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ASYMPTOTIC BEHAVIOR IN SOME POROUS THERMOELASTIC SYSTEMS

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PhD. Thesis presented by

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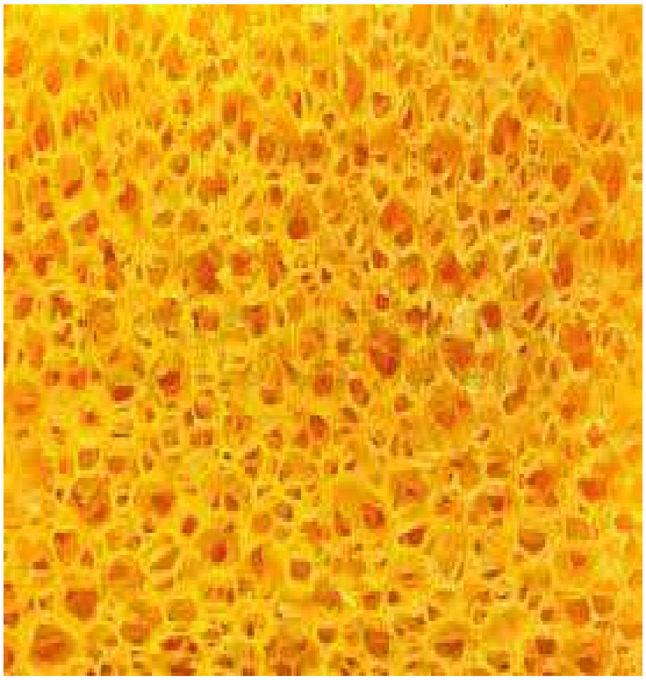

To my mother,

my wife

and the ame of my father

which dead when I have eight years.

Contents

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Published papers

1) General decay for a porous thermoelastic system with memory:the case of equal speeds, Salim A. Messaoudi, Abdelfeteh Fareh, Nonlinear Analysis, TMA, **74** (2011), 6895-6905.

2) Energy decay for a Timoshenko-type ot thermoelasticity type III with different propagation speeds, Salim A. Messaoudi, Abdelfeteh Fareh, Arabian Journal of mathematics **1** (2013).

3) General decay for a porous thermoelastic systemwith memory: The case of non-equal speeds, Salim A. Messaoudi, Abdelfeteh Fareh, Acta Mathematica Scientia, **33** (2013) 23-40.

4) Energy decay for a linear damped porous thermoelastic system of type III, Salim A. Messaoudi, Abdelfeteh Fareh, Prociding of International colloquy of Mathematics, Al Ain 11-14 March 2012.

Chapter 1

Introduction

The subject of this thesis is the study of the asymptotic behavior of solutions of some porous thermoelastic problems. In this regards, several results concerning decay of solutions in classical porous thermoelasticity as well as thermoelasticity of type III have been established. This study extends and improves several earlier results. We begin by a short summary of the theory of porous thermoelasticity and thermoelasticity of type III.

Porous thermoelasticity

The theory of porous materials is an important generalization of the classical theory of elasticity for the treatment of porous solids in which the skeletal materials is thermoelastic and the interstices are void of material. This theory deals with materials containing small pores or voids. The basic premise underlying this theory is the concept that the bulk density is the product of two fields, the matrix material density field and the volume fraction field. This representation of the bulk density introduces an additional degree of kinematic freedom in the theory and was employed previously by Goodman and Cowin [13] to overcome the failure of the classical theory of elasticity to describe the deformation produced by the microstructure contribution. The theory of granular materials developed by Goodeman and Cowin [13], equally valid for porous materials, was motivated by physical grounds. In this theory they introduced a higher order stress and body force to account for energy flux and energy supply associated with the time rate of volume fraction. Terms of this type are also contained in the higher order elasticity theories developed by Mindlin [44], Toupin [67] and Green and Rivlin [19].

Nunziato and Cowin [50] employed the same balance equations developed by Goodman and Cowin [13] and presented a nonlinear theory for the behavior of porous solids. This theory admits both finite deformations and nonlinear constitutive relations. Jarić and Golubović [29] and Jarić and Ranković [30] studied the nonlinear theory of thermoelastic materials with voids. Cowin and Nunziato [9] developed a linear theory of elastic materials with voids to study mathematically the mechanical behavior of porous solids. An extension of this theory to linear thermoelastic bodies was proposed by Ieşan [24]. In addition, Ieşan [25],[26] added the microtemperature elements to this theory.

On the basis of micromorphic continua theory, Grot [20] developed a theory of thermodynamics of elastic material with inner structure whose microelements, in addition to microdeformations, possess microtemperatures. The importance of materials with microstructure has been demonstrated by the huge number of papers appeared in different fields of applications such as petroleum industry, material science, biology and many others.

Since this type of material has both microscopic and macroscopic structures, scientists have investigated the coupling and how strong it is. In addition, an increasing interest has been paid by mathematicians to analyze the longtime behavior of the solutions of thermoelastic and porous problems. One of the first studies, in this sense, was the thermoelastic coupling proposed by Slemrod [61]. As a result it was seen that in the one-dimensional case the solutions decay exponentially. Since then, many problems were studied by considering different dissipation mechanisms at the microscopic and/or the macroscopic levels. Many papers have been published where the authors tried to determine the type, as well as, the rate of decay of solutions in porous elasticity with voids.

In one dimensional thermoelasticity theory, Muñoz Rivera [45] , considered the linear thermoelastic system

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x = 0 \quad & (0, L) \times (0, T) \\ \theta_t - \theta_{xx} + \beta u_{tx} = 0 \quad & (0, L) \times (0, T) \end{cases}$$

with the initial and boundary conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x) \ \text{ in } \ (0, L)$$
$$u(0, t) = u(L, t) = \theta(0, t) = \theta(L, t) = 0 \ \ \forall t \in (0, T)$$

where u is the displacement, θ is the temperature difference and α and β are coupling constants. He used the energy method and proved that the dissipation induced by the heat equation is strong enough to stabilize the system exponentially. Also, Muñoz Rivera and Racke [47] studied the linear Timoshenko type system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } \ (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 \quad \text{in } \ (0,L)\times(0,\infty), \\ \rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0 \quad \text{in } \ (0,L)\times(0,\infty), \end{cases} \tag{1.1}$$

where φ is the displacement and ψ is the rotation angle of filament of the beam and $\rho_1, \rho_2, \rho_3, k, \alpha, \gamma, \kappa$ are constitutive constants. They showed that, for the boundary conditions

$$\varphi\,(x,t) = \psi_x\,(x,t) = \theta\,(x,t) = 0 \text{ for } x = 0, L \text{ and } t \geq 0, \tag{1.2}$$

the energy of system (1.1) decays exponentially if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{\alpha} \tag{1.3}$$

and that condition (1.3) suffices to stabilize system (1.1) exponentially for the boundary conditions

$$\varphi\,(x,t) = \psi\,(x,t) = \theta_x(x,t) = 0, \quad x = 0, L \text{ and } t \geq 0.$$

Guesmia *et al.* [23] established a polynomial decay result for (1.1)-(1.2) in the case of non-equal wave speed propagation, provided that the initial data are regular enough. They also discuss the case when the system (1.1) is supplemented with the boundary conditions

$$\varphi_x\,(x,t) = \psi\,(x,t) = \theta_x(x,t) = 0, \quad x = 0, L \text{ and } t \geq 0.$$

and they established a non-exponential decay results for the case when (1.3) does not hold.

In the isothermal case, (1.1) reduces to

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) = 0 \quad \text{in } (0,L)\times(0,\infty), \end{cases} \tag{1.4}$$

This system is conservative and it would be interesting to add some king of damping that may help in stabilizing such a system. Different types of dampings have been introduced and several stability results have been obtained by Kim and Renardy [31], Raposo *et al.* [60], Soufyane and Wahbe [66] and Muñoz Rivera and Racke, [48], [49]. Alabau-Boussouira [1] extended the results of [49] to the case of nonlinear feedback $\alpha(\psi_t)$, instead of $d\psi_t$, where α is a globally Lipchitz function satisfying some growth conditions at the origin.

A weaker type of dissipation was considered by Ammar-Khodja *et al.* [3] by introducing the memory term $\int_0^t g(t-s)\,\varphi_{xx}(x,s)\,ds$ in the rotation angle equation of (1.4). They used the multiplier techniques and showed that the system is uniformly stable if and only if (1.3) holds and the kernel g decays uniformly. Precisely, they proved that the rate of decay is exponential (polynomial) if g decays exponentially (polynomially). Guesmia and Messaoudi [21] obtained the same uniform decay result under weaker conditions on the regularity and growth of the relaxation function g. More general decay estimate was obtained by Messaoudi and Mustafa [40] for a wider class of relaxation functions. This latter result has been improved by Guesmia and Messaoudi [22] to accommodate systems, were frictional and viscoelastic dampings are cooperating.

Fernàndez Sare and Rivera [12] replaced the finite memory term in [3] by an infinite memory term $\int_0^\infty g(s)\psi_{xx}(t-s,.)\,ds$ and showed that if g is of exponential decay, the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. Messaoudi and Said-Houari [43] extended the results of [12] to polynomially decaying relaxation functions and without any restriction on g'' as in [12].

The analysis of temporal decay in one-dimensional porous-elasticity was first studied by Quintanilla [58] which considered the system

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta\varphi_x, \quad x \in (0,\pi), \ \ t > 0 \\ \rho_0 \kappa \varphi_{tt} = \alpha\varphi_{xx} - \beta u_x - \tau\varphi_t - \xi\varphi \quad x \in (0,\pi), \ \ t > 0 \\ u(x,0) = u_0(x), \ \ \varphi(x,0) = \varphi_0(x), \quad x \in (0,\pi) \\ u_t(x,0) = u_1(x), \ \ \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,\pi) \\ u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = 0, \ \ t \geq 0. \end{cases}$$

where, φ is the volume fraction, $\rho_0 > 0$ is the mass density, $\kappa > 0$ is the equilibrated inertia, and $\mu, \alpha, \alpha, \tau, \xi$ are the constitutive constants which are positive and satisfy $\mu\xi > \beta^2$. He showed that the damping in the porous equation $(-\tau\varphi_t)$ is not strong enough to obtain an exponential decay. In this case, only the slow decay has been proved. Magaña and Quintanilla [36] proved that the presence of viscoelastic dissipation is not powerful enough to stabilize a porous-elastic system exponentially, only slow decay has been established. They also, showed that neither the addition of temperature to viscoelastic-porous problem nor the addition of microtemperature to elastic-viscoporous problem can stabilize the system exponentially. However, the combination of viscoelasticity and porous dissipation or the addition of microtemperature in porous elastic problems lead the solution to decay exponentially.

Casas and Quintanilla [4] considered the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x = 0, \ \text{in}[0,\pi] \times (0,\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta - \tau\varphi_t, \ \text{in}[0,\pi] \times (0,\infty), \\ c\theta_t = k\theta_{xx} - \beta u_{tx} - m\varphi_t, \ \text{in}[0,\pi] \times (0,\infty), \end{cases}$$

with Dirichlet-Neumann- Neumann boundary conditions. They proved that, under same conditions on the constitutive constants, the sum of two slow decay processes (elastic-viscoporous and thermal viscoelastic) determine a process that decays exponentially. Several results concerning the rate of decay of solutions for thermoelastic and porous-thermoelastic systems where obtained in [4, 36, 32, 34, 51]. In those papers, the authors clarified the type of decay we obtain by combining different dissipations via temperature, elastic viscosity, porous viscosity and microtemperature. In particular, we quote the work of Magaña and Quintanilla [36] in which they discussed the time behaviors of several systems with quasi-static microvoids ($\varphi_{tt} \approx 0$) and established different slow and exponential decay results. These results can be summarized by the help of the following scheme:

$$\boxed{\begin{array}{l}\text{Thermal effect}\\ \\ \\ \text{Viscoelastic effect}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Elasticity}\\ \updownarrow \\ \text{Porosity}\end{array}} \longleftarrow \boxed{\begin{array}{l}\text{Microthermal effect}\\ \\ \\ \text{Viscoporous effect}\end{array}}$$

If we take simultaneously one effect from the right square and another one from the left square or more than two effect then we get exponential stability. However, two simultaneous effects from one square only lead to slow decay. Perhaps it is worth recalling the main difference between the concepts of exponential and slow decay. In a thermomechanical point of view, if the decay is exponential, then after a short period of time, the thermomechanical displacements are very small and can be neglected. However, if the decay is slow, then the solution is weaken in the way that thermomechanical displacements could be appreciated in the system after some time.

Muñoz-Rivera and Quintanilla [46] considered some cases where the decay is slow and proved that the energy associated to the solutions decays polynomially. Precisely, $E\left(t\right) \leq \dfrac{C}{t^{\alpha}}$ for some positive constants C and α.

Soufyane [63] was the first who proposed a porous-thermoelastic problem with a dissipation of memory type. He considered the system

$$\begin{cases} u_{tt} = u_{xx} + \varphi_x - \theta_x \text{ in}(0,L) \times \mathbb{R}_+ \\ \varphi_{tt} = \varphi_{xx} - u_x - \varphi - \theta + \displaystyle\int_0^t g(t-s)\varphi_{xx}(x,s)ds \text{ in}(0,L) \times \mathbb{R}_+ \\ \theta_t = \theta_{xx} - u_{tx} - \varphi_t, \text{ in}(0,L) \times \mathbb{R}_+ \\ u(x,t) = \varphi(x,t) = \theta(x,t) = 0 \quad x = 0, L, \qquad t > 0, \\ u(x,0) = u_0(x), \quad \varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in (0,L), \\ u_t(x,0) = u_1(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,L), \end{cases}$$

where g is a positive nonincreasing function, and proved that the decay is exponential (respect. polynomial) when the relaxation function is of exponential (respect.. polynomial) decay. A similar result was also obtained by Soufyane et al. [64] and [65], for the above system with the viscoelastic damping $-\int_0^t g\left(t-s\right)\varphi_{xx}\left(x,s\right)ds$ replaced by two boundary viscoelastic dissipations of the form

$$\begin{cases} u(L,t) = -\displaystyle\int_0^t g_1(t-s)[\mu u_x(L,s) + b\varphi(L,s)]ds \\ \varphi(L,t) = -\displaystyle\int_0^t g_2(t-s)\varphi(L,s)ds, \end{cases}$$

where g_1 and g_2 are positive nonincreasing functions. Recently, Pamplona et al. [52] treated the following one-dimensional porous elastic problem with history

$$\begin{cases} \rho u_{tt} = (\mu + f(0))u_{xx} + (b + h(0))\varphi_x + \displaystyle\int_0^{+\infty} f'(s)u_{xx}(t-s)ds \\ \qquad\qquad + \displaystyle\int_0^{+\infty} h'(0)\varphi_x(t-s)ds \qquad\qquad\qquad\qquad , \\ J\varphi_{tt} = (\delta + g(0))\varphi_{xx} - (b + h(0))u_x - (\xi + k(0))\varphi + \displaystyle\int_0^{+\infty} g'(s)\varphi_{xx}(t-s)ds \\ \qquad\qquad - \displaystyle\int_0^{+\infty} h'(s)u_x(t-s)ds - \int_0^{+\infty} k'(s)\varphi(t-s)ds, \\ u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = 0, \\ u(x,0) = u_0(x), \quad u_x(x,0) = u_1(x), \quad \varphi(x,0) = \varphi_0(x), \quad \varphi_x(x,0) = \varphi_1(x), \end{cases}$$

in $[0,\pi]\times\mathbb{R}_+$, where ρ and J are positive constants, μ,δ,ξ,b satisfy $\mu>0, \mu\xi-b^2>0, \delta>0$ and f,g,h,k are the memory kernel functions. They proved the lack of exponential stability if only porous dissipation are present and g does not satisfy

$$\frac{J\mu}{\rho} - \delta - g(0) = 0,$$

or only elastic dissipation are present and f does not satisfy

$$\frac{\rho\delta}{J} - \mu - f(0) = 0, \tag{1.5}$$

or if both elastic and porous dissipation are present but the porous dissipation is weak and (1.5) does not hold. Otherwise, exponential stability will be obtained.

We recall that problems involving viscoelastic damping given by a memory, or a past history term have attracted the attention of a lot of scientists in the last two decades. The obtained decay results depended on the rate of decay of the relaxation function and it is exponential for g satisfying: $g'(t) \leq -\xi g(t)$ for all $t \geq 0$ and some positive constant ξ. However, only polynomial decay result was proved for relaxation functions satisfying $g'(t) \leq -\xi g^p(t), \forall t \geq 0$ and $1 < p \leq \frac{3}{2}$, see [3], [21],[52], [63, 64, 65].

A considerable efforts are devoted to enlarging the space of admissible relaxation functions leading to strong or slow decay. Messaoudi and Mustafa [40] considered the system

$$\begin{cases} u_{tt} - (u_x + \varphi)_x = 0 \quad \text{in} \quad (0,1) \times \mathbb{R}_+ \\ \varphi_{tt} - \varphi_{xx} + u_x + \varphi + \displaystyle\int_0^t g(t-s)\varphi_{xx}(x,s)ds = 0 \quad \text{in} \quad (0,1) \times \mathbb{R}_+ \\ u(0,t) = u(1,t) = \varphi(0,t) = \varphi(1,t) = 0 \quad t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x) \quad \text{in } (0,1). \end{cases} .$$

and assumed that the relaxation function g satisfies the inequality

$$g' \leq -\xi(t)g(t) \ \forall t \geq 0.$$

They established a more general decay result, from which the exponential and polynomial decay rates are only special cases.

We also remind the contribution of Alabau-Boussouira and Cannarsa [2] in which decay result was obtained for relaxation functions

$$g' \leq -\xi(t)\chi(g(t)), \ \forall t \geq 0,$$

for some strictly increasing function χ.

Nonclassical thermoelasticity

The classical thermoelasticity is concerned with the effect of heat on the deformation of an elastic solid and with the inverse effect of deformation on the thermal state of the solid. It is formulated on the principle of the classical theory of heat conduction, namely, Fourier's law

$$q + k\nabla\theta = 0$$

where θ is the difference temperature, q is heat conduction vector and k is the coefficient of thermal conductivity. Consequently, the heat equation is of parabolic type. As a result, this theory predicts an infinite speed of heat propagation. That is, any thermal disturbance at a point has an instantaneous effect elsewhere in the body. This is physically unrealistic and experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in finite speed. This phenomenon in crystals dialectic is called second sound. To overcome the deficiency of this theory many theories were developed, nonclassical thermoelasticity theories involving hyperbolic-type heat transport equations admitting finite speeds for thermal signals have been formulated either by incorporating a flux-rate term into Fourier's law or by including temperature-rate among the constitutive variables.

By the end of last century Green and Naghdi [14, 16, 17] introduced three new types of thermoelastic theories based on replacing the usual entropy inequality with an entropy balance law. In each of these theories, the heat flux is given by a different constitutive assumption. As a result three theories were obtained and respectively called thermoelasticity type I, type II and type III. When the theory of type I is linearized we obtain the classical system of thermoelasticity. The systems arising in thermoelasticity of type III are of dissipative nature whereas those of type II do not sustain energy dissipation. To understand these new theories and their applications, several mathematical and physical contributions have been made; see for example, Chandrasekharaiah [6, 7, 8], Quintanilla [53, 54, 55] and Quintanilla and Racke [59]. In particular, we mention the survey paper of Chandrasekharaiah [6], in which the author focussed attention on the work done during two decades. He reviewed the theory of thermoelasticity with thermal relaxation and the temperature rate dependent thermoelasticity. He also described the thermoelasticity without dissipation and clarified its properties.

Thermoelasticity of type II was introduced by Green and Naghdi [14] as an alternative theory to describe thermomechanical interactions in elastic materials. They proposed the use of the thermal displacement variable

$$\psi = \int_{t_0}^{t} \theta(x, s)ds + \psi_0 = 0$$

instead of the difference temperature variable θ. In this theory, called thermoelasticity without energy dissipation, the heat is allowed to propagate by means of thermal waves but without energy dissipation. This theory has been the subject of some interesting works in the last two decades.

Cicco and Diaco [11] derived a linear theory of thermoelastic materials with voids that does not sustain energy dissipation and established uniqueness and continuous dependence theorems. Quintanilla [56] proved the well posedness of the linear theory of thermoelasticity without energy dissipation by means of semigroup theory. Ieşan and Quintanilla [28] derived a linear theory of thermoelastic bodies with microstructure and microtemperature based on Green and Naghdi balance

Recently, Leseduarte *et al.* [33] studied the system of thermoelasticity without energy dissipation of the form

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + \gamma \phi_x - \beta \psi_{tx}, & \text{in} \quad (0,\pi) \times \mathbb{R}_+ \\ J\phi_{tt} = b\phi_{xx} + m\psi_{xx} - \xi\phi + d\psi_t - \tau\phi_t - \gamma u_x, & \text{in} \quad (0,\pi) \times \mathbb{R}_+ \\ \alpha\psi_{tt} = k\psi_{xx} + m\phi_{xx} - d\phi_t - \beta u_{tx}, & \text{in} \quad (0,\pi) \times \mathbb{R}_+ \end{cases}$$

where u and ϕ are the displacement and the volume fraction respectively, and proved that when m and β are not vanish the system is exponentially stable. However, if one of the parameters m or β vanishes, then we lost the exponential decay. Note that β relates the displacement and the temperature and m relates the volume fraction with the thermal displacement, these parameters are responsible of the strong coupling among the variables.

The theory of thermoelasticity of type III is characterized by the heat-flux constitutive equation

$$q + \kappa^* \tau_x + \widetilde{\kappa} \theta_x = 0$$

where τ denotes the thermal displacement which satisfies $\tau_t = \theta$ and $\kappa^*, \widetilde{\kappa}$ are positive constants.

In [59] Quintanilla and Racke considered a system of thermoelasticity of type III of the form

$$\begin{cases} u_{tt} - \alpha u_{xx} + \beta \theta_x = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}_+ \\ \theta_{tt} - \delta \theta_{xx} + \gamma u_{ttx} - \kappa \theta_{txx} = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}_+ \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(x),\ \theta(0, x) = \theta_0(x),\ \theta_t(0, x) = \theta_1(x), \end{cases}$$

where u is the displacement, θ is the temperature difference and $\alpha, \beta, \delta, \gamma, \kappa$ are constitutive positive constants. They used the spectral analysis method and the energy method to establish an exponential stability in the one-dimension setting for different boundary conditions. We recall also the contribution of Quintanilla [57], in which he proved that solutions of thermoelasticity of type III converge to solutions of the classical thermoelasticity as well as to solutions of thermoelasticity without energy dissipation.

Zhung and Zuazua [68] studied the long time behavior of the solution of the system

$$\begin{cases} u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla (\operatorname{div} u) + \nabla \theta = 0 \quad \text{in} \ \ \Omega \times (0, +\infty) \\ \theta_{tt} - \Delta \theta - \Delta \theta_t + \operatorname{div} u_{tt} = 0 \quad \text{in} \ \ \Omega \times (0, +\infty) \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(0),\ \theta(0, x) = \theta_0(x),\ \theta_t(0, x) = \theta_1(x), \quad \text{in} \ \ \Omega, \\ u(t, x) = \theta(t, x) = 0 \qquad \text{on} \ \ \partial\Omega \times (0, \infty) \end{cases}$$

and proved that under suitable conditions on the domain the energy of the system decays exponentially. But for most domains in two dimension space, the energy of smooth solutions decays in a polynomial rate.

Messaoudi and Said Houari [41] considered the system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x+\psi) - \beta\theta_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \theta(x,0) = \theta_0(x), \quad 0 \le x \le 1 \\ \varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta_t(x,0) = \theta_1(x), \quad 0 \le x \le 1 \\ \varphi(x,t) = \psi(x,t) = \theta_x(x,t) = 0, \quad x = 0,1, \quad t > 0. \end{cases}$$

and proved an exponential decay result similar to the one proved by Munõz Rivera and Racke [47] for classical thermoelasticity. Also, Messaoudi and Said Houari [42] considered a Timoshenko-type system of type III of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) \\ \qquad\qquad + \displaystyle\int_0^{+\infty} g(s)\,\psi_{xx}(x,t-s)\,ds + \beta\theta_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \theta(x,0) = \theta_0(x), \quad 0 \le x \le 1 \\ \varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta_t(x,0) = \theta_1(x), \quad 0 \le x \le 1 \\ \varphi(x,t) = \psi(x,t) = \theta_x(x,t) = 0, \quad x = 0,1, \quad t > 0. \end{cases}$$

and proved that the above system decays exponentially (respectively polynomially) if g

decays exponentially (respectively polynomially) in the case of equal speed $\left(\frac{\rho_1}{K} = \frac{\rho_2}{b}\right)$. However, the decay is of polynomial type otherwise $\left(\frac{\rho_1}{K} \neq \frac{\rho_2}{b}\right)$. This result has been improved recently by Guesmia et al. [21] and a general decay rate was obtained.

This thesis

In this thesis we studied some problems which arising in porous thermoelasticity and nonclassical thermoelasticity theories. Several decay results has been established which improve and extend earlier ones.

This thesis is divided into four chapters. In chapter two we give a short summary of the derivation of the equations in porous thermoelasticity.

In chapter 3 we studied a porous thermoelastic system of memory type, namely,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x = 0 \text{ in } (0,1) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - \alpha\varphi_{xx} + k(\varphi_x + \psi) + \theta + \int_0^t gt - s)\varphi_{xx}(x,s)ds = 0 \text{ in } (0,1) \times \mathbb{R}_+, \\ \rho_3\theta_t - \kappa\theta_{xx} + \varphi_{tx} + \psi_t = 0 \text{ in } (0,1) \times \mathbb{R}_+, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0 \quad t \geq 0, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), x \in (0,), \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x) \quad x \in (0,1), \end{cases} \tag{1.6}$$

where φ, ψ, θ are the longitudinal displacement, the volume fraction and the temperature difference respectively, $\rho_1, \rho_2, \rho_3, k, \alpha, \kappa$ are positive constants and g is nonincreasing function. We adopted some argument of [52, 63] and established a general decay result of the solutions of (1.6) for which exponential and polynomial decay rates of [63] are merely special cases. Two papers were appear, one concerning the case of equal speed the other concerning the case of nonequal speed.

Chapter 4 is devoted to the study of a porous problem in non classical thermoelasticity. A linear damped system in one-dimensional porous thermoelasticity of type III was considered and several decay rate results have obtained.

Precisely, we have study the following system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x+\psi) - \theta + a\psi_t = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_{tt} - \kappa\theta_{xx} + \varphi_{xtt} + \psi_{tt} - k\theta_{txx} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \theta(x,0) = \theta_0(x), \quad 0 \le x \le 1 \\ \varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta_t(x,0) = \theta_1(x), \quad 0 \le x \le 1 \\ \varphi(x,t) = \psi(x,t) = \theta(x,t) = 0, \quad x = 0,1, \quad t > 0. \end{cases} \tag{1.7}$$

We use the multiplier techniques and proved that the energy of system (1.7) decays exponentially if (1.3) holds and that the rate of decay is of polynomial type otherwise.

Also, in chapter 4, we improved an earlier result obtained by Messaoudi and Houari [41] for the following Timoshenko-type system with thermoelasticity type III

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x = 0 \quad \text{in} \quad (0,\infty)\times(0,1), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \gamma\theta_x = 0 \quad \text{in} \quad (0,\infty)\times(0,1), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 \quad \text{in} \quad (0,\infty)\times(0,1). \end{cases} \tag{1.8}$$

They established an exponential decay result for the weak solutions of (1.7) under the condition In the present thesis we consider (1.8), for the case and prove a polynomial decay result for strong solutions.

Chapter 2

Derivation of equations

In this chapter we shall give a short summary of the three-dimensional theory of porous solids in the framework of thermal conduction.

Consider a homogeneous porous thermoelastic body $\mathcal{B}_0$ which at some instant t_0, occupies the reference configuration Ω, a bounded region of the euclidean three-dimensional space $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. The motion of the body is referred to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. We denote by $\mathcal{B}_t$ the configuration of the body at time $t \geq t_0$, by $u\,(x, t) = X\,(x, t) - x$ the displacement vector of a material point with reference configuration x and by $F = I + \nabla u = \left[\dfrac{\partial X_i}{\partial x_j}\right]$ the deformation gradient, where $X = X\,(x, t)$ is the position of this point at time t. We assume that the motion equation is invertible, such that $\det F \neq 0$. In this chapter, summation convention over repeated subscripts are used as well as the comma followed by subscripts for partial differentiation over the space coordinates while the differentiation over t is denoted by a dot over the function.

We adopt the approach of Goodman and Cowin [13], where, for each t, the region $\mathcal{B}_t$ is endowed with a structure given by two real valued set fuctions $\mathcal{M}_t$ and $\mathcal{V}_t$ which represent the distributed mass and the distributed volume a time t, respectively and satisfy the following axioms

- $\mathcal{M}_t$ and $\mathcal{V}_t$ are non-negative measures defined,
- $\mathcal{V}_t(\mathcal{P}_t) \leq V(\mathcal{P}_t)$ for all $\mathcal{P}_t \subset \mathcal{B}_t$, where V is the Lebesgue volume measure,
- $\mathcal{M}_t$ is absolutely continuous with respect to $\mathcal{V}_t$.

Thus, for all Borel subsets $\mathcal{P}_t \subset \mathcal{B}_t$, we have

$$\mathcal{V}_t(\mathcal{P}_t) = \int_{\mathcal{P}_t} \nu dV \tag{2.1}$$

and

$$\mathcal{M}_t(\mathcal{P}_t) = \int_{\mathcal{P}_t} \gamma\nu d\mathcal{V}_t = \int_{\mathcal{P}_t} \rho dV, \tag{2.2}$$

where ν is the volume fraction, γ is the matrix density, $\rho = \gamma\nu$ is the bulk density and dV is the image, at time t, of an element dV_0 of the bulk volume in the reference configuration. Moreover, the function ν has the property that, for almost every $X \in \mathcal{B}_t$,

$$0 \leq \nu(X,t) \leq 1.$$

Remark. Let dV_0 be the infinitesimal volume element with sides dx, dy and dz in the reference configuration and let dV be the image of dV_0 in the current configuration. For every side $dw = dx, dy$ or dz let dw_1, dw_2, dw_3 be the components of the vector dw, that is

$$dw = (dw_1, dw_2, dw_3)^T, \quad \text{for} \quad w = x, y, z.$$

It is well known that

$$\begin{aligned} dV_0 &= dz \cdot (dx \wedge dy) \\ &= \det(dz, dx, dy) = \varepsilon_{ijk} dz_i dx_j dy_k, \end{aligned}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{for } (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1, & \text{for } (i,j,k) = (1,3,2), (2,1,3), (3,1,2) \\ 0, & \text{otherwise.} \end{cases}$$

Let dX, dY and dZ be the images of dx, dy and dz at time t. Thus,

$$(dW_1, dW_2, dW_3)^T = F(dw_1, dw_2, dw_3)^T, W = X, Y, Z \text{ and } w = x, y, z,$$

or

$$dW_i = F_{ik} dw_k.$$

The element of bulk volume in the current configuration dV is given by

$$\begin{aligned} dV &= \varepsilon_{ijk} dZ_i dX_j dY_k = \varepsilon_{ijk} F_{il} dz_l F_{jm} dx_m F_{kn} dy_n \\ &= \varepsilon_{ijk} F_{il} F_{jm} F_{kn} dz_l dx_m dy_n = \varepsilon_{lmn} |\det F| dz_l dx_m dy_n, \end{aligned}$$

which yields

$$dV = J dV_0,$$

where $J = |\det F| > 0$.

By differentiation of (2.1), it follows that an element of distributed volume in the instantaneous configuration is related to an element of the bulk volume by the relation

$$d\mathcal{V}_t = \nu dV.$$

Similarly, in the reference configuration

$$d\mathcal{V}_0 = \nu_0 dV_0.$$

Thus, an element of distributed volume transforms according to the relation

$$d\mathcal{V}_t = \frac{\nu}{\nu_0} J d\mathcal{V}_0. \tag{2.3}$$

From (2.3) the constraint of incompressible distributed volume can be expressed by the equation

$$\frac{\nu}{\nu_0} J = 1.$$

Time material derivative of time dependent volume integral

Let $\phi(t, X(t))$ be a scalar function, differentiable with respect to t and X_i in $(0, +\infty) \times \mathcal{B}_t$, then

$$\frac{d}{dt} \int_{\mathcal{P}_t} \phi(t, X(t))\, dV = \int_{\mathcal{P}_t} \left(\frac{\partial \phi(t, X(t))}{\partial t} + \frac{\partial \phi(t, X(t))}{\partial X_i} \frac{\partial X_i}{\partial t} \right) dV + \int_{\partial \mathcal{P}_t} \phi(t, X(t)) \frac{dX_i}{dt} . n_i dS$$

where $n = (n_1, n_2, n_3)$ is the outward normal vector on the surface $\partial \mathcal{P}_t$ and dS is a surface element of $\partial \mathcal{P}_t$. Using the divergence theorem we get

$$\frac{d}{dt} \int_{\mathcal{P}_t} \phi(t, X(t))\, dV = \int_{\mathcal{P}_t} \left(\frac{\partial \phi(t, X(t))}{\partial t} + \frac{\partial \phi(t, X(t))}{\partial X_i} \frac{\partial X_i}{\partial t} \right) dV + \int_{\mathcal{P}_t} \phi(t, X(t)) \operatorname{div} \upsilon(t)\, dV,$$

$$= \int_{\mathcal{P}_t} \left(\frac{\partial \phi (t, X(t))}{\partial t} + \frac{\partial \phi (t, X(t))}{\partial X_i} \upsilon_i (t) + \phi (t, X(t)) \operatorname{div} \upsilon (t) \right) dV,$$

which can be written

$$\frac{d}{dt} \int_{\mathcal{P}_t} \phi (t, x(t)) \, dV = \int_{\mathcal{P}_t} \left(\dot{\phi} + \phi \operatorname{div} \upsilon \right) dV,$$

where $\dot{\phi} = \frac{d\phi}{dt}$ and $\upsilon = \frac{\partial X}{\partial t}$ is the velocity of the material point X.

Mass conservation law

Let ρ be the mass density per unit of volume in a fixed region $\mathcal{P}_t$, then, differentiating integral (2.2), the change in mass inside $\mathcal{P}_t$ will be

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho dV = \int_{\mathcal{P}_t} \frac{\partial \rho}{\partial t} dV.$$

The rate of mass flow out the volume $\mathcal{P}_t$ is

$$- \int_{\partial \mathcal{P}_t} \rho \upsilon . n dA,$$

where dA is an area element in $\partial \mathcal{P}_t$ and n is the unit outward normal vector to dA.

The mass conservation law states that the rate of change in mass within a fixed volume mast be equal to rate of flow through the boundaries. Therefore

$$\int_{\mathcal{P}_t} \frac{\partial \rho}{\partial t} dV = - \int_{\partial \mathcal{P}_t} \rho \upsilon . n dA$$

which, using the divergence theorem, can be written

$$\int_{\mathcal{P}_t} \left(\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \upsilon) \right) dV = 0.$$

For $\rho = \gamma\nu$ with fixed γ, the last formula rewritten

$$\int_{\mathcal{P}_t} \gamma \left(\frac{\partial \nu}{\partial t} + \nabla \nu . \upsilon + \nu \operatorname{div} \upsilon \right) dV = 0$$

for all subset $\mathcal{P}_t \subset \mathcal{B}_t$, therefore,

$$\dot{\nu} + \nu \operatorname{div} \upsilon = 0,$$

where

$$\dot{\nu} = \frac{\partial \nu}{\partial t} + \nabla \nu . \upsilon$$

is the material derivative of ν.

As in Goodman and Cowin [13] and Ieşan [24, 27] we postulate an energy balance at time t, for an arbitrary region $\mathcal{P}_t$ of the body $\mathcal{B}_t$, in the form

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{P}_t} \rho \left(|\upsilon|^2 + \kappa \left| \dot{\nu} \right|^2 \right) dV + \frac{d}{dt} \int_{\mathcal{P}_t} \rho \varepsilon dV = \int_{\mathcal{P}_t} \rho \left(f_i \upsilon_i + l \dot{\nu} + s \right) dV + \int_{\partial \mathcal{P}} \left(T_{ji} \upsilon_i + h_j \dot{\nu} + q_j \right) n_j dA$$

where T_{ij} is the first Piola-Kirchhoff stress tensor, ε is the internal energy, κ is the equilibrated inertia, f_i is the body force, l is the extrinsic equilibrated body force, s is the heat supply, h_j is the equilibrated stress vector, q_j is the heat flux vector across the surface $\partial\mathcal{P}$.

The above formula mains that the change of the kinetic and internal energy in $\mathcal{P}_t$ for the interval time dt is equal to the work of mechanical forces and the change of heat in the same time dt,

$$\frac{d \left(E_{kin} + U \right)}{dt} = \frac{dW}{dt} + \frac{dQ}{dt}$$

where,

$$\frac{dE_{kin}}{dt} = \frac{1}{2}\frac{d}{dt}\int_{\mathcal{P}_t} \rho\left(|\upsilon|^2 + \kappa\left|\dot{\nu}\right|^2\right) dV = \int_{\mathcal{P}_t} \rho\left(\upsilon_i\dot{\upsilon}_i + \kappa\dot{\nu}\ddot{\nu}\right) dV$$

is the time derivative of kinetic energy,

$$\frac{dU}{dt} = \frac{d}{dt}\int_{\mathcal{P}_t} \rho\varepsilon dV = \int_{\mathcal{P}_t} \rho\dot{\varepsilon} dV$$

is the time derivative of internal energy,

$$\frac{dW}{dt} = \int_{\mathcal{P}_t} \rho\left(f_i\upsilon_i + l\dot{\nu}\right) dV + \int_A \left(T_{ji}\upsilon_i + h_j\dot{\nu}\right) n_j dA$$

is time derivative of work, and

$$\frac{dQ}{dt} = \int_{\mathcal{P}_t} \rho s dV + \int_A q_j n_j dA$$

is time derivative of the heat.

In the absence of the body force, the heat supply and the extrinsic equilibrated body force the energy balance has the form

$$\int_{\mathcal{P}_t} \rho\left(\upsilon_i\dot{\upsilon}_i + \kappa\dot{\nu}\ddot{\nu}\right) dV + \int_{\mathcal{P}_t} \rho\dot{\varepsilon} dV = \int_A \left(T_{ji}\upsilon_i + h_j\dot{\nu} + q_j\right) n_j dA. \tag{2.4}$$

Following Green and Revilin [18] we consider a second motion which differs from the given motion only by a constant superposed rigid body translational velocity, the body occupying the same position a time t and we assume that $\dot{\varepsilon}, h, T_{ij}$ and q are unaltered by such superposed rigid body velocity. Thus, (2.4) is also true when υ_i is replaced by $\upsilon_i + a_i$, where a_i are arbitrary constants, all others terms being unaltered. By subtraction we get

$$\left[\int_{\mathcal{P}_t} \rho\dot{\upsilon}_i dV - \int_A T_{ji} n_j dA\right] a_i = 0$$

for all arbitrary constants a_i. It follows that

$$\int_{\mathcal{P}_t} \rho \dot{\upsilon}_i dV = \int_A T_{ji} n_j dA$$

which, using the divergence theorem, gives,

$$T_{ji,j} = \rho \ddot{X}_i, \tag{2.5}$$

where $\dot{\upsilon}_i = \ddot{X}_i$. Using the divergence theorem, again, we get

$$\int_A \left(T_{ji}\upsilon_i + h_j \dot{\nu} + q_j \right) n_j dA = \int_{\mathcal{P}_t} \left(T_{ji,j}\upsilon_i + T_{ji}\upsilon_{i,j} + h_{j,j}\dot{\nu} + h_j \dot{\nu}_{,j} + q_{j,j} \right) dV \tag{2.6}$$

In view of (2.5), (2.6), the relation (2.4) reduces to

$$\int_{\mathcal{P}_t} \rho \left(\dot{\varepsilon} + \kappa \dot{\nu} \ddot{\nu} \right) dV = \int_{\mathcal{P}_t} \left(T_{ji}\upsilon_{i,j} + q_{j,j} + h_{j,j}\dot{\nu} + h_j \dot{\nu}_{,j} \right) dV \tag{2.7}$$

which holds for any region $\mathcal{P}_t \subset \mathcal{B}_t$. Thus,

$$\rho \dot{\varepsilon} = T_{ji}\upsilon_{i,j} + q_{j,j} + h_j \dot{\nu}_{,j} - f \dot{\nu} \tag{2.8}$$

where

$$f = \rho \kappa \ddot{\nu} - h_{j,j} \tag{2.9}$$

is a dependent constitutive variable called the intrinsic equilibrated body force.

We consider a motion of the body which differs from the given motion only by superposed uniform rigid body angular velocity, the body occupying the same position at time t and assume that $\dot{\varepsilon}, T_{ij}, h_j$ and q_j are unaltered by such motion. The equation (2.7) holds when

$\upsilon_{i,j}$ is replaced by $\upsilon_{i,j}+\sigma_{ij}$, where σ_{ij} is a constant skew symmetric tensor representing a constant rigid body angular velocity. It follows that

$$\sigma_{ij}\int_{\mathcal{P}_t} T_{ji}dV = 0$$

for all skew symmetric tensors σ_{ij}. Since $\int_{\mathcal{P}_t} T_{ji}dV$ is independent of σ_{ij}, it follows that

$$\sigma_{ij}\int_{\mathcal{P}_t} \left(T_{ji}-T_{ij}\right)dV = 0$$

for all arbitrary region $\mathcal{P}_t$, so that $T_{ij}=T_{ji}$. Thus (2.8) becomes

$$\rho\dot{\varepsilon} = \frac{1}{2}T_{ji}\frac{d}{dt}\left(X_{i,j}+X_{j,i}\right)+q_{j,j}+h_j\dot{\nu}_{,j}-f\dot{\nu}.$$

If we denote by

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right),$$

(2.8) reduces to

$$\rho\dot{\varepsilon} = T_{ji}\dot{e}_{ij}+q_{j,j}+h_j\dot{\nu}_{,j}-f\dot{\nu}. \tag{2.10}$$

To derive the temperature equation, we postulate an entropy production inequality,

$$\int_{\mathcal{P}_t}\rho\dot{\eta}dV-\int_A\frac{q}{T}dA\geq 0, \tag{2.11}$$

where η is the specific entropy and T is the absolute temperature which is assumed to be always positive.

We apply this inequality to a region which in the reference state was a tetrahedron bounded by coordinate planes through the point X and by a plane whose unit normal is n, we obtain

$$q=q_in_i.$$

Consequently, (2.11) can be written

$$\int_{\mathcal{P}_t} \rho\dot{\eta} dV - \int_{\partial\mathcal{P}_t} \left(\frac{q}{T}\right)_i n_i dA \geq 0$$

and divergence theorem yields

$$\int_{\mathcal{P}_t} \rho\dot{\eta} dV - \int_{\mathcal{P}_t} \frac{q_{j,j}T - q_j T_{,j}}{T^2} dV \geq 0.$$

The fact that this inequality holds for all arbitrary region $\mathcal{P}_t$, lead to

$$\rho T\dot{\eta} - q_{j,j} + \frac{1}{T} q_j T_{,j} \geq 0. \tag{2.12}$$

The Helmholtz free energy is defined by

$$\psi = \varepsilon - T\eta \tag{2.13}$$

and we introduce

$$\varphi = \nu - \nu_0, \qquad \theta = T - T_0,$$

where ν_0 is the volume fraction filed in the reference configuration and T_0 is the constant absolute temperature of the body in the reference configuration.

We restrict our attention to the linear theory of thermoelastic materials where the constitutive variables are $e_{ij}, \varphi, \varphi_{,i}, \theta$ and $\theta_{,i}$ which are invariant under superposed rigid body motions. Consistent with this theory, it is assumed that the overall response of a porous material depends on $e_{ij}, \varphi, \varphi_{,i}, \theta$ and $\theta_{,i}$, then, at each point $X \in \Omega$ and for all $t \geq 0$, we have

$$
\begin{aligned}
\sigma &= \widehat{\sigma}\left(e_{ij}, \varphi, \varphi_{,i}, \theta, \theta_{,i}\right) \\
T_{ij} &= \widehat{T_{ij}}\left(e_{ij}, \varphi, \varphi_{,i}, \theta, \theta_{,i}\right) \\
f &= \widehat{f}\left(e_{ij}, \varphi, \varphi_{,i}, \theta, \theta_{,i}\right) \\
h &= \widehat{h}\left(e_{ij}, \varphi, \varphi_{,i}, \theta, \theta_{,i}\right), \\
q &= \widehat{q}\left(e_{ij}, \varphi, \varphi_{,i}, \theta, \theta_{,i}\right),
\end{aligned}
\tag{2.14}
$$

where, $\sigma = \rho\psi$. With the help of (2.10) and (2.13), the inequality (2.12) becomes

$$
-\rho\eta\dot{\theta} + T_{ji}\dot{e}_{ij} - f\dot{\varphi} - \dot{\sigma} + h_i\dot{\varphi}_{,i} + \frac{1}{T}q_i\theta_{,i} \geq 0. \tag{2.15}
$$

The material derivative $\dot{\sigma}$ of σ is given by

$$
\dot{\sigma} = \frac{\partial\sigma}{\partial t} + \frac{\partial\sigma}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\sigma}{\partial\varphi}\dot{\varphi} + \frac{\partial\sigma}{\partial\varphi_{,i}}\dot{\varphi}_{,i} + \frac{\partial\sigma}{\partial\theta}\dot{\theta} + \frac{\partial\sigma}{\partial\theta_{,i}}\dot{\theta}_{,i}. \tag{2.16}
$$

Thanks to (2.16), the insertion of constitutive functions (2.14) in the inequality (2.15), yields

$$
\begin{gathered}
-\frac{\partial\sigma}{\partial t} + \left(T_{ji} - \frac{\partial\sigma}{\partial e_{ij}}\right)\dot{e}_{ij} - \left(\frac{\partial\sigma}{\partial\varphi} + f\right)\dot{\varphi} + \left(h_i - \frac{\partial\sigma}{\partial\varphi_{,i}}\right)\dot{\varphi}_{,i} \\
-\left(\rho\eta + \frac{\partial\sigma}{\partial\theta}\right)\theta - \frac{\partial\sigma}{\partial\theta_{,i}}\dot{\theta}_{,i} + \frac{1}{T}q_i\theta_{,i} \geq 0.
\end{gathered}
\tag{2.17}
$$

In the theory of Green and Naghdi [14],[15],[16] the entropy inequality (2.17) must be satisfied identically for all processes and will place restrictions on the constitutive functions. Thus, (2.17) leads to

$$
\sigma = \widehat{\sigma}\left(e_{ij}, \varphi, \varphi_{,i}, \theta\right),
$$

$$T_{ji} = \frac{\partial \sigma}{\partial e_{ij}}, \qquad f = -\frac{\partial \sigma}{\partial \varphi}$$
$$h_i = \frac{\partial \sigma}{\partial \varphi_{,i}}, \qquad \rho\eta = -\frac{\partial \sigma}{\partial \theta} \tag{2.18}$$

and

$$q_i \theta_{,i} \geq 0. \tag{2.19}$$

The inequality (2.19) implies that

$$q_i = 0, \text{ if } \theta_{,i} = 0. \tag{2.20}$$

Assuming that the initial body is free from stresses and has zero intrinsic body force and that the porous medium is of viscoelastic type, the linear theory yields

$$\begin{aligned} \sigma = & \frac{1}{2} C_{ijrs} e_{ij} e_{rs} - \beta_{ij} e_{ij} \theta - \frac{1}{2} \alpha \theta^2 + \frac{1}{2} A_{ij} \varphi_{,i} \varphi_{,j} + B_{ij} \varphi e_{ij} \\ & + D_{ijk} e_{ij} \varphi_{,k} + d_i \varphi \varphi_{,i} + \frac{1}{2} \xi \varphi^2 - m \theta \varphi - \alpha_i \varphi_{,i} \theta \\ & + \frac{1}{2} \int_0^t g(t-s) \varphi_{,i}^2 (s) \, ds. \end{aligned} \tag{2.21}$$

We note that experiments showed that when subject to sudden changes, the viscoelastic response not only depends on the current state of stress but also on all past states of stress. This leads to a constitutive relationship, between the stress and the stain, given by a memory term which appears in the form of a convolution of the stain with a relaxation function.

The constitutive coefficients in (2.21) have the following symmetries

$$C_{ijrs} = C_{rsij} = C_{jirs}, \qquad \beta_{ij} = \beta_{ji},$$
$$D_{ijk} = D_{jik}, \qquad A_{ij} = A_{ji}, \qquad B_{ij} = B_{ji}.$$

Using (2.18), (2.20) and (2.21) we obtain the following constitutive equations

$$\begin{aligned}
T_{ji} &= C_{ijrs}e_{rs} + B_{ij}\varphi + D_{ijk}\varphi_{,k} - \beta_{ij}\theta, \\
h_i &= A_{ij}\varphi_{,j} + D_{rsi}e_{rs} + d_i\varphi - \alpha_i\theta + \int_0^t g\left(t-s\right)\varphi_{,i}\left(s\right)ds, \\
f &= -B_{ij}e_{ij} - \xi\varphi - d_i\varphi_{,i} + m\theta, \\
\rho\eta &= \beta_{ij}e_{ij} + a\theta + m\varphi + \alpha_i\varphi_{,i} \\
q_i &= \theta_{,i}
\end{aligned} \tag{2.22}$$

where g is the relaxation function. In the case of an isotropic materials, the constitutive equations (2.22) becomes

$$\begin{aligned}
T_{ji} =& \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\varphi\delta_{ij} - \beta\theta\delta_{ij}, \\
h_j =& \alpha\varphi_{,j} + \int_0^t g\left(t-s\right)\varphi_{,j}\left(s\right)ds, \\
f =& -be_{rr} - \xi\varphi + m\theta, \\
\rho\eta =& \beta e_{rr} + a\theta + m\varphi, \\
q_i =& \theta_{,i},
\end{aligned} \tag{2.23}$$

where δ_{ij} is the Kronecker's delta, λ, μ are the Lamé moduli and $b, \alpha, \beta, \xi, m, a$ are constitutive coefficients.

In the linear theory, the insertion of

$$\rho\dot{\varepsilon} = \dot{\sigma} + \rho T\dot{\eta} + \rho\dot{T}\eta$$

in (2.10) using (2.18), reduces the equation of energy (2.10), to

$$\rho T_0\dot{\eta} = q_{i,i}. \tag{2.24}$$

Also, the equation of motion (2.5) can be written

$$\rho\ddot{u}_i = T_{ji,j}.$$

For $j = i$, T_{ji} we obtain

$$\begin{aligned} T_{ii} &= \lambda u_{r,r} + 2\mu u_{i,r}\delta_{ir} + b\varphi - \beta\theta \\ &= (\lambda + 2\mu)\, u_{i,k}\delta_{ik} + b\varphi - \beta\theta + \lambda\,(1 - \delta_{ik})\, u_{k,k} \end{aligned}$$

and for $j \neq i$, we have

$$T_{ji} = \mu\,(u_{i,j} + u_{j,i})\,,$$

then,

$$\rho\ddot{u}_i = T_{ji,j} = \mu u_{i,jj} + \mu u_{j,ij} + \lambda u_{r,ri} + 2\mu u_{i,ii} + b\varphi_{,i} - \beta\theta_{,i}$$

which can be written in the form

$$\rho\ddot{u}_i = \mu\Delta u_i + (\lambda + \mu)\,(\operatorname{div} u)_{,i} + b\varphi_{,i} - \beta\theta_{,i}.$$

The equation (2.9) becomes

$$\rho\kappa\ddot{\varphi} = f + h_{j,j}.$$

By the insertion of constitutive equations (2.23), the above equation takes the form

$$\rho\kappa\ddot{\varphi} = \alpha\varphi_{,jj} - bu_{j,j} - \xi\varphi + m\theta + \int_0^t g\,(t-s)\,\varphi_{,jj}\,(s)\,ds$$

which also written

$$\rho\kappa\ddot{\varphi} = \alpha\Delta\varphi - b\,\mathrm{div}\,u - \xi\varphi + m\theta + \int_0^t g\,(t-s)\,\Delta\varphi\,(x,s)\,ds.$$

The insertion of (2.23) in (2.24) gives the heat equation

$$a\dot{\theta} = \theta_{,ii} - \rho T_0\beta\dot{u}_{j,j} - \rho T_0 m\dot{\varphi}$$

which is also written

$$a\dot{\theta} = \Delta\theta - \rho T_0\beta div\dot{u} - \rho T_0 m\dot{\varphi}.$$

Finally the linearized system takes the form

$$\begin{cases} \rho u_{tt} = \mu\Delta u + (\lambda+\mu)\,\nabla\,(\mathrm{div}\,u) + b\nabla\varphi - \beta\nabla\theta, \\ \rho\kappa\varphi_{tt} = \alpha\Delta\varphi - b\,\mathrm{div}\,u - \xi\varphi + m\theta + \int_0^t g\,(t-s)\,\Delta\varphi\,(x,s)\,ds, \\ a\theta_t = \Delta\theta - \rho T_0\beta\,\mathrm{div}\,u_t - \rho T_0 m\varphi_t. \end{cases} \tag{2.25}$$

In the one-dimensional case system (2.25) takes the form

$$\begin{cases} \rho u_{tt} = (\lambda+2\mu)\,u_{xx} + b\varphi_x - \beta\theta_x, \\ \rho\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta + \int_0^t g\,(t-s)\,\varphi_{xx}\,(x,s)\,ds, \\ a\dot{\theta} = \theta_{xx} - \rho T_0\beta u_{tx} - \rho T_0 m\varphi_t \end{cases}$$

which is the system studied in chapter 3.

Chapter 3

General decay in porous thermoelasticity

3.1 Introduction

In this chapter we investigate the asymptotic behavior of the solutions of a one-dimensional porous thermoelastic problem with two dissipations, porous dissipation of memory type and thermal dissipation arising in the heat equation.

It is well known that the combination of a thermal or viscoelastic dissipation with microthermal or porous dissipation leads to an uniform stability. In addition, if dissipation of memory type is present then the rate of decay depends on the relaxation function, Precisely, the decay is of exponential rate if the kernel decays exponentially and the decay is polynomial if the kernel decays polynomially.

Ammar Khodja *et al.*[3] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, \quad \text{in}(0,L)\times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds + k(\varphi_x+\psi) = 0, \quad \text{in}(0,L)\times \mathbb{R}_+, \end{cases}$$

together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\dfrac{\rho_1}{k} = \dfrac{\rho_2}{b}\right)$ and g decay uniformly. Guesmia and Messaoudi [21] obtained the same uniform decay result under weaker conditions on the regularity and the growth of the relaxation function. Also Messaoudi and Mustafa [40] established a more general decay estimate for a wider class of relaxation function.

In porous thermoelasticity, Soufyane [63] considered the following one-dimensional porous thermoelastic system of memory type

$$\begin{cases} \varphi_{tt} = \varphi_{xx} + \psi_x - \theta_x, \quad \text{in} \quad (0,L)\times \mathbb{R}_+, \\ \psi_{tt} = \psi_{xx} - \varphi_x - \psi + \theta - \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds, \quad \text{in} \quad (0,L)\times \mathbb{R}_+, \\ \theta_t = \theta_{xx} - \varphi_{tx} - \psi_t, \quad \text{in} \quad (0,L)\times \mathbb{R}_+, \end{cases}$$

with Dirichlet boundary conditions and proved that the solutions decay exponentially (polynomially) if the relaxation function g decays exponentially (polynomially).

In this chapter we are concerned with the following system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) - \theta + \int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_t - \kappa\theta_{xx} + \varphi_{xt} + \psi_t = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \geq 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in (0,1) \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,1). \end{cases} \tag{1.1}$$

where $\rho_1, \rho_2, \rho_2, k, \alpha, \kappa$ are positive constants and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function. We will adopt some arguments of [63] and [40] to establish a general decay of "weak" solutions, from which the exponential and polynomial decay results of [63] are only special cases.

3.2 Preliminaries

In this section, we present some notations and material needed in our work then, we present our hypotheses and state, without proof, a global existence result and prove several technical lemmas. For the relaxation function g we assume

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function satisfying

$$g(0) > 0, \qquad \alpha - \int_0^\infty g(s)ds = l > 0$$

(H2) There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0.$$

Example. We give here an example of a function which satisfies (H1) and (H2) and is not of exponential or polynomial decay

$$g(t) = ae^{-b(1+t)^{\nu}}, \qquad 0 < \nu < 1,$$

and a and b to chosen properly.

Proposition 2.1. *Assume that (H1), (H2) hold, then, for any* $((\varphi_0, \varphi_1), (\psi_0, \psi_1)) \in (H_0^1(0;1) \times L^2(0;1))^2$ *and* $\theta_0 \in H_0^1(0;1)$, *problem* (1.1) *has a unique global solution*

$$\begin{gathered} (\varphi, \psi) \in (C(\mathbb{R}_+; H_0^1(0;1)) \cap C^1(\mathbb{R}_+; L^2(0;1)))^2, \\ \theta \in C(\mathbb{R}_+; L^2(0;1)) \cap L^2(\mathbb{R}_+; H_0^1(0;1)). \end{gathered} \tag{2.1}$$

Moreover, if

$$((\varphi_0, \varphi_1), (\psi_0, \psi_1)) \in \left(H^2(0;1) \cap H_0^1(0;1) \times H_0^1(0;1)\right)^2, \quad \theta_0 \in H_0^1(0;1)$$

then (1.1) *has a unique (strong) solution*

$$(\varphi, \psi) \in \left(C\left(\mathbb{R}_+; H^2(0;1) \cap H_0^1(0;1)\right) \cap C^1\left(\mathbb{R}_+; H_0^1(0;1)\right) \cap C^2\left(\mathbb{R}_+; L^2(0;1)\right)\right)^2,$$

$$\theta \in C\left(\mathbb{R}_+; H_0^1(0;1)\right) \cap C^1\left(\mathbb{R}_+; L^2(0;1)\right).$$

Proof. The proof of this proposition can be established by using the well-known Galerkin method. □

We introduce the first order energy of Problem (1.1) by

$$\begin{aligned} E(t) =& \frac{1}{2} \int_0^1 \left[\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_3 \theta^2 + \left(\alpha - \int_0^t g(s)ds\right) \psi_x^2 + k\left(\varphi_x + \psi\right)^2\right] \\ &+ \frac{1}{2} g \circ \psi_x \end{aligned} \tag{2.2}$$

where for $v \in L^2(0;1)$,

$$(g \circ v)(t) = \int_0^1 \int_0^t g(t-s)(v(x,t)-v(x,s))^2 dsdx.$$

It is clear that, by *(H1)*, $E(t) \geq 0$.

Lemma 2.1. *Under assumptions (H1) and (H2), we have*

$$\begin{aligned}\int_0^1 \psi_t(t) \int_0^t g(t-s)\psi_{xx}(s)dsdx =& \frac{1}{2}\frac{d}{dt}\left[g \circ \psi_x - \left(\int_0^t g(s)ds\right)\int_0^1 \psi_x^2(t)dx\right] \\ &- \frac{1}{2}g' \circ \psi_x + \frac{1}{2}g(t)\int_0^1 \psi_x^2(t)dx.\end{aligned} \tag{2.3}$$

Proof. Integrating by parts and using the boundary conditions we get

$$\int_0^1 \psi_t(t) \int_0^t g(t-s)\psi_{xx}(s)dsdx = -\int_0^1 \int_0^t g(t-s)\psi_{xt}(t)\psi_x(s)dsdx$$

$$= -\int_0^1 \int_0^t g(t-s)\psi_{xt}(t)\left[\psi_x(s)-\psi_x(t)\right]dsdx - \int_0^1 \int_0^t g(t-s)\psi_{xt}(t)\psi_x(t)dsdx$$

$$= \frac{1}{2}\frac{d}{dt}\int_0^1 \int_0^t \left[g(t-s)[\psi_x(s)-\psi_x(t)]^2 ds\right]dx$$

$$-\frac{1}{2}\int_0^1 \int_0^t g'(t-s)\left[\psi_x(s)-\psi_x(t)\right]^2 dsdx - \frac{1}{2}\left(\int_0^t g(s)ds\right)\int_0^1 \frac{d}{dt}\psi_x^2(t)dx.$$

$$= \frac{1}{2}\frac{d}{dt}\left[g \circ \psi_x - \left(\int_0^t g(s)ds\right)\int_0^1 \psi_x^2(t)\right] - \frac{1}{2}g' \circ \psi_x + \frac{1}{2}g(t)\int_0^1 \psi_x^2(t). \quad \square$$

Lemma 2.2. *Assume that (H1) holds. Then*

$$\int_0^1 \left(\int_0^t g(t-s)(v(t)-v(s))ds\right)^2 dx \leq c_0 g \circ v_x$$

for all $v \in H_0^1(0;1)$,*where* c_0, *here and throughout this section, denotes a generic positive constant.*

Proof. By using Schwarz and Poincaré's inequalities, we get

$$\int_0^1 \left(\int_0^t g(t-s)(v(t)-v(s))ds\right)^2 dx = \int_0^1 \left(\int_0^t g^{\frac{1}{2}}(t-s)g^{\frac{1}{2}}(t-s)(v(t)-v(s))ds\right)^2 dx$$

$$
\begin{aligned}
&\leq \int_0^1 \left(\int_0^t g(s)ds\right)\left(\int_0^t g(t-s)(v(t)-v(s))^2 ds\right)dx \\
&\leq \left(\int_0^t g(s)ds\right)\int_0^1\int_0^t g(t-s)(v(t)-v(s))^2 dsdx \\
&\leq \left(\int_0^t g(s)ds\right)\int_0^t g(t-s)\left(\int_0^1 (v(t)-v(s))^2 dx\right)ds \\
&\leq \left(\int_0^t g(s)ds\right)\int_0^t g(t-s)\left(C\int_0^1 (v_x(t)-v_x(s))^2 dx\right)ds \\
&\leq c_0 \int_0^1\int_0^t g(t-s)(v_x(t)-v_x(s))^2 dsdx = c_0 g \circ v_x. \quad \square
\end{aligned}
$$

Remark 2.2. Similarly, we have

$$\int_0^1 \left(\int_0^t g(t-s)(v_x(t)-v_x(s))ds\right)^2 dx \leq c_0 g \circ v_x.$$

and

$$\int_0^1 \left(\int_0^t -g'(t-s)(v(t)-v(s))ds\right)^2 dx \leq -c_0 g' \circ v_x.$$

Lemma 2.3. *There exists a constant $c_0 > 0$ such that*

$$\int_0^1 \left(\psi_x - \int_0^t g\left(t-s\right)\psi_x\left(s\right)ds\right)^2 dx \leq c_0 \int_0^1 \psi_x^2 dx + c_0 g \circ \psi_x$$

Proof. Using the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ and Remark 2.2, we obtain

$$
\begin{aligned}
\int_0^1 \left(\psi_x - \int_0^t g\left(t-s\right)\psi_x\left(s\right)ds\right)^2 dx &\leq 2\int_0^1 \psi_x^2 dx + 2\int_0^1 \left(\int_0^t g\left(t-s\right)\psi_x\left(s\right)ds\right)^2 dx \\
&\leq 2\int_0^1 \psi_x^2 dx + 2\int_0^1 \left(\int_0^t g\left(t-s\right)\left(\psi_x\left(s\right)-\psi_x\left(t\right)+\psi_x\left(t\right)\right)ds\right)^2 dx
\end{aligned}
$$

$$\leq 2\int_0^1 \psi_x^2 dx + 4\int_0^1 \psi_x^2(t)\left(\int_0^t g(t-s)\,ds\right)^2 dx$$
$$+4\int_0^1 \left(\int_0^t g(t-s)\left(\psi_x(s)-\psi_x(t)\right)ds\right)^2 dx$$
$$\leq c_0\int_0^1 \psi_x^2 dx + c_0 g\circ\psi_x. \quad \square$$

Lemma 2.4. *Assume that (H1) holds. Then, there exists a positive constant c_0 such that*

$$\int_0^1 u\int_0^t g(t-s)v_x(s)dsdx \leq \varepsilon\int_0^1 u^2 dx + \frac{c_0}{\varepsilon}\int_0^1 v_x^2 dx + \frac{c_0}{\varepsilon} g\circ v_x$$

for all $\varepsilon > 0$ and for all $u \in L^2(0,1)$ and $v \in H^1(0,1)$.

Proof. Using Young's inequality, we get

$$\begin{aligned}\int_0^1 u\int_0^t g(t-s)v_x(s)dsdx \leq& \varepsilon\int_0^1 u^2 dx + \frac{c_0}{\varepsilon}\int_0^1\left(\int_0^t g(t-s)v_x(s)ds\right)^2 dx\\ \leq& \varepsilon\int_0^1 u^2 dx + \frac{c_0}{\varepsilon}\int_0^1\left(\int_0^t g(s)ds\right)^2 v_x^2(t)\,dx\\ &+\frac{c_0}{\varepsilon}\int_0^1\left(\int_0^t g(t-s)\left[v_x(s)-v_x(t)\right]ds\right)^2 dx.\end{aligned}$$

Then, the result is obtained by using Remark 2.2. $\square$

3.3 General decay (equal-speed case)

In this section we state and prove the main general decay result for solutions of system (1.1). All calculations are done for strong solutions, the results remain valid for weak solutions by simple density arguments. We start with the following

Theorem 3.1. Let $((\varphi_0,\varphi_1),(\psi_0,\psi_1)) \in \left(H_0^1(0,1)\times L^2(0,1)\right)^2$ *and* $\theta_0 \in H_0^1(0,1)$.

Assume that (H1) and (H2) hold and that

$$\frac{k}{\rho_1} = \frac{\alpha}{\rho_2}. \tag{3.1}$$

Then, there exist two positive constants ω and λ, for which the solution of Problem (1.1) *satisfies*

$$E(t) \leq \lambda e^{-\omega \int_0^t \xi(s)\, ds} \qquad \forall t \geq 0.$$

The proof of this theorem will be established through several lemmas.

Lemma 3.2. *Under assumptions (H1), (H2), the energy satisfies, along the solution of* (1.1),

$$E'(t) = -\frac{\kappa}{2}\int_0^1 \theta_x^2 dx + \frac{1}{2} g' \circ \psi_x - \frac{1}{2} g(t) \int_0^1 \psi_x^2(x,t) dx \leq 0. \tag{3.2}$$

Proof. Multiplying the first equation of (1.1) by φ_t, the second by ψ_t and the third by θ and integrating over $(0,1)$, we get

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \frac{k}{2}\frac{d}{dt}\int_0^1 \varphi_x^2 dx - k\int_0^1 \psi_x \varphi_t dx + \int_0^1 \theta_x \varphi_t dx = 0,$$

$$\frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx + k\int_0^1 \varphi_x \psi_t dx + \frac{k}{2}\frac{d}{dt}\int_0^1 \psi^2$$

$$-\int_0^1 \theta \psi_t dx + \int_0^1 \psi_t(t) \int_0^t g(t-s)\psi_{xx}(x,s) ds dx = 0,$$

and

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \frac{\kappa}{2}\int_0^1 \theta_x^2 dx + \int_0^1 \varphi_{xt}\theta dx + \int_0^1 \psi_t \theta dx = 0,$$

By summation of theses three identities, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_3\theta^2+k\left(\varphi_x+\psi\right)^2+\alpha\psi_x^2\right]dx+\frac{\kappa}{2}\int_0^1\theta_x^2dx \\ +\int_0^1\psi_t\left(t\right)\int_0^t g(t-s)\psi_{xx}(x,s)dsdx=0.\end{aligned} \tag{3.3}$$

Thus, (2.3) leads to equality (3.2) for regular solutions. This equality remains valid for solutions (2.1) by simple density argument. □

Lemma 3.3. *Under assumptions (H1), (H2), the functional*

$$I(t):=-\int_0^1\rho_2\psi_t\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx$$

satisfies, along the solution of (1.1)*, the estimate*

$$\begin{aligned}I'(t)\le-\left(\rho_2\int_0^t g(s)ds-\delta\right)\int_0^1\psi_t^2dx+\delta\int_0^1(\varphi_x+\psi)^2dx \\ +\delta c_0\int_0^1\psi_x^2dx+\delta c_0\int_0^1\theta_x^2dx-\frac{c_0}{\delta}g'\circ\psi_x+c_0\left(\delta+\frac{1}{\delta}\right)g\circ\psi_x,\end{aligned} \tag{3.4}$$

for all $\delta>0$.

Proof. A direct differentiation using (1.1), gives

$$\begin{aligned}I'(t)=&-\int_0^1\left[\alpha\psi_{xx}-k\varphi_x-k\psi+\theta-\int_0^t g(t-s)\psi_{xx}(s)ds\right]\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \\ &-\int_0^1\rho_2\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx-\int_0^t g(s)ds\int_0^1\rho_2\psi_t^2(t)dx \\ =&\int_0^1\alpha\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx+k\int_0^1(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \\ &-\int_0^1\theta\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \\ &-\int_0^1\int_0^t g(t-s)\psi_x(x,s)ds\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx\end{aligned}$$

$$-\int_0^1 \rho_2\psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx - \int_0^t g(s)ds \int_0^1 \rho_2\psi_t^2(t)dx$$

By using Young's and Poincaré's inequalities, Lemma 2.2. and Remark 2.2, we get, for all $\delta > 0$,

$$-\int_0^1 \rho_2\psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx \leq \delta \int_0^1 \psi_t^2 dx - \frac{c_0}{\delta} g' \circ \psi_x,$$

$$\int_0^1 \alpha\psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx \leq \delta \int_0^1 \psi_x^2 dx + \frac{c_0}{\delta} g \circ \psi_x,$$

$$k\int_0^1 (\varphi_x+\psi) \int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \leq \delta \int_0^1 (\varphi_x+\psi)^2 + \frac{c_0}{\delta} g \circ \psi_x,$$

$$-\int_0^1 \theta \int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \leq \delta \int_0^1 \theta_x^2 + \frac{c_0}{\delta} g \circ \psi_x,$$

and

$$\begin{aligned} -\int_0^1 \int_0^t g(t-s)\psi_x(s)ds \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx \\ \leq \delta' \int_0^1 \left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t)+\psi_x(t))ds\right)^2 dx \\ +\frac{c_0}{\delta'} \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \end{aligned}$$

$$\leq 2\delta' \left(\int_0^t g(s)ds\right)^2 \int_0^1 \psi_x^2 dx + \left(2\delta' + \frac{c_0}{\delta'}\right) \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx$$

$$\leq c_0\delta' \int_0^1 \psi_x^2 dx + c_0\left(\delta' + \frac{1}{\delta'}\right) g \circ \psi_x \leq \delta \int_0^1 \psi_x^2 dx + \ c_0\left(\delta + \frac{1}{\delta}\right) g \circ \psi_x,$$

By combining all above estimates the assertion of the lemma is established.□

Lemma 3.4. *Under assumptions (H1), (H2), the functional*

$$J(t) := -\int_0^1 (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)dx$$

satisfies, along the solution of (1.1), *the estimate*

$$\begin{aligned} J'(t) \leq & -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + k(1+\varepsilon)\int_0^1 (\varphi_x+\psi)^2\,dx \\ & +c_0\int_0^1 \psi_x^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \theta_x^2 dx + c_0 g\circ\psi_x, \end{aligned} \tag{3.5}$$

for all $\varepsilon > 0$.

Proof. Direct differentiation, using (1.1), leads to

$$\begin{aligned} J'(t) = & -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx - \int_0^1 \varphi(k\varphi_{xx} + k\psi_x - \theta_x)dx \\ & -\int_0^1 \psi\left[\alpha\psi_{xx} - k\varphi_x - k\psi + \theta - \int_0^t g(t-s)\psi_{xx}(s)ds\right]dx \end{aligned} \tag{3.6}$$

$$= -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + k\int_0^1 \varphi_x(\varphi_x+\psi)dx - \int_0^1 \varphi_x\theta$$

$$+\alpha\int_0^1 \psi_x^2 dx + k\int_0^1 \psi(\varphi_x+\psi)dx - \int_0^1 \psi\theta dx - \int_0^1 \psi_x\int_0^t g(t-s)\psi_x(s)dsdx$$

$$= -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + k\int_0^1 (\varphi_x+\psi)^2 dx + \alpha\int_0^1 \psi_x^2 dx$$

$$-\int_0^1 (\varphi_x+\psi)\,\theta dx - \int_0^1 \psi_x\int_0^t g(t-s)\psi_x(s)dsdx.$$

Using Lemma 2.4, we obtain,

$$-\int_0^1 \psi_x\int_0^t g(t-s)\psi_x(x,s)ds)dx \leq c_0\int_0^1 \psi_x^2(t)\,dx + c_0 g\circ\psi_x.$$

Also, exploiting Young's and Poincaré's inequalities, we get, for all $\varepsilon > 0$,

$$-\int_0^1 (\varphi_x+\psi)\,\theta dx \leq \varepsilon k\int_0^1 (\varphi_x+\psi)^2\,dx + \frac{c_0}{\varepsilon}\int_0^1 \theta_x^2 dx.$$

Substituting these estimates in (3.6), we obtain (3.5). □

Lemma 3.5. *Assume that (H1), (H2) and* (3.1) *hold. Then, the functional*

$$K_1(t) := \int_0^1 \rho_1\varphi_t\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)dsdx\right) + \int_0^1 \rho_2 k\psi_t(\varphi_x+\psi)\,dx$$

satisfies, along the solution of (1.1)*, and for any* $\varepsilon > 0$*, the estimate*

$$\begin{aligned} K_1'(t) &\le k\left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(x,s)ds\right)\right]_{x=0}^{x=1} \\ &+k\rho_2\int_0^1 \psi_t^2dx - k^2(1-\varepsilon c_0)\int_0^1(\varphi_x+\psi)^2dx \\ &+\varepsilon\int_0^1\varphi_t^2dx + \frac{c_0}{\varepsilon}\int_0^1\theta_x^2dx + \frac{c_0}{\varepsilon}\int_0^1\psi_x^2dx + \varepsilon g\circ\psi_x - \frac{c_0}{\varepsilon}g'\circ\psi_x. \end{aligned} \tag{3.7}$$

Proof. Direct differentiation, using (1.1), gives

$$\rho_1\alpha\frac{d}{dt}\int_0^1\varphi_t\psi_xdx = \alpha\int_0^1(k\varphi_{xx}+k\psi_x-\theta_x)\psi_xdx + \rho_1\alpha\int_0^1\varphi_t\psi_{xt}dx$$

$$= \alpha k[\varphi_x\psi_x]_{x=0}^{x=1} - \alpha k\int_0^1\varphi_x\psi_{xx}dx + \alpha k\int_0^1\psi_x^2dx - \alpha\int_0^1\psi_x\theta_xdx + \rho_1\alpha\int_0^1\varphi_t\psi_{xt}dx.$$

Also,

$$-\frac{d}{dt}\int_0^1\rho_1\varphi_t\int_0^t g(t-s)\psi_x(s)dsdx = -\int_0^1(k\varphi_{xx}+k\psi_x-\theta_x)\int_0^t g(t-s)\psi_x(s)dsdx$$

$$-\rho_1 g(0)\int_0^1\varphi_t\psi_xdx - \rho_1\int_0^1\varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx.$$

Finally,

$$k\rho_2\frac{d}{dt}\int_0^1\psi_t(\varphi_x+\psi)\,dx = k\rho_2\int_0^1\psi_t(\varphi_{xt}+\psi_t)\,dx$$

$$+k\int_0^1\left(\alpha\psi_{xx} - k\varphi_x - k\psi + \theta - \int_0^t g(t-s)\psi_{xx}(s)ds\right)(\varphi_x+\psi)\,dx$$

$$= -k\rho_2\int_0^1\psi_{xt}\varphi_tdx + k\rho_2\int_0^1\psi_t^2dx + \alpha k\int_0^1\psi_{xx}\varphi_xdx - \alpha k\int_0^1\psi_x^2dx$$

$$-k^2\int_0^1(\varphi_x+\psi)^2\,dx+k\int_0^1\theta(\varphi_x+\psi)\,dx$$
$$+k\int_0^1\varphi_{xx}\int_0^t g(t-s)\psi_x(s)dsdx-k\left[\varphi_x(x,t)\int_0^t g(t-s)\psi_x(x,s)ds\right]_{x=0}^{x=1}$$
$$+k\int_0^1\psi_x\int_0^t g(t-s)\psi_x(s)dsdx.$$

Therefore, a combination of all above estimates and use of (3.1) lead to

$$\begin{aligned}K_1'(t)=&k\left[\varphi_x\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(x,s)ds\right)\right]_{x=0}^{x=1}-\alpha\int_0^1\psi_x\theta_x dx\\&+k\rho_2\int_0^1\psi_t^2dx-k^2\int_0^1(\varphi_x+\psi)^2\,dx+k\int_0^1\theta(\varphi_x+\psi)\,dx\\&+\int_0^1\theta_x\int_0^t g(t-s)\psi_x(s)dsdx-g(0)\int_0^1\rho_1\varphi_t\psi_x dx\\&-\int_0^1\rho_1\varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx.\end{aligned}\tag{3.8}$$

Using Young's inequality, Remark 2.2 and Lemma 2.4, we get

$$-\alpha\int_0^1\psi_x\theta_x dx\le\varepsilon\int_0^1\psi_x^2+\frac{c_0}{\varepsilon}\int_0^1\theta_x^2dx$$

$$k\int_0^1\theta(\varphi_x+\psi)\,dx\le\frac{c_0}{\varepsilon}\int_0^1\theta_x^2dx+\varepsilon k^2c_0\int_0^1(\varphi_x+\psi)^2\,dx$$
$$\int_0^1\theta_x\int_0^t g(t-s)\psi_x(s)dsdx\le\frac{c_0}{\varepsilon}\int_0^1\theta_x^2(t)\,dx+\varepsilon g\circ\psi_x$$
$$\int_0^1\rho_1\varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx\le\varepsilon\int_0^1\varphi_t^2(t)\,dx-\frac{c_0}{\varepsilon}g'\circ\psi_x$$
$$g(0)\int_0^1\rho_1\varphi_t\psi_x dx\le\varepsilon\int_0^1\varphi_t^2(t)\,dx+\frac{c_0}{\varepsilon}\int_0^1\psi_x^2.$$

A substitution of these estimates in (3.8), inequality (3.7) occurs immediately. □

As in [47], to handle the boundary terms in (3.7), we let $M \in C^1([0;1])$ be a function satisfying

$$M(1) = -M(0) = -2.$$

Lemma 3.6. *Under assumptions (H1), (H2), the functionals K_2 and K_3 defined by*

$$K_2(t) := \int_0^1 \rho_2 M(x)\, \psi_t \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) dx$$

$$K_3(t) := \int_0^1 \rho_1 M(x)\, \varphi_t \varphi_x dx$$

satisfy, along the solution of (1.1), and for any $\varepsilon > 0$, the estimates

$$\begin{aligned} K_2'(t) \leq & -\left(\alpha\psi_x(1,t) - \int_0^t g(t-s)\psi_x(1,s)ds \right)^2 \\ & - \left(\alpha\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds \right)^2 + \varepsilon k \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \frac{c_0}{\varepsilon} \int_0^1 \psi_x^2 + c_0 \int_0^1 \psi_t^2 dx + c_0 \int_0^1 \theta_x^2 + \frac{c_0}{\varepsilon} g \circ \psi_x - c_0 g' \circ \psi_x. \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} K_3'(t) \leq & -k\left(\varphi_x^2(1,t) + \varphi_x^2(0,t) \right) \\ & + c_0 \left(\int_0^1 \varphi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \theta_x^2 dx \right). \end{aligned} \tag{3.10}$$

Proof. Direct differentiation, using (1.1), yields

$$\begin{aligned} K_2'(t) = & \int_0^1 M(x) \left(\alpha\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds \right) \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) dx \\ & - \int_0^1 M(x)(k\varphi_x + k\psi - \theta) \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) dx \end{aligned}$$

$$+\int_0^1 \rho_2 M(x)\,\psi_t\left(\alpha\psi_{xt}-g(0)\,\psi_x-\int_0^t g'(t-s)\psi_x(s)ds\right)dx$$

$$=-\left(\alpha\psi_x(1,t)-\int_0^t g(t-s)\psi_x(1,s)ds\right)^2-\left(\alpha\psi_x(0,t)-\int_0^t g(t-s)\psi_x(0,s)ds\right)^2$$

$$-\frac{1}{2}\int_0^1 M'(x)\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2dx$$

$$-k\int_0^1 M(x)(\varphi_x+\psi)\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)dx$$

$$+\int_0^1 M(x)\,\theta\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)dx-\frac{\rho_2\alpha}{2}\int_0^1 M'(x)\,\psi_t^2dx$$

$$+\int_0^1 \rho_2 M(x)\,\psi_t\left(-g(t)\,\psi_x+\int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)dx.$$

Using Young's inequality and Remark 2.2 we get

$$\int_0^1 \rho_2 M(x)\,\psi_t\left(-g(t)\,\psi_x+\int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)dx$$

$$\leq c_0\int_0^1 \psi_t^2dx+c_0\int_0^1 \psi_x^2dx-c_0g'\circ\psi_x.$$

Again, Young's inequality and Lemma 2.3, lead to

$$-k\int_0^1 M(x)(\varphi_x+\psi)\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)dx$$

$$\leq \varepsilon k\int_0^1 (\varphi_x+\psi)^2dx+\frac{c_0}{\varepsilon}\int_0^1 \psi_x^2+\frac{c_0}{\varepsilon}g\circ\psi_x.$$

The use of the same arguments for the other terms yields the desired result. □

Next,

$$K_3(t):=\int_0^1 \rho_1 M(x)\,\varphi_t\varphi_x dx$$

$$K_3'(t) = \int_0^1 M(x)(k\varphi_{xx} + k\psi_x - \theta_x)\varphi_x dx + \int_0^1 \rho_1 M(x)\varphi_t\varphi_{xt}dx$$

$$= -k\left(\varphi_x^2(1,t) + \varphi_x^2(0,t)\right) - \frac{k}{2}\int_0^1 M'(x)\varphi_x^2 dx + k\int_0^1 M(x)\psi_x\varphi_x dx$$

$$-\int_0^1 M(x)\theta_x\varphi_x dx - \frac{\rho_1}{2}\int_0^1 M'(x)\varphi_t^2 dx.$$

Similar estimates lead to (3.10).

Lemma 3.7. *Under assumptions (H1), (H2), the functional*

$$K_4 := \frac{1}{k}K_1 + \frac{1}{4\varepsilon}K_2 + \frac{\varepsilon}{k}K_3,$$

satisfies along the solution of (1.1) *and for any* $\varepsilon > 0$, *the estimate*

$$K_4'(t) \le -\left(\frac{3}{4}k - \varepsilon c_0\right)\int_0^1 (\varphi_x + \psi)^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \psi_t^2 + \varepsilon c_0\int_0^1 \varphi_t^2 dx$$
$$+\frac{c_0}{\varepsilon^2}\int_0^1 \psi_x^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \theta_x^2 + \frac{c_0}{\varepsilon^2} g\circ\psi_x - \frac{c_0}{\varepsilon} g'\circ\psi_x. \tag{3.11}$$

Proof. By using the inequality

$$\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right) \le \varepsilon\varphi_x^2 + \frac{1}{4\varepsilon}\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2$$

and substituting (3.7), (3.9) and (3.10) in the expression of K_4', we obtain

$$K_4'(t) \le \rho_2\int_0^1 \psi_t^2 dx - k(1-\varepsilon c_0)\int_0^1 (\varphi_x + \psi)^2 dx$$

$$+\varepsilon\int_0^1 \varphi_t^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \theta_x^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \psi_x^2 dx + \varepsilon g\circ\psi_x - \frac{c_0}{\varepsilon} g'\circ\psi_x$$

$$+\frac{k}{4}\int_0^1 (\varphi_x + \psi)^2 dx + \frac{c_0}{\varepsilon^2}\int_0^1 \psi_x^2 + \frac{c_0}{\varepsilon}\int_0^1 \psi_t^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \theta_x^2 + \frac{c_0}{\varepsilon^2} g\circ\psi_x$$

$$-\frac{c_0}{\varepsilon}g' \circ \psi_x + \varepsilon c_0 \left(\int_0^1 \varphi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_x^2 dx + \int_0^1 \theta_x^2 dx \right).$$

By recalling

$$\varphi_x^2 dx \leq 2 \left(\varphi_x + \psi\right)^2 + 2\psi^2$$

and Poincaré's inequality, we get

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 \left(\varphi_x + \psi\right)^2 + c_0 \int_0^1 \psi_x^2.$$

Thus (3.11) is proven. □

Lemma 3.8. *Under assumptions (H1), (H2), the functional*

$$K_5 := \frac{2c_0\varepsilon}{\rho_1} J + K_4,$$

satisfies along the solution of (1.1) *and for a fixed* ε *small enough, the estimate*

$$K_5'(t) \leq -\frac{k}{2} \int_0^1 \left(\varphi_x + \psi\right)^2 dx - \mu \int_0^1 \varphi_t^2 dx + c_0 \int_0^1 \psi_t^2$$
$$+ c_0 \int_0^1 \psi_x^2 dx + c_0 \int_0^1 \theta_x^2 + c_0 g \circ \psi_x - c_0 g' \circ \psi_x, \tag{3.12}$$

for some (fixed) $\mu > 0$.

Proof. Direct differentiation, using (3.5), (3.11) gives

$$K_5'(t) \leq -\left(\frac{3}{4}k - \varepsilon c_0\right) \int_0^1 \left(\varphi_x + \psi\right)^2 dx - c_0\varepsilon \int_0^1 \varphi_t^2 dx + \left(\frac{c_0}{\varepsilon} - 2\varepsilon \frac{c_0\rho_2}{\rho_1}\right) \int_0^1 \psi_t^2$$
$$+ \left(\varepsilon c_0 + \frac{c_0}{\varepsilon^2}\right) \int_0^1 \psi_x^2 dx + c_0 \int_0^1 \theta_x^2 + c_0 g \circ \psi_x - \frac{c_0}{\varepsilon} g' \circ \psi_x.$$

Fixing ε small enough, such that $\frac{3}{4}k - \varepsilon c_0 \geq \frac{k}{2}$ and $\frac{c_0}{\varepsilon} - 2\varepsilon \frac{c_0\rho_2}{\rho_1} > 0$, we arrive at (3.10)(3.10), with $\mu = c_0\varepsilon$. □

As in [64], we use the multiplier w given by the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0. \tag{3.13}$$

Lemma 3.9. *The solution of* (3.13) *satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx, \quad \int_0^1 w_t^2 dx \leq C \int_0^1 \psi_t^2 dx.$$

Proof. Using integration by parties and (3.13) we obtain

$$\int_0^1 w_x^2 dx = \int_0^1 \psi_x w dx = -\int_0^1 w_x \psi dx.$$

Thus, Cauchy Schwarz inequality leads to

$$\int_0^1 w_x^2 dx \leq \int_0^1 |\psi w_x| \, dx \leq \left(\int_0^1 \psi^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 w_x^2 dx \right)^{\frac{1}{2}},$$

then,

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Similarly,

$$\int_0^1 w_{tx}^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Thus, Poincaré's inequality yields

$$\int_0^1 w_t^2 dx \leq C \int_0^1 w_{tx}^2 dx \leq C \int_0^1 \psi_t^2 dx. \quad \square$$

Lemma 3.10. *Under assumptions (H1), (H2), the functional*

$$K_6(t) := \int_0^1 \left(\rho_1 w \varphi_t + \rho_2 \psi_t \psi \right) dx$$

satisfies, along the solution of (1.1) *and for* $0 < \varepsilon_1 < \frac{l}{2}$, *the estimate*

$$K_6'(t) \leq -\frac{l}{2}\int_0^1 \psi_x^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx + \frac{c_0}{\varepsilon_1}\int_0^1 \theta_x^2 dx + \frac{c_0}{\varepsilon_1}\int_0^1 \psi_t^2 dx + c_0 g \circ \psi_x, \tag{3.14}$$

where l is defined in *(H1)*.

Proof. Direct differentiation, using (1.1), yields

$$\begin{aligned}
K_6'(t) &= \int_0^1 w\left(k\varphi_{xx} + k\psi_x - \theta_x\right)dx + \rho_1\int_0^1 w_t\varphi_t dx \\
&+ \int_0^1 \left(\alpha\psi_{xx} - k\varphi_x - k\psi + \theta - \int_0^t g(t-s)\psi_{xx}(s)ds\right)\psi dx + \rho_2\int_0^1 \psi_t^2 dx \\
&= -k\int_0^1 w_x\varphi_x dx + k\int_0^1 w_x^2 dx + \int_0^1 w_x\theta dx + \rho_1\int_0^1 w_t\varphi_t dx \\
&-\alpha\int_0^1 \psi_x^2 dx - k\int_0^1 \varphi_x\psi dx - k\int_0^1 \psi^2 dx + \int_0^1 \theta\psi dx + \rho_2\int_0^1 \psi_t^2 dx \\
&-\int_0^1 \psi_x\left(\int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right)ds\right)dx + \left(\int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx \\
&= k\left(\int_0^1 w_x^2 dx - \int_0^1 \psi^2 dx\right) + \int_0^1 w_x\theta dx + \rho_2\int_0^1 \psi_t^2 dx + \rho_1\int_0^1 w_t\varphi_t dx \\
&-\alpha\int_0^1 \psi_x^2 dx - k\int_0^1 \left(w_x + \psi\right)\varphi_x dx + \int_0^1 \theta\psi dx \\
&-\int_0^1 \psi_x\left(\int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right)ds\right)dx + \left(\int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx.
\end{aligned}$$

Integration by parts, using (3.13), gives

$$k\int_0^1 \left(w_x + \psi\right)\varphi_x dx = 0.$$

Consequently, Lemma 3.9 yields

$$K_6'(t) \leq -\alpha\int_0^1 \psi_x^2 dx + \int_0^1 w_x\theta dx + \rho_2\int_0^1 \psi_t^2 dx + \rho_1\int_0^1 w_t\varphi_t dx + \int_0^1 \theta\psi dx$$

$$-\int_0^1 \psi_x \left(\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)dx+\left(\int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx.$$

Young's inequality, Remark 2.2, Poincaré's inequality and Lemma 3.9 then yield

$$K_6'(t) \leq -\left(\alpha-\int_0^t g(s)ds-\varepsilon_1-\varepsilon c_0\right)\int_0^1 \psi_x^2 dx+\varepsilon_1\int_0^1 \varphi_t^2 dx$$
$$+\frac{c_0}{\varepsilon_1}\int_0^1 \psi_t^2 dx+\frac{c_0}{\varepsilon} g\circ\psi_x+\frac{c_0}{\varepsilon_1}\int_0^1 \theta_x^2 dx, \qquad \forall\varepsilon>0.$$

Recalling (H1), we deduce

$$K_6'(t) \leq -(l-\varepsilon_1-\varepsilon c_0)\int_0^1 \psi_x^2 dx+\varepsilon_1\int_0^1 \varphi_t^2 dx+\frac{c_0}{\varepsilon_1}\int_0^1 \theta_x^2 dx+\frac{c_0}{\varepsilon_1}\int_0^1 \psi_t^2 dx+\frac{c_0}{\varepsilon} g\circ\psi_x.$$

Thus, for $0<\varepsilon_1<\frac{l}{2}$ we can fix ε small enough such that (3.14) is established. □

Proof of Theorem 3.1.

For $N_1, N_2, N_3>0$, we define the Lyapunov functional by

$$\mathcal{L}(t)=N_1 E(t)+N_2 I(t)+N_3 K_6(t)+\frac{1}{k}K_5(t).$$

By combining (2.3), (3.2), (3.12) and (3.14), we have

$$\mathcal{L}'(t) \leq -\left(\frac{N_3 l}{2}-c_0-N_2\delta c_0\right)\int_0^1 \psi_x^2 dx-\left(\frac{N_1\kappa}{2}-c_0-N_2\delta c_0-\frac{N_3 c_0}{\varepsilon_1}\right)\int_0^1 \theta_x^2 dx$$
$$-(\mu-N_3\varepsilon_1)\int_0^1 \varphi_t^2 dx-\left[\left(\rho_2\int_0^t g(s)ds-\delta\right)N_2-c_0-N_3\frac{c_0}{\varepsilon_1}\right]\int_0^1 \psi_t^2 dx$$
$$-\left(\frac{1}{2}-\delta N_2\right)\int_0^1 (\varphi_x+\psi)^2 dx+\left(N_2 c_0(\delta+\frac{1}{\delta})+N_3 c_0+c_0\right)g\circ\psi_x$$
$$+\left(\frac{N_1}{2}-N_2\frac{c_0}{\delta}-c_0\right)g'\circ\psi_x.$$

Let $\delta = \dfrac{1}{4N_2}$ and $g_0 = \rho_2 \displaystyle\int_0^{t_0} g(s)ds$ for some fixed $t_0 > 0$. Then for all $t \geq t_0$, and $0 < \varepsilon_1 < \dfrac{l}{2}$ we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\frac{N_3 l}{2} - \frac{5c_0}{4}\right)\int_0^1 \psi_x^2 dx - \left(\frac{N_1\kappa}{2} - \frac{5c_0}{4} - \frac{N_3 c_0}{\varepsilon_1}\right)\int_0^1 \theta_x^2 dx \\
& - (\mu - N_3\varepsilon_1)\int_0^1 \varphi_t^2 dx - \left(N_2 g_0 - \frac{1}{4} - \frac{N_3 c_0}{\varepsilon_1} - c_0\right)\int_0^1 \psi_t^2 dx \\
& -\frac{1}{4}\int_0^1 (\varphi_x + \psi)^2 dx + \left(4N_2^2 + N_3 c_0 + \frac{5c_0}{4}\right) g \circ \psi_x \\
& + \left(\frac{N_1}{2} - 4N_2^2 c_0 - c_0\right) g' \circ \psi_x.
\end{aligned} \tag{3.15}$$

Next, we choose N_3 large enough so that

$$c_1 = \frac{N_3 l}{2} - \frac{5c_0}{4} > 0,$$

then ε_1 small enough so that

$$c_2 = \mu - N_3\varepsilon_1 > 0.$$

After that, we pick N_2 large enough such that

$$c_3 = N_2 g_0 - \frac{1}{4} - c_0 - \frac{N_3 c_0}{\varepsilon_1} > 0.$$

Finally, we take N_1 large enough so that

$$c_4 = \frac{N_1\kappa}{2} + c_0 - \frac{N_3 c_0}{\varepsilon_1} > 0 \text{ and } \; c_5 = \frac{N_1}{2} - 4N_2^2 c_0 - c_0 > 0.$$

Therefore, (3.15) takes the form

$$\mathcal{L}'(t) \leq -c_1 \int_0^1 \psi_x^2 dx - c_2 \int_0^1 \varphi_t^2 dx - c_3 \int_0^1 \psi_t^2 dx - c_4 \int_0^1 \theta_x^2 dx$$

$$-\frac{1}{4}\int_0^1 (\varphi_x+\psi)^2\,dx + c_5 g'\circ\psi_x + \left(4N_2^2 + N_3 c_0 + \frac{5c_0}{4}\right) g\circ\psi_x.$$

Thus, for two positive constants λ and C, we have

$$\mathcal{L}'(t) \leq -\lambda E(t) + Cg\circ\psi_x, \quad \forall t\geq t_0 \tag{3.16}$$

On the other hand, by choosing N_1 even large (if needed), we get $\mathcal{L}\sim E$.

Multiplying (3.16) by $\xi(t)$ and using (H1), (H2) and (3.2) we arrive at

$$\begin{aligned}\xi(t)\,\mathcal{L}' &\leq -\lambda\xi(t)\,E(t) + C\xi(t)\,g\circ\psi_x\\ &\leq -\lambda\xi(t)\,E(t) - Cg'\circ\psi_x\\ &\leq -\lambda\xi(t)\,E(t) - CE'(t).\end{aligned}$$

By using the fact that $\xi'(t)\leq 0$, we obtain

$$\frac{d}{dt}\left(\xi(t)\,\mathcal{L}(t) + CE(t)\right) \leq -\lambda\xi(t)\,E(t), \quad \forall t\geq t_0.$$

Again, by noting that

$$F = \xi\mathcal{L} + CE \sim E(t),$$

we obtain, for some positive constant ω,

$$F'(t) \leq -\omega\xi(t)\,F(t), \quad \forall t\geq t_0.$$

Integrating over (t_0,t) we easily see that

$$\begin{aligned}F(t) &\leq F(t_0)\,e^{-\omega\int_{t_0}^t \xi(s)\,ds}\\ &\leq Ce^{-\omega\int_0^t \xi(s)\,ds}, \quad \forall t\geq t_0.\end{aligned}$$

The assertion of Theorem 3.1 is then obtained by virtue of the boundedness of E and ξ and the fact that $F \sim E$. □

3.4 General decay (nonequal-speed case)

In this section we consider (1.1) for the case of different propagation speeds, that is (3.1) does not holds.We will establish a general decay result which depends on the asymptotic behavior of g, and the regularity of solutions.

It is worth mentioning that in the case of nonzero history, the integral in (1.1) will be infinite (from 0 to infinity) instead of the finite integral. This situation was discussed by Rivera and others (see references [63], [65], [52]) for only exponential decaying relaxation functions. However, in this work, we are concerned with more general relaxation functions and our analysis cannot be applied directly to the situation of nonzero history.

In this section we present our hypotheses and state, without proof, a global existence result as well as some well-known lemmas.

We recall that the first-order energy of the system (1.1) is given by

$$E\left(t\right)=\frac{1}{2}\int_{0}^{1}\left[\rho_{1}\varphi_{t}^{2}+\rho_{2}\psi_{t}^{2}+\rho_{3}\theta^{2}+\left(\alpha-\int_{0}^{t}g(s)ds\right)\psi_{x}^{2}+k\left(\varphi_{x}+\psi\right)^{2}\right]dx$$

$$+\frac{1}{2}g\circ\psi_{x},$$

where, for any $v \in L^2(0;1)$,

$$(g \circ v)(t) := \int_0^1 \int_0^t g(t-s)(v(x,t) - v(x,s))^2 \, ds dx$$

and note that, throughout this section, c denotes a generic positive constant.

Lemma 4.1. *Assume that* $(H1)$, $(H2)$ *hold. Then we have, for all* $t > 0$,

$$\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx \leq g_0(t)\, g \circ \psi_x, \tag{4.1}$$

where $g_0(t) := \int_0^t g(s)\, ds$.

Proof. By using Schwarz inequality, we get

$$\begin{aligned} &\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\ &= \int_0^1 \left(\int_0^t g^{\frac{1}{2}}(t-s) g^{\frac{1}{2}}(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\ &\leq \left(\int_0^t g(s) ds \right) \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 ds \right) dx. \quad \square \end{aligned}$$

Remark 4.2. For $-g'$ instead of g, (4.1) becomes

$$\int_0^1 \left(\int_0^t -g'(t-s)(\psi(t) - \psi(s)) ds \right)^2 dx \leq -g(0)\, g' \circ \psi_x.$$

Now, we state the main theorem in this section.

Theorem 4.3. *Let* $(\varphi_0, \psi_0) \in \left[H^2(0,1) \cap H_0^1(0;1)\right]^2, \theta_0 \in H_0^1(0;1)$ *and* $(\varphi_1, \psi_1) \in \left[H_0^1(0;1)\right]^2$ *and suppose that (H1), (H2) hold and* $\dfrac{k}{\rho_1} \neq \dfrac{\alpha}{\rho_2}$. *Then, for any* t_0, *there exists a positive constant* λ *for which the "strong" solution of problem* (1.1) *satisfies*

$$E(t) \leq \frac{\lambda}{\int_0^t \xi(s)\, ds}, \qquad \forall t \geq t_0. \tag{4.2}$$

Examples. To illustrate our general estimate (4.2), we give here some examples of functions which satisfy (H1) and (H2).

1) Let $g(t) = \frac{a}{(1+t)^{\nu}}$, for $\nu > 1$ and $0 < a < \alpha(1+\nu)$, then (H2) is satisfied with $\xi(t) = \frac{\nu}{1+t}$ and (4.2) becomes

$$E(t) \leq \frac{C}{\ln(1+t)}, \quad t > 0.$$

2) Let $g(t) = ae^{-(1+t)^{\nu}}$, for $0 < \nu < 1$ and $a > 0$ to be chosen such that (H1) is satisfied, then (H2) is satisfied with $\xi(t) = \nu(1+t)^{\nu-1}$ and (4.2) becomes

$$E(t) \leq \frac{C}{(1+t)^{\nu}}, \quad t > 0.$$

3) Let $g(t) = ae^{-(\ln(1+t))^{q}}$, for $q > 1$ and $a > 0$ small enough so that (H1) is satisfied, then (H2) is satisfied with $\xi(t) = \frac{q}{1+t}(\ln(1+t))^{q-1}$ and (4.2) becomes

$$E(t) \leq \frac{C}{(\ln(1+t))^{q}}, \quad t > 0.$$

Proof of Theorem 4.3.

The proof of Theorem 4.3 will be established through several lemmas.

Lemma 4.4. *Under the assumptions (H1), (H2), the functional*

$$I_1(t) := -\rho_2 \int_0^1 \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))\, ds dx$$

satisfies, along the solution of (1.1)*, the estimate*

$$I_1'(t) \leq -\rho_2 \left(\int_0^t g(s)\, ds - \delta \right) \int_0^1 \psi_t^2 dx + \delta \int_0^1 \psi_x^2 dx + \delta \int_0^1 \theta_x^2 dx$$

$$+\delta\int_0^1(\varphi_x+\psi)^2\,dx+c_\delta g_0(t)\,g\circ\psi_x-c_\delta g'\circ\psi_x, \tag{4.3}$$

where c_δ is a constant depending on δ.

Proof. By differentiating I_1 and using (1.1), we arrive at

$$\begin{aligned}
I_1'(t)=&-\rho_2\int_0^1\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx-\rho_2\left(\int_0^t g(s)\,ds\right)\int_0^1\psi_t^2dx\\
&+\alpha\int_0^1\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\\
&+k\int_0^1(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&-\int_0^1\theta\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\\
&-\int_0^1\int_0^t g(t-s)\psi_x(s)\,ds\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx.
\end{aligned}$$

Using Young's and Poincaré's inequalities, Remark 4.2 and (4.1), we get

$$-\rho_2\int_0^1\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,dsdx\le\rho_2\delta\int_0^1\psi_t^2dx-c_\delta g'\circ\psi_x,$$

$$\begin{aligned}
\alpha\int_0^1\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\le&\frac{\delta}{2}\int_0^1\psi_x^2dx\\
&+c_\delta\int_0^1\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2dx\\
\le&\frac{\delta}{2}\int_0^1\psi_x^2dx+c_\delta g_0(t)\,g\circ\psi_x,
\end{aligned}$$

$$k\int_0^1(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\le\delta\int_0^1(\varphi_x+\psi)^2\,dx+c_\delta g_0(t)\,g\circ\psi_x,$$

$$-\int_0^1\theta\int_0^t g(t-s)(\psi(t)-\psi(s))\,dsdx\le\delta\int_0^1\theta_x^2dx+c_\delta g_0(t)\,g\circ\psi_x.$$

Finally, the last term

$$-\int_0^1\int_0^t g(t-s)\psi_x(s)\,ds\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,dsdx\le$$

$$\frac{\delta}{4\left(\int_0^\infty g\left(s\right)ds\right)^2}\int_0^1\left(\int_0^t g\left(t-s\right)\psi_x\left(s\right)ds\right)^2 dx$$
$$+\,c_\delta\int_0^1\left(\int_0^t g\left(t-s\right)\left(\psi_x\left(t\right)-\psi_x\left(s\right)\right)ds\right)^2 dx.$$

By using

$$\int_0^1\left(\int_0^t g\left(t-s\right)\psi_x\left(s\right)ds\right)^2 dx \le \int_0^1\left(\int_0^t g\left(t-s\right)\left[\psi_x\left(s\right)+\psi_x\left(t\right)-\psi_x\left(t\right)\right]ds\right)^2 dx$$

$$\le 2\int_0^1\left(\int_0^t g\left(t-s\right)\psi_x\left(t\right)ds\right)^2 dx+2\int_0^1\left(\int_0^t g\left(t-s\right)\left[\psi_x\left(t\right)-\psi_x\left(s\right)\right]ds\right)^2 dx$$
$$\le 2\left(\int_0^\infty g\left(s\right)ds\right)^2\int_0^1\psi_x^2\left(t\right)dx+2\int_0^1\left(\int_0^t g\left(t-s\right)\left[\psi_x\left(t\right)-\psi_x\left(s\right)\right]ds\right)^2 dx$$

and recalling Lemma 4.1, we get

$$-\int_0^1\int_0^t g\left(t-s\right)\psi_x\left(s\right)ds\int_0^t g\left(t-s\right)\left(\psi_x\left(t\right)-\psi_x\left(s\right)\right)dsdx \le$$

$$\le \frac{\delta}{2}\int_0^1\psi_x^2\left(t\right)dx+\left[\frac{\delta}{2\left(\int_0^\infty g\left(s\right)ds\right)^2}+c_\delta\right]\int_0^1\left(\int_0^t g\left(t-s\right)\left(\psi_x\left(s\right)-\psi_x\left(t\right)\right)ds\right)^2 dx$$
$$\le \frac{\delta}{2}\int_0^1\psi_x^2\left(t\right)dx+c_\delta g_0\left(t\right)g\circ\psi_x.$$

Combining all the above inequalities, we obtain the desired estimate. □

Lemma 4.5. *Under the assumptions* $(H1)$, $(H2)$, *the functional*

$$I_2(t):=-\int_0^1(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t)dx$$

satisfies, along the solution of (1.1), *the estimate*

$$I_2'(t)\le-\int_0^1(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx+2k\int_0^1\left(\varphi_x+\psi\right)^2dx+c\int_0^1\psi_x^2dx$$
$$+c\int_0^1\theta_x^2dx+cg_0\left(t\right)g\circ\psi_x. \tag{4.4}$$

Proof: Similar to proof of Lemma 3.4. $\square$

Lemma 4.6. *Under the assumptions* $(H1)$, $(H2)$, *the functional*

$$I_3(t) := \frac{\rho_1}{k}\int_0^1 \varphi_t\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(x,s)dsdx\right) + \int_0^1 \rho_2\psi_t(\varphi_x+\psi)\,dx$$

satisfies, along the solution of (1.1), *the following estimate*

$$\begin{aligned} I_3'(t) \le \frac{1}{2\varepsilon}&\left(\alpha\psi_x(1,t) - \int_0^t g(t-s)\psi_x(1,s)ds\right)^2 \\ +\frac{1}{2\varepsilon}\left(\alpha\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2 &+ \frac{\varepsilon}{2}\left(\varphi_x^2(1,t)+\varphi_x^2(0,t)\right) \\ +\rho_2\int_0^1 \psi_t^2dx - k(1-\varepsilon c)\int_0^1(\varphi_x+\psi)^2dx &+ \varepsilon\int_0^1\varphi_t^2dx + \frac{c}{\varepsilon}\int_0^1\theta_x^2dx \\ +\frac{c}{\varepsilon}\int_0^1\psi_x^2dx + \varepsilon g\circ\psi_x - \frac{c}{\varepsilon}g'\circ\psi_x &+ \left(\frac{\alpha\rho_1}{k}-\rho_2\right)\int_0^1\varphi_t\psi_{xt}dx, \end{aligned} \tag{4.5}$$

for all $\varepsilon > 0$.

Proof. Using (1.1), we get

$$\frac{d}{dt}\left(\frac{\rho_1}{k}\int_0^1 \alpha\varphi_t\psi_xdx\right) = \frac{\alpha}{k}\int_0^1\left(k(\varphi_x+\psi)_x - \theta_x\right)\psi_xdx + \frac{\rho_1\alpha}{k}\int_0^1\varphi_t\psi_{xt}dx$$

$$= \alpha\left[\varphi_x\psi_x\right]_{x=0}^{x=1} - \alpha\int_0^1\varphi_x\psi_{xx}dx + \alpha\int_0^1\psi_x^2dx - \frac{\alpha}{k}\int_0^1\psi_x\theta_xdx + \frac{\rho_1\alpha}{k}\int_0^1\varphi_t\psi_{xt}dx,$$

$$\frac{d}{dt}\left(\int_0^1\rho_2\psi_t(\varphi_x+\psi)\,dx\right) = \rho_2\int_0^1\psi_t(\varphi_{xt}+\psi_t)\,dx$$

$$+\int_0^1\left(\alpha\psi_{xx} - k(\varphi_x+\psi) + \theta - \int_0^t g(t-s)\psi_{xx}(x,s)ds\right)(\varphi_x+\psi)\,dx$$

$$= -\rho_2\int_0^1\psi_{xt}\varphi_tdx + \rho_2\int_0^1\psi_t^2dx + \alpha\int_0^1\psi_{xx}\varphi_xdx - \alpha\int_0^1\psi_x^2dx - k\int_0^1(\varphi_x+\psi)^2dx$$

$$+\int_0^1\theta(\varphi_x+\psi)\,dx + \int_0^1\varphi_{xx}(t)\int_0^t g(t-s)\psi_x(x,s)dsdx$$

$$-\left[\varphi_x(t)\int_0^t g(t-s)\psi_x(x,s)ds\right]_{x=0}^{x=1}+\int_0^1\psi_x(t)\int_0^t g(t-s)\psi_x(x,s)dsdx,$$

and

$$\begin{aligned}
-\frac{\rho_1}{k}\left(\frac{d}{dt}\int_0^1\varphi_t\int_0^t g(t-s)\psi_x(x,s)dsdx\right) &=\\
-\int_0^1\left((\varphi_x+\psi)_x-\frac{1}{k}\theta_x\right)&\int_0^t g(t-s)\psi_x(x,s)dsdx\\
-g(0)\frac{\rho_1}{k}\int_0^1\varphi_t\psi_x dx-\frac{\rho_1}{k}&\int_0^1\varphi_t\int_0^t g'(t-s)\psi_x(x,s)dsdx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_3'(t)=&\left[\varphi_x(t)\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(x,s)ds\right)\right]_{x=0}^{x=1}\\
&-\frac{\alpha}{k}\int_0^1\psi_x\theta_x dx+\left(\frac{\rho_1\alpha}{k}-\rho_2\right)\int_0^1\varphi_t\psi_{xt}dx\\
&+\rho_2\int_0^1\psi_t^2dx-k\int_0^1(\varphi_x+\psi)^2dx+\int_0^1\theta(\varphi_x+\psi)dx\\
&+\int_0^1\varphi_{xx}(t)\int_0^t g(t-s)\psi_x(x,s)dsdx+\int_0^1\psi_x(t)\int_0^t g(t-s)\psi_x(x,s)dsdx\\
&-\int_0^1\left((\varphi_x+\psi)_x-\frac{1}{k}\theta_x\right)\int_0^t g(t-s)\psi_x(x,s)dsdx\\
&-g(0)\frac{\rho_1}{k}\int_0^1\varphi_t\psi_x(x,t)dx-\frac{\rho_1}{k}\int_0^1\varphi_t\int_0^t g'(t-s)\psi_x(x,s)dsdx.
\end{aligned}$$

Using Young's and Poincaré's inequalities and Lemma 2.4, we obtain

$$\left[\varphi_x\left(\alpha\psi_x-\int_0^t g(t-s)\psi_x(x,s)ds\right)\right]_{x=0}^{x=1}\leq\frac{\varepsilon}{2}\left(\varphi_x^2(1,t)+\varphi_x^2(0,t)\right)$$

$$+\frac{1}{2\varepsilon}\left(\alpha\psi_x(1,t)-\int_0^t g(t-s)\psi_x(1,s)ds\right)^2+\frac{1}{2\varepsilon}\left(\alpha\psi_x(0,t)-\int_0^t g(t-s)\psi_x(0,s)ds\right)^2,$$

$$\int_0^1\theta(\varphi_x+\psi)dx\leq\varepsilon c\int_0^1(\varphi_x+\psi)^2dx+\frac{c}{\varepsilon}\int_0^1\theta_x^2dx,$$

$$-\frac{\alpha}{k}\int_0^1 \psi_x\theta_x dx \le \varepsilon\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon}\int_0^1 \theta_x^2 dx,$$

$$-g\left(0\right)\frac{\rho_1}{k}\int_0^1 \varphi_t\psi_x(x,t)dx \le \varepsilon\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 \psi_x^2 dx,$$

$$\frac{1}{k}\int_0^1 \theta_x\int_0^t g(t-s)\psi_x(x,s)dsdx \le \frac{c}{\varepsilon}\int_0^1 \theta_x^2 dx + \varepsilon\int_0^1 \psi_x^2 dx + \varepsilon g\circ\psi_x$$

and

$$-\frac{\rho_1}{k}\int_0^1 \varphi_t\int_0^t g'(t-s)\psi_x(x,s)dsdx \le \varepsilon\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 \psi_x^2 dx - \frac{c}{\varepsilon}g'\circ\psi_x.$$

Combining all the above inequalities yields (4.5) for all $\varepsilon > 0$. $\square$

To estimate the boundary terms in (4.5) we need the following

Lemma 4.7. *Let $m\left(x\right) = 2 - 4x$ and (φ, ψ, θ) be the strong solution of* (1.1)*. Then for any $\varepsilon > 0$, the functionals*

$$I_4(t) := \rho_2\int_0^1 m\left(x\right)\psi_t(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds)dx$$

and

$$I_5(t) := \rho_1\int_0^1 m\left(x\right)\varphi_t\varphi_x dx$$

satisfy

$$\begin{aligned} I_4'(t) \le & -\left(\alpha\psi_x\left(1,t\right) - \int_0^t g(t-s)\psi_x(1,s)ds\right)^2 - \left(\alpha\psi_x\left(0,t\right) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2 \\ & +\varepsilon k\int_0^1 \left(\varphi_x+\psi\right)^2 dx + c\left(1+\frac{1}{\varepsilon}\right)\left(\int_0^1 \varphi_x^2 dx + g_0\left(t\right)g\circ\psi_x\right) \\ & +c\int_0^1 \left(\psi_t^2+\theta_x^2\right)dx - cg'\circ\psi_x \end{aligned} \tag{4.6}$$

and

$$I_5'(t) \le -k\left(\varphi_x^2(1,t) + \varphi_x^2(0,t)\right) + c\int_0^1 \left(\varphi_t^2 + \varphi_x^2 + \psi_x^2 + \theta_x^2\right)dx. \tag{4.7}$$

Proof. Differentiating I_4, using (1.1) and properties of m, we obtain

$$\begin{aligned}
I_4'(t) = &\int_0^1 m(x)\left(\alpha\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - k(\varphi_x+\psi) + \theta\right)\\
&\times\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx\\
&+\rho_2\int_0^1 m(x)\psi_t\\
&\times\left(\alpha\psi_{xt} + \psi_x(t)\int_0^t g'(t-s)\,ds - g(t)\,\psi_x(t) - \int_0^t g'(t-s)\psi_x(s)ds\right)dx\\
= &\int_0^1 m(x)\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx\\
&-\int_0^1 m(x)\left(k(\varphi_x+\psi) - \theta\right)\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx\\
&+\alpha\rho_2\int_0^1 m(x)\psi_t\psi_{xt}dx + \rho_2\int_0^1 m(x)\psi_t\int_0^t g'(t-s)\left(\psi_x(t) - \psi_x(s)\right)dsdx\\
&-\rho_2 g(t)\int_0^1 m(x)\psi_t\psi_x(t)dx
\end{aligned}$$

$$\begin{aligned}
= &-\left(\alpha\psi_x(1,t)\,dx - \int_0^t g(t-s)\psi_x(1,s)ds\right)^2 - \left(\alpha\psi_x(0,t)\,dx - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2\\
&+2\int_0^1\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx + 2\rho_2\alpha\int_0^1 \psi_t^2 dx\\
&-k\int_0^1 m(x)\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x+\psi)\,dx\\
&+\int_0^1 m(x)\,\theta\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx\\
&+\rho_2\int_0^1 m(x)\psi_t\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))dsdx - \rho_2 g(t)\int_0^1 m(x)\psi_t\psi_x dx.
\end{aligned}$$

By using Lemma 4.1 and the fact that $(a+b)^2 \leq 2a^2 + 2b^2$, we get

$$2\int_0^1 \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx =$$

$$2\int_0^1 \left(\left(\alpha - \int_0^t g(s)\,ds\right)\psi_x(t) + \int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right)ds\right)^2 dx$$
$$\leq c\int_0^1 \psi_x^2 dx + c\int_0^1 \left(\int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right)ds\right)^2 dx$$
$$\leq c\left(\int_0^1 \psi_x^2 dx + g_0(t)\, g \circ \psi_x\right).$$

Similarly, we have

$$-k\int_0^1 m(x)\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x + \psi)\,dx$$

$$\leq \varepsilon k\int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\varepsilon}\int_0^1 \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx$$
$$\leq \varepsilon k\int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\varepsilon}\left(\int_0^1 \psi_x^2 dx + g_0(t)\, g \circ \psi_x\right),$$
$$\int_0^1 m(x)\,\theta\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx$$
$$\leq c\int_0^1 \theta_x^2 dx + \frac{c}{\varepsilon}\left(\int_0^1 \psi_x^2 dx + g_0(t)\, g \circ \psi_x\right)$$

and

$$\rho_2\int_0^1 m(x)\,\psi_t\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))ds dx - \rho_2 g(t)\int_0^1 m(x)\,\psi_t\psi_x dx$$

$$\leq c\int_0^1 \left(\psi_t^2 + \psi_x^2\right)dx - cg' \circ \psi_x.$$

A combination of all the above estimate leads to (4.6).

To prove (4.7), we differentiate I_5, we obtain

$$\begin{aligned} I_5'(t) =& \int_0^1 m(x)\left(k(\varphi_x+\psi)_x-\theta_x\right)\varphi_x dx+\rho_1\int_0^1 m(x)\,\varphi_t\varphi_{xt}dx \\ =& \frac{k}{2}\left[m(x)\,\varphi_x^2(x)\right]_0^1+2k\int_0^1\varphi_x^2(x)\,dx+2\rho_1 k\int_0^1\varphi_t^2dx \\ &+k\int_0^1 m(x)\,\psi_x\varphi_x dx-k\int_0^1 m(x)\,\theta_x\varphi_x dx \end{aligned}$$

then, Young's inequality, for the last two integrals, yields (4.7). □

Lemma 4.8. *Under the assumptions* $(H1)$, $(H2)$, *the functional*

$$I_6 := I_3+\frac{1}{2\varepsilon}I_4+\frac{\varepsilon}{2k}I_5$$

satisfies, along the strong solution of (1.2), *the estimate*

$$\begin{aligned} I_6'(t) \le& -\left(\frac{k}{2}-\varepsilon c\right)\int_0^1(\varphi_x+\psi)^2dx+\frac{c}{\varepsilon}\int_0^1\left(\psi_t^2+\theta_x^2\right)dx+c\varepsilon\int_0^1\varphi_t^2dx \\ &+\frac{c}{\varepsilon^2}\int_0^1\psi_x^2dx+\frac{c}{\varepsilon^2}g_0(t)\,g\circ\psi_x-\frac{c}{\varepsilon}g'\circ\psi_x+\left(\frac{\alpha\rho_1}{k}-\rho_2\right)\int_0^1\varphi_t\psi_{xt}dx, \end{aligned} \tag{4.8}$$

for any $0<\varepsilon<1$.

Proof. Substitution of (4.5)-(4.7) in the expression of $I_6'(t)$ and use of

$$\int_0^1\varphi_x^2 \le 2\int_0^1(\varphi_x+\psi)^2dx+2\int_0^1\psi^2dx,$$

together with Poincaré's inequality lead to (4.8). □

Lemma 4.9. *Under the assumptions* $(H1)$, $(H2)$, *the functional*

$$G_1 := I_6+\frac{1}{16}I_2$$

satisfies, along the strong solution of (1.1) *and for* ε *small enough, the estimate*

$$G_1'(y) \le -\frac{k}{4}\int_0^1(\varphi_x+\psi)^2dx-\frac{\rho_1}{32}\int_0^1\varphi_t^2dx+c\left(\int_0^1\psi_t^2dx+\int_0^1\psi_x^2dx+\int_0^1\theta_x^2dx\right)$$

$$+c\left(g_0\left(t\right)g\circ\psi_x-g'\circ\psi_x\right)+\left(\frac{\alpha\rho_1}{k}-\rho_2\right)\int_0^1\varphi_t\psi_{xt}dx. \tag{4.9}$$

Proof. By exploiting (4.4) and (4.8) we arrive at

$$G_1'(t)\le-\left(\frac{3k}{8}-\varepsilon c\right)\int_0^1\left(\varphi_x+\psi\right)^2dx-\left(\frac{\rho_1}{16}-c\varepsilon\right)\int_0^1\varphi_t^2dx+\left(\frac{c}{\varepsilon^2}+c\right)\int_0^1\psi_x^2dx$$

$$+\left(\frac{c}{\varepsilon^2}+c\right)g_0\left(t\right)g\circ\psi_x-\frac{c_0}{\varepsilon}g'\circ\psi_x+\left(\frac{\alpha\rho_1}{k}-\rho_2\right)\int_0^1\varphi_t\psi_{xt}dx$$
$$+\left(\frac{c}{\varepsilon}-\frac{\rho_2}{16}\right)\int_0^1\psi_t^2dx+\left(c+\frac{c}{\varepsilon}\right)\int_0^1\theta_x^2dx.$$

By choosing ε small enough such that

$$\frac{3k}{8}-\varepsilon c\ge\frac{k}{4},\quad\frac{\rho_1}{16}-c\varepsilon\ge\frac{\rho_1}{32},\quad\frac{c}{\varepsilon}-\frac{\rho_2}{16}\ge0,$$

we obtain (4.9). $\square$

Now, we introduce the function

$$\omega\left(x\right):=-\int_0^x\psi\left(y,t\right)dy+\left(\int_0^1\psi\left(y,t\right)dy\right)x.$$

One can easily see that

$$\omega_x\left(x\right)=-\psi\left(x,t\right)+\int_0^1\psi\left(y,t\right)dy$$

and

$$\omega_t\left(x\right):=-\int_0^x\psi_t\left(y,t\right)dy+\left(\int_0^1\psi_t\left(y,t\right)dy\right)x.$$

A simple calculation then yields

$$
\begin{aligned}
\int_0^1 \omega_x^2 dx &\leq 2\int_0^1 \psi^2(x,t)\,dx + 2\left(\int_0^1 \psi(y,t)\,dy\right)^2 dx \\
&\leq 4\int_0^1 \psi^2 dx \\
\int_0^1 \omega_t^2 dx &\leq 4\int_0^1 \psi_t^2 dx.
\end{aligned} \tag{4.10}
$$

Lemma 4.10. *Under the assumptions* $(H1)$, $(H2)$, *the functional*

$$
G_2(t) := \int_0^1 (\rho_2\psi\psi_t + \rho_1\omega\varphi_t)\,dx
$$

satisfies, along the solution of (1.1), *the estimate*

$$
\begin{aligned}
G_2'(t) \leq& -\frac{l}{2}\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon}\int_0^1 \psi_t^2 dx \\
&+\varepsilon\int_0^1 \varphi_t^2 dx + cg_0(t)\, g\circ\psi_x + \frac{1}{l}\int_0^1 \theta_x^2 dx.
\end{aligned} \tag{4.11}
$$

Proof. A differentiation of G_2 gives

$$
\begin{aligned}
G_2'(t) =& \int_0^1 \left(\rho_2\psi_t^2 + \rho_1\omega_t\varphi_t\right)dx + \int_0^1 \omega\left(k(\varphi_x+\psi)_x - \theta_x\right)dx \\
&+\int_0^1 \psi\left(\alpha\psi_{xx} - k(\varphi_x+\psi) + \theta - \int_0^t g(t-s)\psi_{xx}(x,s)ds\right)dx \\
=& \int_0^1 \left(\rho_2\psi_t^2 + \rho_1\omega_t\varphi_t\right)dx - \alpha\int_0^1 \psi_x^2 dx + \int_0^1 \psi\theta dx + \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(x,s)dsdx \\
&-k\int_0^1 (\omega_x+\psi)(\varphi_x+\psi)dx - \int_0^1 \omega\theta_x dx \\
=& \rho_2\int_0^1 \psi_t^2 dx + \rho_1\int_0^1 \omega_t\varphi_t dx - k\int_0^1 (\omega_x+\psi)(\varphi_x+\psi)dx + \int_0^1 (\omega_x+\psi)\,\theta dx \\
&-\left(\alpha - \int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx + \int_0^1 \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right)dsdx.
\end{aligned}
$$

By noting that

$$\omega_x + \psi = \int_0^1 \psi\left(y, t\right) dy,$$

we get

$$-k\int_0^1 \left(\omega_x + \psi\right)\left(\varphi_x + \psi\right) dx = -k\left(\int_0^1 \psi\left(y, t\right) dy\right)^2 \leq 0,$$

$$-\left(\alpha - \int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx \leq -l\int_0^1 \psi_x^2 dx.$$

By using Young's, Poincaré's inequalities, Lemma 4.1 and (4.10), we arrive at

$$\rho_1\int_0^1 \omega_t\varphi_t dx \leq \varepsilon\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 \psi_t^2 dx,$$

$$\begin{aligned}\int_0^1 \left(\omega_x + \psi\right)\theta dx &= \left(\int_0^1 \psi\left(y, t\right) dy\right)\int_0^1 \theta dx \\ &\leq \frac{l}{4}\int_0^1 \psi_x^2 dx + \frac{1}{l}\int_0^1 \theta_x^2 dx\end{aligned}$$

and

$$\int_0^1 \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x\left(t\right)\right) ds dx \leq \frac{l}{4}\int_0^1 \psi_x^2 dx + cg_0\left(t\right) g\circ\psi_x.$$

Combining all the above estimate, (4.11) is established. □

Next, we would like to deal with the second-order energy. For this reason, we first note that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_0^t g\left(t-s\right)\psi_{xx}\left(s\right) ds\right) &= \frac{\partial}{\partial t}\int_0^t g\left(s\right)\psi_{xx}\left(t-s\right) ds \\ &= g\left(t\right)\psi_{0xx}\left(x\right) + \int_0^t g\left(s\right)\psi_{xxt}\left(t-s\right) ds.\end{aligned}$$

Therefore, by differentiating system (1.1) with respect to t, we arrive at

$$\begin{cases} \rho_1\varphi_{ttt} - k(\varphi_x+\psi)_{xt} + \theta_{tx} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{ttt} - \alpha\psi_{txx} + k(\varphi_{tx}+\psi_t) - \theta_t + g(t)\,\psi_{0xx}(x) \\ \qquad\qquad + \displaystyle\int_0^t g(t-s)\,\psi_{xxt}(s)\,ds = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_{tt} - \kappa\theta_{txx} + \varphi_{xtt} + \psi_{tt} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+, \\ \varphi_t(0,t) = \varphi_t(1,t) = \psi_t(0,t) = \psi_t(1,t) = \theta_t(0,t) = \theta_t(1,t) = 0, \ t \geq 0. \end{cases} \tag{4.12}$$

We also, define the second-order energy by

$$E_*(t) = \frac{1}{2} g \circ \psi_{xt} + \frac{1}{2}\int_0^1 \left(\alpha - \int_0^t g(s)\,ds\right)\psi_{xt}^2(t)\,dx$$
$$+\frac{1}{2}\int_0^1 \left[\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + \rho_3\theta_t^2 + k(\varphi_{xt}+\psi_t)^2\right]dx.$$

Lemma 4.11. *Let $\psi \in C^2(\mathbb{R}_+; H_0^1(0;1))$ and g satisfies (H1), then*

$$\int_0^1 \psi_{tt}(t)\int_0^t g(t-s)\,\psi_{xxt}(s)\,dsdx = -\frac{1}{2}g' \circ \psi_{xt} + \frac{1}{2}g(t)\int_0^1 \psi_{xt}^2(t)\,dx$$
$$+\frac{1}{2}\frac{d}{dt}\left[g\circ\psi_{xt} - \left(\int_0^t g(s)\,ds\right)\int_0^1 \psi_{xt}^2(t)\,dx\right]. \tag{4.13}$$

Proof. Using integration by parts with respect to x, we get

$$\int_0^1 \psi_{tt}(t)\int_0^t g(t-s)\,\psi_{xxt}(s)\,dsdx = -\int_0^1 \psi_{xtt}(t)\int_0^t g(t-s)\,\psi_{xt}(s)\,dsdx$$

$$= -\int_0^1 \psi_{xtt}(t)\int_0^t g(t-s)\left(\psi_{xt}(s) - \psi_{xt}(t)\right)dsdx - \int_0^1 \psi_{xtt}(t)\int_0^t g(t-s)\,\psi_{xt}(t)\,dsdx$$
$$= \frac{1}{2}\int_0^1\int_0^t g(t-s)\frac{d}{dt}\left(\psi_{xt}(s) - \psi_{xt}(t)\right)^2 dsdx - \frac{1}{2}\left(\int_0^t g(s)\,ds\right)\frac{d}{dt}\int_0^1 \psi_{xt}^2(t)\,dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^t g(t-s)(\psi_{xt}(s)-\psi_{xt}(t))^2\,dsdx - \frac{1}{2}\frac{d}{dt}\left(\left(\int_0^t g(s)\,ds\right)\int_0^1 \psi_{xt}^2(t)\,dx\right)$$
$$-\frac{1}{2}\int_0^1\int_0^t g'(t-s)(\psi_{xt}(s)-\psi_{xt}(t))^2\,dsdx + \frac{1}{2}g(t)\int_0^1 \psi_{xt}^2(t)\,dx$$
$$= \frac{1}{2}\frac{d}{dt}\left[g\circ\psi_{xt} - \left(\int_0^t g(s)\,ds\right)\int_0^1 \psi_{xt}^2(t)\,dx\right] - \frac{1}{2}g'\circ\psi_{xt} + \frac{1}{2}g(t)\int_0^1 \psi_{xt}^2(t)\,dx. \quad \square$$

Lemma 4.12. *Let* (φ,ψ,θ) *be the strong solution of* (1.1), *Then the energy* $E_*(t)$ *satisfies, for all* $t \geq 0$,

$$E_*'(t) \leq \frac{1}{2}g'\circ\psi_{xt} - g(t)\int_0^1 \psi_{tt}\psi_{0xx}(x)\,dx \tag{4.14}$$

and

$$E_*(t) \leq M, \qquad \forall t \geq 0. \tag{4.15}$$

Proof. Multiplying the equations of system (4.12) by φ_{tt}, ψ_{tt} and θ_t respectively and integrating over $(0,1)$ using boundary conditions, (H1), (4.13) and the assumptions on the constitutive constants, we obtain (4.14) for regular solutions. This inequality remains valid for strong solutions of (1.1) by a simple density argument.

To prove (4.15), we use the fact that

$$\frac{1}{2}g(t)\int_0^1 \left(\sqrt{\rho_2}\psi_{tt} + \frac{1}{\sqrt{\rho_2}}\psi_{0xx}\right)^2 dx \geq 0, \quad \forall t \geq 0;$$

which implies that

$$\frac{1}{2}g(t)\int_0^1 \left(\rho_2\psi_{tt}^2 + \frac{1}{\rho_2}\psi_{0xx}^2\right) dx \geq -g(t)\int_0^1 \psi_{tt}\psi_{0xx}(x)\,dx.$$

Thus,

$$E_*^{'}(t) \leq \frac{1}{2} g(t) \int_0^1 \left(\rho_2 \psi_{tt}^2 + \frac{1}{\rho_2} \psi_{0xx}^2 \right) dx$$
$$\leq g(t) E_*(t) + \frac{1}{2\rho_2} g(t) \int_0^1 \psi_{0xx}^2 dx.$$

Consequently,

$$\left(E_*^{'}(t) - g(t) E_*(t) \right) e^{-\int_0^t g(s)\, ds} \leq \frac{e^{-\int_0^t g(s)\, ds}}{2\rho_2} g(t) \int_0^1 \psi_{0xx}^2 dx,$$

which implies that

$$\frac{d}{dt} \left(E_*(t) e^{-\int_0^t g(s)\, ds} \right) \leq \frac{1}{2\rho_2} g(t) \int_0^1 \psi_{0xx}^2 dx.$$

A simple integration yields

$$E_*(t) e^{-\int_0^t g(s)\, ds} - E_*(0) \leq \frac{1}{2\rho_2} \left(\int_0^t g(s)\, ds \right) \int_0^1 \psi_{0xx}^2 dx,$$

then,

$$E_*(t) e^{-\int_0^{+\infty} g(s)\, ds} \leq E_*(t) e^{-\int_0^t g(s)\, ds} \leq E_*(0) + \frac{1}{2\rho_2} (\alpha - l) \int_0^1 \psi_{0xx}^2 dx$$

which gives (4.15). $\square$

Now let $t_0 > 0$, set $g_1 = \displaystyle\int_0^{t_0} g(s)\, ds$ and define, for $N_1, N_2, N_3 > 0$, the Lyapunov functional

$$\mathcal{L}(t) := N_1 (E(t) + E_*(t)) + N_2 I_1(t) + N_3 G_2(t) + G_1(t).$$

By noting that

$$E'(t) \leq -\kappa \int_0^1 \theta_x^2 dx + \frac{1}{2} g' \circ \psi_x, \quad E_*'(t) \leq -g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x)\, dx$$

and combining the estimates (4.3), (4.9) and (4.11), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\frac{lN_3}{2} - \delta N_2 - c\right) \int_0^1 \psi_x^2 dx - \left(\frac{\rho_1}{32} - \varepsilon N_3\right) \int_0^1 \varphi_t^2 dx \\
& -\left(\rho_2 N_2 g_1 - \rho_2 N_2 \delta - \frac{cN_3}{\varepsilon} - c\right) \int_0^1 \psi_t^2 dx - \left(\frac{k}{4} - \delta N_2\right) \int_0^1 (\varphi_x + \psi)^2 dx \\
& -\left(\kappa N_1 - \delta N_2 - c - \frac{1}{l}\right) \int_0^1 \theta_x^2 dx + \left(\frac{\alpha \rho_1}{k} - \rho_2\right) \int_0^1 \varphi_t \psi_{xt} dx \\
& + C_{N_2 N_3} g_0(t)\, g \circ \psi_x + \left(\frac{N_1}{2} - N_2 c_\delta - c\right) g' \circ \psi_x - N_1 g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x)\, dx.
\end{aligned}$$

By taking $\delta = \dfrac{k}{8N_2}$, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\frac{lN_3}{2} - (\frac{k}{8} + c)\right) \int_0^1 \psi_x^2 dx - \left(\frac{\rho_1}{32} - \varepsilon N_3\right) \int_0^1 \varphi_t^2 dx \\
& -\left(\rho_2 N_2 g_1 - \frac{cN_3}{\varepsilon} - (\frac{k\rho_2}{8} + c)\right) \int_0^1 \psi_t^2 dx - \frac{k}{8} \int_0^1 (\varphi_x + \psi)^2 dx \\
& -\left(\kappa N_1 - \frac{k}{8} - c - \frac{1}{l}\right) \int_0^1 \theta_x^2 dx + \left(\frac{\alpha \rho_1}{k} - \rho_2\right) \int_0^1 \varphi_t \psi_{xt} dx \\
& C_{N_2 N_3} g_0(t)\, g \circ \psi_x + \left(\frac{N_1}{2} - N_2 c - c\right) g' \circ \psi_x - N_1 g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x)\, dx
\end{aligned}$$

for all $t \geq t_0$.

We choose N_3 large enough such that $\dfrac{lN_3}{2} - (\dfrac{k}{8} + c) > 0$, then we choose $\varepsilon < \dfrac{\rho_1}{32N_3}$ to get $\dfrac{\rho_1}{32} - \varepsilon N_3 > 0$. Next we choose N_2 large enough so that $\rho_2 N_2 g_1 - \dfrac{cN_3}{\varepsilon} - c > 0$.

Consequently, we have for all $t \geq t_0$,

$$\begin{aligned} \mathcal{L}'(t) \leq & -c\left(\int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx + \int_0^1 (\varphi_x + \psi)^2 dx\right) \\ & - \left(\kappa N_1 - c - \frac{1}{l}\right)\int_0^1 \theta_x^2 dx + cg_0(t)\, g \circ \psi_x + \left(\frac{N_1}{2} - c\right) g' \circ \psi_x \\ & + \left(\frac{\alpha\rho_1}{k} - \rho_2\right)\int_0^1 \varphi_t \psi_{xt} dx - N_1 g(t)\int_0^1 \psi_{tt}\psi_{0xx}(x)\, dx. \end{aligned} \tag{4.16}$$

To estimate the term $\left(\frac{\alpha\rho_1}{k} - \rho_2\right)\int_0^1 \varphi_t \psi_{xt} dx$, we prove the following lemma.

Lemma 4.13. *Let (φ, ψ, θ) be the strong solution of (1.1), then for any $\varepsilon > 0$ and $t \geq t_0$, we have*

$$\left(\frac{\alpha\rho_1}{k} - \rho_2\right)\int_0^1 \varphi_t \psi_{xt} dx \leq \varepsilon\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon}\left(g_0(t)\, g \circ \psi_{xt} - g' \circ \psi_x\right) + \frac{c}{\varepsilon}E(0)\, g(t). \tag{4.17}$$

Proof. We have, for all $t \geq t_0$,

$$\begin{aligned} \left(\frac{\alpha\rho_1}{k} - \rho_2\right)\int_0^1 \varphi_t \psi_{xt} dx = & \left(\frac{\frac{\alpha\rho_1}{k} - \rho_2}{\int_0^t g(s)\, ds}\right)\int_0^1 \varphi_t \int_0^t g(t-s)\, \psi_{xt}(s)\, ds dx \\ & + \left(\frac{\frac{\alpha\rho_1}{k} - \rho_2}{\int_0^t g(s)\, ds}\right)\int_0^1 \varphi_t \int_0^t g(t-s)\left(\psi_{xt}(t) - \psi_{xt}(s)\right) ds dx. \end{aligned}$$

By using Young's inequality, the fact that

$$\frac{\left(\frac{\alpha\rho_1}{k} - \rho_2\right)}{\int_0^t g(s)\, ds} \leq \frac{1}{g_1}\left(\frac{\alpha\rho_1}{k} - \rho_2\right)$$

and Lemma 4.1 (for ψ_{xt}) we get, for all $\varepsilon > 0$,

$$\frac{\left(\frac{\alpha\rho_1}{k} - \rho_2\right)}{\int_0^t g(s)\, ds}\int_0^1 \varphi_t \int_0^t g(t-s)\left(\psi_{xt}(t) - \psi_{xt}(s)\right) ds dx \leq \frac{\varepsilon}{2}\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon} g_0(t)\, g \circ \psi_{xt}. \tag{4.18}$$

On the other hand, by integrating by parts with respect to t, using Young's inequality and the fact that $(\psi_x(t)-\psi_{0x})^2 \le 2\psi_x^2(t)+2\psi_{0x}^2$, we get

$$\begin{aligned}
&\frac{\left(\frac{\alpha\rho_1}{k}-\rho_2\right)}{\int_0^t g(s)\,ds}\int_0^1 \varphi_t \int_0^t g(t-s)\,\psi_{xt}(s)\,dsdx \\
&= \frac{\left(\frac{\alpha\rho_1}{k}-\rho_2\right)}{\int_0^t g(s)\,ds}\int_0^1 \varphi_t\left(g(0)\,\psi_x(t)-g(t)\,\psi_{0x}-\int_0^t g'(t-s)\,\psi_x(s)\,ds\right)dx \\
&= \frac{\left(\frac{\alpha\rho_1}{k}-\rho_2\right)}{\int_0^t g(s)\,ds}\int_0^1 \varphi_t\left(g(t)\,\psi_x(t)-g(t)\,\psi_{0x}-\int_0^t g'(t-s)\,(\psi_x(t)-\psi_x(s))\,ds\right)dx \\
&\le \frac{\varepsilon}{2}\int_0^1 \varphi_t^2dx+\frac{c}{\varepsilon}g^2(t)\int_0^1\left(\psi_x^2(t)+\psi_{0x}^2\right)dx-\frac{c}{\varepsilon}g'\circ\psi_x.
\end{aligned}$$

Noting that

$$\int_0^1 \psi_x^2(t)\,dx \le \frac{1}{l}E(t)\le cE(0), \qquad \forall t\ge 0.$$

Consequently, the boundedness of g yields

$$\frac{c}{\varepsilon}g^2(t)\int_0^1\left(\psi_x^2(t)+\psi_{0x}^2\right)dx \le \frac{c}{\varepsilon}g(t)\,E(0), \quad \forall t\ge 0.$$

Thus,

$$\frac{\left(\frac{\alpha\rho_1}{k}-\rho_2\right)}{\int_0^t g(s)\,ds}\int_0^1 \varphi_t \int_0^t g(t-s)\,\psi_{xt}(s)\,dsdx \le \frac{\varepsilon}{2}\int_0^1\varphi_t^2dx+\frac{c}{\varepsilon}g(t)\,E(0)-\frac{c}{\varepsilon}g'\circ\psi_x. \tag{4.19}$$

A combination of all the above leads to (4.17). $\square$

Lemma 4.14. *Assume that (φ,ψ,θ) is the strong solution of (1.1). Then for all $t\ge t_0$, we have*

$$\begin{aligned}
\mathcal{L}'(t) \le& -c\left(\int_0^1\psi_x^2dx+\int_0^1\varphi_t^2dx+\int_0^1\psi_t^2dx+\int_0^1(\varphi_x+\psi)^2\,dx+\int_0^1\theta_x^2dx\right) \\
&+cg_0(t)\,(g\circ\psi_x+g\circ\psi_{tx})+cg(t)\left(E(0)+E_*(0)+\int_0^1\psi_{0xx}^2dx\right).
\end{aligned} \tag{4.20}$$

Proof. Young's inequality and (4.15) yield

$$-\int_0^1 \psi_{tt}\psi_{0xx}dx \leq \frac{1}{2}\int_0^1 \left(\psi_{tt}^2 + \psi_{0xx}^2\right) dx \leq c. \tag{4.21}$$

We then insert (4.17) and (4.21) in (4.16) to obtain

$$\begin{aligned}\mathcal{L}'(t) \leq & -c\left(\int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx + \int_0^1 (\varphi_x + \psi)^2 dx\right) \\ & - \left(\kappa N_1 - c - \frac{1}{l}\right)\int_0^1 \theta_x^2 dx + \varepsilon \int_0^1 \varphi_t^2 dx + cg_0(t)\, g \circ \psi_x \\ & + \left(\frac{N_1}{2} - \frac{c}{\varepsilon} - c\right) g' \circ \psi_x + \frac{c}{\varepsilon} g_0(t)\, g \circ \psi_{xt} + \frac{c}{\varepsilon} g(t) + N_1 c g(t).\end{aligned}$$

We choose ε small enough and N_1 large enough so that

$$\kappa N_1 - c - \frac{1}{l} > 0,\ \frac{N_1}{2} - \frac{c}{\varepsilon} - c > 0,\ \text{and } \mathcal{L}(t) \geq cE(t),$$

thus, we obtain (4.20).

The fact that

$$-c\left(\int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx + \int_0^1 (\varphi_x + \psi)^2 dx + \int_0^1 \theta_x^2 dx\right)$$

$$\leq -E(t) + cg \circ \psi_x$$

allows (4.20) to be written

$$\mathcal{L}'(t) \leq -cE(t) + c\left[(1 + g_0(t))\, g \circ \psi_x + g_0(t)\, g \circ \psi_{xt}\right] + cg(t), \qquad \forall t \geq t_0. \tag{4.22}$$

Recalling $(H1)$, $(H2)$, one can easily see that

$$\xi(t)\, g \circ \psi_x(t) \leq -g' \circ \psi_x,\ \text{and } \xi(t) g \circ \psi_{xt}(t) \leq -g' \circ \psi_{xt}.$$

Consequently, multiplying (4.22) by $\xi(t)$, we obtain

$$\xi(t)E(t) \leq -c\xi(t)\mathcal{L}'(t) - c\left[(1+g_0(t))g' \circ \psi_x + g_0(t)g' \circ \psi_{xt}\right] + cg(t)\xi(t). \quad (4.23)$$

The non-increasingness of ξ leads, for $v \in \{\psi_x, \psi_{tx}\}$, to

$$\xi(t)(g \circ v) = \int_0^1 \xi(t)\int_0^t g(t-s)(v(t)-v(s))^2 dsdx$$

$$\leq \int_0^1\int_0^t \xi(t-s)g(t-s)(v(t)-v(s))^2 dsdx \leq -g' \circ v,$$

Thus, Integrating (4.23) over $[t_0, t]$, using (4.14), (4.15) and the fact that $2E'(t) \leq g' \circ \psi_x$, we get

$$\begin{aligned}
&\int_{t_0}^t \xi(s)E(s)\,ds \\
&\leq c\left(\xi(t_0)\mathcal{L}(t_0) - \xi(t)\mathcal{L}(t) + \int_{t_0}^t \xi'(s)\mathcal{L}(s)\,ds\right) \\
&-c\int_{t_0}^t \left[(1+g_0(s))E'(s) + g_0(s)\left(E_*'(s) + g(s)\int_0^1 \psi_{tt}\psi_{0xx}(x)\,dx\right)\right]ds \\
&+c\int_{t_0}^t g(s)\xi(s)\,ds.
\end{aligned} \quad (4.24)$$

Using $(H1)$, we easily see that

$$\mathcal{L}(t_0) \leq c\left[E(t_0) + E_*(t_0)\right] \leq c,$$

$$\int_{t_0}^t \xi'(s)\mathcal{L}(s)\,ds - \xi(t)\mathcal{L}(t) \leq 0, \qquad g_0(s) \leq \int_0^\infty g(s)\,ds = \alpha - l$$

$$\int_{t_0}^t g(s)\xi(s)\,ds \leq \xi(0)\int_0^\infty g(s)\,ds = \xi(0)(\alpha - l),$$

$$\text{and } \int_{t_0}^t g_0(s)g(s)\,ds \leq \int_{t_0}^t \left(\int_0^\infty g(\tau)\,d\tau\right)g(s)\,ds \leq (\alpha - l)\int_0^{+\infty} g(s)\,ds.$$

Therefore, using (4.21), estimate (4.24) becomes

$$\begin{aligned}\int_{t_0}^{t}\xi(s)\,E(s)\,ds &\leq c-c\int_{t_0}^{t}\left[E'(s)+E'_{*}(s)\right]ds+c\int_{t_0}^{t}g_0(s)\,g(s)\,ds \qquad (4.25)\\ &\leq c+E(t_0)+E_{*}(t_0)-E_{*}(t)+(\alpha-l)\int_{0}^{+\infty}g(s)\,ds\leq c.\end{aligned}$$

Thus, recalling that E is positive and non-increasing, (4.25) gives

$$\begin{aligned}E(t)\int_0^t\xi(s)\,ds \leq \int_0^t\xi(s)\,E(s)\,ds &= \int_0^{t_0}\xi(s)\,E(s)\,ds+\int_{t_0}^{t}\xi(s)\,E(s)\,ds\\ &\leq t_0\xi(0)\,E(0)+c,\end{aligned}$$

which yields

$$E(t)\leq\frac{\lambda}{\int_0^t\xi(s)\,ds}\quad\forall t\geq t_0.$$

This completes the proof of Theorem 4.3.

Chapter 4

Thermoelasticity type III

4.1 Introduction

The Fourier's law of heat conduction,

$$q + k\nabla\theta = 0,$$

gives, along with the law of conservation of energy and in absence of internal heat source, the heat transport equation

$$\theta_t = k\Delta\theta.$$

A direct consequence is, that any thermal disturbance at a point, in material conducting heat, has an instantaneous effect elsewhere in the body. This is physically unrealistic. To overcome this deficiency many theories were developed, one of which is proposed by

Cattaneo (1948-1958) which generalized Fourier's law by

$$\tau q_t + q + k\nabla\theta = 0,$$

where τ is nonnegative constant. The corresponding heat equation reads thus,

$$\tau\theta_{tt} + \theta_t = k\Delta\theta.$$

This equation is of hyperbolic type and allows heat to propagate as wave with finite speed. Green and Naghdi [14]- [17] proposed three thermoelastic theories based on an entropy equality rather than the usual entropy inequality. In each of these theories the heat flux is given by a different constitutive assumption. Three theories were obtained and called respectively, thermoelasticity type I, type II and type III.

In this chapter we are concerned with thermoelasticity of type III. In section 2 we study a linear damped porous thermoelastic system. However, section 3 is devoted to the study of Timoshenko type system.

4.2 Porous thermoelasticity

We consider the following one-dimensional linear damped porous thermoelastic system of type III

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) - \theta + a\psi_t = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_{tt} - \kappa\theta_{xx} + \varphi_{xtt} + \psi_{tt} - k\theta_{txx} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \theta(x,0) = \theta_0(x), \quad 0 \le x \le 1 \\ \varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta_t(x,0) = \theta_1(x), \quad 0 \le x \le 1 \\ \varphi(x,t) = \psi(x,t) = \theta(x,t) = 0, \quad x = 0,1, \quad t \ge 0, \end{cases} \tag{2.1}$$

where φ is the displacement, ψ is the volume fraction, θ is the temperature difference and $\rho_1, \rho_2, \rho_3, \alpha, \kappa, k$ and a are constitutive constants. In this section we will investigate how strong is it the damping given by $a\psi_t$ $(a > 0)$ to stabilize system (2.1) uniformly. The type as well as the rate of decay of this system will be determined following the wave's propagation speed.

Many papers have appeared where the authors used different dissipative mechanisms at the microscopic and/or macroscopic levels to stabilize the vibrations. Messaoudi and Said-Houari [41] considered a type III linear thermoelastic system of Timoshenko type and proved an exponential decay result. Also, Messaoudi and Said-Houari [42] considered a Timoshenko-type system of type III of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \displaystyle\int_0^\infty g\left(s\right)\psi_{xx}\left(x,t-s\right)ds + \beta\theta_x = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, \quad \text{in } (0,1)\times \mathbb{R}_+ \\ \varphi\left(x,0\right) = \varphi_0\left(x\right), \quad \psi\left(x,0\right) = \psi_0\left(x\right), \quad \theta\left(x,0\right) = \theta_0\left(x\right), \quad 0\le x\le 1 \\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right), \quad \psi_t\left(x,0\right) = \psi_1\left(x\right), \quad \theta_t\left(x,0\right) = \theta_1\left(x\right), \quad 0\le x\le 1 \\ \varphi\left(x,t\right) = \psi\left(x,t\right) = \theta_x\left(x,t\right) = 0, \quad x=0,1, \quad t>0. \end{cases} \tag{2.2}$$

and proved that the system (2.2) decays exponentially (respectively polynomially) if g decays exponentially (respectively polynomially) in the case of equal speed $(\frac{\rho_1}{K} = \frac{\rho_2}{b})$. However, the decay is of polynomial rate otherwise $\left(\frac{\rho_1}{K} \neq \frac{\rho_2}{b}\right)$.

In this section several decay results depending on the wave's propagation speeds, will be obtained for system (2.1).

In the aim of exhibiting the dissipative nature of (2.1), we differentiate the first and the second equations of (2.1) with respect to t and introduce new dependent variables

$\phi = \varphi_t$ and $\Psi = \psi_t$. So system (2.1) takes the form

$$\begin{cases} \rho_1 \phi_{tt} - k(\phi_x + \Psi)_x + \theta_{tx} = 0, & \text{in } (0,1) \times \mathbb{R}_+ \\ \rho_2 \Psi_{tt} - \alpha \Psi_{xx} + k(\phi_x + \Psi) - \theta_t + a\Psi_t = 0, & \text{in } (0,1) \times \mathbb{R}_+ \\ \rho_3 \theta_{tt} - \kappa \theta_{xx} + \phi_{xt} + \Psi_t - k\theta_{txx} = 0, & \text{in } (0,1) \times \mathbb{R}_+ \\ \phi(x,0) = \phi_0(x), \quad \Psi(x,0) = \Psi_0(x), \quad \theta(x,0) = \theta_0(x), & 0 \le x \le 1, \\ \phi_t(x,0) = \phi_1(x), \quad \Psi_t(x,0) = \Psi_1(x), \quad \theta_t(x,0) = \theta_1(x), & 0 \le x \le 1 \\ \phi(x,t) = \Psi(x,t) = \theta(x,t) = 0, \quad x = 0,1, & t \ge 0. \end{cases} \tag{2.3}$$

The first energy associated to (2.3) is given by

$$E_1(t) = E(\phi, \Psi, \theta) = \frac{1}{2} \int_0^1 \left(\rho_1 \phi_t^2 + \rho_2 \Psi_t^2 + k(\phi_x + \Psi)^2 + \alpha \Psi_x^2 + \rho_3 \theta_t^2 + \kappa \theta_x^2 \right) dx.$$

4.2.1 Uniform decay

The main result of this section is the following

Theorem 2.1. *Let* $((\phi_0, \phi_1), (\Psi_0, \Psi_1), (\theta_0, \theta_1)) \in \left(H_0^1(0,1) \times L^2(0,1)\right)^3$ *be given and suppose that*

$$\frac{\rho_1}{k} = \frac{\rho_2}{\alpha} \tag{2.4}$$

holds. Then, there exist two positive constants ω *and* λ, *independent of the initial data and* t, *for which the solution of* (2.3) *satisfies*

$$E_1(t) \le \lambda e^{-\omega t}, \quad t > 0. \tag{2.5}$$

The proof of our result will be established through several lemmas.

Lemma 2.2. *The energy* $E_1(t)$ *satisfies*

$$E_1'(t) = -a\int_0^1 \Psi_t^2 dx - k\int_0^1 \theta_{xt}^2 dx \leq 0. \tag{2.6}$$

Proof. Multiplying the equations (2.3) by ϕ_t, Ψ_t and θ_t respectively, integrating over $(0,1)$, we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \phi_t^2 + \frac{k}{2}\frac{d}{dt}\int_0^1 \phi_x^2 + k\int_0^1 \Psi\phi_{xt} + \int_0^1 \theta_{xt}\phi_t = 0,$$

$$\frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \Psi_t^2 + \frac{\alpha}{2}\frac{d}{dt}\int_0^1 \Psi_x^2 + k\int_0^1 \phi_x\Psi_t + \frac{k}{2}\frac{d}{dt}\int_0^1 \Psi^2 - \int_0^1 \theta_t\Psi_t + a\int_0^1 \Psi_t^2 = 0$$

and

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta_t^2 + \frac{\kappa}{2}\frac{d}{dt}\int_0^1 \theta_x^2 - \int_0^1 \phi_t\theta_{xt} + \int_0^1 \Psi_t\theta_t + k\int_0^1 \theta_{xt}^2 = 0.$$

Thus, summing up gives

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\phi_t^2 + \rho_2\Psi_t^2 + \rho_3\theta_t^2 + k(\phi_x + \Psi)^2 + \alpha\Psi_x^2 + \kappa\theta_x^2\right] dx.$$

$$= -a\int_0^1 \Psi_t^2 dx - k\int_0^1 \theta_{xt}^2 dx. \quad \square$$

Let ω be the solution of

$$w_{xx} = -\Psi_x, \quad w(0) = w(1) = 0.$$

Lemma 2.3. *The function ω satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \Psi^2 dx \leq \int_0^1 \Psi_x^2 dx, \tag{2.7}$$

$$\int_0^1 w_t^2 dx \leq \int_0^1 w_{tx}^2 dx \leq \int_0^1 \Psi_t^2 dx, \tag{2.8}$$

$$-\int_0^1 \Psi \left(w_x + \Psi\right) dx = -\int_0^1 \Psi^2 dx + \int_0^1 w_x^2 dx \leq 0. \tag{2.9}$$

Proof. See chapter II, Lemma 3.9. $\square$

Lemma 2.4. *The functional*

$$I(t) := \int_0^1 \left(\rho_1 \phi_t w + \rho_2 \Psi \Psi_t + \frac{a}{2}\Psi^2\right) dx \tag{2.10}$$

satisfies, for all $\varepsilon > 0$, the estimate

$$I'(t) \leq -\frac{\alpha}{2}\int_0^1 \Psi_x^2 dx + \frac{1}{\alpha}\int_0^1 \theta_{tx}^2 dx + \left(\rho_2 + \frac{\rho_1}{4\varepsilon}\right)\int_0^1 \Psi_t^2 + \varepsilon\rho_1 \int_0^1 \phi_t^2 dx. \tag{2.11}$$

Proof. A differentiation of (2.10), using (2.3), leads to

$$\begin{aligned}
I'(t) = &-\int_0^1 k(\phi_x + \Psi)\left(w_x + \Psi\right) dx + \int_0^1 \theta_t w_x dx \\
&- \alpha \int_0^1 \Psi_x^2 dx + \int_0^1 \Psi\theta_t dx + \int_0^1 \rho_1 \phi_t w_t + \int_0^1 \rho_2 \Psi_t^2 \\
= &-\int_0^1 k\Psi\left(w_x + \Psi\right) dx + \int_0^1 \theta_t w_x dx - \alpha\int_0^1 \Psi_x^2 dx + \int_0^1 \Psi\theta_t dx \\
&+ \int_0^1 \rho_1\phi_t w_t + \int_0^1 \rho_2 \Psi_t^2 \\
\leq &-\alpha\int_0^1 \Psi_x^2 dx + \int_0^1 \theta_t w_x dx + \int_0^1 \Psi\theta_t dx + \int_0^1 \rho_1\phi_t w_t + \int_0^1 \rho_2\Psi_t^2.
\end{aligned}$$

By using (2.7)-(2.9), Young's and Poincaré's inequalities, (2.11) is established. $\square$

Lemma 2.5. *Under Condition* (2.4), *the functional*

$$J\left(t\right) := \rho_2 \int_0^1 \Psi_t \left(\phi_x + \Psi\right) dx + \rho_2 \int_0^1 \Psi_x \phi_t dx + \frac{a}{2} \int_0^1 \Psi^2 \tag{2.12}$$

satisfies

$$J'\left(t\right) \leq \left[\alpha \phi_x \Psi_x\right]_{x=0}^{x=1} - \frac{k}{2} \int_0^1 \left(\phi_x + \Psi\right)^2 + \left(\rho_2 + \frac{2a^2}{k}\right) \int_0^1 \Psi_t^2$$

$$+ \left(\varepsilon_1 + \frac{k}{4}\right) \int_0^1 \Psi_x^2 + \left(\frac{1}{k} + \frac{\rho_2^2}{4\varepsilon_1 \rho_1^2}\right) \int_0^1 \theta_{tx}^2, \tag{2.13}$$

for all $\varepsilon_1 > 0$.

Proof. Differentiating (2.12) and using (2.3), we arrive at

$$J'\left(t\right) = \int_0^1 \left(\alpha \Psi_{xx} - k(\phi_x + \Psi) + \theta_t - a\Psi_t\right) \left(\phi_x + \Psi\right) dx + \rho_2 \int_0^1 \Psi_t \left(\phi_{tx} + \Psi_t\right) dx$$

$$+\rho_2 \int_0^1 \Psi_{tx} \phi_t dx + \frac{\rho_2}{\rho_1} \int_0^1 \Psi_x \left(k(\phi_x + \Psi)_x - \theta_{tx}\right) + a \int_0^1 \Psi \Psi_t$$

$$= \left[\alpha \phi_x \Psi_x\right]_{x=0}^{x=1} - \int_0^1 k(\phi_x + \Psi)^2 + \int_0^1 (\phi_x + \Psi)\theta_t - a \int_0^1 \Psi_t \phi_x dx$$

$$+\rho_2 \int_0^1 \Psi_t^2 - \frac{\rho_2}{\rho_1} \int_0^1 \Psi_x \theta_{tx}.$$

To obtain (2.13), it suffuses to use Young's inequality and the fact that

$$\int_0^1 \phi^2 dx \leq \int_0^1 \phi_x^2 dx \leq 2 \int_0^1 (\phi_x + \Psi)^2 dx + 2 \int_0^1 \Psi_x^2 dx. \quad \square \tag{2.14}$$

To handle the boundary term in (2.13) we introduce the function $m\left(x\right) = 2 - 4x$.

Lemma 2.6. *Let* (ϕ, Ψ, θ) *be a solution of* (2.3) *. Then, we have, for all* $\varepsilon_1 > 0$,

$$J'(t) + \frac{\varepsilon_1}{k}\frac{d}{dt}\int_0^1 \rho_1 m\phi_t\phi_x + \frac{\alpha\rho_2}{4\varepsilon_1}\frac{d}{dt}\int_0^1 m\Psi_t\Psi_x$$

$$\begin{aligned}\leq -\left(\frac{k}{2} - \frac{\varepsilon_1 k^2}{4} - 8\varepsilon_1\right)\int_0^1 (\phi_x + \Psi)^2 + \left(\frac{3\alpha^2}{4\varepsilon_1} + \frac{\alpha^2}{4\varepsilon_1^3} + 10\varepsilon_1 + \frac{k}{4}\right)\int_0^1 \Psi_x^2 \\ + \frac{2\varepsilon_1\rho_1}{k}\int_0^1 \phi_t^2 + \left(\rho_2 + \frac{2a^2}{k} + \frac{\alpha\rho_2}{2\varepsilon_1} + \frac{a^2}{2\varepsilon_1}\right)\int_0^1 \Psi_t^2 \\ + \left(\frac{1}{k} + \frac{\rho_2^2}{4\varepsilon_1\rho_1^2} + \frac{1}{2\varepsilon_1} + \frac{\varepsilon_1}{k^2}\right)\int_0^1 \theta_{tx}^2.\end{aligned} \tag{2.15}$$

Proof. Using (2.3) and Young's inequality, we easily see that

$$\frac{d}{dt}\int_0^1 \rho_1 m\phi_t\phi_x = \int_0^1 m\left(k(\phi_x+\Psi)_x - \theta_{tx}\right)\phi_x + \int_0^1 \rho_1 m\phi_t\phi_{tx}$$

$$= -k\left[\phi_x^2(0) + \phi_x^2(1)\right] + 2k\int_0^1 \phi_x^2 + k\int_0^1 m\Psi_x\phi_x - \int_0^1 m\theta_{tx}\phi_x + 2\rho_1\int_0^1 \phi_t^2$$

$$\leq -k\left[\phi_x^2(0) + \phi_x^2(1)\right] + 4k\int_0^1 \phi_x^2 + k\int_0^1 \Psi_x^2 + \frac{1}{k}\int_0^1 \theta_{tx}^2 + 2\rho_1\int_0^1 \phi_t^2. \tag{2.16}$$

Similarly,

$$\begin{aligned}\alpha\rho_2\frac{d}{dt}\int_0^1 m\Psi_t\Psi_x =& \alpha\int_0^1 m\left(\alpha\Psi_{xx} - k(\phi_x+\Psi) + \theta_t - a\Psi_t\right)\Psi_x + \int_0^1 \alpha\rho_2 m\Psi_t\Psi_{tx} \\ =& \left[\frac{\alpha^2}{2}m\Psi_x^2\right]_{x=0}^{x=1} + 2\alpha^2\int_0^1 \Psi_x^2 + 2\alpha\rho_2\int_0^1 \Psi_t^2 - \int_0^1 \alpha k m(\phi_x+\Psi)\Psi_x \\ &+ \int_0^1 \alpha m\theta_t\Psi_x - a\int_0^1 \alpha m\Psi_t\Psi_x \\ \leq& -\alpha^2\left[\Psi_x^2(0) + \Psi_x^2(1)\right] + 3\alpha^2\int_0^1 \Psi_x^2 + 2\alpha\rho_2\int_0^1 \Psi_t^2 \\ &+ \frac{\alpha^2}{\varepsilon_1^2}\int_0^1 \Psi_x^2 + \varepsilon_1^2 k^2\int_0^1 (\phi_x+\Psi)^2 + 2\int_0^1 \theta_t^2 + 2a^2\int_0^1 \Psi_t^2.\end{aligned} \tag{2.17}$$

Combining (2.16), (2.17) and

$$[\alpha\phi_x\Psi_x]_{x=0}^{x=1} \leq \varepsilon_1\left[\phi_x^2(0)+\phi_x^2(1)\right]+\frac{\alpha^2}{4\varepsilon_1}\left[\Psi_x^2(0)+\Psi_x^2(1)\right],$$

we arrive at

$$\begin{aligned}
[\alpha\phi_x\Psi_x]_{x=0}^{x=1} \leq & -\frac{\varepsilon_1}{k}\frac{d}{dt}\int_0^1 \rho_1 m\phi_t\phi_x - \frac{\alpha\rho_2}{4\varepsilon_1}\frac{d}{dt}\int_0^1 m\Psi_t\Psi_x + 4\varepsilon_1\int_0^1 \phi_x^2 \\
& + \varepsilon_1\int_0^1 \Psi_x^2 + \frac{\varepsilon_1}{k^2}\int_0^1 \theta_{tx}^2 + \frac{2\varepsilon_1\rho_1}{k}\int_0^1 \phi_t^2 + \frac{1}{2\varepsilon_1}\int_0^1 \theta_t^2 \qquad (2.18)\\
& + \left(\frac{3\alpha^2}{4\varepsilon_1}+\frac{\alpha^2}{4\varepsilon_1^3}\right)\int_0^1 \Psi_x^2 + \left(\frac{\alpha\rho_2}{2\varepsilon_1}+\frac{a^2}{2\varepsilon_1}\right)\int_0^1 \Psi_t^2 + \frac{\varepsilon_1 k^2}{4}\int_0^1 (\phi_x+\Psi)^2.
\end{aligned}$$

A substitution of (2.18) in (2.13) leads to (2.15). □

Lemma 2.7. *The functional*

$$K_1(t) := -\rho_1\int_0^1 \phi_t\phi dx - \rho_2\int_0^1 \Psi_t\Psi dx - \frac{a}{2}\int_0^1 \Psi^2 dx$$

satisfies, along the solution of (2.3) , *the estimate*

$$\begin{aligned}
K_1'(t) \leq & -\rho_1\int_0^1 \phi_t^2 dx - \rho_2\int_0^1 \Psi_t^2 dx + \left(k+\frac{1}{4}\right)\int_0^1 (\phi_x+\Psi)^2 dx \qquad (2.19)\\
& + \left(\alpha+\frac{1}{2}\right)\int_0^1 \Psi_x^2 dx + 3\int_0^1 \theta_{tx}^2 dx.
\end{aligned}$$

Proof. A simple differentiation of K_1, using (2.3), gives

$$\begin{aligned}
K_1'(t) = & -\rho_1\int_0^1 \phi_t^2 dx - \int_0^1 \phi\left(k(\phi_x+\Psi)_x - \theta_{tx}\right)dx - \rho_2\int_0^1 \Psi_t^2 dx - a\int_0^1 \Psi_t\Psi dx \\
& - \int_0^1 \Psi\left(\alpha\Psi_{xx} - k(\phi_x+\Psi) + \theta_t - a\Psi_t\right)dx \\
= & -\rho_1\int_0^1 \phi_t^2 dx - \rho_2\int_0^1 \Psi_t^2 dx + \int_0^1 k(\phi_x+\Psi)^2 dx + \alpha\int_0^1 \Psi_x^2 dx + \int_0^1 \phi\theta_{tx} dx - \int_0^1 \Psi\theta_t dx.
\end{aligned}$$

The desired estimate then follows, by using Young's and Poincaré's inequalities, (2.7) and (2.14). □

Lemma 2.8. *Along the solution of (2.3), the functional*

$$K_2(t) := \int_0^1 \left(\rho_3\theta_t\theta + \frac{k}{2}\theta_x^2 + (\phi_x + \Psi)\theta\right) dx.$$

satisfies, for all $\varepsilon_2 > 0$, *the estimate*

$$K_2'(t) \leq -\kappa \int_0^1 \theta_x^2 dx + \left(\rho_3 + \frac{1}{4\varepsilon_2}\right) \int_0^1 \theta_{tx}^2 dx + \varepsilon_2 \int_0^1 (\phi_x + \Psi)^2 dx. \tag{2.20}$$

Proof. A simple differentiation of K_2, using (2.3), leads to

$$\begin{aligned} K_2'(t) &= \rho_3 \int_0^1 \theta_t^2 dx + \int_0^1 (\kappa\theta_{xx} - \phi_{xt} - \Psi_t + k\theta_{txx})\,\theta dx \\ &\quad + k \int_0^1 \theta_x\theta_{tx} dx + \int_0^1 (\phi_{tx} + \Psi_t)\theta dx + \int_0^1 (\phi_x + \Psi)\theta_t dx \\ &= \rho_3 \int_0^1 \theta_t^2 dx - \kappa \int_0^1 \theta_x^2 dx + \int_0^1 (\phi_x + \Psi)\theta_t dx. \end{aligned}$$

The desired estimate is then obtained, using Young's and Poincaré's inequalities. □

To finalize the proof of Theorem 2.1, we define the Lyapunov functional by

$$\begin{aligned} F(t) &:= NE_1(t) + N_1 I(t) + \delta K_1(t) + K_2(t) \\ &\quad + \left[J(t) + \frac{\varepsilon_1}{k}\frac{d}{dt}\int_0^1 \rho_1 m\phi_t\phi_x + \frac{\alpha\rho_2}{4\varepsilon_1}\frac{d}{dt}\int_0^1 m\Psi_t\Psi_x\right]. \end{aligned}$$

A combination of (2.6), (2.11), (2.15), (2.19) and (2.20) gives

$$\begin{aligned} F'(t) \leq & -\left[Nk - \frac{N_1}{\alpha} - \left(\frac{\rho_2^2 + 2\rho_1^2}{4\varepsilon_1\rho_1^2} + \frac{k+\varepsilon_1}{k^2}\right) - 3\delta - \rho_3 - \frac{1}{4\varepsilon_2}\right]\int_0^1 \theta_{tx}^2 \\ & - \left[\frac{N_1\alpha}{2} - \left(\frac{3\alpha^2}{4\varepsilon_1} + \frac{\alpha^2}{4\varepsilon_1^3} + 10\varepsilon_1 + \frac{k}{4}\right) - \delta\left(\alpha + \frac{1}{2}\right)\right]\int_0^1 \Psi_x^2 \\ & - \rho_1\left[\delta - \varepsilon N_1 - \frac{2\varepsilon_1}{k}\right]\int_0^1 \phi_t^2 - \kappa\int_0^1 \theta_x^2 \\ & - \left[\frac{k}{2} - \varepsilon_2 - \delta\left(k + \frac{1}{4}\right) - \left(\frac{k^2}{4} + 8\right)\varepsilon_1\right]\int_0^1 (\phi_x + \Psi)^2 \\ & - \left[Na - N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon}\right) - (1-\delta)\rho_2 - \frac{2a^2}{k} - \frac{\alpha\rho_2}{2\varepsilon_1} - \frac{a^2}{2\varepsilon_1}\right]\int_0^1 \Psi_t^2. \end{aligned} \tag{2.21}$$

Now we have to choose our constants carefully. First we take

$$\delta = \frac{k}{8\left(k + \frac{1}{4}\right)}, \qquad \varepsilon_2 = \frac{k}{4}$$

and then ε_1 so small that

$$\frac{k}{2} - \varepsilon_2 - \delta\left(k + \frac{1}{4}\right) - \left(\frac{k^2}{4} + 8\right)\varepsilon_1 = \frac{k}{8} - \left(\frac{k^2}{4} + 8\right)\varepsilon_1 > 0$$

$$\delta - \frac{2\varepsilon_1}{k} > \frac{\delta}{2}.$$

Next, we choose N_1 large enough such that

$$\frac{N_1\alpha}{2} - \left(\frac{3\alpha^2}{4\varepsilon_1} + \frac{\alpha^2}{4\varepsilon_1^3} + 10\varepsilon_1 + \frac{k}{4}\right) - \delta\left(\alpha + \frac{1}{2}\right) > 0$$

and ε so small that

$$\delta - \varepsilon N_1 - \frac{2\varepsilon_1}{k} > 0.$$

Finally, we pick N large enough such that

$$Nk - \frac{N_1}{\alpha} - \left(\frac{\rho_2^2 + 2\rho_1^2}{4\varepsilon_1\rho_1^2} + \frac{k+\varepsilon_1}{k^2}\right) - 3\delta - \rho_3 - \frac{1}{4\varepsilon_2} > 0$$

and

$$Na - N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon}\right) - (1-\delta)\,\rho_2 - \frac{2a^2}{k} - \frac{\alpha\rho_2}{2\varepsilon_1} - \frac{a^2}{2\varepsilon_1} > 0.$$

Consequently, there exist two positive constants η and C such that

$$\begin{aligned} F'(t) &\leq -\eta\left(\int_0^1 \phi_t^2 + \int_0^1 \Psi_t^2 + \int_0^1 (\phi_x + \Psi)^2 + \int_0^1 \Psi_x^2 + \int_0^1 \theta_{tx}^2 + \int_0^1 \theta_x^2\right) \\ &\leq -CE_1(t). \end{aligned} \tag{2.22}$$

where we have used $\int_0^1 \theta_t^2 \leq \int_0^1 \theta_{tx}^2$. Moreover, we may choose N even large (if needed) so that

$$F(t) \sim E_1(t). \tag{2.23}$$

A combination of (2.22) and (2.23) yields

$$F'(t) \leq -\omega F(t), \quad t \geq 0, \tag{2.24}$$

for a positive constant ω.

A simple integration of (2.24) leads to

$$F(t) \leq F(0)\, e^{-\omega t}, \quad t \geq 0. \tag{2.25}$$

Again, a use of (2.23) and (2.25) yield (2.5). □

4.2.2 Polynomial decay

In this subsection, we discuss the case of non-equal speed of wave propagation. We have the following result.

Theorem 2.9. *Let* $((\phi_0, \phi_1), (\Psi_0, \Psi_1), (\theta_0, \theta_1)) \in \left[H_0^1(0,1) \cap H^2(0,1) \times H_0^1(0,1)\right]^3$ *be given and suppose that*

$$\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}. \tag{2.26}$$

holds. Then, there exists a positive constant ξ*, independent of* t *and the initial data, such that the energy of system* (2.3) *satisfies*

$$E_1(t) \leq \frac{\xi\,(E_1(0) + E_2(0))}{t}, \quad t > 0,$$

where $E_2(t) := E(\phi_t, \Psi_t, \theta_t)$ *is the second energy of the system* (2.3)*.*

Remark 2.10. The energy $E_2(t)$ satisfies along the strong solution of (2.3) the estimate

$$E_2'(t) = -a\int_0^1 \Psi_{tt}^2 dx - k\int_0^1 \theta_{ttx}^2 dx \leq 0. \tag{2.27}$$

To prove Theorem 2.9 we need the following lemmas.

Lemma 2.11. *Let* (ϕ, Ψ, θ) *be a solution of* (2.3) *. Assume that* (2.26) *holds. Then the functional*

$$J_1(t) := \rho_2\int_0^1 \Psi_t(\phi_x + \Psi)\,dx + \frac{\rho_1\alpha}{k}\int_0^1 \Psi_x\phi_t dx + \frac{a}{2}\int_0^1 \Psi^2 \tag{2.28}$$

satisfies

$$\begin{aligned} J_1'(t) \leq{}& [\alpha\phi_x\Psi_x]_{x=0}^{x=1} - \frac{k}{2}\int_0^1 (\phi_x + \Psi)^2 + \left(\rho_2 + \frac{2a^2}{k}\right)\int_0^1 \Psi_t^2 \\ &+ \left(\varepsilon_1 + \frac{k}{4}\right)\int_0^1 \Psi_x^2 + \left(\frac{1}{k} + \frac{\rho_2^2}{4\varepsilon_1\rho_1^2}\right)\int_0^1 \theta_{tx}^2 \\ &+ \left(\rho_2 - \frac{\rho_1\alpha}{k}\right)\int_0^1 \Psi_t\phi_{tx}dx, \end{aligned} \tag{2.29}$$

for all $\varepsilon_1 > 0$.

Proof. To obtain (2.29), it suffuses to differentiate (2.28) and use (2.3), Young's and Poincaré's inequalities as in Lemma 2.5. □

Lemma 2.12. *Let* (ϕ, Ψ, θ) *be a solution of* (2.3), *then the functional*

$$J_2(t) := \left(\rho_2 - \frac{\rho_1 b}{k}\right)\left[\kappa \int_0^1 \Psi_x \theta_x + k \int_0^1 \Psi_x \theta_{tx}\right]$$

satisfies

$$J_2'(t) + \left(\rho_2 - \frac{\rho_1 b}{K}\right)\int_0^1 \phi_t \Psi_{tx} \leq \left(\int_0^1 \Psi_t^2 + \int_0^1 \Psi_x^2\right) + c\left(\int_0^1 \theta_{tx}^2 + \int_0^1 \theta_{ttx}^2\right), \qquad (2.30)$$

where c *is a positive constant.*

Proof. By using $(2.3)_3$ we get

$$\int_0^1 \phi_{tx}\Psi_t = \int_0^1 \Psi_t\left(\kappa\theta_{xx} - \rho_3\theta_{tt} - \Psi_t + k\theta_{txx}\right)$$

$$= -\int_0^1 \Psi_t^2 dx - \kappa\int_0^1 \Psi_{tx}\theta_x - k\int_0^1 \Psi_{tx}\theta_{tx} - \rho_3\int_0^1 \Psi_t\theta_{tt}$$

$$\begin{aligned}&= -\int_0^1 \Psi_t^2 dx - \kappa\frac{d}{dt}\int_0^1 \Psi_x\theta_x + \kappa\int_0^1 \Psi_x\theta_{tx} - k\frac{d}{dt}\int_0^1 \Psi_x\theta_{tx}\\ &+ k\int_0^1 \Psi_x\theta_{ttx} - \rho_3\int_0^1 \Psi_t\theta_{tt},\end{aligned}$$

which implies

$$\begin{aligned}&\int_0^1 \phi_{tx}\Psi_t + \kappa\frac{d}{dt}\int_0^1 \Psi_x\theta_x + k\frac{d}{dt}\int_0^1 \Psi_x\theta_{tx}\\ &= -\int_0^1 \Psi_t^2 dx + \kappa\int_0^1 \Psi_x\theta_{tx} + k\int_0^1 \Psi_x\theta_{ttx} - \rho_3\int_0^1 \Psi_t\theta_{tt}.\end{aligned}$$

Consequently,

$$J_2'(t)+\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1\phi_t\Psi_{tx}$$
$$=\left(\frac{\rho_1 b}{K}-\rho_2\right)\left[-\int_0^1\Psi_t^2dx+\kappa\int_0^1\Psi_x\theta_{tx}+k\int_0^1\Psi_x\theta_{ttx}-\rho_3\int_0^1\Psi_t\theta_{tt}\right].$$

Use of Young's and Poincaré's inequalities yield (2.30). □

To complete the proof of Theorem 2.9, we define the functional

$$G(t):=N\left[E_1(t)+E_2(t)\right]+N_1I(t)+\delta K_1(t)+K_2(t)+$$
$$\left[J_1(t)+J_2(t)+\frac{\varepsilon_1}{k}\frac{d}{dt}\int_0^1\rho_1 m\phi_t\phi_x+\frac{\alpha\rho_2}{4\varepsilon_1}\frac{d}{dt}\int_0^1 m\Psi_t\Psi_x\right].$$

A combination of (2.6), (2.11), (2.18), (2.19), (2.20), (2.27), (2.29) and (2.30) gives

$$\begin{aligned}
G'(t)\leq &-\left[Nk-\frac{N_1}{\alpha}-\left(\frac{\rho_2^2+2\rho_1^2}{4\varepsilon_1\rho_1^2}+\frac{k+\varepsilon_1}{k^2}\right)-3\delta-\rho_3-\frac{1}{4\varepsilon_2}-c\right]\int_0^1\theta_{tx}^2\\
&-\left[\frac{N_1\alpha}{2}-\left(\frac{3\alpha^2}{4\varepsilon_1}+\frac{\alpha^2}{4\varepsilon_1^3}+10\varepsilon_1+\frac{k}{4}\right)-\delta\left(\alpha+\frac{1}{2}\right)-1\right]\int_0^1\Psi_x^2\\
&-\rho_1\left[\delta-\varepsilon N_1-\frac{2\varepsilon_1}{k}\right]\int_0^1\phi_t^2-\kappa\int_0^1\theta_x^2\\
&-\left[\frac{k}{2}-\varepsilon_2-\delta\left(k+\frac{1}{4}\right)-\left(\frac{k^2}{4}+8\right)\varepsilon_1\right]\int_0^1(\phi_x+\Psi)^2\\
&-\left[Na-N_1\left(\rho_2+\frac{\rho_1}{4\varepsilon}\right)-(1-\delta)\rho_2-\frac{2a^2}{k}-\frac{\alpha\rho_2}{2\varepsilon_1}-\frac{a^2}{2\varepsilon_1}-1\right]\int_0^1\Psi_t^2\\
&-[Nk-c]\int_0^1\theta_{ttx}^2dx-Na\int_0^1\Psi_{tt}^2dx.
\end{aligned}\tag{2.31}$$

At this point, we choose our constants similarly to the previous section, to arrive at

$$\begin{aligned}
G'(t)&\leq-\eta\left(\int_0^1\phi_t^2+\int_0^1\Psi_t^2+\int_0^1(\phi_x+\Psi)^2+\int_0^1\Psi_x^2+\int_0^1\theta_{tx}^2+\int_0^1\theta_x^2\right)\\
&\leq -CE_1(t).
\end{aligned}$$

Recalling that E_1 is nonincreasing, integration of the last inequality leads to

$$tE_1(t) \leq \int_0^t E_1(s)\,ds \leq \frac{1}{C}(G(0) - G(t)) \leq \frac{G(0)}{C}.$$

Thus,

$$E_1(t) \leq \frac{G(0)}{Ct} \leq \frac{\xi (E_1(0) + E_2(0))}{t}, \quad \forall t > 0.$$

This, completes the proof of Theorem 2.9.

4.3 Timoshenko system with thermoelasticity type III

Messaoudi and Said-Houari [42] considered the following Timoshenko-type system with thermoelasticity type III

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 & \text{in } (0,\infty) \times (0,1), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0 & \text{in } (0,\infty) \times (0,1), \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \beta\psi_{ttx} - k\theta_{txx} = 0 & \text{in } (0,\infty) \times (0,1), \end{cases} \tag{3.1}$$

together with initial and boundary conditions, and showed, under the condition $\dfrac{K}{\rho_1} = \dfrac{b}{\rho_2}$, that weak solutions decay exponentially. In this section we consider (3.1), for the case $\dfrac{K}{\rho_1} \neq \dfrac{b}{\rho_2}$ and prove a polynomial decay result for strong solutions.

In order to exhibit the dissipative nature of system (3.1), we introduce the new variables $\phi = \ \varphi_t$ and $\Psi = \psi_t$, So, system (3.1) takes the form

$$\begin{cases} \rho_1 \phi_{tt} - K\left(\phi_x + \Psi\right)_x = 0 & \text{in } (0,\infty)\times(0,1) \\ \rho_2 \Psi_{tt} - b\Psi_{xx} + K\left(\phi_x + \Psi\right) + \beta\theta_{tx} = 0 & \text{in } (0,\infty)\times(0,1) \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \beta\Psi_{tx} - \kappa\theta_{txx} = 0 & \text{in } (0,\infty)\times(0,1)\,. \end{cases} \tag{3.2}$$

We supplement (3.2) with the following initial and boundary conditions

$$\begin{cases} \phi\left(.,0\right) = \phi_0,\ \phi_t\left(.,0\right) = \phi_1,\ \Psi\left(.,0\right) = \Psi_0,\ \Psi_t\left(.,0\right) = \Psi_1 \\ \theta\left(.,0\right) = \theta_0,\ \theta_t\left(.,0\right) = \theta_1 \\ \phi_x\left(0,t\right) = \phi_x\left(1,t\right) = \Psi\left(0,t\right) = \Psi\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right) = 0. \end{cases} \tag{3.3}$$

From equations $(3.2)_1$, $(3.2)_3$ and (3.3), we easily verify that

$$\frac{d^2}{dt^2}\int_0^1 \phi(x,t)dx = 0 \qquad \text{and} \qquad \frac{d^2}{dt^2}\int_0^1 \theta(x,t)dx = 0.$$

So, if we set

$$\begin{aligned} \overline{\phi}(x,t) &= \phi(x,t) - t\int_0^1 \phi_1(x)dx - \int_0^1 \phi_0(x)dx \\ \overline{\theta}\left(x,t\right) &= \theta\left(x,t\right) - t\int_0^1 \theta_1\left(x\right)dx - \int_0^1 \theta_0\left(x\right)dx \end{aligned}$$

then simple substitution shows that $(\overline{\phi}, \Psi, \overline{\theta})$ satisfies (3.2), the boundary conditions in

(3.3), and more importantly

$$\int_0^1 \overline{\phi}(x,t)dx = 0 \quad \text{and} \quad \int_0^1 \overline{\theta}(x,t)dx = 0, \quad \forall\, t \geq 0. \tag{3.4}$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$ and $\overline{\phi}$. In the sequel, we work with $\overline{\phi}$ and $\overline{\theta}$ but for convenience, we write ϕ and θ instead.

Remark 3.1. Our main objective is to prove stability result. The existence of weak and strong solutions can be established by using the standard Galerkin method.

To state our decay result, we introduce the first and second-order energy functionals:

$$E_1\left(t\right) = \int_0^1 \left(\rho_1 \phi_t^2 + \rho_2 \Psi_t^2 + \rho_3 \theta_t^2 + K\left|\phi_x + \Psi\right|^2 + b\Psi_x^2 + \delta\theta_x^2\right) dx, \tag{3.5}$$

$$E_2\left(t\right) = \int_0^1 \left(\rho_1 \phi_{tt}^2 + \rho_2 \Psi_{tt}^2 + \rho_3 \theta_{tt}^2 + K\left|\phi_{xt} + \Psi_t\right|^2 + b\Psi_{xt}^2 + \delta\theta_{xt}^2\right) dx. \tag{3.6}$$

Theorem 3.1. *Let (ϕ, Ψ, θ) be the strong solution of (3.2), (3.3) then there exists a positive constant k, independent of t and the initial data, such that the energy E satisfies the estimate*

$$E_1(t) \leq \frac{k\left(E_1(0) + E_2(0)\right)}{t}, \quad \forall t > 0. \tag{3.7}$$

The proof of our result will be established through several lemmas.

Lemma 3.2. *Let (ϕ, Ψ, θ) be the strong solution of (3.2), (3.3). Then, we have*

$$E_1'\left(t\right) = -\kappa \int_0^1 \theta_{tx}^2 dx \leq 0 \tag{3.8}$$

and

$$E_2'(t) = -\kappa \int_0^1 \theta_{ttx}^2 dx \le 0. \tag{3.9}$$

Proof. Multiplying equations (3.2) by ϕ_t, Ψ_t and θ_t, respectively, integrating over $(0,1)$ and summing up we obtain (3.8). Then, differentiating (3.2) with respect to t and multiplying the resulting equations by ϕ_{tt}, Ψ_{tt} and θ_{tt}, respectively, integrating over $(0,1)$ and summing up we obtain (3.9). □

Lemma 3.3. *Let* (ϕ, Ψ, θ) *be the strong solution of* (3.2), (3.3). *Then the functional*

$$I_1(t) := \rho_2 \int_0^1 \Psi_t \Psi - \rho_1 \int_0^1 \phi_t \left(\int_0^x \Psi(y,t) dy \right)$$

satisfies, for all $\varepsilon_1 > 0$,

$$I_1'(t) \le -\frac{b}{2} \int_0^1 \Psi_x^2 + \varepsilon_1 \int_0^1 \phi_t^2 + \left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_1} \right) \int_0^1 \Psi_t^2 + \frac{\beta^2}{2b} \int_0^1 \theta_{tx}^2. \tag{3.10}$$

Proof. By taking a derivative of I_1 and using (3.2), (3.3), we conclude

$$I_1'(t) = -b \int_0^1 \Psi_x^2 + \rho_2 \int_0^1 \Psi_t^2 - \beta \int_0^1 \Psi \theta_{tx} - \rho_1 \int_0^1 \phi_t \left(\int_0^x \Psi_t(y,t) dy \right).$$

By using Young's inequality and

$$\left(\int_0^x \Psi_t(y,t) dy \right)^2 \le \int_0^1 \Psi_t^2 \quad \text{and} \quad \int_0^1 \Psi^2 \le \int_0^1 \Psi_x^2,$$

estimate (3.10) is established. □

Lemma 3.4. *Let* (ϕ, Ψ, θ) *be the strong solution of* (3.2), (3.3). *Then the functional:*

$$I_2(t) := \rho_2 \rho_3 \int_0^1 \Psi_t \left(\int_0^x \theta_t(y,t) \, dy \right) - \delta \rho_2 \int_0^1 \theta_x \Psi$$

satisfies, for all $\varepsilon_2 > 0$,

$$I_2'(t) \leq -\frac{\beta\rho_2}{2}\int_0^1 \Psi_t^2 + \varepsilon_2 \int_0^1 (\Psi_x^2 + \phi_x^2) + C_{\varepsilon_2}\int_0^1 \theta_{tx}^2. \tag{3.11}$$

Proof. By taking a derivative of I_2 and using (3.2), (3.3), we get

$$I_2'(t) = \rho_3 \int_0^1 (b\Psi_{xx} - K(\phi_x + \Psi) - \beta\theta_{tx}) \left(\int_0^x \theta_t(y,t)\, dy\right) dx$$

$$+\rho_2 \int_0^1 \Psi_t \left(\int_0^x \delta\theta_{xx} - \beta\Psi_{tx} - \kappa\theta_{txx}\right) dx - \delta\rho_2 \int_0^1 \theta_{tx}\Psi dx - \delta\rho_2 \int_0^1 \theta_x \Psi_t dx$$

$$\begin{aligned} I_2'(t) = {} & \beta\rho_3 \int_0^1 \theta_t^2 - \rho_3 b \int_0^1 \theta_t \Psi_x + \rho_3 K \int_0^1 \theta_t \phi - \rho_3 K \int_0^1 \Psi \left(\int_0^x \theta_t(y,t)\, dy\right) \\ & - \kappa\rho_2 \int_0^1 \theta_{tx}\Psi_t - \beta\rho_2 \int_0^1 \Psi_t^2 - \delta\rho_2 \int_0^1 \Psi\theta_{tx}. \end{aligned}$$

The assertion of the lemma then follows, using Young's and Poincaré's inequalities. □

Lemma 3.5. *Let* (ϕ, Ψ, θ) *be the strong solution of* (3.2), (3.3). *Then the functional:*

$$I_3(t) := \rho_2 \int_0^1 \Psi_t(\phi_x + \Psi) + \frac{b\rho_1}{K}\int_0^1 \Psi_x \phi_t$$

satisfies

$$I_3'(t) \leq -\frac{K}{2}\int_0^1 (\phi_x + \Psi)^2 + \rho_2 \int_0^1 \Psi_t^2 + \frac{\beta^2}{2K}\int_0^1 \theta_{tx}^2 + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^1 \Psi_{tx}\phi_t. \tag{3.12}$$

Proof. A differentiation of I_3, taking in account (3.2), (3.3), gives

$$I_3'(t) = -K\int_0^1 (\phi_x + \Psi)^2 - \beta\int_0^1 (\phi_x + \Psi)\theta_{tx} + \rho_2 \int_0^1 \Psi_t^2 + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^1 \Psi_{tx}\phi_t.$$

Consequently, (3.12) follows by Young's inequality. □

Lemma 3.6. *Let* (ϕ, Ψ, θ) *be the strong solution of* (3.2), (3.3). *Then the functional:*

$$I_4(t) := -\rho_1 \int_0^1 \phi_t \phi - \rho_2 \int_0^1 \Psi_t \Psi$$

satisfies

$$I_4'(t) \leq -\rho_1 \int_0^1 \phi_t^2 - \rho_2 \int_0^1 \Psi_t^2 + \left(b + \frac{1}{2}\right) \int_0^1 \Psi_x^2 \qquad (3.13)$$
$$+ K \int_0^1 (\phi_x + \Psi)^2 + \frac{\beta^2}{2} \int_0^1 \theta_{tx}^2.$$

Proof. A differentiation of I_4, taking in account (3.2), (3.3), gives

$$I_4'(t) = -\rho_1 \int_0^1 \phi_t^2 - \rho_2 \int_0^1 \Psi_t^2 + b \int_0^1 \Psi_x^2 + K \int_0^1 (\phi_x + \Psi)^2 + \beta \int_0^1 \Psi\theta_{tx}.$$

Using Young's and Poincaré's inequalities for the last term, (3.13) follows. □

Lemma 3.7. *Let* (ϕ, Ψ, θ) *be the strong solution of* (3.2),(3.3). *Then the functional:*

$$I_5(t) := \rho_3 \int_0^1 \theta_t\theta + \frac{k}{2} \int_0^1 \theta_x^2 + \beta \int_0^1 \Psi_x\theta,$$

satisfies, for all $\varepsilon_2 > 0$,

$$I_5'(t) \leq -\delta \int_0^1 \theta_x^2 + \left(\rho_3 + \frac{\beta^2}{4\varepsilon_2}\right) \int_0^1 \theta_t^2 + \varepsilon_2 \int_0^1 \Psi_x^2. \qquad (3.14)$$

Proof. A simple differentiation of I_5, taking in account (3.2),(3.3) , leads to

$$I_5'(t) = \rho_3 \int_0^1 \theta_t^2 - \delta \int_0^1 \theta_x^2 + \beta \int_0^1 \Psi_x\theta_t$$

Finally, by Young's inequality, (3.14) is obtained. □

Proof of Theorem 3.1. We define the Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := N(E_1(t) + E_2(t)) + N_1 I_1 + N_2 I_2 + I_3(t) + \frac{1}{4} I_4(t) + I_5(t).$$

A combination of (3.8)-(3.14), and use of

$$\int_0^1 \theta_t^2 \leq \int_0^1 \theta_{tx}^2, \qquad \int_0^1 \phi_x^2 dx \leq 2 \int_0^1 (\phi_x + \Psi)^2 + 2 \int_0^1 \Psi_x^2, \qquad (3.15)$$

give

$$\begin{aligned}\mathcal{L}'(t) \leq &-\left(\frac{1}{4}\rho_1 - N_1\varepsilon_1\right)\int_0^1 \phi_t^2 - \left(\frac{K}{4} - 2\varepsilon_2 N_2\right)\int_0^1 (\phi_x + \Psi)^2 \\ &- \left[\frac{N_2\beta\rho_2}{2} - N_1\left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_1}\right) - \frac{3}{4}\rho_2\right]\int_0^1 \Psi_t^2 \\ &- \left[\frac{N_1 b}{2} - 3\varepsilon_2 N_2 - \frac{1}{4}\left(\frac{1}{2} + b\right) - \varepsilon_2\right]\int_0^1 \Psi_x^2 - \delta\int_0^1 \theta_x^2 \\ &- \int_0^1 \theta_t^2 - N\kappa\int_0^1 \theta_{xtt}^2 - (N\kappa - \lambda)\int_0^1 \theta_{xt}^2 + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^1 \Psi_{tx}\phi_t\end{aligned} \tag{3.16}$$

where λ is a positive constant independent of N.

At this point, we choose our constants carefully. First, let's take N_1 large enough such that $\frac{N_1 b}{4} - \frac{1}{4}\left(\frac{1}{2} + b\right) > 0$, then pick ε_1 so small that

$$\frac{1}{4}\rho_1 - N_1\varepsilon_1 > 0.$$

We then choose N_2 large enough so that

$$\frac{N_2\beta\rho_2}{2} - N_1\left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_1}\right) - \frac{3}{4}\rho_2 > 0.$$

Finally, we select ε_2 so small that

$$\frac{N_1 b}{2} - 3\varepsilon_2 N_2 - \frac{1}{4}\left(\frac{1}{2} + b\right) - \varepsilon_2 > 0 \text{ and } \frac{K}{4} - 2\varepsilon_2 N_2 > 0.$$

Therefore (3.16) takes the form

$$\mathcal{L}'(t) \leq -2\eta E_1(t) - N\beta\kappa\int_0^1 \theta_{xtt}^2 - (N\beta\kappa - \lambda)\int_0^1 \theta_{xt}^2 + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^1 \Psi_{tx}\phi_t \tag{3.17}$$

for some constant $\eta > 0$.

Now, we handle the last term in the right-hand side of (3.17), using $(3.2)_3$ as follows:

$$\begin{aligned}\int_0^1 \psi_{xt}\phi_t &= \frac{1}{\beta}\int_0^1 \phi_t\left(\kappa\theta_{xxt}+\delta\theta_{xx}-\rho_3\theta_{tt}\right)\\ &= -\frac{\rho_3}{\beta}\int_0^1 \phi_t\theta_{tt} - \frac{\delta}{\beta}\frac{d}{dt}\int_0^1 \theta_x\phi_x + \frac{\delta}{\beta}\int_0^1 \theta_{xt}\phi_x\\ &\quad - \frac{\kappa}{\beta}\frac{d}{dt}\int_0^1 \theta_{xt}\phi_x + \frac{\kappa}{\beta}\int_0^1 \theta_{xtt}\phi_x.\end{aligned}$$

Multiplying by $\dfrac{\rho_1 b}{K} - \rho_2$, we get

$$\begin{aligned}\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1 \psi_{xt}\phi_t &= -\frac{d}{dt}\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1\left(\frac{\delta}{\beta}\theta_x\phi_x + \frac{\kappa}{\beta}\theta_{xt}\phi_x\right)\\ &\quad + \left(\frac{\rho_1 b}{K}-\rho_2\right)\left(\frac{\delta}{\beta}\int_0^1 \theta_{xt}\phi_x + \frac{\kappa}{\beta}\int_0^1 \theta_{xtt}\phi_x - \frac{\rho_3}{\beta}\int_0^1 \phi_t\theta_{tt}\right).\end{aligned}$$

Therefore, recalling Young's inequality and (3.15), we get, $\forall\varepsilon_3 > 0$,

$$\begin{aligned}\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1 \psi_{xt}\phi_t &\le -\left(\frac{\rho_1 b}{K}-\rho_2\right)\frac{d}{dt}\int_0^1\left(\frac{\delta}{\beta}\theta_x\phi_x + \frac{\kappa}{\beta}\theta_{xt}\phi_x\right)\\ &\quad + \varepsilon_3\int_0^1\left(\phi_t^2 + \Psi_x^2 + (\phi_x+\Psi)^2\right)\\ &\quad + \frac{C\left(\frac{\rho_1 b}{K}-\rho_2\right)^2}{\varepsilon_3}\int_0^1\left(\theta_{xt}^2+\theta_{xtt}^2\right).\end{aligned} \tag{3.18}$$

where C is a positive constant depending on $\delta, \beta, \kappa, \rho_3$ only. We then define

$$L(t) := \mathcal{L}(t) + \left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1\left(\frac{\delta}{\beta}\theta_x\phi_x + \frac{\kappa}{\beta}\theta_{xt}\phi_x\right)$$

to get, from (3.17) and (3.18),

$$\begin{aligned}L'(t) &\le -2\eta E_1(t) + \varepsilon_3\int_0^1\left(\phi_t^2+\Psi_x^2+(\phi_x+\Psi)^2\right)\\ &\quad - \left(N\kappa - \frac{C'}{\varepsilon_3}\right)\int_0^1 \theta_{xtt}^2 - \left(N\kappa - \lambda - \frac{C'}{\varepsilon_3}\right)\int_0^1 \theta_{xt}^2.\end{aligned} \tag{3.19}$$

where $C' = C\left(\frac{\rho_1 b}{K} - \rho_2\right)^2$. By using (3.5), choosing ε_3 small enough and taking N large enough so that L is positive and $N\beta\kappa - \lambda - \frac{C'}{\varepsilon_3} > 0$, (3.19) takes the form

$$L'(t) \leq -\eta E_1(t).$$

Simple integration, recalling that E_1 is non-increasing, leads to

$$tE_1(t) \leq \int_0^t E_1(s)ds \leq \frac{1}{\eta}\left(L(0) - L(t)\right) \leq \frac{L(0)}{\eta}$$

Consequently,

$$E_1(t) \leq \frac{L(0)/\eta}{t} \leq \frac{k\left(E_1(0) + E_2(0)\right)}{t}, \quad \forall t > 0.$$

This completes the proof. $\square$.

Remark 3.2. Similar results can be established for boundary conditions of the form

$$\phi\left(0, t\right) = \phi\left(1, t\right) = \Psi_x\left(0, t\right) = \Psi_x\left(1, t\right) = \theta_x\left(0, t\right) = \theta_x\left(1, t\right).$$

Prospects

The results we obtained encouraging us to extend our study to a wide class of dissipation mechanisms.

In classical porous thermoelasticity we will study problem (1.1) of chapter 3 with the dissipation $\int_0^\infty g(s)\psi_{xx}(t-s)dx$ instead of the dissipation of memory type. We will also, investigate system (1.1) in the absence of the viscoelstic dissipation ($g \equiv 0$), and try to obtain some type of uniform decay.

In thermoelasticity of type III, we will investigate problem (2.1) of chapter 4 when the dissipation is given by $h(\psi_t)$ for some convex function h and try to obtain some stability results similar at those obtained by damping term $a\psi_t$.

Finally, we will extend all results to problems of thermoelasticity with second sound.

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ملخص

هذه الرسالة تتناول دراسة جملة معادلات حرارية مسامية لزجة و أخرى لتيموشينكو من الصنف III.

في المسألتان الأوليان تمت دراسة جملة معادلات حرارية مسامية لزجة تحتوي على نوعين من التخميد أحداهما حراري و الآخر مسامي متعلق بحد ذو ذاكرة، حيث تمّ إثبات استقرارعام للطاقة يتعلق بالتابع المعرف لحد الذاكرة و الذي يشمل الاستقرارين الجبري و الأسي كحالات خاصة منه.

وفي الموضوع الثالث تمت دراسة مسألة حرارية مسامية لزجة من الصنف III و أثبتنا أن التخميدين الحراري و الخطي المسامي كافيان لدفع الجملة إلى الاستقرار الأسي أو الجبري حسب العلاقة بين سرعات انتشارالأمواج.

أمّا المسألة الرابعة فتناولت جملة معادلات لتيموشينكو مع وجود حرارة من الصنف III وذلك في حالة عدم تساوي سرعتي انتشارالأمواج و أثبتنا أن الطاقة تتجه إلى استقرار جبري عندما يؤول الزمن إلى ما لا نهاية.

Résumé

Dans cette thèse on a étudié quelques problèmes de thérmoelasticité des milieux poreux et de Timoshenko de type III.

Le premier travail est un problème en thérmoelasticité classique. Un système de milieux poreux a été considéré où le contrôle est donné par deux dissipations, thermique dans l'équation d'élasticité et viscoélastique dans l'équation de poreusité. En utilisant la méthode des multiplicateurs, on a établi un résultat de stabilité générale pour les deux cas, égalité et non égalité des vitesses de propagation. Les stabilités exponentielle et polynomiale sont des cas particuliers de cette stabilité générale.

Le second travail est un problème poreux de type III. Deux forces de contrôles ont été considéré, thermique et un amortissement linéaire. Une stabilité exponentielle a été établie dans le cas de vitesses égales quand au cas de vitesses différentes la stabilité est polynomiale.

Le troisième travail est un problème de type Timoshenko de type III. Le cas de vitesses différentes a été considéré et une stabilité polynomiale a été établie.

Abstract

In this thesis we consider some one-dimensional porous thermo-elastic and Timoshenko-type problems.

The first work is concerned with a classical porous thermoelastic system in the presence of thermal and porous dissipations. We used the multiplier techniques and established a general decay rate which depends on the decay of the relaxation function of the memory term. Both the equal and non-equal speed of wave propagation were considered.

The second problem is a porous thermo-elastic system of type III which has two dissipative mechanisms being present in the elastic equation by a thermal dissipation and a in the porous equation by a linear frictional damping. An exponential decay rate was established for the equal-speed of propagation case, whereas in the case of non-equal speeds, only a polynomial decay rate was obtained.

The third problem is concerned with a Timoshinko-type problem of thermoelasticity of type III. A polynomial stability result was established for the non-equal wave propagation speed case.