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IBNR with dependent accident years for Solvency II

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DEDICATION

This thesis is lovingly dedicated to my mother who was always my support in the moments when there was no one to answer my queries, and all my invisible helpers, watching over and helping me from above during this process and over the years. Thank you

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"Nobody said it would be easy, but
I wanted to rise the challenge.."

ولسوف يعطيك ربك فترضى

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IBNR Avec Années d'Accident Dépendantes Pour Solvabilité II

Résumé : L'objectif principal de cette thèse est de présenter les résultats de nos recherches: *the Stochastic Incremental Approach For Modeling The claims Reserves* [2] et *the Chain-ladder with Bivariate Development factors*, pour cela nous présentons d'abord les théories mathématiques nécessaires.

Notre résultat principal généralise le modèle de Mack [18] pour la bien connue Méthode de Chain Ladder dans les calculs IBNR, en incluant la dépendance des paiements de sinistres entre les années d'accident et de développement. De même pour la méthode Chain Ladder, nous définissons les facteurs de développements bivariées (BDF) α et β pour la dépendance sur les années d'accident, resp. les années de développement. Le cas $\alpha \equiv 0$ correspond au modèle de Mack. Nous montrons comment estimer les BDF (α, β) par un système d'équations linéaires. L'estimateur est implicitement sans biais. Cela conduit à des estimations des provisions. Pour intégrer le modèle de Mack, une transition lisse à partir de notre modèle est établi. Des exemples sont donnés. Finalement, nous nous adaptons la méthode England-Verrall bootstraping pour le nouveau modèle sous l'hypothèse des distributions Gamma indépendantes avec un paramètre de forme. De cette manière, on définit les résidus multiplicatifs indépendants de la ré-échantillonnage qui donne la base de la procédure bootstrapping. Un exemple comparatif a [7] est donnée.

Pour le deuxieme resultat l'approche stochastique incrémental nous formulons les hypothèses en modifiant le modèle de Mack [18]. Nous estimons les facteurs de développement (β) et les provisions en utilisant uniquement des données incrémentaux. Nous nous concentrons après sur un point de vue d'année calendaire du triangle de développement, pour clarier plus, nous proposons une nouvelle forme de tabulation, puis nous appliquons le CDR [25] pour notre modèle. En utilisant la vision incrémentale nous évitons des étapes de calcul, et nous obtenons des résultats identiques avec des formules plus simple ce qui apporte beaucoup davantages pour les compagnies d'assurance.

Mots clés : Calcul stochastique des provision pour sinistres , IBNR, Chain Ladder avec années d'accident dépendantes, Facteurs de développement bivariées, Paiements incrémentaux, Bootstrap, CDR.

IBNR With Dependent Accident Years For Solvency II

Abstract : The main objective of this thesis is to presents the basic results of our researches which are *the Stochastic Incremental Approach For Modeling The claims Reserves* [2], and *the Chain-ladder with Bivariate Development factors*, for that we first present the mathematical theories needed for our different results.

Our main result generalize the model of Th. Mack [18] for the well known Chain-Ladder method in IBNR calculations, by including dependencies of the loss payments between accident and development years. Similarly to the Chain-Ladder method, we define bivariate developments factors (BDF) α and β for the dependence on the accident, resp. the development years. The case $\alpha \equiv 0$ corresponds to Mack's model. We show how to estimate the BDFs (α, β) by a linear system of equations. The estimator turns out to be implicitly unbiased. This leads to estimations of the yearly and total provisions. To integrate Mack's model, a smooth transition from it to our model is established. Examples are given. Finally, we adapt the England-Verrall bootstrapping method to the new model under the assumption of independent Gamma distributions with a uniform shape parameter. In this way, we define independent multiplicative residuals the resampling of which yields the basis of the bootstrapping procedure. An example comparative to the one in [7] is given.

For the second result the stochastic incremental approach we make the assumptions by modifying Mack's model [18]. We show how to estimate the development factors (β) using only incremental data, this again leads to estimations the provisions. We next concentrate on the calendar year view of the development triangle, to clarify more we propose a new tabulation form, then we apply the CDR [25] for our model. By using the incremental vision we avoid steps of calculation and we got identical results with easier formulae which brings lot of advantages for insurance companies.

Key Words : Stochastic Claims Reserving, IBNR, Chain Ladder with dependent accident years, Dependent bivariate development factors, Incremental Payments, Bootstrap, CDR.

Publications

[2] Chorfi, I., and Remita, M. R. Stochastic incremental approach for modelling the claims reserves. *International Mathematical Forum*, Vol. 8, no. 17, 807-828 (2013).
[5] Eisele, K. T., Chorfi, I. Bivariate chain-ladder with dependent development factors.(2014).

Seminars, Congresses and Schools

- Internationales Days Solvabilité II. Casablanca, Maroc (September 2010)
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- Journées jeunes chercheurs. Annaba, Algeria (October 2011)
- Journées Internationales de Probabilités et Statistique. Alger, Algeria (November 2011)
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- International Conference RAMA '8. Tipaza, Algeria. (November 2012)
- International School on Recent advances in partial differential equations and applications. Milan, Italy (June 2013)

Scientific training

- Training period from 01/01/2013 to 31/01/2013 in the Institut de Recherche Mathématique Avancée at Université de Strasbourg, France. Hosted by Professor Karl-Théodor Eisele.
- Training period from 26/12/2013 to 26/01/2014 in Mannhiem, Germany. Hosted by Professor Karl-Théodor Eisele.

Contents

Introduction

1 Background

We begin with necessary background that motivates the study of the problems presented in this thesis, where we consider claims reserving for a branch of insurance called **Non-Life Insurance**

Though loss reserves are by far the largest liability on the balance sheet of an insurance company, yet by nature these positions are heavily based on estimates. Even for the responsible actuary, it is often very demanding to quantify the inherent uncertainty. This is mainly a consequence of the complexity of estimating the variability of the reserve estimates. Correlation across several dimensions makes statistical measurement of uncertainty difficult. For a comprehensive bibliography on loss reserving we refer to [30].

The Chain-Ladder method is unquestionably the most widely used technique in claims reserving (see for instance the presentation in [33]). Being at the beginning a purely deterministic calculation, considerable attention has been given to the relationship between the Chain-Ladder technique and various stochastic models since a number of years. In particular, the pioneering work of Th. Mack was the starting point in this research (see [17], [18], [20]). Other works can be found e.g. in [21], [28], [36] and [37]. Many of these stochastic models have been established with the aim of producing in the mean exactly the same reserve estimates as the traditional deterministic Chain-Ladder model. Besides the better insight on the randomness inherent to the future liabilities which these methods reveal, they help also to get a feeling of the importance of the assumptions underlying the Chain-Ladder model. With respect to the Solvency II program in Europe, a transition from traditional reserving methods to stochastic one is more and more demanded. For more recent papers in this direction, see e.g. [38] and [39].

In this thesis we prefer to describe a bootstrap procedure in analogy to the methods introduced by P. England and R. Verrall (see [6], [7], [8], and [9], for a general presentation see [3]). It's essentially due to them that bootstrapping has become very popular in stochastic claims reserving, because of the simplicity and flexibility of the approach. Other applications of the bootstrap technique to claims reserving can be found in [16] and [33].

2 Aims and outlines

The main objective of this thesis is to present the basic results of our researches which are a various stochastic models based on the determinist chain-ladder method which is probably the most popular loss reserving technique.

The thesis is set out as follows. First we start with the mathematical framework and notation we use in the thesis. The two first chapters presents the mathematical theory we needs in our different results. Chapter 1 describes methods of type Chain-Ladder, we present first the Mack's model (see [17], [18], [20]), followed by the Claims Development Result (see [25], [24] and [40]), after that we present Bootstrap method. The next chapter describes the Generalized Linear Models theory and the uses of GLM in claim reserving methods. The Chapter 3 presents our first result, the stochastic incremental approach [2] which produces also the same reserve estimates as Chain-Ladder model by using only incremental payments, we propose also a new tabulation form for claims reserves, and we calculate the CDR for our model (for more details about CDR see [25], [24] and [40]). The main result of our thesis is in Chapter 4, that presents a mathematical model of loss reserves which starts from ideas similar to those of Mack's model, but which includes dependencies of the loss payments between accident **and** development years. In so far, it represents a generalization with respect to the works of Mack. Finally a general conclusion.

3 Mathematical framework and Notations

In this section we introduce Mathematical framework for claims reserving and the notations used in this thesis. In most cases outstanding loss liabilities are estimated in so-called claims development triangles which separates claims on two time axis: accident years $i \in \{1, ..., I\}$ and development years $j \in \{1, ..., J\}$. We assume that the last development period is given by J, i.e. $X_{i,j} = 0$ for $j > J$, Moreover for simplicity, we assume that $I = J$. Of course, all formulas similarly hold true for $I > J$(development trapezoids). We regard the matrix of random variables

$$X := (X_{i,j})_{1 \leq i,j \leq I} \tag{1}$$

for some $I \geq 2$. We assume that the variable $X_{i,j}$ denotes incremental payments of development year j for claims occurred in accident year i. In a claims development triangle accident years

are usually on the vertical line whereas development periods are on the horizontal line. Usually the loss development tables split into two parts the upper triangle where we have observations $\widehat{X}_{i,j} \in \mathbb{R}$ and the lower triangle where we want to estimate the outstanding payments. On the diagonals we always see the accounting years.
Hence the claims data have the following structure ($I = J$):

accident	development years j							
year i	1	2	3	$\cdots$	j	$\cdots$	J-1	J
1								
2		realizations of r.v. $\widehat{C}_{i,j}$, $\widehat{X}_{i,j}$						
3		(observations)						
$\vdots$								
i								
$\vdots$					predicted $C_{i,j}$, $X_{i,j}$			
I-1								
I								

Data can be shown in cumulative form or in non-cumulative (incremental) form.
We regard a matrix of random variables

$$C := (C_{i,j})_{1 \le i,j \le I} \tag{2}$$

for some $I \geq 2$. The variable $C_{i,j}$ is interpreted as the cumulated claims amount of accident year i till the development year j, the cumulative data are given by

$$C_{i,j} = \sum_{l=1}^{j} X_{i,l}. \tag{3}$$

Within the standard theory of claim reserving, also called IBNR-theory, it is assumed that pairs (i, j) of accident and development years with $i + j \leq I + 1$ describe past years for which the **run-off triangle** of real observations $\widehat{C}_{i,j} \in \mathbb{R}$ are available.

$$\widehat{C} := \left(\widehat{C}_{i,j}\right)_{i+j \le I+1} \tag{4}$$

Pairs (i, j) with $i + j > I + 1$ refer to future years with unknown results for the corresponding claim amounts.

Chapter 1

Methods of type Chain-Ladder

The Chain-Ladder Method is probably the most popular loss reserving technique. Despite its popularity, there are weaknesses inherent to this method, the primary weakness is that it is a deterministic algorithm, which implies that nothing is known about the variability of the actual outcome. To amend this shortcoming, stochastic models have been developed which provide the same estimates as in the Chain-Ladder method. These models make it possible to find the variability of the estimate. A stochastic model can also be used to assess whether the Chain-Ladder method is suitable for a given data set.

In this chapter we give different derivations for the Chain-Ladder method that we need in the next chapters. We first introduce the Mack's model presented in [18], [19] and [20], then the difference between two successive predictions for the total ultimate claim: it's about the Claims Development Result (see Merz and Wüthrich [24], [25] and [40]). Finally we introduce the Bootstrap Method.

1 Around Mack's model

After having briefly presented Mack's model, we will describe secondly the difference between two successive predictions of the total ultimate claim: The Claims Development Result.

1.1 The distribution-free chain-ladder model

The distribution-free chain-ladder model (Mack's model) is the stochastic version of the deterministic Chain-Ladder method. In fact, the estimated amount of reserves is identical. It is a non-parametric conditional model applies to run-off triangle of real observations $\widehat{C}_{i,j}$. However, it has the advantage that we are also able to give an estimator for the conditional mean square error of prediction for Chain-Ladder estimator. It's based on three assumptions:

Model Assumptions 1.1.

- *$C_{i,j}$ and $C_{k,j}$ are stochastically independent for $i \neq k$.*
- *There exist constants f_j and σ_j^2, such that for $1 \leq i \leq I, 1 \leq j \leq I-1$ we have*

$$\mathbb{E}\left[C_{i,j+1}\middle|C_{i,1},\dots,C_{i,j}\right] = f_j \cdot C_{i,j} \tag{1.1}$$

and

$$Var\left[C_{i,j+1}\middle|C_{i,1},\dots,C_{i,j}\right] = \sigma_j^2 \cdot C_{i,j} \tag{1.2}$$

The coefficients $(f,\sigma^2) := (f_j,\sigma_j^2)_{j=1,\dots,I-1}$ are called respectively Chain-Ladder development factors and variance parameters.

Mack has proved in his article [18] the following results:

Theorem 1.2.
(i) We have:

$$\mathbb{E}\left[C_{i,I}\middle|\mathscr{F}_I\right] = C_{i,I-i+1} \cdot \prod_{j=I-i+1}^{I-1} f_j \tag{1.3}$$

with

$$\mathscr{F}_I := \sigma\{C_{i,j}, i+j \leq I+1\}.$$

(ii) The Chain-Ladder development factors estimators

$$\widetilde{f}_j = \frac{\sum\limits_{i=1}^{I-j} C_{i,j+1}}{\sum\limits_{i=1}^{I-j} C_{i,j}}. \tag{1.4}$$

the Chain-Ladder development factors estimators $\widetilde{f}_j$ are an unbiased estimators of f_j, and they are uncorrelated.

Theorem 1.3.
(i) The estimator $\widetilde{C}_{i,I} = C_{i,I-i+1} \cdot \widetilde{f}_{I-i+1} \cdots \widetilde{f}_{I-1}$ is an unbiased estimator of $\mathbb{E}\left[C_{i,I}\middle|\mathscr{F}_I\right]$.
(ii) The estimator $\widetilde{R}_i$ of the yearly claim reserve $\mathbb{E}\left[R_i\middle|\mathscr{F}_I\right]$ with $R_i = C_{i,I} - C_{i,I-i+1}$ defined by $\widetilde{R}_i = \widetilde{C}_{i,I} - C_{i,I-i+1}$ is an unbiased estimator.

Theorem 1.4. *An estimator of σ_j^2 for $j = 1, \cdots, I-2$ is given by :*

$$\widetilde{\sigma}_j^2 = \frac{1}{I-j-1} \sum_{i=1}^{I-j} C_{i,j} \cdot \left(\frac{C_{i,j+1}}{C_{i,j}} - \widetilde{f}_j \right)^2, \tag{1.5}$$

$\widetilde{\sigma}_j^2$ is an unbiased estimator of σ_j^2 with, moreover,

$$\widetilde{\sigma}_{I-1}^2 = min \left(\frac{\widetilde{\sigma}_{I-2}^4}{\widetilde{\sigma}_{I-3}^2}, min(\widetilde{\sigma}_{I-3}^2, \widetilde{\sigma}_{I-2}^2) \right). \tag{1.6}$$

The uncertainty in the estimation of $C_{i,j}$ par $\widetilde{C}_{i,j}$ is traditionally measured by the mean square deviation (Conditionally to $\mathscr{F}_I$):

$$\begin{aligned} MSEP(\widetilde{C}_{i,I}) &= \mathbb{E}\left[(C_{i,I} - \widetilde{C}_{i,I})^2 \middle| \mathscr{F}_I \right] \\ &= \underbrace{Var(C_{i,I}|\mathscr{F}_I)}_{\text{Process variance}} + \underbrace{\left(\mathbb{E}[C_{i,I}|\mathscr{F}_I] - \widetilde{C}_{i,I} \right)^2}_{\text{Estimation error}}. \end{aligned} \tag{1.7}$$

This writing decomposes the MSEP into two terms, the stochastic error (process variance) which measures the internal variability of the model and the estimation error, related to the estimation real Chain-Ladder development factors f_j. The process variance comes from the stochastic character of the process $C_{i,I}$ and can be explicitly calculated. The estimation error comes from the fact that we estimated the real Chain-Ladder coefficients f_j with $\widetilde{f}_j$. It is from this latest error which in fact poses the major difficulties. The MSEP can be estimated using the following formula:

Theorem 1.5.
(i) Under Mack's assumptions (1.1) the mean squared error $MSEP(\widetilde{R}_i)$ can be estimated by

$$\widetilde{MSEP}(\widetilde{R}_i) = \widetilde{C}_{i,I}^2 \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}_j^2}{\widetilde{f}_j^2} \left[\frac{1}{\widetilde{C}_{i,j}} + \frac{1}{\sum_{k=1}^{I-j} C_{k,j}} \right]. \tag{1.8}$$

This expression is the sum of the two principal terms, which the first corresponds to the process variance and the second to the estimation error.

(ii) An estimator of the aggregate MSEP of the total reserves is given by :

$$\widetilde{MSEP}(\widetilde{R}) = \sum_{i=2}^{I}\left[MSEP(\widetilde{R}_i) + 2\widetilde{C}_{i,I}\left(\sum_{k=i+1}^{I}\widetilde{C}_{k,I}\right)\sum_{j=I-i+1}^{I-1}\frac{\widetilde{\sigma}_j^2}{\widetilde{f}_j^2\sum_{l=1}^{I-j}\widetilde{C}_{l,j}}\right]. \tag{1.9}$$

1.2 Claims Development Result (CDR)

We follow the approach developed by M. Merz and M. Wüthrich in [24], [25] and [40]. We assume that the claims liability process satisfies the distribution-free chain-ladder model assumptions. For claims reserving we predict the total ultimate claim at time I and after updating the information available at time $I+1$. The difference between these two successive best estimate predictions for the ultimate claim is so-called claims development result (CDR) for accounting year $(I, I+1]$. The realization of this claims development result has a direct impact on the profit & loss (P&L) statement and on the financial strength of the insurance company. Therefore, it also needs to be studied for solvency purposes.
The outstanding loss liabilities for accident year $i \in \{1,\ldots,I\}$ at time $t = I$ are given by

$$R_i^I = C_{i,I} - C_{i,I-i+1}, \tag{1.10}$$

and at time $t = I+1$ they are given by

$$R_i^{I+1} = C_{i,I} - C_{i,I-i+2}, \tag{1.11}$$

Let

$$\mathscr{F}_I := \sigma\{C_{i,j}, i+j \leq I+1\} \tag{1.12}$$

denote the data available at time $t = I$ and

$$\mathscr{F}_{I+1} := \sigma\{C_{i,j}, i+j \leq I+2\} = \mathscr{F}_I \cup \{\widehat{C}_{i,I-i+2}\} \tag{1.13}$$

denote the claims data available one period later, at time $t = I+1$. That is, If we go one step ahead in time, we obtain new observations $\{\widehat{C}_{i,I-i+2}\}$ on the new diagonal. More formally, this means that we get an enlargement of the $\sigma-$algebra generated by the observations $\mathscr{F}_I$ to the $\sigma-$algebra generated by the observations $\mathscr{F}_{I+1}$, .i.e. $\sigma(\mathscr{F}_I) \to \sigma(\mathscr{F}_{I+1})$.

Model Assumptions 1.6.

- *$C_{i,j}$ and $C_{k,j}$ are stochastically independent for $i \neq k$.*

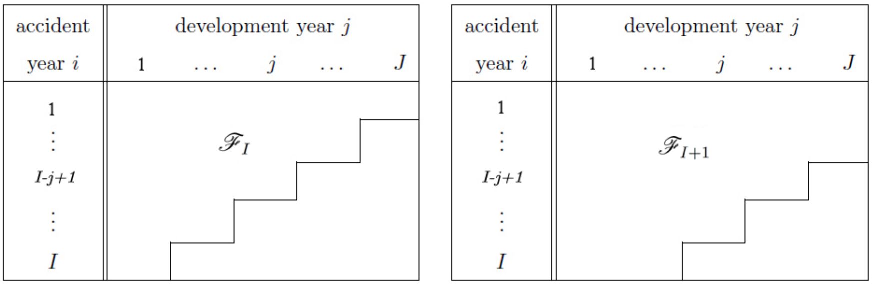

Figure 1.1 – Loss development triangle at time $t = I$ and $t = I + 1$

- *There exist constants $f_j > 0$ and $\sigma_j^2 > 0$, such that for $1 \leq i \leq I, 1 \leq j \leq I - 1$ we have*

$$\mathbb{E}\left[C_{i,j+1} \middle| C_{i,j}\right] = f_j \cdot C_{i,j}, \tag{1.14}$$

and

$$Var\left[C_{i,j+1} \middle| C_{i,j}\right] = \sigma_j^2 \cdot C_{i,j}. \tag{1.15}$$

Remark 1.7.

- We require stronger assumption than the ones used in the original work of Mack [19] and [18], namely the Markov process assumption was replaced by an assumption only on the first two moments of $C_{i,j+1}$ depend only of $C_{i,j}$ and not of $C_{i,l}$ for $l < j + 1$. (see also Wüthrich-Merz [39]).
- The Assumptions (1.6) satisfy the model assumptions of the Mack's chain ladder model.

and that we have:

$$\mathbb{E}\left[C_{i,I} \middle| \mathscr{F}_I\right] = C_{i,I-i+1} \cdot \prod_{j=I-i+1}^{I-1} f_j \quad \text{and} \quad \mathbb{E}\left[C_{i,I} \middle| \mathscr{F}_{I+1}\right] = C_{i,I-i+2} \cdot \prod_{j=I-i+2}^{I-1} f_j \tag{1.16}$$

As usual in the distribution-free Chain-Ladder model the Chain-Ladder development factors f_j are estimated as follows:

$$\widetilde{f}_j^I = \frac{\sum_{i=1}^{I-j} C_{i,j+1}}{\sum_{i=1}^{I-j} C_{i,j}} \quad \text{and} \quad \widetilde{f}_j^{I+1} = \frac{\sum_{i=1}^{I-j+1} C_{i,j+1}}{\sum_{i=1}^{I-j+1} C_{i,j}} \tag{1.17}$$

Mack [18] has proved that these are unbiased estimators for f_j, and moreover that $\widetilde{f}_j^m$ and $\widetilde{f}_l^m$ ($m = I$ or $I + 1$) are uncorrelated random variables for $j \neq l$ (see Theorem 2 in [18]).

This immediately implies that, given $C_{i,I-i+1}$,

$$\widetilde{C}_{i,I}^{I} = C_{i,I-i+1} \cdot \widetilde{f}_{I-i+1}^{I} \cdots \widetilde{f}_{I-1}^{I} \tag{1.18}$$

is an unbiased estimator of $\mathbb{E}\left[C_{i,I} \middle| \mathscr{F}_I\right]$ with $j \geq I-i+1$, and given $C_{i,I-i+2}$,

$$\widetilde{C}_{i,I}^{I+1} = C_{i,I-i+2} \cdot \widetilde{f}_{I-i+2}^{I+1} \cdots \widetilde{f}_{I-1}^{I+1} \tag{1.19}$$

is an unbiased estimator of $\mathbb{E}\left[C_{i,I} \middle| \mathscr{F}_{I+1}\right]$ with $j \geq I-i+2$.
The estimator $\widetilde{R}_i$ of the yearly claim reserve $\mathbb{E}\left[R_i \middle| \mathscr{F}_I\right]$ with $R_i = C_{i,I} - C_{i,I-i+1}$ defined by $\widetilde{R}_i = \widetilde{C}_{i,I} - C_{i,I-i+1}$ is an unbiased estimator.
We denote equally that, given $C_{i,I-i+1}$,

$$\widetilde{R}_i^{\mathscr{F}_I} = \widetilde{C}_{i,I}^{I} - C_{i,I-i+1} \qquad (2 \leq i \leq I) \tag{1.20}$$

is an unbiased estimator of $\mathbb{E}\left[R_i^I \middle| \mathscr{F}_I\right]$, and given $C_{i,I-i+2}$,

$$\widetilde{R}_i^{\mathscr{F}_{I+1}} = \widetilde{C}_{i,I}^{I+1} - C_{i,I-i+2} \qquad (3 \leq i \leq I) \tag{1.21}$$

is an unbiased estimator of $\mathbb{E}\left[R_i^{I+1} \middle| \mathscr{F}_{I+1}\right]$.

Definition 1.8 (True CDR for a single accident year)**.** The **true** CDR for accident year $i \in \{1, ..., I\}$ in accounting year $(I, I+1]$ is given by

$$\begin{aligned} CDR_i(I+1) &= \mathbb{E}\left[R_i^I \middle| \mathscr{F}_I\right] - \left(X_{i,I-i+2} + \mathbb{E}\left[R_i^{I+1} \middle| \mathscr{F}_{I+1}\right]\right) \\ &= \mathbb{E}\left[C_{i,I} \middle| \mathscr{F}_I\right] - \mathbb{E}\left[C_{i,I} \middle| \mathscr{F}_{I+1}\right]. \end{aligned} \tag{1.22}$$

Definition 1.9 (Observable CDR, estimator for true CDR)**.** The **observable** CDR for accident year $i \in \{1, ..., I\}$ in accounting year $(I, I+1]$ is given by

$$\widetilde{CDR}_i(I+1) = \widetilde{R}_i^{\mathscr{F}_I} - \left(X_{i,I-i+2} + \widetilde{R}_i^{\mathscr{F}_{I+1}}\right) = \widetilde{C}_{i,I}^{I} - \widetilde{C}_{i,I}^{I+1}. \tag{1.23}$$

True CDR ($CDR_i(I+1)$) is estimated by the CDR observable $\left(\widetilde{CDR}_i(I+1)\right)$ (see [40]).

2 Bootstrap method

Obtaining only the first two moments is a disadvantage of mack's model. The bootstrap procedure adapted for provisioning allows to have more information, because it produces the complete distribution of the claims reserves, and thus the moments of any order.

2.1 Methodology

With the framework of generalized linear models, we focused on the model described by Renshaw and Verrall (1998), who proposed modelling the incremental claims using an over-dispersed Poisson distribution. If the incremental claims for origin year i in development year j are denoted X_{ij} , then

$$\mathbb{E}[X_{ij}] = m_{ij} \quad \text{and} \quad Var[X_{ij}] = \phi\mathbb{E}[X_{ij}] = \phi m_{ij}, \tag{1.24}$$

$$log(m_{ij}) = \eta_{ij}, \tag{1.25}$$

$$\eta_{ij} = c + \alpha_i + \beta_j, \quad \alpha_1 = \beta_1 = 0. \tag{1.26}$$

equations (1.24, 1.26) define a generalised linear model in which the response is modelled with a logarithmic link function and the variance is proportional to the mean (hence over-dispersed Poisson). The parameter ϕ is an unknown scale parameter estimated as part of the fitting procedure. With certain positivity constraints, predicted values and reserve estimates from this model are exactly the same as those from the chain ladder model.
Therefore, Chain Ladder method is applied to each stage of the bootstrap procedure with back and forth cumulative increments. More specifically, the bootstrap method applied to provisioning is performed by completing the following steps:

- Obtain the standard chain ladder development factors from cumulative data.
- Obtain cumulative fitted values for the past triangle by backwards recursion, as described in Appendix A of England and Verrall (1999).
- Obtain incremental fitted values for the past triangle by differencing.
- Calculate the unscaled Pearson residuals for the past triangle using:

$$r_{ij}^{(p)} = \frac{X_{ij} - \tilde{X}_{ij}}{\sqrt{\tilde{X}_{ij}}}. \tag{1.27}$$

It is unscaled in the sense that it does not include the scale parameter ϕ which is not needed when performing the bootstrap calculations, but is needed when considering the process error.

- Adjust the Pearson residuals using:

$$r_{ij}^{adj(p)} = \sqrt{\frac{n}{n-p}} r_{ij}^{(p)}.$$

To enable the adjustment to follow through to the predictive distribution automatically, it is suggested that the residuals are adjusted prior to implementing the procedure. That is, replace $r_{ij}^{(p)}$ by $r_{ij}^{adj(p)}$, for more details see England (2002).

- Calculate the Pearson scale parameter, ϕ, using:

$$\phi^{(p)} = \frac{\sum_{i+j \le I+1} (r_{ij}^{(p)})^2}{n-p},$$

where n is the number of data points in the sample, p the number of parameters estimated and the summation is over the number (n) of residuals.

- Begin iterative loop, to be repeated N times ($N = 1000$, say):
  - Resample the adjusted residuals with replacement, creating a new past triangle of residuals.
  - For each cell in the past triangle, solve the equation (1.27) for $\widehat{X}_{ij}^{*}$, giving a set of pseudo-incremental data for the past triangle.
  - Create the associated set of pseudo-cumulative data.
  - Fit the standard chain ladder model to the pseudo-cumulative data.
  - Project to form a future triangle of cumulative payments.
  - Obtain the corresponding future triangle of incremental payments in each cell (i, j) by differencing, to be used as the mean $(\widetilde{m}_{ij}^{*})$ when simulating from the process distribution.
  - For each cell (i, j) in the future triangle, simulate a payment from the process distribution with mean $\widetilde{m}_{ij}^{*}$ (obtained at the previous step), and variance $\phi\widetilde{m}_{ij}^{*}$, using equation (1.24) and the value of ϕ calculated previously.

- ◦ Sum the simulated payments in the future triangle by origin year and overall to give the origin year and total reserve estimates, respectively.
- ◦ Store the results, and return to start of iterative loop.

The set of stored results forms the predictive distribution. The mean of the stored results should be compared to the standard chain ladder reserve estimates to check for bias. The standard deviation of the stored results gives an estimate of the prediction error.
It can be seen that essentially the bootstrap procedure provides a distribution of "means" in the future triangle, and the process error is replicated by sampling from the underlying distribution conditional on those means. The result is a simulated predictive distribution of future payments which when summed appropriately provides a predictive distribution of reserve estimates, from which summary statistics can be obtained.

Chapter 2

Generalized Linear Models

Generalized Linear Models have been introduced by J. Nelder and R.Wedderburn in 1972. Many papers set out in details the underlying statistical theory. One of the best reference is Mc Cullag and Nelder, 1989 [23]. Generalized linear modeling is a methodology for modeling relationships between variables. It generalizes the classical normal linear model, by relaxing some of its restrictive assumptions, and provides methods for the analysis of non-normal data (de Jong, Heller, 2008). The exponential family of distributions is one of the key constructs in generalized linear models (GLM) which are important in the analysis of insurance data. With insurance data, the assumptions of the normal model are frequently not applicable.

1 The Structure of Generalized Linear Models

A generalized linear model (or GLM) consists of three components:

(i) **A random component**, We consider independent random variable (response) $y_i(i = 1, ..., n)$ with probability distributions belonging to the exponential family such as the Gaussian (normal), binomial, Poisson, gamma, or inverse-Gaussian families of distributions. The density[1] function of $y_i(i = 1, ..., n)$ is defined by

$$f(y_i; \theta_i, \phi) = exp\left\{\frac{y_i\theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i, \phi)\right\}. \tag{2.1}$$

where θ_i is the canonical parameter, $\phi > 0$ is a dispersion parameter (eventually known) independent of i and j, and $b(\theta_i)$ and $c(y_i, \phi)$ are known functions. In all models considered

[1] True probability density function in the continuous case, simple probability in the discrete case.

in these notes the function $a_i(\phi)$ has the form

$$a_i(\phi) = \frac{\phi}{w_i},$$

where w_i is a known *prior weight*, Usually 1.
It can be shown that if Y_i has a distribution in the exponential family then it has mean and variance

$$E(Y_i) = \mu_i = b'(\theta_i) \quad (2.2)$$
$$var(Y_i) = \sigma_i^2 = b''(\theta_i)a_i(\phi), \quad (2.3)$$

where $b'(\theta_i)$ and $b''(\theta_i)$ are the first and second derivatives of $b(\theta_i)$. When $a_i(\phi) = \phi/w_i$ the variance has the simpler form

$$var(Y_i) = \sigma_i^2 = b''(\theta_i)\phi/w_i.$$

(ii) **Systematic Component** refers to the explanatory variables $(x_1, \ldots, x_p)$ as a combination of linear predictors,

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

The deterministic component of the model can also be defined as follow:

$$\eta = X\beta$$

where β is a vector of unknown parameters of size k, and X is the design matrix of explanatory variables of size $n \times p$.

(iii) **Link function g(.)**, provides the relationship between the mean of the distribution function $\mu_i = E(Y_i)$ and the linear predictor, in other words it links the random and deterministic parts of the model:

$$g(\mu_i) = \eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

- For a GLM where the response follows an exponential distribution we have

$$g(\mu_i) = g(b'(\theta_i)) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

The **canonical link** is defined as

$$\begin{aligned} g &= (b')^{-1} \\ g(\mu_i) &= \theta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} \end{aligned}$$

2 Maximum Likelihood Estimation

We are interested here in the estimation of the vector of parameters β, of dimension k, coefficients of the linear combination of covariables (or explanatory variables) to explain the Y vector. We describe, for this, briefly the classical procedure ML (Maximum Likelihood) allowing to achieve the maximum likelihood estimator.

With the assumption of independence of the components of Y, the likelihood of the canonical parameter vector θ is written:

$$L(\theta; y) = \prod_{i=1}^{n} f(y_i; \theta_i, \phi) = \prod_{i=1}^{n} exp\left[\frac{y_i\theta_i - b(\theta_i)}{\phi/p_i} + c(y_i, \phi)\right]$$

Rather than maximising this product which can be quite tedious, we often use the fact that the logarithm is an increasing function so it will be equivalent to maximise the log-likelihood:

$$\ell(\theta; y) = \sum_{i=1}^{n} log\left[f(y_i; \theta_i, \phi)\right] = \sum_{i=1}^{n} \frac{y_i\theta_i - b(\theta_i)}{\phi/p_i} + c(y_i, \phi)$$

To obtain the equations of maximum likelihood estimation β, we Differentiating the log-likelihood ℓ of the vector parameters β with respect to its various components.

$$\frac{\partial \ell_i}{\partial \beta_j} = \frac{\partial \ell_i}{\partial \theta_i}\frac{\partial \theta_i}{\partial \mu_i}\frac{\partial \mu_i}{\partial \eta_i}\frac{\partial \eta_i}{\partial \beta_j} = \frac{y_i - \mu_i}{a_i(\phi)}\frac{a_i(\phi)}{var(Y_i)}\frac{1}{g'(\mu_i)}x_{ij}$$

The maximum likelihood estimates are obtained numerically by solving the following equations:

$$\frac{\partial L}{\partial \beta_j} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)x_{ij}}{var(Y_i)g'(\mu_i)} = 0 \qquad j = 1, \ldots, n,$$

using iterative algorithms such as the Newton-Raphson or Fisher scoring methods (for details see McCullagh and Nelder, 1989).

3 Generalised Linear Models and claim reserving methods

The use of GLM in actuarial science is well developed and broadly accepted. Not only does the framework of GLM allow for flexibility in parameter and model selection, in some cases, such as with the CL method, GLM recovers traditional methods for claims reserves estimation. Following Renshaw and Verrall (1994) we can formulate most of the stochastic models for claim reserving by means of a particular family of generalised linear models (see McCullagh and Nelder (1989)). The structure of those GLM will be given by

(i) $Y_{ij} \sim f(y; \mu_{ij}, \phi)$ with independent Y_{ij}, $\mu_{ij} = E(Y_{ij})$ and where $f(.)$, the density(probability) function of Y_{ij} belongs to the exponential family. ϕ is a scale parameter.

(ii) $\eta_{ij} = g(\mu_{ij})$

(iii) $\eta_{ij} = c + \alpha_i + \beta_j$ with $\alpha_1 = \beta_1 = 0$ to avoid over-parametrization.

It is common in claim reserving to consider three possible distributions for the variable C_{ij}: Lognormal, Gamma or Poisson. For models based on Gamma or Poisson distributions, the relations $(i) - (iii)$ define a GLM with $Y_{ij} = C_{ij}$ denoting the incremental claim amounts. The link function is $\eta_{ij} = ln(\mu_{ij})$.

3.1 The Poisson model

When finding a stochastic model that reproduces chain-ladder estimates, some assumptions must be made about the insurance claims. It is possible either to specify the distribution of the insurance claims, or merely state the two first moments (England, Verrall, 2002).

There is a wide range of stochastic reserving models and they can be divided as chain ladder "type" and as extensions to the chain ladder. The chain-ladder "type" models may reproduce the chain ladder results exactly or can have a similar structure to chain-ladder without giving the exactly the same results. We consider the Poisson model which reproduces reserve estimates given by the chain-ladder technique.

Remark 2.1 (England, Verrall, 2002). Renshaw & Verrall (1998) were not the first to notice the link between the chain-ladder technique and the Poisson distribution, but were the first to implement the model using standard methodology in statistical modelling.

Already in 1975 a stochastic model corresponding to Poisson model, which leads to the chain-ladder technique, was discovered. This model works on the incremental amounts

$$\begin{aligned} C_{ij} &= D_{ij}, \text{ if } j = 1, \\ C_{ij} &= D_{ij} - D_{ij-1}, \text{ if } j > 1. \end{aligned}$$

The model makes the following assumptions:

(P1) $E(C_{ij}) = x_i y_j$ with unknown parameters x_i and y_i.

(P2) Each incremental amount C_{ij} has a Poisson distribution.

(P3) All incremental amounts C_{ij} are independent.

Here x_i is the expected ultimate claims amount (up to the latest development year so far observed) and y_j is the proportion of ultimate claims to emerge in each development year with restriction $\sum_{k=1}^{n} y_k = 1$. The restriction immediately follows from the fact that y_j is interpreted as the proportion of claims reported in development year j. Obviously, the aggregate proportion over all periods has to be 1.
We estimate the unknown parameters x_i and y_j from the triangle of known data (notation $\triangle$ is used for that) with the maximum likelihood method. The estimation procedure and results are given with the following lemma.

Lemma 2.2. *Assume that all C_{ij} are independent with a Poisson distribution and $E(C_{ij}) = x_i y_j$ holds. Then the maximum likelihood estimators for x_i and y_j are given by:*

$$x_i = \frac{\sum_{j \in \triangle_i} C_{ij}}{\sum_{j \in \triangle_i} y_j}$$

$$y_i = \frac{\sum_{i \in \triangle_j} C_{ij}}{\sum_{i \in \triangle_j} x_i}.$$

Proof. We derive the maximum likelihood estimates for the unknown parameters x_i and y_j with the likelihood function

$$L = \prod_{i,j \in \triangle} \frac{(x_i y_j)^{C_{ij}}}{C_{ij}!} exp(-x_i y_j).$$

Therefore the log-likelihood function is

$$\ell = ln(L) = -\sum_{i,j \in \triangle} x_i y_j + \sum_{i,j \in \triangle} C_{ij} ln(x_i y_j) - \sum_{i,j \in \triangle} ln(C_{ij}!),$$

where the summation is for all i, j where C_{ij} is known. The maximum likelihood estimator are those values x_i, y_j which maximize L or equivalently $ln(L)$. They are given by the equations

$$\begin{aligned} 0 &= \frac{\partial \ell}{\partial x_i} = -\sum_{j \in \triangle_i} y_j + \sum_{j \in \triangle_i} C_{ij} \frac{1}{x_i} \\ 0 &= \frac{\partial \ell}{\partial y_j} = -\sum_{i \in \triangle_j} x_i + \sum_{i \in \triangle_j} C_{ij} \frac{1}{y_j}, \end{aligned}$$

thus the likelihood estimator for x_i and y_j are given, respectively, by

$$x_i = \frac{\sum_{j \in \triangle_i} C_{ij}}{\sum_{j \in \triangle_i} y_j}$$

$$y_i = \frac{\sum_{i \in \triangle_j} C_{ij}}{\sum_{i \in \triangle_j} x_i}.$$

The lemma is proved. □

Thus, the proportion factors y_j express the ratio of the sum of observed incremental values for certain development year j with respect to certain ultimate claims, i.e. y_i denotes the proportion of claims reported in development year j. The parameters x_i refer to the ratio of the sum of observed incremental values for certain origin year i with respect to corresponding proportion factors, i.e. if the incremental claim amounts and respective proportions factors are known, it is simple to derive the corresponding ultimate claim x_i for origin year i. One can note the principal similarities with chain-ladder technique, where development factors are also outcomes of certain ratios.

The mean given as $E(C_{ij}) = x_i y_j$ in assumption $(P1)$ has a multiplicative structure, i.e. it is the product of the row effect and the column effect. Both the row and the column effect have specific interpretations and it is sometimes useful to preserve the model in this form. Nevertheless, for estimation purposes, sometimes it is better to reparameterise the model so that the mean has a linear form. The Poisson model can be cast into the form of a GLM and to linearise the multiplicative model we need to choose the logarithm as a link function so that

$$E(C_{ij}) = exp(lnx_i + lny_j),$$

or, equivalently,

$$ln(E(C_{ij})) = \alpha_i + \beta_j, \qquad (2.4)$$

where $\alpha_i = ln(x_i)$ and $\beta_j = ln(y_j)$ and structure of linear predictor (2.4) is still a chain-ladder type, because parameters for each row i and each column j are given. Hence, the structure (2.4) is defined as a generalised linear model, in which the incremental values C_{ij} are modelled as Poisson random variables with a logarithmic link function and linear predictor

$$\eta_{i,j} = c + \alpha_i + \beta_j. \tag{2.5}$$

In any case, a constraint

$$\alpha_1 = \beta_1 = 0 \tag{2.6}$$

is needed to estimate the remaining model parameters c, α_i, β_j and to avoid overparametrization. Considering a single incremental payment C_{ij} with origin year i and claim payments in development year j (yet to be observed), we obtain the estimates of future payments from the parameter estimates by inserting them into equation (2.4) and exponentiating, resulting as

$$\widetilde{C}_{ij} = \widetilde{x}_i \widetilde{y}_j = exp(\widetilde{\eta}_{i,j}). \tag{2.7}$$

Given the equation (2.7), the reserve estimates for origin year and overall estimates can be easily derived by summation:

$$\widetilde{R}_i = \widetilde{x}_i \widetilde{y}_{n+2-i} + \ldots + \widetilde{x}_i \widetilde{y}_n \tag{2.8}$$

From the assumptions $(\mathbf{P1}) - (\mathbf{P3})$, the maximum likelihood estimator (2.8) of the claims reserve for origin year i, $R_i = C_{i,I+2-i} + \ldots + C_{iI} = D_{iI} - D_{i,I+1-i}$, gives the same prediction $\widetilde{D}_{iI} = D_{i,I+1-i} + \widetilde{R}_i$ as the chain-ladder method. According to the assumption (P3), $D_{i,I+1-i} + \widetilde{R}_i$ is an estimator of the conditional expectation $E(D_{iI}|D_{i1}, \ldots, D_{i,I+1-i})$ and assumption (P2) constrains all incremental amounts C_{ij} to be non-negative integers.

In the Poisson model for loss reserving it is assumed that the incremental claims are independent and Poisson distributed with expectations being the product of two factors, depending on the occurrence year and the development year, respectively. It is well-known that maximum-likelihood estimation in the Poisson model yields the chain-ladder estimators of the expected ultimate aggregate claims (Schmidt, 2002). Moreover, Renshaw & Verrall (1998) pointed out that this is also true for overdispersed Poisson models.

We recall, that the only distributional assumptions used in GLMs are the functional relationship between variance and mean and the fact, that distribution belongs to the exponential family. In case of Poisson, the mentioned relationship is $Var(C_{ij}) = \mathbb{E}(C_{ij})$ and it can be generalized to $Var(C_{ij}) = \phi\mathbb{E}(C_{ij})$ without any change in form and solution of the likelihood equations

(Mack, Venter, 1999). This kind of generalisation allows for more dispersion in the data. For the solution of the likelihood equations it is not needed incremental values C_{ij} to be non-negative or integers and this leads to an over-dispersed Poisson model and to quasi-likelihood equations, since the range of the underlying distribution is not important anymore.

3.2 The over-dispersed Poisson model

The model we consider in this section is based on the Poisson distribution. The specification of the Poisson modelling distribution does not mean that the model can only be applied to data which are positive integers; it is easy to write down a quasi-likelihood which has all the characteristics of a Poisson likelihood, without actually referring directly to the probability function for the Poisson random variable (Renshaw, Verrall, 1998). This means that the model can be applied to non-integer data, positive and negative.

The over-dispersed Poisson (ODP) model is different from the distribution-free chain-ladder model of Mack (1993), but both methods reproduce the historical chain-ladder estimator for the claims reserve and these models are the only ones known that lead to the same estimators for D_{in} as the chain-ladder algorithm. However, only the Mack's distribution-free model is close enough to the chain-ladder algorithm in enough aspects so it qualifies to be called the stochastic model underlying the chain-ladder algorithm, because the Poisson model deviates from the historical chain-ladder algorithm in several aspects that the Mack's distribution-free model does not (Mack, Venter, 1999).

The ODP distribution differs from the Poisson distribution in that the variance is not equal to the mean, but is proportional to the mean. It is shown (in Schmidt, 2002) that every ODP model can be transformed into the Poisson model by dividing all incremental claims by the parameter. The over-dispersed Poisson model assumes that the incremental claims C_{ij} are distributed as independent over-dispersed Poisson random variables and the general form for the overdispersed Poisson chain-ladder model can be given as follows:

$$\begin{aligned} E(C_{ij}) &= x_i y_j, \\ Var(C_{ij}) &= \phi x_i y_j, \end{aligned}$$

where

$$\sum_{k=1}^{I} y_k = 1.$$

Over-dispersion is introduced through the parameter ϕ, which is unknown and estimated from the data.

Remark 2.3. The parameter y_j appears in variance, so the restriction that y_j must be positive is automatically imposed. This leads to the limitation of the model that the sum of incremental claims in column j must be positive. Some negative incremental values are allowed, as long as any column sum is not negative.

The over-dispersed Poisson model makes the following assumptions:

(ODP1) $E(C_{ij}) = x_i y_j$ with unknown parameters x_i and y_i.

(ODP2) The distribution of C_{ij} belongs to the exponential family with $Var(C_{ij}) = \phi x_i y_j$, where ϕ is an unknown parameter.

(ODP3) All C_{ij} are independent.

The resulting quasi-likelihood equations are

$$\sum_{j=1}^{I+1-i} x_i y_j = \sum_{j=1}^{I+1-i} C_{ij}, \quad i = 1, \ldots, I,$$

$$\sum_{i=1}^{I+1-j} x_i y_j = \sum_{i=1}^{I+1-j} C_{ij}, \quad j = 1, \ldots, I,$$

Mack (1991) has shown that these equations have the unique solution (if all $\tilde{f}_j$ are well defined and are not equal to zero, but without any restrictions on the row sums or column sums over C_{ij}):

$$\begin{aligned} \tilde{x}_i \tilde{y}_j &= D_{i,I+1-i} \tilde{f}_{I+2-i} \cdot \ldots \cdot \tilde{f}_{j-1} (\tilde{f}_{j-1} - 1) \text{ for } j > I + 1 - i, \\ \tilde{x}_i \tilde{y}_j &= D_{i,I+1-i} ((\tilde{f}_{j+1} \cdot \ldots \cdot \tilde{f}_{I+1-i})^{-1} (\tilde{f}_j \cdot \ldots \cdot \tilde{f}_{I+1-i})^{-1}) \text{ for } j \leq I + 1 - i, \end{aligned}$$

with $\tilde{f}_j$ from the chain-ladder algorithm.
Because $(\tilde{f}_j - 1) + \tilde{f}_j \cdot (\tilde{f}_{j+1} - 1) = \tilde{f}_j \cdot \tilde{f}_{j+1} - 1$, we obtain an estimation for the reserve of origin year i

$$\tilde{R}_i = \tilde{x}_i \tilde{y}_{I+2-i} + \ldots + \tilde{x}_i \tilde{y}_j = D_{i,I+1-i} (\tilde{y}_{I+2-i} \cdot \ldots \cdot \tilde{y}_I - 1).$$

This shows us that the solution of the quasi-likelihood equations of the overdispersed Poisson model gives the same estimator for ultimate claim D_{iI} as the chain-ladder algorithm (the distribution-free model of Mack).

3.3 Gamma model

Mack (1991) proposed further model with a multiplicative parametric structure for the mean incremental claims amounts which are modelled as Gamma response variables. As Renshaw & Verrall (1998) note, the same model can be fitted using the GLM described in over-dispersed Poisson model, but in which the incremental claim amounts are modelled as independent Gamma response variables, with a logarithmic link function and the same linear predictor, and just replacing $Var(C_{ij}) = \phi\mu_{ij}$ by $Var(C_{ij}) = \phi\mu_{ij}^2$. As it was with log-normal model, the predicted values provided by Gamma model are usually close to chain-ladder estimates, but it cannot be guaranteed.

Remark 2.4 (England, Verrall, 1998)**.** The Gamma model implemented as a generalised linear model gives exactly the same reserve estimates as the Gamma model implemented by Mack (1991), which is comforting rather than surprising.

To obtain predictions and prediction errors for the Gamma model simply requires a small change in the ODP model. The Gamma model is given with the mean

$$\mathbb{E}(C_{ij}) = \mu_{ij},$$

and with variance

$$Var(C_{ij}) = \phi(\mathbb{E}(C_{ij}))^2 = \phi\mu_{ij}^2,$$

so the variance in this model is proportional to the mean squared, not proportional to the mean as in the case of ODP model.

Remark 2.5. We need to impose that each incremental value should be nonnegative if we work with gamma (Poisson) models. This restriction can be overcome using a quasi-likelihood approach.

Using the chain-ladder type predictor structure

$$\eta_{ij} = c + \alpha_i + \beta_j, \alpha_1 = \beta_1 = 0,$$

$$log(\mu_{ij}) = \eta_{ij},$$

it is straightforward to obtain parameter estimates and predicted values using GLM.

Chapter 3

Stochastic Incremental Approach For Modeling The claims Reserves

This chapter, with some development and modifications is the text of an article [2] appeared in Journal of International Mathematical Forum in which we focuses on calculation of claims reserves using a stochastic incremental approach (stochastic Chain Ladder using only incremental payments), and for that we establish the require formulae and we put our assumptions. We will next concentrate on the calendar year view of the development triangle, then we apply the CDR for incremental approach and calendar year view.

Introduction

Payment of claims does not always take place at once, in the same accident year. The regulation of claims is done over time, and it is necessary to establish reserves to honor future liabilities. As the claims amount, that will be finally paid are unknown currently. The amount to put in reserve is also unknown and must be estimated. These estimations can be calculate by the so-called technical IBNR (Incurred But Not Reported), which are based on the past claims payments to estimate its future development. In this chapter we will study a new approaches, more formally a modern and sophistical vision to old techniques, for estimation of loss reserves. So we will first present the so-called stochastic incremental approach which calculate the reserves and the ultimate claims using only incremental payments of development triangle, and the second approach is a modern presentation of an old vision, the incremental approach in calendar year view (t) which is already mentioned by Buchwalder-Bühlmann-Merz-Wüthrich[1], Eisele-Artzner[4], Partrat-Pey-Schilling[26], in our case we study the approach in details, establish the formulae and propose a new tabulation form to give a clear view for the calendar years.

A claims development triangle has three directions:

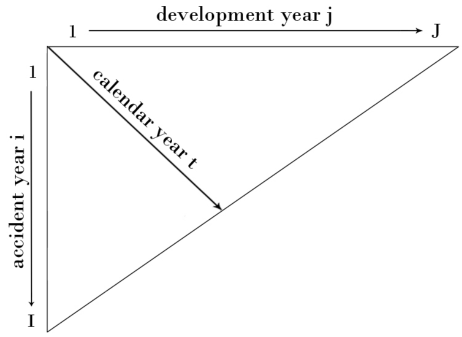

Figure 3.1 – Three directions of claims development triangle

The two directions, development year and accident year, are orthogonal, but the calendar year direction is not orthogonal neither to the accident year direction nor to the development year direction. The chapter is organized as follows. In the first section, we will present the stochastic incremental approach which is the incremental case of the Mack's model, so we will reformulate the assumptions and establish the formulae require basing only on incremental payments, which include the error estimation on calculation of reserves, then we will apply the CDR for this approach. Section 2 presents the incremental approach in calendar year view, so we propose a new form of tabulation figure 3.4 to clarify this vision, follow by the apply of the CDR. To illustrate the results, a numerical example with conclusion is provided in Section 3, the example show the uses of the two approaches studied, and the results obtained are identical of those using traditional claims reserving techniques for run-off triangles, in addition of these results we can get the reserve of each calendar year (t).

1 Presentation of the incremental approach

In this section we will base only on the incremental payments data $\mathscr{D}_{i,j}$ to estimate the parameters. This mean we don't need to go a step forward to calculate the cumulated payments as it's done in traditional claims reserving techniques, otherwise said, we want show that we can find the same results of traditional claims reserving techniques based on cumulated development triangle, directly by the incremental development triangle and for that we present the approach as follow, first we will add some specific notations. Secondly we will present the assumptions of the approach by modifying Mack's Model, and show that the development will be conditioned to the incremental payments data, then we will establish the formulae for esti-

mating the parameters. Next we will estimate the error in calculation of reserves. After that we will make comparison between Mack's formulae and the formulae given by the incremental approach, finally we apply the CDR for this approach. We begin with adding so notations we need is this chapter, and we denote:
Provisioning is a prediction problem, conditioned by the information available at time t=I.

For that we denote $\mathscr{D}_{i,j}$ the set of all data available at time t=I, more formally

$$\mathscr{D}_{i,j} = \sigma\{X_{i,j} \mid i + j \leq I + 1\}.$$

If we focus on the set of all data observed until the development year j,we note

$$\mathscr{D}_j = \sigma\{X_{i,l} \mid i + l \leq I + 1, l \leq j\}.$$

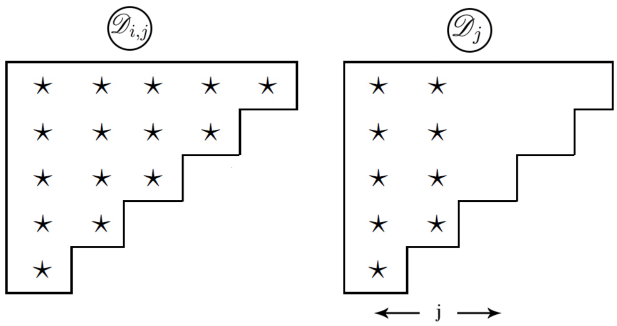

Figure 3.2 – The information available to make the prediction.

1.1 Assumptions' modification of Mack's model

This new approach is also based on three assumptions, the two first assumptions are:
H1 : There exist constants f_j, such that for $1 \leq i \leq I, 1 \leq j \leq I - 1$ we have

$$\mathbb{E}\left[X_{i,j+1} \middle| X_{i,1}, \ldots, X_{i,j}\right] = S_{i,j} \cdot (f_j - 1), \text{ with } S_{i,j} = \sum_{l=1}^{j} X_{i,l}$$

H2 : $X_{i,j}$ and $X_{k,j}$ are stochastically independent for $i \neq k$.

1.2 Estimation of parameters

1.2.1 The development factors

For each development year $j \in \{1, ..., I-1\}$, the development factors f_j are estimated by

$$\widetilde{f}_j = \frac{\sum_{i=1}^{I-j} X_{i,j+1}}{\sum_{i=1}^{I-j} S_{i,j}} + 1. \tag{3.1}$$

We can not guess the true values of the development factors $f_1, \ldots, f_{I-1}$ from data because the whole run-off triangular is not yet known at time $t = I$. They only can be estimated using incremental payments $X_{i,j}$ as we showed in the formula (3.1), One prominent property of a good estimator is that the estimator should be unbiased.

Theorem 3.1. *Under the assumptions $H1$ and $H2$, the development factors estimators $\widetilde{f}_1, \ldots, \widetilde{f}_{I-1}$ defined by (3.1) are unbiased and uncorrelated.*

***Proof A*.** First we demonstrate the unbiasedness of development factors estimators (the Chain Ladder Development factors), i.e. that $\mathbb{E}[\widetilde{f}_j] = f_j$. Because of the iterative rule for expectations,

$$\mathbb{E}[\widetilde{f}_j] = \mathbb{E}\left[\mathbb{E}[\widetilde{f}_j \mid \mathscr{D}_j]\right]. \tag{A1}$$

We have

$$\begin{aligned} \mathbb{E}[X_{i,j+1} \mid \mathscr{D}_j] &= \mathbb{E}\left[X_{i,j+1} \mid \; X_{i,1}, \ldots, X_{i,j}\right], \; \textit{by } \mathbf{H2} \\ &= S_{i,j} \cdot (f_j - 1), \; \text{by } \mathbf{H1}. \end{aligned} \tag{A2}$$

This implies that

$$\begin{aligned} \mathbb{E}[\widetilde{f}_j \mid \mathscr{D}_j] &= \mathbb{E}[\frac{\sum_{i=1}^{I-j} X_{i,j+1}}{\sum_{i=1}^{I-j} S_{i,j}} + 1 \mid \mathscr{D}_j] \\ &= \frac{\sum_{i=1}^{I-j} \mathbb{E}[X_{i,j+1} | \mathscr{D}_j]}{\sum_{i=1}^{I-j} S_{i,j}} + 1, \text{ because } S_{i,j} \text{ is } \mathscr{D}_j\text{-measurable} \\ &= \frac{(f_j - 1) . \sum_{i=1}^{I-j} S_{i,j}}{\sum_{i=1}^{I-j} S_{i,j}} + 1 = f_j. \end{aligned} \tag{A3}$$

$\mathbb{E}[\widetilde{f}_j] = \mathbb{E}\left[\mathbb{E}[\widetilde{f}_j \mid \mathscr{D}_j]\right] = \mathbb{E}[f_j] = f_j$. Which shows that the estimators $\widetilde{f}_j$ are unbiased.

We turn now to the no correlation between the estimators of the development factors, i.e. that $\mathbb{E}[\widetilde{f}_{j_1} \cdot \widetilde{f}_{j_2}] = \mathbb{E}[\widetilde{f}_{j_1}] \cdot \mathbb{E}[\widetilde{f}_{j_2}]$ for $j_1 < j_2$,

$$\begin{aligned}
\mathbb{E}[\widetilde{f}_{j_1} \cdot \widetilde{f}_{j_2}] &= \mathbb{E}\left[\mathbb{E}[\widetilde{f}_{j_1} \cdot \widetilde{f}_{j_2} \mid \mathscr{D}_{j_2}]\right] \\
&= \mathbb{E}\left[\widetilde{f}_{j_1} \cdot \mathbb{E}[\widetilde{f}_{j_2} \mid \mathscr{D}_{j_2}]\right], \text{ because } j_1 < j_2 \\
&= \mathbb{E}[\widetilde{f}_{j_1}] \cdot f_{j_2} \\
&= \mathbb{E}[\widetilde{f}_{j_1}] \cdot \mathbb{E}[\widetilde{f}_{j_2}], \text{ because} \widetilde{f}_j \text{ are unbiased.}
\end{aligned} \tag{A4}$$

□

1.2.2 The ultimate claims amount

The aim of the Chain Ladder method and every claims reserving method is the estimation of the ultimate claims amount Z_i for the accident years $i \in \{2, ..., I\}$. In our approach we can calculate $\widetilde{Z}_i$ using two formulae, the first one by using the summation of the i^{th} line of the matrix of random variables $(X_{i,j})_{1 \leq i,j \leq I}$

$$\begin{aligned}
\widetilde{Z}_i &= \sum_{j=1}^{I-i+1} X_{i,j} + \sum_{j=I-i+2}^{I} \widetilde{X_{i,j}} \qquad (3.2) \\
&= Z_i^0 + \widetilde{R}_i.
\end{aligned}$$

Where the first term represents a known part (the yearly past obligations) Z_i^0, and the second term corresponds the outstanding claims reserves $\widetilde{R}_i$, or we can simply using the classic form given by

$$\widetilde{Z}_i = Z_i^0 \cdot (\widetilde{f}_{I-i+1} \ldots \widetilde{f}_{I-1}). \tag{3.3}$$

Theorem 3.2. *Under the assumptions $H1$ and $H2$, the estimation of the ultimate claims amount $\widetilde{Z}_i$ defined by (3.2) and (3.3) are unbiased.*

Proof B.

$$\begin{aligned}\mathbb{E}[\widetilde{Z}_i] =&\mathbb{E}\left[Z_i^0 \cdot \widetilde{f}_{I-i+1} \dots \widetilde{f}_{I-1}\right]\\ =&\mathbb{E}\left[\mathbb{E}[Z_i^0 \cdot \widetilde{f}_{I-i+1} \dots \widetilde{f}_{I-1} \mid \mathscr{D}_{I-1}]\right]\\ =&\mathbb{E}\left[Z_i^0 \cdot \widetilde{f}_{I-i+1} \dots \widetilde{f}_{I-2} \cdot \mathbb{E}[\widetilde{f}_{I-1} \mid \mathscr{D}_{I-1}]\right]\\ =&\mathbb{E}[Z_i^0 \cdot \widetilde{f}_{I-i+1} \dots \widetilde{f}_{I-2}] \cdot f_{I-1}\\ &\text{by repeated the operation we get}\\ \mathbb{E}[\widetilde{Z}_i] =&Z_i^0 \cdot f_{I-i+1} \dots f_{I-2} \cdot f_{I-1} = Z_i.\end{aligned}$$

□

1.2.3 Yearly claims reserves

One of the advantages of our new approach is that we can easily calculate the outstanding claims reserves estimators (in other words, what is left to pay for claims incurred in year i), by a simple sum of the incremental payments estimators for $i \in \{2, ..., I\}$

$$\widetilde{R}_i = \sum_{j=I-i+2}^{I} \widetilde{X}_{i,j}. \tag{3.4}$$

We can also obtain $\widetilde{R}_i$ using the classic principle, i.e.(make the difference between the estimators of the ultimate claims amount $\widetilde{Z}_i$ and the past obligations Z_i^0 has already been paid up to now), and we denote for $i \in \{2, ..., I\}$

$$\widetilde{R}_i = \widetilde{Z}_i - Z_i^0 = Z_i^0 \cdot \left[\prod_{j=I-i+1}^{I-1} \widetilde{f}_j \ - 1\right], \tag{3.5}$$

and their sum will give

$$\begin{aligned}\widetilde{R} = \sum_{i=2}^{I}\widetilde{R}_i =&\sum_{i=2}^{I}\left[\widetilde{Z}_i - Z_i^0\right] \\ =&\sum_{i=2}^{I} Z_i^0 \cdot \left[\prod_{j=I-i+1}^{I-1} \widetilde{f}_j \ - 1\right].\end{aligned} \tag{3.6}$$

we observe here that we can also get the overall reserve by a simple summation of all the future incremental payments in the lower part of the matrix of random variables $(X_{i,j})_{1\leq i,j\leq I}$,

$$\widetilde{R} = \sum_{i=2}^{I} \sum_{j=I-i+2}^{I} \widetilde{X}_{i,j}. \tag{3.7}$$

Theorem 3.3. *Under the assumptions $H1$ and $H2$, the estimation of the outstanding claims reserves $\widetilde{R}_i$ and the estimation overall reserve $\widetilde{R}$ are unbiased.*

Proof C.

$$\begin{aligned} \mathbb{E}[\widetilde{R}_i] &= \mathbb{E}[\widetilde{Z}_i - Z_i^0] \\ &= \mathbb{E}[\widetilde{Z}_i] - \mathbb{E}[Z_i^0] \\ &= Z_i - Z_i^0 = R_i. \end{aligned} \tag{C1}$$

and

$$\mathbb{E}[\widetilde{R}] = \mathbb{E}[\sum_{i=2}^{I} \widetilde{R}_i] = \sum_{i=2}^{I} R_i = R. \tag{C2}$$

□

1.3 Calculation of standard error

The formula for estimating the standard error is based on Mack's model, and in this subsection we will measure the uncertainty using only the incremental payments.

1.3.1 The estimation error of the yearly claims reserves

$\widetilde{Z}_i$ provides an estimator but not the exact value of Z_i. Here we are interested in the average distance between the estimator and the true value. The mean square error $MSEP(\widetilde{Z}_i)$ is defined by

$$MSEP(\widetilde{Z}_i) = \mathbb{E}[(\widetilde{Z}_i - Z_i)^2 \mid \mathscr{D}_{i,j}].$$

We are interested in a conditional mean based on the specific incremental payments data set $\mathscr{D}_{i,j}$ to obtain the mean deviation between $\widetilde{Z}_i$ and Z_i only due to future randomness.
The standard error $SE(\widetilde{Z}_i)$ is defined as equal to $\sqrt{MSEP(\widetilde{Z}_i)}$. If we are interested in the error on the provision, we must calculate

$$MSEP(\widetilde{R}_i) = \mathbb{E}\left[(\widetilde{R}_i - R_i)^2 \mid \mathscr{D}_{i,j}\right],$$

such as $\widetilde{R}_i = \widetilde{Z}_i - Z_i^0$ and $R_i = Z_i - Z_i^0$ we finally obtain

$$\begin{aligned} MSEP(\widetilde{R}_i) &= \mathbb{E}[(\widetilde{R}_i - R_i)^2 \mid \mathscr{D}_{i,j}] \\ &= \mathbb{E}[(\widetilde{Z}_i - Z_i)^2 \mid \mathscr{D}_{i,j}] = MSEP(\widetilde{Z}_i). \end{aligned}$$

The error in estimating the yearly claims reserves is equal to the estimation error of the ultimate claims amount.
In order to calculate $MSEP(\widetilde{Z}_i)$ we'll decompose it according to the following formula:

$$MSEP(\widetilde{R}_i) = Var(Z_i \mid \mathscr{D}_{i,j}) + (\mathbb{E}[Z_i \mid \mathscr{D}_{i,j}] - \widetilde{Z}_i)^2. \tag{3.8}$$

This formula shows that we therefore need to estimate the variance of $X_{i,j}$, this will lead us to identify a third assumption for our new approach derived from Mack's Model,

$$\textbf{H3} : Var(X_{i,j+1} \mid X_{i,1}, \ldots, X_{i,j}) = S_{i,j} \cdot \sigma_j^2, \quad \text{for } 1 \leq i \leq I, 1 \leq j \leq I-1.$$

The parameters σ_j^2 are unknown and must be estimated.

Property 3.4. *Under $H1, H2$ and $H3$,*

$$\widetilde{\sigma}_j^2 = \frac{1}{I-j-1} \sum_{i=1}^{I-j} S_{i,j} \cdot \left(\frac{X_{i,j+1}}{S_{i,j}} + 1 - \widetilde{f}_j\right)^2 , \ \textit{for } 1 \leq j \leq I-2 \tag{D1}$$

is an unbiased estimator of σ_j^2 and

$$\widetilde{\sigma}_{I-1}^2 = min\left(\frac{\widetilde{\sigma}_{I-2}^4}{\widetilde{\sigma}_{I-3}^2}, min(\widetilde{\sigma}_{I-3}^2, \widetilde{\sigma}_{I-2}^2)\right), \ \textit{see Mack [19].} \tag{D2}$$

Proof D. The formula $(D1)$ Comes from

$$\begin{aligned}
\sigma_j^2 &= \frac{Var(X_{i,j+1} \mid X_{i,1},\dots,X_{i,j})}{S_{i,j}} \\
&= Var(\frac{X_{i,j+1}}{\sqrt{S_{i,j}}} \mid X_{i,1},\dots,X_{i,j}) \\
&= \frac{1}{I-j-1}\sum_{i=1}^{I-j}\left(\frac{X_{i,j+1}}{\sqrt{S_{i,j}}} - \mathbb{E}[\frac{X_{i,j+1}}{\sqrt{S_{i,j}}} \mid X_{i,1},\dots,X_{i,j}]\right)^2 \\
&= \frac{1}{I-j-1}\sum_{i=1}^{I-j}\left(\frac{X_{i,j+1}}{\sqrt{S_{i,j}}} - \frac{\mathbb{E}[X_{i,j+1} \mid X_{i,1},\dots,X_{i,j}]}{\sqrt{S_{i,j}}}\right)^2 \\
&= \frac{1}{I-j-1}\sum_{i=1}^{I-j} S_{i,j}(\frac{X_{i,j+1}}{S_{i,j}} + 1 - f_j)^2 \text{ by } H1.
\end{aligned}$$

We get $\widetilde{\sigma}_j^2$ by replacing the unknown parameters f_j, with their unbiased estimators $\widetilde{f}_j$.

$$\widetilde{\sigma}_j^2 = \frac{1}{I-j-1}\sum_{i=1}^{I-j} S_{i,j} \cdot (\frac{X_{i,j+1}}{S_{i,j}} + 1 - \widetilde{f}_j)^2.$$

□

Theorem 3.5. *Under the assumptions $H1, H2$ and $H3, \widetilde{\sigma}_j^2$ are unbiased.*

Proof E. The definition of $\widetilde{\sigma}_j^2$ can be rewritten as

$$\begin{aligned}
(I-j-1)\cdot\widetilde{\sigma}_j^2 &= \sum_{i=1}^{I-j} S_{i,j}\cdot(\tfrac{S_{i,j}+X_{i,j+1}}{S_{i,j}} - \widetilde{f}_j)^2 \\
&= \sum_{i=1}^{I-j} \tfrac{(S_{i,j}+X_{i,j+1})^2}{S_{i,j}} - 2\cdot(S_{i,j}+X_{i,j+1})\times\widetilde{f}_j + S_{i,j}\cdot\widetilde{f}_j^2 \qquad (E1) \\
&= \sum_{i=1}^{I-j} \tfrac{(S_{i,j}+X_{i,j+1})^2}{S_{i,j}} - \sum_{i=1}^{I-j} S_{i,j}\cdot\widetilde{f}_j^2,
\end{aligned}$$

then

$$\mathbb{E}[(I-j-1)\cdot\widetilde{\sigma}_j^2 \mid \mathscr{D}_j] = \sum_{i=1}^{I-j}\underbrace{\mathbb{E}[(S_{i,j}+X_{i,j+1})^2 \mid \mathscr{D}_j]}_{(E3)}/S_{i,j} - \sum_{i=1}^{I-j} S_{i,j}\cdot\underbrace{\mathbb{E}[\widetilde{f}_j^2 \mid \mathscr{D}_j]}_{(E4)}, \qquad (E2)$$

because $S_{i,j}$ is $\mathscr{D}_j$-measurable, we calculate first $(E3)$ by

$$\begin{aligned}\mathbb{E}\left[(S_{i,j}+X_{i,j+1})^2 \mid \mathscr{D}_j\right] &= \mathbb{E}\left[(S_{i,j}+X_{i,j+1})^2 \mid X_{i,1},\ldots,X_{i,j}\right], \text{ by } H2\\ &= Var\left[(S_{i,j}+X_{i,j+1}) \mid X_{i,1},\ldots,X_{i,j}\right]\\ &\quad +\mathbb{E}\left[(S_{i,j}+X_{i,j+1}) \mid X_{i,1},\ldots,X_{i,j}\right]^2\\ &= Var(X_{i,j+1} \mid X_{i,1},\ldots,X_{i,j})\\ &\quad +(\mathbb{E}[X_{i,j+1} \mid X_{i,1},\ldots,X_{i,j}]+S_{i,j})^2\\ &= S_{i,j}\cdot\sigma_j^2+(S_{i,j}\cdot f_j)^2, \text{ by } H1 \text{ and } H3.\end{aligned} \tag{E3}$$

We turn now to $(E4)$ for that we using $(F6)$ and $(A3)$ we obtain

$$\mathbb{E}[\widetilde{f}_j^2 \mid \mathscr{D}_j] = Var(\widetilde{f}_j \mid \mathscr{D}_j) + (\mathbb{E}[\widetilde{f}_j \mid \mathscr{D}_j])^2 = \frac{\sigma_j^2}{\sum_{i=1}^{I-j} S_{i,j}} + f_j^2. \tag{E4}$$

Inserting $(E3)$ and $(E4)$ into $(E2)$ we obtain

$$\mathbb{E}[(I-j-1)\cdot\tilde{\sigma}_j^2 \mid \mathscr{D}_j] = \sum_{i=1}^{I-j}(\sigma_j^2+S_{i,j}\cdot f_j^2) - \sum_{i=1}^{I-j}\frac{S_{i,j}\cdot\sigma_j^2}{\sum_{i=1}^{I-j} S_{i,j}} + S_{i,j}\cdot f_j^2 = (I-j-1)\cdot\sigma_j^2.$$

□

The following theorem gives the formula for estimating $MSEP(\widetilde{R}_i)$.

Theorem 3.6. *Under the assumptions $H1, H2$ and $H3$, $MSEP(\widetilde{R}_i)$ can be estimated by*

$$\widetilde{MSEP}(\widetilde{R}_i) = \widetilde{Z}_i^2 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}_j^2}{\widetilde{f}_j^2}\left(\frac{1}{\widetilde{S}_{i,j}} + \frac{1}{\sum_{l=1}^{I-j} S_{l,j}}\right). \tag{3.9}$$

Proof F. We use the abbreviations

$$\mathbb{E}_i[Y] = \mathbb{E}[Y \mid X_{i,1},...,X_{i,I-i+1}], \tag{F1}$$

$$Var_i[Y] = Var[Y \mid X_{i,1},...,X_{i,I-i+1}]. \tag{F2}$$

The theorem of conditional variance

$$Var(Y) = \mathbb{E}[Var(Y|X)] + Var[\mathbb{E}[Y|X]], \tag{F3}$$

we have the formula (3.8) given by

$$MSEP(\widetilde{R}_i) = \underbrace{Var(Z_i \mid \mathscr{D}_{i,j})}_{\text{Error Process}} + \underbrace{(\mathbb{E}[Z_i \mid \mathscr{D}_{i,j}] - \widetilde{Z}_i)^2}_{\text{Error Estimators}}. \tag{F4}$$

Using assumption $H2$ and then repeatedly applying the formula of the assumptions $H1$ and $H3$, the first term of $MSEP(\widetilde{R}_i)$ can be written as:

$$\begin{aligned}
Var(Z_i \mid \mathscr{D}_{i,j}) &= Var_i(Z_i), H2 \\
&= \mathbb{E}_i[Var(Z_i \mid X_{i,1}, \dots, X_{i,I-1})] + Var_i[\mathbb{E}[Z_i \mid X_{i,1}, \dots, X_{i,I-1}]] \\
&= \mathbb{E}_i[S_{i,I-1} \cdot \sigma^2_{I-1}] + Var_i[S_{i,I-1} + S_{i,I-1} \cdot (f_{I-1} - 1)], \text{ by } \ H3, H1 \\
&= \mathbb{E}_i[S_{i,I-1}] \cdot \sigma^2_{I-1} + Var_i[S_{i,I-1}] \cdot f^2_{I-1} \\
&= \mathbb{E}_i[S_{i,I-2} \cdot (f_{I-2} - 1) + S_{i,I-2}] \cdot \sigma^2_{I-1} + f^2_{I-1} \cdot \mathbb{E}_i[S_{i,I-2} \cdot \sigma^2_{I-2}] \\
&\quad + f^2_{I-1} \cdot Var_i\left[S_{i,I-2} + S_{i,I-2} \cdot (f_{I-2} - 1)\right] \\
&= \mathbb{E}_i[S_{i,I-2}] \cdot f_{I-2} \cdot \sigma^2_{I-1} + \mathbb{E}_i[S_{i,I-2}] \cdot f^2_{I-1} \cdot \sigma^2_{I-2} \\
&\quad + Var_i[S_{i,I-2}] \cdot f^2_{I-1} \cdot f^2_{I-2} \\
&\qquad \vdots \\
&= Z_i^0 \cdot \sum_{j=I+1-i}^{I-1} (f_{I+1-i} \dots f_{j-1}) \times \sigma^2_j \cdot \left(f^2_{j+1} \dots f^2_{I-1}\right).
\end{aligned}$$

As we don't know the parameters f_j et σ^2_j, we replace them by their estimators $\widetilde{f}_j$ et $\widetilde{\sigma}^2_j$, that is to say that we estimate the first term of the expression ($F4$) for $MSEP(\widetilde{R}_i)$ by

$$\begin{aligned}
\widetilde{Var(Z_i \mid \mathscr{D}_{i,j})} &= Z_i^0 \cdot \sum_{j=I-i+1}^{I-1} \widetilde{f}_{I-i+1} \dots \widetilde{f}_{j-1} \cdot \widetilde{\sigma}^2_j \cdot \widetilde{f}^2_{j+1} \dots \widetilde{f}^2_{I-1} \\
&= Z_i^0 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{f}^2_{I-i+1} \cdots \widetilde{f}^2_{j-1} \cdot \widetilde{f}^2_j \cdot \widetilde{\sigma}^2_j \cdot \widetilde{f}^2_{j+1} \cdots \widetilde{f}^2_{I-1}}{\widetilde{f}_{I-i+1} \cdots \widetilde{f}_{j-1} \cdot \widetilde{f}^2_j} \\
&= (Z_i^0)^2 \cdot \sum_{j=I+1-i}^{I-1} \frac{\widetilde{f}^2_{I-i+1} \cdots \widetilde{f}^2_{I-1} \cdot \widetilde{\sigma}^2_j}{Z_i^0 \cdot \widetilde{f}_{I-i+1} \cdots \widetilde{f}_{j-1} \cdot \widetilde{f}^2_j} \\
&= (Z_i^0)^2 \cdot \widetilde{f}^2_{I-i+1} \dots \widetilde{f}^2_{I-1} \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}^2_j / \widetilde{f}^2_j}{Z_i^0 \cdot \widetilde{f}_{I-i+1} \cdots \widetilde{f}_{j-1}}.
\end{aligned}$$

So

$$\widetilde{Var(Z_i \mid \mathscr{D}_{i,j})} = (\widetilde{Z}_i)^2 \cdot \sum_{j=I+1-i}^{I-1} \frac{\widetilde{\sigma}^2_j / \widetilde{f}^2_j}{\widetilde{S}_{i,j}}. \tag{F5}$$

We turn now to the second term of the expression ($F4$) for $MSEP(\widetilde{R}_i)$

$$(\mathbb{E}[Z_i \mid \mathscr{D}_{i,j}] - \widetilde{Z}_i)^2 = (Z_i^0)^2 \cdot \ (f_{I-i+1} \dots f_{I-1} - \widetilde{f}_{I-i+1} \dots \widetilde{f}_{I-1})^2$$

To estimate the second term we cannot simply replace f_j by their estimator because this will lead to cancel it. We therefore use another approach. We assume that,

$$F = f_{I-i+1} \ldots f_{I-1} - \widetilde{f}_{I-i+1} \ldots \widetilde{f}_{I-1} = T_{I-i+1} + \ldots + T_{I-1},$$

$$\text{where } \; T_j = \widetilde{f}_{I-i+1} \ldots \widetilde{f}_{j-1} \cdot (f_j - \widetilde{f}_j) \; \cdot f_{j+1} \ldots f_{I-1},$$

so we have

$$F^2 = (T_{I+1-i} + \ldots + T_{I-1})^2 = \sum_{j=I-i+1}^{I-1} T_j^2 + 2\sum_{l<j} T_l T_j$$

Then we approximate T_j^2 by $\mathbb{E}(T_j^2 \mid \mathscr{D}_j)$ et T_l, T_j par $\mathbb{E}[T_l T_j \mid \mathscr{D}_j]$.
As, $\mathbb{E}[f_j - \widetilde{f}_j \mid \mathscr{D}_j] = 0$ (because $\widetilde{f}_j$ is an unbiased), we have

$$\mathbb{E}[T_l T_j \mid \mathscr{D}_j] = 0, \;\; for \;\; l < j,$$

so

$$F^2 = \sum_{j=I+1-i}^{I-1} \mathbb{E}[T_j^2 \mid \mathscr{D}_j] + 2\sum_{i \langle j} \mathbb{E}[T_i T_j \mid \mathscr{D}_j] = \sum_{j=I+1-i}^{I-1} \mathbb{E}[T_j^2 \mid \mathscr{D}_j].$$

or

$$\mathbb{E}[T_j^2 \mid \mathscr{D}_j] = \widetilde{f}_{I+1-i}^2 \ldots \widetilde{f}_{j-1}^2 \cdot \mathbb{E}[(f_j - \widetilde{f}_j)^2 \mid \mathscr{D}_j] \; \cdot f_{j+1}^2 \ldots f_{I-1}^2.$$

So we should calculate $\mathbb{E}[(f_j - \widetilde{f}_j)^2 \mid \mathscr{D}_j]$,

$$\begin{aligned}
\mathbb{E}[(f_j - \widetilde{f}_j)^2 \mid \mathscr{D}_j] &= Var(\widetilde{f}_j \mid \mathscr{D}_j) \\
&= Var(\frac{\sum_{i=1}^{I-j} X_{i,j+1}}{\sum_{i=1}^{I-j} S_{i,j}} + 1 \mid \mathscr{D}_j) \\
&= \frac{\sum_{i=1}^{I-j} Var\left(X_{i,j+1} \mid \mathscr{D}_j\right)}{\left(\sum_{i=1}^{I-j} S_{i,j}\right)^2} = \frac{\sigma_j^2}{\sum_{i=1}^{I-j} S_{i,j}}.
\end{aligned}$$

So

$$\mathbb{E}[(f_j - \widetilde{f}_j)^2 \mid \mathscr{D}_j] = Var(\widetilde{f}_j \mid \mathscr{D}_j) = \frac{\sigma_j^2}{\sum_{i=1}^{I-j} S_{i,j}}. \qquad (F6)$$

We obtain

$$\mathbb{E}[T_j^2 \mid \mathscr{D}_j] = \frac{\widetilde{f}_{I+1-i}^2 \cdots \widetilde{f}_{j-1}^2 \cdot \ \sigma_j^2 \ \cdot f_{j+1}^2 \cdots f_{I-1}^2}{\sum_{i=1}^{I-j} S_{i,j}},$$

so we estimate F^2 by $\sum_{j=I-i+1}^{I-1} \mathbb{E}(T_j^2 \mid \mathscr{D}_j)$ and we can now replace the parameters f_j et σ_j^2 by their unbiased estimators $\widetilde{f}_j$ et $\widetilde{\sigma}_j^2$, that is to say that we estimate the second term of $MSEP(\widetilde{R}_i)$,

$$\begin{aligned}(\mathbb{E}[Z_i \mid \widetilde{\mathscr{D}_{i,j}]} - \widetilde{Z}_i)^2 &= (\widetilde{Z}_i)^2 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{f}_{I+1-i}^2 \cdots \widetilde{f}_{j-1}^2 \cdot \widetilde{\sigma}_j^2 \cdot \widetilde{f}_{j+1}^2 \cdots \widetilde{f}_{I-1}^2}{\sum_{i=1}^{I-j} S_{i,j}} \\ &= (\widetilde{Z}_i)^2 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}_j^2 / \widetilde{f}_j^2}{\sum_{i=1}^{I-j} S_{i,j}}. \end{aligned} \tag{F7}$$

By adding the expressions ($F5$) and (F7), we find the formula proposed for $\widetilde{MSEP(\widetilde{R}_i)}$,

$$\begin{aligned}\widetilde{MSEP(\widetilde{R}_i)} &= Var(\widetilde{Z_i \mid \mathscr{D}_{i,j}}) + (\mathbb{E}[Z_i \mid \widetilde{\mathscr{D}_{i,j}]} - \widetilde{Z}_i)^2 \\ &= (\widetilde{Z}_i)^2 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}_j^2 / \widetilde{f}_j^2}{\widetilde{S}_{i,j}} + (\widetilde{Z}_i)^2 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}_j^2 / \widetilde{f}_j^2}{\sum_{i=1}^{I-j} S_{i,j}} \\ &= (\widetilde{Z}_i)^2 \cdot \sum_{j=I-i+1}^{I-1} \frac{\widetilde{\sigma}_j^2}{\widetilde{f}_j^2} \left(\frac{1}{\widetilde{S}_{i,j}} + \frac{1}{\sum_{i=1}^{I-j} S_{i,j}} \right). \end{aligned}$$

□

1.3.2 The estimation error of the overall reserves

It is also interesting to calculate the error on the estimated overall reserve $\widetilde{R} = \widetilde{R}_2 + ... + \widetilde{R}_I$. We can not simply sum the errors $MSEP(\widetilde{R})$ because they are correlated by the same estimators $\widetilde{f}_j$ and $\widetilde{\sigma}_j^2$, but we can use the following theorem.

Theorem 3.7. *Under the assumptions* $H1$*,* $H2$ *and* $H3$*,* $MSEP(\widetilde{R})$ *can be estimated by*

$$\widetilde{MSEP}(\widetilde{R}) = \sum_{i=2}^{I}\left[MSEP(\widetilde{R}_i) + \widetilde{Z}_i\left(\sum_{k\,=\,i+1}^{I}\widetilde{Z}_k\right)\sum_{j=I-i+1}^{I-1}\frac{2\,\widetilde{\sigma}_j^2/\widetilde{f}_j^2}{\sum_{k=1}^{I-j}S_{k,j}}\right]. \tag{3.10}$$

Proof G. This proof is analogous to that in (Proof F). The explanations will therefore be brief.

$$\begin{aligned}
MSEP(\sum_{i=2}^{I}\widetilde{R}_i) &= \mathbb{E}[(\sum_{i=2}^{I}\widetilde{R}_i - \sum_{i=2}^{I}R_i)^2 \mid \mathscr{D}_{i,j}] \\
&= \mathbb{E}[(\sum_{i=2}^{I}\widetilde{Z}_i - \sum_{i=2}^{I}Z_i)^2 \mid \mathscr{D}_{i,j}] \\
&= Var(\sum_{i=2}^{I}Z_i \mid \mathscr{D}_{i,j}) + (\mathbb{E}[\sum_{i=2}^{I}Z_i \mid \mathscr{D}_{i,j}] - \sum_{i=2}^{I}\widetilde{Z}_i)^2.
\end{aligned}$$

We have been calculated this terms on (Proof F),

$$MSEP(\sum_{i=2}^{I}\widetilde{R}_i) = \sum_{i=2}^{I}MSEP(\widetilde{R}_i) + \sum_{2\leqslant i<k\leqslant I} 2\cdot Z_i^0 Z_k^0 \cdot F_iF_k.$$

We therefore need only develop an estimator for F_iF_k. We immediately get the final result of $\widetilde{MSEP}(\widetilde{R})$

$$\widetilde{MSEP}(\widetilde{R}) = \sum_{i=2}^{I}\left[MSEP(\widetilde{R}_i) + \widetilde{Z}_i\left(\sum_{k\,=\,i+1}^{I}\widetilde{Z}_k\right)\sum_{j=I+1-i}^{I-1}\frac{2\,\widetilde{\sigma}_j^2/\widetilde{f}_j^2}{\sum_{k=1}^{I-j}S_{k,j}}\right].$$

□

1.4 Comparison between Mack's model and incremental approach

We can see that this new approach using only incremental claims which results easier formulae. Proceeding from the formulae of our approach we can automatically find the same formulae of

Mack's model.

Mack's Model	Incremental Approach
$\widetilde{f}_j = \frac{\sum_{i=1}^{I-j} C_{i,j+1}}{\sum_{i=1}^{I-j} C_{i,j}}$	$\widetilde{f}_j = \frac{\sum_{i=1}^{I-j} X_{i,j+1}}{\sum_{i=1}^{I-j} S_{i,j}} + 1$
$\widetilde{C}_{i,I} = C_{i,I-i+1} \cdot \prod_{j=I-i+1}^{I-1} \widetilde{f}_j$	$\widetilde{Z}_i = \sum_{j=1}^{I-i+1} X_{i,j} + \sum_{j=I-i+2}^{I} \widetilde{X}_{i,j}$
$\widetilde{R}_i = \widetilde{C}_{i,I} - C_{i,I-i+1}$	$\widetilde{R}_i = \sum_{j=I-i+2}^{I} \widetilde{X}_{i,j}$
$\widetilde{R} = \sum_{i=2}^{I} \widetilde{R}_i$	$\widetilde{R} = \sum_{i=2}^{I} \widetilde{R}_i$

1.5 Claim development result

We will calculate the CDR using the incremental claims data, for that we will change the notation of the incremental claims data available at time $t = I$ to

$$\mathscr{D}_{i,j}^{I} = \sigma\{X_{i,j}| \ \ i + j \leq I + 1\},$$

and for the incremental claims data available one period later, at time $t = I + 1$ by

$$\mathscr{D}_{i,j}^{I+1} = \{X_{i,j}| \ i + j \leq I + 2, \ i \leq I\} = \mathscr{D}_{i,j}^{I} \cup \{X_{i,I-i+2}, \ i \leq I\},$$

1.5.1 Estimators

The development factors

$$\widetilde{f}_j^{I} = \frac{\sum_{i=1}^{I-j} X_{i,j+1}}{\sum_{i=1}^{I-j} S_{i,j}} + 1 \qquad \text{and} \qquad \widetilde{f}_j^{I+1} = \frac{\sum_{i=1}^{I-j+1} X_{i,j+1}}{\sum_{i=1}^{I-j+1} S_{i,j}} + 1. \tag{3.11}$$

The ultimate claims amount

$$\widetilde{Z}_i^{I} = \sum_{j=1}^{I-i+1} X_{i,j} + \sum_{j=I-i+2}^{I} \widetilde{X}_{i,j} = Z_i^0 + \widetilde{R}_i. \tag{3.12}$$

$$\widetilde{Z}_i^{I+1} = \sum_{j=1}^{I-i+2} X_{i,j} + \sum_{j=I-i+3}^{I} \widetilde{X}_{i,j} = \mathbf{Z}_i^0 + \widetilde{R}_i^{\mathscr{D}_{i,j}^{I+1}}. \tag{3.13}$$

Yearly claims reserves

$$\widetilde{R}_i^{\mathscr{D}_{i,j}^I} = \sum_{j=I-i+2}^{I} \widetilde{X}_{i,j} = \widetilde{Z}_i^I - Z_i^0. \tag{3.14}$$

$$\widetilde{R}_i^{\mathscr{D}_{i,j}^{I+1}} = \sum_{j=I-i+3}^{I} \widetilde{X}_{i,j} = \widetilde{Z}_i^{I+1} - \mathbf{Z}_i^0. \tag{3.15}$$

These estimators are unbiased

Definition 3.8 (True CDR according to the incremental approach)**.**
The true CDR for accident year $i \in \{1, ..., I\}$ in accounting year $(I, I+1]$ is given by

$$CDR_i(I+1) = \mathbb{E}[\widetilde{R}_i^I \mid \mathscr{D}_{i,j}^I] - (X_{i,I-i+2} + \mathbb{E}[\widetilde{R}_i^{I+1} \mid \mathscr{D}_{i,j}^{I+1}]) \tag{3.16}$$

our approach will permit us to define the CDR as the difference between the matrix of all incremental payments data at time $t = I$ minus the matrix of all incremental payments data at time $t = I + 1$.

$$CDR_i(I+1) = \mathbb{E}[Z_i \mid \mathscr{D}_{i,j}^I] - \mathbb{E}[Z_i \mid \mathscr{D}_{i,j}^{I+1}]. \tag{3.17}$$

Definition 3.9 (Observable CDR according to the incremental approach)**.**
The observable CDR for accident year $i \in \{1, ..., I\}$ in accounting year $(I, I+1]$ in the chain ladder method is given by

$$\widetilde{CDR}_i(I+1) = \widetilde{R}_i^{\mathscr{D}_{i,j}^I} - (X_{i,I-i+2} + \widetilde{R}_i^{\mathscr{D}_{i,j}^{I+1}}) = \widetilde{Z}_i^I - \widetilde{Z}_i^{I+1}. \tag{3.18}$$

CDR real $(CDR_i(I+1))$ is estimated by the CDR observable $(\widetilde{CDR}_i(I+1))$. See [24], [25]

1.5.2 The one-year bootstrap

The stochastic process is applied only to the diagonal of the calendar year $I + 1$, this new diagonal has been projected by our approach. Because we use only the incremental claims in this approach, the Bootstrap simulation will be more easy to calculate since the Pearson residuals are based on incremental payments. So we will avoid 3 steps in calculations and we directly calculate the data incremental payments backwards $X_{i,j}^B$ by the following formula:

$$X_{i,j}^B = \frac{S_{i,I-i+2}}{\prod_{l=j}^{I-i+1} f_l} - \frac{S_{i,I-i+2}}{\prod_{k=j-1}^{I-i+1} f_k}. \tag{3.19}$$

then we calculate the Pearson residues, and apply the bootstrap n times only for the diagonal we want simulate, using classic formulae see [8].

2 The incremental approach in the calendar years view

We will focus our interest from the development years j to the calendar years t which give us a new view for the past obligation of each calendar year t and also the estimation of the obligations left to pay of each calendar year t.

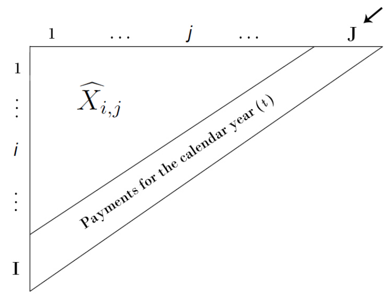

Figure 3.3 – The triangle liquidation in calendar years view

To more clarify this vision we proposed new tabulation form instead of the development triangle.

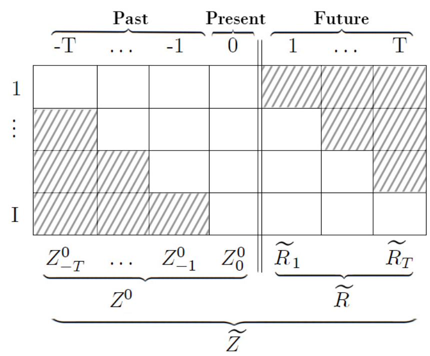

Figure 3.4 – The new form of the triangle liquidation

In this new tabulation, we always represent the accident years by $i \in \{1, ..., I\}$ and the calender years by $t \in \{-T, ..., T\}$, for $t \in \{-T, ..., 0\}$ we have $i \leq t + I$ and for the future part $t \in \{1, ..., T\}$ we have $i \geq t + 1$. We used negative index $t \in \{-T, ..., -1\}$ to represent the past and the current time by $t = 0$ and finally the future time by the positive index $t \in \{1, ..., T\}$. Such as $T = I - 1$. In this view we can calculate Z^0_t, $\widetilde{R}_t$ of each calender year and the sum of all payments of the development lozenge will give the ultimate claims amount Z. We can complete the right part of the lozenge (future part) of the incremental payments by,

$$\mathbb{E}[X_{i,t+1} \mid X_{i,i-I}, \dots, X_{i,t}] = \ S_{i,t} \cdot (f_{-i+t+1} - 1). \tag{3.20}$$

2.1 Estimation of parameters in the calendar years view

2.1.1 The development factors at time t

For each development year $t \in \{-T, ..., -1\}$, the development factors are estimated by

$$\widetilde{f}_t = \frac{\sum_{i=1}^{-t} X_{i,i+t}}{\sum_{i=1}^{-t} S_{i,i+t-1}} + 1, \quad \text{where } S_{i,t} = \sum_{l=i-I}^{t} X_{i,l}. \tag{3.21}$$

We observe that the development factors are indexed by the calendar year (t), but they calculate always the proportion between the development years.

2.1.2 The past obligations at time t

The process of past obligations for each $t \in \{-T, ..., 0\}$, i.e the past claims amount $X_{i,j}$, has already been paid for each calendar year till the current date $t = 0$, which is defined by

$$Z_t^0 = \sum_{i=1}^{t+I} X_{i,t}. \tag{3.22}$$

We can obtain the overall past obligations by

$$Z^0 = \sum_{t=-T}^{0} Z_t^0.$$

2.1.3 The ultimate claims amount

We can calculate the yearly ultimate claims amount by

$$\widetilde{Z}_i = \sum_{t=-T+i-1}^{0} X_{i,t} + \sum_{t=1}^{i-1} \widetilde{X}_{i,t} \tag{3.23}$$

The process $\widetilde{Z}$ defined by

$$\widetilde{Z} = \sum_{i=1}^{I} \widetilde{Z}_i = \sum_{t=-T}^{0} Z_t^0 + \sum_{t=1}^{T} \widetilde{R}_t. \tag{3.24}$$

2.1.4 Claims reserves at time t

To obtain the estimators of the outstanding claims reserves for each calendar year t, we denote for $t \in \{1, ..., T\}$

$$\widetilde{R}_t = \sum_{i=t+1}^{I} \widetilde{X}_{i,t}. \tag{3.25}$$

also the reserves for each year i, for $i \in \{2, ..., I\}$

$$\widetilde{R}_i = \sum_{t=1}^{i-1} \widetilde{X}_{i,t}. \tag{3.26}$$

and their sum will give the overall reserves

$$\widetilde{R} = \sum_{t=1}^{T} \widetilde{R}_t = \sum_{i=2}^{I} \widetilde{R}_i.$$

2.2 Claim development result in the calendar years view

We can apply the same definitions of the CDR presented above to get the observable $\widetilde{CDR}_i(I+1)$

$$\widetilde{CDR}_i(I+1) = \widetilde{R}_i^{\mathscr{D}_{i,t}^{I}} - (X_{i,1} + \widetilde{R}_i^{\mathscr{D}_{i,t}^{I+1}}) = \widetilde{Z}_i^I - \widetilde{Z}_i^{I+1}. \tag{3.27}$$

Where $X_{i,1}$ is simulated by a one year bootstrapping, with $X_{1,1} = 0$.

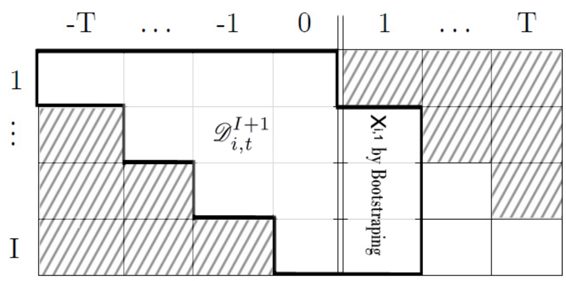

Figure 3.5 – Claim Development Result in the calendar years view

The development factors

$$\widetilde{f}_t^I = \frac{\sum_{i=1}^{-t} X_{i,i+t}}{\sum_{i=1}^{-t} S_{i,i+t-1}} + 1 \quad \text{and} \quad \widetilde{f}_t^{I+1} = \frac{\sum_{i=1}^{-t+1} X_{i,i+t}}{\sum_{i=1}^{-t+1} S_{i,i+t-1}} + 1.$$

The ultimate claims amount

$$\widetilde{Z}_i^I = \sum_{t=-T+i-1}^{0} X_{i,t} + \sum_{t=1}^{i-1} \widetilde{X}_{i,t} \quad \text{and} \quad \widetilde{Z}_i^{I+1} = \sum_{t=-T+i-1}^{1} X_{i,t} + \sum_{t=2}^{i-1} \widetilde{X}_{i,t}.$$

Yearly claims reserves

$$\widetilde{R}_i^{\mathscr{D}_{i,j}^I} = \sum_{t=1}^{i-1} \widetilde{X}_{i,t} \quad \text{and} \quad \widetilde{R}_i^{\mathscr{D}_{i,j}^{I+1}} = \sum_{t=2}^{i-1} \widetilde{X}_{i,t}.$$

The aggregate CDR is given by

$$\widetilde{CDR}(I+1) = \widetilde{Z}^I - \widetilde{Z}^{I+1} = \sum_{i=2}^{I} \widetilde{CDR}_i(I+1).$$

3 Numerical example and conclusions

For our numerical example we use the data set given in table 3.1. The table contains incremental payments for accident years $i \in \{1, ..., 9\}$. We will apply our approach (Stochastic Chain ladder using incremental payments) on the incremental observations $\widehat{X}_{i,j}$ data $X_{i,j}$, we first calculate the development factors according to formula (3.1) then we complete the lower triangle, and finally we obtain the amounts we seek to put in reserve.

i	Development year j 1	2	3	4	5	6	7	8	9	Reserve R_i	Ultimate Z_i
1	2 202 584	1 007 865	257 673	76 948	76 557	23 009	24 376	5 499	4 122	0	3 678 633
2	2 350 650	1 202 373	230 823	56 221	25 120	13 557	19 537	4 144		4 378	3 906 803
3	2 321 885	1 102 305	276 686	97 322	56 557	24 238	19 832			9 347	3 908 172
4	2 171 487	993 787	230 567	70 612	49 250	32 718				28 392	3 576 813
5	2 140 328	1 016 751	242 183	101 258	85 292					51 444	3 637 256
6	2 290 664	1 047 533	212 135	90 704						111 811	3 752 847
7	2 148 216	1 071 559	208 560							187 084	3 615 419
8	2 143 728	1 014 853								411 864	3 570 445
9	2 144 738									1 433 505	3 578 243
f_j	1,4759	1,0719	1,0232	1,0161	1,0063	1,0056	1,0013	1,0011		2 237 825	33 224 631

Table 3.1 – Run-off triangle incremental payments, in Euro 1000

The results are identical to those given by the classic Chain ladder using cumulated payments, so we can calculate our reserves simply without need to calculate the cumulated triangle, by avoiding a step of calculation and we can got a clear image about information we have in the triangle, and the formula we use here are more easy by applying simple summations. Let's move at present to the new tabulation form for the calendar years view and we can recalculate the reserves as in table 3.2.

	Past calendar year t								Present	Future calendar year t								Z_i
	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	Ultimate
i=1	2 202 584	1 007 865	257 673	76 948	76 557	23 009	24 376	5 499	4 122									3 678 633
i=2		2 350 650	1 202 373	230 823	56 221	25 120	13 557	19 537	4 144	**4 378**								3 906 803
i=3			2 321 885	1 102 305	276 686	97 322	56 557	24 238	19 832	**4 968**	**4 379**							3 908 172
i=4				2 171 487	993 787	230 567	70 612	49 250	32 718	**19 837**	**4 547**	**4 008**						3 576 813
i=5					2 140 328	1 016 751	242 183	101 258	85 292	**22 572**	**20 173**	**4 624**	**4 076**					3 637 256
i=6						2 290 664	1 047 533	212 135	90 704	**58 732**	**23 289**	**20 814**	**4 771**	**4 205**				3 752 847
i=7							2 148 216	1 071 559	208 560	**79 368**	**56 581**	**22 436**	**20 052**	**4 596**	**4 051**			3 615 419
i=8								2 143 728	1 014 853	**227 107**	**78 380**	**55 878**	**22 157**	**19 802**	**4 539**	**4 001**		3 570 445
i=9									2 144 738	**1 020 741**	**227 603**	**78 551**	**56 000**	**22 205**	**19 845**	**4 549**	**4 010**	3 578 243
	2 202 584	3 358 515	3 781 931	3 581 563	3 543 579	3 683 433	3 603 034	3 627 204	3 604 963	1 437 703	414 953	186 311	107 055	50 809	28 435	8 550	4 010	33 224 631
	Z^0_{-8}	Z^0_{-7}	Z^0_{-6}	Z^0_{-5}	Z^0_{-4}	Z^0_{-3}	Z^0_{-2}	Z^0_{-1}	Z^0_0	$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$	$\tilde{R}_4$	$\tilde{R}_5$	$\tilde{R}_6$	$\tilde{R}_7$	$\tilde{R}_8$	

$Z^0 = 30\,986\,806$ $\quad$ $\tilde{Z} = Z^0 + \tilde{R} = 33\,224\,631$ $\quad$ $\tilde{R} = 2\,237\,825$

f_t	1,4759	1,0719	1,0232	1,0161	1,0063	1,0056	1,0013	1,0011

Table 3.2 – Run-off triangle in calendar view (incremental payments, in Euro 1000) for time I = 9

We observe here that we can calculate reserves of each calendar year (which is represented by diagonals in classic triangle) by calculating first the development factors for each calendar year t using formula (3.21), then we can find the overall reserve. By using the new tabulation form we find the same development factors, also we got identical overall reserves and ultimate claims amounts comparing with those calculated by our incremental approach (stochastic chain ladder using incremental payments). Now we will calculate CDR for the two cases (incremental approach and for calendar year view), the results are given in the following table,

	$\widetilde{Z}_i^I$	$\widetilde{Z}_i^{I+1}$	$\widetilde{CDR}_i(I+1)$
1	3 678 633	3 678 633	0
2	3 906 803	3 906 699	104
3	3 908 172	3 906 113	2059
4	3 576 813	3 572 156	4657
5	3 637 256	3 635 610	1646
6	3 752 847	3 771 609	-18762
7	3 615 419	3 641 266	-25847
8	3 570 445	3 575 394	-4949
9	3 578 243	3 566 649	11594
Total	33 224 631	33 254 129	-29498

Table 3.3 – Realization of the observable CDR at time t = I + 1, in Euro 1000

Conclusion

This work has examined new ways to estimate the amounts place in reserve to perform in the future payments related to claims incurred.

First, the calculation of claims reserves using a stochastic incremental approach by modifying Mack's model, so we establish the require formulae using only incremental payments which avoid calculation of the cumulative triangle, and we can got identical results see Comparison [1.4]. We calculate the CDR for this case, and we apply the bootstrap by developed a new formula see (3.19) which avoid 3 steps in calculations.

Second, we examined the incremental approach in calendar year view, so we propose a new form of tabulation Figure 3.4, so we can easily calculate the past obligation of each calendar years (t), the claims reserves of each accident year (i) and the overall reserve.

The incremental approach give us the advantage to avoid lot of steps in different cases, and we can observe the simplicity of the formulae used to give identical results.

Chapter 4

Chain-ladder with Bivariate Development factors

This Chapter presents a mathematical model of loss reserves which represents a generalization of Mack's Model [18], it includes dependencies of the loss payments between accident and development years.

More precisely, we assume that the expectations of the loss payments in accident year i and development year j, conditioned on the information of previous years, depend linearly on the payments of the proceeding accident and the proceeding development year. We get development factors α between accident years and factors β between development years: the bivariate development factors. In this way, the whole run-off 'triangle' is covered by a net of dependencies.

Here, the usual stationarity assumption in IBNR methods is replaced by a condition on the upper boundary of the IBNR-triangle which reminds the von-Neumann boundary condition in partial differential equations. The right-hand boundary of the triangle is naturally subject to a Dirichlet-type condition since no claims settlements can be done before the accident has occurred.

If by brut force we suppress the development factors α between accident years, we are pushed back to Mack's model and the Chain-Ladder development factors β^{CL}. However, there is a smooth transition between our full model and Mack's one by varying a continuous parameter between 1 and 0 (see Section 2). The parameter transition allows also to find first order perturbation terms, both for the bivariate development factors and the claims provisions.

Simple examples show the influence of these boundary conditions on the final provision. In an other example, we compare our results to those resulting from the example of Wüthrich in [38].

We deliberately do not require any assumptions on the variances, as they can be found in [18] and have further be analyzed in [12]. If the consequences of these assumptions are used to calculate variances at the level of the provisions, one ends up with most complicated formulas and proofs.

In our approach to the bootstrap method, we first apply a linear regression estimator for the mean of the boundary values with special attention to the critical value on the lower corner of the triangle. We then choose a multiplicative version for the residuals with an identical Gamma distribution (see Section 4). The numerical example in Section 5 uses the same initial data as the one in [7], thus allowing an easy comparison with their results.

1 The Bivariate Development Factors Model

Throughout this chapter we use the standard conventions that empty sums or products are equal to the additive, resp. multiplicative, neutral elements, i. e. for $\kappa > \ell$ we have $\sum_{\mu=\kappa}^{\ell} \cdots = 0$ and $\prod_{\mu=\kappa}^{\ell} \cdots = 1$.

We regard a matrix of random variables

$$C := (C_{i,j})_{1\leq i\leq I, 1\leq j\leq J} \tag{4.1}$$

for some $I \geq J \geq 2$. The variable $C_{i,j}$ is interpreted as the cumulated claims amount of accident year i till the development year j. For convenience we set $C_{i,0} = 0$.

Within the standard theory of claim reserving, also called IBNR-theory, it is assumed that pairs (i, j) of accident and development years with $i + j \leq I + 1$ describe past years for which the **run-off 'triangle'** of real observations $\widehat{C}_{i,j} \in \mathbb{R}$ are available:

$$\widehat{C} := \left(\widehat{C}_{i,j}\right)_{(i,j)\in\mathcal{P}} \tag{4.2}$$

where $\mathcal{P} := \left\{(i,j)\middle|1 \leq i \leq I, 1 \leq j \leq (I-i+1) \wedge J\right\}$ is the index set of the 'triangle'. Pairs (i, j) with $i + j > I + 1$ refer to future years with unknown results for the corresponding claim amounts. It is the task of the IBNR-theory to provide estimations for the outstanding claims to be paid, i.e. estimators for the random total provision

$$P := \sum_{i=I-J+2}^{I} C_{i,J} - C_{i,I+1-i} = \sum_{i=I-J+2}^{I} P_i \tag{4.3}$$

with $P_i := C_{i,J} - C_{i,I+1-i}$ the random provision for accident year $i \geq I - J + 2$.

1.1 Assumption Model

In generalization of the theory of Th. Mack we make the following assumption:

Assumption 4.1.

(i) For all $1 \leq i \leq I-1$, $1 \leq j \leq J-1$ *there exist constants* α_i *and* β_j *with*

$$\mathbb{E}\left[C_{i+1,j+1}\middle|\,\mathcal{F}_{i+j}\right] = \alpha_i \cdot C_{i,j+1} + \beta_j \cdot C_{i+1,j} \tag{4.4}$$

where we have set

$$\mathcal{F}_k := \sigma\left\{C_{i,j},\ i+j \leq k+1\right\} \quad \textit{for } 1 \leq k \leq I+J-1 \quad \textit{and } \mathcal{F}_0 := \sigma\{\emptyset, \Omega\}. \tag{4.5}$$

(ii) At the accident year boundary $(i = 0)$ *we replace condition* (4.4) *by*

$$\mathbb{E}\left[C_{1,j+1}\middle|\,\mathcal{F}_j\right] = \alpha_0 \cdot \mathbb{E}\left[C_{0,j+1}\middle|\,\mathcal{F}_{j-1}\right] + \beta_j \cdot C_{1,j} \tag{4.6}$$

for $1 \leq j \leq J-1$ *and we assume a von-Neumann kind stationary boundary condition for the unknown expectation on the right-hand side of* (4.6)*:*

$$\mathbb{E}\left[C_{1,j+1}\right] = \mathbb{E}\left[C_{0,j+1}\right]. \tag{4.7}$$

The coefficients $(\alpha, \beta) := (\alpha_0, \ldots, \alpha_{I-1}, \beta_1, \ldots, \beta_{J-1})$ *are called **bivariate development factors**, short **BDFs**.*

Remark 4.2.

(i) The case of $\alpha_i = 0$ for all $0 \leq i < I$ is exactly Mack's model (see [18]) for the Chain-Ladder estimations. The estimated Chain-Ladder development factors $\widetilde{\beta}_j^{CL}$ are defined via

$$\sum_{m=1}^{I-j} C_{m,j+1} = \widetilde{\beta}_j^{CL} \cdot \sum_{m=1}^{I-j} C_{m,j}. \tag{4.8}$$

(ii) The first example of section 3 of this chapter shows that the boundary condition (4.7) is important to get reasonable results.

1.2 Estimation of Bivariate Development Factors

We sum up equations (4.4) over i, resp. j within the region of the run-off triangle and take expectations. Together with $C_{i,0} = 0$ for all i, this leads to the following system of equations for $0 \leq i \leq I-1$ and $1 \leq j \leq J-1$:

$$\begin{aligned} \mathbb{E}\left[\sum_{n=1}^{J\wedge(I-i)} C_{i+1,n}\right] &= \alpha_i \cdot \sum_{n=1}^{J\wedge(I-i)} \mathbb{E}\left[C_{i,n}\right] + \sum_{n=2}^{J\wedge(I-i)} \beta_{n-1} \cdot \mathbb{E}\left[C_{i+1,n-1}\right] \\ \mathbb{E}\left[\sum_{m=1}^{I-j} C_{m,j+1}\right] &= \sum_{m=1}^{I-j} \alpha_{m-1} \cdot \mathbb{E}\left[C_{m-1,j+1}\right] + \beta_j \cdot \sum_{m=1}^{I-j} \mathbb{E}\left[C_{m,j}\right]. \end{aligned} \tag{4.9}$$

We use (4.9) to derive the following system of linear equations for the estimations of the dependent development factors (α, β). Here, the boundary condition (4.7) is naturally transformed into $C_{0,n} := C_{1,n}$ for $1 \leq n \leq J$.

$$\begin{aligned} &\text{for } i = 0, \ldots, I-J \\ &\qquad \sum_{n=1}^{J} C_{i+1,n} = \widetilde{\alpha}_i \cdot \sum_{n=1}^{J} C_{i,n} + \sum_{n=2}^{J} \widetilde{\beta}_{n-1} \cdot C_{i+1,n-1}, \\ &\text{for } i = I-J+1, \ldots, I-1 \\ &\qquad \sum_{n=1}^{I-i} C_{i+1,n} = \widetilde{\alpha}_i \cdot \sum_{n=1}^{I-i} C_{i,n} + \sum_{n=2}^{I-i} \widetilde{\beta}_{n-1} \cdot C_{i+1,n-1}, \\ &\text{and for } j = 1, \ldots, J-1 \\ &\qquad \sum_{m=1}^{I-j} C_{m,j+1} = \sum_{m=1}^{I-j} \widetilde{\alpha}_{m-1} \cdot C_{m-1,j+1} + \widetilde{\beta}_j \cdot \sum_{m=1}^{I-j} C_{m,j}. \end{aligned} \tag{4.10}$$

This defines the implicit **BDF-estimators** $(\widetilde{\alpha}, \widetilde{\beta})^{tr} := (\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{I-1}, \widetilde{\beta}_1 \ldots, \widetilde{\beta}_{J-1})^{tr}$.

To rewrite (4.10) in matrix form, we introduce the stochastic matrix Z by

$$Z := \begin{pmatrix} Z_{1,1} & Z_{1,2} \\ Z_{2,1} & Z_{2,2} \end{pmatrix} \tag{4.11}$$

with the sub-matrices

$$Z_{1,1} := \begin{pmatrix} \sum_{n=1}^{J} C_{0,n} & 0 & 0 & \cdots & & & 0 \\ 0 & \sum_{n=1}^{J} C_{1,n} & 0 & \cdots & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & & \sum_{n=1}^{J} C_{I-J,n} & \cdots & & \vdots \\ 0 & \cdots & & & \sum_{n=1}^{J-1} C_{I-J+1,n} & \cdots & \\ & & & & & \ddots & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & C_{I-1,1} \end{pmatrix},$$

$$Z_{1,2} := \begin{pmatrix} C_{1,1} & \cdots & & & C_{1,J-1} \\ C_{2,1} & \cdots & & & C_{2,J-1} \\ \vdots & & & & \vdots \\ C_{I-J+1,1} & \cdots & & & C_{I-J+1,J-1} \\ C_{I-J+2,1} & \cdots & & C_{I-J+2,J-2} & 0 \\ \vdots & & & & \vdots \\ C_{I-1,1} & 0 & \cdots & & 0 \\ 0 & \cdots & & & 0 \end{pmatrix},$$

$$Z_{2,1} := \begin{pmatrix} C_{0,2} & C_{1,2} & \cdots & C_{I-J,2} & \cdots & C_{I-2,2} & 0 \\ C_{0,3} & C_{1,3} & \cdots & C_{I-J,3} & \cdots & 0 & 0 \\ \vdots & & & & \vdots & & \\ C_{0,J-1} & C_{1,J-1} & \cdots & C_{I-J,J-1} & C_{I-J+1,J-1} & \cdots & 0 \\ C_{0,J} & C_{1,J} & \cdots & C_{I-J,J} & 0 & \cdots & 0 \end{pmatrix},$$

and

$$Z_{2,2} := \begin{pmatrix} \sum_{m=1}^{I-1} C_{m,1} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \sum_{m=1}^{I-J+1} C_{m,J-1} \end{pmatrix}. \tag{4.12}$$

Further we set

$$X := \left(\sum_{n=1}^{J} C_{1,n}, \sum_{n=1}^{J} C_{2,n}, \ldots, \sum_{n=1}^{J} C_{I-J+1,n}, \sum_{n=1}^{J-1} C_{I-J+2,n}, \ldots, C_{I,1} \right)^{tr}$$

$$Y := \left(\sum_{m=1}^{I-1} C_{m,2}, \sum_{m=1}^{I-2} C_{m,3}, \ldots, \sum_{m=1}^{I-J+1} C_{m,J} \right)^{tr}, \tag{4.13}$$

so that the linear equation system (4.10) can be rewritten as

$$Z \cdot \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{4.14}$$

An application of the general Hadamard's invertibility criterion shows that the system (4.14) has always a unique solution:

Theorem 4.3.
Let's assume that $C_{i,j} > 0$ a.s. Then the stochastic matrix Z is a.s. invertible.

Proof.
Let $\widetilde{Z} = (z_{\mu,\nu})_{1\leq\mu,\nu\leq I+J-1,\mu\neq I,\nu\neq I}$ be the matrix obtained from Z by suppressing the I^{th} column and the I^{th} row. Since all suppressed values are equal zero except for the diagonal element $z_{I,I} = C_{I-1,1} > 0$, the matrix Z is invertible if and only if $\widetilde{Z}$ is invertible.

By simple inspection the stochastic matrix $\widetilde{Z}$ clearly satisfies a.s. condition (ii) of definition D.11. It remains only to show the irreducibility of $\widetilde{Z}$. We remark that if $1 \leq \mu \leq I-1$ then

$$\begin{aligned} &z_{\mu,I+1}\, z_{I+1,\nu} > 0 && \text{for } 1 \leq \nu \leq I-1 \\ &z_{\mu,I+J-\mu}\, z_{I+J-\mu,1}\, z_{1,\nu} > 0 && \text{for } I+1 \leq \nu \leq I+J-1, \end{aligned}$$

and if $I+1 \leq \mu \leq I+J-1$ then

$$\begin{aligned} &z_{\mu,I+J-\mu}\, z_{I+J-\mu,I+1}\, z_{I+1,\nu} > 0 && \text{for } 1 \leq \nu \leq I-1 \\ &z_{\mu,1}\, z_{1,\nu} > 0 && \text{for } I+1 \leq \nu \leq I+J-1. \end{aligned}$$

This shows that all elements of $\widetilde{Z}^3$ are strictly positive. Hence, $\widetilde{Z}$ is irreducible and the proof is complete. □

Though the stochastic matrix Z is a.s. invertible, we can not expect the BDF-estimator $(\widetilde{\alpha}, \widetilde{\beta})$ to be unbiased in an explicit form, since the inverse matrix Z^{-1} is strongly non-linear. However, we have the following result of implicit unbiasedness:

Theorem 4.4.
Under the condition of theorem 4.3 the BDF-estimator $(\widetilde{\alpha}, \widetilde{\beta})^{tr}$ satisfies the following property of implicit unbiasedness:

$$\mathbb{E}\left[Z \cdot \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{pmatrix}\right] = \mathbb{E}\left[Z\right] \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{4.15}$$

Proof.

By (4.14), (4.9), (4.7), and (4.4), we get

$$\mathbb{E}\left[Z\cdot\begin{pmatrix}\widetilde{\alpha}\\ \widetilde{\beta}\end{pmatrix}\right] = \mathbb{E}\left[\begin{pmatrix}X\\ Y\end{pmatrix}\right] = \begin{pmatrix}\alpha_0\sum_{n=1}^{J}\mathbb{E}[C_{0,n}]+\sum_{m=2}^{J}\beta_{n-1}\mathbb{E}[C_{1,n-1}]\\ \vdots\\ \alpha_{I-1}\mathbb{E}[C_{I-1,1}]\\ \sum_{m=1}^{I-1}\alpha_{m-1}\mathbb{E}[C_{m-1,2}]+\beta_1\sum_{m=1}^{I-1}\mathbb{E}[C_{m,1}]\\ \vdots\\ \sum_{m=1}^{I-J+1}\alpha_{m-1}\mathbb{E}[C_{m-1,J}]+\beta_{J-1}\sum_{m=1}^{I-J+1}\mathbb{E}[C_{m,J-1}]\end{pmatrix}$$

$$= \mathbb{E}\left[Z\right]\cdot\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

□

If we replace in the definitions (4.10) to (4.13) of X, Y, and Z, the random variables $C_{i,j}$ by their observations $\widehat{C}_{i,j}$ from (4.2), we get the observed items $\widehat{X}, \widehat{Y}$, and $\widehat{Z}$. The *estimations* $(\widehat{\alpha},\widehat{\beta})^{tr} := (\widehat{\alpha}_0,\ldots,\widehat{\alpha}_{I-1},\widehat{\beta}_1\ldots,\widehat{\beta}_{J-1})^{tr}$ are then defined by the linear system

$$\widehat{Z}\cdot\begin{pmatrix}\widehat{\alpha}\\ \widehat{\beta}\end{pmatrix} = \begin{pmatrix}\widehat{X}\\ \widehat{Y}\end{pmatrix}. \tag{4.16}$$

1.3 Estimators For Future Claims Amounts

For the claims amount of future years we find the following mean result:

Proposition 4.5.
Under the assumptions 4.1 the conditional mean of the claim amount of future years are given as follows:
For all $I-J+1<i\leq I$, $1<j\leq J$ and $i\vee j\leq k\leq i+j-1$ we have

$$\mathbb{E}\left[C_{i,j}\middle|\mathcal{F}_k\right] = \sum_{\ell=k+1-j}^{i} C_{\ell,k+1-\ell}\binom{i+j-k-1}{\ell+j-k-1}\prod_{\mu=\ell}^{i-1}\alpha_\mu\prod_{\nu=k+1-\ell}^{j-1}\beta_\nu. \tag{4.17}$$

Proof.
We shall prove the equation (4.17) by (4.4) and the induction on the difference between $i+j-1$

and k. If $k = i + j - 1$, then $C_{i,j}$ is $\mathcal{F}_k$-measurable and the left-hand side of (4.17) is equal to $C_{i,j}$. The summation on the right-hand side has only $l = i$ as index and its result turns out to be again $C_{i,j}$.
Now let $1 < i \vee j \leq k < i + j - 1$. By induction we get

$$
\begin{aligned}
&\mathbb{E}[C_{i,j}|\,\mathcal{F}_k] = \mathbb{E}[\mathbb{E}[C_{i,j}|\mathcal{F}_{i+j-2}]|\,\mathcal{F}_k] = \alpha_{i-1}\mathbb{E}[C_{i-1,j}|\,\mathcal{F}_k] + \beta_{j-1}\mathbb{E}[C_{i,j-1}|\,\mathcal{F}_k] \\
&= C_{k+1-j,j}\binom{i+j-k-2}{0}\prod_{\mu=k+1-j}^{i-1}\alpha_\mu\prod_{\nu=j}^{j-1}\beta_\nu \\
&\quad + \sum_{\ell=k+2-j}^{i-1} C_{\ell,k+1-\ell}\left[\binom{i+j-k-2}{\ell+j-k-1} + \binom{i+j-k-2}{\ell+j-k-2}\right]\prod_{\mu=\ell}^{i-1}\alpha_\mu\prod_{\nu=k+1-\ell}^{j-1}\beta_\nu \\
&\quad + C_{i,k+1-i}\binom{i+j-k-2}{i+j-k-2}\prod_{\mu=i}^{i-1}\alpha_\mu\prod_{\nu=k+1-i}^{j-1}\beta_\nu \\
&= \sum_{\ell=k+1-j}^{i} C_{\ell,k+1-\ell}\binom{i+j-k-1}{\ell+j-k-1}\prod_{\mu=\ell}^{i-1}\alpha_\mu\prod_{\nu=k+1-\ell}^{j-1}\beta_\nu
\end{aligned}
$$

where we used the equality $\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}$ for $0 < m \leq n$ from Pascal's triangle. □

Let's assume for the moment that the dependent development factors (α, β) are known. Then for future accident and development years (i, j), i.e. $i + j > I + 1$, proposition 4.5 leads with $k = I$ to the estimator $\widetilde{C}_{i,j}(\alpha, \beta)$ by

$$
\widetilde{C}_{i,j}(\alpha,\beta) := \sum_{\ell=I+1-j}^{i} C_{\ell,I+1-\ell}\binom{i+j-I-1}{\ell+j-I-1}\prod_{\mu=\ell}^{i-1}\alpha_\mu\prod_{\nu=I+1-\ell}^{j-1}\beta_\nu. \tag{4.18}
$$

Proposition 4.6.
If the dependent development factors (α, β) are known, then the estimator $\widetilde{C}_{i,j}(\alpha, \beta)$ for future years $i + j > I + 1$ is conditionally unbiased with respect to $\mathcal{F}_I$.

Proof.
Since the proof of proposition 4.5 needs only assumptions 4.1, the proposition follows from formula (4.18). □

If we combine theorem 4.4 and proposition 4.6 we find the following natural estimator $\widetilde{C}_{i,j}(\widetilde{\alpha}, \widetilde{\beta})$ for the claim amounts in future years $(i + j > I + 1, i \leq I, j \leq J)$:

$$
\widetilde{C}_{i,j}(\widetilde{\alpha},\widetilde{\beta}) := \sum_{\ell=I+1-j}^{i} C_{\ell,I+1-\ell}\binom{i+j-I-1}{\ell+j-I-1}\prod_{\mu=\ell}^{i-1}\widetilde{\alpha}_\mu\prod_{\nu=I+1-\ell}^{j-1}\widetilde{\beta}_\nu \tag{4.19}
$$

Of course, we can no longer expect that the composed estimator $\widetilde{C}_{i,j}(\widetilde{\alpha},\widetilde{\beta})$ is unbiased.

1.4 Estimation of Provisions

For the yearly claim provisions of the accident years $i = I-J+2,\dots,I$ we get

$$\begin{aligned}\widetilde{P}_i(\widetilde{\alpha},\widetilde{\beta}) \;&:=\; \sum_{\ell=I-J+1}^{i-1} C_{\ell,I+1-\ell}\binom{i-(I-J)-1}{\ell-(I-J)-1}\prod_{\mu=\ell}^{i-1}\widetilde{\alpha}_\mu \prod_{\nu=I+1-\ell}^{J-1}\widetilde{\beta}_\nu \\ &\quad + C_{i,I+1-i}\left(\prod_{\nu=I+1-i}^{J-1}\widetilde{\beta}_\nu \;-1\right)\end{aligned} \tag{4.20}$$

and their sum gives for the total provision $\widetilde{P}(\widetilde{\alpha},\widetilde{\beta}) := \sum_{i=I-J+2}^{I}\widetilde{P}_i(\widetilde{\alpha},\widetilde{\beta})$.

Again, if we replace the variables $C_{i,j}$ by their observations $\widehat{C}_{i,j}$ and the estimators $(\widetilde{\alpha},\widetilde{\beta})$ by the real estimations $(\widehat{\alpha},\widehat{\beta})$ we get in (4.19) to (4.20) the estimations $\widehat{C}_{i,j}$ for future years $(i+j > I+1, i \le I, j \le J)$ and the estimations $\widehat{P}_i(\widehat{\alpha},\widehat{\beta})$ of the provisions for year $i \ge I-J+2$, as well as of the total provision $\widehat{P}(\widehat{\alpha},\widehat{\beta})$.

2 Embedding The Univariate Chain-Ladder Into The Bivariate One

In [18] Th. Mack presented a model for IBNR-calculations which gave a statistical background for the well known Chain-Ladder method which is based on the standard estimators for the Chain-Ladder development factors $\widetilde{\beta}^{(0)} := \left(\widetilde{\beta}_j^{(0)}\right)_{1\le j\le J-1} = \widetilde{\beta}^{CL}$ given by (4.8). The two main assumptions of Mack's model are:

Assumption 4.7.

(M1) The sequence $(C_{i,j}, 1 \le j \le J)_{1\le i\le I}$ of families of the accident year variables are mutually independent.

(M2) Among the accident year variables $(C_{i,j})_{1\le j\le J}$ the following Markov-property holds for $j = 1,\dots,J-1$:

$$\mathbb{E}\left[C_{i,j+1}\middle|C_{i,1},\dots,C_{i,j}\right] = \beta_j^{(0)}\cdot C_{i,j} \tag{4.21}$$

with stationary ***development factors*** $\beta^{(0)} := \left(\beta_j^{(0)}\right)_{1\le j\le J-1}$.

It is shown that under Assumptions 4.7 the Chain-Ladder estimators (4.8) are unbiased and uncorrelated estimators for $\beta^{(0)}$.

In this context, we want to give our estimators of the dependent development factors a more general version which include as well the bivariate development factors $(\widetilde{\alpha}, \widetilde{\beta})$ from (4.14) and $\widetilde{\beta}^{(0)}$ from (4.21). For this purpose we set for $\tau \in [0,1]$

$$Z^{(\tau)} := \begin{pmatrix} Z_{1,1} & \tau \cdot Z_{1,2} \\ \tau \cdot Z_{2,1} & Z_{2,2} \end{pmatrix}. \tag{4.22}$$

The extended depended development factors $(\widetilde{\alpha}^{(\tau)}, \widetilde{\beta}^{(\tau)})^{tr} := (\widetilde{\alpha}_0^{(\tau)}, \ldots, \widetilde{\alpha}_{I-1}^{(\tau)}, \widetilde{\beta}_1^{(\tau)}, \ldots, \widetilde{\beta}_{J-1}^{(\tau)})^{tr}$ are now defined by

$$Z^{(\tau)} \cdot \begin{pmatrix} \widetilde{\alpha}^{(\tau)} \\ \widetilde{\beta}^{(\tau)} \end{pmatrix} = \begin{pmatrix} \tau \cdot X \\ Y \end{pmatrix} + \tau(1-\tau) \cdot \begin{pmatrix} Z_{1,2} \cdot \mathbb{1}_{J-1} \\ -Z_{2,1} \cdot \mathbb{1}_I \end{pmatrix}. \tag{4.23}$$

Here, $\mathbb{1}_I := (1, \ldots, 1) \in \mathbb{R}^I$ is the counting vector in $\mathbb{R}^I$ whose components are all 1; similar for $\mathbb{1}_{J-1}$. Notice that we introduced the last, second order term in (4.23) which vanishes for $\tau = 1$ and $\tau = 0$, in order to smooth the transition from $\tau = 1$ to $\tau = 0$ which otherwise could reveal great excursions.

As in the proof of theorem 4.3, Hadamard's criterion applies to the matrix $Z^{(\tau)}$ for all $\tau \in [0,1]$ such that $Z^{(\tau)}$ is a.s. invertible and the implicit estimators $(\widetilde{\alpha}^{(\tau)}, \widetilde{\beta}^{(\tau)})$ have a unique solution.

Obviously, for $\tau = 1$ the matrix Z from (4.11) is equal to $Z^{(1)}$, thus the equality (4.23) is identical to (4.14) and the estimator for the dependent development factors $(\widetilde{\alpha}, \widetilde{\beta})$ is equal to $(\widetilde{\alpha}^{(1)}, \widetilde{\beta}^{(1)})$.

On the other hand, for $\tau = 0$, the matrix $Z^{(0)}$ is diagonal and the estimator $(\widetilde{\alpha}^{(0)}, \widetilde{\beta}^{(0)})$ has an explicit solution with $\widetilde{\alpha}^{(0)} = 0 \in \mathbb{R}^I$ and $\widetilde{\beta}^{(0)} = \widetilde{\beta}^{CL}$ the Chain-Ladder estimators from (4.8). Of course, if we apply the Chain-Ladder estimators $(\widetilde{\alpha}^{(0)}, \widetilde{\beta}^{(0)})$ in the estimators (4.19) and (4.20) we get the standard Chain-Ladder estimator for future claim amounts $\widetilde{C}_{i,j}$, for the accident claim provisions $\widetilde{P}_i$ and the total reserve $\widetilde{P}$ (see [18]). In example 3.2 below, we show how the total provision $\widetilde{P}^{(\tau)}$ develops during the transition from Mack's model ($\tau = 0$) to the dependent model ($\tau = 1$).

Next, we want to establish a first-order perturbation of the univariate Chain-Ladder estimators ($\widetilde{\alpha}^{(0)} = 0, \widetilde{\beta}^{(0)} = \widetilde{\beta}^{CL}$). For this purpose we write (4.23) for $\tau > 0$ as

$$\left(\begin{pmatrix} Z_{1,1} & 0 \\ 0 & Z_{2,2} \end{pmatrix} + \tau \begin{pmatrix} 0 & Z_{1,2} \\ Z_{2,1} & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \widetilde{\alpha}^{(\tau)} \\ \widetilde{\beta}^{(\tau)} \end{pmatrix} = \begin{pmatrix} 0 \\ Y \end{pmatrix} + \tau \begin{pmatrix} X + (1-\tau) Z_{1,2} \cdot \mathbb{1}_{J-1} \\ -(1-\tau) Z_{2,1} \cdot \mathbb{1}_I \end{pmatrix}. \tag{4.24}$$

The first-order perturbation term $(\widetilde{\alpha}^{(0)\prime}, \widetilde{\beta}^{(0)\prime})$ at $\tau = 0$ of $(\widetilde{\alpha}^{(\tau)}, \widetilde{\beta}^{(\tau)})$ is characterized via

$$\begin{pmatrix} \widetilde{\alpha}^{(\tau)} \\ \widetilde{\beta}^{(\tau)} \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\beta}^{(0)} \end{pmatrix} + \tau \begin{pmatrix} \widetilde{\alpha}^{(0)\prime} \\ \widetilde{\beta}^{(0)\prime} \end{pmatrix} + \mathfrak{o}(\tau), \tag{4.25}$$

where $\mathfrak{o}(\tau)$ is the Landau symbol representing any vector function of τ with $\mathfrak{o}(\tau)/\tau \to 0$ for $\tau \searrow 0$. It follows from (4.24) and (4.25) that

$$\begin{pmatrix} Z_{1,1} \cdot \widetilde{\alpha}^{(0)\prime} \\ Z_{2,2} \cdot \widetilde{\beta}^{(0)\prime} \end{pmatrix} = \begin{pmatrix} X + Z_{1,2} \cdot (\mathbb{1}_{J-1} - \widetilde{\beta}^{(0)}) \\ -Z_{2,1} \cdot \mathbb{1}_I \end{pmatrix}. \tag{4.26}$$

Thereof we get the explicit solution, first for $\widetilde{\beta}^{(0)\prime}$, then for $\widetilde{\alpha}^{(0)\prime}$:

$$\widetilde{\beta}^{(0)\prime} = \begin{pmatrix} \widetilde{\beta}_1^{(0)\prime} \\ \vdots \\ \widetilde{\beta}_{J-1}^{(0)\prime} \end{pmatrix} = \begin{pmatrix} -\sum_{m=0}^{I-2} C_{m,2} \Big/ \sum_{m=1}^{I-1} C_{m,1} \\ \vdots \\ -\sum_{m=0}^{I-J} C_{m,J} \Big/ \sum_{m=1}^{I-J+1} C_{m,J-1} \end{pmatrix}. \tag{4.27}$$

$$\widetilde{\alpha}^{(0)\prime} = \begin{pmatrix} \widetilde{\alpha}_0^{(0)\prime} \\ \vdots \\ \widetilde{\alpha}_{I-J}^{(0)\prime} \\ \widetilde{\alpha}_{I-J+1}^{(0)\prime} \\ \vdots \\ \widetilde{\alpha}_{I-2}^{(0)\prime} \\ \widetilde{\alpha}_{I-1}^{(0)\prime} \end{pmatrix} = \begin{pmatrix} \left(\sum_{n=1}^{J-1} C_{1,n}(2 - \widetilde{\beta}_n^{(0)}) + C_{1,J} \right) \Big/ \sum_{n=1}^{J} C_{0,n} \\ \vdots \\ \left(\sum_{n=1}^{J-1} C_{I-J+1,n}(2 - \widetilde{\beta}_n^{(0)}) + C_{I-J+1,J} \right) \Big/ \sum_{n=1}^{J} C_{I-J,n} \\ \left(\sum_{n=1}^{J-2} C_{I-J+2,n}(2 - \widetilde{\beta}_n^{(0)}) + C_{I-J+2,J-1} \right) \Big/ \sum_{n=1}^{J-1} C_{I-J+1,n} \\ \vdots \\ \left(C_{I-1,1}(2 - \widetilde{\beta}_1^{(0)}) + C_{I-1,2} \right) \Big/ \sum_{n=1}^{2} C_{I-2,n} \\ C_{I,1} \Big/ C_{I-1,1} \end{pmatrix} \tag{4.28}$$

Even if one uses the standard Chain-Ladder estimators for future claims amounts or for claims provision, it may be interesting to take a look at the first-order correction terms. Since $\widetilde{\alpha}^{(0)} = 0$, the estimator for the claim amount has the following first-order correction term if $i+j > I+1$:

$$\begin{aligned}\widetilde{C}_{i,j}^{(0)'} &= (i+j-I-1)\cdot C_{i-1,I+2-i}\cdot\widetilde{\alpha}_{i-1}^{(0)'}\cdot\prod_{\nu=I+2-i}^{j-1}\widetilde{\beta}_{\nu}^{(0)}\\ &\quad + C_{i,I+1-i}\cdot\sum_{\nu=I+1-i}^{j-1}\left(\prod_{\rho=I+1-i,\rho\neq\nu}^{j-1}\widetilde{\beta}_{\rho}^{(0)}\right)\widetilde{\beta}_{\nu}^{(0)'} \end{aligned}\tag{4.29}$$

For the provision of the accident year $i > I-J+1$ the correction term is

$$\begin{aligned}\widetilde{P}_i^{(0)'} = &\prod_{\nu=I+1-i}^{J-1}\widetilde{\beta}_{\nu}^{(0)}\cdot\\ &\left[(i-(I-J)-1)\cdot C_{i-1,I+2-i}\cdot\ \widetilde{\alpha}_{i-1}^{(0)'}\Big/\widetilde{\beta}_{I+1-i}^{(0)} + C_{i,I+1-i}\cdot\sum_{\nu=I+1-i}^{J-1}\widetilde{\beta}_{\nu}^{(0)'}\Big/\widetilde{\beta}_{\nu}^{(0)}\right]\end{aligned}\tag{4.30}$$

and the correction term of the total provision is $\widetilde{P}^{(0)'} = \sum_{i=I-J+2}^{I}\widetilde{P}_i^{(0)'}$.

3 Examples

In this section we present a number of examples and applications of the model with dependent development factors. The first example shows the importance of a good choice of the boundary condition for the first accident year.

Example 3.1.
We regard a run-off triangle with $I = 4$, where the cumulative values are linearly decreasing in accident years but constant in development years. Obviously, the classical Chain-Ladder development factors are all equal to one and the provision is zero. If we take $C_{0,\cdot} = (0, 0.5, 1.5, 2.0)$ as boundary condition we get a provision of $P = 2.074$

i	j 1	2	3	4	$\widetilde{\alpha}_i^{(1)}$	BDF model $\widetilde{\mathbf{P}}_i^{(1)}$	Chain Ladder $\widetilde{\mathbf{P}}_i^{(0)}$
0	0,0	0,5	1,5	2,0	12,600		
1	4,0	4,0	4,0	4,0	1,575	0,0	0,0
2	3,0	3,0	3,0		0,900	-12,600	0,0
3	2,0	2,0			0,500	2,610	0,0
4	1,0					12,064	0,0
$\widetilde{\beta}_j^{(1)}$	-0,7	-2,6	-5,3				
$\widetilde{\beta}_j^{CL}$	1,0	1,0	1,0		$\sum \widetilde{P}_i$	2,074	0,0

Table 4.1 – Run-off triangle of cumulative claims $C_{i,j}$ for $I = J = 4$

However, if we apply the boundary condition (4.7) leading to the system (4.14) we find:

i	j 1	2	3	4	$\widetilde{\alpha}_i^{(1)}$	BDF model $\widetilde{\mathbf{P}}_i^{(1)}$	Chain Ladder $\widetilde{\mathbf{P}}_i^{(0)}$
0	4,0	4,0	4,0	4,0	1,000		
1	4,0	4,0	4,0	4,0	0,750	0,0	0,0
2	3,0	3,0	3,0		0,667	0,0	0,0
3	2,0	2,0			0,500	0,0	0,0
4	1,0					0,0	0,0
$\widetilde{\beta}_j^{(1)}$	0,0	0,0	0,0				
$\widetilde{\beta}_j^{CL}$	1,0	1,0	1,0		$\sum \widetilde{P}_i$	0,0	0,0

Table 4.2 – Run-off triangle of cumulative claims $C_{i,j}$ with von-Neumann boundary condition

Example 3.2.

In this example we compare our model with dependent bivariate development factors to the following example of M. Wüthrich given in [38].

	1	2	3	4	5	6	7	8	9	10	11
1	1 196 242	1 838 489	2 144 331	2 255 181	2 355 208	2 539 765	2 761 554	2 922 055	3 045 575	3 184 027	3 291 088
2	1 225 928	1 818 969	2 066 089	2 337 709	2 542 414	2 711 028	2 980 627	3 161 874	3 424 628	3 549 130	3 786 885
3	1 233 323	1 846 154	2 073 088	2 278 136	2 498 793	2 585 022	2 784 254	2 967 520	3 376 057	3 416 823	3 647 386
4	1 267 049	1 923 375	2 169 457	2 330 124	2 593 996	2 839 750	3 241 585	3 679 853	3 795 873	4 117 558	4 126 216
5	1 339 749	2 082 500	2 439 217	2 658 199	2 928 930	3 237 039	3 449 952	3 624 018	3 719 605	3 820 989	
6	1 471 940	2 171 472	2 470 916	2 834 330	3 230 480	3 626 482	4 231 529	4 940 961	5 198 128		
7	1 328 544	1 993 900	2 280 334	2 498 017	2 759 961	3 058 012	3 644 176	3 930 538			
8	1 246 696	1 695 630	1 867 971	2 253 053	2 429 396	2 695 728	2 963 261				
9	1 105 615	1 650 577	2 033 279	2 395 948	2 803 067	3 157 740					
10	1 012 347	1 541 137	1 747 709	2 011 153	2 536 544						
11	1 005 938	1 553 794	1 822 230	2 087 166							
12	1 025 464	1 507 839	1 828 583								
13	1 047 903	1 674 176									
14	1 097 661										

Table 4.3 – Data of the run-off 'triangle' from [38]

We find the development factors of the Chain-Ladder model, resp. of the bivariate model as follows:

Chain-Ladder	bivariat Chain-Ladder	
$\tilde{\beta}^{CL}{}_{i}$	$\tilde{\alpha}_i$	$\tilde{\beta}_j$
	0,9690	
1,5024	1,0417	0,0033
1,1535	0,9398	0,0033
1,1222	1,0823	0,0120
1,1185	1,0176	0,0318
1,0956	1,1487	0,0370
1,1187	0,8405	0,0338
1,0924	0,8465	0,0480
1,0593	1,0611	0,0671
1,0419	0,8765	0,0267
1,0409	1,0200	0,0510
	0,9935	
	1,0731	
	1,0475	

Table 4.4 – The development factors of the Chain-Ladder and the bivariate model

This gives the estimations of the payments in future years:

	1	2	3	4	5	6	7	8	9	10	11
5											4 393 515
6										4 527 763	5 277 711
7									4 632 925	3 929 349	4 636 562
8								3 469 282	4 154 427	3 436 936	4 100 076
9							3 250 820	3 837 011	4 665 444	3 771 186	4 542 725
10						2 861 732	2 946 129	3 504 611	4 324 542	3 420 931	4 156 358
11					2 653 712	3 017 166	3 106 974	3 723 756	4 660 853	3 613 674	4 423 818
12				2 095 594	2 703 119	3 097 525	3 191 358	3 852 583	4 888 933	3 720 525	4 584 764
13			1 967 910	2 272 557	2 973 134	3 434 064	3 540 745	4 304 182	5 535 218	4 140 250	5 131 237
14		1 757 331	2 067 216	2 405 354	3 190 847	3 715 168	3 834 340	4 692 467	6 112 808	4 499 856	5 604 376

Table 4.5 – The estimations for future years

For the provision the different models shows the following results:

provisions	Mack's Chain Ladder	Wüthrich's Bayesian model	BDF model
total provision	12 411 560	12 374 052	18 556 355

Table 4.6 – Comparison of the total provision for different models

The following diagram shows the behaviour of the total provision $\widetilde{P}^{(\tau)}$ during the transition between Mack's model ($\tau = 0$) and ours ($\tau = 1$).

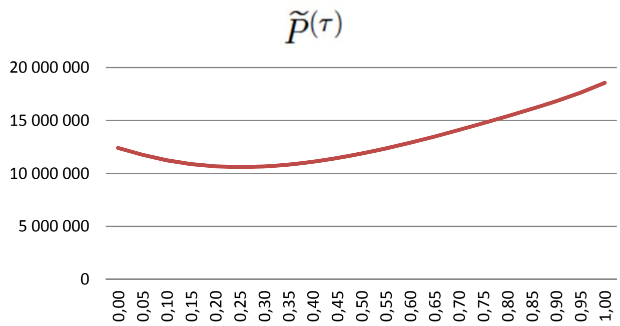

Figure 4.1 – Transition of the total provision between Mack's model in the bivariate one

For the first-order correction terms $\left(\widetilde{\alpha}^{(0)\prime}, \widetilde{\beta}^{(0)\prime}\right)$ of $\left(\widetilde{\alpha}^{(\tau)}, \widetilde{\beta}^{(\tau)}\right)$ and for the correction terms $\widetilde{P}^{(0)\prime}$ at $\tau = 0$, we get:

$\tilde{\alpha}'_i$	$\tilde{\beta}'_j$	$\tilde{P}'_i$
0,9046		
0,9757	-1,5130	
0,8789	-1,1681	
1,0165	-1,1295	
0,9421	-1,1109	
1,0613	-1,0700	133 693
0,7599	-1,1100	-2 316 538
0,7526	-1,0487	149 166
0,9465	-0,9583	-438 167
0,7661	-1,0052	-3 525 736
0,8717	-0,9824	-2 026 580
0,8250		-378 135
0,8667		-3 631 222
1,0475		-5 102 817
		3 203 302
	Sum	-13 933 034

Table 4.7 – First-order correction terms of the development factors and the provision

4 Bootstrap for the BDF Model

New systems of regulation and supervision of insurances, like Solvency II and the Swiss Solvency Test (SST), uses not only a sound estimation of the 'mean value' (or more precisely the market-consistent best estimate) of liabilities, but in addition they require the applications of operators of their whole distribution, like the value at risk (VaR) or the conditional tail expectation (Tail-VaR) (see e.g. [10] for details). In the case of outstanding liabilities, the problem to estimate their distribution is a very demanding task.

One possible solution to this problem — at least approximately — is the application of bootstrapping to the IBNR-data as developed in [7], [8], and [9]. We adapt these methods to the bivariate Chain-Ladder scheme.

The first step is to establish how the mean values $\mathbb{E}[C_{i,j}]$ depends on the left boundary values $(\mathbb{E}[C_{m,1}])_{1\leq m\leq I}$ and the upper boundary values $(\mathbb{E}[C_{0,n}])_{2\leq n\leq J}$. For $i < I$, the answer is given in the following proposition.

Proposition 4.8. *For* $1 \leq i \leq I-1,\ 2 \leq j \leq J$ *we have*

$$\begin{aligned}\mathbb{E}\left[C_{i,j}\right] &= \prod_{\mu=0}^{i-1}\alpha_\mu \sum_{n=2}^{j}\mathbb{E}\left[C_{0,n}\right]\binom{i+j-n-1}{i-1}\prod_{\nu=n}^{j-1}\beta_\nu \\ &\quad+\prod_{\nu=1}^{j-1}\beta_\nu \sum_{m=1}^{i}\mathbb{E}\left[C_{m,1}\right]\binom{i+j-m-2}{j-2}\prod_{\mu=m}^{i-1}\alpha_\mu. \end{aligned} \tag{4.31}$$

Proof. We proceed by induction on $\ell = i+j$. For $\ell = 3$, i.e. $i = 1$ and $j = 2$, we have by (4.6)

$$\mathbb{E}\left[C_{1,2}\right] = \mathbb{E}\left[C_{0,2}\right]\alpha_0 + \mathbb{E}\left[C_{1,1}\right]\beta_1.$$

which is exactly (4.31). Recalling the convention that $\sum_{\mu=\kappa}^{\ell}\cdots = 0$ whenever $\kappa > \ell$, we get for

$\ell > 3$, again by (4.4), resp. (4.6), and the induction hypothesis

$$\begin{aligned}
\mathbb{E}\left[C_{i,j}\right] &= \mathbb{E}\left[C_{i-1,j}\right]\alpha_{i-1} + \mathbb{E}\left[C_{i,j-1}\right]\beta_{j-1} \\
&= \sum_{n=2}^{j} \mathbb{E}\left[C_{0,n}\right] \binom{i-1+j-n-1}{i-2} \left(\prod_{\mu=0}^{i-2} \alpha_\mu\right) \alpha_{i-1} \prod_{\nu=n}^{j-1} \beta_\nu \\
&\quad + \sum_{m=1}^{i-1} \mathbb{E}\left[C_{m,1}\right] \binom{i-1+j-m-2}{j-2} \left(\prod_{\mu=m}^{i-2} \alpha_\mu\right) \alpha_{i-1} \prod_{\nu=1}^{j-1} \beta_\nu \\
&\quad + \sum_{n=2}^{j-1} \mathbb{E}\left[C_{0,n}\right] \binom{i+j-1-n-1}{i-1} \prod_{\mu=0}^{i-1} \alpha_\mu \left(\prod_{\nu=n}^{j-2} \beta_\nu\right) \beta_{j-1} \\
&\quad + \sum_{m=1}^{i} \mathbb{E}\left[C_{m,1}\right] \binom{i+j-1-m-2}{j-3} \prod_{\mu=m}^{i-1} \alpha_\mu \left(\prod_{\nu=1}^{j-1} \beta_\nu\right) \beta_{j-1} \\
&= \sum_{n=2}^{j} \mathbb{E}\left[C_{0,n}\right] \binom{i+j-n-1}{i-1} \prod_{\mu=0}^{i-1} \alpha_\mu \prod_{\nu=n}^{j-1} \beta_\nu \\
&\quad + \sum_{m=1}^{i} \mathbb{E}\left[C_{m,1}\right] \binom{i+j-m-2}{j-2} \prod_{\mu=m}^{i-1} \alpha_\mu \prod_{\nu=1}^{j-1} \beta_\nu .
\end{aligned}$$

□

If for a negative lower index −1, we define the binomial coefficient as the Kronecker symbol:

$$\binom{m}{-1} = \delta_{m,-1} = \begin{cases} 1 & \text{if } m = -1, \\ 0 & \text{else .} \end{cases} \tag{4.32}$$

By this, we see that (4.31) also holds trivially for $j = 1$ and $1 \leq i \leq I - 1$.
In nearly all IBNR-estimations, the left lower corner of the run-off triangle is a critical point, since only one value is available for the accident year I. This is particularly true for the classical Chain-Ladder method where the provision for the accident year I is given by $\widetilde{P}_I^{CL} = \mathbb{E}\left[C_{I,1}\right]\left(\prod_{j=1}^{J-1} \widetilde{\beta}_j^{CL} - 1\right)$.
To stabilize this critical corner, we make use of the fact that Assumption 4.1 (i) make also sense for $j = 1$ since $C_{i+1,0} = 0$ for all $1 \leq i \leq I - 1$. In the arithmetic mean we find

$$\mathbb{E}\left[C_{I,1}\right] = \frac{1}{I} \sum_{m=1}^{I} \prod_{\mu=m}^{I-1} \alpha_\mu \mathbb{E}\left[C_{m,1}\right]. \tag{4.33}$$

Collecting the equations (4.31), together with (4.32) in the case $j = 1$ and the relation (4.33) for $\mathbb{E}\left[C_{I,1}\right]$, we rewrite the dependence of the run-off triangle on the boundary values as follows in a matrix form:

$$\left(\mathbb{E}\left[C_{i,j}\right]\right)_{\substack{1\leq i\leq I\\ 1\leq j\leq (I-i+1)\wedge J}} = W\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \left(A^{(1)}A^{(2)}, B^{(1)}B^{(2)}\right)\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{4.34}$$

Here we have set:

$$V_1 := \left(\mathbb{E}\left[C_{0,j}\right]\right)_{2\leq j\leq J}, \qquad V_2 := \left(\mathbb{E}\left[C_{i,1}\right]\right)_{1\leq i\leq I}, \tag{4.35}$$

$$\begin{aligned} A^{(1)} &:= \left(\delta_{(i,j),(m,n)}\prod_{\mu=0}^{i-1}\alpha_\mu\right)_{(i,j),(m,n)\in\mathcal{P}}, \\ B^{(1)} &:= \left(\delta_{(i,j),(m,n)}\prod_{\nu=1}^{j-1}\beta_\nu\right)_{(i,j),(m,n)\in\mathcal{P}}, \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} A^{(2)} &:= \left(\mathbb{1}_{n\leq j}\binom{i+j-n-1}{i-1}\prod_{\nu=n}^{j-1}\beta_\nu\right)_{(i,j)\in\mathcal{P},2\leq n\leq J}, \\ B^{(2)} &:= \left(B^{(2)}_{(i,j),m}\right)_{(i,j)\in\mathcal{P},1\leq m\leq I} \qquad \text{with} \\ B^{(2)}_{(i,j),m} &:= \begin{cases} \mathbb{1}_{m\leq i}\binom{i+j-m-2}{j-2}\prod_{\mu=m}^{i-1}\alpha_\mu. & \text{for } i<I \\ \frac{1}{I}\prod_{\mu=m}^{I-1}\alpha_\mu & \text{for } i=I \end{cases} \end{aligned} \tag{4.37}$$

Based on the given data $\widehat{C}$ from (4.2), we use the linear regression estimator $(\widetilde{V}_1^{tr}, \widetilde{V}_2^{tr})^{tr}$ to estimate the mean values of the boundaries. We get

$$\begin{pmatrix} \widetilde{V}_1 \\ \widetilde{V}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{C}_{0,2} \\ \vdots \\ \widetilde{C}_{0,J} \\ \widetilde{C}_{1,1} \\ \vdots \\ \widetilde{C}_{I,1} \end{pmatrix} = \left(W^{tr}W\right)^{-1} W^{tr}\,\widehat{C}. \tag{4.38}$$

Having thus found estimator for the mean values of the boundaries we get estimator for the mean values using (4.4) recursively:

$$\widetilde{C}_{i+1,j+1} = \widetilde{\alpha}_i \cdot \widetilde{C}_{i,j+1} + \widetilde{\beta}_j \cdot \widetilde{C}_{i+1,j} \tag{4.39}$$

for $1 \leq i \leq I-1$, $1 \leq j \leq J-1$.

It is well known that in order to apply a bootstrapping procedure, the usual assumptions about the (conditional) means, as they are given in our case by the Assumption 4.1, have to be strengthened. Standard conditions include independent distributions which may be normal, overdispersed Poisson, or Gamma. In our case, it seems to be appropriate to adopt for the last case, i.e. for independent Gamma distributions with a uniform shape parameter.

Assumption 4.9. *(i) For $1 \leq i \leq I$, $1 \leq j \leq J$, the random variables $C_{i,j}$ are Gamma distributed with a common shape parameter ϕ, but dependent scale parameter $1/\psi_{i,j}$. The latter are supposed to satisfy the following linear system*

$$\psi_{i,j} = \sum_{n=2}^{j} \psi_{0,n} \binom{i+j-n-1}{i-1} \prod_{\mu=0}^{i-1} \alpha_\mu \prod_{\nu=n}^{j-1} \beta_\nu + \sum_{m=1}^{i} \psi_{m,1} \binom{i+j-m-2}{j-2} \prod_{\mu=m}^{i-1} \alpha_\mu \prod_{\nu=1}^{j-1} \beta_\nu \tag{4.40}$$

where $1 \leq i \leq I-1$, $2 \leq j \leq J$. Of course, (4.40) is an immediate consequence of the conditions made in Assumption 4.1. Under these assumptions, we have that

$$\mathbb{E}\left[C_{i,j}\right] = \phi\ \psi_{i,j} \qquad \text{and} \qquad var(C_{i,j}) = \phi\ \psi_{i,j}^2. \tag{4.41}$$

Consequently, the multiplicative Pearson residuals $R_{i,j}$, defined by

$$R_{i,j} := \frac{C_{i,j}}{\phi\ \psi_{i,j}}, \tag{4.42}$$

are Gamma variables with mean 1 and variance $1\big/\phi$.

(ii) We assume that the residuals are exchangeable.

Now in analogy to (4.31), the estimation of the boundary values from (4.38) deliver estimations for $\phi\ \psi_{i,j}$ with $1 \leq i \leq I-1, 2 \leq j \leq (I+1-i) \wedge J$; more precisely we have

$$\begin{aligned} \widetilde{\phi\ \psi_{i,j}} := \widetilde{C}_{i,j} \quad &= \quad \sum_{n=2}^{j} \widetilde{C}_{0,n} \binom{i+j-n-1}{i-1} \prod_{\mu=0}^{i-1} \widetilde{\alpha}_\mu \prod_{\nu=n}^{j-1} \widetilde{\beta}_\nu \\ &\quad + \sum_{m=1}^{i} \widetilde{C}_{m,1} \binom{i+j-m-2}{j-2} \prod_{\mu=m}^{i-1} \widetilde{\alpha}_\mu \prod_{\nu=1}^{j-1} \widetilde{\beta}_\nu. \end{aligned} \tag{4.43}$$

According to (4.42), we get the observed multiplicative Pearson residuals $\widehat{R}_{i,j}$ by

$$\widehat{R}_{i,j} := \frac{\widehat{C}_{i,j}}{\widetilde{C}_{i,j}} \tag{4.44}$$

where $1 \leq i \leq I-1, 1 \leq j \leq (I+1-i) \wedge J$. The bootstrapping procedure assumes that the observed residuals $\widehat{R}_{i,j}$ may be treated as observations of iid random variables.

Remark 4.10.

(i) Most bootstrap calculations of the residuals include the case $(i,j) = (I,1)$ where the observed value $\widehat{C}_{I,1}$ is also used as the estimated mean $\widetilde{C}_{I,1}$. As a result, the observed residual is deterministic, either $\widehat{R}_{I,1} = 0$ for additive residuals or $\widehat{R}_{I,1} = 1$ for multiplicative residuals. Our estimator $\widetilde{C}_{i,j}$ of the mean $\mathbb{E}[C_{m,1}]$ include all data $\widehat{C}$ from (4.2), i.e; all observations of the run-off 'triangle'. Therefore our residual $\widehat{R}_{I,1}$ is not deterrministic.

(ii) In Assumption 4.9 we suppose that the random variables $C_{i,j}$ are Gamma distributed. This restricts the application of bootstrapping essentially to IBNR situation stemming from attritional losses. One should clearly avoid to apply the bootstrap procedure to situations where heavy tailed losses, like Pareto distributed ones, are involved.

The *bootstrap procedure* now runs as follows:

(i) To every index-pair (i,j) of the upper triangle, i.e. $1 \leq i \leq I, 1 \leq j \leq (I+1-i) \wedge J$, one of the observed residuals from (4.44) is attributed in an independent random way allowing for repetitions. The result is denoted by $\widehat{R}_{\kappa(i,j)}$.

(ii) The resampled observed cumulative claims amounts are then defined as

$$\widehat{C}^{(\kappa)}_{i,j} := \widetilde{C}_{i,j}\ \widehat{R}_{\kappa(i,j)}. \tag{4.45}$$

(iii) If we replacing $\widehat{C}_{i,j}$ by $\widehat{C}^{(\kappa)}_{i,j}$ in (4.16), we get the resampled estimations $(\widehat{\alpha}^{(\kappa)}, \widehat{\beta}^{(\kappa)})$.

(iv) The last paragraph of Section 1.3 describes how to derive the resampled estimations $\widehat{C}^{(\kappa)}_{i,j}$ for future years and the resampled estimations $\widehat{P}^{(\kappa)}_i(\widehat{\alpha}^{(\kappa)}, \widehat{\beta}^{(\kappa)})$ of the provisions for year $i \geq I-J+2$, as well as of the total provision $\widehat{P}^{(\kappa)}(\widehat{\alpha}^{(\kappa)}, \widehat{\beta}^{(\kappa)})$.

(v) The procedure restarts at (i) by a new sampling of the residuals to deliver new resampled estimations of provisions in (iv).

(vi) With a sufficiently large number of resampled estimations, we can construct empirical distribution, calculate an empirical VaR or TailVaR for the provisions, etc.

5 Example for bootstrapping

For comparison reasons, we apply the bootstrap procedure to the example below given by P. England and R. Verrall (see [6], [7], [8], and [9]).

	1	2	3	4	5	6	7	8	9	10
1	357 848	1 124 788	1 735 330	2 218 270	2 745 596	3 319 994	3 466 336	3 606 286	3 833 515	3 901 463
2	352 118	1 236 139	2 170 033	3 353 322	3 799 067	4 120 063	4 647 867	4 914 039	5 339 085	
3	290 507	1 292 306	2 218 525	3 235 179	3 985 995	4 132 918	4 628 910	4 909 315		
4	310 608	1 418 858	2 195 047	3 757 447	4 029 929	4 381 982	4 588 268			
5	443 160	1 136 350	2 128 333	2 897 821	3 402 672	3 873 311				
6	396 132	1 333 217	2 180 715	2 985 752	3 691 712					
7	440 832	1 288 463	2 419 861	3 483 130						
8	359 480	1 421 128	2 864 498							
9	376 686	1 363 294								
10	344 014									

Table 4.8 – Data of the exemple from [7]

The calculation of the BDFs gives the following table:

$\tilde{\alpha}_i$	$\tilde{\beta}_j$	$\tilde{P}_i$
0,9374		0
1,2531	0,0866	0
0,9467	0,1072	-109804
0,9867	0,1108	391961
0,8116	0,0793	1018326
0,9924	0,0568	1008099
1,0423	0,0483	1508003
1,0753	0,0689	2337711
0,9589	0,0910	3860345
0,9133	0,0637	5566478
		6458544
	Sum	22 039 663

Table 4.9 – The bivariate development factors

For the first multiplicative residuals we find the following results:

	1	2	3	4	5	6	7	8	9	10
1	1,0000	1,0211	1,0548	0,9579	0,9788	1,0385	0,9902	0,9914	0,9885	1,0000
2	1,0046	0,8762	0,9806	1,0655	1,0091	0,9763	1,0125	1,0081	1,0067	
3	0,9509	0,9488	0,9900	1,0023	1,0433	0,9812	1,0174	0,9962		
4	0,9347	1,0336	0,9308	1,0903	0,9968	0,9991	0,9760			
5	0,9531	0,9843	1,0444	0,9586	0,9663	1,0302				
6	0,9485	1,1282	1,0148	0,9221	0,9841					
7	0,9858	1,0141	1,0184	0,9573						
8	0,9924	1,0168	1,0590							
9	1,0022	0,9932								
10	0,8498									

Table 4.10 – The multiplicative residuals

Finally, a thousand simulations yield for the total provisions a histogram of the Diagram 4.2. This empirical distribution is in line with the results in [7], [8], and [9].

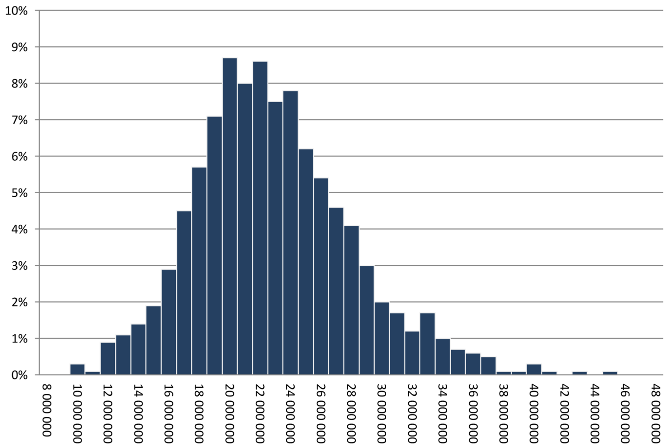

Figure 4.2 – Histogram of the total provisions

General Conclusion

In this thesis we have developed our own models for claims reserves, the main result is a generalization of the Mack's model in the case of $\alpha \equiv 0$, the most popular stochastic model using in the IBNR theory, derived from the well known Chain-Ladder method.

The bivariate development factors model presented in chapter 4 includes dependencies of the loss payments between accident and development years, in this way, the whole run-off triangle is covered by a net of dependencies. as mentioned above the boundary conditions on the final provision has an important influence, of these the usual stationarity assumption in IBNR methods is replaced by a condition on the upper boundary of the IBNR-triangle which reminds the von-Neumann boundary condition in partial differential equations. The right-hand boundary of the triangle is naturally subject to a Dirichlet-type condition since no claims settlements can be done before the accident has occurred. To integrate Mack's model, a smooth transition from it to our model is established. In our approach to the bootstrap method, we first apply a linear regression estimator for the mean of the boundary values with special attention to the critical value on the lower corner of the triangle. We then choose a multiplicative version for the residuals with an identical Gamma distribution.

This new incremental approach – developed by modifying Mack's model assumptions – was aimed to estimate the development factors (β) and provisions using only incremental claims. Starting from our formulae we can proceed the same as in Mack's model, thus the identical results. To apply the bootstrap for our approach we developed a new formula see (3.19) which avoid three steps in calculations. We can also examine the incremental approach in calendar year view, and we focus our interest from the development years (j) to the calendar years (t) which give us a new view for the past obligation of each calendar year (t) and also the estimation of the obligations left to pay of each calendar year (t), accident year (i) and the overall reserve, and to give a clear view we proposed a new form of tabulation Figure 3.4. The incremental approach avoid lot of steps in different cases, with the simplicity of the formulae used to give identical results, which brings lot of advantages for insurance companies.

Appendix

In Theorem 4.3 the Hadamard theorem is used in a general form making us of the following definition:

Definition D.11. A square matrix $A = (a_{i,j})_{1 \leq i,j \leq n} \in \mathbb{C}^{n \times n}$ is said to be *irreducible diagonally dominant* if the following properties hold:

(i) A is irreducible.

(ii) A is diagonally dominant, i.e.

$$\mid a_{i,i} \mid \geq \sum_{1 \leq j \leq n, j \neq i}^{n} \mid a_{i,j} \mid \tag{D.1}$$

and for at least one $i_0 \leq n$, we have a strict inequality in (D.1).

Hadamard's theorem runs now as follows:

Theorem D.12. *A irreducible diagonally dominant matrix A is invertible.*

For the proof, we refer to [15], corollary 6.2.27. It can also be found in [34], theorem 1.21 or in [35], theorem 1.11.

In fact, in a less general form, this theorem goes back to L. Lévy in 1881 and has been found independently by several other mathematicians, for example by O. Taussky [32]. A nice history of the theorem can be found in [31].

Bibliography

[1] Buchwalder, M., Bühlman, H., Merz, M., and Wüthrich, M. Legal valuation portfolio in non-life insurance. *Conference paper presented at the 36th Int. ASTIN Colloquim ETH Zurich* (2005). 23

[2] Chorfi, I., and Remita, M. R. Stochastic incremental approach for modelling the claims reserves. *International Mathematical Forum 8* (2013), 807–828. ii, iii, 2, 23

[3] Efron, B., and Tibshirani, R. J. *An introduction to the bootstrap*. Chapman & Hall, 1993. 1

[4] Eisele, K. T., and Artzner, P. Multiperiod insurance supervision: Top-down models. *European Actuarial Journal 1* (2011), 107–130. 23

[5] Eisele, K. T., and Chorfi, I. Chain-ladder with bivariate development factors. iii

[6] England, P. Addendum to 'analytic and bootsstrap estimates of prediction errors in claims reservering'. *Insurance: Mathematics and Economics 31* (2002), 461–466. 1, 65

[7] England, P., and Verral, R. Analytic and bootsstrap estimates of prediction errors in claims reservering. *Insurance: Mathematics and Economics 25* (1999), 281–293. ii, 1, 46, 60, 65, 66

[8] England, P., and Verral, R. Stochastic claims reservering in general insurance. *British Actuarial Journal 8* (2002), 443–544. 1, 39, 60, 65, 66

[9] England, P., and Verral, R. Predicitve distrbutions of outstanding liabilities in general insurance. *Annals of Actuarial Science 1* (2006), 221–270. 1, 60, 65, 66

[10] Federal Office of Private Insurance. Technisches Dokument zum Swiss Solvency Test. *FOPI, Bern* (2006). 60

[11] Gersgorin, S. Uber die abgrenzung der eigenwerte einer matrix. *Izvestia Akademii Nauk SSSR 7* (1931), 749–754.

[12] Gisler, A., and Wüthrich, M. V. Credibility for chain ladder reserving method. *Astin Bulletin 38* (2008), 565–600. 46

[13] Hadamard, J. *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique*. Hermann, Paris, 1903.

[14] Holmberg, R. D. Correlation and the measurement of loss reserve variability. *Casualty Actuarial Society Forum 1* (1994), 247–278.

[15] Horn, R. A., and Johnson, C. R. *Matrix Analysis*. Cambridge University Press, 1985. 68

[16] Lowe, J. A practical guide to measuring reserve variability using: Bootstrapping, operational time and a distribution free approach. General Insurance Convention, Institute of Actuaries and Faculty of Actuaries, 1994. 2

[17] Mack, T. A simple parametric model for rating automobile insurance or estimating ibnr claims reserves. *ASTIN Bull 21* (1991), 93–109. 1, 2

[18] Mack, T. Distribution-free calculation of the standard error of chain-ladder reserve estimates. *ASTIN Bulletin 23* (1993), 213–225. ii, 1, 2, 4, 5, 8, 45, 46, 47, 53, 54

[19] Mack, T. Measuring the variability of chain ladder reserve estimates. *Casualty Actuarial Society Forum (Spring) 1* (1994), 101–182. 4, 8, 30

[20] Mack, T. Which stochastic model is underlying the chain-ladder method? *Insurance Math. Econom 15* (1994), 133–138. 1, 2, 4

[21] Mack, T., and Venter, G. A comparison of stochastic models that reproduce chain ladder reserve estimates. *Insurance: Mathematics and Economics 26* (2000), 101–107. 1

[22] Marcus, M., and Minc, H. *A survey of matrix theory and matrix inequalities.* Dover Publications, 1992.

[23] McCullagh, P., and Nelder, J. *Generalized Linear Models.* Chapman & Hall, 2^{nd} ed, 1989. 13

[24] Merz, M., and Wüthrich, M. V. Prediction error of the expected claims development result in the chain ladder method. *Bulletin of Swiss Association of Actuaries 1* (2007), 117–137. 2, 4, 7, 38

[25] Merz, M., and Wüthrich, M. V. Modelling the claims development result for solvency purposes. *Casualty Actuarial Society E-Forum* (2008), 542–568. ii, 2, 4, 7, 38

[26] Partrat, C., Pey, N., and Schilling, J. Delta method and reserving. *Conference paper presented at the 36th Int. ASTIN Colloquim ETH Zurich* (2005). 23

[27] Pinheire, P., Andrade e Silve, J., and Centeno, M. Bootstrap methodology in claim reserving. *Journal of Risk and Insurance 70* (2003), 701–714.

[28] Renshaw, A., and Verrall, R. The stochastic model underlying the chain-ladder technique. *Actuar. Res. Paper No. 63.* (1994). 1

[29] Renshaw, A., and Verrall, R. A stochastic model underlying the chain-ladder technique. *British Actuar. J. 4* (1998), 903–923.

[30] Schmidt, K. D. A bibliography on loss reserving. *see: http://www.math.tu-dresden.de/sto/ schmidt* (2013). 1

[31] Schneider, H. Olga Taussky-Todd's influence on matrix theory and matrix theorists. *Linear and Multilinear Algebra 5* (1977), 197–224. 68

[32] Taussky, O. A recurring theorem on determinants. *The American Mathematical Monthly 56* (1949), 672–676. 68

[33] Taylor, G. *Loss Reserving: An Actuarial Perspective.* Kluwer Academic Press, 2000. 1, 2

[34] Varga, R. S. *Matrix Iterative Analysis.* 1962. 68

[35] Varga, R. S. *Gersgorin and His Circles.* Springer Series in Computational Mathematics, Springer Berlin Heidelberg, 2004. 68

[36] Verrall, R. On the estimation of reserves from loglinear models. *Insurance: Mathematics and Economics 10* (1991), 75–80. 1

[37] Verrall, R. An investigation into stochastic claims reserving models and the chain-ladder technique. *Insurance: Mathematics and Economics 26* (2000), 91–99. 1

[38] Wüthrich, M. V. Accounting year effects modelling in the stochastic chain ladder reserving method. *North American Actuarial Journal 14* (2010), 235–255. 1, 45, 57, 58

[39] Wüthrich, M. V., and Merz, M. *Stochastic Claims Reserving Methods in Insurance.* Wiley Finance, 2008. 1, 8

[40] Wüthrich, M. V., Merz, M., and Lysenko, N. Uncertainty of the claims development result in the chain ladder method. *Scandinavian Actuarial Journal 1* (2009), 63–84. 2, 4, 7, 9