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On the Asymptotic Behaviour and Oscillation of the Solutions of Certain Differential Equations and Systems with Delay

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Abstract

The abstract of this thesis, which concerns with the oscillatory behavior of the solutions of differential equations of second order and of partial equations , is as follows:

The first contribution, subject of chapter two, is concerned with a simple generalization of Parhi and Kirane [28]. Our essential contribution in this part of thesis is to consider the coefficients and the delays as functions.

The second contribution, subject of chapter three, is to consider the following equation without forcing term

$$\big[r(t)\psi(x(t))x'(t)\big]' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0 \geq 0, \tag{1}$$

where $t_0 \geq 0$, $r(t) \in C^1([t_0,\infty);(0,\infty))$, $p(t) \in C([t_0,\infty));\mathbb{R})$, $q(t) \in C([t_0,\infty));\mathbb{R})$, $p(t)$ and $q(t)$ are not identical to zero on $[t_\star,\infty[$ for some $t_\star \geq t_0$, $f(x), \psi(x) \in C(\mathbb{R},\mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$.

We do not only prove the oscillation of equation (1), but we also localize its zeros thanks to an idea of Nasr [27].

Our result is obtained under the following conditions:

- (C_1) For some positive constant K, $f(x)/x \geq K > 0$ for all $x \neq 0$.
- (C_2) For some two positive constants C_1, $0 < C \leq \psi(x) \leq C_1$
- (C_3) Suppose further there exists a continuous function $u(t)$ such that $u(a) = u(b) = 0$, $u(t)$ is differentiable on the open set (a,b), $a, b \geq t_\star$, and

$$\int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2C_1 r(t)(u')^2(t)\big]dt \geq 0\,. \tag{2}$$

The condition (C_3) ensures that any solution of (1) admits a zero in $[a,b]$.

This last result is an extension of the result of Kirane and Rogovchenko [21] where the localization of the zeros is not treated.

On the other hand, our method to obtain previous result is very simple in the sense that we use a simple Ricatti's transformation:

$$v(t) = -\frac{x'(t)}{x(t)}, \quad t \in [a,b], \tag{3}$$

with respect to the work of Kirane and Rogovchenko [21] where the following Ricatti's transformation is used:

$$v(t) = \rho \frac{r(t)\psi(x(t))x'(t)}{x(t)}. \tag{4}$$

Therefore, with the Ricatti's transformation (3), the computations could be done easily.
We should mention that, for a given second order equation, there is neither a general rule to choose the Ricatti's transformation nor a Darboux transformation:

$$v(t) = \frac{A(t)x(t) + B(t)x'(t)}{c(t)x(t) + D(t)x'(t)}. \tag{5}$$

The technique used to study the oscillation of (1) is extended to study the oscillation of the equation with forcing term

$$\big[r(t)\psi(x(t))x'(t)\big]' + p(t)x'(t) + q(t)f(x(t)) = g(t), \quad t \geq t_0 \geq 0, \tag{6}$$

with the same conditions cited above for r, ψ, p, and q.
The forcing term g is assumed to satisfy the condition:
(C_4): There exists an interval $[a, b]$, where $a, b \geq t_\star$, such that $g(t) \geq 0$ and there exists $c \in (a, b)$ such that $g(t)$ has different signs on $[a, c]$ and $[c, b]$.
We assume in addition that
$(C_1)^\star$: $\frac{f(x)}{x|x|} \geq K$,
for some positive constant K and for all $x \neq 0$.
Furthermore, we assume that there exists a continuous function $u(t)$ such that $u(a) = u(b) = u(c) = 0$, $u(t)$ is differentiable on the open set $(a, c) \cup (c, b)$, and satisfies the inequalities

$$\int_a^c \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2 - 2C_1 r(t)(u')^2(t)\big]d(t) \geq 0, \tag{7}$$

$$\int_c^b \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2 - 2C_1 r(t)(u')^2(t)\big]d(t) \geq 0\,. \tag{8}$$

Then every solution of equation (1) has a zero in $[a, b]$.
The third contribution, subject of chapter four, in our thesis is a generalization of our work of Magister [3].

We are concerned with the oscillation of the following hyperbolic equation:

$$
\begin{aligned}
u_{tt}\,(x,t) \quad &+ \quad \alpha u_t(x,t) - [\Delta u(x,t) + \sum_{i=1}^{k} b_i(t)\Delta u(x,\sigma_i(t))] \\
&+ \quad C(x,t,u(x,t),u(x,\tau_1(t)),u(x,\tau_2(t)),\ldots,u(x,\tau_m(t))) = f(x,t), \qquad (9)
\end{aligned}
$$

where $(x,t) \in Q = \Omega \times (0,\infty)$ and Ω is a bounded domain of $\mathbb{R}^n$ with a sufficiently regular boundary Γ and Δ is the Laplacian in $\mathbb{R}^n$.
Under some suitable conditions on $\sigma_i(t)$, $\tau_i(t)$, the coefficients $b_i(t)$, as well as some condition on the non–linearity C, we prove that all the solution of (9) are oscillatory. In addition, we provide a localization for the zeros of such oscillatory solution.

Key words: Differential equation of second order, Hyperbolic partial differential equation with delays, Oscillation

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Sommaire

Chapter 1

Introduction

While studying the heat conduction, C. Sturm, in 1936, posed the problem of oscillations of linear differential equations of the form

$$x^{''}(t) + a(t)x(t) = 0. \tag{1.1}$$

Since the work of Sturm, several works have been devoted for the oscillation theory.
The works of Sturm and Liouville, e.g. Sturm separation and comparison theorems, could be found in the standard books of differential equations..
Swanson [38] summarized the classical results of the oscillation theory. We also find a nice review of such theory in Kreith ”Oscillation theory" Springer, 1973.
The oscillation theory of non linear differential equations of second order has also attracted an attention, see for instance the books of Bogolyubov and Mitropolski ”Méthodes asymptotiques en théorie des oscillations non linéaires", 1962, of Roseau ”Vibrations non linéaires", 1966, and of Coddington and Levinson ”Theory of ODEs", 1955. We do not know exactly the first work on the oscillation theory of non linear differential equations of second order.
It seems that a first attempt to study the oscillation theory of differential equation with delays was done by Fite in 1921.
Fite considered the following differential equation of order n with a delayed argument :

$$y^{(n)}(t) + p(t)y(r(t)) = 0, \ t \in \mathbb{R}, \tag{1.2}$$

where $n \geq 1$, $p \in C(\mathbb{R})$, $r(t) = k - t$, $t \in \mathbb{R}$, and k is a positive constant.
We should mention that for differential equations of first order, not considered here, there is

a drastic between an equation of type:

$$y'(t) + p(t)y(t) = 0, \ p \in C(\mathbb{R}^+), \tag{1.3}$$

and an equation with a delay of type

$$y'(t) + y(t - \frac{\pi}{2}) = 0, \tag{1.4}$$

for example. The solutions of (1.3) have constant signs, whereas the equation (1.4) admits an oscillatory solution $y(t) = \sin(t)$. This last oscillatory behaviour is caused by the delay $\frac{\pi}{2}$ which appears in the argument of the second term on the left hand side of (1.4).
This remark shows that the study of equations with delays, even those simple, require a particular attention.
It seems that the study of the oscillatory solutions of ordinary differential equations, with or without delays, or of partial differential equations is important in practice, e.g. in biology, mechanics, electronics, physics of elementary particles,....
Our first contribution in this thesis is concerned with an ordinary non linear differential equation of second order with a forcing term in the general case, whereas the second contribution is concerned with a non linear hyperbolic equation with delays and a forcing term.
The first contribution, the subject of the third chapter and the article [2], is an extension of Kirane and Rogovchenko [20]. In [20], the authors show only that the solutions are oscillatory, whereas in our work we provide with a criterion allowing us to localize the zeros of such oscillatory solutions. These results are important in practice.
In [2], we suggest a simple Ricatti's transformation, in contrast of [20], to study the oscillation of the equation under consideration. This simple transformation makes our computations easy.
We should mention that, for a given second order equation, there is neither a general rule to choose the Ricatti's transformation nor a Darboux transformation:

$$v(t) = \frac{A(t)x(t) + B(t)x'(t)}{c(t)x(t) + D(t)x'(t)}.$$

The second contribution, the subject of the fourth chapter, does not only provide us with some criterion for the oscillation behaviour of the solutions of an hyperbolic partial differential equation, but also allows us to localize the zeros of such oscillatory solutions.
In the end of thesis, we suggest some problems which could be good paths of research to be followed in the future.

1.1 Some examples of oscillatory solutions

1.1.1 The case of ordinary differential equations

Consider the following differential equation:

$$x''(t) + q(t)x(t) = 0, \ t \in [t_0, +\infty), \tag{1.5}$$

where the function q is locally integrable.

The following criterion is due to Wintner [39]: if the following condition

$$\lim_{t\to+\infty} \frac{1}{t}\int_{t_0}^{t} dr \int_{t_0}^{r} q(s)ds = +\infty \tag{1.6}$$

is fulfilled, then the solutions of equation (1.5) are oscillatory. Hartman [17] showed that the condition (1.6) could be replaced by an upper limit.

Theorem 1.1.1. *(cf. Kamenev [23])*

Let the function $t^{1-n}A_n(t)$, where A_n is the n–th primitive of the function q, be not bounded above for some $n > 2$ (not necessarily integral). Then the solutions of (1.5) are oscillatory.

Proof We can remark that

$$A_n = \frac{1}{\Gamma(n)}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds, \tag{1.7}$$

so one could write the condition of Theorem 1.1.1 as

$$\limsup_{t\to+\infty} t^{1-n}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds = +\infty. \tag{1.8}$$

If we set

$$w = \frac{x'}{x}, \tag{1.9}$$

Then equation (1.5) is transformed into

$$w' + w^2 + q = 0, \tag{1.10}$$

which implies that

$$\int_{t_0}^{t}(t-s)^{n-1}w^{'}(s)ds+\int_{t_0}^{t}(t-s)^{n-1}w^{2}(s)ds=-\int_{t_0}^{t}(t-s)^{n-1}q(s)ds. \tag{1.11}$$

Thanks to an integration by parts, we have

$$\int_{t_0}^{t}(t-s)^{n-1}w^{'}(s)ds=(n-1)\int_{t_0}^{t}(t-s)^{n-2}w(s)ds-w(t_0)(t-t_0)^{n-1}. \tag{1.12}$$

Inserting this in (1.11) and multiplying by t^{1-n}, we get

$$\begin{aligned}
t^{1-n}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds &= -t^{1-n}\int_{t_0}^{t}\big((n-1)(t-s)^{n-2}w(s)+(t-s)^{n-1}w^2(s)\big)\,ds \\
&+ w(t_0)(\frac{t-t_0}{t})^{n-1} \\
&= -t^{1-n}\int_{t_0}^{t}\{(t-s)^{\frac{n-1}{2}}w(s)+\frac{n-1}{2}(t-s)^{\frac{n-3}{2}}\}^2 \\
&+ \frac{(n-1)^2(t-t_0)^{n-2}}{4(n-2)t^{n-1}} \\
&+ w(t_0)(\frac{t-t_0}{t})^{n-1} \\
&\leq \frac{(n-1)^2(t-t_0)^{n-2}}{4(n-2)t^{n-1}}+w(t_0)(\frac{t-t_0}{t})^{n-1} \\
&\leq C_1, \qquad (1.13)
\end{aligned}$$

for all $t \geq t_0$, which contradicts assumption (1.8). ■

Remark 1.1.1. *It is useful to mention that assumption (1.6) implies assumption (1.8) for $n=3$. Therefore the Wintner's criterion for the oscillation of equation (1.5) could be a particular application of Theorem 1.1.1.*

1.1.2 Oscillation of the solution of some time dependent equations in higher dimensions

In the previous subsection, we quoted some criteria for the oscillation of some one dimensional differential equations. These criteria are given in [23] and [39].
In this subsection, we quote some criteria for the oscillation of some differential equations in higher dimensions. The models and the criteria we will quote in this subsection are given in Parhi and Kirane [28]. These models are important from the point of view that they appear

in biology.
We will focus our attention on the following “delay " equations:

$$\begin{aligned} u_{tt}(x,t) + \beta u_{tt}(x,t-\rho) + \gamma u_t(x,t-\theta) - \{\Delta u(x,t) + \alpha \Delta u(x,t-\tau)\} \\ + \quad c(x,t,u(x,t),u(x,t-\sigma)) = f(x,t), \ (x,t) \in \mathcal{Q} = \Omega \times (0,\infty), \end{aligned} \tag{1.14}$$

where Ω is a bounded domain in $\mathbb{R}^d$, with regular boundary $\Gamma = \partial\Omega$.
Of course, to get the well–posedness of (1.14), we need additional conditions. Some of these conditions are concerned with the boundary conditions like

$$u(x,t) = \psi(x,t), \ (x,t) \in \Gamma \times (0,\infty), \ \text{Dirichlet boundary conditions.} \tag{1.15}$$

$$\nabla u(x,t) \cdot \mathbf{n}(x) = \tilde{\psi}(x,t), \ (x,t) \in \Gamma \times (0,\infty), \ \text{Neumann boundary conditions.} \tag{1.16}$$

$$\nabla u(x,t) \cdot \mathbf{n}(x) + \mu u(x,t) = 0, \ (x,t) \in \Gamma \times (0,\infty),, \ \text{Robin boundary conditions.} \tag{1.17}$$

(Where we have denoted, as usual, $\mathbf{n}(x)$, $x \in \Gamma$, the unit vector normal to the boundary Γ on the point x, outward to Ω.)
Here

$$\Delta u(x) = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}, \ x = (x_1, x_2, x_3, \ldots, x_d). \tag{1.18}$$

The functions ψ, $\tilde{\psi}$ in (1.15)–(1.16) are given functions, and μ in (1.17) is positive.

The constants $\alpha, \beta, \gamma, \theta, \tau, \sigma$ which appear in (1.14) are positive. In addition to this, we assume that

$$f \in \mathcal{C}(\overline{\mathcal{Q}}). \tag{1.19}$$

We need the following assumption on the function c which appears (1.14):

Assumption 1. *We assume that the function c satisfies*

$\mathcal{H}_1$ *: $c(x,t,\xi,\eta)$ is a real valued continuous function on $\mathcal{Q}\times\mathbb{R}\times\mathbb{R}$.*

$\mathcal{H}_2$ *: $c(x,t,\xi,\eta)\geq 0$ for all $(x,t,\xi,\eta)\in\mathcal{Q}\times\mathbb{R}^+\times\mathbb{R}^+$.*

$\mathcal{H}_3$ *: $c(x,t,-\xi,-\eta)=-c(x,t,\xi,\eta)$ is a real valued continuous function on $\mathcal{Q}\times\mathbb{R}^+\times\mathbb{R}^+$.*

To analyze the oscillation of the time dependent equations (1.14), we need to use an eigenfunction for Laplace operator; it is known that the first eigenvalue λ_1 of the following spectral problem,

$$-\Delta\bar{\omega}(x)=\lambda\bar{\omega}(x),\ \ x\in\Omega, \tag{1.20}$$

with homogenous Dirichlet boundary condition

$$\bar{\omega}(x)=0,\ \ x\in\Gamma, \tag{1.21}$$

is positive and the corresponding eigenfunction is either positive or negative. One could remark that if $\bar{\omega}$ is an eigenfunction corresponding to an eigenvalue λ, than $-\bar{\omega}$ is also an eigenfunction corresponding to an eigenvalue λ, one could choose the eigenfunction, denoted by φ, corresponding to the first value λ_1 such that $\varphi(x)>0$, for all $x\in\Omega$.
We also use the following notations, for all $u\in\mathcal{C}^2(\mathcal{Q})\cap\mathcal{C}^1(\overline{\mathcal{Q}})$:

$$U(t)=\int_\Omega u(x,t)\varphi(x)dx, \tag{1.22}$$

$$\tilde{U}(t)=\int_\Omega u(x,t)dx, \tag{1.23}$$

$$F(t)=\int_\Omega f(x,t)\varphi(x)dx, \tag{1.24}$$

$$\tilde{F}(t)=\int_\Omega f(x,t)dx, \tag{1.25}$$

$$\Psi(t)=\int_\Gamma \psi(x,t)\nabla\varphi(x)\cdot\mathbf{n}(x)dx, \tag{1.26}$$

$$\tilde{\Psi}(t)=\int_\Gamma \tilde{\psi}(x,t)dx. \tag{1.27}$$

The following Theorems, see [3], give sufficient conditions for the oscillation of equation (1.14) with different boundary conditions.

Theorem 1.1.2. *(Oscillation of (1.14) with Dirichlet boundary condition) Assume that Assumption 1 and the following conditions are fulfilled:*

1.

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left(1-\frac{s}{t}\right) \left(F(s)-\Psi(s)-\alpha\Psi(s-\tau)\right) ds = -\infty, \tag{1.28}$$

2.

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(1-\frac{s}{t}\right) \left(F(s)-\Psi(s)-\alpha\Psi(s-\tau)\right) ds = +\infty, \tag{1.29}$$

for a sufficiently large t_0, then each solution of (1.14)–(1.15) is oscillatory on Q.

Theorem 1.1.3. *(Oscillation of (1.14) with Neumann boundary conditions) Assume that Assumption 1 and the following conditions are fulfilled:*

3.

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left(1-\frac{s}{t}\right) \left(\tilde{F}(s)+\tilde{\Psi}(s)+\alpha\tilde{\Psi}(s-\tau)\right) ds = -\infty, \tag{1.30}$$

4.

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(1-\frac{s}{t}\right) \left(\tilde{F}(s)+\tilde{\Psi}(s)+\tilde{\alpha}\Psi(s-\tau)\right) ds = +\infty, \tag{1.31}$$

for a sufficiently large t_0, then each solution of (1.14) with (1.16) is oscillatory on Q.

Theorem 1.1.4. *(Oscillation of (1.14) with Robin boundary conditions) Assume that Assumption 1 and the following conditions are fulfilled:*

5.

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left(1-\frac{s}{t}\right) \tilde{F}(s) ds = -\infty, \tag{1.32}$$

6.

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(1-\frac{s}{t}\right) \tilde{F}(s) ds = +\infty, \tag{1.33}$$

for a sufficiently large t_0, then each solution of (1.14) with (1.17) is oscillatory on Q.

The Proofs of Theorems 1.1.2–1.1.4 are similar, we only present the Proof of Theorem 1.1.2.

Proof of Theorem 1.1.2:
Assume that the solution u is not oscillatory, then there exists $a > 0$ such that u is either positive or negative on $\mathcal{Q}_a$ (recall that $\mathcal{Q}_a = \Omega \times (a, \infty)$).
We assume that

$$u(x,t) > 0, \ \forall (x,t) \in \mathcal{Q}_a. \tag{1.34}$$

Multiplying (1.14) by φ (Recall that φ is the positive eigenfunction corresponding to the first positive eigenvalue λ_1 of (1.20).) and integrating the result over $x \in \Omega$, we get

$$\begin{aligned} U_{tt}(t) + \beta U_{tt}(t-\rho) + \gamma U_t(t-\theta) = F(t) \\ + \int_\Omega \Delta u(x,t)\varphi(x)dx + \alpha \int_\Omega \Delta u(x,t-\tau)\varphi(x)dx \\ - \int_\Omega c(x,t,u(x,t),u(x,t-\sigma))\varphi(x)dx, \ \forall t \in (0,\infty). \end{aligned} \tag{1.35}$$

We first remark that, thanks to Assumption 1, and the fact that u and φ are positive

$$-\int_\Omega c(x,t,u(x,t),u(x,t-\sigma))\varphi(x) \le 0, \ \forall t \in (a+\sigma,\infty). \tag{1.36}$$

On the other hand, thanks to an integration by parts, we get

$$\begin{aligned} \int_\Omega \Delta u(x,t)\varphi(x)dx &= -\int_\Gamma u(x,t)\nabla\varphi(x)\cdot\mathbf{n}(x)dx + \int_\Omega \Delta\varphi(x)u(x,t)dx \\ &= -\int_\Gamma \psi(x,t)\nabla\varphi(x)\cdot\mathbf{n}(x)dx - \lambda_1\int_\Omega \varphi(x)u(x,t)dx \\ &\le -\Psi(t), \end{aligned} \tag{1.37}$$

and by the same way, we have

$$\int_\Omega \Delta u(x,t-\tau)\varphi(x)dx \le -\Psi(t-\tau), \ \forall t \in (a+\tau,\infty). \tag{1.38}$$

From (1.35)–(1.38), we deduce that, for any $t \in (a+\max(\tau,\sigma),\infty)$

$$U_{tt}(t) + \beta U_{tt}(t-\rho) + \gamma U_t(t-\theta) \le F(t) - \Psi(t) - \alpha\Psi(t-\tau). \tag{1.39}$$

To simplify the notation, we set

$$g(t) = F(t) - \Psi(t) - \alpha\Psi(t-\tau), \ \forall t \in (T_0, \infty)\,, \tag{1.40}$$

where, for the sake of simplicity of the notations

$$T_0 = a + \max(\tau, \sigma). \tag{1.41}$$

Thanks to (1.39) and definition (1.40) of g, we have

$$U_{tt}(t) + \beta U_{tt}(t-\rho) + \gamma U_t(t-\theta) \le g(t), \ \forall t \in (T_0, \infty)\,. \tag{1.42}$$

Integrating inequality (1.42) over (T_0, ∞), we get

$$U_t(t) + \beta U_t(t-\rho) + \gamma U(t-\theta) \le \int_{T_0}^{t} g(s)ds + d_1, \ \forall t \in (T_0, \infty)\,, \tag{1.43}$$

where $d_1 \in \mathbb{R}$.
Integrating again (1.43) over (T_1, ∞), where $T_1 = a + \max(\tau, \sigma, \theta, \sigma, \rho)$, we get since U and γ are positive

$$\begin{aligned} U(t) + \beta U(t-\rho) &\le \int_{T_1}^{t}\int_{T_0}^{r} g(s)dsdr + d_1(t-T_1) - \gamma\int_{T_1}^{t} U(r-\theta)dr + d_2 \\ &\le \int_{T_1}^{t}\int_{T_0}^{r} g(s)dsdr + d_1(t-T_1) + d_2, \end{aligned} \tag{1.44}$$

where $d_2 \in \mathbb{R}$.
One could remark that $\int_{T_1}^{t}\int_{T_0}^{r} g(s)dsdr = \int_{T_1}^{t} (t-s)g(s)ds + \zeta(t-T_1)$ for some $\zeta \in \mathbb{R}$, one could deduce from the previous inequality that

$$U(t) + \beta U(t-\rho) \le \int_{T_1}^{t} (t-s)g(s)ds + \bar{d}_1(t-T_1) + d_2, \tag{1.45}$$

for some $\bar{d}_1 \in \mathbb{R}$.
Since U is positive on (T_1, ∞) (recall that $T_1 = a + \max(\tau, \sigma, \theta, \sigma, \rho)$), we have

$$\liminf_{t\to\infty} \frac{1}{t-T_1}\int_{T_1}^{t} \{U(s) + \beta U(s-\rho)\}ds \ge 0, \tag{1.46}$$

which implies, using (1.45), that

$$\liminf_{t\to\infty} \frac{1}{t-T_1}\int_{T_1}^{t}(t-s)g(s)ds \geq 0. \tag{1.47}$$

On the other hand, thanks to assumption (1.28) of Theorem 1.1.2, we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t-T_1}\int_{T_1}^{t}(t-s)g(s)ds &= \liminf_{t\to\infty} \frac{t}{t-T_1}\int_{T_1}^{t}(1-\frac{s}{t})g(s)ds \\ &= \liminf_{t\to\infty} \int_{T_1}^{t}(1-\frac{s}{t})g(s)ds \\ &= -\infty, \end{aligned} \tag{1.48}$$

which is a contradiction with (1.47).
So far, we proved, under the assumptions of Theorem 1.1.2, that on any interval (a,∞) u can not be only positive, i.e., for each $a > 0$, there exists some $\hat{t} \in (a,\infty)$ such that $u(\hat{t}) \leq 0$. To conclude now the Proof of Theorem 1.1.2, we should prove that on any interval (a,∞) u can not be only negative, i.e., for each $a > 0$, there exists some $\bar{t} \in (a,\infty)$ such that $u(\bar{t}) \geq 0$. This will allow us to confirm that for each interval (a,∞), there exists some $t_1 \in (a,\infty)$ such that $u(t_1) = 0$.
Assume then that there exists $a > 0$ such that

$$u(x,t) < 0,\ \forall (x,t) \in \mathcal{Q}_a. \tag{1.49}$$

Set

$$v(x,t) = -u(x,t),\ \forall (x,t) \in \mathcal{Q}_a, \tag{1.50}$$

which implies, using (1.49)

$$v(x,t) > 0,\ \forall (x,t) \in \mathcal{Q}_a. \tag{1.51}$$

Multiplying (1.35) by -1, we get

$$\begin{aligned} V_{tt}(t) + \beta V_{tt}(t-\rho) &+ \gamma V_t(t-\theta) = -F(t) + \int_\Omega \Delta v(x,t)\varphi(x)dx + \alpha\int_\Omega \Delta v(x,t-\tau)\varphi(x)dx \\ &+ \int_\Omega c(x,t,-v(x,t),-v(x,t-\sigma))\varphi(x)dx,\ \forall t \in (0,\infty), \end{aligned} \tag{1.52}$$

where

$$V(t) = -U(t),\ \forall t > 0. \tag{1.53}$$

Using now hypothesis $\mathcal{H}_3$ of Assumption 1, equation (1.52) with (1.51) leads to

$$V_{tt}(t)+\beta V_{tt}(t-\rho)+\gamma V_t(t-\theta) \leq -F(t)+\int_\Omega \Delta v(x,t)\varphi(x)dx+\alpha\int_\Omega \Delta v(x,t-\tau)\varphi(x)dx, \quad (1.54)$$

for any $t \in (0,\infty)$.
On the other hand, thanks to an integration by parts, we get

$$\begin{aligned}\int_\Omega \Delta v(x,t)\varphi(x)dx &= -\int_\Gamma v(x,t)\nabla\varphi(x)\cdot\mathbf{n}(x)dx+\int_\Omega \Delta\varphi(x)v(x,t)dx \\ &= \int_\Gamma \psi(x,t)\nabla\varphi(x)\cdot\mathbf{n}(x)dx-\lambda_1\int_\Omega \varphi(x)v(x,t)dx \\ &\leq \Psi(t), \end{aligned} \quad (1.55)$$

and by the same way, we have

$$\int_\Omega \Delta u(x,t-\tau)\varphi(x)dx \leq \Psi(t-\tau),\ \forall t\in(a+\tau,\infty). \quad (1.56)$$

From (1.54)–(1.56), we deduce that

$$V_{tt}(t)+\beta V_{tt}(t-\rho)+\gamma V_t(t-\theta) \leq h(t),\ \forall t\in(a+\max(\tau,\sigma),\infty), \quad (1.57)$$

where h is defined by

$$h(t) = -F(t)+\Psi(t)+\alpha\Psi(t-\tau). \quad (1.58)$$

On the other hand, thanks to assumption (1.29) of Theorem 1.1.2, for a sufficiently large T_1

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t-T_1}\int_{T_1}^t (t-s)h(s)ds &= \liminf_{t\to\infty}\frac{t}{t-T_1}\int_{T_1}^t (1-\frac{s}{t})h(s)ds \\ &= \liminf_{t\to\infty}\int_{T_1}^t (1-\frac{s}{t})h(s)ds \\ &= -\infty. \end{aligned} \quad (1.59)$$

This allows us to apply the same techniques used in (1.42)–(1.48), and consequently we get a contradiction, which completes the proof. ■

1.2 Oscillation of some nonlinear second order equations

Among the results related to our principal contribution given in chapter three, are those of the article [20].
The article [20] presents new oscillation creteria for a nonlinear second order differential equation with a damping term. An essential result in [20] is the nondecreasing property of the nonlinearity.
Kirane and Rogovchenko [20] studied the oscillatory solutions of the equation

$$\left[r(t)\psi(x(t))x'(t)\right]' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0, \tag{1.60}$$

where $t_0 \geq 0$, $r(t) \in C^1([t_0,\infty);(0,\infty))$, $p(t) \in \mathcal{C}([t_0,\infty);\mathbb{R})$, $q(t) \in \mathcal{C}([t_0,\infty);(0,\infty))$, $q(t)$ is not identical to zero on $[t_*,\infty)$ for some $t_* \geq t_0$, $f(x), \psi(x) \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$.

As usual, a function $x : [t_0, t_1) \to (-\infty, \infty)$ with $t_1 > t_0$ is called a solution of equation (1.60) if $x(t)$ satisfies Equation (1.60) for all $t \in [t_0, t_1)$. The authors consider only proper solutions $x(t)$ of (1.60) in the sense that $x(t)$ is non-constant solutions which exist for all $t \geq t_0$ and

$$\sup\{x(t); \forall t \geq t_0\} > 0. \tag{1.61}$$

A proper solution $x(t)$ of (1.60) is called oscillatory if it has arbitrarily large zeroes; otherwise it is called nonoscillatory. Finally, an equation (1.60) is said to be oscillatory if all its proper solutions are oscillatory.

Oscillatory and nonoscillatory behavior of solutions for different classes of linear and nonlinear second order differential equations has been studied by many authors (see, for example, [1–26] and the references quoted therein). Some papers [12, 13, 15, 16, 21, 23] are concerned with particular cases of equation (1.60) such as linear equations:

$$x''(t) + q(t)x(t) = 0, \tag{1.62}$$

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0, \tag{1.63}$$

and the nonlinear equation

$$(r(t)x'(t))' + q(t)f(x(t)) = 0. \tag{1.64}$$

The main idea to deal with Equations (1.62)–(1.64) uses the average behavior of the integral of $q(t)$ and originates from the techniques used in Wintner [39], Hartman [17] and Kamenev [23]. For more details, we refer to Yan [42], Philos [29], and Li [24] , where one can follow the refinement of the ideas and methods cited above, see also corrections to the later paper in Rogovchenko [31].

The purpose of [20] is to derive new oscillation criteria for equation (1.60) which complements and extends those in [9], [11], [31], [40],[42].

More precisely, the techniques used [20] are similar to that used in Grace [11], Kirane and Rogovchenko [21], Philos [29], Rogovchenko [32], [33], and Yan [42]. Their results are as follows

Theorem 1.2.1. *Assume that for some constants K, C, C_1 and for all $x \neq 0$, $f(x)/x \geq K > 0$ and $0 < C \leq \psi(x) \leq C_1$. Let h, $H \in \mathcal{C}(D, R)$, where $D = \{(t,s) : t \geq s \geq t_0\}$, be such that*

(i) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ in $D_0 = \{(t,s) : t > s \geq t_0\}$

(ii) H has a continuous and non-positive partial derivative in D_0 with respect to the second variable, and

$$-\frac{\partial H}{\partial s} = h(t,s)\sqrt{H(t,s)} \tag{1.65}$$

for all $(t,s) \in D_0$.

If there exists a function $\rho \in C^1([t_0, \infty); (0, \infty))$ such that

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)\Theta(s) - \frac{C_1}{4}\rho(s)r(s)Q^2(t,s)]ds = \infty\,, \tag{1.66}$$

where

$$\Theta(t) = \rho(t)\Big(Kq(t) - (\frac{1}{C} - \frac{1}{C_1})\frac{p^2(t)}{4r(t)}\Big),$$
$$Q(t,s) = h(t,s) + [\frac{p(s)}{C_1 r(s)} - \frac{\rho'(s)}{\rho(s)}]\sqrt{H(t,s)},$$

then (1.60) is oscillatory.

Proof Assume that a solution $x(t)$ for (1.60) is not oscillatory, therefore there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \in (T_0, +\infty)$.
Let us consider the function $v(t)$ defined by

$$v(t) = \rho(t)\frac{r(t)\psi(x(t))x^{'}(t)}{x(t)}, \forall t \in (T_0, +\infty). \tag{1.67}$$

Since $x(t) \neq 0$ for all $t \in (T_0, +\infty)$, then $v(t)$ is well defined.
Differentiating (1.67) and using (1.60), we get

$$\begin{aligned} v^{'}(t) &= \rho^{'}(t)\frac{r(t)\psi(x(t))x^{'}(t)}{x(t)} \\ &+ \rho(t)\frac{[r(t)\psi(x(t))x^{'}(t)]^{'}x(t) - r(t)\psi(x(t))[x^{'}(t)]^2}{x^2(t)} \\ &= \rho^{'}(t)\frac{r(t)\psi(x(t))x^{'}(t)}{x(t)} \\ &- \rho(t)\frac{p(t)x'(t)}{x(t)} - \rho(t)\frac{q(t)f(x(t))}{x(t)} \\ &- \rho(t)\frac{r(t)\psi(x(t))[x^{'}(t)]^2}{x^2(t)} \\ &= -\rho(t)\frac{q(t)f(x(t))}{x(t)} \\ &+ \left\{\rho^{'}(t)r(t)\psi(x(t)) - \rho(t)p(t)\right\}\frac{x^{'}(t)}{x(t)} - \rho(t)r(t)\psi(x(t))[\frac{x^{'}(t)}{x(t)}]^2 \end{aligned} \tag{1.68}$$

On the other hand, thanks to (1.67), we get

$$\frac{x^{'}(t)}{x(t)} = \frac{v(t)}{\rho(t)r(t)\psi(x(t))}. \tag{1.69}$$

Inserting this in (1.68), we get

$$\begin{aligned} v^{'}(t) &= -\rho(t)\frac{q(t)f(x(t))}{x(t)} \\ &+ \left\{\rho^{'}(t)r(t)\psi(x(t)) - \rho(t)p(t)\right\}\frac{v(t)}{\rho(t)r(t)\psi(x(t))} - \rho(t)r(t)\psi(x(t))[\frac{v(t)}{\rho(t)r(t)\psi(x(t))}]^2 \\ &= -\rho(t)\frac{q(t)f(x(t))}{x(t)} + \frac{\rho^{'}(t)}{\rho(t)}v(t) \\ &- \frac{1}{r(t)\psi(x(t))}\left\{p(t)v(t) + \frac{1}{\rho(t)}v^2(t)\right\} \\ &= -\rho(t)\frac{q(t)f(x(t))}{x(t)} + \frac{\rho^{'}(t)}{\rho(t)}v(t) + \frac{p^2(t)\rho(t)}{4r(t)\psi(x(t))} \\ &- \frac{1}{r(t)\psi(x(t))}\left\{\frac{p(t)\sqrt{\rho(t)}}{2} + \frac{1}{\sqrt{\rho(t)}}v(t)\right\}^2. \end{aligned} \tag{1.70}$$

Thanks to the hypothesis $f(x)/x \geq K > 0$ of Theorem 1.2.1 , for $x \neq 0$, and since $x(t) \neq 0$, for all $t \in (T_0, +\infty)$, ρ and q are positive, we have the following estimate for the first term on the right–hand side of the previous inequality

$$-\rho(t)\frac{q(t)f(x(t))}{x(t)} \leq -K\rho(t)q(t), \ \forall t \in (T_0, +\infty). \tag{1.71}$$

Thanks to the hypothesis $C_1 \geq \psi(x) \geq C > 0$ of Theorem 1.2.1, and since $x(t) \neq 0$, for all $t \in (T_0, +\infty)$, and ρ is positive, we have the following estimate for the third term on the right–hand side of inequality (1.70)

$$\frac{p^2(t)\rho(t)}{4r(t)\psi(x(t))} \leq \frac{p^2(t)\rho(t)}{4Cr(t)}, \ \forall t \in (T_0, +\infty). \tag{1.72}$$

and the following estimate for the fourth term on on the right–hand side of inequality (1.70)

$$\begin{aligned} -\frac{1}{r(t)\psi(x(t))}\left\{\frac{p(t)\sqrt{\rho(t)}}{2} + \frac{1}{\sqrt{\rho(t)}}v(t)\right\}^2 &\leq -\frac{1}{C_1 r(t)}\left\{\frac{p(t)\sqrt{\rho(t)}}{2} + \frac{1}{\sqrt{\rho(t)}}v(t)\right\}^2 \\ &= -\frac{1}{C_1 r(t)}\left\{p(t)v(t) + \frac{1}{\rho(t)}v^2(t)\right\} \\ &- \frac{p^2(t)\rho(t)}{4C_1 r(t)}, \end{aligned} \tag{1.73}$$

for any $t \in (T_0, +\infty)$.

Combining now (1.72) and (1.73) with (1.71), we get

$$\begin{aligned}
v^{'}(t) &\leq -K\rho(t)q(t) + \frac{\rho^{'}(t)}{\rho(t)}v(t) + \frac{1}{4p^2(t)\rho(t)r(t)}(\frac{1}{C} - \frac{1}{C_1}) \\
&- \frac{1}{C_1 r(t)}\left\{p(t)v(t) + \frac{1}{\rho(t)}v^2(t)\right\} \\
&= -\Theta(t) - \left(\frac{p(t)}{C_1 r(t)} - \frac{\rho^{'}(t)}{\rho(t)}\right)v(t) - \frac{1}{C_1 r(t)\rho(t)}v^2(t), \qquad (1.74)
\end{aligned}$$

for any $t \in (T_0, +\infty)$.
where $\Theta(t)$ is defined in Theorem 1.2.1.
Multiplying both sides of (1.74) by $H(t,s)$, integrating the result over (T,t), where $t \geq T \geq T_0$, by an integration by parts, using the fact that $H(t,t) = 0$ and (1.65), we get

$$\begin{aligned}
\int_T^t H(t,s)\Theta(s)ds &\leq -\int_T^t H(t,s)v^{'}(s)ds - \int_T^t H(t,s)\left(\frac{p(s)}{C_1 r(s)} - \frac{\rho^{'}(s)}{\rho(s)}\right)v(s)ds \\
&- \int_T^t H(t,s)\frac{1}{C_1 r(s)\rho(s)}v^2(s)ds \\
&= -\int_T^t H(t,s)v^{'}(s)ds - \int_T^t H(t,s)\left(\frac{p(s)}{C_1 r(s)} - \frac{\rho^{'}(s)}{\rho(s)}\right)v(s)ds \\
&- \int_T^t \frac{H(t,s)}{C_1 r(s)\rho(s)}v^2(s)ds \\
&= -\int_T^t H(t,s)v^{'}(s)ds - \int_T^t Q(t,s)\sqrt{H(t,s)}v(s)ds - \int_T^t \frac{\partial H}{\partial s}(t,s)v(s)ds \\
&- \int_T^t \frac{H(t,s)}{C_1 r(s)\rho(s)}v^2(s)ds \\
&= H(t,T)v(T) - \int_T^t Q(t,s)\sqrt{H(t,s)}v(s)ds - \int_T^t \frac{H(t,s)}{C_1 r(s)\rho(s)}v^2(s)ds \\
&= H(t,T)v(T) \\
&- \int_T^t \left\{\sqrt{\frac{H(t,s)}{C_1 r(s)\rho(s)}}v(s) + \frac{1}{2}\sqrt{C_1 r(s)\rho(s)}Q(t,s)\right\}^2 ds \\
&+ \frac{C_1}{4}\int_T^t r(s)\rho(s)Q^2(t,s)ds, \qquad (1.75)
\end{aligned}$$

which implies that

$$\int_T^t \left\{ H(t,s)\Theta(s) - \frac{C_1 r(s)\rho(s)Q^2(t,s)}{4} \right\} ds \le H(t,T)v(T)$$
$$- \int_T^t \left\{ \sqrt{\frac{H(t,s)}{C_1 r(s)\rho(s)}} v(s) + \frac{1}{2}\sqrt{C_1 r(s)\rho(s)} Q(t,s) \right\}^2 ds \tag{1.76}$$

and therefore, since H is a decreasing function with respect to the second variable and $t \ge t_0$

$$\int_T^t \left\{ H(t,s)\Theta(s) - \frac{C_1 r(s)\rho(s)Q^2(t,s)}{4} \right\} ds \le H(t,t_0)|v(T)|. \tag{1.77}$$

Using the fact that $H(t,s) \ge 0$ for all $s \in [t_0, t]$ and again the fact that H is a decreased function with respect to the second variable, (1.77) leads to

$$\int_{t_0}^t \left\{ H(t,s)\Theta(s) - \frac{C_1 r(s)\rho(s)Q^2(t,s)}{4} \right\} ds = \int_{t_0}^T H(t,s)\Theta(s)ds$$
$$- \int_{t_0}^T \frac{C_1 r(s)\rho(s)Q^2(t,s)}{4} ds + \int_T^t \left\{ H(t,s)\Theta(s) - \frac{C_1 r(s)\rho(s)Q^2(t,s)}{4} \right\} ds$$
$$\le H(t,t_0) \left\{ \int_{t_0}^T \Theta(s)ds + |v(T)| \right\}, \tag{1.78}$$

which implies that

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left\{ H(t,s)\Theta(s) - \frac{C_1 r(s)\rho(s)Q^2(t,s)}{4} \right\} ds \le \int_{t_0}^T \Theta(s)ds + |v(T)|, \tag{1.79}$$

which contradicts the assumption (1.66) of Theorem 1.2.1, and consequently $x(t)$ is oscillatory. ■ From Theorem 1.2.1, we could deduce the following useful Corollary.

Corollary 1.2.2. *The assumption (1.66) of Theorem 1.2.1 could be replaced by the following two conditions together:*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\Theta(s)ds = \infty, \tag{1.80}$$

and

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t r(s)\rho(s)Q^2(t,s)ds < \infty. \tag{1.81}$$

It is useful to give some choices for H and h which satisfy conditions of Theorem 1.2.1. We define

$$H(t,s) = (t-s)^{n-1},\ \forall (t,s) \in D, \tag{1.82}$$

where $D = \{(t,s),\ t \geq s \geq t_0\}$ and n is an integer such that $n > 2$.
We have

- $H(t,t) = 0,$

- $H(t,s) \geq 0,\ \forall (t,s) \in D,$

- $H(t,s) \geq 0,\ \forall (t,s) \in D_0 = D \cap \{(t,s),\ t \neq s\}.$

The function h could then be chosen as

$$\begin{aligned} h(t,s) &= -\frac{\frac{\partial H}{\partial s}(t,s)}{\sqrt{H(t,s)}} \\ &= (n-1)(t-s)^{(n-3)/2},\ \forall (t,s) \in D. \end{aligned} \tag{1.83}$$

With the choices (1.82) and (1.83), Theorem 1.2.1 becomes:

Corollary 1.2.3. *Under the conditions of Theorem 1.2.1, if there exists a function* $\rho \in C^1([t_0,\infty);(0,\infty))$ *such that*

$$\limsup_{t\to+\infty} t^{1-n} \int_0^t \left\{ (t-s)^{n-1}\Theta(s) - \frac{C_1}{4}\rho(s)r(s)(t-s)^{n-3}A(s) \right\} ds = \infty, \tag{1.84}$$

where Θ *is defined as in Theorem 1.2.1 and*

$$A(s) = \left\{ n-1+\left(\frac{p(s)}{C_1 r(s)} - \frac{\rho'(s)}{\rho(s)} \right)(t-s) \right\}^2,$$

then (1.60) *is oscillatory.*

Example: Consider the following non–linear differential equation:

$$\begin{aligned} &\left\{ (1+\cos^2 t)\frac{2+x^2(t)}{1+x^2(t)}x'(t) \right\}' + 2(\sin t\cos t)x'(t) \\ &+ \ (20+\cos^2 t)x(t)\left\{ 1+\frac{18}{1+x^2(t)} \right\} = 0,\ t \geq 1. \end{aligned} \tag{1.85}$$

The equation (1.85) is of the form (1.60) where:

- $t_0 = 1$,
- $f(x) = x(t)\left\{1 + \dfrac{18}{1 + x^2(t)}\right\}$,
- $q(t) = 20 + \cos^2 t$,
- $p(t) = 2(\sin t \cos t)$,
- $r(t) = 1 + \cos^2 t$,
- $\psi(x) = \dfrac{2 + x^2}{1 + x^2}$.

First, one remarks that for all $x \in \mathbb{R}$ that

$$\psi(x) = 1 + \frac{1}{1 + x^2}, \tag{1.86}$$

one could deduce that

$$1 \leq \psi(x) \leq 2, \ \forall x \in \mathbb{R}, \tag{1.87}$$

which gives $C = 1$ and $C_1 = 2$.
We also remark that

$$f(x)/x = 1 + \frac{18}{1 + x^2(t)} \geq 1, \ \forall x \in \mathbb{R}. \tag{1.88}$$

Let $n = 3$ and $\rho(t) = 1$, for all $t \in [1, +\infty)$. Therefore the function Θ, given in Theorem 1.2.1, is defined by

$$\begin{aligned} \Theta(t) &= 20 + \cos^2 t - (1 - \frac{1}{2})\frac{4\sin^2 t \cos^2 t}{4(1 + \cos^2 t)} \\ &= 20 + \cos^2 t - \frac{\sin^2 t \cos^2 t}{2(1 + \cos^2 t)}. \end{aligned} \tag{1.89}$$

Inserting the previous value of Θ in the left–hand side on (1.84), we get

$$\begin{aligned}
\limsup_{t\to+\infty} t^{-2} \quad & \int_0^t (t-s)^2 \left(20+\cos^2 s - \frac{\sin^2 s\cos^2 s}{2(1+\cos^2 s)}\right) ds \\
- \quad & \int_0^t \frac{1}{2}(1+\cos^2 s)\left\{2+\left(\frac{\sin s\cos s}{1+\cos^2 s}\right)(t-s)\right\}^2 ds \\
\geq \quad & \limsup_{t\to+\infty} t^{-2}\int_0^t (t-s)^2\left(20-\frac{1}{2}\right) ds \\
- \quad & \int_0^t \frac{1}{2}(1+\cos^2 s)\left\{2+\left(\frac{\sin s\cos s}{1+\cos^2 s}\right)(t-s)\right\}^2 ds \\
\geq \quad & \limsup_{t\to+\infty} t^{-2}\int_0^t \left\{(t-s)^2\left(\frac{39}{2}\right)-(2+t-s)^2\right\} ds \\
= \quad & \limsup_{t\to+\infty} t^{-2}\left\{-\frac{39}{6}(t-s)^3-4s+2(t-s)^2+\frac{1}{3}(t-s)^3\right\}\Big|_0^t \\
= \quad & \limsup_{t\to+\infty} t^{-2}\left\{\frac{37}{6}t^3-4t-2t^2\right\} \\
= \quad & +\infty. \qquad (1.90)
\end{aligned}$$

Thanks to Corollary 1.2.3, equation (1.85) is oscillatory.

Chapter 2

On the zeros of the solutions of certain coupled hyperbolic problems with delays

2.1 Introduction

The study of the oscillatory behavior of the solutions of hyperbolic differential equations of neutral type had an increasing interest these last years.
It seems that the first attempt in this direction was made by Mishev and Bainov [25]. In [25], they have obtained sufficient conditions for the oscillation of all solutions of a class of neutral hyperbolic equations with conditions at the boundary of the Neumann type. Yosida [7] had obtained sufficient conditions which garantee the existence of bounded domains in which each solution of a neutral hyperbolic equation with boundary conditions of the Dirichlet, Neumann ou Robin type has a zero.

The oscillatory behavior of the solutions of differential equations of neutral type are studied recently by many authors (see for instance [28], [26], [45], and [46], and the references quoted therein.).
Let us cite for example the attempt of Parhi and Kirane [28] who studied the oscillatory behavior of equations of type

$$\begin{aligned} & u_{tt}(x,t) + \beta u_{tt}(x,t-\rho) + \gamma u_t(x,t-\theta) - [\delta \triangle \mu(x,t) + \alpha \triangle u(x,t-z)] \\ + \quad & c(x,t,u(x,t,u(x,t),u(x,t-\sigma))) = f(x,t) \end{aligned} \tag{2.1}$$

and other generalizations on the coefficients and on the delays, as well as the work of Parhi and Kirane [28], who studied oscillatory behavior of the following coupled problem of neutral type:

$$\begin{aligned} u_{tt}(x,t) &+ \delta_1 u_{tt}(x,t-\rho_1) + \gamma_1 u_t(x,t-\theta_1) - \{\alpha_1 \Delta u(x,t) + \alpha_2 \Delta u(x,t-z_1) + \alpha_3 \Delta v(x,t) \\ &+ \alpha_4 \Delta v(x,t-z_2)\} + c_1(x,t,u(x,t),u(x,t-\sigma_1),v(x,t),v(x,t-\sigma_2)) \\ &= f_1(x,t), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v_{tt}(x,t) &+ \delta_2 v_{tt}(x,t-\rho_2) + \gamma_2 v_t(x,t-\theta_2) - (\beta_1 \Delta v(x,t) + \beta_2 \Delta u(x,t-z_3) + \beta_3 \Delta v(x,t) \\ &+ \beta_4 \triangle v(x,t-z_4)) + c_2(x,t,u(x,t),u(x,t-\sigma_3),v(x,t),v(x,t-\sigma_4) \\ &= f_2(x,t). \end{aligned} \tag{2.3}$$

This work is concerned with a simple generalization of Parhi and Kirane [28]. Our essential contribution is to consider the coefficients and the delays as functions. Consequently we consider the following coupled problem

$$\begin{aligned} u_{tt}(x,t) &+ \delta_1(t) u_{tt}(x,\rho_1(t)) + \gamma_1(t) u_t(x,\theta_1(t)) \\ &- \{\alpha_1(t)\Delta u(x,t) + \alpha_2(t)\Delta u(x,z_1(t)) + \alpha_3 \Delta v(x,t) + \alpha_4 \Delta v(x,z_2(t))\} \\ &+ c_1(x,t,u(x,t),u(x,\sigma_1^1(t)),...,u(x,\sigma_1^r(t)),v(x,t),v(x,\sigma_2^1(t)), \\ &,...,v(x,\sigma_2^s(t))) = f_1(x,t) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} v_{tt}(x,t) &+ \delta_2(t) v_{tt}(x,\rho_2(t)) + \gamma_2(t) v_t(x,\theta_2(t)) \\ &- \{\beta_1(t)\Delta u(x,t) + \beta_2(t)\Delta u(x,z_3(t)) + \beta_3(t)\Delta v(x,t) + \beta_4(t)\Delta v(x,z_4)\} \\ &+ c_2(x,t,u(x,t),u(x,\sigma_3^1(t)),...,u(x,\sigma_3^k(t)),v(x,t),v(x,\sigma_4^1(t)), \\ &...,v(x,\sigma_4^l(t))) = f_2(x,t) \end{aligned} \tag{2.5}$$

where $(x,t) \in Q = \Omega \times (0,\infty)$ where Ω is a bounded domain of $\mathbb{R}^n$ with a sufficiently regular boundary Γ and Δ is the Laplacian in $\mathbb{R}^n$.

This problem is posed on Ω with one of the following types of boundary conditions:

- Neumann boundary conditions:

$$(B_1): \ \forall (x,t) \in \Gamma \times (0,\infty), \begin{cases} \nabla u(x,t) \cdot \mathbf{n}(x) = \psi_1(x,t), \\ \\ \nabla v(x,t) \cdot \mathbf{n}(x) = \psi_2(x,t). \end{cases} \tag{2.6}$$

- Robin boundary conditions:

$$(B_2): \ \forall (x,t) \in \Gamma \times (0,\infty), \begin{cases} \nabla u(x,t) \cdot \mathbf{n}(x) + \mu_2 u(x,t) = 0, \\ \\ \nabla v(x,t) \cdot \mathbf{n}(x) + \mu_2 v(x,t) = 0. \end{cases} \tag{2.7}$$

- Dirichlet boundary conditions:

$$(B_3): \ \forall (x,t) \in \Gamma \times (0,\infty), \begin{cases} u(x,t) = \tilde{\psi}_1(x,t), \\ \\ v(x,t) = \tilde{\psi}_2(x,t). \end{cases} \tag{2.8}$$

where ψ_i and $\widetilde{\psi}_i \left(i = \overline{1,2}\right)$ are real valued functions on $\Gamma \times (0,\infty)$, μ_1 and μ_2 are positive continuous functions on $\Gamma \times (0,\infty)$ and $\mathbf{n}$ denotes the unit normal vector to Γ outward to Ω. Some times, in order to simplify the notations, we denote by

$$\nabla u(x,t) \cdot \mathbf{n}(x) = \frac{\partial u}{\partial \nu}(x,t), \tag{2.9}$$

and

$$\nabla v(x,t) \cdot \mathbf{n}(x) = \frac{\partial v}{\partial \nu}(x,t). \tag{2.10}$$

In order to get the oscillation behavior, we assume that :

$$(H_1) \begin{cases} \delta_i, \sigma_i, \alpha_j \text{ and } \beta_j \in \mathcal{C}\left((0,\infty);(0,\infty)\right) \text{ for } i = \overline{1,2} \quad \text{and} \quad j = \overline{1,4} \\ c_1 \in \mathcal{C}\left(\overline{Q} \times \mathbb{R} \times \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^s; \mathbb{R}\right), \\ c_2 \in \mathcal{C}(\overline{Q} \times \mathbb{R} \times \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^\ell; \mathbb{R}). \end{cases}$$

$$(H_2)\begin{cases} \rho_i, \theta_i \in \mathcal{C}((0,\infty);\mathbb{R}) \quad \text{for} \quad i=\overline{1,2} \quad \text{and} \quad z_i \in \mathcal{C}((0,\infty);\mathbb{R}) \text{ for } \quad i=\overline{1,4} \\ \sigma_i^1, \sigma_j^2, \sigma_k^3, \sigma_L^4 \in \mathcal{C}((0,\infty);\mathbb{R}) \quad \text{for} \quad i=\overline{1,r} \quad \text{and} \\ j=\overline{1,s},\ k=\overline{1,k},\ L=\overline{1,\ell}, \\ \text{and for each } \ j: \lim\limits_{t\to+\infty} \rho_i(t) = \lim\limits_{t\to+\infty} \theta_i(t) = \lim\limits_{t\to+\infty} \sigma_i^j(t) = +\infty \\ \text{and } \rho_i(t), \theta_i(t), \sigma_i^j \le t, \forall t>0 \end{cases}$$

$$(H_3)\begin{cases} (i) \quad c_1\left(x,t,\xi,\xi_1,...,\xi_r,\eta,\eta_1,...,\eta_s\right) \left[\begin{array}{l} \ge 0 \text{ if } \ \xi \text{ and } \xi_i > 0, \forall\ i=\overline{1,r} \\ \\ \le 0 \text{ if } \ \xi \text{ and } \xi_i < 0, \forall\ i=\overline{1,r} \end{array}\right. \\ \\ (ii) \quad c_2\left(x,t,\xi,\xi_1,...,\xi_k,\eta,\eta_1,...,\eta_\ell\right) \left[\begin{array}{l} \ge 0 \ \ si \ \ \eta \text{ and } \eta_i > 0, \forall\ i=\overline{1,\ell} \\ \\ \le 0 \ \ si \ \ \eta \text{ and } \eta_i < 0, \forall\ i=\overline{1,\ell} \end{array}\right. \end{cases}$$

$$(H_4)\begin{cases} \alpha_2,\alpha_3,\alpha_4,\beta_4,\delta_1,\delta_2,\gamma_1,\gamma_2 \text{ are nonnegative real valued fiunctions} \\ \text{and} \\ \alpha_1 \ et \ \ \beta_3 \quad \text{are positive functions.} \end{cases}$$

$$(H_5)\begin{cases} \alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_4,\delta_1,\delta_2,\gamma_1,\gamma_2 \quad \text{are non–negative functions} \\ \text{and} \\ \alpha_1 \ et \ \ \beta_3 \ \text{ are positive functions.} \end{cases}$$

For our need we define:

$$\widehat{t}\,(s) = \min\left(\min_{1\le i\le 2}\left(\inf_{t\ge s}\rho_i\,(t)\right), \min_{1\le i\le z}\left(\inf_{t\ge s}\theta_i\,(t)\right), \min_{1\le i\le 4}\left(\inf_{t\ge t} z_i\,(t)\right); \min_{i,\partial_J}\left(\inf_{t\ge s}\left(\sigma_i^J\,(t)\right)\right)\right)$$

We say that the couple of functions (u,v) is a solution of ((2.4),(2.5)) with a boundary condition B_{i_0}, $i_0 \in \{1,2,3\}$ if the couple of functions (u,v) satisfies the coupled equation (2.4)–(2.5) and B_{i_0}.

A function $w(x,t)$ is said oscillatory in Q if w has a zero (or vanishes) in Q_a for each $a \ge 0$, with $Q_a = \Omega \times (a,\infty)$.

A solution (u,v) of the coupled problem (2.4)–(2.5) with one type of the boundary conditions $(B_i)\,, i=1,2,3$ is said to be oscillatory in Q if u or v oscillates.

A solution (u,v) of (2.4)–(2.5), with one type of the boundary conditions $(B_i)\,, i=1,2,3$ is said to be strongly oscillatory in Q if u,v oscillates at the some time.
We assume that:

$$(i) \qquad (\alpha_1+\beta_3)^2 \geq 4\left(\alpha_1\ \beta_3-\alpha_3\ \beta_1\right).$$

$$(ii) \qquad \alpha_1\ \beta_3>\beta_1\ \alpha_3$$

which ensures the total hyperbolicity of equations (2.4)–(2.5) (see [6]).

The following notations will be used in what follows.
For each

$$u,\ v\in \mathcal{C}^2\left(Q\right)\bigcap \mathcal{C}^2\left(\overline{Q}\right),$$

We write:

$$U\left(t\right)=\int_{\Omega} u\left(x,t\right)\ dx,\ V\left(t\right)=\int_{\Omega} v\left(x,t\right)\ dx,$$

$$\widetilde{U}\left(t\right)=\int_{\Omega} \varphi\left(x\right)\ u\left(x,t\right)\ dx,\ \widetilde{V}\left(t\right)=\int_{\Omega} \varphi\left(x\right) v\left(x,t\right)\ dx.$$

In addition, for all $i\in\{1,2\}$, we denote by

$$\Psi_i\left(t\right)=\int_{\Gamma} \psi_i\left(x,t\right)\ d\gamma(x),$$

$$\widetilde{\Psi}_i\left(t\right)=\int_{\Gamma} \widetilde{\Psi}_i\left(x,t\right)\frac{\partial\varphi}{\partial v}\ d\gamma(x)$$

$$F_i\left(t\right)=\int_{\Omega} f_i\left(x,t\right)\ dx$$

$$\widetilde{F_i}\left(t\right)=\int_{\Omega} \varphi\left(x\right) f_i\left(x,t\right)\ dx,$$

where $d\gamma$ denotes the integration symbol for $(n-1)$–dimensional Lebesgue measure on the considered hyperplane.

2.2 The coupled hyperbolic problem

2.2.1 Existence theorem of oscillations

The main results of our paper are the following Theorems:

Theorem 2.2.1. *We assume that the conditions $(H_1),(H_2),(H_3),(H_5)$ hold. If in addition*

$$(A_1)\quad \begin{cases} \lim\limits_{t\to\infty}\inf \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[F_1(s)+\alpha_1\Psi_1(s)+\alpha_2\Psi_1(s-z_1(s))+\right. \\ \left. +\ \alpha_3\Psi_2(s)+\alpha_4\Psi_2(s-z_2(s))\right]\, ds=-\infty, \end{cases}$$

and

$$(A_2)\quad \begin{cases} \lim\limits_{t\to\infty}\sup \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[F_1(s)+\alpha_1\Psi_1(s)+\alpha_2\Psi_1(s-z_1(s))+\right. \\ \left. +\alpha_3\Psi_2(s)+\alpha_4\Psi_2(s-z_2(s))\right]\, ds=\infty, \end{cases}$$

or if

$$(A_3)\quad \begin{cases} \lim\limits_{t\to\infty}\inf \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[F_2(s)+\beta_1\Psi_1(s)+\beta_2\Psi_1(s-z_3(s))+\right. \\ \left. +\ \beta_3\Psi_2(s)+\beta_4\Psi_2(s-z_4(s))\right]\, ds=-\infty \end{cases}$$

and

$$(A_4)\quad \begin{cases} \lim\limits_{t\to\infty}\sup \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[F_2(s)+\beta_1\Psi_1(s)+\beta_2\Psi_1(s-z_3(s))+\right. \\ \left. +\ \beta_3\Psi_2(s)+\beta_4\Psi_2(s-z_4(s))\right]\, ds=\infty \end{cases}$$

for any $t_0\geq 0$, then each solution (u,v) of the coupled problem ((2.4),(2.5)) with (B_1) oscillates in Q.

Theorem 2.2.2. *We assume that the conditions $(H_1),(H_2),(H_5),(A_1)-(A_4)$ hold. Then each solution of problem ((2.4),(2.5)) , (B_1) oscillates in Q.*

Theorem 2.2.3. *Assume that the conditions $(H_1),(H_2),(H_4)$ are satisfied and $(A_1)-(A_4)$ hold, then each solution of problem ((2.4),(2.5)) , (B_2) oscillates in Q.*

Theorem 2.2.4. *We assume that the conditions $(H_1),(H_2),(H_5)$ are satisfied. If*

$$\lim_{t\to\infty}\inf\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[\widetilde{F}_1(s)-\alpha_1\widetilde{\Psi}_1(s)-\alpha_2\widetilde{\Psi}_1(s-z_1(s))-\right.$$

$$\left.-\ \alpha_3\widetilde{\Psi}_2(s)-\alpha_4\widetilde{\Psi}_2(s-z_2(s))\right]\,ds=-\infty,$$

$$\lim_{t\to\infty}\inf\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[\widetilde{F}_1(s)-\beta_3\widetilde{\Psi}_1(s)-\beta_2\widetilde{\Psi}_1(s-z_3(s))-\right.$$

$$\left.-\ \beta_3\widetilde{\Psi}_2(s)-\beta_4\widetilde{\Psi}_2(s-z_4(s))\right]\,ds=-\infty,$$

$$\lim_{t\to\infty}\sup\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[\widetilde{F}_1(s)-\alpha_1\widetilde{\Psi}_1(s)-\alpha_2\widetilde{\Psi}_1(s-z_1(s))-\right.$$

$$\left.-\ \alpha_3\widetilde{\Psi}_2(s)-\alpha_4\widetilde{\Psi}_2(s-z_2(s))\right]\,ds=\infty,$$

and

$$\lim_{t\to\infty}\sup\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\ \left[\widetilde{F}_2(s)-\beta_3\widetilde{\Psi}_1(s)-\beta_2\widetilde{\Psi}_1(s-z_3(s))-\right.$$

$$\left.-\ \beta_4\widetilde{\Psi}_2(s-z_4(s))\right]\,ds=\infty,$$

for each $t_0\geq 0$, then each solution of problem (2.4)–(2.5), (B_3) oscillates in Q.

In order to prove Theorems 2.2.2–2.2.4, we need to use the following technical Lemmata:

Lemma 2.2.1. *We assume that the conditions $(H_1),(H_2),(H_3)(i),(H_5)$ are satisfied. If (u,v) is a solution of problem ((2.4),(2.5)), (B_1) and $u(x,t)>0$ in Q_{t_0}. Then the function U satisfies the following differential inequality of neutral type:*

$$y''(t)+\delta_1\ y''(\rho_1(t))+\gamma_1\ y'(\theta_1(t))\leq F_1(t)+$$
$$+\ \alpha_1\Psi_1+\alpha_2\Psi_1(z_1(t))+\alpha_3\Psi_2(t)+\alpha_4\Psi_2(z_2(t)) \tag{2.11}$$

for a sufficiently large t.

Proof Integrating equation (2.4) over the domain Ω, we get:

$$U''+\delta_1(t)\,U''(\rho_1(t))+\gamma_1(t)\,U'(\theta_1(t))-$$

$$[\alpha_1(t)\int_\Omega \triangle\ u(x,t)\ dx+\alpha_2(t)\int_\Omega \triangle\ u\left(x,z_1(t)\right)\ dx+$$

$$+\alpha_3(t)\int_\Omega \triangle\ v(x,t)\ dx+\alpha_4(t)\int_\Omega \triangle\ v(x,z_2(t))\ dx]+$$

$$+\int_\Omega c_1(x,t,u(x,t),u(x,\sigma_1^1(t)),...,u(x,\sigma_1^r(t)),v(x,t),v(x,\sigma_2^1(t)),...,v(x,\sigma_2^s(t)))dx=F_1\left(t\right)$$

An integration by parts yields:

$$\begin{aligned}\int_\Omega \triangle\ u\left(x,t\right)\ dx &= \int_\Gamma \frac{\partial u}{\partial\nu}(x,t)\ d\gamma(x)\\ &= \int_\Gamma \psi_1\left(x,t\right)\ d\gamma(x)\\ &= \Psi_1\left(t\right)\end{aligned} \tag{2.12}$$

and

$$\int_\Omega \triangle\ u\left(x,z_1\left(t\right)\right)\ dx=\Psi_1\left(z_1\left(t\right)\right).$$

Assumption (H_2) implies that $\lim\limits_{t\to+\infty}\sigma_1^1\left(t\right)=+\infty$, and then there exists some $A>0$ such that for all $t>A$, we have

$$\sigma_1^1(t)>t_0. \tag{2.13}$$

This together with the hypothesis $u(x,t)>0$ yields that

$$u(x,\sigma_1^1(t))>0, \tag{2.14}$$

for a sufficiently large t.

By the same manner, we justify that

$$u(x,\sigma_1^i(t))>0,\ \forall i=\overline{1,r}, \tag{2.15}$$

for a sufficiently large t.

Inequality (2.15) together with hypothesis $(H_3)(i)$ implies that

$$c_1(x,t,u(x,t),u(x,\sigma_1^1(t)),...,u(x,\sigma_1^r(t)),v(x,t),v(x,\sigma_2^1(t)),...,v(x,\sigma_2^s(t)))>0, \tag{2.16}$$

for a sufficiently large t.
Which implies

$$U''(t)+\delta_1(t)U''(\rho_1(t))+\delta_1(t)U'(\theta_1(t))-\alpha_1(t)\Psi_1(t)-\alpha_2(t)\Psi_1(z_1(t))-$$

$$\alpha_3(t)\Psi_2(t)-\alpha_4(t)\Psi_2(z_2(t))\leq F_1(t),$$

and therefore

$$U''(t)+\delta_1(t)U''(\rho_1(t))+\gamma_1(t)U'(\theta_1(t))\leq$$

$$F_1(t)+\alpha_1\Psi_1(t)+\alpha_2\Psi_1(z_1(t))+\alpha_3\Psi_2(t)+\alpha_4\Psi_2(z_2(t)).$$

■

Lemma 2.2.2. *We assume that the conditions $(H_1),(H_2),(H_3)(ii),(H_5)$ are satisfied, if (u,v) is a solution of ((2.4),(2.5)),(B_1) such that $v(x,t)>0$ on Q_{t_0}, then the solution $V(t)$ satisfies the following neutral ordinary differential inequality:*

$$y''(t)+\delta_2(t)\ y''(\rho_2(t))+\gamma_2\ y'(\theta_2(t))\leq F_2(t)+\beta_1\Psi_1(t)+$$
$$\beta_2\Psi_1(z_3(t))+\beta_3(t)\Psi_2(t)+\beta_4(t)\Psi_2(z_4(t)) \tag{2.17}$$

for a sufficiently large t.

The proof of this Lemma is similar to the one of Lemma 2.2.1, so we omit it.

Lemma 2.2.3. *Let us suppose that the conditions $(H_1),(H_2),(H_3)(i),(H_4)$ are satisfied, and (u,v) is a solution of the coupled problem ((2.4),(2.5)), (B_2) such that $u(x,t)>0$ and $v(x,t)>0$ on Q_{t_0} for some $t_0>0$.*
Then the solution U satisfies inequality (2.11) for a sufficiently large t.
If $u<0$ and $v<0$ on Q_{t_0}, then the function $-U$ satisfies the following neutral ordinary differential inequality:

$$y''(t)+\delta_1(t)\ y''(\rho_1(t))+\gamma_1\ y'(\theta_1)\leq$$

$$-\left[-F_1(t)+\alpha_1(t)\Psi_1(t)+\alpha_2(t)\Psi_1(z_1(t))+\alpha_3(t)\Psi_2(t)+\alpha_4(t)\Psi_2(z_2(t))\right] \quad (2.18)$$

for a sufficiently large t.

Lemma 2.2.4. *Assume that the conditions (H_1), (H_2), $(H_3)(ii)$, (H_4) are satisfied, if (u,v) is a solution of problems ((2.4),(2.5)),(B_2).*
If $u<0$ and $v>0$ on a Q_{t_0}, then $V(t)$ satisfies the ordinary differential inequality (2.17) for a sufficiently large t.
If $u>0$ and $v<0$ on Q_{t_0}, then $-V(t)$ satisfies inequality:

$$y''(t)+\delta_2(t)\ y''(\rho_2(t))+\gamma_2\ y'(\theta_2(t))\leq$$

$$-\left[-F_2(t)+\beta_1(t)\Psi_1(t)+\beta_2(t)\Psi_1(z_3(t))+\beta_3(t)\Psi_2(t)+\beta_4(t)\Psi_2(z_4(t))\right] \quad (2.19)$$

for sufficiently large t.

The proofs of Lemmata 2.2.3 and 2.2.4 are similar to that of Lemma 2.2.1, so we omit them.

Lemma 2.2.5. *Let us suppose that the conditions (H_1), $(H_3)(i)$, (H_4) are satisfied, if (u,v) is a solution of problem ((2.4),(2.5)),(B_3). If $u(x,t)>0$ and $v(x,t)>0$ on a Q_{t_0}, then the function $\widetilde{U}(t)$ verifies the following differential inequality of neutral type:*

$$y''(t)+\delta_1(t)\ y''(\rho_1(t))+\gamma_1\ y'(\theta_1(t))\leq$$

$$\widetilde{F}_1(t)-\alpha_1(t)\widetilde{\Psi}_1(t)-\alpha_2(t)\widetilde{\Psi}_1(z_1(t))-\alpha_3(t)\widetilde{\Psi}_2(t)+\alpha_4(t)\widetilde{\Psi}_2(z_2(t)) \quad (2.20)$$

for a sufficiently large t. If $u(x,t)>0$ and $v(x,t)<0$ on a Q_{t_0}, then the function $-\widetilde{U}(t)$ satisfies the following inequality:

$$y''(t)+\delta_1(t)\ y''(\rho_1(t))+\delta_1\ y'(\theta_1(t))$$

$$\leq -\widetilde{F}_1(t)-\alpha_1(t)\widetilde{\Psi}_1(t)-\alpha_2(t)\widetilde{\Psi}_1(z_1(t))-\alpha_3(t)\Psi_2(t)+\alpha_4(t)\widetilde{\Psi}_2(z_2(t)) \quad (2.21)$$

Proof Multiplying both sides of equation (2.4) by the function $\varphi(x)$, integrating the result over Ω, using (H_2) and $(H_3)(ii)$, to obtain:

$$\widetilde{U}''(t) + \delta_1(t)\,U''(\rho_1(t)) + \gamma_1(t)\,\widetilde{U}'(\theta_1(t))$$

$$\leq \widetilde{F}_1(t) + \alpha_1(t)\int_\Omega \varphi(x)\triangle\ u(x,t)\ dx + \alpha_2(t)\int_\Omega \varphi(x)\triangle\ u(x,z_1(t))\ dx +$$

$$+\alpha_3(t)\int_\Omega \varphi(x)\triangle\ v(x,t)\ dx + \alpha_4(t)\int_\Omega \varphi(x)\triangle\ v(x,z_2(t))\ dx,$$

for a sufficiently large t.
An application of the Green's formula yields, since $\varphi = 0$ in Γ and $\Delta\varphi = -\lambda_1\varphi$:

$$\int_\Omega \varphi(x)\triangle\ u(x,t)\ dx = \int_\Gamma \varphi(x)\nabla u(x)\cdot \mathbf{n}(x)d\gamma(x) - \int_\Gamma u(x)\nabla\varphi(x)\cdot\mathbf{n}(x)d\gamma(x) + \int_\Omega u(x)\Delta\varphi(x)dx$$

$$= -\int_\Gamma u(x)\nabla\varphi(x)\cdot\mathbf{n}(x)d\gamma(x) - \lambda_1\int_\Omega u(x)\varphi(x)dx$$

$$= -\widetilde{\Psi}_1(t) - \lambda_1\tilde{U}(t)$$

and then, thanks to (H_1), $u > 0$, $\varphi > 0$, and $\lambda_1 > 0$:

$$\widetilde{U}''(t) + \delta_1(t)\,U''(\rho_1(t)) + \gamma_1(t)\,\widetilde{U}'(\theta_1(t))$$

$$\leq \widetilde{F}_1(t) - \alpha_1(t)\,\widetilde{\Psi}_1(t) - \alpha_2(t)\,\widetilde{\Psi}_1(z_1(t)) - \alpha_3(t)\,\Psi_2(t) + \alpha_4(t)\,\widetilde{\Psi}_2(z_2(t))\,.$$

Hence the first part of the Lemma is proved. The second part of the Lemma can handled as above.

■

Lemma 2.2.6. *Assume that the conditions $(H_1),(H_3)(ii),(H_4)$ are satisfied, if (u,v) is a solution of problem ((2.4),(2.5)),(B_3). If $u(x,t) < 0$ and $v(x,t) > 0$ on some Q_{t_0}, then $\widetilde{V}(t)$ verifies the following differential inequality of neutral type:*

$$y'' + \delta_2(t)\ y''(\rho_2(t)) + \delta_2\ y'(\theta_2(t))$$

$$\leq \widetilde{F}_2(t) - \beta_1(t)\widetilde{\Psi}_1(t) - \beta_2(t)\widetilde{\Psi}_1(z_3(t)) - \beta_3(t)\widetilde{\Psi}_2(t) - \beta_4(t)\widetilde{\Psi}_2(z_4(t)) \qquad (2.22)$$

for sufficiently large t.
If $u(x,t) > 0$ and $v(x,t) < 0$ on some Q_{t_0}, then the function $-\widetilde{V}(t)$ satisfies the following inequality:

$$y''(t) + \delta_2(t)\ y''(\rho_2(t)) + \delta_2\ y'(\theta_2(t)) \leq$$

$$-\left[\widetilde{F}_2(t) - \beta_1(t)\widetilde{\Psi}_1(t) - \beta_2(t)\widetilde{\Psi}_1(z_3(t)) - \beta_3(t)\widetilde{\Psi}_2(t) - \beta_4(t)\widetilde{\Psi}_2(z_4(t))\right]$$

for a sufficiently large t.

The proof of this Lemma is similar to that one of Lemma 2.2.5, so we omit it.

Theorem 2.2.5. *Assume that the conditions $(H_1) - (H_5)$ are fulfilled. If the differential inequatities (2.11) and (2.18) or the differential inequalities (2.17) and (2.19) do not admit positive solutions for a sufficiently large t, then all solutions of problem ((2.4),(2.5)),(B_1) oscillate in Q.*

Proof Let (u, v) be a solution of problem ((2.4),(2.5)),(B_1) which does not oscillates in Q. Then, there exists t_0 such that

$$u(x,t) \neq 0 \text{ and } v(x,t) \neq 0 \text{ in } Q_{t_0}.$$

Assume that (2.11) and (2.18) do not admit positive solutions for a sufficently large t.
Since $u(x,t) \neq 0$ in Q_{t_0}, one has $u(x,t) > 0$ or $u(x,t) < 0$ in Q_{t_0}.
If $u(x,t) > 0$ in Q_{t_0}, then thanks to Lemma 2.2.1, U is a positive solution of (2.11) for a sufficently large t, which is a contradiction.
If $u(x,t) < 0$ on Q_{t_0}, we set $\hat{u}(x,t) = -u(x,t)$ on Q, and then $(\hat{u}, v)$ is a solution of the following problem:

$$u_{tt}(x,t) + \delta_1(t)\ u_{tt}(x,\rho_1(t)) + \gamma_1(t)u_t(x,\theta_1(t)) -$$

$$[\alpha_1(t) \triangle u\ (x,t) + \alpha_2(t) \triangle u(x,z_1(t)) - \alpha_3(t) \triangle v(x,t) - \alpha_4(t) \triangle v(x,z_2(t))]$$

$$-\ c_1\left(x,t,-u(x,t), -\ u\left(x,\sigma_1^1(t)\right)\right), ..., -u\left(x,\sigma_1^r(t)\right), v(x,t),$$

$$v\left(x,\sigma_2^1(t)\right),...,v\left(x,\sigma_2^s(t)\right)=-f_1(x,t)$$

$$v_{tt}(x,t)+\delta_2(t)\ v_{tt}(x,\rho_2(t))+\gamma_2\ v_t(x,\theta_2(t))-$$

$$[\beta_1\triangle\ u\ \ (x,t)+\beta_2(t)\triangle\ u(x,z_3(t))-\beta_3\triangle\ v(x,t)-\beta_4(t)\,\Delta\ v(x,z_4(t))]$$

$$-\ c_2\left(x,t,-\ \hat{u}(x,t),-\ \hat{u}\left(x,\sigma_2^1(t)\right)\right),...,-\ \hat{u}\left(x,\sigma_3^r(t)\right),v(x,t),$$

$$v\left(x,\sigma_4^1(t)\right),...,v\left(x,\sigma_4^s(t)\right)=f_2(x,t),$$

and

$$\frac{\partial\hat{u}}{\partial\nu}=-\psi_1\quad\text{and}\quad\frac{\partial v}{\partial\nu}=\psi_2\quad\text{on}\quad\Gamma\times(0,\infty).$$

Proceeding as in the proof of Lemma 2.2.1 one may prove that $\hat{U}$ is a positive solution of (2.18), where

$$\hat{U}=\int_{\Omega}\hat{u}(x,t)dx.$$

we get a contradiction.
If the differential inequalities (2.17) and (2.19) do not admit positive solutions for large t, then we proceed as above considering $v(x,t)\neq 0$ in some Q_{t_0} to arrive at necessary contradictions. This completes the proof of the Theorem.

■

Theorem 2.2.6. *Assume that conditions* $(H_1)-(H_5)$ *are satisfied. Suppose that none of the differential inequalities (2.11), (2.17), (2.18) and (2.19) admit a positive solution for large* t*. Then all solutions of problem ((2.4),(2.5)),* (B_1) *oscillates strongly in* Q.

Proof Assume the contrary, so there exists a solution $(u(x,t),v(x,t))$ of problem ((2.4),(2.5)) (B_1) which does not oscillate strongly in Q. This means that u or v does not oscillate.
If u does not oscillate on Q, then there exists some t_0 such that $u(x,t)>0$ or $u(x,t)<0$ in Q_{t_0}. If $u(x,t)>0$ in Q_{t_0}, then thanks to Lemma 2.2.1 it follows that $U(t)=\int_\Omega u(x,t)dx$ is a positive solution of inequality (2.11), a contradiction.
If $u(x,t)<0$ in Q_{t_0}, then by setting $\hat{u}(x,t)=-u(x,t)$ and proceeding as in Lemma 2.2.1 it

may be proven that $\hat{U} = \int\limits_{\Omega} \hat{u}\,(x,t)\,dx$ is a positive solution of (2.18), contradiction. Similar contradictions may be obtained thanks to 2.2.2 if $v(x,t)$ does not osciallate in Q. The proof of the Theorem is complete. ■

Theorem 2.2.7. *Let us suppose that $(H_1)-(H_4)$ are satisfied, if inequalities (2.11), (2.17), (2.18) and (2.19) do not admit positive solutions for sufficiently large t, then each solution of problem ((2.4),(2.5)), (B_2) oscillates in Q.*

The proof of this Theorem is similar to that of the previous Theorem.
Similarly we prove:

Theorem 2.2.8. *Assume that $(H_1)-(H_4)$ are satisfied. If the inequalities (2.11), (2.17), (2.18) and (2.19) do not admit positive solutions for sufficiently large t, then each solution of problem ((2.4),(2.5)), (B_3) oscillates in Q.*

In the previous section, we remarked that the oscillation of problem ((2.4),(2.5)) with one of the boundary conditions (B_i), where $i \in \{1,2,3\}$, depends on the fact if (2.23), given below, admit or not positive solutions.
For this reason, we will devote the following sections to give some sufficient conditions, see Lemma 2.2.7 given below, in order that inequality (2.23) does not admit positive solution for large t.
Let

$$y''(t) + \lambda_1(t)\;\; y''(\rho(t)) + \lambda_2(t)\;\; y'(\theta(t)) \leq g(t) \tag{2.23}$$

where

- ρ is a positive increasing function
- ρ has an inverse ξ such that ξ' is an increasing function
- $\lim_{t\to+\infty} \rho(t) = +\infty$
- the same properties satisfied by ρ should be satisfied by θ. The inverse of θ will be denoted by χ.

In addition, λ_1 and λ_2 are decreasing positive functions and there derivatives are increasing functions.

Lemma 2.2.7. *If the following limit holds*

$$\varlimsup_{t\to+\infty}\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\;g(s)\,ds=-\infty \tag{2.24}$$

for each $t_0>0$, then inequality (2.23) does not admit positive solution for large t.

Proof Assume the contrary, which means that there exists a positive solution $y(t)$ for (2.23) for some $t>t_0>0$.
Let us condsider $t>t_1>t_0$ such that $\rho(t_1)>t_0$ and $\theta(t_1)>t_0$ (this is possible since $\lim_{t\to+\infty}\rho(t)=+\infty$ and $\lim_{t\to+\infty}\theta(t)=+\infty$).
Integrating (2.23) over (t_1,t) to get

$$\int_{t_1}^{t}y''(s)\;ds+\int_{t_1}^{t}\lambda_1(s)\,y''(\rho(s))\;ds+\int_{t_1}^{t}\lambda_2(s)\,y''(\theta(s))\;ds\le\int_{t_1}^{t}g(s)\,ds. \tag{2.25}$$

Integrations by parts, the previous inequality yields that

$$y'(t)+M(\rho(t))\,y'(\rho(t))-M'(\rho(t))\,y(\rho(t))+\int_{\rho(t_1)}^{\rho(t)}M''(s)\;y(s)\;ds+$$

$$+\,m(\theta(t))\;y'(\theta(t))-m'(\theta(t))\;y(\theta(t))+\int_{\theta(t_1)}^{\theta(t)}m''(s)\;y\;(s)\;ds+c_1\le\int_{t_1}^{t}g(s)\;ds$$

with $c_1\in\mathbb{R}$, $M(s)=\lambda_1(\xi(s))\xi'(s)$ and $m(s)=\lambda_2(\chi(s))\chi'(s)$.
A second integration from t to t_1, yields:

$$y(t)+\int_{t_1}^{t}M(\rho(\alpha))\;y'(\rho(\alpha))\;d\alpha-\int_{t_1}^{t}M'(\rho(\alpha))\,y(\rho(\alpha))\;d\alpha+$$

$$\int_{t_1}^{t}\left(\int_{\rho(t_1)}^{\rho(\alpha)}M''(s)\;y(s)\;ds\right)\;d\alpha+\int_{t_1}^{t}m(\theta(\alpha))\;y'(\theta(\alpha))\;d\alpha-\int_{t_1}^{t}m'(\theta(\alpha))\;y'(\theta(\alpha))\;d\alpha+$$

$$+\left(\int_{\theta(t_1)}^{\theta(\alpha)}m\text{''}(s)\;y(s)\;ds\right)d\alpha+\;c_1(t-t_1)\le\int_{t_1}^{t}(t-s)\,g(s)\;ds,$$

which yields

$$y(t) + H(\rho(t))\, y(\rho(t)) + c - \int_{\rho(t_1)}^{\rho(t)} H'(x)\ y(x)\ dx - \int_{t_1}^{t} M'(\rho(\alpha))\, y(\rho(\alpha))\ d\alpha +$$

$$+\int_{t_1}^{t} \left(\int_{\rho(t_1)}^{\rho(\alpha)} M"(s)\ y(s)\ ds \right) d\alpha + h(\theta(t))\ y(\theta(t)) - \int_{\theta(t_1)}^{\theta(t)} h'(x)\ y(x)\ dx -$$

$$-\int_{t_1}^{t} m'(\theta(\alpha))\ y(\theta(\alpha))\ d\alpha + \ c_1(t - t_1) \leq \int_{t_1}^{t} (t-s)\, g(s)\ ds$$

with

$$\left(\begin{array}{l} M(x) = \lambda_1(\rho^{-1}(x)), \\ \\ m(x) = \lambda_2(\theta^{-1}(x)), \end{array} \right. \quad \text{and} \quad \left(\begin{array}{l} H(x) = M(x)\left(\rho^{-1}(x)'\right), \\ \\ h(x) = m(x)\left((\theta^{-1}(x))'\right). \end{array} \right.$$

We have

$$\left(\begin{array}{c} H(\rho(t)) \geq 0 \\ \text{and} \\ h(\theta(t)) \geq 0 \end{array} \right. \quad \text{and} \quad \left(\begin{array}{l} H \ \text{ decreases} \Rightarrow H' \leq 0 \\ \\ h \ \text{ decreases} \Rightarrow h' \leq 0 \end{array} \right.$$

Now:

$$M'(x) = \left(\rho^{-1}(x)\right)' \lambda_1' \left(\rho^{-1}(x)\right).$$

and then

$$M''(x) = \left(\rho^{-1}(x)\right)'' \lambda_1' \left(\rho^{-1}(x)\right) + \left[\left(\rho^{-1}(x)'\right)\right]^2 \lambda_1" \left(\rho^{-1}(x)\right) \leq 0$$

By the same way, we justify that

$$m'' \leq 0. \tag{2.26}$$

Therefore

$$c\,(t-t_1)+c_2 \le \int\limits_{t_1}^{t} (t-s)\ g\,(s)\,ds$$

$$\Longrightarrow c_1 + \frac{c_2}{t-t_1} \le \int\limits_{t_1}^{t} (t-s)\ g\,(s)\,ds$$

$$\Longrightarrow \varliminf_{t\to+\infty} \frac{1}{t-t_1} \int\limits_{t_1}^{t} (t-s)\ g\,(s)\,ds \ge c$$

■

Come back to prove Theorems 2.2.1, 2.2.2, 2.2.3, and 2.2.4.

Proof of Theorem 2.2.1. It follows from Lemma 2.2.7 and Theorem 2.2.5.
Proof of Theorem 2.2.2. It follows from Lemma 2.2.7 and Theorem 2.2.6.
Proof of Theorem 2.2.3. It follows from Lemma 2.2.7 and Theorem 2.2.7.
Proof of Theorem 2.2.4 . We use Lemma 2.2.7 and Theorem 2.2.8.

Remark 2.2.1. *If*

$$\varliminf_{t\to+\infty} \frac{1}{t} \int\limits_{0}^{t} (t-s)g(s)ds = -\infty,$$

then:

$$\varliminf_{t\to+\infty} \frac{1}{t-t_0} \int\limits_{t_0}^{t} (t-s)\ g\,(s)\ ds = -\infty.$$

Indeed

$$\frac{1}{t-t_0} \int\limits_{t_0}^{t} (t-s)\ g\,(s)\ ds = \frac{1}{t-t_0} \int\limits_{0}^{t} (t-s)\ g\,(s)\ ds - \int\limits_{0}^{t_0} (t-s)\,g\,(s)\ ds\ \frac{1}{t-t_0}$$

now:

$$\frac{1}{t-t_0} \int\limits_{t_0}^{t_0} (t-s)\ g\,(s)\ ds = \frac{t}{t-t_0} \int\limits_{0}^{t_0} g\,(s)\ ds - \int\limits_{0}^{t_0} s\ g\,(s)\ ds\ \frac{1}{t-t_0} \xrightarrow{t\to+\infty} \int\limits_{0}^{t_0} g\,(s)\ ds$$

$$\Longrightarrow \varliminf_{t\to+\infty} \frac{1}{t-t_0} \int\limits_{t_0}^{t} (t-s)\ g\,(s)\ ds = \varliminf_{t\to+\infty} \frac{1}{t-t_0}.\frac{1}{t} \int\limits_{0}^{t} (t-s)\ g\,(s)\ ds$$

$$= \lim_{t\to+\infty} \frac{1}{t} \int_0^t (t-s)\ g(s)\ ds = -\infty.$$

The following examples, even not describing phenomena of physics, of elasticity or other sciences, though illustrate our results .

2.2.2 Examples

Example 1

Let us consider the problem:

$$u_{tt}(x,t) + \frac{1}{2}\ u_{tt}(x,t-\pi) + u_t(x,t-\pi) -$$

$$-\left[\triangle\ u(x,t) + \triangle\ u(x,t-\pi) + \triangle\ v(x,t) + \triangle\ v\left(x,t-\frac{\pi}{2}\right)\right] +$$

$$+\ u(x,t) + u(x,t-\pi) = 2\left(e^{-x}-1\right)\ e^t \sin t \sin x + \left(2-3e^{-x}\right)\ e^t \cos t \sin x +$$

$$+\ e^t \sin t \cos x - e^{-\frac{x}{2}} e^t \cos t \cos x \tag{2.27}$$

and

$$v_{tt}(x,t) + \frac{1}{2}\ v_{tt}(x,t-\pi) + v_t(x,t-\pi)$$

$$-\left[\triangle\ u(x,t) + \triangle\ u(x,t-\pi) + \triangle\ v(x,t) + \triangle\ v\left(x,t-\frac{\pi}{2}\right)\right]$$

$$+\ v(x,t) + v\left(x,t-\frac{\pi}{2}\right) = 2\left(1-e^{-\pi}-e^{-\frac{\pi}{2}}\right)\ e^t \cos t \cos x$$

$$+\left(2-e^{-\pi}\right) e^t \sin t \cos x + (1-e^{-\pi}) e^t \cos t \sin x \tag{2.28}$$

$(x,t) \in (0,\pi) \times (0,\pi)$,
with boundary conditions:

$$-\ u_x(0,t) = u_x(\pi,t) = -e^t\ cos\ t \tag{2.29}$$

and

$$-\ v_x(0,t) = v_x(\pi,t) = 0$$

So

$$\Omega = (0, \pi), \psi_1(x,t) = -\ e^t \cos t \text{ and } \psi_2(x,t) = 0.$$

Therefore

$$\Psi_1(t) = -\ 2\ e^t \cos t\ and \Psi_2(t) = 0, t > 0$$

and

$$F_1(t) = \int_0^\pi f_1(x,t)\ dx = 4\left(e^{-\pi} - 1\right)\ e^t\ sin\ t + 2\left(2 - 3\ e^{-\pi}\right)\ e^t\ cos\ t$$

and

$$F_2(t) = \int_0^\pi f_2(x,t)\ dx = 2\left(1 - e^{-\pi}\right)\ e^t\ cos\ t.$$

Now

$$I_1(t) = \frac{1}{t}\int_0^t (t-s)\ \left[F_1(s) + \Psi_1(s) + \Psi_1(s-\pi)\right] ds$$

$$= \frac{1}{t}\int_0^t (t-s)\ \left[4\left(e^{-\pi} - 1\right)\ e^s \sin s + 2\left(1 - 2\ e^{-\pi}\right)\ e^s\ \cos s\right] ds$$

$$= 2\left(e^{-\pi} - 1\right)\frac{1}{t}\left(1 + t -\ e^t\ \cos t\right) + \left(1 - 2\ e^{-\pi}\right) 1t\left(-t + e^t\ sin\ t\right)$$

and

$$I_2(t) = \frac{1}{t}\int_0^t (t-s)\ \left[F_2(s) + \Psi_1(s) + \Psi_1(s-\pi)\right]\ ds = 0$$

It is clear that:

$$\liminf_{t\to+\infty} I_1(t) = -\infty \quad , \quad \limsup_{t\to+\infty} I_1(t) = +\infty.$$

According to Theorem 2.2.1, any solution of (2.27), (2.28), (2.29) oscillates on $(0, \pi) \times (0, \infty)$.

In particular the solution (e^t *cos t sin x,* e^t *sin t cos x)* oscillates.

Example 2

Let us consider the problem:

$$u_{tt}(x,t) + u_{tt}(x, t-\pi) + u_t\left(x, t - \frac{\pi}{2}\right)$$

$$-\left[\triangle\ u\left(x,t\right)+\triangle\ u\left(x,t-\pi\right)+\triangle\ v\left(x,t\right)+\triangle\ v\left(x,t-2\pi\right)\right]$$

$$+\ u\left(x,t\right)+u\left(x,t-\frac{\pi}{2}\right)=2\left(-1+e^{-\pi}+e^{-\frac{\pi}{2}}\right)e^{t}\sin t\sin x$$

$$+\left(2+e^{-\frac{x}{2}}-e^{-x}\right)e^{t}\cos t\sin x+2(t-\pi)\sin x \tag{2.30}$$

and

$$v_{tt}\left(x,t\right)+v_{tt}\left(x,t-\frac{\pi}{2}\right)+v_{t}\left(x,t-2\pi\right)$$

$$-\left[\triangle\ u\left(x,t\right)+\triangle\ u\left(x,t-\pi\right)+\triangle\ v\left(x,t\right)+\triangle\ v\left(x,t-\frac{2}{\pi}\right)\right]$$

$$+v\left(x,t\right)+v\left(x,t-2\pi\right)$$

$$=\left(\frac{2-5\pi}{2}\right)\sin x+4t\sin x+\left(e^{-\pi}-1\right)\ e^{t}\cos t\sin x \tag{2.31}$$

$(x,t)\in(0,\pi)\times(0,\infty)$ with the boundary conditions:

$$u=(x,t)=0\quad\text{and}\quad v(x,t)=0,\ \ (x,t)\in\{0,\pi\}\times(0,\infty). \tag{2.32}$$

So

$$\Omega=(0,\pi),\widetilde{\Psi}_1\left(x,t\right)=0\quad et\quad\widetilde{\Psi}_2\left(x,t\right)=0.$$

Therefore

$$\widetilde{\Psi}_1\left(t\right)=\widetilde{\Psi}_2\left(t\right)=0\quad\text{for}\quad t>0\quad\text{and}\quad\varphi\left(x\right)=sin\ x\quad\text{and}\quad\lambda_1=1,$$

so:

$$\widetilde{F}_1\left(t\right)=\int_0^{\pi}f_1\left(x,t\right)\sin\ x\ dx$$

$$= a_1 e^t \; sin\; t + a_2 \; e^t \; cos\; t + \pi t - \pi^2$$

with:

$$a_1 = \pi \left(-1 + e^{-\pi} + e^{-\frac{\pi}{2}} \right), \; a_2 = \frac{\pi}{2} \left(2 + e^{-\frac{\pi}{2}} - e^{-\pi} \right)$$

and:

$$\widetilde{F}_2 (t) = \int_0^\pi f_2 (x, t) \; sin\; x \; dx$$

$$= (2 - 5\pi) \frac{\pi}{4} + 2xt + \frac{\pi}{2} \left(e^{-\pi} - 1 \right) \; e^t \; cos\; t.$$

and:

$$I_1 (t) = \frac{1}{t} \int_0^t (t - s) \; \widetilde{F}_1 (s) \; ds$$

$$= \frac{e^t}{2t} \; \left[a_1 \left(e^{-t} + t\; e^{-t} + cos\; t \right) + a_2 \left(-t\; e^{-t} + sin\; t \right) + \frac{\pi}{3} \left(t^3 e^{-t} - \pi^2 t^2 e^{-t} \right) \right]$$

and:

$$I_2 (t) = \frac{e^t}{t} \; \left[\frac{\pi}{3} \; t^3 e^{-t} + (2 - 5\pi) \frac{x}{8} \; t^2 \; e^{-t} + \left(e^{-\pi} - 1 \right) \frac{\pi}{4} \left(-t\; e^{-t} + sin\; t \right) \right]$$

it is clear that:

$$\liminf_{t \to +\infty} I_1(t) = -\infty \quad and \quad \limsup_{t \to +\infty} I_1(t) = \infty,$$

$$\liminf_{t \to +\infty} I_2(t) = -\infty \quad and \quad \limsup_{t \to +\infty} I_2(t) = \infty$$

According to Theorem 2.2.4 any solution of problem (2.30), (2.31), (2.32) oscillates on $(0, \pi) \times (0, \infty)$. In particular $(e^t \; cos\; t\; sin\; x, \; t\; sin\; x)$ oscillates.

Chapter 3

Sufficient conditions for the oscillation of solutions to nonlinear second-order differential equations

3.1 Introduction

Kirane and Rogovchenko [20] studied the oscillatory solutions of the equation

$$\left[r(t)\psi(x(t))x'(t)\right]' + p(t)x'(t) + q(t)f(x(t)) = g(t), \quad t \geq t_0, \tag{3.1}$$

where $t_0 \geq 0$, $r(t) \in C^1([t_0, \infty); (0, \infty))$, $p(t) \in C([t_0, \infty); \mathbb{R})$, $q(t) \in C([t_0, \infty); (0, \infty))$, $q(t)$ is not identical zero on $[t_*, \infty)$ for some $t_* \geq t_0, f(x), \psi(x) \in C(\mathbb{R}, \mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$. Their results read as follows

Theorem 3.1.1. *Case $g(t) \equiv 0$: Assume that for some constants K, C, C_1 and for all $x \neq 0$, $f(x)/x \geq K > 0$ and $0 < C \leq \Psi(x) \leq C_1$. Let h, $H \in C(D, R)$, where $D = \{(t, s) : t \geq s \geq t_0\}$, be such that*

(i) $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ in $D_0 = \{(t, s) : t > s \geq t_0\}$

(ii) H has a continuous and non-positive partial derivative in D_0 with respect to the second variable, and

$$-\frac{\partial H}{\partial s} = h(t, s)\sqrt{H(t, s)}$$

for all $(t, s) \in D_0$.

If there exists a function $\rho \in C^1([t_0,\infty);(0,\infty))$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)\Theta(s) - \frac{C_1}{4}\rho(s)r(s)Q^2(t,s)]ds = \infty\,,$$

where

$$\Theta(t) = \rho(t)\Big(Kq(t) - \big(\frac{1}{C} - \frac{1}{C_1}\big)\frac{p^2(t)}{4r(t)}\Big),$$
$$Q(t,s) = h(t,s) + \big[\frac{p(s)}{C_1 r(s)} - \frac{\rho'(s)}{\rho(s)}\big]\sqrt{H(t,s)},$$

then (3.1) *is oscillatory.*

Theorem 3.1.2. *Case* $g(t) \neq 0$*: Let the assumptions of Theorem 3.1.1 be satisfied and suppose that the function* $g(t) \in C([t_0,\infty);\mathbb{R})$ *satisfies*

$$\int^{\infty} \rho(s)|g(s)|ds = N < \infty\,.$$

Then any proper solution $x(t)$ *of* (3.1)*; i.e, a non-constant solution which exists for all* $t \geq t_0$ *and satisfies* $\sup_{t\geq t_0}|x(t)| > 0$*, satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0.$$

Note that localization of the zeros is not given in the work by Kirane and Rogovchenko [20]. Here we intend to give conditions that allow us to localize the zeros of solutions to (3.1). Observe that in contrast to [20] where a Ricatti type transform,

$$v(t) = \rho\frac{r(t)\psi(x(t))x'(t)}{x(t)},$$

is used, here we simply use a usual Ricatti transform.

3.2 Solution of nonlinear second–order differential equation

3.2.1 Differential equation without a forcing term

Consider the second-order differential equation

$$\big[r(t)\psi(x(t))x'(t)\big]' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0 \tag{3.2}$$

where $t_0 \geq 0$, $r(t) \in C^1([t_0,\infty);(0,\infty))$, $p(t) \in C([t_0,\infty));\mathbb{R})$, $q(t) \in C([t_0,\infty));\mathbb{R})$, $p(t)$ and $q(t)$ are not identical zero on $[t_\star,\infty)$ for some $t_\star \geq t_0$, $f(x), \psi(x) \in C(\mathbb{R},\mathbb{R})$ and $\psi(x) > 0$ for $x \neq 0$. The next theorem follows the ideas in Nasr [27].

Theorem 3.2.1. *Assume that for some constants K, C, C_1 and for all $x \neq 0$,*

$$\frac{f(x)}{x} \geq K \geq 0, \tag{3.3}$$

$$0 < C \leq \psi(x) \leq C_1\,. \tag{3.4}$$

Suppose further there exists a continuous function $u(t)$ such that $u(a) = u(b) = 0$, $u(t)$ is differentiable on the open set (a,b), $a, b \geq t_\star$, and

$$\int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2C_1 r(t)(u')^2(t)\big]dt \geq 0\,. \tag{3.5}$$

Then every solution of (3.2) *has a zero in* $[a,b]$.

Proof Let $x(t)$ be a solution of (3.2) that has no zero on $[a,b]$. We may assume that $x(t) > 0$ for all $t \in [a,b]$ since the case when $x(t) < 0$ can be treated analogously. Let

$$v(t) = -\frac{x'(t)}{x(t)}, \quad t \in [a,b]. \tag{3.6}$$

Multiplying this equality by $r(t)\psi(x(t))$ and differentiate the result. Using (3.2) we obtain

$$\begin{aligned}(r(t)\psi(x(t))v(t))' &= -\frac{(r(t)\psi(x(t))x'(t))'}{x(t)} + r(t)\psi(x(t))v^2(t)\\ &= -p(t)v(t) + q(t)\frac{f(x(t))}{x(t)} + r(t)\psi(x(t))v^2(t)\\ &= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v^2(t) - 2\frac{p(t)}{r(t)\psi(x(t))v(t)}\big)\\ &\quad + q(t)\frac{f(x(t))}{x(t)}\\ &= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{(r(t)\psi(x(t))}\big)^2\\ &\quad - \frac{p^2(t)}{2r(t)\psi(x(t))} + q(t)\frac{f(x(t))}{x(t)}\,.\end{aligned}$$

Using (3.3)-(3.4) and the fact that

$$\frac{(r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))}\big)^2 \geq 0,$$

we have

$$(r(t)\psi(x(t))v(t))' \geq \frac{(r(t)\psi(x(t))}{2}v^2(t) - \frac{p^2(t)}{2Cr(t)} + Kq(t) \tag{3.7}$$

Multiplying both sides of this inequality by $u^2(t)$ and integrating on $[a,b]$. Using integration by parts on the left side, the condition $u(a)=u(b)=0$ and (3.4), we obtain

$$\begin{aligned}
0 &\geq \int_a^b \frac{r(t)\psi(x(t))}{2} v^2(t)u^2(t)dt + 2\int_a^b r(t)\psi(x(t))v(t)u(t)u'(t)dt \\
&\quad + \int_a^b Kq(t)u^2(t)dt - \int_a^b \frac{p^2(t)}{2Cr(t)}u^2(t)dt \\
&\geq \int_a^b \frac{r(t)\psi(x(t))}{2}(v^2(t)u^2(t) + 4v(t)u(t)u'(t))dt \\
&\quad + \int_a^b Kq(t)u^2(t)dt - \int_a^b \frac{p^2(t)u^2(t)}{2Cr(t)}dt \\
&\geq \int_a^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2 dt - 2\int_a^b r(t)\psi(x(t))u'^2(t)dt \\
&\quad + \int_a^b Kq(t)u^2(t)dt - \int_a^b \frac{p^2(t)}{2Cr(t)}u^2(t)dt \\
&\geq \int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2r(t)\psi(x(t))u'^2(t)\big]dt \\
&\quad + \int_a^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2 dt\,.
\end{aligned}$$

Now, from (3.4) we have

$$\begin{aligned}
0 \geq &\int_a^b \big[\big(Kq(t) - \frac{p^2(t)}{2Cr(t)}\big)u^2(t) - 2r(t)C_1u'^2(t)\big]dt \\
&+ \int_a^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2 dt.
\end{aligned}$$

If the first integral on the right-hand side of the inequality is greater than zero, then we have directly a contradiction. If the first integral is zero and the second is also zero then $x(t)$ has the same zeros as $u(t)$ at the points a and b; $(x(t) = ku^2(t))$, which is again a contradiction with our assumption. ■

Corollary 3.2.2. *Assume that there exist a sequence of disjoint intervals* $[a_n, b_n]$*, and a sequence of functions* $u_n(t)$ *defined and continuous an* $[a_n, b_n]$*, differentiable on* (a_n, b_n) *with* $u_n(a_n) = u_n(b_n) = 0$*, and satisfying assumption* (3.5)*. Let the conditions of Theorem 3.2.1. hold. Then* (3.2) *is oscillatory.*

3.2.2 Differential equation with a forcing term

Consider the differential equation

$$\big[r(t)\psi(x(t))x'(t)\big]' + p(t)x'(t) + q(t)f(x(t)) = g(t), \quad t \ge t_0 \tag{3.8}$$

where $t_0 \ge 0$, $g(t) \in C([t_0,\infty);\mathbb{R})$ $r(t) \in C^1([t_0,\infty);(0,\infty))$, $p(t) \in C([t_0,\infty));\mathbb{R})$, $q(t) \in C([t_0,\infty));\mathbb{R})$, $p(t)$ and $q(t)$ are not identical zero on $[t_\star,\infty[$ for some $t_\star \ge t_0$, $f(x),\psi(x) \in C(\mathbb{R},\mathbb{R})$ and $\psi(x) > 0$ for $x \ne 0$. Assume that there exists an interval $[a,b]$, where $a,b \ge t_\star$, such that $g(t) \ge 0$ and there exists $c \in (a,b)$ such that $g(t)$ has different signs on $[a,c]$ and $[c,b]$. Without loss of generality, let $g(t) \le 0$ on $[a,c]$ and $g(t) \ge 0$ on $[c,b]$.

Theorem 3.2.3. *Let* (3.4) *hold and assume that*

$$\frac{f(x)}{x|x|} \ge K, \tag{3.9}$$

for a positive constant K and for all $x \ne 0$. Furthermore assume that there exists a continuous function $u(t)$ such that $u(a) = u(b) = u(c) = 0$, $u(t)$ differentiable on the open set $(a,c) \cup (c,b)$, and satisfies the inequalities

$$\int_a^c \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2 - 2C_1 r(t)(u')^2(t)\big]d(t) \ge 0, \tag{3.10}$$

$$\int_c^b \big[\big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\big)u^2 - 2C_1 r(t)(u')^2(t)\big]d(t) \ge 0\,. \tag{3.11}$$

Then every solution of equation (3.8) *has a zero in* $[a,b]$.

Proof Assume to the contrary that $x(t)$, a solution of (3.8), has no zero in $[a,b]$. Let $x(t)<0$ for example. Using the same computations as in the first part, we obtain:

$$\begin{aligned}(r(t)\psi(x(t))v(t))' &= -\frac{(r(t)\psi(x(t))x'(t))'}{x(t)} + r(t)\psi(x(t))v^2(t) - \frac{g(t)}{x(t)}\\ &= -p(t)v(t) + q(t)\frac{f(x(t))}{x(t)} + r(t)\psi(x(t))v^2(t) - \frac{g(t)}{x(t)}\\ &= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v^2(t) - 2\frac{p(t)v(t)}{r(t)\psi(x(t))}\big)\\ &\quad + q(t)\frac{f(x(t))}{x(t)} - \frac{g(t)}{x(t)}\\ &= \frac{(r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))v(t)}\big)^2\\ &\quad - \frac{p^2(t)}{2r(t)\psi(x(t))} + q(t)\frac{f(x(t))}{x(t)} - \frac{g(t)}{x(t)}\end{aligned}$$

For $t\in[c,b]$ we have

$$\begin{aligned}(r(t)\psi(x(t))v(t))' = \frac{r(t)\psi(x(t))}{2}v^2(t) + \frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))}\big)^2\\ - \frac{p^2(t)}{2r(t)\psi(x(t))} + q(t)\frac{f(x(t))}{x(t)|x(t)|}|x(t)| + \frac{|g(t)|}{|x(t)|}\end{aligned}$$

From (3.9), and using the fact that

$$\frac{r(t)\psi(x(t))}{2}\big(v(t) - \frac{p(t)}{r(t)\psi(x(t))}\big)^2 \geq 0$$

we deduce

$$(r(t)\psi(x(t))v(t))' \geq \frac{(r(t)\psi(x(t))}{2}v^2(t) - \frac{p^2(t)}{2r(t)\psi(x(t))} + Kq(t)|x(t)| + \frac{|g(t)|}{|x(t)|}\,. \tag{3.12}$$

Using the Hölder inequality in (3.12) we obtain

$$(r(t)\psi(x(t))v(t))' \geq \frac{(r(t)\psi(x(t))}{2}v^2(t) + \sqrt{Kq(t)|g(t)|} - \frac{p^2(t)}{2r(t)\psi(x(t))}\,. \tag{3.13}$$

Multiplying both sides of this inequality by $u^2(t)$ and integrating on $[c, b]$, we obtain after using integration by parts on the left-hand side and the condition $u(c) = u(b) = 0$,

$$\begin{aligned}
0 \geq & \int_c^b \frac{r(t)\psi(x(t))}{2} v^2(t)u^2(t)dt + \int_c^b \sqrt{Kq(t)|g(t)|}u^2(t)dt \\
& - \int_c^b \frac{p^2(t)u^2(t)}{2r(t)\psi(x(t))}dt + 2\int_c^b r(t)\psi(x(t))v(t)u(t)u'(t)dt \\
\geq & \int_c^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) - 2u'(t)]^2 dt - 2\int_c^b r(t)\psi(x(t))u'^2(t)dt \\
& + \int_c^b \sqrt{Kq(t)|g(t)|}u^2(t)dt - \int_c^b \frac{p^2(t)u^2(t)}{2r(t)\psi(x(t))}dt.
\end{aligned}$$

Assumption (3.4) allows us to write

$$\begin{aligned}
0 \geq & \int_c^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2 dt - 2\int_c^b C_1 r(t)(u')^2(t)dt \\
& + \int_c^b \sqrt{Kq(t)|g(t)|}u^2(t)dt - \int_c^b \frac{p^2(t)u^2(t)}{2Cr(t)}dt \\
\geq & \int_c^b \frac{r(t)\psi(x(t))}{2}[v(t)u(t) + 2u'(t)]^2 dt \\
& + \int_c^b \Big[\Big(\sqrt{Kq(t)g|(t)|} - \frac{p^2(t)}{2Cr(t)}\Big)u^2(t) - 2C_1 r(t)(u')^2(t)\Big]dt.
\end{aligned}$$

This leads to a contradiction as in Theorem 3.2.1; the proof is complete. ■

Corollary 3.2.4. *Assume that there exist a sequence of disjoint intervals* $[a_n, b_n]$ *a sequences of points* $c_n \in (a_n, b_n)$, *and a sequence of functions* $u_n(t)$ *defined and continuous on* $[a_n, b_n]$, *differentiable on* $(a_n, c_n) \cup (c_n, b_n)$*with* $u_n(a_n) = u_n(b_n) = u_n(c_n) = 0$, *and satisfying assumptions* (3.10)-(3.11). *Let the conditions of Theorem 3.2.3 hold. Then* (3.8) *is oscillatory.*

Chapter 4

Some results concerning the oscillations of solutions of a hyperbolic equation with delays

4.1 Introduction

In this chapter, we extend the results of [3] to the equation

$$\begin{aligned} u_{tt}(x,t) &+ \alpha u_t(x,t) - [\Delta u(x,t) + \sum_{i=1}^{k} b_i(t)\Delta u(x,\sigma_i(t))] \\ &+ C(x,t,u(x,t),u(x,\tau_1(t)),u(x,\tau_2(t)),\ldots,u(x,\tau_m(t))) = f(x,t), \end{aligned} \tag{4.1}$$

where $(x,t) \in Q = \Omega \times (0,\infty)$ and Ω is a bounded domain of $\mathbb{R}^n$ with a sufficiently regular boundary Γ and Δ is the Laplacian in $\mathbb{R}^n$.

This problem is posed on Ω with one of the following types of boundary conditions:

- Dirichlet boundary conditions:

$$\text{(DBC)}: \quad u(x,t) = \psi(x,t),\ \Gamma \times (0,+\infty) \tag{4.2}$$

- Robin boundary conditions:

$$\text{(RBC)}: \quad \nabla u(x,t)\cdot \mathbf{n}(x) + \mu\, u(x,t) = \tilde{\psi},\ \Gamma \times (0,+\infty) \tag{4.3}$$

where $\psi, \tilde{\psi}, \mu \in \mathcal{C}(\Gamma \times (0,\infty))$ and $\mu \geq 0$ on $\Gamma \times (0,\infty)$.
We assume that

(H1) $b_i \in \mathcal{C}([0,\infty);[0,\infty))$, $i = 1,\dots,k$, $f(x,t) \in \mathcal{C}(\bar{Q};\mathbb{R})$, and α is constant

(H2) $\sigma_i \in \mathcal{C}([0,\infty);\mathbb{R})$, $i = 1,\dots,k$, $\tau_i \in \mathcal{C}([0,\infty;\mathbb{R})$, $i = 1,\dots,m$.

$$\lim_{t\to\infty} \sigma_i(t) = \infty,\ \sigma_i(t) \leq t,\ \forall t \in [0,\infty),\ i = 1,\dots,k \tag{4.4}$$

$$\lim_{t\to\infty} \tau_i(t) = \infty,\ \tau_i(t) \leq t,\ \forall t \in [0,\infty),\ i = 1,\dots,m \tag{4.5}$$

(H3) $C(x,t,\xi,\eta_1,\dots,\eta_m) \in \mathcal{C}(\bar{Q} \times \mathbb{R} \times \mathbb{R}^m;\mathbb{R})$,

$$C(x,t,\xi,\eta_1,\dots,\eta_m) \geq K_0^2\xi,\ \forall (x,t) \in Q,\ \xi \geq 0,\ \eta_i \geq 0, \tag{4.6}$$

$$C(x,t,\xi,\eta_1,\dots,\eta_m) \leq K_0^2\xi,\ \forall (x,t) \in Q,\ \xi \leq 0,\ \eta_i \leq 0, \tag{4.7}$$

where K_0 is some positive constant.

We define the following definitions:

$$t^\star(s) = \min_{1\leq i\leq k}\{\inf_{t\geq s} \sigma_i(t)\}, \tag{4.8}$$

$$t^{\star\star}(s) = \min_{1\leq i\leq m}\{\inf_{t\geq s} \tau_i(t)\}, \tag{4.9}$$

$$T^\star = \min\{t^\star(s), t^{\star\star}(s)\}. \tag{4.10}$$

It is useful to mention that the generalization of this section does not only consider the delay as function but also the number of delays is increased. It is also useful to mention that, in contrast of the previous section, the ((RBC) is not homogeneous.The notion of a solution is given in the following definition:

Definition 1. *A function $u \in \mathcal{C}^2(\bar{\Omega} \times (t^\star(0),\infty);\mathbb{R}) \cap \mathcal{C}(\bar{\Omega} \times (t^{\star\star}(0),\infty);\mathbb{R})$ satisfying (4.1), with (4.3) (resp. (4.2)) is said to be a solution of* (4.1) *with (4.3) (resp. (4.2)).*

4.2 Oscillation of the problem (4.1)

4.2.1 Oscillation of the problem (4.1) with Dirichlet boundary condition

Theorem 4.2.1. *Assume that* $\alpha^2-4(\lambda_1+K_0^2) < 0$, *and let* $\omega_1 = \frac{\alpha}{2}$ *and* $\omega_2 = \frac{1}{2}\sqrt{4(\lambda_1 + K_0^2) - \alpha^2}$. *If there exists a number* s *such that* $T^\star(s) \geq 0$ *and*

$$H(s) = \int_s^{s+\frac{\pi}{\omega_2}} R(t)e^{\omega_1(t-s)} \sin(\omega_2(t-s))dt = 0, \tag{4.11}$$

where

$$R(s) = F(s) - \Psi(s) - \sum_{i=1}^{k} b_i(s)\Psi(\sigma_i(s)), \tag{4.12}$$

where Ψ *is given by* (1.26).
Then any solution u *of problem (4.1) with (4.2) has a zero in* $\Omega \times (T^\star, s + \frac{\pi}{\omega_2})$.

Proof Assume the contrary. So there exists a solution u with no zero in $\Omega \times (T^\star, s + \frac{\pi}{\omega_2})$. Assume then that $u > 0$ on $\Omega \times (T^\star, s + \frac{\pi}{\omega_2})$.
Multiplying both sides of (4.1) by φ and integrating over Ω, we get for $t > 0$

$$\begin{aligned} U_{tt}(t) + \alpha U_t(t) - \int_\Omega [\Delta(x,t) + \sum_{i=1}^{k} b_i(t)\Delta U(x, \sigma_i(t))]\varphi(x)dx \\ + \int_\Omega C(x,t,u(x,t),u(x,\tau_1(t)),u(x,\tau_2(t)),\ldots,u(x,\tau_m(t)))\varphi(x)dx = F(t), \end{aligned} \tag{4.13}$$

where $F(t)$ is given by (1.24).
Using the fact that

$$\int_\Omega C(x,t,u(x,t),u(x,\tau_1(t)),\ldots,u(x,\tau_m(t)))\varphi(x)dx \geq K_0^2 U(t), \tag{4.14}$$

for all $t \in [s, s + \frac{\pi}{\omega_2})$.
Combining (4.14) with (4.13), we get

$$U_{tt}(t) + \alpha U_t(t)(\lambda_1 + K_0^2)U(t) + \lambda_1 \sum_{i=1}^{k} b_i(t)U(\sigma_i(t)) \leq F(t), \tag{4.15}$$

for $t \in [s, s + \frac{\pi}{\omega_2})$.

Since $u > 0$ on $(T^\star, s + \frac{\pi}{\omega_2})$, then $U(\sigma_i(t)) \geq 0$, for $i \in \{1, \dots, k\}$, on $[s, s + \frac{\pi}{\omega_2})$ thanks to the definition of $T^\star(s)$.
Therefore $U(t)$ is a positive solution of the following differential inequation

$$U_{tt}(t) + \alpha U_t(t)(\lambda_1 + K_0^2)U(t) \leq R(t),\ t \in [s, s + \frac{\pi}{\omega_2}), \tag{4.16}$$

where $R(t)$ is given by (4.12).
which is equivalent to

$$U_{tt}(t) + 2\omega_1 U_t(t)(\omega_1^2 + \omega_2^2)U(t) \leq R(t),\ t \in [s, s + \frac{\pi}{\omega_2}). \tag{4.17}$$

Multiplying both sides of the previous inequality by $e^{\omega_1(t-s)}\sin(\omega_2(t-s))$ and integrating both sides of the resulting inequation over $(s, s + \frac{\pi}{\omega_2})$, we get

$$0 < U(s + \frac{\pi}{\omega_2})e^{\frac{\omega_1}{\omega_2}\pi} + U(s) \leq \frac{1}{\omega_2}\int_s^{s+\frac{\pi}{\omega_2}} R(t)e^{\omega_1(t-s)}\sin(\omega_2(t-s))dt \tag{4.18}$$

Using the definition of $H(s)$ given by (4.11), we get

$$0 < U(s + \frac{\pi}{\omega_2})e^{\frac{\omega_1}{\omega_2}\pi} + U(s) \leq \frac{1}{\omega_2}H(s). \tag{4.19}$$

This with (4.11) leads to a contradiction.
If $u < 0$, we set $v = -u$ and obtain, for $V(t) = -\int_\Omega u(x,t)\varphi(x)dx$, the following inequation

$$V_{tt}(t) + 2\omega_1 V_t(t)(\omega_1^2 + \omega_2^2)V(t) \leq -R(t),\ t \in [s, s + \frac{\pi}{\omega_2}). \tag{4.20}$$

We can obtain a contradition by following the steps of the previous proof. ■

4.2.2 Oscillation of the problem (4.1) with Robin boundary condition

The following theorem gives a sufficient condition for the oscillation of (4.1) with (4.3).

Theorem 4.2.2. *Let $K_0 > 0$ and assume that $\alpha^2 - 4K_0^2 < 0$, and let $\omega_3 = \frac{1}{2}\sqrt{4K_0^2 - \alpha^2})$. If there exists a number s such that $T^\star(s) \geq 0$ and*

$$H^\star(s) = \int_s^{s+\frac{\pi}{\omega_2}} \tilde{R}(t)e^{\omega_1(t-s)}\sin(\omega_3(t-s))dt = 0, \tag{4.21}$$

where

$$\tilde{R}(s) = \tilde{F}(s) + \tilde{\Psi}(s) - \sum_{i=1}^{k} b_i(s)\tilde{\Psi}(\sigma_i(s)). \tag{4.22}$$

Then any solution u of problem (4.1) with (4.3) has a zero in $\Omega \times (T^\star, s + \frac{\pi}{\omega_3})$.

Proof The proof of this Theorem is similar to that of Theorem 4.2.1 ■

4.3 Example

Let us consider the following one dimensional problem

$$u_{tt}(x,t)+2u_t(x,t)-[u_{xx}(x,t)+2u_{xx}(x,t-\pi)]+u(x,t) = 4\sin x\cos 2t,\ \ (x,t) \in (0,\pi)\times(0,\infty), \tag{4.23}$$

with the Dirichlet boundary condition

$$u(0,t) = u(\pi,t) = 0,\ \ t > 0. \tag{4.24}$$

Here $n = 1$, $\Omega = (0,\pi)$, $\alpha = 2$, $k = 1$, $b_1(t) = 2$, $\sigma_1(t) = t - \pi$, $K_0 = 1$, $\psi = 0$, and $f(x,t) = 4\sin x\cos 2t$.
It is easily seen that $\omega_1 = \omega_2 = 1$, $T^\star(s) = \inf_{t\geq s}(t-s) = s-\pi$ and $F(x,t) = \int_0^\pi 4\sin x\cos 2t dx = 2\pi\cos 2t$.
On the other hand, since $\psi(t) = 0$, we have $R(t) = F(t)$.
Some computation gives us

$$\begin{aligned} H(s) &= \int_s^{s+\frac{\pi}{\omega_2}} 4\cos 2te^{t-s}\sin(t-s)dt \\ &= \frac{1}{\sqrt{5}}(e^\pi + 1)\sin(2s - \theta_0), \end{aligned} \tag{4.25}$$

where $\theta_0 = tg^{-1}(\frac{1}{2})$ $(0 < \theta_0 < \frac{\pi}{2})$.
Therefore $H(s) = 0$ for $s = s_n = \frac{\theta_0}{2} + n\frac{\pi}{2}$ $(n = 2, 3, \ldots)$.
Theorem 4.2.1 implies that any solution of (4.23)–(4.24) has a zero on $\Omega \times (s_n - \pi, s_n + \pi)$.
Such solution is $u(x,t) = \sin x \sin 2t$.

Perspectives

Although a huge number of articles has been devoted to the oscillation theory of differential equations, some interesting problems have not attracted the attention it merits yet.
Let us quote some of these problems:

1. Find conditions for non existence of oscillatory solutions of functional differential equations (FDEs).
   It is well known that there is a considrable number of articles devoted to find the sufficient conditions such that all the solutions of the following second order ordinary differential equation are not oscillatory:
$$y'' + p(t)f(y) = 0, \tag{1}$$
   whereas there is lack of such results concerning the second order FDE:
$$y''(t) + p(t)f(y(r(t))) = 0. \tag{2}$$
   The technique used to look for the sufficient conditions in order that all the solutions of (1) are oscillatory is based in general on the constuction of the energy function and on the use of the uniqueness of the zero solution.
   It is well known that all the solutions of
$$y'(t) + q(t)y(t) + p(t)y([t]) = 0, \tag{3}$$
   are nonoscillatory, if $p(t) \leq 0$ or $p(t) \geq 0$ and $\limsup_{n\to+\infty} \int_n^{n+1} p(t)exp\left(\int_n^t q(s)ds\right) dt < 1$ where $[t]$ denotes the integer part function.
   There is no such result even for
$$y'(t) + q(t)y(t) + p(t)y(\tau(t)) = 0, \tag{4}$$

where $\tau(t) \leq t$ ($\tau(t) \not\equiv t$) and $\lim_{t\to\infty} \tau(t) = \infty$.

2. Find conditions for the existence of oscillatory solutions of FDE.

3. Study oscillation and non oscillation problems for delay systems.

4. Find conditions for the non oscillation caused by delays.
   It is easy to see that every solution $(x(t), y(t))$ of

$$\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = x(t) + y(t) \end{cases} \tag{5}$$

   is oscillatory. If we consider the corresponding delay system of the form

$$\begin{cases} x'(t) = 2x(t) - y\left(t - \frac{1}{3}\log 4\right) \\ y'(t) = x\left(t - \frac{1}{3}\log 4\right) + y(t), \end{cases} \tag{6}$$

   then it is easy to find that (6) has a non oscillatory solution $x(t) = \exp(\frac{3}{2}t)$, $y(t) = \exp(\frac{3}{2}t)$.
   This non oscillation is caused by the delay.
   But there is no further result concerning this problem till now.

5. Find further relation between the oscillation theory of second order FDE and the corresponding boundary value problems.

6. Study the distribution of zeros, the variations of amplitude and the asymptotic behavior of oscillatory solutions.

7. Study some special FDE, which are posed by practical applications. For example, consider the oscillation problems for the equations with delays, which depends on the states, such as

$$y^{(n)}(t) + p(t)y\left(t - r(t)\right) = 0. \tag{7}$$

   Even the case $n = 1$ is interesting to be considered.
   There are also other paths to be followed like partial differential equations and systems of equations with delays depending on the states.

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