وزارة التعليم العالي والبحث العلمي

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THESE

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Option : Probabilités et Statistique

Ordre stochastique des produits scalaires avec application

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Dédicace

Je dédié ce travail de longues années d'étude à

La lumière de ma vie, au cœur le plus tendre et le plus doux, à celle qui s'est tellement sacrifiée pour me voir toujours meilleure : ma très chère mère

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ملخص

في هذه الأطروحة نقترح لمحة عامة عن الترتيبات العشوائية للأخطار وتطبيقاتها. بمعنى أدق سنتطرق إلى مشكلة تعظيم قيمة الفوائد لمحفظة التأمين، إضافة إلى تطبيقات حول التوزيع الأمثل لحدود واقتطاعات بوليصة التأمين وعلاقتها ببعض المواضيع الإكتوارية الأخرى منها: مقارنة copulas، نماذج عن المخاطر الفردية والجماعية وعقود إعادة التأمين، إلخ.

الكلمات الرئيسية: Comonotonicité، الترتيب المحدب، حدود واقتطاعات بوليصة التأمين.

Résumé

Nous nous proposons, dans cette thèse, un aperçu sur les ordres stochastiques et ses applications. Plus précisément, nous étudions le problème de maximisation pour certaines fonctionnelles sur les portefeuilles d'assurance. Des applications sur l'allocation optimale des limites de police et des déductibles sont obtenus, et quelques relations avec d'autres sujets actuariels principaux (comparaison des copules, les modèles de risque individuels et collectifs, des contrats de réassurance, etc.) sont également étudiés.

Mots clés: Comonotonicité, Ordre Convexe, Les limites de police et déductibles.

Abstract

We propose in this thesis an overview on several types of stochastic orders affecting random variables and linear combinations of random variables. We study the problem of finding maximal expected utility for some functionals on insurance portfolios involving some additional (independent) randomization. Applications in policy limits and deductible are obtained, and some relationships with other actuarial main topics (comparison of copulas, individual and collective risk models, reinsurance contracts, etc.) are studied too.

Key Words: Comonotonicity, Convex Order, Policy Limits and Deductibles.

Contents

List of Figures

BIBLIOGRAPHIE PERSONNELLE

Articles publiés

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2. Bouhadjar, M. Zeghdoudi, H. Remita, M.R, Stochastic Order Relationship and Its Applications in Actuarial Science. *Global Journal of Pure and Applied Mathematics, ISSN **0973-1768** Volume **11**, Number **6**, pp. 4395-4403 (2015).*

3. Bouhadjar, M. Zeghdoudi, H. Remita, M.R, On Stochastic Orders and Applications : Policy Limits and Deductibles. *Applied Mathematics & Information Sciences, Volume **10**, Number **4**, pp. 1-8 Published online: (1 Jul 2016).*

Introduction générale en français

La science actuarielle moderne et la théorie de risque jouent un rôle important dans l'économie et la finance. Un des principaux objectifs de la profession actuarielle est la comparaison de variables aléatoires (risques). Habituellement, le critère probabiliste «moyenne - variance» ne suffit pas toujours à comparer des variables aléatoires. Cependant, il arrive souvent qu'on possède des informations plus détaillées en utilisant les fonctions de répartition des variables aléatoires pour les comparés. Pour cela, il est préférable de faire une comparaison basée sur les distributions que celle basée uniquement sur deux statistiques. La méthode utilisée pour comparer deux distributions est nommée «ordre stochastique». Tout d'abord, nous donnons un aperçu historique de ce terme. Depuis les années 70, le concept de dominance stochastique, introduit par Rothschild-Stieglitz, permet de comparer des distributions de probabilité. Plus récemment, les ordres stochastiques qui généralisent la dominance stochastique sont utilisés de façon accélérée dans plusieurs domaines, notamment la finance, science actuarielle et l'économie. En faite, des ordres stochastiques particuliers aient déjà été étudiés par Karamata en 1932, par Lehmann en 1955, et par Littlewood et Polya en 1967. Enfin, les premières études presque complètes des ordres stochastiques ont été données par Stoyan dans les années 1977 et 1983, et par Mosler en 1982. En finance et en économie, une des raisons principales pour comparer des variables aléatoires (risques) en utilisant des ordres stochastiques est le fait que ces derniers utilisent toute l'information sur la fonction de répartition afin d'établir une comparaison adéquate entre deux variables aléatoires.

Notre travail est structuré de la manière suivante :

Le chapitre 2 présente et examine de façon systématique les ordres stochastiques univariés les plus utilisés dans la littérature. Par ailleurs, des définitions, notations et propriétés sont établies. Par exemple, la fonction d'utilité, ordre de majorisation, valeur at risque, et la fonction de distribution inverse.

De ce fait, le chapitre 3 traite la comonotonicté, à savoir les ensembles comonotones, les vecteurs aléatoires comonotones, la somme comonotone des variables aléatoires et les bornes convexes pour la somme des variables aléatoires.

Enfin, le dernier chapitre présente la contribution originale de notre travail dont nous introduisons un nouveau modéle de l'allocation optimale des limites de police et des déductibles. Il s'agit d'une extension et complémemt du résultat de Cheung [6], Hua and Cheung [29] and Zhuang et al.[58]. Des applications sur l'allocation optimale des limites de police et des déductibles sont obtenus, et quelques relations avec d'autres sujets actuariels principaux (comparaison des copules, les modèles de risque individuels et collectifs, des contrats de réassurance, etc.) sont également étudiés.

Cette contribution est couronnée par (03) publications scientifiques dans des revues de renommées internationales à savoir:

- Bouhadjar, M. Zeghdoudi, H. Remita, M.R, Ordering of the Optimal Allocation of Policy Limits in general model. *European Journal of Scientific Research,* *ISSN **3** Volume **134**, 317-324 (2015).*

- Bouhadjar, M. Zeghdoudi, H. Remita, M.R, Stochastic Order Relationship and Its Applications in Actuarial Science. *Global Journal of Pure and Applied Mathematics, ISSN **0973-1768** Volume **11**, Number **6**, pp. 4395-4403 (2015).*

- Bouhadjar, M. Zeghdoudi, H. Remita, M.R, On Stochastic Orders and Applications : Policy Limits and Deductibles. *Applied Mathematics & Information Sciences, Volume **10**, Number **4**, pp. 1-8 Published online: (1 Jul 2016).*

Chapter 1

Introduction

Modern actuarial science and risk theory play a crucial role in the economy and finance. One of the principal objectives of the actuarial profession is the comparison of random variables (risks). Usually, the probabilistic criterion «mean - variance» is not always enough to compare random variables. However, it often happens that we have more detailed information by using the distribution functions of the random variables for compared.

This work is innovative in many respects. It integrates the theory of stochastic orders, one of the methodological cornerstones of risk theory and the theory of stochastic dependence, which has become increasingly important as new types of risks emerge. More precisely, risk measures will be used to generate stochastic orderings, by identifying pairs of risks about which a class of risk measures agree. Stochastic orderings are then used to define positive dependence relationships.

In several works, orderings of optimal allocations of policy limits and deductibles were established by maximizing the expected utility of wealth of the policyholder. In this work, we study the problems of optimal allocation of policy limits and deductibles for general model,by using some characterizations of stochastic ordering relations, we reconsider the new general model and obtain some new results on orderings of optimal allocations of policy limits and deductibles. To this end, we obtain the ordering of the optimal allocation of policy limits, deductibles for this model and we extend the above results in Cheung $(2007, 2008)$.

We consider for the following model :

$$S_N = X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n) \tag{M1}$$

where :$Y_i = \delta_i T_i$, S_N is total discounted loss, X_i are loss due to the i-th risk, T_i are time of occurrence of i-th insured risk and δ_i are discount rate capture the impact of financial environment (X_i, T_i are independent non-negative random variables and δ_i are non-random numbers). Also, we will make the following assumptions

1. $f(Y_i) \geq 0; \forall Y_i$ and $\lim\limits_{Y_i \to \infty} f(Y_i) = 0$.
2. $f(Y_i)$ is decreasing and convex function.
3. $Y_1, Y_2, ..., Y_n$ are mutually independent.
4. A policyholder exposed to risks $X_1, X_2, ..., X_n$ is granted a total of l dollars ($l > 0$) as the policy limit with which (s)he can allocate arbitrarily among the n risks.

Remark 1.1 *A very good property of the model* $(M1)$ *is that* X_i*'s characterize the* *scales of the losses while $f(Y_i)$ characterize the chances of the losses.*

In this situation, if some risk occurs, the insurer will make the payment right after the event of the loss and the insurance coverage for this risk will terminate. However the insurance coverage for the other risks is still in effect. If $(l_1, ..., l_n)$ are the allocated policy we have $\forall i : l_i \geq 0$ and $\sum_{i=1}^{n} l_i = l$. When l is n-tuple admissible and $\mathcal{A}_n(l)$ denote the class of all such n-tuples. If $\mathbf{l} = (l_1, ..., l_n) \in \mathcal{A}_n(l)$ is chosen, then the discounted value of benefits obtained from the insurer would be

$$\sum_{i=1}^{n} (X_i \wedge l_i)\, f(Y_i) \tag{1.1}$$

If we take expected utility of wealth as the criterion for the optimal allocation, then the problem of the optimal allocation of policy limits is

$$\text{Problem } L : \max_{\mathbf{l} \in \mathcal{A}_n(l)} \mathbb{E}\left[u\left(w - \sum_{i=1}^{n} \left[X_i - (X_i \wedge l_i)\right] f(Y_i)\right)\right]. \tag{1.2}$$

where u is the utility function of the policyholder and w is the wealth (after premium). Similarly, instead of policy limits, the policyholder may be granted a total of d dollars ($d > 0$) as the policy deductible with which (s)he can allocate arbitrarily among the n risks. If $\mathbf{d} = (d_1, ..., d_n) \in \mathcal{A}_n(d)$ are the allocated deductibles, then $\forall i : d_i \geq 0, \sum_{i=1}^{n} d_i = d$, and the discounted value of benefits obtained from the insurer would be

$$\sum_{i=1}^{n} (X_i - d_i)_+\, f(Y_i) \tag{1.3}$$

Then the problem of the optimal allocation of policy deductibles is

$$\text{Problem } D: \max_{\mathbf{d}\in\mathcal{A}_n(d)} \mathbb{E}\left[u\left(w-\sum_{i=1}^{n}\left[X_i-(X_i-d_i)_+\right]f(Y_i)\right)\right]. \tag{1.4}$$

Chapter 2

Preliminaries and Notations

In this chapter, we will present some definitions like: Utility function, Majorization Order, Arrangement Increasing...etc.

Throughout this work, we define $\mathcal{I}_n = \{(a_1, ..., a_n) \in \mathbb{R}^n : a_1 \leq ... \leq a_n\}$ and $\mathcal{D}_n = \{(a_1, ..., a_n) \in \mathbb{R}^n : a_1 \geq ... \geq a_n\}$. In addition, we noted that $x_{[i]}$ and $x_{(i)}$ are the i-th largest and the i-th smallest element of $\mathbf{x}$ respectively. The notation $\mathbf{x} \uparrow$ will be used to indicate the increasing rearrangement $(x_{(1)}, ..., x_{(n)}) \in \mathcal{I}_n$ and $\mathbf{x} \downarrow$ will be used to indicate the decreasing rearrangement $(x_{[1]}, ..., x_{[n]}) \in \mathcal{D}_n$ for any vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$. We represent a permutation of the set $\{1, 2, ..., n\}$ by τ, then the permuted vector $(x_{\tau_{(1)}}, ..., x_{\tau_{(n)}})$ will be denoted as $\mathbf{x} \circ \tau$.

2.1 Utility function

In economics, utility is a measure of preferences over some set of goods and services. A utility function, $u(x)$, can be described as a function which measures the value, or utility, that an individual (or institution) attaches to the monetary amount x. Throughout this work we assume that a utility function satisfies the conditions

$$u'(x) > 0 \text{ and } u''(x) < 0. \tag{2.1}$$

Mathematically, the first condition says that u is an increasing function, while the second says that u is a concave function. Simply put, the first states that an individual whose utility function is u prefers amount y to amount z provided that $y > z$, that is the individual prefers more money to less! The second states that as the individual's wealth increases, the individual places less value on a fixed increase in wealth.

An individual whose utility function satisfies the conditions in (2.1) is said to be risk averse, and risk aversion can be quantified through the coefficient of risk aversion defined by

$$r(x) = \frac{-u''(x)}{u'(x)} \tag{2.2}$$

The expected utility criterion

Decision making using a utility function is based on the expected utility criterion. This criterion says that a decision maker should calculate the expected utility of resulting wealth under each course of action, then select the course of action that gives the greatest value for expected utility of resulting wealth. If two courses of action yield the same expected utility of resulting wealth, then the decision maker has no preference between these two courses of action. To illustrate this concept, let us consider an investor with utility function u who is choosing between two investments which will lead to random net gains of X_1 and X_2 respectively. Suppose that the investor has current wealth W, so that the result of investing in Investment i is $W + X_i\ for\ i = 1$ and 2. Then, under the expected utility criterion, the investor would choose Investment 1 over Investment 2 if and only if

$$\mathbb{E}[u(W + X_1)] > \mathbb{E}[u(W + X_2)]$$

Further, the investor would be indifferent between the two investments if

$$\mathbb{E}[u(W + X_1)] = \mathbb{E}[u(W + X_2)]$$

Types of utility function

It is possible to construct a utility function by assigning different values to different levels of wealth. In the following table we consider some mathematical functions which may be regarded as having suitable forms to be utility functions.

Types	*Form*	*Use*
Exponential	$u(x){=}-\exp\{-\beta x\},\ where\ \beta{>}0$	*Decisions do not depend on the individualʼs wealth*
Quadratic	$u(x){=}\beta\log x,\ for\ x{>}0\ and\ \beta{>}0$	*Restricted by the constraint* $x{<}\frac{1}{2\beta}$, *which is required* $u'(x){>}0$
Logarithmic	$u(x){=}\beta\log x,\ for\ x{>}0\ and\ \beta{>}0$	*For positive values of* x
Fractional power	$u(x){=}x^{\beta},\ for\ x{>}0\ and\ 0{<}\beta{<}1$	*For positive values of* x

2.2 Majorization Order

Definition 2.1 ([6]) *Given any two vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

1. $\mathbf{b}$ *is said to be majorized by* $\mathbf{a}$ *(denoted by* $\mathbf{b} \prec \mathbf{a}$*),if*

$$\begin{cases} \sum_{i=1}^{n} b_{[i]} = \sum_{i=1}^{n} a_{[i]} \\ \sum_{i=1}^{m} b_{[i]} \le \sum_{i=1}^{m} a_{[i]} \quad m = 1, ..., n-1. \end{cases} \tag{2.3}$$

2. $\mathbf{b}$ *is said to be weakly majorized by* $\mathbf{a}$ *(denoted by* $\mathbf{b} \prec\prec \mathbf{a}$), *if*

$$\sum_{i=1}^{m} b_{[i]} \le \sum_{i=1}^{m} a_{[i]}, \qquad m = 1, ..., n. \tag{2.4}$$

If $\mathbf{b} \prec \mathbf{a}$, then b is also said to be smaller than a in the majorization order. In literature, there are two different versions of weak majorization. The definition given above is commonly known as weak submajorization. We refer to Marshall and Olkin (1979) and Tong (1980) as standard references for the theory of majorization.

The following is the definition of a T-transform, which is a very useful tool in the study of the majorization order. After the definition, we will collect some useful properties of a T-transform.

Definition 2.2 ([6]) *A T-transform is a kind of linear transformation (from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*) whose transformation matrix has the form*

$$T = \lambda \mathbf{I} + (1-\lambda)\mathbf{Q} \tag{2.5}$$

where $0 \le \lambda \le 1$, $\mathbf{I}$ *is the identity matrix and* $\mathbf{Q}$ *is a permutation matrix that interchanges two coordinates.*

Lemma 2.3 ([6]) *If* $\mathbf{b} \prec \mathbf{a}$, *then* $\mathbf{b}$ *can be derived from* $\mathbf{a}$ *by successive applications of a finite number of T-transforms, and any T-transform can preserve the majorization order. i.e.,* $\mathbf{b} \prec T(\mathbf{a}) \prec \mathbf{a}$, *where* T *is a T-transform.*

For a proof of this result, we refer to Marshall and Olkin (1979) and Hardy et al.(1934, 1952). Since each T-transform will only modify two coordinates in a vector, Lemma (2.3) shows that it is often sufficient to check the case $n = 2$ in proving results concerning majorization order. The next lemma provides a refinement when both a and b are in $\mathcal{I}_n$:

Lemma 2.4 ([6]) *Suppose that* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors in* $\mathcal{I}_n$. *If* $\mathbf{b} \prec \mathbf{a}$, *then* $\mathbf{b}$ *can be derived from* $\mathbf{a}$ *by successive applications of a finite number of T-transforms of the form*

$$T(\mathbf{a}) = (a_1, ..., a_{i-1}, \lambda a_i + (1-\lambda) a_j, a_{i+1}, ..., ..., a_{j-1}, \lambda a_j + (1-\lambda) a_i, a_{j+1}, ..., a_n), \quad (2.6)$$

where $\frac{1}{2} \leq \lambda \leq 1$, *so that* $T(\mathbf{a}) \in \mathcal{I}_n$ *and* $\mathbf{b} \prec T(\mathbf{a}) \prec \mathbf{a}$.

Proof. If we change each $\mathcal{I}_n$ in the above result to $\mathcal{D}_n$ and keep the rest unchanged, then it is a fact given in [40]. Since

$$\begin{aligned} \mathbf{a}, \mathbf{b} &\in \mathcal{D}_n, \mathbf{b} \prec \mathbf{a} \Rightarrow -\mathbf{a}, -\mathbf{b} \in \mathcal{I}_{\mathbf{n}}, \text{ and} \quad (2.7) \\ T(\mathbf{a}) &\in \mathcal{D}_n \Rightarrow T(-\mathbf{a}) = -T(\mathbf{a}) \in \mathcal{I}_{\mathbf{n}}, \end{aligned}$$

the result follows. □

An interesting property of the above T-transform is that it not only preserves the majorization order but also preserves the ordering of the vector's components.

Lemma 2.5 ([6]) *If* $\mathbf{b} \prec \mathbf{a}$ *and* $\mathbf{a} \in \mathcal{I}_n$, *then*

$$\sum_{i=1}^{n} a_i x_{[i]} \leq \sum_{i=1}^{n} b_i x_{[i]} \leq \sum_{i=1}^{n} a_i x_{[n-i+1]} \tag{2.8}$$

Lemma 2.6 ([6]) $\mathbf{b} \prec\prec \mathbf{a}$ *if and only if there exists a* $\mathbf{c}$ *such that* $\mathbf{c} \prec \mathbf{a}$ *and* $\mathbf{b} \leq \mathbf{c}$ $(i.e., b_i \leq c_i \ for \ each \ i)$

Lemma (2.5) can be derived from the well-known rearrangement inequality; a proof is given in [41]. Lemma (2.6) characterizes the weak majorization order in terms of the majorization order, its proof can be found in Marshall and Olkin (1979, page 123).

2.3 Arrangement Increasing

Definition 2.7 ([6]) *A function* $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *is said to be arrangement increasing [decreasing], if for all* i *and* j *such that* $1 \leq i < j \leq n$,

$$(x_i - x_j)\{f(x_1, ..., x_i, ..., x_j, ..., x_n) - f(x_1, ..., x_j, ..., x_i, ..., x_n)\} \leq [\geq] 0. \tag{2.9}$$

One major example is given by the joint density function of mutually independent random variables that are ordered by the likelihood ratio order.

Lemma 2.8 ([6]) *If $X_1, ..., X_n$ are mutually independent and $X_1 \leq_{lr} ... \leq_{lr} X_n$, then the joint density function of $(X_1, ..., X_n)$ is arrangement increasing.*

Definition 2.9 ([6]) *A function $g(\mathbf{x}, \boldsymbol{\lambda}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be an arrangement increasing (AI) function if*

1. *g is permutation invariant, i.e., $g(\mathbf{x}, \boldsymbol{\lambda}) = g(\mathbf{x} \circ \tau, \boldsymbol{\lambda} \circ \tau)$ for any permutation τ, and*

2. *g exhibits permutation order, i.e., $g(\mathbf{x} \downarrow, \boldsymbol{\lambda} \uparrow) \leq g(\mathbf{x} \downarrow, \boldsymbol{\lambda} \circ \tau) \leq g(\mathbf{x} \downarrow, \boldsymbol{\lambda} \downarrow)$ for any permutation τ.*

The following lemmas give us two examples of AI functions. Proofs can be found in [6].

Lemma 2.10 ([29]) *The function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$g(\mathbf{x}, \boldsymbol{\lambda}) = -\sum_{i=1}^{n} (x_i - \lambda_i)_+ \text{and } g(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} (x_i \wedge \lambda_i) \tag{2.10}$$

are an AI function.

Proofs of the following lemmas can be found in [29].

Lemma 2.11 ([29]) *Suppose that the function $\phi(x, \lambda) : \mathbb{R}^2 \to \mathbb{R}$ is increasing both in x and λ. If the function*

$$g(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} \phi(x_i, \lambda_i)$$

from $\mathbb{R}^n \times \mathbb{R}^n$ *to* $\mathbb{R}$ *is an AI function, then*

$$\hat{\phi}(\mathbf{x}\downarrow, \boldsymbol{\lambda}\uparrow) \prec\prec \hat{\phi}\left(\mathbf{x}\downarrow, \boldsymbol{\lambda}\circ\tau\right) \prec\prec \hat{\phi}\left(\mathbf{x}\downarrow, \boldsymbol{\lambda}\downarrow\right) \tag{2.11}$$

for any permutation τ.

Lemma 2.12 ([29]**)** *Suppose that the function* $\phi(x, \lambda) : \mathbb{R}^2 \to \mathbb{R}$ *is increasing in one variable and decreasing in the other. If the function*

$$g(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} \phi\left(x_i, \lambda_i\right)$$

from $\mathbb{R}^n \times \mathbb{R}^n$ *to* $\mathbb{R}$ *is an AI function, then*

$$-\hat{\phi}(\mathbf{x}\downarrow, \boldsymbol{\lambda}\downarrow) \prec\prec -\hat{\phi}\left(\mathbf{x}\downarrow, \boldsymbol{\lambda}\circ\tau\right) \prec\prec -\hat{\phi}\left(\mathbf{x}\downarrow, \boldsymbol{\lambda}\uparrow\right) \tag{2.12}$$

for any permutation τ.

2.4 Value-at-Risk

These last years, several experts saw importance of the quantiles of the probability distributions Since quantiles especially have an easy interpretation in practice of risk management in the form of the concept of value-at-risk (VaR). This concept was introduced to answer the following question: how much can we expect to lose in one day, week, year, with a given probability? The VaR is given in Jorion (2000). In the following definition we defined VaR .

Definition 2.13 ([21]) *Given a risk X and a probability level $p \in (0,1)$, the corresponding VaR, denoted by $VaR\,[X;p]$, is defined as*

$$VaR\,[X;p] = F_X^{-1}\,(p)\,. \tag{2.13}$$

Note that the VaR risk measure reduces to the percentile principle of Goovaerts, De Vijlderand Hazendonck (1984).

2.4.1 Tail Value-at-Risk

A single VaR at a predetermined level p does not give any information about the thickness of the upper tail of the distribution function. This is a considerable shortcoming since in practice a regulator is not only concerned with the frequency of default, but also with the severity of default. Also shareholders and management should be concerned with the question 'how bad is bad?' when they want to evaluate the risks at hand in a consistent way. Therefore, one often uses another risk measure, which is called the tail value-at-risk ($TVaR$) and defined next.

Definition 2.14 ([21]) *Given a risk X and a probability level p, the corresponding TVaR, denoted by $TVaR\,[X;p]$, is defined as*

$$TVaR\,[X;p] = \frac{1}{1-p}\int_p^1 VaR\,[X;\xi]\,d\xi, \qquad 0 < p < 1. \tag{2.14}$$

We thus see that $TVaR\,[X;p]$ can be viewed as the 'arithmetic average' of the VaRs of X, from p on.

2.5 Sotachastic orders

2.5.1 Stochastic Dominance

Stochastic dominance and risk measures

Stochastic dominance and VaRs In order to compare a pair of risks X and Y, it seems natural to resort to the concept of VaR, and to consider X as less dangerous than Y if $VaR[X;\alpha_0] \leq VaR[Y;\alpha_0]$ for some prescribed probability level α_0. However, it is sometimes difficult to select such an α_0, and it is conceivable that $VaR[X;\alpha_0] < VaR[Y;\alpha_0]$ and $VaR[X;\alpha_1] > VaR[Y;\alpha_1]$ simultaneously for two probability levels α_0 and α_1. In this case, what can we conclude? It seems reasonable to adopt the following criterion: we place X before Y if the $VaRs$ for X are smaller than the corresponding $VaRs$ for Y, for any probability level.

Definition 2.15 *Let X and Y be two rvs. Then X is said to be smaller than Y in stochastic dominance, denoted as $X \preceq_{ST} Y$, if the inequality $VaR[X;p] \leq VaR[Y;p]$ is satisfied for all $p \in [0,1]$.*

Standard references for $\preceq_{ST}$ are the books of Lehmann (1959), Marshall and Olkin (1979), Ross (1983) and Stoyan (1983).

Stochastic dominance can also be characterized by the relative inverse distribution function, defined as

$$x \longmapsto VaR[X;F_Y(x)] \tag{2.15}$$

It is nothing more than the VaR of X at probability level $p = F_Y(x)$.

Stochastic dominance and monotonicity An important characterization of $\preceq_{ST}$ is given in the next result. It essentially states that if $X \preceq_{ST} Y$ holds then there exist rvs $\tilde{X}$ and $\tilde{Y}$, distributed as X and Y for which $\Pr\left[\tilde{X} \leq \tilde{Y}\right] = 1$. In such a case, $\tilde{Y}$ is larger than $\tilde{X}$ according to Kaas et al.(2001, Definition 10.2.1). This proposition shows that $\preceq_{ST}$ is closely related to pointwise comparison of rvs.

Proposition 2.16 *Two rvs X and Y satisfy $X \preceq_{ST} Y$ if, and only if, there exist two rvs $\tilde{X}$ and $\tilde{Y}$ such that $X =_d \tilde{X}$, $Y =_d \tilde{Y}$ and $\Pr\left[\tilde{X} \leq \tilde{Y}\right] = 1$. For a proof, see Kaas et al.(2001, Theorem 10.2.3). The construction of $\tilde{X}$ and $\tilde{Y}$ involved in Proposition* (2.16) *is known as a coupling technique (see Lindvall 1992). Proposition* (2.16) *can be rewritten as follows.*

Proposition 2.17 *Two rvs X and Y satisfy $X \preceq_{ST} Y$ if, and only if, there exists a comonotonic random couple (X^c, Y^c) such that $\Pr[X^c \leq Y^c] = 1$, $X =_d X^c$ and $Y =_d Y^c$.*

Stochastic dominance and stop-loss premiums The following result relates $\preceq_{ST}$ to stop-loss transforms. (The function $\pi_X(t) = \mathbb{E}\left[(X-t)_+\right]$ is called the stop-loss transform of X (See Kaas 1993)).

Proposition 2.18 *Given two rvs X and Y, $X \preceq_{ST} Y$ if, and only if, $\pi_Y - \pi_X$ is non-increasing on $\mathbb{R}$.*

Proof. Clearly, if $X \preceq_{ST} Y$, $\bar{F}_Y - \bar{F}_X$ is non-negative so that the result follows from $\pi_X(t) = \int_t^{+\infty} \bar{F}_X(\xi)\, d\xi$. Now assume the non-increasingness of the difference $\pi_Y - \pi_X$. Differentiating this expression with respect to t yields $\bar{F}_X(t) - \bar{F}_Y(t) \leq 0$, whence $X \preceq_{ST} Y$ follows. □

It is interesting to give an actuarial interpretation for Proposition (2.18). It basically states that when $X \preceq_{ST} Y$ holds, the difference between their respective stop-loss premiums decreases with the level t of retention. Since

$$\lim_{t\to+\infty} \left(\mathbb{E}\left[(Y-t)_+\right] - \mathbb{E}\left[(X-t)_+\right]\right) = 0$$

we thus have that the largest difference occurs for $t = 0$ (and equals $\mathbb{E}(Y) - \mathbb{E}(X)$) and that this difference decreases with the level of retention. The pricing of stop-loss treaties will thus lead to very different premiums for small retentions, but this difference will decrease as the retention increases.

2.5.2 Sotachastic order

There are several references for stochastic orders include Denuit et al.(2002, 2005), Kaas et al.(1994, 2001), Müller and Stoyan (2002), Shaked and Shanthikumar (1994, 2007). We assume all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and all expectations mentioned exist.

Definition 2.19 *Let X and Y be two random variables,*

1. *X is said to be smaller than Y in the usual stochastic order (resp. increasing*

convex order, decreasing convex order, convex order), denoted by $X \leq_{st} Y$ *(resp.* $X \leq_{icx} Y, X \leq_{dcx} Y, X \leq_{cx} Y$*), if*

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}\left[\phi(Y)\right] \tag{2.16}$$

for all increasing (resp. increasing convex, decreasing convex, convex) function ϕ.

2. X *is said to be smaller than* Y *in the likelihood ratio order, denoted by* $X \leq_{lr} Y$*, if*

$$f_X(x)g_Y(y) \geq f_X(y)g_Y(x) \text{ for all } x \leq y \tag{2.17}$$

where f_X *and* g_Y *are the density functions of* X *and* Y*, respectively.*

2.5.3 Convex ordering random variables

In this subsection we give the definition of the Stop-loss premium by $\mathbb{E}[(X-d)_+] = \int_d^\infty (1 - F_X(x))dx, -\infty < d < +\infty$. Hence, we use the notation S for the sum of the random vector $(X_1, X_2, ..., X_n) : S = X_1 + X_2 + ... + X_n$. Moreover, we will present the convex order and his characterization using Stop-loss premium .

We start by define the Stop-loss order between random variables

Definition 2.20 ([9]) *(Stop-loss order). Consider two random variables* X *and* Y*. then* X *is said to precede* Y *in the stop-loss order sense, notation* $X \leq_{st} Y$ *if and only if* X *has lower stop-loss premiums than* Y*:*

$$\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]; -\infty < d < +\infty \tag{2.18}$$

with $(X-d)_+ = max(X-d,0)$.

Definition 2.21 ([9]) *(convex order). Consider two random variables* X *and* Y *such that* $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, *for all convex functions* ϕ,*provided expectation exit. Then* X *is said to be smaller than* Y *in the convex order denoted as* $X \leq_{cx} Y$.

Definition 2.22 ([9]) *(Convex order characterization using stop-loss premium). Consider two random variables* X *and* Y. *Then* X *is said to precede* Y *in convex order sense if and only if*

$$\mathbb{E}[X] = \mathbb{E}[Y] \tag{2.19}$$

$$\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]; -\infty < d < +\infty$$

Where

$$(X-d)_+ = \max(X-d,0)$$

An equivalent definition can be derived from the following relation

$$\mathbb{E}[(X-d)_+] - \mathbb{E}[(d-X)_+] = \mathbb{E}(X) - d$$

For the random variable Y the same relation is given by

$$\mathbb{E}[(Y-d)_+] - \mathbb{E}[(d-Y)_+] = \mathbb{E}(Y) - d$$

Now assume $X \leq_{cx} Y$, which implies that

$$\mathbb{E}[X] = \mathbb{E}[Y]$$

and

$$\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]; -\infty < d < +\infty$$

hence

$$\mathbb{E}[(d - X)_+] \leq \mathbb{E}[(d - Y)_+]$$

Therefore, a definition equivalent to the definition here is

$$\mathbb{E}[X] = \mathbb{E}[Y]$$

$$\mathbb{E}[(d - X)_+] \leq \mathbb{E}[(d - Y)_+]; -\infty < d < +\infty$$

Properties of Convex Ordering of Random Variables

1. If X precedes Y in convex order sense i.e if $X \leq_{cx} Y$, then $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbf{V}[X] \leq \mathbf{V}[Y]$, where $\mathbf{V}[X]$ is variance of X. [9]

2. If $X \leq_{cx} Y$ and Z is independent of X and Y then $X + Z \leq_{cx} Y + Z$. [9]

3. Let X and Y be two random variables, then $X \leq_{cx} Y \Rightarrow -X \leq_{cx} -Y$. [15]

4. Let X and Y be two random variables such that $\mathbb{E}[X] = \mathbb{E}[Y]$. Then $X \leq_{cx} Y$ if and only if $\mathbb{E}|X - a| \leq_{cx} \mathbb{E}|Y - a|, \forall a \in \mathbb{R}$. [15]

5. The convex order is closed under mixtures: Let X, Y and Z be random variables such that $[X \mid Z = z] \leq_{cx} [Y \mid Z = z] \, \forall z$ in the support of Z. Then $X \leq_{cx} Y$. [33]

6. The convex order is closed under Convolution: Let $X_1, X_2, ..., X_m$ be a set of independent random variables and $Y_1, Y_2, ..., Y_m$ be another set of independent random variables. If $X_j \leq_{cx} Y_j$,for $i = 1, ..., m$, then $\sum_{j=1}^{m} X_j \leq_{cx} \sum_{j=1}^{m} Y_j$. [9]

7. Let X be a random variable with finite mean. Then $X + \mathbb{E}[X] \leq_{cx} 2X$. (It suffices to use the Definition (2.22))

8. Let $X_1, X_2, ..X_n$ and Y be $(n+1)$ random variables. If $X_i \leq_{cx} Y, i = 1, ..., n$, then $\sum_{i=1}^{n} a_i X_i \leq_{cx} Y$, whenever $a_i \geq 0, i = 1, ..., n$ and $\sum_{i=1}^{n} a_i = 1$. (It suffices to use the property 6)

9. Let X and Y be two random variables. Then $X \leq_{cx} Y$ if and only if $\mathbb{E}[\Phi(X, Y)] \leq \mathbb{E}[\Phi(Y, X)]$,$\forall \Phi \in \Psi_{cx}$ where

$$\Psi_{cx} = \left\{ \Phi : \mathbb{R}^2 \to \mathbb{R} : \Phi(X, Y) - \Phi(Y, X) \text{ is convex for all } x \in y \right\}. \tag{2.20}$$

(It suffices to use the Definition (2.22))

10. Let X_1 and X_2 be a pair of independent random variables and let Y_1 and Y_2 be another pair of independent random variables. If $X_i \leq_{cx} Y_i$, $i = 1, 2$ then $X_1 X_2 \leq_{cx} Y_1 Y_2$. [9]

11. For all convex function υ, then $X \leq_{cx} Y$ if and only $\mathbb{E}[\upsilon(X)] = \mathbb{E}[\upsilon(Y)]$. [33]

12. Let X, Y and Z be random variables such that $X \leq_{cx} Y$ and $Y \leq_{cx} Z$, then $X \leq_{cx} Z$. (It suffices to use the Definition (2.22))

13. If $X \leq_{lr} Y$ and ϕ is any decreasing function, then $\phi(X) \geq_{lr} \phi(Y)$. [50]

14. Let $\mathbf{X} \in \mathbb{R}^n_+$ and $X_1 \leq_{lr} ... \leq_{lr} X_n$ are mutually independent. If $\mathbf{b}$ is weakly majorized by $\mathbf{a}$ (denoted by $\mathbf{b} \prec\prec \mathbf{a}$) and $\mathbf{a} \in \mathcal{I}_n$, then $\sum\limits_{i=1}^{n} b_i X_i \leq_{icx} \sum\limits_{i=1}^{n} a_i X_i$. [29]

2.5.4 Lorenz Order

Lorenz curves

The Lorenz order is defined by means of pointwise comparison of the Lorenz curves. These are used in economics to measure the inequality of incomes. More precisely, let X be a non-negative rv with df F_X. The Lorenz curve LC_X associated with X is then defined by

$$LC_X = \frac{1}{\mathbb{E}(X)} \int_{\xi=0}^{p} VaR[X;\xi]\, d\xi, \quad p \in [0,1]. \tag{2.21}$$

In an economics framework, when X models the income of the individuals in some population, LC_X maps $p \in [0,1]$ to the proportion of the total income of the population which accrues to the poorest $100p$ % of the population. An interpretation of LC_X in insurance business is the following: $LC_X(p)$ can be thought of as being the fraction of the aggregate claims caused by the $100p$ % of the treaties with the lowest claim size.

Lorenz order

Now, we define the lorenz order

Definition 2.23 ([9]) *Consider two risks X and Y. Then, X is said to be smaller than Y in the Lorenz order, henceforth denoted by $X \preceq_{Lorenz} Y$, when $LC_X(p) \geq LC_Y(p)$ for all $p \in [0,1]$.*

A standard reference on $\preceq_{Lorenz}$ is Arnold (1987). We mention that $<_3$ in Heilmann (1985) is in fact Lorenz. We also mention that the Lorenz order is the k-order introduced by Heilmann (1986) (see also Heilmann and Schröter 1991, Remark 5)

Lorenz and Convex orders

There is a closely relation between the convex and Lorenz orders.

Property Given two risks X and Y,

$$X \preceq_{Lorenz} Y \Leftrightarrow \frac{X}{\mathbb{E}[X]} \preceq_{cx} \frac{Y}{\mathbb{E}[Y]}. \tag{2.22}$$

Obviously, when $\mathbb{E}[X] = \mathbb{E}[Y]$, we have

$$X \preceq_{Lorenz} Y \Leftrightarrow X \preceq_{cx} Y \tag{2.23}$$

since the convex order is scale-invariant. It is worth mentioning that other approaches to comparing rvs with unequal means via $\preceq_{cx}$ have been proposed. The dilation order, for instance, compares $X - \mathbb{E}[X]$ to $Y - \mathbb{E}[Y]$.

2.6 Inverse distribution functions

The cdf $F_X(x) = \mathbb{P}[X \leq x]$ of a random variable X is a right-continuous (further abbreviated as r.c.) non-decreasing function with

$$F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0, F_X(+\infty) = \lim_{x\to+\infty} F_X(x) = 1.$$

The usual definition of the inverse of a distribution function is the non-decreasing and left-continuous (l.c.) function $F_X^{-1}(p)$ defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in [0, 1] \tag{2.24}$$

with $\inf \emptyset = +\infty(\sup \emptyset = -\infty)$ by convention. For all $x \in \mathbb{R}$ and $p \in [0; 1]$, we have

$$F_X^{-1}(p) \leq x \Leftrightarrow p \leq F_X(x). \tag{2.25}$$

In this work, we will use a more sophisticated definition for inverses of distribution functions. For any real $p \in [0, 1]$, a possible choice for the inverse of F_X in p is any point in the closed interval

$$[\inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}],$$

where, as before, $\inf \emptyset = +\infty$, and also $\sup \emptyset = -\infty$. Taking the left hand border of this interval to be the value of the inverse cdf at p, we get $F_X^{-1}(p)$.

Similarly, we define $F_X^{-1+}(p)$ as the rihgt hand border of the interval:

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}, \quad p \in [0,1] \tag{2.26}$$

which is a non-decreasing and r.c. function. Note that $F_X^{-1}(0) = -\infty$, $F_X^{-1+}(1) = +\infty$ and that all the probability mass of X is contained in the interval $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$. Also note that $F_X^{-1+}(p)$ are finite for all $p \in (0,1)$. In the sequel, we will always use p as a variable ranging over the open interval $(0,1)$, unless stated otherwise.

For any $\alpha \in [0,1]$, we define the α-mixed inverse function of F_X as follows:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p), \quad p \in (0,1), \tag{2.27}$$

which is a non-decreasing function. In particular, we find $F_X^{-1(0)}(p) = F_X^{-1+}(p)$ and $F_X^{-1(1)}(p) = F_X^{-1}(p)$. One immediately finds that for all $\alpha \in [0,1]$,

$$F_X^{-1}(p) \leq F_X^{-1(\alpha)}(p) \leq F_X^{-1+}(p), \quad p \in (0,1). \tag{2.28}$$

Note that only values of p corresponding to a horizontal segment of F_X lead to different values of $F_X^{-1}(p)$, $F_X^{-1+}(p)$ and $F_X^{-1(\alpha)}(p)$. This phenomenon illustrated in figure 5.1.

Now led d be such that $0 < F_X(d) < 1$. Then $F_X^{-1}(F_X(d))$ and $F_X^{-1+}(F_X(d))$ are finite, and $F_X^{-1}(F_X(d)) \leq d \leq F_X^{-1+}(F_X(d))$. So for some value $\alpha_d \in [0,1]$, d can be expressed as $d = \alpha_d F_X^{-1}(F_X(d)) + (1-\alpha_d) F_X^{-1+}(F_X(d)) = F_X^{-1(\alpha_d)}(F_X(d))$. This implies that for any random variable X and any d with $0 < F_X(d) < 1$, there exists an $\alpha_d \in [0,1]$ such that $F_X^{-1(\alpha_d)}(F_X(d)) = d$.

In the following theorem, we state the relation between the inverse didtribution functions of the random variables X and $g(X)$ For a monotone function g.

Theorem 2.24 *Let X and $g(X)$ be real-valued random variables, and let $0 < p < 1$.*

(a) *If g is non-decreasing and l.c., then*

$$F_{g(X)}^{-1}(p) = g\left(F_X^{-1}(p)\right). \tag{2.29}$$

(b) *If g is non-decreasing and r.c., then*

$$F_{g(X)}^{-1+}(p) = g\left(F_X^{-1+}(p)\right). \tag{2.30}$$

(c) *If g is non-increasing and l.c., then*

$$F_{g(X)}^{-1+}(p) = g\left(F_X^{-1}(1-p)\right). \tag{2.31}$$

(d) *If g is non-increasing and r.c., then*

$$F_{g(X)}^{-1}(p) = g\left(F_X^{-1+}(1-p)\right). \tag{2.32}$$

Proof. We will prove (a). Then other results can be proven similarly. Let $0 < p < 1$ and consider a non-decreasing and left-continuous function g. For any real x we find from (2.25) that

$$F_{g(X)}^{-1}(p) \leq x \Leftrightarrow p \leq F_{g(X)}(x).$$

As g is l.c., we have that

$$g(z) \leq x \Leftrightarrow z \leq \sup\{y \mid g(y) \leq x\}$$

holds for all real z and x. Hence,

$$p \leq F_{g(X)}(x) \Leftrightarrow p \leq F_X\left[\sup\left\{y \mid g(y) \leq x\right\}\right]$$

If $\sup\left\{y \mid g(y) \leq x\right\}$ is finite then we find from (2.25) and the equivalence above

$$p \leq F_X\left[\sup\left\{y \mid g(y) \leq x\right\}\right] \Leftrightarrow F_X^{-1}(p) \leq \sup\left\{y \mid g(y) \leq x\right\}.$$

In case $\sup\left\{y \mid g(y) \leq x\right\}$ is $+\infty$ or $-\infty$, we cannot use (2.25), but one can verify that the equivalence above also holds in this case. Indeed, if the supreum equals $-\infty$, then the equivalence becomes $p \leq 1 \Leftrightarrow F_X^{-1}(p) \leq -\infty$.

Because g is non-decreasing and l.c., we get that

$$F_X^{-1}(p) \leq \sup\left\{y \mid g(y) \leq x\right\} \Leftrightarrow g\left(F_X^{-1}(p)\right) \leq x$$

Compining the equivalences, we finally find that

$$F_{g(X)}^{-1}(p) \leq x \Leftrightarrow g\left(F_X^{-1}(p)\right) \leq x$$

holds for all values of x, which means that (a) must hold. $\Box$

For the special cases that X and $g\left(X\right)$ are continuous and strictly increasing on $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$, a simple proof is possible. Indeed, in this case we have that $F_{g(X)}(x) = \left(F_X \circ g^{-1}\right)(x)$, which is a continuous and strictly increasing function of x. The results (a) and (b) then follow by inversion of this relation. A similar proof holds for (c) and (d) if g and F_X are both continuous, while g is strictly decreasing and F_X is strictly increasing.

Hereafter, we will reserve the notation U for a uniform $(0,1)$ random variable, i.e. $F_u(p) = p$ and $F_u^{-1}(p) = p$ for all $0 < p < 1$. We can prove that for all $\alpha \in [0,1]$,

$$X \stackrel{d}{=} F_X^{-1}(U) \stackrel{d}{=} F_X^{-1+}(U) \stackrel{d}{=} F_X^{-1(\alpha)}(U). \tag{2.33}$$

The first distributional equality is known as the quantile transform theorem and follows immediately from (2.25). It states that a sample of random numbers from a general distribution function F_X can be generated from a sample of uniform random numbers. Note that F_X has at most a countable number of horizontal segments, implying that the last three random variables in (2.33) only differ in a null-set of values of U. This implies that these random variables are equal with probability one.

□

Chapter 3

Comonotonicity

In this chapter we will discuss the comonotonicity notion for sums of dependent random variables whose marginal distributions are known, but with an unknown or complicated joint distribution. Considering comonotonic random vectors essentially reduces the multidimensional problem to a univariate one since then all components depend on the same variable. Also, we give a result to present a new theorem with proposition of convex bounds and the comonotonic upper bound for S_N.

3.1 Comonotonic sets and random vectors

In this section we give a total $S = \sum_{i=1}^{n} X_i$ where the terms X_i are not mutually independent, and also we have the multivariate distribution function of the random vector $\underline{X}= (X_1, X_2, ..., X_n)$. We will find the dependence structure for the random vector $(X_1, ..., X_n)$. Then we use the joint distribution in the convex orders sense.

Now, we define the comonotonicity of a set of $n-$vectors in $\mathbb{R}^n$. Let $n-$vector $(x_1, x_2, .., x_n)$ be denoted by $\underline{x}$. For two $n-$vectors $\underline{x}$ and $\underline{y}$, the notation $\underline{x} \leq \underline{y}$ is used for the componentwise order which is defined by $x_i \leq y_i$ for all $i = 1, 2, ..., n$.

Definition 3.1 *(Comonotonic set). The set $A \subseteq \mathbb{R}^n$ is comonotonic if for any x and y in A, either $\underline{x} \leq \underline{y}$ or $\underline{y} \leq \underline{x}$ holds.*

So, a set $A \subseteq \mathbb{R}^n$ is comonotonic if for any $\underline{x}$ and $\underline{y}$ in A, if $x_i < y_i$ for some i,then $\underline{x} \leq \underline{y}$ must hold. Hence, a comonotonic set is simultaneously non-decreasing in each component. Notice that a comonotonic set is a 'thin' set: it cannot contain any subset of dimension larger than 1. Any subset of a comonotonic set is also comonotonic.

we will denote the (i, j)-projection of a set A in $\mathbb{R}^n$ by $A_{i,j}$. It is defined by

$$A_{i,j} = \{(x_i, x_j) \mid \underline{x} \in A\} \tag{3.1}$$

Lemma 3.2 *$A \subseteq \mathbb{R}^n$ is comonotonic if and only if $A_{i,j}$ is comonotonic for all $i \neq j$ in $\{1, 2, ..., n\}$.*

The proof of lemma (3.2) is straightforward.

For a general set A, comonotonicity of the $(i, i+1)$-projection $A_{i,i+1}$,$(i = 1, 2, ..., n-1)$, will not nessarily imply that A is comonotonic. As an example, consider the set

$$A = \{(x_1, 1, x_3) \mid 0 < x_1, x_3 < 1\}.$$

This set is not comonotonic, although $A_{1,2}$ and $A_{2,3}$ are comonotonic. Next, we will define the notion of support of an n-dimensional random vector $\underline{X} = (X_1, ..., X_n)$.

Any subset $A \subseteq \mathbb{R}^n$ will be called a support of $\underline{X}$ if $\Pr[\underline{X} \subseteq A] = 1$ holds true. In general we will be interested in support of wich are "as small as possibe". Informally, the smallest support of a random vector $\underline{X}$ is the subset of $\mathbb{R}^n$ that is obtained by subtracting of $\mathbb{R}^n$ all points which have a zero-probability neighborhood (with respect to $\underline{X}$). This support can be interpreted as the set of all possible outcomes $\underline{X}$. Next, we will define comonotonicity of random vectors.

Definition 3.3 *(Comonotonic random vector). A random vector* $\underline{X} = (X_1, ..., X_n)$ *is comonotonic if it has a comonotonic support.*

From the definition, we can conclude that comonotonicity is a very strong positive dependency structure. Indeed, if $\underline{x}$ and $\underline{y}$ are elements of the (comonotonic) support of $\underline{X}$, i.e. $\underline{x}$ and $\underline{y}$ are possible outcomes of $\underline{X}$, then they must be ordered componentwise. This explains why the term comonotonic (common monotonic) is used.

Comonotonicity of a random vector $\underline{X}$ implies that the higher the value of one component X_j, the higher the value of any other component X_k. This means that comonotonicity entails that no X_j is in any way a "hedge", perfect or imperfect, for another component X_k.

In the following theorem, some equivalent characterizations are given for comonotonicity of a random vector.

Theorem 3.4 *(Equivalent conditions for comonotonicity)*

A random vector $\underline{X} = (X_1, X_2, ..., X_n)$ *is comonotonic if and only if one of the following equivalent conditions holds:*

1. $\underline{X}$ has a comonotonic support;

2. $\underline{X}$ has a comonotonic copula, i.e. for all $\underline{x} = (x_1, x_2, ..., x_n)$, we have

$$F_{\underline{X}}(\underline{x}) = \min\left\{F_{X_1}(x_1), F_{X_1}(x_1), ..., F_{X_n}(x_n)\right\}; \tag{3.2}$$

3. For $U \sim Uniform(0,1)$, we have

$$\underline{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U)\right); \tag{3.3}$$

4. A random variable Z and non-decreasing functions $f_i(i = 1, ..., n)$ exist such that

$$\underline{X} \stackrel{d}{=} \left(f_1(Z), f_2(Z), ..., f_n(Z)\right). \tag{3.4}$$

Proof. (1) $\Rightarrow$ (2) : Assum that $\underline{X}$ has comonotonic support B. Let $\underline{x} \in \mathbb{R}^n$ and let A_j be defined by

$$A_j = \left\{\underline{y} \in B \mid y_j \leq x_j\right\}, \quad j = 1, 2, ..., n.$$

Because of the comonotonicity of B, there exists an i such that $A_i = \cap_{j=1}^{n} A_j$

Hence, we find

$$F_{\underline{X}}(\underline{x}) = \Pr\left(\underline{X} \in \cap_{j=1}^{n} A_j\right) = \Pr(\underline{X} \in A_i) = F_{X_i}(x_i)$$

$$= \min\left\{F_{X_1}(x_1), F_{X_1}(x_1), ..., F_{X_n}(x_n)\right\}.$$

The last equality follows from $A_i \subset A_j$ so that $F_{X_i}(x_i) \leq F_{X_j}(x_j)$ holds for all values of j.

(2) $\Rightarrow$ (3) : Now assume that $F_{\underline{X}}(\underline{x}) = \min\{F_{X_1}(x_1), F_{X_1}(x_1), ..., F_{X_n}(x_n)\}$ for all $\underline{x} = (x_1, x_2, ..., x_n)$. Then we find by (2.25)

$$
\begin{aligned}
& \Pr\left[F_{X_1}^{-1}(U) \leq x_1, ..., F_{X_n}^{-n}(U) \leq x_n\right] \\
= \; & \Pr\left[U \leq F_{X_1}(x_1), ..., U \leq F_{X_n}(x_n)\right] \\
= \; & \Pr\left[U \leq \min_{j=1,...,n} \left\{F_{X_j}(x_j)\right\}\right] \\
= \; & \min_{j=1,...,n} \left\{F_{X_j}(x_j)\right\}
\end{aligned}
$$

(3) $\Rightarrow$ (4) : straightforward.

(4) $\Rightarrow$ (1) : Assume that there exists a random variable Z with support B, and non-decreasing functions f_i, $(i = 1, 2, ..., n)$, such that

$$\underline{X} \stackrel{d}{=} (f_1(Z), f_2(Z), ..., f_n(Z)).$$

The set of possible outcomes of $\underline{X}$ is $\{f_1(z), f_2(z), ..., f_2(z) \mid z \in B\}$ which is obviously comonotonic, which implies that $\underline{X}$ is indeed comonotonic. $\square$

From (3.2) we see that, in order to find the probability of all the outcomes of n comonotonic risks X_i being less than x_i $(i = 1, ..., n)$ one simply takes the probability of the least likely of these n events. It is obvious that for any random vector $(X_1, ..., X_n)$, not necessarily comonotonic, the following inequality holds:

$$\Pr[X_1 \leq x_1, ..., X_n \leq x_n] \leq \min\{F_{X_1}(x_1), ..., F_{X_n}(x_n)\}, \tag{3.5}$$

and since Hoeffding [28] and Fréchet [20] it is known that the function

$\min\{F_{X_1}(x_1), ..., F_{X_n}(x_n)\}$ is indeed the multivariate cdf of a random vector, i.c. $\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U)\right)$, which has the same marginal distributions as $(X_1, ..., X_n)$. Inequality (3.5) states that in the class of all random vectors $(X_1, ..., X_n)$ with the same marginal distributions, the probability that all X_i simultaneously realize 'small' values is maximized if the vector is comonotonic, suggesting that comonotonicity is indeed a very strong positive dependency structure.

From (3.3) we find that in the special case that all marginal distribution functions F_{X_i} are identical, comonotonicity of $\underline{X}$ is equivalent to saying that $X_1 = X_2 = ... = X_n$ holds almost surely.

A standard way of modelling situations where individual random variables $X_1, ..., X_n$ are subject to the same external mechanism is to use a secondary mixing distribution. The uncertainty about the external mechanism is then described by a structure variable z, which is a realization of a random variable Z and acts as a (random) parameter of the distribution of $\underline{X}$. The aggregate claims can then be seen as a two-stage process: first, the external parameter $Z = z$ is drawn from the distribution function F_Z of z. The claim amount of each individual risk X_i is then obtained as a realization from the conditional distribution function of X_i given $Z = z$. A special type of such a mixing model is the case where given $Z = z$, the claim amounts X_i are degenerate on x_i, where the $x_i = x_i(z)$ are non-decreasing in z. This means that $(X_1, ..., X_n) \overset{d}{=} (f_1(Z), ..., f_n(Z))$ where all functions f_i are non-decreasing. Hence, $(X_1, ..., X_n)$ is comonotonic. Such a model is in a sense an extreme form of a mixing model, as in this case the external parameter $Z = z$ completely determines the aggregate claims. As the random vectors $\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U)\right)$ and $\left(F_{X_1}^{-1(\alpha_1)}(U), F_{X_2}^{-1(\alpha_2)}(U), ..., F_{X_n}^{-1(\alpha_n)}(U)\right)$ are equal with probability one, we find that comonotonicity of $\underline{X}$ can be charcterized by

$$\underline{X} \overset{d}{=} \left(F_{X_1}^{-1(\alpha_1)}(U), F_{X_2}^{-1(\alpha_2)}(U), ..., F_{X_n}^{-1(\alpha_n)}(U)\right) \tag{3.6}$$

For $U \sim Uniform(0,1)$ and given real numbers $\alpha_i \in [0,1]$.

If $U \sim Uniform(0,1)$, then also $1 - U \sim Uniform(0,1)$. This implies that comonotonicity of $\underline{X}$ can also be characterized by

$$\underline{X} \overset{d}{=} \left(F_{X_1}^{-1}(1-U), F_{X_2}^{-1}(1-U), ..., F_{X_n}^{-1}(1-U)\right) \tag{3.7}$$

0ne can prove that $\underline{X}$ is comonotonic if and only if there exist a random variable Z and non-increasing functions f_i, $(i = 1, 2, ..., n)$, such that

$$\underline{X} \overset{d}{=} \left(f_1(Z), f_2(Z), ..., f_n(Z)\right).$$

The proof is similar to the proof of the characterization (4) in theorem (3.4).

In the sequel, for any random vector $(X_1, ..., X_n)$, the notation $(X_1^c, ..., X_n^c)$ or $\left(\tilde{X}_1, ..., \tilde{X}_n\right)$ will be used to indicate a comonotonic random vector with the marginals as $(X_1, ..., X_n)$.

From (3.3), we find that for any random vector $\underline{X}$ the outcome of its comonotonic counterpart $\underline{X}^c = (X_1^c, ..., X_n^c)$ is with probability 1 in the following set

$$\left\{\left(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p), ..., F_{X_n}^{-1}(p) \mid 0 < p < 1\right)\right\}. \tag{3.8}$$

This support of $\underline{X}^c$ is not necessarily a connected curve. Indeed, all horizontal segments of the cdf of X_i lead to "missing pieces" in this curve. This support can be seen to be a series of ordered connected curves. Now by connecting the endpoints of consecutive curves by straigh lines, we obtain a comonotonic connected curve in $\mathbb{R}^n$. Hence, it may be traversed in a direction which is upwards for all components simultaneously. we will call this set the connected support of $\underline{X}^c$. It might be parmeterized as follows:

$$\left\{ \left(F_{X_1}^{-1(\alpha)}(U), F_{X_2}^{-1(\alpha)}(U), ..., F_{X_n}^{-1(\alpha)}(U) \right) \mid 0 < p < 1, 0 < \alpha < 1 \right\}. \tag{3.9}$$

Observe that this parameterization is not necessarily unique: there may be elements in the connected support which can be characterized by different values of α.

Theorem 3.5 *(Pairwise comonotonicity)*

A random vector $\underline{X}$ is comonotonic if and only if the couples (X_i, X_j) are comonotonic for all i and j in $\{1, 2, ..., n\}$.

3.2 Examples

- **Continuous Distributions** [15]. Let $X \backsim$Uniform on the set $\left(0, \frac{1}{2}\right) \cup \left(0, \frac{3}{2}\right)$, $Y \backsim$Beta $(2, 2)$, hence $F_Y(y) = 3y^2 - 2y^3$ on $(0, 1)$, and $Z \backsim$Normal $(0, 1)$. If X, Y and Z are mutually independent, then the support of (X, Y, Z) is the set

$$\left\{(x, y, z) \mid x \in \left(0, \frac{1}{2}\right) \cup \left(0, \frac{3}{2}\right), y \in (0, 1), z \in \mathbb{R}\right\}.$$

The support of the comonotonic random vector (X^c, Y^c, Z^c) is given by

$$\left\{\left(F_X^{-1}(p), F_Y^{-1}(p), F_Z^{-1}(p)\right) \mid 0 < p < 1\right\},$$

See Figure 5.2. Actually, not all of this support is depicted. The part left out corresponds to $p \notin (\Phi(-2), \Phi(2))$ and extends along the vertical asymptotes $(0, 0, z)$ and $\left(\frac{3}{2}, 1, z\right)$. The thick continuous line is the support of $\underline{X}^c$, while the dotted line is the straight line needed to transform this support into the connected support. Note that F_X has a horizontal segment between $\frac{1}{2}$ and 1. The projection of the connected curve along the $z-$axis can also be seen to constitute an in increasing curve, as projections along the other axes.

- **Discrete Distributions** [15]. We take $X \backsim$Uniform$\{0, 1, 2, 3\}$ and $Y \backsim$Binomial $\left(3, \frac{1}{2}\right)$. It is easy to verify that

$$\begin{aligned}
\left(F_X^{-1}(p), F_Y^{-1}(p)\right) &= (0,0) \text{ for } 0 < p < \frac{1}{8}, \\
&= (0,1) \text{ for } \frac{1}{8} < p < \frac{2}{8}, \\
&= (1,1) \text{ for } \frac{2}{8} < p < \frac{4}{8}, \\
&= (2,2) \text{ for } \frac{4}{8} < p < \frac{6}{8}, \\
&= (3,2) \text{ for } \frac{6}{8} < p < \frac{7}{8}, \\
&= (3,3) \text{ for } \frac{7}{8} < p < \frac{1}{8},
\end{aligned}$$

The support of (X^c, Y^c) is just these six points, and the connected support arises by simply connecting them consecutively with straight lines, the dotted lines in Figure 5.3. The straight line connecting $(1,1)$ and $(2,2)$ is not along, one of the axes. This happens because at level $p = \frac{1}{2}$, both $F_X(y)$ and $F_Y(y)$ have horizontal segments. Note that any non-decreasing curve connecting $(1,1)$ and $(2,2)$ would have led to a feasible connected curve. These two points have probability $\frac{2}{3}$, the other points $\frac{1}{8}$.

3.3 Sums of comonotonic random variables

Notice that S^c is the sum of the components of the comonotonic counterpart $(X_1^c, X_2^c, ..., X_n^c)$ of a random vector $(X_1, X_2, ..., X_n)$:

$$S^c = X_1^c + X_2^c + ... + X_n^c \tag{3.10}$$

In this section, we will prove the fowllowing theorems which we give the approximation of the distribution function of $S = X_1 + X_2 + ... + X_n$ by the distribution function of the comonotonic sum S^c is a prudent strategy in the sense that $S \leq_{cx} S^c$ and determining the marginal distribution functions of the terms in the sum.

In the next theorem we prove that the inverse disrtibution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions.

Theorem 3.6 *The $\alpha-$inverse distribution $F_{S^c}^{-1(\alpha)}$ of a sum S^c of comonotonic random variables $(X_1^c, X_2^c, ..., X_n^c)$ is given by*

$$F_{S^c}^{-1(\alpha)}(p) = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(p), \quad 0 < p < 1, \quad 0 \leq \alpha \leq 1. \tag{3.11}$$

Proof. Consider the random vector $(X_1, X_2, ..., X_n)$ and its comonotonic counterpart $(X_1^c, X_2^c, ..., X_n^c)$. Then $S^c = X_1^c + X_2^c + ... + X_n^c \overset{d}{=} g\,(U)\,,$ with U uniformly distributed on $(0, 1)$ and with the function g defined by

$$g\,(u) = \sum_{i=1}^{n} F_{X_i}^{-1}(u), \quad 0 < u < 1.$$

It is clear that g is non-decreasing and left-continuous. Application of Theorem 2.24(a) leads to

$$F_{S^c}^{-1}(p) = F_{g(U)}^{-1}(p) = g\left(F_U^{-1}(p)\right) = g(p), \quad 0 < p < 1,$$

So the inverse distribution function of S^c can be computed from

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \quad 0 < p < 1.$$

Similarly, from Theorem 2.24(b), we find that

$$F_{S^c}^{-1+}(p) = \sum_{i=1}^{n} F_{X_i}^{-1+}(p), \quad 0 < p < 1.$$

Multiplying the last two equalities by α and $1-\alpha$ respectively, and adding up, we find the desired result. □

Note that

$$S^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(U). \tag{3.12}$$

By the theorem above, we find that the connected support of S^c is given by

$$\left\{F_{S^c}^{-1(\alpha)}(p) \mid 0 < p < 1, 0 \leq \alpha \leq 1\right\}$$
$$\left\{\sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(p) \mid 0 < p < 1, 0 \leq \alpha \leq 1\right\}.$$

This implies

$$F_{S^c}^{-1+}(0) = \sum_{i=1}^{n} F_{X_i}^{-1+}(0), \tag{3.13}$$

$$F_{S^c}^{-1}(1) = \sum_{i=1}^{n} F_{X_i}^{-1}(1). \tag{3.14}$$

Hence, The minimal value of the comonotonic sum equals the sum of the minimal values of each term. Similarly, the maximal value of the comonotonic sum equals the sum of the maximal values of each term. The number $\sum_{i=1}^{n} F_{X_i}^{-1+}(0)$, which is either finite or $-\infty$ (if any the terms in the sum is $-\infty$), is the minimum possible value of S^c, and $\sum_{i=1}^{n} F_{X_i}^{-1}(1)$ is the maximum.

Also note that

$$\begin{aligned}
F_{S^c}^{-1+}(1) &= \sum_{i=1}^{n} F_{X_i}^{-1+}(1) = +\infty, \\
F_{S^c}^{-1}(0) &= \sum_{i=1}^{n} F_{X_i}^{-1+}(0) = -\infty.
\end{aligned}$$

For any $(X_1, X_2, ..., X_n)$, we have that $S = X_1 + X_2 + ... + X_n \geq \sum_{i=1}^{n} F_{X_i}^{-1+}(0)$ must hold with probability 1. This implies

$$\sum_{i=1}^{n} F_{X_i}^{-1+}(0) \leq F_S^{-1+}(0). \tag{3.15}$$

Similarly, we find

$$F_S^{-1}(1) \leq \sum_{i=1}^{n} F_{X_i}^{-1}(1). \tag{3.16}$$

This means that the sum S of the components of any random vector $(X_1, X_2, ..., X_n)$ has a support that is contained in the interval $\left[\sum_{i=1}^{n} F_{X_i}^{-1+}(0), \sum_{i=1}^{n} F_{X_i}^{-1}(1)\right]$. The minimal value of S is larger than or equal to the one of S^c, since by comonotonicity all terms of the latter are small simultaneously.

Given the inverse functions F_X^{-1}, the cdf of $S^c = X_1^c + X_2^c + ... + X_n^c$ can be determined as follows:

$$\begin{aligned} F_{S^c}(x) &= \sup\{p \in (0,1) \mid F_{S^c}(x) \geq p\} \qquad (3.17) \\ &= \sup\left\{p \in (0,1) \mid F_{S^c}^{-1}(p) \leq x\right\} \\ &= \sup\left\{p \in (0,1) \mid \sum_{i=1}^{n} F_{X_i}^{-1}(p) \leq x\right\}. \end{aligned}$$

In the sequel, for any random variables X, the expression "F_X increasing" should always be interpreted as "F_X is strictly increasing on $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$".

Observe that for any random variable X, the following equivalences hold:

$$F_X \text{ is strictly increasing} \Leftrightarrow F_X^{-1} \text{is continuous on } (0,1), \qquad (3.18)$$

and also

$$F_X \text{ is continuous} \Leftrightarrow F_X^{-1} \text{ is strictly increasing on } (0,1). \qquad (3.19)$$

Now assume that the marginal distribution functions F_{X_i},$i = 1, ..., n$ of the comonotonic random vector $(X_1^c, X_2^c, ..., X_n^c)$ are strictly increasing and continuous. Then each inverse distribution function $F_{X_i}^{-1}$ is continuous on $(0,1)$, wich implies that $F_{S^c}^{-1}$ is continuous on $(0,1)$ because $F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p)$ holds for $0 < p < 1$. This in turn implies that F_{S^c} is strictly increasing on $\left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$. Further, by a similar reasoning we find that F_{S^c} is continuous.

Hence, in case of strictly increasing and continuous marginals, for any $F_{S^c}^{-1+}(0) < x < F_{S^c}^{-1}(1)$, the probability $F_{S^c}(x)$ is uniquely determined by $F_{S^c}^{-1}(F_{S^c}(x)) = x$, or equivalently,

$$\sum_{i=1}^{n} F_{S^c}^{-1}\left(F_{S^c}(x)\right) = x, \quad F_{S^c}^{-1+}(0) < x < F_{S^c}^{-1}(1). \tag{3.20}$$

It suffices thus to solve the latter equation to get $F_{S^c}(x)$.

In the following theorem, we prove that also the stop-loss premiums of a sum of comonotonic random variables can be obtained from the stop-loss premimiums of the terms.

Theorem 3.7 *The stop-loss premiums of the sum S^c of the components of the comonotonic random vector $(X_1^c, X_2^c, ..., X_n^c)$ are given by*

$$\mathbb{E}\left[(S^c - d)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[(X_i - d_i)_+\right], \quad \left(F_{S^c}^{-1+}(0) < d < F_{S^c}^{-1}(1)\right), \tag{3.21}$$

with the d_i given by

$$d_i = F_{X_i}^{-1(\alpha_d)}(F_{S^c}(d)), \quad (i = 1, ..., n) \tag{3.22}$$

and $\alpha_d \in [0, 1]$ determined by

$$F_{S^c}^{-1(\alpha_d)}(F_{S^c}(d)) = d. \tag{3.23}$$

Proof. Let $d \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, hence $0 < F_{S^c}(d) < 1$.

As the connected support of $\underline{X}^c$ as defined in (3.9) is comonotonic, it can have at most one point of intersection with the hyperplane $\{\underline{x} \mid x_1 + ... + x_n = d\}$. This is because the hyperplane contains no different points $\underline{x}$ and $\underline{y}$ such that $\underline{x} \leq \underline{y}$ or $\underline{x} \geq \underline{y}$ holds.

Now we will prove that the vector $\underline{d} = (d_1, d_2, ..., d_n)$ as defined above is the unique point of this intersection. As $0 < F_{S^c}(d) < 1$ must hold, we know from Section (2.6) that there exists an $\alpha_d \in [0.1]$ that fulfils condition (3.23). Also note that by Theorem (3.6), we have that $\sum_{i=1}^{n} d_i = d$. Hence, the vector $\underline{d}$ with the d_i defined in (3.22) and (3.23) is an element of both the connected support of $\underline{X}^c$ and the hyperplane $\{\underline{x} \mid x_1 + ... + x_n = d\}$.

We can conclude that $\underline{d}$ is the unique element of the intersection of the connected support and the hyperplane. Let $\underline{x}$ be an element of the connected support of $\underline{X}^c$. Then the following equality holds:

$$(x_1 + x_2 + ... + x_n - d)_+ \equiv (x_1 - d_1)_+ + (x_2 - d_2)_+ + ... + (x_n - d_n)_+ .$$

This is because $\underline{x}$ and $\underline{d}$ are both elements of the connected support of $\underline{X}^c$, and hence, if there exists any j such that $x_j > d_j$ holds, then we also have $x_k \geq d_k$ for all k, and the left hand side equals the right hand side because $\sum_{i=1}^{n} d_i = d$. On the other hand, when all $x_j \leq d_j$, obviously the left hand side is 0 as well.

Now replacing constants by the corresponding random variables in the equality above and taking expectations, we find (3.21). □

Note that we also find that

$$\mathbb{E}\left[(S^c - d)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right] - d, \quad \text{if } d \leq F_{S^c}^{-1+}(0) \tag{3.24}$$

and

$$\mathbb{E}\left[(S^c - d)_+\right] = 0, \quad \text{if } d \geq F_{S^c}^{-1}(1). \tag{3.25}$$

So from (3.13), (3.14), (3.24), (3.25) and Theorem (3.7) we can conclude that for any real d, there exist d_i with $\sum_{i=1}^n d_i = d$, such that $\mathbb{E}\left[(S^c - d)_+\right] = \sum_{i=1}^n \mathbb{E}\left[(X_i - d_i)_+\right]$ holds.

The expression for the stop-loss premiums of a comonotonic sum S^c can also be written in terms of the usual inverse distribution functions. Indeed, for any retention $d \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(X_i - F_{X_i}^{-1(\alpha_d)}\left(F_{S^c}(d)\right)\right)_+\right] \\
&= \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)_+\right] - \left(F_{X_i}^{-1(\alpha_d)}\left(F_{S^c}(d)\right) - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)\left(1 - F_{S^c}(d)\right)
\end{aligned}$$

Summing over i, and taking into account the definition of α_d, we find the expression derived in Dhaene, Wang, Young & Goovaerts (2000), where the random variables were assumed to be non-negative. This expression holds for any retention $d \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$:

$$\begin{aligned}
\mathbb{E}\left[(S^c - d)_+\right] &= \sum_{i=1}^n \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)_+\right] \\
&\quad - \left(d - F_{S^c}^{-1}\left(F_{S^c}(d)\right)\right)\left(1 - F_{S^c}(d)\right).
\end{aligned} \tag{3.26}$$

In case the marginal cdf's F_{X_i} are strictly increasing, (3.26) reduces to

$$\mathbb{E}\left[(S^c - d)_+\right] = \sum_{i=1}^n \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)_+\right], \quad d \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right). \tag{3.27}$$

From Theorem (3.7), we can conclude that ant stop-loss premium of a sum of comonotonic random variables can be written as the sum of stop-loss premiums for the individual random variables involved. The theorem provided an algorithm for directly computing stop-loss premiums of sums of comonotonic random variables, without having to compute the stop-loss premium with retention d, we only need to know $F_{S^c}(d)$, which can be computed directly from (3.17).

Application of the relation $\mathbb{E}\left[(X-d)_+\right] = \mathbb{E}\left[(d-X)_+\right] + \mathbb{E}[X] - d$ for S^c and the X_i in relation (3.21) leads to the following expression for the lower tails of a sum of comonotonic random variables:

$$\mathbb{E}\left[(d-S^c)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[(d_i - X_i)\right], \quad F_{S^c}^{-1+}(0) < d < F_{S^c}^{-1}(1), \tag{3.28}$$

with the d_i as defined in (3.22) and (3.23).

The comonotonic upper bound for $\sum_{i=1}^{n} X_i$

Theorem 3.8 ([15]) *For any vector $(X_1, X_2, ..., X_n)$ we have*

$$X_1 + X_2 + ... + X_n \leq_{cx} X_1^c + X_2^c + ... + X_n^c. \tag{3.29}$$

3.4 The New Results

The main results of this work are the following theorem, and proposition.

3.4.1 Convex bounds and the comonotonic upper bound for S_N

In risk theory and finance, one is often interested in distribution of the sums $S = X_1 + ... + X_n$ or the form $S_N = X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n)$(our model) of individual risks of a portfolio **X**. In this subsection we give a short overview of these stochastic ordering results. For proofs and more details on the presented results, we refer to the overview paper of Dhaene et al.[9] and Zeghdoudi and Remita [56].

Theorem 3.9 (M.Bouhadjar et al.) *We note that:*

$$\tilde{S}_N = \tilde{X}_1 f(Y_1) + \tilde{X}_2 f(Y_2) + ... + \tilde{X}_n f(Y_n) \tag{3.30}$$

For any random vector $X = (X_1, ..., X_n)$ *and* $f(Y_i), i = 1, ..., n$ *we have*

$$S_N \leq_{cx} \tilde{S}_N. \tag{3.31}$$

Proof. It is suffices to prove stop-loss order because $\mathbb{E}(S_N) = \mathbb{E}\left(\tilde{S}_N\right)$. Hence, we have to prove that

$$\mathbb{E}[(S_N - d)_+] \leq \mathbb{E}[(\tilde{S}_N - d)_+]$$

The following holds for all $(X_1 f(Y_1), X_2 f(Y_2), ..., X_n f(Y_n))$ when $d_1 + d_2 + ... + d_n = d$

$$\begin{aligned}
& (X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n) - d)_+ \\
= \; & (X_1 f(Y_1) - d_1 + X_2 f(Y_2) - d_2 + ... + X_n f(Y_n) - d_n)_+ \\
\leq \; & \left((X_1 f(Y_1) - d_1)_+ + (X_2 f(Y_2) - d_2)_+ + ... + (X_n f(Y_n) - d_n)_+\right)_+ \\
= \; & (X_1 f(Y_1) - d_1)_+ + (X_2 f(Y_2) - d_2)_+ + ... + (X_n f(Y_n) - d_n)_+
\end{aligned}$$

Now taking expectations, we get that

$$\mathbb{E}\left[(X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n) - d)_+\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[(X_i f(Y_i) - d_i)_+\right]$$

According to [15] we have

$$\mathbb{E}[(\tilde{S}_N - d)_+] = \sum_{i=1}^{n} \mathbb{E}\left[(X_i f(Y_i) - d_i)_+\right]$$

Then,

$$S_N \leq_{cx} \tilde{S}_N.$$

□

Proposition 3.10 *For any random vector $X = (X_1, ..., X_n)$, any random variable Λ and for $U \backsim Uniform(0,1)$, which is assumed to be a function of X and for $f(Y_i) \geq 1, i = 1, ..., n$, we have,*

(a)

$$S \leq_{cx} S_N \tag{3.32}$$

(b)

$$\tilde{S} \leq_{cx} \tilde{S}_N \tag{3.33}$$

(c)

$$\sum_{j=1}^{n} \mathbb{E}\left[X_i \mid \Lambda\right] \leq_{cx} S_N \tag{3.34}$$

(d)

$$\sum_{j=1}^{n} \mathbb{E}\left[\tilde{X}_i \mid \Lambda\right] \leq_{cx} \tilde{S}_N \tag{3.35}$$

Proof.

(**a**) We have $f(Y_i) \geq 1, i = 1, ..., n$ and we used property 10 and 6, we obtain

$$X_1 + X_2 + ... + X_n \leq_{cx} X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n)$$

thus

$$S \leq_{cx} S_N.$$

(**b**) We will omit the proof here because the idea is very similar to the proof in (**a**).

(**c**) According to Dhaene et al.[9] we have, $\sum_{j=1}^{n} \mathbb{E}\left[X_i \mid \Lambda\right] \leq_{cx} S$ and (**a**), we deduce that

$$\sum_{j=1}^{n} \mathbb{E}\left[X_i \mid \Lambda\right] \leq_{cx} S_N.$$

(**d**) According to Zeghdoudi and Remita [56] we have $\sum_{j=1}^{n} \mathbb{E}\left[\tilde{X}_i \mid \Lambda\right] \leq_{cx} \tilde{S}$, using property 12 and (**b**), we obtain

$$\sum_{j=1}^{n} \mathbb{E}\left[\tilde{X}_i \mid \Lambda\right] \leq_{cx} \tilde{S}_N.$$

□

In addition, if $f(Y_i) \leq 1, i = 1, ..., n$, we can check easily that

$$S_N \leq_{cx} \tilde{S}_N \leq_{cx} S \leq_{cx} \tilde{S}. \tag{3.36}$$

Chapter 4

Policy Limits and Deductibles

If the sum of policy limits(deductible) is fixed, then $X_i \leq_{st} X_j \Rightarrow\ l_i^* \leq l_j^*$ and $d_i^* \geq d_j^*$ when ($X_1, X_2, ..., X_n$) is comonotonic, where l_i^* : optimal policy limit and d_i^* : optimal deductible allocated to i-th risk.

In this chapter we present the problem of the optimal allocation of policy limits and deductibles. For make the new general model analytically tractable, we will make the following assumptions :

1. the policyholder is risk-averse, and therefore the utility function is increasing and concave;

2. the random vector $\mathbf{X} = (X_1, ..., X_n)$, which represents the loss severities, and random vector $\mathbf{Y} = (Y_1, ..., Y_n)$, which represents the time of occurrence of losses, are independent; moreover, $Y_1, ..., Y_n$ are mutually independent;

3. dependence structure of the severities of the risks is unknown.

4.1 Policy limits with unknown dependent structures

The first problem to be considered is to maximize the expected utility of wealth:

$$\max_{\mathbf{l}\in\mathcal{A}_n(l)}\min_{\mathbf{X}\in\mathcal{R}}\mathbb{E}\left[u\left(w-\sum_{i=1}^{n}\left[X_i-(X_i\wedge l_i)_+\right]f(Y_i)\right)\right] \tag{4.1}$$

where u and w are the utility function (increasing and concave), the wealth (after premium) respectively and $\tilde{u}$ is an increasing convex function. The problem is equivalent to

$$\min_{\mathbf{l}\in\mathcal{A}_n(l)}\max_{\mathbf{X}\in\mathcal{R}}\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+f(Y_i)\right)\right] \tag{4.2}$$

Lemma 4.1 (B) *If $(\tilde{X}_1,...,\tilde{X}_n)\in\mathcal{R}$ comonotonic, then*

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+f(Y_i)\right)\right]\leq\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+f(Y_i)\right)\right] \tag{4.3}$$

for any $(l_1,...,l_n)\in\mathcal{A}_n(l)$ and $(X_1,...,X_n)\in\mathcal{R}$ independent of $\mathbf{Y}$.

Proof. Let $\tilde{X}=(\tilde{X}_1,...,\tilde{X}_n)\in\mathcal{R}$ be comonotonic and independent of $\mathbf{Y}$. For any fixed constants $y_1,...,y_n$, Theorem (3.4) implies that

$$\left(\left(\tilde{X}_1-l_1\right)_+f(y_1),...,\left(\tilde{X}_n-l_n\right)_+f(y_n)\right)$$

is still comonotonic. Therefore, by Theorem (3.8) and Theorem (3.9), we have

$$\sum_{i=1}^{n}(X_i-l_i)_+f(y_i)\leq_{cx}\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+f(y_i)$$

because $\tilde{u}$ is increasing and convex. Then by the independence of $\mathbf{X}$ and $\mathbf{Y}$,

$$\begin{aligned}
\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(Y_i)\right)\right] &= \mathbb{E}\left[\mathbb{E}\left\{\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(Y_i)\right) \mid Y_1,...,Y_n\right\}\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left\{\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(Y_i)\right) \mid Y_1,...,Y_n\right\}\right] \\
&= \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(Y_i)\right)\right].
\end{aligned}$$

and hence

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(y_i)\right)\right] \leq \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(y_i)\right)\right]$$

□

Now, the initial problem becomes

Problem L' : $\left\{ \min_{\mathbf{l}\in\mathcal{A}_n(l)} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(Y_i)\right)\right]\right.$

Proposition 4.2 *Let $\mathbf{l}^* = (l_1^*,...,l_n^*)$ be the solution to Problem L', then*

$$Y_i \geq_{lr} Y_j, X_i \leq_{st} X_j \Rightarrow l_i^* \leq l_j^*. \tag{4.4}$$

Proof. Assume that $l_i \leq l_j$. Since $x \to f(Y_i)$ is decreasing , by property 13

$$Y_i \geq_{lr} Y_j \Rightarrow f(Y_i) \leq_{lr} f(Y_j)$$

Since (X_i, X_j) is comonotonic and $X_i \leq_{st} X_j$, $X_i(\omega) \leq X_j(\omega)$ for any $\omega \in \Omega$. By the independence of $\mathbf{X}$ and $\mathbf{Y}$, we can hereafter fix an outcome of $(X_1,...,X_i,...,X_j,...,X_n)$ as $(x_1,...,x_i,...,x_j,...,x_n)$ with $x_i \leq x_j$. As $g(\mathbf{x},\mathbf{l}) = -\sum_{i=1}^{n}(x_i-l_i)_+$ is an AI function by Lemma (2.10) and the function $(x,l) \to -(x-l)_+$ is increasing in l but decreasing in x, then by Lemma (2.12)

$$((x_i - l_i)_+, (x_j - l_j)_+) \prec\prec ((x_i - l_j)_+, (x_j - l_i)_+)$$

Since wehave $(x_i - l_j)_+ \leq (x_j - l_i)_+$, then by property 14 we have

$$(x_i - l_i)_+ f(Y_i) + (x_j - l_j)_+ f(Y_j) \leq_{icx} (x_i - l_j)_+ f(Y_i) + (x_j - l_i)_+ f(Y_j).$$

Morever, for the increasing convex function $\tilde{u}$,

$$\begin{aligned} & \mathbb{E}\left[\tilde{u}(\left(x_i - l_i)_+ f(Y_i) + (x_j - l_j)_+ f(Y_j) + \sum_{k \neq i,j} (x_k - l_k)_+ f(Y_k)\right)\right] \\ \leq \ & \mathbb{E}\left[\tilde{u}\left((x_i - l_j)_+ f(Y_i) + (x_j - l_i)_+ f(Y_j) + \sum_{k \neq i,j} (x_k - l_k)_+ f(Y_k)\right)\right] \end{aligned}$$

By taking expectations conditional on $\mathbf{X}$, we obtain

$$\begin{aligned} & \mathbb{E}\left[\tilde{u}\left((X_i - l_i)_+ f(Y_i) + (X_j - l_j)_+ f(Y_j) + \sum_{k \neq i,j} (X_k - l_k)_+ f(Y_k)\right)\right] \\ \leq \ & \mathbb{E}\left[\tilde{u}\left((X_i - l_j)_+ f(Y_i) + (X_j - l_i)_+ f(Y_j) + \sum_{k \neq i,j} (X_k - l_k)_+ f(Y_k)\right)\right] \end{aligned}$$

The result follows. $\square$

4.2 Policy deductibles with unknown dependent structures

The same thing is made that the of policy limits, we consider the problem of the optimal allocation of deductibles :

$$\max_{\mathbf{d}\in\mathcal{A}_n(d)}\min_{\mathbf{X}\in\mathcal{R}} \mathbb{E}\left[u\left(w-\sum_{i=1}^{n}\left[X_i-(X_i-d_i)_+\right]f(Y_i)\right)\right] \tag{4.5}$$

which we have

$$\min_{\mathbf{d}\in\mathcal{A}_n(d)}\max_{\mathbf{X}\in\mathcal{R}} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i\wedge d_i)_+ f(Y_i)\right)\right] \tag{4.6}$$

Lemma 4.3 *If $(\tilde{X}_1,...,\tilde{X}_n)\in\mathcal{R}$ is comonotonic and independent of $\mathbf{Y}$, then*

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i\wedge d_i)_+ f(Y_i)\right)\right] \leq \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i\wedge d_i\right)_+ f(Y_i)\right)\right] \tag{4.7}$$

for any $(d_1,...,d_n)\in\mathcal{A}_n(d)$ and $(X_1,...,X_n)\in\mathcal{R}$ independent of $\mathbf{Y}$.

The proof is similar to the proof of Lemma B.

From the above lemma, our problem becomes

Problem D' : $\left\{ \min_{\mathbf{d}\in\mathcal{A}_n(d)} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i\wedge d_i)_+ f(Y_i)\right)\right]\right.$

Proposition 4.4 *Let $\mathbf{d}^*=(d_1^*,...,d_n^*)$ be the solution to Problem D', then*

$$Y_i \geq_{lr} Y_j, X_i \leq_{st} X_j \Rightarrow d_i^* \geq d_j^*. \tag{4.8}$$

Proof. Assume that $d_i \geq d_j$. As in the proof of Proposition (4.2), we have

$$Y_i \geq_{lr} Y_j \Rightarrow f(Y_i) \leq_{lr} f(Y_j),$$

And we can fix an outcome of $(X_1,...,X_i,...,X_j,...,X_n)$ as $(x_1,...,x_i,...,x_j,...,x_n)$ with $x_i \leq x_j$. As $g(\mathbf{x},\mathbf{d})=\sum_{i=1}^{n}(x_i\wedge d_i)$ is an AI function by Lemma (2.10) and the function $(x, d) \to x \wedge d$ is increasing both in x and d, then by Lemma (2.11),

$$((x_i \wedge d_i), (x_j \wedge d_j)) \prec\prec ((x_i \wedge d_j), (x_j \wedge d_i)).$$

Since we also have $(x_i \wedge d_j) \leq (x_j \wedge d_i)$, then by property 14 we have

$$(x_i \wedge d_i)f(Y_i) + (x_j \wedge d_j)f(Y_j) \leq_{icx} (x_i \wedge d_j)f(Y_i) + (x_j \wedge d_i)f(Y_j)$$

By independence convolution, we have

$$\begin{aligned} & (x_i \wedge d_i)f(Y_i) + (x_j \wedge d_j)f(Y_j) + \sum_{k \neq i,j} (x_k \wedge d_k)_+ f(Y_k) \\ \leq & {}_{icx} (x_i \wedge d_j)f(Y_i) + (x_j \wedge d_i)f(Y_j) + \sum_{k \neq i,j} (x_k \wedge d_k)_+ f(Y_k) \end{aligned}$$

Morever, for the increasing convex function $\tilde{u}$,

$$\begin{aligned} & \mathbb{E}\left[\tilde{u}\left((x_i \wedge d_i)f(Y_i) + (x_j \wedge d_j)f(Y_j) + \sum_{k \neq i,j} (x_k \wedge d_k)_+ f(Y_k)\right)\right] \\ \leq \; & \mathbb{E}\left[\tilde{u}\left((x_i \wedge d_j)f(Y_i) + (x_j \wedge d_i)f(Y_j) + \sum_{k \neq i,j} (x_k \wedge d_k)_+ f(Y_k)\right)\right] \end{aligned}$$

With a same manner we find the result on X. $\square$

4.3 Some examples and Application

In this section we will describe several examples that show how distribution function of the sum of random variables can be approximated by convex order of random variable (see Rüschendorf [48]) for lower convex order of random variables and comparison of two families of copulas.

4.3.1 Lower Bound Approximations of the Distribution Sum of Random Variables with Convex Ordering

Example 4.5 *(Approximation of distribution sum of two independent standard normal random variables)[22]*

Suppose X and Y be independent $N(0,1)$ random variables. We want to derive lower bounds for $S = X + Y$. In this case we know the exact distribution of S, i.e $S \backsim N(0,2)$. Let us see how lower bound approximation works in this case. Let $Z = X + aY$ for some real a. Then $Z \backsim N(0, 1 + a^2)$. Therefore, for some choices of a, we get the following distribution for the lower bound for S :

$$
\begin{aligned}
a &= 0 \text{ gives } N(0,1) \leq_{cx} S = X + Y \backsim N(0,2) \\
a &= 1 \text{ gives } N(0,2) \leq_{cx} S = X + Y \backsim N(0,2) \\
a &= -1 \text{ gives } N(0,2) \leq_{cx} S = X + Y \backsim N(0,2)
\end{aligned}
$$

Thus in this case best lower bound is obtained for $a = 1$ which is the exact distribution. The variance of the lower bound can be seen to have a maximum at $a = 1$ and a minimum at $a = -1$.

Example 4.6 *[15]*

As a theoretical example, consider an insurance portfolio consisting of n risks. The payments to be made by the insurer are described by a random vector $(X_1 +$

$X_2 + ... + X_n$), where X_i is the claim amount of policy i during the insurance period. We assume that all payments have to be done at the end of the insurance period $[0, 1]$. In a deterministic financialsetting, the present value at time 0 of the aggregate claims $X_1 + X_2 + ... + X_n$ to be paid by the insurer at time 1 is determined by

$$S = (X_1 + X_2 + ... + X_n)\upsilon$$

where $\upsilon = (1+r)^{-1}$ is the deterministic discount factor and r is the technical interest rate. This will be chosen in a conservative way (i.e.sufficiently low), if the insurer doesn't want to underestimate his future obligations. To demonstrate the effect of introducing random interest on insurance business, we look at the following special case. Assume all risks X_i to be non-negative, independent and identically distributed,and let $X \stackrel{d}{=} X_i$, where the symbol $\stackrel{d}{=}$ is used to indicate equality in distribution. The average payment $\frac{S}{n}$ has mean and variance

$$\mathbb{E}\left(\frac{S}{n}\right) = \upsilon\mathbb{E}(X); \mathbb{V}\left(\frac{S}{n}\right) = \frac{\upsilon^2}{n}\mathbb{V}(X)$$

The stability necessary for both insureds and insurer is maintained by the Law of Large Numbers, provided that n is indeed 'large'and that the risks are mutually independent and rather well-behaved, not describing for instance risks of catastrophic nature for which the variance might be very large or even infinite.

Example 4.7 *[15]*

Let us examine the consequences of introducing stochastic discounting. Replacing the fixed discount factor v by a random variable Y, representing the stochastic amount to be invested at time 0 with value 1 at the end of the period $[0,1]$, the present value of the aggregate claims becomes

$$S = (X_1 + X_2 + ... + X_n)Y$$

If we assume that the discount factor is independent of the payments,we find that the average payment per policy S_n has mean and variance

$$\mathbb{E}\left(\frac{S}{n}\right) = \mathbb{E}(X)\mathbb{E}(Y); \mathbb{V}\left(\frac{S}{n}\right) = \frac{\mathbb{V}(X)}{n}\mathbb{E}(Y^2) + \mathbb{E}^2(\mathbb{X})\mathbb{V}(Y)$$

Assuming that $\mathbb{E}[X]$ and $\mathbb{V}[Y]$ are positive, the Law of Large Numbers no longer eliminates the risk involved. This is because for $n \to \infty, \mathbb{V}\left[\frac{S}{n}\right]$ converges to its second term. So to evaluate the total risk, both the distributions of insurance risk and financial risk are needed. Risk pooling and large portfolios are no longer sufficient tools to eliminate or reduce the average risk associated with a portfolio. This observation implies that the introduction of stochastic financial aspects in actuarial models immediately leads to the necessity of determining distribution functions of sums of dependent random variables. Under the assumption that the vectors $\underline{X} = (X_1 + X_2 + ... + X_n)$ and $\underline{Y} = (Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$ are mutually independent and that the marginal distributions of the X_i and the $Y_{t_i}$are given, the problem of determining bounds for the distribution function of $S = \sum_{i=1}^{n} X_i Y_{t_i}$ can be reduced to determining bounds for the distribution function of a sum $S = Z_1 + Z_2 + ... + Z_n$of random variables $Z_1, Z_2, ..., Z_n$ with given marginaldistributions, but of which the joint distribution is either unspecified or too cumbersome to work with. The unknown or complex nature of the dependence between the random variables Z_i is the reason why it is impossible to derive the distribution function of S exactly.

4.3.2 Individual and collective risk model

The classical individual and collective model of risk theory has the form $X_{Ind} = \sum_{i=1}^{n} b_i I_i$, $X_{Coll} = \sum_{i=1}^{n} b_i N_i$, where $I_i \backsim Bernoulli(p_i)$ and $N_i \backsim poisson(\lambda_i)$. With probability p_i contract i will yield a claim of size $b_i \geq 0$ for any of the n policies. As an application of stochastic and stop-loss ordering we get that the collective risk model X_{Coll} leads to an overestimate of the risks and, therefore, also to an increase of the corresponding risk premiums for the whole portfolio

$$X_{Ind} \leq_{sl(cx)} X_{Coll}.$$

4.3.3 Reinsurance contracts

We consider reinsurance contracts $I(X)$ for a risk X, where $0 \leq I(X) \leq X$ is the reinsured part of the risk X and $X - I(X)$ is the retained risk of the insurer. Consider the stop-loss reinsurance contract $I_a(X) = (X - a)_+$, where a is chosen such that $EI_a(X) = EI(X)$. Then for any reinsurance contract $I(X)$

$$X - I_a(X) \leq_{sl(cx)} X - I(X).$$

4.3.4 Dependent portfolios increase risk

Let $Y_i = \sum_{i=1}^m \alpha_i X_i$, where α_i and $X_i \backsim Bernoulli$ with $\sum\limits_{i=1}^{m} \alpha_i = 1$,then $Y_i \backsim$ $Bernoulli$. It is interesting to compare the total risk $T_n = \sum_{i=1}^n Y_i$ in the mixed model (X_i) with the total risk $S_n = \sum_{i=1}^n W_i$ in an independent portfolio model (W_i), where $W_i \backsim Bernoulli$ are distributed identical to X_i. Then we obtain

$$S_n \leq_{sl(cx)} T_n.$$

4.3.5 Applications of the theory of comonotonicity

Derivatives pricing and hedging

Several European options have a pay-off written on one or multiple underlyings combined in a weighted sum of non-independent random variables expressing asset prices at the time of maturity or at different time points before and at maturity. Examples of this type of options with positive weights are Asian options, basket options and Asian basket options. When the weights can be both positive and negative, one refers to these options as spread options, Asian spread options, basket spread options and Asian basket spread options. Pricing and hedging of these products by means of comonotonicity bounds has been studied in a model dependent as well as in a model independent framework. As mentioned before, early references to this topic are Rogers and Shi (1995), Simon et al.(2000) and Dhaene et al.(2002).

Risk management : risk sharing, optimal investment, capital allocation

Dhaene et al.(2009a) investigate the influence of the dependence between random losses on the shortfall and on the diversification benefit that arises from merging these losses. They prove that increasing the dependence between losses, expressed in terms of correlation order, has an increasing e ect on the shortfall, expressed in terms of an appropriate integral stochastic order. Furthermore, increasing the dependence between losses decreases the diversification benefit. In particular, they consider merging comonotonic losses and show that even in this extreme case a non-negative diversification benefit may arise. Also, Embrechts et al.(2005) prove that comonotonicity gives rise to the on-average-most-adverse Value-at-Risk (VaR) scenario for a function of dependent risks, when the marginal distributions are known but the dependence structure between the risks is unknown. Dhaene et al.(2005) investigate multiperiod portfolio selection problems in a Black and Scholes type market where a basket of one risk free and m risky securities are traded continuously. They look for the optimal allocation of wealth within the class of constant-mix portfolios.

The Enterprise Risk Management process of a financial institution usually contains a procedure to allocate, or subdivide, the total risk capital of the company into its different business units. In Dhaene et al.(2003), an optimization argument is used to nd an optimal rule for allocating the aggregate capital of a financial firm to its business units The optimal allocation can be found using general results from the theory on comonotonicity. Dhaene et al.(2009b) generalize the approach of Dhaene et al.(2003) and develop a unifying framework for allocating the aggregate capital by considering more general deviation measures.

Life Insurance and pensions

In the classical approach to the theory of life contingencies, discounting factors and mortality tables are assumed to be deterministic. In view of the long durations of life annuity contracts it is more realistic to take the stochastic nature of investment returns and mortality into account when investigating the risks related to annuity portfolios. Over the last two decades, a large number of papers have been published covering this stochastic approach of returns and/or mortality. For more details we can see: (Koch and De Schepper (2007), Darkiewicz et al.(2009), Hoedemakers et al.(2005) and Ahcan et al.(2006), Zhang et al.(2006), Denuit and Dhaene (2007) and Denuit (2007, 2008, 2009), Spreeuw (2006)).

4.3.6 Comparison of two families of copulas

Definition 4.8 (copulas) *$C(u_1, ..., u_n)$ is distribution function whose marginals are all uniformly distributed, (See Nelson [45]).*

Now we consider two risks X and Y with given survival functions $\bar{F}$ and $\bar{G}$. A sufficient condition of the stop-loss order is given by:

Cut-criterion(Karlin and Novikoff [35]). Let X and Y be two risks with $\mathbb{E}\,(X) \leq \mathbb{E}\,(Y)$. If there exists a constant c such that

$$\begin{cases} \bar{F}\,(x) \geq \bar{G}\,(x) & \text{for all} \quad x < c, \\ \bar{F}\,(x) \leq \bar{G}\,(x) & \text{for all} \quad x \geq c, \end{cases}$$

then

$$X \preceq_{st} Y$$

Definition 4.9 (Bivariate orthant convex order) *Given non-negative random vectors* $\mathbf{X} = (X_1, X_2)$ *and* $\mathbf{Y} = (Y_1, Y_2)$*. We say that* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the orthant convex order denoted as* $\mathbf{X} \preceq_{uo-cx} \mathbf{Y}$ *if the inequalities*

$$\mathbb{E}\,[v_1(X_1)v_2(X)_2] \leq \ \mathbb{E}\,[v_1(Y_1)v_2(Y)_2]$$

holds for all non-decreasing convex function v_1 *and* v_2 *.*

Characterization. $\mathbf{X} \preceq_{uo-cx} \mathbf{Y}$ if and only, if

1. $\mathbb{E}[(X_i - d_i)_+] \leq \mathbb{E}[(Y_i - d_i)_+]$ for all $d_i > 0$, $i = 1, 2$

2. $\mathbb{E}[(X_1 - d_1)_+(X_2 - d_2)_+] \leq \mathbb{E}[(Y_1 - d_1)_+(Y_2 - d_2)_+]$ for all $d_1, d_2 > 0$.

Consequently:

$$\mathbf{X} \preceq_{uo-cx} \mathbf{Y} \Rightarrow X_i \preceq_{st} Y_i, i = 1, 2.$$

This shows that $\preceq_{uo-cx}$can be viewed as bivariate extension of stop-loss order.

Crossing condition for the bivariate orthant convex order

Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be non-negative random vectors with survival functions $\bar{F}$ and $\bar{G}$. Let h be a level curve defined by

$$\bar{F}\left(x, h\left(x\right)\right) - \bar{G}\left(x, h\left(x\right)\right) = 0, x \geq 0.$$

Let

$$C = \left\{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : y \leq h\left(x\right)\right\}$$

we denote by $\bar{C}$ the complement of C in $\mathbb{R}^+ \times \mathbb{R}^+$.

Comparison of two families of copulas

The concordance order is used to compare members of a given copula family C_θ when the dependence parameter varies:

$$\theta_1 \leq \theta_2 \Rightarrow C_{\theta_1} \preceq_C C_{\theta_2}$$

Remark 4.10 *There is no comparison between a copulas from different families with* $\preceq_C$:

$$C_{\theta_1} \npreceq_c C_{\theta_2} \text{ and } C_{\theta_2} \npreceq_C C_{\theta_1}$$

Example 4.11 *Let C_{θ_1} be a Clayton copula with parameter $\theta_1 = 1$ and C_{θ_2} be a Frank copula with parameter $\theta_2 = 2$. Since $\preceq_{uo-cx}$ is weaker than $\preceq_C$. Thus one can expect to rank the copulas C_{θ_1} and C_{θ_2} with respect to $\preceq_{uo-cx}$ instead of $\preceq_C$. Therefore, one* *can use our cut-criterion to establish a such comparison with respect* $\preceq_{uo-cx}$. *To this end, we can see that* $C_{\theta_1} \preceq_{uo-cx} C_{\theta_2}$. *This means that the upper orthant convex order can be more convenient for compare the concordance between two different families of copulas.*

Conclusion and Perspectives

In this work, we give a synthesis on the theory of stochastic orderings, comonotonicity and their applications. Also, we study the problems of optimal allocation of policy limits and deductibles. By using some characterizations of stochastic ordering relations, we reconsider the new general model and obtain some new results on orderings of optimal allocations of policy limits and deductibles. Moreover, we obtain an convex upper and lower bound in terms of comonotonic portfolios for $S_N = X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n)$(our model).

For future studies, we may try to explore the following directions. First, we can relax the condition imposed on $f(Y_i)$ and introduce financial risks to the model. Second, we can remakes same work for obtain the optimal allocation of policy limits and deductibles in a model with mixture and discount factors.

Chapter 5

Appendix

Proof of Theorem 3.5

The proof of the "$\Rightarrow$"-implication is straightforward.

For the proof of the "$\Leftarrow$"-implication, consider the set A in $\mathbb{R}^n$ defined by

$$A = \left\{ \left(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p), ..., F_{X_n}^{-1}(p) \mid 0 < p < 1 \right) \right\}$$

Its (i, j)-projections are give by

$$A_{i,j} = \left\{ \left(F_{X_i}^{-1}(p), F_{X_j}^{-1}(p) \right) \right\}$$

The event "$\underline{X} \in A$" is equivalent with the event "$(X_i, X_j) \in A_{i,j}$ for all (i, j)". Because of the comonotonicity of the pairs (X_i, X_j), the latter event is the certain event. Hence we find that $\Pr\left[\underline{X} \subseteq A\right] = 1$, so that the comonotonic random vector.

$\square$

The theorem states that comonotonicity of a random vector is equivalent with pairwise comonotonicity.

Consider the random vector $(U, 1, V)$ where U and V are mutually independent random variables that are both uniformly distributed on the unitinterval $(0, 1)$. It is clear that $(U, 1)$ and $(1, V)$ are both comonotonic pairs, but $(U, 1, V)$ isn't comonotonic. Hence, for a general random vector $\underline{X}$, comonotonicity of the pairs (X_i, X_{i+1}), $(i = 1, 2, ..., n-1)$, will not necessary imply comonotonicity of $\underline{X}$.

Figures

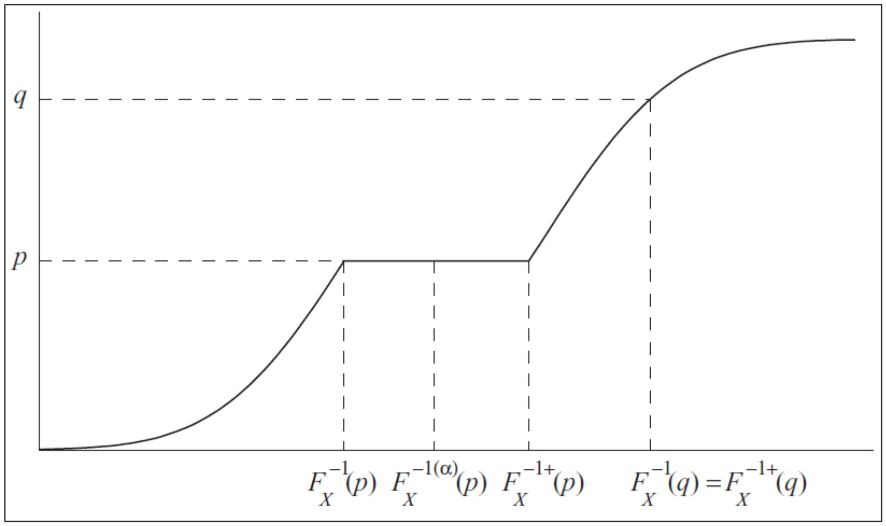

Figure 5.1: Graphical definition of F_X^{-1}, F_X^{-1+} and $F_X^{-1(\alpha)}$.

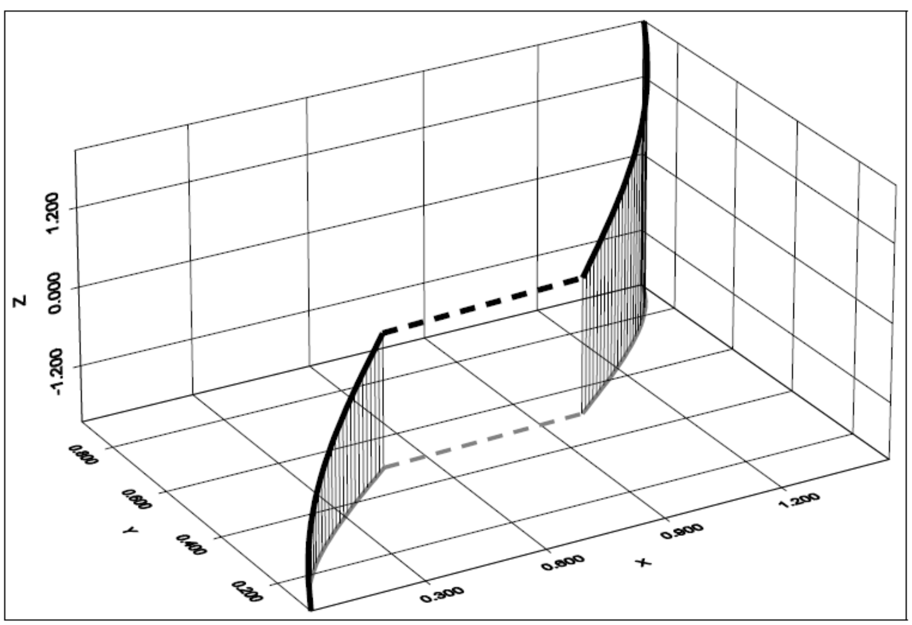

Figure 5.2: A continuous example with $n = 3$.

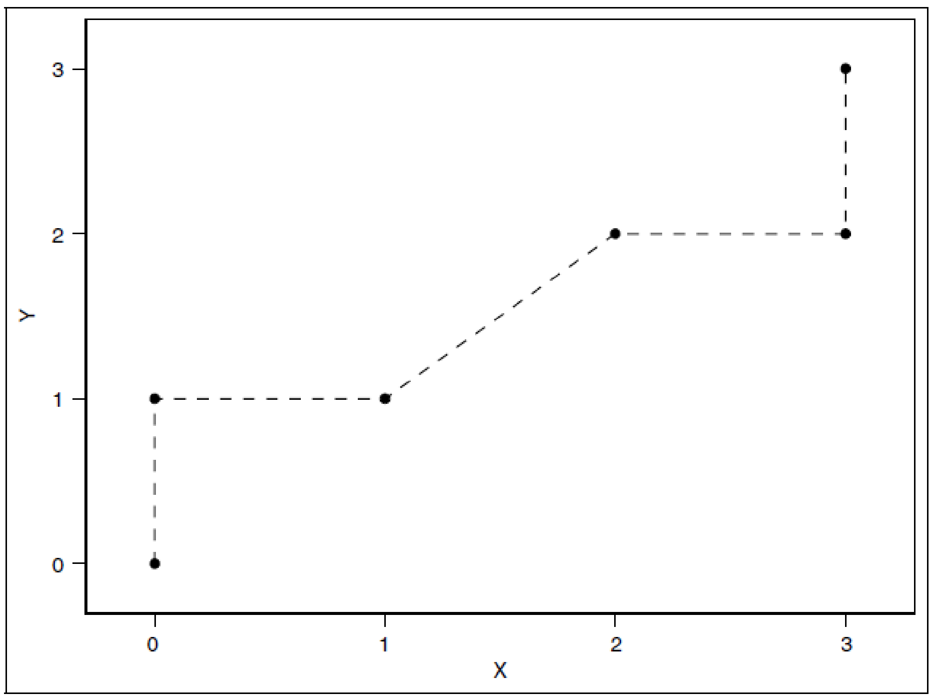

Figure 5.3: A discrete example.

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