{h(t_1)}\right| \tilde{f}(u, \varepsilon, \varepsilon)\, du \\
&\quad + \int_L^{t_2} \frac{K(t_2-u)}{h(t_2-u)} \tilde{f}(u, \varepsilon, \varepsilon)\, du + \int_L^{t_1} \frac{K(t_1-u)}{h(t_1-u)} \tilde{f}(u, \varepsilon, \varepsilon)\, du \\
&\leqslant \frac{\varepsilon_1}{3} + 2M_2 \int_L^\infty \tilde{f}(u, \varepsilon, \varepsilon)\, du \leqslant \varepsilon_1.
\end{aligned}$$

Hence the required conclusion is true.

Step 3. we claim that $B : \Im(\varepsilon) \to E$ is a contraction mapping.

In fact, for any $x, y \in \Im(\varepsilon)$, from (4.6)-(4.8), we obtain that

$$\begin{aligned}
\sup_{t\geqslant 0}\left|\frac{Bx(t)}{h(t)} - \frac{By(t)}{h(t)}\right| &= \sup_{t\geqslant 0}\left\{\left|\int_0^t \frac{e^{-k(t-u)}}{h(t)} (kx(u) + g(u, x(u-\tau(u))))\, du \right.\right. \\
&\quad \left.\left. - \int_0^t \frac{e^{-k(t-u)}}{h(t)} (ky(u) + g(u, y(u-\tau(u))))\, du\right|\right\} \\
&\leqslant \sup_{t\geqslant 0} |k| \int_0^t \frac{e^{-k(t-u)}}{h(t-u)} \frac{|x(u)-y(u)|}{h(u)} du \\
&\quad + \sup_{t\geqslant 0} \int_0^t \frac{e^{-k(t-u)}}{h(t-u)} \frac{|g(u, x(u-\tau(u))) - g(u, y(u-\tau(u)))|}{h(u)} du \\
&\leqslant |k| \int_0^t \frac{e^{-k(t-u)}}{h(t-u)} du \left(1 + \frac{L_g}{|k|}\right) \|x-y\| \\
&\leqslant \beta_1 \left(1 + \frac{L_g}{|k|}\right) \|x-y\| < \|x-y\|.
\end{aligned}$$

By Theorem 2.8, we know that there exists at least one fixed point of the operator $A+B$ in $\Im(\varepsilon)$. Finally, for any $\varepsilon_2 > 0$, if $0 < \delta_1 \leqslant \frac{\left(1-\beta_1\left(1+\frac{L_g}{|k|}\right)-\beta_2\right)|k|}{|k|M_1+(1+M_1)(1+L_g)}\varepsilon_2$, then $|\phi(t)| + |x_1| \leqslant \delta_1$

implies that

$$\begin{aligned}
\|x\| &= \sup_{t\geqslant 0}\left\{\left|\phi(0)\frac{e^{-kt}}{h(t)} + \frac{1-e^{-kt}}{kh(t)}\left(x_1 - g(0,\phi(-\tau(0)))\right)\right.\right. \\
&+ \int_0^t \frac{e^{-k(t-s)}}{h(t)}\left(kx(u) + g(u, x(u-\tau(u)))\right) du \\
&+ \left.\left.\int_0^t \frac{K(t-u)}{h(t)} f(u, x(u), x(u-\tau(u)))du\right|\right\} \\
&\leqslant \sup_{t\geqslant 0}\left\{\frac{e^{-kt}}{h(t)}\phi(0) + \frac{\left|1-e^{-kt}\right|}{|k|\,h(t)}\left(|x_1| + L_g\left|\phi(-\tau(0))\right|\right)\right. \\
&+ |k|\int_0^t \frac{e^{-k(t-u)}}{h(t-u)h(u)}\left(|x(u)| + \frac{L_g}{|k|}|x(u)|\right) du \\
&+ \left.\int_0^t \frac{K(t-u)}{h(t-u)}\frac{|f(u,x(u),x(u-\tau(u)))|}{h(u)} du\right\} \\
&\leqslant M_1\delta_1 + \frac{1+M_1}{|k|}\left(\delta_1 + L_g\delta_1\right) + \beta_1\left(1+\frac{L_g}{|k|}\right)\|x\| + \beta_2\|x\| \\
&\leqslant \frac{|k|\,M_1 + (1+M_1)(1+L_g)}{\left(1-\beta_1\left(1+\frac{L_g}{|k|}\right)-\beta_2\right)|k|}\delta_1 \leqslant \varepsilon_2.
\end{aligned}$$

Thus, we know that trivial solution of (4.1) is stable in Banach space E. ■

Theorem 4.2 *Suppose that all conditions of Theorem 4.1 are satisfied,*

$$\lim_{t\to\infty} e^{-kt}/h(t) = 0, \tag{4.19}$$

and for any $r > 0$, there exists a function $\varphi_r(t) \in L^1([0,+\infty))$, $\varphi_r(t) > 0$ such that $|u|, |v| \leqslant r$ implies

$$|f(t,u,v)|/h(t) \leqslant \varphi_r(t), \ \ a.e.\ t \in [0,+\infty). \tag{4.20}$$

Then the trivial solution of (4.1) is asymptotically stable.

Proof. First, it follows from Theorem 4.1 that the trivial solution of (4.1) is stable in the Banach space E. Next, we shall show that the trivial solution $x = 0$ of (4.1) is attractive. For any $r > 0$, defining

$$\Im_*(r) = \left\{x \in \Im(r),\ \lim_{t\to\infty} x(t)/h(t) = 0\right\}.$$

We only need to prove that $Ax + By \in \Im_*(r)$ for any $x, y \in \Im_*(r)$, i.e.

$$\frac{Ax(t)+By(t)}{h(t)} \to 0 \text{ as } t \to \infty,$$

where

$$
\begin{aligned}
Ax(t)+By(t) &= \phi(0)e^{-kt}+\frac{1-e^{-kt}}{k}\left(x_1-g(0,\phi(-\tau(0)))\right)\\
&+\int_0^t e^{-k(t-s)}\left(ky(u)+g(u,y(u-\tau(u)))\right)du\\
&+\int_0^t K(t-u)f(u,x(u),x(u-\tau(u)))du.
\end{aligned}
$$

In fact, for $x,y\in\Im_*(r)$, based on the fact that used in the proof of Theorem 4.1 (Step2), it follows from (4.6)-(4.8) and (4.19) that

$$
\int_0^t \frac{e^{-k(t-u)}}{h(t-u)}\frac{(ky(u)+g(u,y(u-\tau(u))))}{h(u)}du\to 0,
$$

and

$$
\frac{K(t-u)}{h(t-u)}=\frac{\int_u^t \frac{e^{-k(t-s)}}{h(t-u)}(s-u)^{\alpha-2}ds}{\Gamma(\alpha-1)}\to 0,
$$

as $t\to\infty$. Together with the hypothesis $\varphi_r(t)\in L^1([0,+\infty))$, we obtain that

$$
\int_0^t \frac{K(t-u)}{h(t-u)}\frac{|f(u,x(u),x(u-\tau(u)))|}{h(u)}du\leqslant\int_0^t\frac{K(t-u)}{h(t-u)}\varphi_r(u)du\to 0,
$$

as $t\to\infty$. Thus we get the conclusion. ■

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